Ionization for Three Dimensional Time-dependent Point Interactions

Michele Correggi* Gianfausto Dell'Antonio† Rodolfo Figari‡ Andrea Mantile§

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Abstract

We study the time evolution of a three dimensional quantum particle under the action of a time-dependent point interaction fixed at the origin. We assume that the "strength" of the interaction $\alpha(t)$ is a periodic function with an arbitrary mean. Under very weak conditions on the Fourier coefficients of $\alpha(t)$, we prove that there is complete ionization as $t \to \infty$, starting from a bound state at time $t = 0$. Moreover we prove also that, under the same conditions, all the states of the system are scattering states.

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1 Introduction

We shall study the time evolution of a three dimensional system with time-dependent Hamiltonian given by

$$H(t) = H_0 + H_I(t)$$

where the "perturbation" $H_I(t)$ is a zero-range interaction with time-dependent (periodic) "strength". In particular we are interested in proving complete ionization of the system as $t \to \infty$, starting from an initial condition at $t = 0$ given by a bound state of the system. By complete ionization one can mean two different statements. The weaker one is that the survival probability of the

*International School of Advanced Studies SISSA/ISAS, Trieste, Italy. E-mail address: correggi@sissa.it

†Centro Linceo Interdisciplinare. On leave from Dipartimento di Matematica, Università di Roma, "La Sapienza", Italy. E-mail address: gianfa@sissa.it

‡Università di Napoli "Federico II", Italy. E-mail address: rodolfo.figari@unina.it

§Università di Napoli "Federico II", Italy. E-mail address: andrea.mantile@dma.unina.it

bound state, i.e. the scalar product of the state at time t with the bound state, goes to zero as $t \to \infty$. The stronger one is that every state Ψ in the Hilbert space of the system is a scattering state (see for example [11, 13]) of $H(t)$, i.e. for every compact set $S \subset \mathbb{R}^3$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \int_S d^3\vec{x} \, \left|\Psi_\tau(\vec{x})\right|^2 = 0$$

Ψ_t denoting the time evolution of the state Ψ. The last statement is related to the absence of eigenvalues of the Floquet operator associated to $H(t)$ (see [14, 18]).

The usual way to deal with problems of this kind is by means of time-dependent perturbation theory and Fermi's golden rule, which gives for the survival probability the well known exponential decay for each term of order n of the perturbative expansion. On the other hand simple examples of regular perturbations show that the survival probability decays to zero as a power-law (i.e. the limits $t \to \infty$ and $n \to \infty$ can not be interchanged). When the perturbation is not small, it is in general very difficult to solve the problem and find the law of decay. Therefore it is interesting to find models in which a non-perturbative solution exists and study the survival probability. In this paper we study one such model, in which $H_I(t)$ is given by a three dimensional point interaction. We shall see that it is possible to prove asymptotic complete ionization and find a power law decay (with exponent $-3/2$) for the survival probability, under generic condition on the scattering length[1].

The one-dimensional version of the same problem has been widely analyzed in [4, 5, 6, 7], where complete ionization is proved under a suitable and very weak condition on the Fourier coefficients of the strength of the interaction. We shall see that the same *genericity* condition is also sufficient in the three dimensions to have complete ionization of the system.

From a physical point of view, the model we are going to study is related to the strong laser ionization of Rydberg atoms[2], showing many features of experimental data. Indeed, despite of the simplicity of the model, as in the one-dimensional case, it is possible to reproduce many effects of multiphoton ionization of excited hydrogen atoms by microwave field, with a good agreement with experiments (see [8]).

2 The model

The model we are going to study is a quantum particle subjected to a time-dependent point interaction fixed at the origin in three dimensions, namely a system defined by the time-dependent self-adjoint Hamiltonian $H_{\alpha(t)}$,

$$\mathcal{D}(H_{\alpha(t)}) = \Big\{ \Psi \in L^2(\mathbb{R}^3) \;\big|\; \exists\, q_\lambda(t) \in \mathbb{C}, \big(\Psi(\vec{x}) - q_\lambda(t)\, \mathcal{G}^\lambda(\vec{x})\big) \in H^2(\mathbb{R}^3), \\ \big(\Psi - q_\lambda(t)\, \mathcal{G}^\lambda\big)\big|_{\vec{x}=0} = \Big(\alpha(t) + \frac{\lambda}{4\pi}\Big)\, q_\lambda(t) \Big\} \tag{2.1}$$

[1] In three dimensions the parameter $\alpha(t)$ is proportional to the inverse of the scattering length.

[2] See the discussion contained in [4, 8] and references therein.

$$\big(H_{\alpha(t)} + \lambda\big)\Psi = \big(H_0 + \lambda\big)\big(\Psi - q(t)\,\mathcal{G}^\lambda\big) \tag{2.2}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$,

$$\mathcal{G}^\lambda(\vec{x} - \vec{x}') = \frac{e^{-\sqrt{\lambda}|\vec{x}-\vec{x}'|}}{4\pi|\vec{x} - \vec{x}'|}$$

is the Green function of the free Hamiltonian $H_0 = -\Delta$.
The operator[3] (2.2) has absolutely continuous spectrum if $\alpha(t)$ is positive, while, when $\alpha(t) < 0$, there exists exactly one negative eigenvalue $-(4\pi\alpha(t))^2$, , with normalized eigenfunction

$$\varphi_{\alpha(t)}(\vec{x}) \equiv \frac{\sqrt{2|\alpha(t)|}\, e^{4\pi\alpha(t)|\vec{x}|}}{|\vec{x}|} \tag{2.3}$$

It is well known (see [9, 10, 12, 16, 17]) that the operator (2.2) defines a time propagation $U(t,s)$ given by a two-parameters unitary family, solving the time-dependent Shrödinger equation

$$i\frac{\partial \Psi_t}{\partial t} = H_{\alpha(t)}\Psi_t \tag{2.4}$$

and

$$\Psi_t(\vec{x}) = U(t,s)\,\Psi_s(\vec{x}) = U_0(t-s)\Psi_s(\vec{x}) + i\int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x}) \tag{2.5}$$

where $U_0(t) = \exp(-iH_0 t)$, $U_0(t;\vec{x})$ is the kernel associated to the free propagator and the charge $q(t)$ satisfies a Volterra integral equation for $t \geq s$,

$$q(t) + 4\sqrt{\pi i}\int_s^t d\tau\, \frac{\alpha(\tau)q(\tau)}{\sqrt{t-\tau}} = 4\sqrt{\pi i}\int_s^t d\tau\, \frac{\big(U_0(\tau)\Psi_s\big)(0)}{\sqrt{t-\tau}} \tag{2.6}$$

We are interested in studying complete ionization of system defined by (2.2) and (2.4), starting by initial conditions

$$\Psi_0(\vec{x}) = \varphi_{\alpha(0)}(\vec{x}) \tag{2.7}$$

$\varphi_{\alpha(0)}(\vec{x})$ being the bound state[4] of $H_{\alpha(0)}$.
The meaningful parameter of the system is the lower bound of $\alpha(t)$. Indeed, if $\inf(\alpha(t)) \geq 0$, it is not difficult to see (see the remark at the end of section 5) that the asymptotic complete ionization is a straightforward consequence of the explicit expression of the evolution. Therefore we shall assume that the minimum of $\alpha(t)$ is negative and, since the choice of the initial time is arbitrary, we can require that the following conditions hold:

$$\begin{array}{ll} 1. & \alpha(t),\ t \geq 0,\ \text{is a continuous periodic function with period } T \\ 2. & \alpha(t) \in \mathbb{R}\,,\ \forall t \in \mathbb{R}^+ \\ 3. & \alpha(0) = \inf\limits_{t\in\mathbb{R}}(\alpha(t)) < 0 \end{array} \tag{2.8}$$

Condition 1 guarantees that $\alpha(t)$ can be decomposed in a Fourier series, for each $t \in \mathbb{R}^+$, and the series converges uniformly on every compact subset of the real

[3] For a general review about point interactions see [1, 2] and references therein.
[4] In order to do this analysis we shall require that $\alpha(0) < 0$.

line, since $\alpha(t)$ is continuous, so that we can rewrite conditions (2.8) in terms of Fourier coefficients of $\alpha(t)$,

$$\begin{aligned}
&1. \quad \alpha(t) = \sum_{n\in\mathbb{Z}} \alpha_n \, e^{-in\omega t} \ , \ \{\alpha_n\} \in \ell_2(\mathbb{Z}) \\
&2. \quad \alpha_n = \alpha^*_{-n} \\
&3. \quad \sum_{n\in\mathbb{Z}} \alpha_n < 0
\end{aligned} \tag{2.9}$$

Equation (2.6) is of the form $q = \mathcal{L}q + f$ and, considering the spectral properties of $\mathcal{L}$ in $L^2(\mathbb{R}^+)$, it is easy to prove that the charge $q(t)$ has at most an exponential behavior as $t \to \infty$, i.e. asymptotically $q(t) \le Ae^{bt}$:

Proposition 2.1 *There exists $A > 0$ such that, $\forall\, b > b_0$ and $\forall\, t \in \mathbb{R}^+$,*

$$|q(t)| \le Ae^{bt} \tag{2.10}$$

where

$$b_0 \equiv \big(4\pi\|\alpha\|_\infty\big)^2 = \big(4\pi \sup_{t\in\mathbb{R}^+} |\alpha(t)|\big)^2$$

Proof: Let us do the substitution $q(t) = r(t)e^{bt}$ in equation (2.6):

$$r(t) = -4\sqrt{\pi i} \int_0^t ds \, \frac{e^{-b(t-s)}}{\sqrt{t-s}} \, \alpha(s)r(s) + f(t)e^{-bt}$$

From the explicit expression of the function $f(t)$, we can easily obtain that $g(t) \equiv f(t)e^{-bt}$ is a bounded function for each $t \in \mathbb{R}^+$, i.e.

$$\|g\|_\infty = C < \infty$$

so that, from the equation above, one has

$$\|r\|_\infty \le 4\sqrt{\pi} \sup_{t\in\mathbb{R}^+} \left| \int_0^\infty ds \, \frac{\theta(t-s)e^{-(t-s)}}{\sqrt{t-s}} \, \alpha(s)r(s) \right| + C$$

and

$$\sup_{t\in\mathbb{R}^+} \left| \int_0^\infty ds \, \frac{\theta(t-s)e^{-b(t-s)}}{\sqrt{t-s}} \, \alpha(s)r(s) \right| \le A(b) \, \|\alpha\|_\infty \|r\|_\infty$$

where

$$A(b) = \int_0^\infty ds \, \frac{e^{-bs}}{\sqrt{s}} = \sqrt{\frac{\pi}{b}}$$

In conclusion

$$\left[1 - \frac{4\pi\|\alpha\|_\infty}{\sqrt{b}}\right] \|r\|_\infty \le C$$

and therefore r is bounded, if $b > (4\pi\|\alpha\|_\infty)^2$.

$\square$

Therefore the Laplace transform of $q(t)$, denoted by

$$\tilde{q}(p) \equiv \int_0^\infty dt\, e^{-pt} q(t)$$

exists analytic at least for $\Re(p) > b_0$. Hence, applying the Laplace transform to equation (2.6), one has

$$\tilde{q}(p) = -4\pi\sqrt{\frac{i}{p}} \sum_{k\in\mathbb{Z}} \alpha_k\, \tilde{q}(p+i\omega k) + f(p) \tag{2.11}$$

where

$$f(p) \equiv \frac{2\sqrt{2|\alpha(0)|}}{\pi} \sqrt{\frac{i}{p}} \int_0^\infty dt\, e^{-pt} \int_{\mathbb{R}^3} d^3\vec{k}\, \frac{e^{-ik^2 t}}{k^2 + (4\pi\alpha(0))^2} =$$

$$= 8\sqrt{\frac{2|\alpha(0)|}{ip}} \int_0^\infty dk\, \frac{k^2}{(k^2+(4\pi\alpha(0))^2)(k^2-ip)} =$$

$$= -4\pi\sqrt{\frac{2i|\alpha(0)|}{p}}\, \frac{4\pi|\alpha(0)| - \sqrt{-ip}}{(4\pi\alpha(0))^2 + ip}$$

In the following sections we shall prove asymptotic complete ionization of the system under generic conditions on $\alpha(t)$. Although the result does not depend on the sign of the mean α_0 of $\alpha(t)$, we have to discuss separately the case $\alpha_0 < 0$ and $\alpha_0 \geq 0$, because of the slightly different form of equation (2.11).

3 CASE I: $\alpha_0 < 0$

Since $\alpha(0) < 0$, changing the energy scale, it is always possible to assume that $\alpha(t)$ satisfies the normalization

$$4. \quad \alpha(0) = \sum_{n\in\mathbb{Z}} \alpha_n = -\frac{1}{4\pi} \tag{3.1}$$

Moreover we introduce another condition we shall use later on: let $\mathcal{T}$ the right shift operator on $\ell_2(\mathbb{N})$, i.e.

$$\big(\mathcal{T}a\big)_n \equiv a_{n+1} \tag{3.2}$$

we say that $\alpha = \{\alpha_n\} \in \ell_2(\mathbb{Z})$ is *generic* with respect to $\mathcal{T}$, if $\tilde{\alpha} \equiv \{\alpha_n\}_{n>0} \in \ell_2(\mathbb{N})$ satisfies the following condition

$$e_1 = \big(1,0,0,\dots\big) \in \overline{\bigvee_{n=0}^{\infty} \mathcal{T}^n \tilde{\alpha}} \tag{3.3}$$

For a detailed discussion of genericity condition see [4].
If (3.1) holds, equation (2.11) becomes (at least for $\Re(p) > b_0$)

$$\tilde{q}(p) = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{-ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, \tilde{q}(p+i\omega k) - \frac{2\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{-ip}}\, \frac{1-\sqrt{-ip}}{1+ip} \tag{3.4}$$

with the choice of the branch cut for the square root along the negative real line: if $p = \varrho\, e^{i\vartheta}$,

$$\sqrt{p} = \sqrt{\varrho}\ e^{i\vartheta/2} \tag{3.5}$$

with $-\pi < \vartheta \leq \pi$.

3.1 Analyticity on the (open) right half plane

Let us extend equation (3.4) on the whole open right half plane: we are going to prove that the solution exists and is analytic for $\Re(p) > 0$.
Setting $q_n(p) \equiv \tilde{q}(p + i\omega n)$, we obtain a sequence of functions on the strip $\mathcal{I} = \{p \in \mathbb{C},\ 0 \leq \Im(p) < \omega\}$ and calling

$$q(p) \equiv \{q_n(p)\}_{n\in\mathbb{Z}}$$

equation (3.4) can be rewritten

$$q(p) = \mathcal{L}(p)\, q(p) + g(p) \tag{3.6}$$

where

$$\big(\mathcal{L}q\big)_n(p) \equiv -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, q_{n+k}(p) \tag{3.7}$$

and $g(p) = \{g_n(p)\}_{n\in\mathbb{Z}}$ with

$$g_n(p) \equiv -\frac{2\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{-ip}}\,\frac{1-\sqrt{-ip}}{1+ip} \tag{3.8}$$

Proposition 3.1 *For $p \in \mathcal{I}$, $\Re(p) > 0$, $\mathcal{L}(p)$ is an analytic operator-valued function and $\mathcal{L}(p)$ is a compact operator on $\ell_2(\mathbb{Z})$.*

Proof: Analyticity for $\Re(p) > 0$ easily follows from the explicit expression of the operator.
Moreover $\mathcal{L}(p)$ can be written

$$\mathcal{L}(p) = c(p) \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, \mathcal{T}^{n+k}$$

where $c(p)$ is the operator

$$(cq)_n(p) \equiv c_n(p)\, q_n(p) = \frac{4\pi q_n(p)}{4\pi\alpha_0 + \sqrt{\omega n - ip}}$$

and $\mathcal{T}$ is the right shift operator on $\ell_2(\mathbb{Z})$.
Since $\|\mathcal{T}\| = 1$, the series converges strongly to a bounded operator. Moreover $c(p)$ is a compact operator for $\Re(p) > 0$: $c(p)$ is the norm limit of a sequence of finite rank operators, because $\lim_{n\to\infty} c_n(p) = 0$. Hence the result follows for example from Theorem VI.12 and VI.13 of [15].

□

Proposition 3.2 *The solution $\tilde{q}(p)$ of (3.4) exists and is analytic for $\Re(p) > 0$.*

Proof: The key point will be the application of the analytic Fredholm theorem to the operator $\mathcal{L}(p)$ (Theorem VI.14 of [15]), in order to prove that $(I-\mathcal{L}(p))^{-1}$ exists for $\Re(p)>0$.
So let us begin with the analysis of the homogeneous equation associated to (3.6),

$$q(p)=\mathcal{L}(p)\,q(p)$$

and suppose that there exists a nonzero solution $Q(p)=\{Q_n(p)\}_{n\in\mathbb{Z}}$. Multiplying both sides of the equation by Q_n^* and summing over $n\in\mathbb{Z}$, $n\neq 0$ we have

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(4\pi\alpha_0+\sqrt{\omega n-ip}\right)|Q_n|^2=-4\pi\sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq 0}}Q_n^*\,\alpha_{k-n}\,Q_k$$

but, since the right hand side is real, because of condition 2 in (2.9), it follows that

$$\Im\Bigg(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(4\pi\alpha_0+\sqrt{\omega n-ip}\right)|Q_n|^2\Bigg)=0$$

On the open right half plane, the imaginary part of $\sqrt{\omega n-ip}$ is strictly negative $\forall n\in\mathbb{Z}$: $\Im(\omega n-ip)=-\Re(p)<0$, so that $\omega n-ip=\varrho e^{i\vartheta}$ with $-\pi<\vartheta<0$ and

$$\Im\left(\sqrt{\omega n-ip}\right)=\Im\left(\sqrt{\varrho}\,e^{i\vartheta/2}\right)=\sqrt{\varrho}\,\sin(\vartheta')<0$$

because $-\pi/2<\vartheta'=\vartheta/2<0$. In conclusion $Q_n=0$, for every $n\in\mathbb{Z}$, $n\neq 0$ and then by (3.7) $Q_0=0$ too.
Since there is no nonzero solution of the homogeneous equation associated to (3.6) and $\mathcal{L}$ is compact on the whole open right half plane, analytic Fredholm theorem applies and the result then easily follows, because $g(p)\in\ell_2(\mathbb{Z})$ and each $g_n(p)$ is analytic for $\Re(p)>0$.

□

3.2 Behavior on the imaginary axis at $p\neq 0$

From equation (3.6),

$$q_n(p)=-\frac{4\pi}{4\pi\alpha_0+\sqrt{\omega n-ip}}\left\{\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\alpha_k\,q_{n+k}(p)+\frac{1-\sqrt{\omega n-ip}}{\sqrt{2\pi}\,(1+ip-\omega n)}\right\}\tag{3.9}$$

and discussion contained in Propositions 3.1 and 3.2, it is clear that the solution can have poles only at $p=((4\pi\alpha_0)^2-\omega n_0)i$, for some $n_0\in\mathbb{Z}$.
Since $\Im(p)\in[0,\omega)$, one has

$$\frac{(4\pi\alpha_0)^2}{\omega}-1<n_0\leq\frac{(4\pi\alpha_0)^2}{\omega}\tag{3.10}$$

and then the singularity appears at most in the equation for q_{n_0} (there is only one integer which satisfies the previous inequality) at $p=((4\pi\alpha_0)^2-\omega n_0)i$. For example if $\omega>(4\pi\alpha_0)^2$, the pole may be at $p=(4\pi\alpha_0)^2 i$ in the equation for

q_0.
Actually we have to distinguish the so called (see [4]) resonant case, i.e. when

$$(4\pi\alpha_0)^2 = N\omega$$

for some $N \in \mathbb{N}$, because in that case we can have a pole only at $p = 0$ and then the solution is immediately seen to be analytic on the whole imaginary axis except at most for $p = 0$.
In the following we are going to prove that, under suitable condition on the Fourier coefficients of $\alpha(t)$, the solution has actually no pole singularity on the imaginary axis (except at most a pole at $p = 0$): let us start with a preliminary useful result

Lemma 3.1 *Let (2.9) and the genericity condition (3.3) be satisfied by $\{\alpha_n\}$. The system of equations*

$$r_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \Bigg\{ \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k r_{n+k} + h_n(p) \Bigg\} \tag{3.11}$$

has a unique solution $\{r_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$ in a pure imaginary neighborhood of $p = (4\pi\alpha_0)^2 i$, for every $h_n(p)$ such that

$$h'_n(p) \equiv \frac{h_n(p)}{4\pi\alpha_0 + \sqrt{\omega n - ip}}$$

belongs to $\ell_2(\mathbb{Z} \setminus \{0\})$.
Moreover, if $h_n(p)$ is analytic in a neighborhood of $p = (4\pi\alpha_0)^2 i$, the solution is analytic in the same neighborhood.

Proof: Equation (3.13) is of the form

$$r = \mathcal{L}'r + h'$$

where $h' \equiv \{h'_n\}$ belongs to $\ell_2(\mathbb{Z} \setminus \{0\})$ and $\mathcal{L}'$ is a compact operator (see Proposition 3.1).
In order to apply analytic Fredholm theorem to the operator $\mathcal{L}'$, we need to prove that there is no nonzero solution in a neighborhood of $p = (4\pi\alpha_0)^2 i$ of the homogeneous equation. Suppose that the contrary is true, so that $\{R_n\} \in \ell_2(\mathbb{Z} \setminus \{0\}$ is a nonzero solution of

$$R_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k R_{n+k}$$

Multiplying both sides of equation above by R_n^* and summing over $n \in \mathbb{Z} \setminus \{0\}$, one has

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(4\pi\alpha_0 + \sqrt{\omega n - ip}\right) \left|R_n\right|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq 0}} {R_n}^* \alpha_{k-n} R_k$$

The right hand side is real, so

$$\Im\Bigg(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(4\pi\alpha_0 + \sqrt{\omega n - ip}\right) |R_n|^2 \Bigg) = 0$$

Since we are interested in the behavior of the solution on the imaginary axis we can take $s = -i(p - (4\pi\alpha_0)^2 i)$ in a real neighborhood of 0 and, setting $n_0 = \frac{-s-(4\pi\alpha_0)^2}{\omega}$, we have

$$\Im\big(\sqrt{(4\pi\alpha_0)^2 + s + \omega n}\big) = 0$$

if $n \geq n_0$ and

$$\Im\big(\sqrt{(4\pi\alpha_0)^2 + s + \omega n}\big) < 0$$

otherwise, with the choice (3.5) for the branch cut of $\sqrt{p}$.
Therefore $R_n = 0$, $\forall n < n_0$, in a real neighborhood of $s = 0$.
Suppose that $R \neq 0$ and let $N \in \mathbb{N}$ be such that $R_n = 0$, $n < N$, and $R_N \neq 0$ (hence $N \leq n_0$). For each $n < N$,

$$\sum_{\substack{k\in\mathbb{Z}\\ k\geq N, k\neq 0}} \alpha_{k-n} R_k = 0$$

or, setting $k = N - 1 + k'$, for $n \geq 0$,

$$\sum_{\substack{k'\in\mathbb{Z}\\ k'\geq 1, k'\neq 1-N}} \alpha_{k'+n} R_{N-1+k'} = 0$$

which implies (see (2.9) and(3.1)), for each $n \geq 0$,

$$\Big(R' \,, \mathcal{T}^n \alpha\Big)_{\ell_2(\mathbb{N})} = 0$$

where $R'_n = R^*_{N-1+n}$ and $(\cdot\,,\,\cdot)$ stands for the standard scalar product on $\ell_2(\mathbb{N})$.
If $\{\alpha_n\}$ satisfies the genericity condition (3.3), R' has to be orthogonal also to e_1 and then $R_N = 0$, which is a contradiction. Therefore $R = 0$.
The first part of the Lemma then follows from analyticity of $\mathcal{L}'(p)$ and analytic Fredholm theorem. Moreover if $\{h_n(p)\}$ is analytic in a neighborhood of $p = (4\pi\alpha_0)^2 i$, analyticity of the solution is a straightforward consequence.

□

Proposition 3.3 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition with respect to $\mathcal{T}$ (3.3), the solution of (3.6) has no pole for $p = i\lambda$, $0 < \lambda < \omega$.*

Proof: First let us assume that $\omega > (4\pi\alpha_0)^2$: the case $\omega < (4\pi\alpha_0)^2$ and the resonant one will be analyzed in the end.
The strategy of the proof is to remove the contribution of the term q_0, which may be singular, from (3.9), $n \neq 0$, using a change of variable and then separately prove that in fact q_0 is analytic in a neighborhood of $p = (4\pi\alpha_0)^2 i$.
By Lemma 3.1 there is a unique solution of the system

$$t_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k t_{n+k} - \frac{4\pi\alpha_{-n}}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \tag{3.12}$$

Setting $q_n = r_n + t_n q_0$, $n \neq 0$, on (3.9), one has

$$r_n + t_n q_0 = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}}\Big\{\alpha_{-n} q_0 + \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k \big(r_{n+k} + t_{n+k} q_0\big)\Big\} +$$

$$-\frac{2\sqrt{2\pi}}{4\pi\alpha_0+\sqrt{\omega n-ip}}\frac{1-\sqrt{\omega n-ip}}{1+ip-\omega n}$$

and therefore the equation for $\{r_n\}$, $n\neq 0$, becomes

$$r_n=-\frac{4\pi}{4\pi\alpha_0+\sqrt{\omega n-ip}}\left\{\sum_{\substack{k\in\mathbb{Z}\\k\neq 0,-n}}\alpha_k r_{n+k}+\frac{1-\sqrt{\omega n-ip}}{\sqrt{2\pi}(1+ip-\omega n)}\right\} \tag{3.13}$$

while q_0 satisfies the equation

$$q_0=-\frac{4\pi}{4\pi\alpha_0+\sqrt{-ip}}\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\alpha_k\big(r_k+t_kq_0\big)-\frac{2\sqrt{2\pi}}{4\pi\alpha_0+\sqrt{-ip}}\frac{1-\sqrt{-ip}}{1+ip}$$

or

$$\left[4\pi\alpha_0+\sqrt{-ip}+4\pi\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\alpha_kt_k\right]q_0=-4\pi\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\alpha_kr_k-\frac{2\sqrt{2\pi}(1-\sqrt{-ip})}{1+ip}$$

Since $\frac{1-\sqrt{-ip}}{1+ip}$ is analytic in a neighborhood of $p=(4\pi\alpha_0)^2i$ and $\{t_n\}$, $\{r_n\}\in\ell_2(\mathbb{Z}\setminus\{0\})$ are both analytic, as it follows applying Lemma 3.1 above to (3.12) and (3.13), it is sufficient to prove that

$$\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\alpha_k\tilde{t}_k\neq 0$$

where

$$\tilde{t}_n\equiv t_n(p)\big|_{p=(4\pi\alpha_0)^2i}$$

Assume that the contrary is true: from equation (3.12) we obtain

$$\sum_{\substack{n\in\mathbb{Z}\\n\neq 0}}\big(4\pi\alpha_0+\sqrt{\omega n+(4\pi\alpha_0)^2}\big)\left|\tilde{t}_n\right|^2=-4\pi\sum_{\substack{n,k\in\mathbb{Z}\\n,k\neq 0}}\tilde{t}_n^*\alpha_{k-n}\tilde{t}_k-4\pi\sum_{\substack{n\in\mathbb{Z}\\n\neq 0}}\alpha_n^*\tilde{t}_n^*=$$

$$=-4\pi\sum_{\substack{n,k\in\mathbb{Z}\\n,k\neq 0}}\tilde{t}_n^*\alpha_{k-n}\tilde{t}_k$$

where we have used condition 2 in (2.9). The previous equation implies (the right hand side is real) $\tilde{t}_n=0$, $\forall n<N_0=-\frac{(4\pi\alpha_0)^2}{\omega}$ and, from (3.12) we have, $\forall n<N_0$,

$$\sum_{\substack{k\leq N_0\\k\neq 0}}\alpha_{k-n}\tilde{t}_k+\alpha_{-n}=0$$

Now setting $T_n=\tilde{t}_n$, $n\neq 0$, and $T_0=1$, we obtain, at least $\forall n\geq 0$,

$$\sum_{k\geq N_0}^{\infty}\alpha_{k+n}T_k=0$$

and using the genericity condition (3.3) (as in the proof of Lemma 3.1) we get $\{T_n\} = 0$, which is a contradiction.
In conclusion q_0 is analytic in a neighborhood of $p = (4\pi\alpha_0)^2 i$: analyticity of q_n, $n \neq 0$ is then a straightforward consequence of analyticity of $\{r_n\}$, $\{t_n\}$ and decomposition $q_n = r_n + t_n q_0$. The proof is then completed, since r_n and t_n belong to $\ell_2(\mathbb{Z} \setminus \{0\})$ in a neighborhood of $p = (4\pi\alpha_0)^2 i$.
The non-resonant case $\omega < (4\pi\alpha_0)^2$ can be treated in the same way: let n_0 be the unique integer that satisfies (3.10), setting $q'_n = q_{n+n_0}$, $n \in \mathbb{Z}$ and $p' = p + i\omega n_0$, it is easy to see that $\{q'_n(p')\}$ satisfies the system of equations (3.9) with $p \to p'$, the possible pole is at $p' = (4\pi\alpha_0)^2 i$ and it may explicitly appear only in the equation for q'_0. We can then apply the same argument and show that the result still holds.
On the other hand in the resonant case $(4\pi\alpha_0)^2 = N\omega$, for some $N \in \mathbb{N}$, all the coefficients of equation (3.9) are analytic on the open subset of the imaginary axis $0 < \Im(p) < \omega$ and the proof of Proposition 3.2 applies.

□

3.3 Behavior at $p = 0$

We shall now study the behavior of the solution of (3.9) on the imaginary axis at the origin. With the choice (3.5) for the branch cut of the square root, it is clear that we must expect branch points of $\tilde{q}(p)$, solution of (3.4), at $p = i\omega n$, $n \in \mathbb{Z}$, which should imply a branch point at $p = 0$ for each q_n in (3.9).
We are going to show that q_n, $n \in \mathbb{N}$ has a branch point at $p = 0$. The non-resonant case and the resonant one will be treated separately.

Proposition 3.4 (non-resonant case)
If $(4\pi\alpha_0)^2 \neq N\omega$, $\forall N \in \mathbb{N}$ and $\{\alpha_n\}$ satisfies (2.9) and (3.3) (genericity condition), the solution of equation (3.9) has the form $q_n(p) = c_n(p) + d_n(p)\sqrt{p}$, $n \in \mathbb{Z}$, in an imaginary neighborhood of $p = 0$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$.

Proof: Setting $q_n = r_n + t_n q_0$, $n \neq 0$ and choosing a non-zero solution $\{t_n\}$ of the system of equations (3.12), we obtain that $\{r_n\}$ must satisfy (3.13). It is easy to see that the result of Lemma 3.1 holds also in a neighborhood of $p = 0$, so that $\{r_n\}, \{t_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$ are unique and analytic at $p = 0$.
Thus it is sufficient to prove that q_0, which is solution of

$$\left[4\pi\alpha_0 + \sqrt{-ip} + 4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k t_k\right] q_0 = -4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k r_k - \frac{2\sqrt{2\pi}(1 - \sqrt{-ip})}{1 + ip}$$

has the required behavior near $p = 0$.
First, setting $t_n^0 = t_n(p = 0)$, we have to prove that

$$\sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k t_k^0 \neq -\alpha_0$$

but, assuming that the contrary is true and multiplying both sides of equation (3.12) by ${t_n^0}^*$ and summing over $n \in \mathbb{Z}$, $n \neq 0$, one has

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(4\pi\alpha_0 + \sqrt{\omega n}\right)|t_n^0|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq 0}} {t_n^0}^* \alpha_{k-n} t_k^0 + 4\pi\alpha_0$$

The right hand side is still real so that, assuming that the genericity condition is satisfied by $\{\alpha_n\}$ and applying the argument about $\{\tilde{t}_n\}$ contained in the proof of Proposition 3.3, we immediately obtain $\{t_n^0\} = 0$, which is a contradiction. Now, calling

$$F \equiv 4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k t_k$$

and

$$G \equiv -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k r_k$$

we have

$$\left[4\pi\alpha_0 + \sqrt{-ip} + F\right] q_0 = G + \frac{2\sqrt{2\pi}(1-\sqrt{-ip})}{1+ip}$$

and

$$q_0 = F' + \sqrt{p}\ G'$$

where F' is analytic in a neighborhood of $p = 0$, because of analyticity of F and G, and

$$G' \equiv \frac{2\sqrt{-2\pi i}\,(4\pi\alpha_0 + F + 1) - \sqrt{-i}\,(1+ip)\,G}{(1+ip)[(4\pi\alpha_0 + F)^2 + ip]} \tag{3.14}$$

□

The resonant case, i.e. $(4\pi\alpha_0)^2 = \omega N$ for some $N \in \mathbb{N}$, is not so different from the non-resonant one and we shall prove that the solution has the same behavior at the origin. The proof is slightly different because we need to show the absence of a pole at $p = 0$: from (3.9) one has

$$q_N(p) = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega N - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, q_{n+k}(p) + \frac{1-\sqrt{\omega N - ip}}{\sqrt{2\pi}\,(1+ip-\omega N)} \right\}$$

and it is easy to see that we may have a pole at $p = 0$ in the solution. We are going to prove that this is not the case: proceeding as in the proof of Proposition 3.3, let us begin with a preliminary result, which take the place of Lemma 3.1:

Lemma 3.2 *Let (2.9) and the genericity condition (3.3) be satisfied by $\{\alpha_n\}$. The system of equations*

$$r_n = \frac{4\pi}{\omega N - \sqrt{\omega n - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k r_{n+k} + h_n(p) \right\} \tag{3.15}$$

has a unique solution $\{r_n\} \in \ell_2(\mathbb{Z}\setminus\{0\})$ *in a pure imaginary neighborhood of* $p=0$*, for every* $h_n(p)$ *such that*

$$h'_n(p) \equiv \frac{h_n(p)}{\omega N - \sqrt{\omega n - ip}}$$

belongs to $\ell_2(\mathbb{Z}\setminus\{0\})$*.*
Moreover, if $h_n(p)$ *is analytic in a neighborhood of* $p=0$*, the solution is analytic in the same neighborhood.*

Proof: We shall proceed as in the proof of Proposition 3.3, separating the contribution of q_N, which may be singular, from the other equations: setting $r_n = u_n + v_n q_N$, $n \neq 0, N$, on (4.2), one has

$$u_n + v_n q_N = \frac{4\pi}{\omega N - \sqrt{\omega n - ip}}\Big\{\alpha_{N-n} q_N + \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n,N-n}} \alpha_k\big(u_{n+k} + v_{n+k} q_N\big)\Big\} +$$

$$+\frac{2\sqrt{2\pi}}{\omega N - \sqrt{\omega n - ip}}\,\frac{1-\sqrt{\omega n - ip}}{1+ip-\omega n}$$

and requiring for $\{v_n\}$, $n\neq 0, N$,

$$v_n = \frac{4\pi}{\omega N - \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n,N-n}} \alpha_k v_{n+k} + \frac{4\pi\alpha_{N-n}}{\omega N - \sqrt{\omega n - ip}} \tag{3.16}$$

the equation for $\{u_n\}$, $n\neq 0, N$, becomes

$$u_n = \frac{4\pi}{\omega N - \sqrt{\omega n - ip}}\Big\{ \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n,N-n}} \alpha_k u_{n+k} + \frac{1-\sqrt{\omega n - ip}}{\sqrt{2\pi}(1+ip-\omega n)}\Big\} \tag{3.17}$$

Moreover q_N satisfies

$$q_N = \frac{4\pi}{\omega N - \sqrt{\omega N - ip}}\Big\{ \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,N}} \alpha_k\big(u_k + v_k q_N\big) + \frac{1-\sqrt{\omega N - ip}}{\sqrt{2\pi}(1+ip-\omega N)}\Big\}$$

or

$$\Big[\omega N - \sqrt{\omega N - ip} - 4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,N}} \alpha_k t_k\Big]\, q_0 = 4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,N}} \alpha_k r_k + \frac{1-\sqrt{\omega N - ip})}{\sqrt{2\pi}(1+ip-\omega N)}$$

Applying the discussion contained in the proof of Lemma 3.1, it is not difficult to see that the solutions of equations (3.17) and (3.16) are analytic in a neighborhood of the origin and belong to $\ell_2(\mathbb{Z}\setminus\{0,N\})$. Therefore it remains to prove that (setting $v_n^0 = v_n(p=0)$)

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,N}} \alpha_k v_k^0 \neq 0$$

but the argument in the proof of Proposition 3.3 excludes this possibility, if $\{\alpha_n\}$ satisfies the genericity condition. The proof is then completed, because analyticity of r_N implies analyticity of all r_n, $n\neq 0, N$.

□

Proposition 3.5 (resonant case)
If $(4\pi\alpha_0)^2 = N\omega$, for some $N \in \mathbb{N}$ and $\{\alpha_n\}$ satisfies (2.9) and (3.3) (genericity condition), the solution of equation (3.9) has the form $q_n(p) = c_n(p) + d_n(p)\sqrt{p}$, $n \in \mathbb{Z}$, in an imaginary neighborhood of $p = 0$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$.

Proof: See the proof of Proposition 3.4 and Lemma 3.2 above.

□

3.4 Complete ionization in the generic case

Summing up the results about the behavior of the Laplace transform $\tilde{q}(p)$ of $q(t)$ we can state the following

Theorem 3.1 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$q(t) \sim A\, t^{-\frac{3}{2}} + R(t) \tag{3.18}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.

Proof: Proposition 3.2, 3.3 and 3.4 guarantee that $\tilde{q}(p)$ is analytic on the closed right half plane, except branch point singularities on the imaginary axis at $p = i\omega n$, $n \in \mathbb{Z}$.
Therefore we can chose a integration path for the inverse of Laplace transform of $\tilde{q}(q)$ along the imaginary axis like in [4].
Proposition 3.4 implies that the contribution of the branch point at $p = 0$ is given by the integral

$$2\sqrt{i} \int_0^\infty dp\, \sqrt{p}\; G'(-p)\, e^{-pt}$$

where G', defined in (3.14), is a bounded analytic function on the negative real line: from explicit expression of F and G and equations (3.13) and (3.12), it is clear that G' is analytic and $\lim_{p\to\infty} G'(-p) = 0$ on the real line. So that the corresponding asymptotic behavior as $t \to \infty$ is

$$\left| \int_0^\infty dp\, \sqrt{p}\; G'(-p)\, e^{-pt} \right| \leq C \int_0^\infty dp\, \sqrt{p}\; e^{-pt} = A\, t^{-\frac{3}{2}}$$

Let us consider now the contribution of branch points at $p = i\omega n$, $n \neq 0$: from equation (3.9) follows that, in a neighborhood of $p = 0$,

$$q_n(p) = F_n(p) + \frac{4\pi\alpha_{-n}}{1 - \sqrt{\omega n - ip}}\, \sqrt{-ip}$$

where $F_n(p) \in \ell_2(\mathbb{Z} \setminus \{0\})$ is analytic at $p = 0$, like the coefficient of $\sqrt{p}$. Since $q_n(p) = \tilde{q}(p + i\omega n)$, the contribution of singularities at $p = i\omega n$, $n \neq 0$, is then given by

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \int_{i\omega n-\infty}^{i\omega n} dp\, \frac{4\pi\alpha_{-n}}{1-\sqrt{-ip}} \sqrt{-ip-\omega n}\; e^{pt} =$$

$$= \int_0^\infty dp \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{4\pi\alpha_{-n}}{1-\sqrt{\omega n + ip}} \sqrt{-ip}\; e^{i\omega nt} e^{-pt}$$

and

$$\left| \int_0^\infty dp \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{4\pi\alpha_{-n}}{1-\sqrt{\omega n + ip}} \sqrt{ip}\; e^{i\omega nt} e^{-pt} \right| \leq C' \int_0^\infty dp \, \sqrt{p}\; e^{-pt} = A'\, t^{-\frac{3}{2}}$$

because $\{\alpha_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$:

$$\left| \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{4\pi\alpha_{-n}}{1-\sqrt{\omega n + ip}} e^{i\omega nt} \right| \leq \left[C \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |\alpha_n|^2 \right]^{\frac{1}{2}} < \infty$$

Finally the rest function $R(t)$ is given by the contribution of poles outside the imaginary axis and then shows an exponential decay as $t \to \infty$.

□

A straightforward consequence of Theorem 3.1 is that the survival probability of the bound state (initial datum)

$$\theta(t) \equiv \Big(\varphi_{\alpha(0)} \,, \Psi_t \Big)_{L^2(\mathbb{R}^3)} \tag{3.19}$$

tends to 0 when $t \to \infty$:

Corollary 3.1 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3) with respect to $\mathcal{T}$, the system shows asymptotic complete ionization,*

$$\lim_{t\to\infty} \theta(t) = 0$$

Proof: We are going to prove that $\forall \varepsilon > 0$, there exists a $t_0 > 0$ such that $|\theta(t)| \leq \varepsilon$, $\forall t > t_0$.
Using the decomposition of the wave function at time t defined by (2.4), we can break the scalar product in two pieces: let us consider first the second part which is given by

$$\left| \left(\varphi_{\alpha(0)}(\vec{x}) \,, \int_s^t d\tau \; q(\tau)\, U_0(t-\tau;\vec{x}) \right)_{L^2(\mathbb{R}^3)} \right| =$$

$$= \left| \int_s^t d\tau \; q(\tau) \left(e^{-iH_0(t-\tau)} \varphi_{\alpha(0)} \right)(0) \right| \leq$$

$$\leq \int_s^t d\tau \, \big|q(\tau)\big| \left| \int_0^\infty dk \, \frac{Ck^2}{k^2+1} \, e^{-ik^2(t-\tau)} \right| \leq C \int_s^t d\tau \, \frac{|q(\tau)|}{\sqrt{t-\tau}}$$

but for each $s > s_0(\varepsilon)$ and $\forall t \geq s$,

$$C \int_s^t d\tau \, \frac{|q(\tau)|}{\sqrt{t-\tau}} \leq \frac{\varepsilon}{2}$$

since $\frac{|q(\tau)|}{\sqrt{t-\tau}}$ is a L^1-function by Theorem (3.1). Let us now fix $s > s_0(\varepsilon)$ and consider the first term in the scalar product:

$$\left(\varphi_{\alpha(0)} \, , e^{-iH_0(t-s)} \Psi_s \right)_{L^2(\mathbb{R}^3)}$$

where $\Psi_s \in L^2(\mathbb{R}^3)$. Since the free propagator $e^{-iH_0 t}$ tends weakly to 0 when $t \to \infty$, choosing $t > s + \delta(\varepsilon)$, with $\delta(\varepsilon)$ sufficiently great, one has

$$\left| \left(\varphi_{\alpha(0)} \, , e^{-iH_0(t-s)} \Psi_s \right)_{L^2(\mathbb{R}^3)} \right| \leq \frac{\varepsilon}{2}$$

so that the whole expression is bounded by ε, $\forall t > s_0(\varepsilon) + \delta(\varepsilon)$.

□

In the following we shall prove a stronger result about complete ionization of the system, namely that every state $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state[5] for the operator $H_{\alpha(t)}$, i.e.

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t d\tau \, \left\| F(|\vec{x}| \leq R) U(\tau, 0) \Psi \right\|^2 = 0 \tag{3.20}$$

where $F(S)$ is the multiplication operator by the characteristic function of the set $S \subset \mathbb{R}^3$ and $U(t,s)$ the unitary two-parameters family associated to $H_{\alpha(t)}$ (see (2.4)).
In order to prove (3.20), we first need to study the evolution of a generic initial datum in a suitable dense subset of $L^2(\mathbb{R}^3)$ and then we shall extend the result to every state using the unitarity of the evolution defined by (2.4) (see e.g. [9]).

Proposition 3.6 *Let $\Psi \in C_0^\infty(\mathbb{R}^3)$ a smooth function with compact support and $q(t)$ be the solution of equation (2.6) with initial condition $\Psi_0 = \Psi$. If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$q(t) \sim A \, t^{-\frac{3}{2}} + R(t) \tag{3.21}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.

Proof: It is easy to see that the result contained in Proposition 2.1 holds also for every (sufficiently smooth) initial datum Ψ, so that the Laplace transform $\tilde{q}(p)$ of $q(t)$ exists analytic $\forall p$ with $\Re(p)$ sufficiently large.
Hence we can consider the Laplace transform of equation (2.6), which has the form (2.11) with

$$f(p) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{i}{p}} \int_0^\infty dt \, e^{-pt} \int_{\mathbb{R}^3} d^3\vec{k} \ \hat{\Psi}(\vec{k}) \, e^{-ik^2 t}$$

where $\hat{\Psi}(\vec{k})$ is the Fourier transform of $\Psi(\vec{x})$.
The equation for $\tilde{q}(p)$ is then given by

$$\tilde{q}(p) = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{-ip}} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k \, \tilde{q}(p + i\omega k) + \frac{g(p)}{4\pi\alpha_0 + \sqrt{-ip}}$$

[5]For the definition of scattering states of a time-dependent operator see e.g. [11, 13].

where

$$g(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty dt\, e^{-pt} \int_{\mathbb{R}^3} d^3\vec{k}\ \ \hat{\Psi}(\vec{k})\, e^{-ik^2 t}$$

It is clear that for every smooth function Ψ, $g(p)$ is analytic for $\Re(p) > 0$. Hence Proposition 3.1 and 3.2 still apply and then the solution is unique and analytic in the right open half plane.
It is now sufficient to show that the solution $\tilde{q}(p)$ is also analytic on the imaginary axis except at most square root branch points at $p = i\omega n$ as in the discussion of section 3.2 and 3.3.
For every smooth function Ψ with compact support, $\hat{\Psi}(\vec{k})$ is a smooth function with an exponential decay as $k \to \infty$, so that

$$g(is) = \lim_{r \to 0^+} \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} d^3\vec{k}\ \frac{\hat{\Psi}(\vec{k})}{r + (s + k^2)i} = -i\sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} d^3\vec{k}\ \frac{\hat{\Psi}(\vec{k})}{s + k^2}$$

is a bounded function for $s > 0$. Hence the function $g(p)$ has no pole for $\Im(p) \in (0, \omega)$ and therefore the result contained in Proposition 3.3 still holds. Moreover

$$g(0) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} d^3\vec{k}\ \ \hat{\Psi}(\vec{k}) \int_0^\infty dt\, e^{-ik^2 t} = -i\sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} d^3\vec{k}\ \frac{\hat{\Psi}(\vec{k})}{k^2}$$

which is again bounded, so that $g(p)$ has at the origin at most a branch point singularity of the form $a(p) + b(p)\sqrt{p}$: following the proofs of Proposition 3.4 and 3.5, we can show that $\tilde{q}(p)$ has the same behavior at the origin.
In conclusion the solution is analytic on the closed right half plane except branch points at $p = i\omega n$, $n \in \mathbb{N}$, of the form $a(p)+b(p)\sqrt{p - i\omega n}$. The proof of Theorem 3.1 then implies that $q(t)$ as the prescribed behavior as $t \to \infty$.

□

Theorem 3.2 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3) with respect to $\mathcal{T}$, every $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state of $H_{\alpha(t)}$, i.e.*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t d\tau\, \left\| F(|\vec{x}| \leq R) U(\tau, 0)\Psi \right\|^2 = 0$$

Proof: We shall restrict the proof to the dense subset of $L^2(\mathbb{R}^3)$ given by smooth functions with compact support and then we shall extend the result to every state using the unitarity of the evolution defined by (2.4) (see e.g. [9]). Actually we are going to prove an equivalent but slightly different statement, i.e. $\forall \varepsilon > 0$, there exists t_0 such that $\forall t > t_0$,

$$\left\| F(|\vec{x}| \leq R) U(t, 0)\Psi \right\| \leq \varepsilon$$

The evolution of an initial state Ψ according to (2.4) is given by

$$\Psi_t(\vec{x}) = U(t,s)\Psi_s(\vec{x}) = U_0(t-s)\Psi_s(\vec{x}) + i \int_s^t d\tau\, q(\tau)\, U_0(t-\tau; \vec{x}) \qquad (3.22)$$

Moreover, since $\Psi_t \in \mathcal{D}(H_{\alpha(t)})$, the following decomposition holds

$$\Psi_t(\vec{x}) = \varphi_t(\vec{x}) + \frac{q(t)}{4\pi|\vec{x}|} \qquad (3.23)$$

where $q(t)$ is the solution of (2.6), $\varphi_t \in H^2_{\text{loc}}(\mathbb{R}^3)$ and

$$\varphi_t(0) = \alpha(t) q(t)$$

We are going to show that, if $q(t) \in L^1(\mathbb{R}^+)$, Ψ_t satisfies the required property. Let us start analyzing the second term in (3.22): imposing the unitarity condition of the evolution we have

$$\|\Psi_s\|^2 = \|\Psi_t\|^2 = \left\| U_0(t-s)\Psi_s(\vec{x}) + i\int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x}) \right\|^2$$

and then

$$\left\| \int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x}) \right\|^2 = 2\Im\Big(\int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x})\,,\, U_0(t-s)\Psi_s(\vec{x}) \Big) =$$

$$= 2\Im\left[\int_s^t d\tau\, q^*(\tau) \Big(e^{-iH_0(\tau-s)}\Psi_s \Big)(0) \right]$$

but, using the decomposition (3.23),

$$\Big(e^{-iH_0(s-\tau)}\Psi_s \Big)(0) = \Big(e^{-iH_0(s-\tau)}\varphi_s \Big)(0) + \int_{\mathbb{R}^3} d^3\vec{k}\, e^{-ik^2(\tau-s)}\, \frac{q(s)}{(2\pi)^3 k^2} =$$

$$= \Big(e^{-iH_0(s-\tau)}\varphi_s \Big)(0) + \frac{q(s)}{4\pi\sqrt{\pi i}\sqrt{\tau-s}}$$

Since $\varphi_s \in H^2_{\text{loc}}$, the absolute value of the first term on the right hand side is bounded by a constant $c(\tau,s) < \infty$ such that $c(s,s) = q(s)$ and

$$\lim_{\tau\to\infty} c(\tau,s) = 0$$

Hence there exists $s_1(\varepsilon) > 0$ such that, $\forall s > s_1$,

$$2\left| \int_s^t d\tau\, q^*(\tau) \Big(e^{-iH_0(s-\tau)}\varphi_s \Big)(0) \right| \leq \frac{2\varepsilon^2}{9}$$

if $q(t) \in L^1(\mathbb{R}^+)$. Moreover by the same reason there exists $s_2(\varepsilon) > 0$ such that $\forall s > s_2$,

$$2\left| \int_s^t d\tau\, q^*(\tau) \frac{q(s)}{4\pi\sqrt{\pi i}\sqrt{\tau-s}} \right| \leq \frac{2\varepsilon^2}{9}$$

Setting $s_0(\varepsilon) = \max(s_1(\varepsilon), s_2(\varepsilon))$, one has $\forall s > s_0$

$$\left\| \int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x}) \right\| \leq \frac{2\varepsilon}{3} \tag{3.24}$$

so that the whole L^2-norm of the second term in decomposition (3.22) is suitably small for $s > s_0$.
On the other hand the first term in (3.22) is the free evolution of a L^2-function and hence there exists $\delta(\varepsilon) > 0$ such that $\forall t > s + \delta$ and $\forall R < \infty$,

$$\big\| F(|\vec{x}| \leq R) U(t-s)\Psi_s \big\| \leq \frac{\varepsilon}{3} \tag{3.25}$$

Setting $t_0(\varepsilon) = s_0(\varepsilon) + \delta(\varepsilon)$, from (3.22), (3.24) and (3.25) one has

$$\left\| F(|\vec{x}| \leq R) \Psi_t \right\| \leq \varepsilon$$

$\forall t > t_0$, if $q(t) \in L^1(\mathbb{R}^+)$.
By Proposition 3.6 the inequality is then satisfied by every $\Psi \in C_0^\infty(\mathbb{R}^3)$: unitarity of the family $U(t,s)$ allows to extend the result to the whole Hilbert space $L^2(\mathbb{R}^3)$.

$\square$

Corollary 3.2 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition with respect to $\mathcal{T}$ (3.3), the discrete spectrum of the Floquet operator associated to $H_{\alpha(t)}$,*

$$K \equiv -i\frac{\partial}{\partial t} + H_{\alpha(t)}$$

is empty.

Proof: The result is a straightforward consequence of Theorem 3.2: every eigenvector of K is necessarily a periodic function and hence can not satisfy (3.20).

$\square$

4 CASE II: $\alpha_0 = 0$

If $\alpha(t) = \alpha_0 = 0$ does not depend on time, the problem has a simple solution: the spectrum of $H_{\alpha(t)}$ is absolutely continuous and equal to the positive real line, with a resonance at the origin; hence there is no bound state and the system shows complete ionization independently on the initial datum.
On the other hand if $\alpha(t)$ is a zero mean function, we shall see that the genericity condition (3.3) is still needed to have complete ionization.
So let us assume that $\alpha_0 = 0$, the normalization (3.1) holds and the initial datum is given by (2.7): equation (2.11) then becomes

$$\tilde{q}(p) = -4\pi\sqrt{\frac{i}{p}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k \, \tilde{q}(p + i\omega k) - 2\sqrt{\frac{2\pi i}{p}} \, \frac{1-\sqrt{-ip}}{1+ip} \tag{4.1}$$

with the choice (3.5) for the branch cut of $\sqrt{p}$.

4.1 Analyticity on the (open) right half plane

Setting $q_n(p) \equiv \tilde{q}(p + i\omega n)$, $p \in \mathcal{I} = [0, \omega)$, as in Section 3.1, equation (4.1) assumes the form (3.6),

$$q(p) = \mathcal{M}(p) \, q(p) + o(p) \tag{4.2}$$

with

$$\big(\mathcal{M}q\big)_n(p) \equiv -\frac{4\pi}{\sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k \, q_{n+k}(p) \tag{4.3}$$

and $o(p) = \{o_n(p)\}_{n\in\mathbb{Z}}$,

$$o_n(p) \equiv -\frac{2\sqrt{2\pi}}{\sqrt{\omega n - ip}\ (1+\sqrt{\omega n - ip})} \tag{4.4}$$

Proposition 4.1 *For $p \in \mathcal{I}$, $\Re(p) > 0$, $\mathcal{M}(p)$ is an analytic operator-valued function and $\mathcal{M}(p)$ is a compact operator on $\ell_2(\mathbb{Z})$.*

Proof: See the proof of Proposition 3.1.

□

Proposition 4.2 *The solution $\tilde{q}(p)$ of (4.1) exists and is analytic for $\Re(p) > 0$.*

Proof: See the proof of Proposition 3.2.

□

4.2 Singularities on the imaginary axis

From (4.2) and explicit expressions of $\mathcal{M}$ and o_n, it follows that the solution can not have a pole on the imaginary axis[6] and the only singularities are branch points at $p = 0$: with the choice (3.5) of the branch cut of the square root, the branch points of $\sqrt{\omega n - ip}$ are at $p = i\omega n$, $n \in \mathbb{Z}$, but, since $0 \le \Im(p) < \omega$, we can have only one branch point at $p = 0$, independently on the value of ω.

Proposition 4.3 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3), the solution of equation (4.2) has the form $q_n(p) = c_n(p) + d_n(p)\sqrt{p}$, $n \in \mathbb{Z}$, in a neighborhood of $p = 0$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$.*

Proof: Let us proceed as in the proof of Proposition 3.4: setting $q_n = r_n + t_n q_0$, $n \in \mathbb{Z}\setminus\{0\}$, where $\{t_n\}$ is the solution of

$$t_n = -\frac{4\pi}{\sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k t_{n+k} - \frac{4\pi\alpha_{-n}}{\sqrt{\omega n - ip}} \tag{4.5}$$

Lemma 3.1 guarantees that the solution $\{t_n\} \in \ell_2(\mathbb{Z}\setminus\{0,-n\})$ is unique and analytic at $p = 0$.
By means of this substitution we obtain

$$r_n = -\frac{4\pi}{\sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k r_{n+k} - \frac{2\sqrt{2\pi}}{\sqrt{\omega n - ip}\ (1+\sqrt{\omega n - ip})} \tag{4.6}$$

and

$$q_0 = -\frac{4\pi}{\sqrt{-ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k \big(r_k + t_k q_0\big) - \frac{2\sqrt{2\pi}}{\sqrt{-ip}\ (1+\sqrt{-ip})}$$

[6]The absence of non zero solutions of the homogeneous equation associated to (4.2) follows from the genericity condition as in the proof of Proposition 3.3.

or

$$\left(\sqrt{-ip}+F\right)q_0 = -G - \frac{2\sqrt{2\pi}}{1+\sqrt{-ip}}$$

where (like in the proof of Proposition 3.4)

$$F \equiv 4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k t_k$$

and

$$G \equiv 4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k r_k$$

Moreover $F(0) \neq 0$: suppose that the contrary is true, from equation (4.5) we have

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \sqrt{\omega n}\ |t_n^0|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq 0}} {t_n^0}^* \alpha_{k-n} t_k^0$$

The right hand side is real while $\Im(\sqrt{\omega n}) > 0$, $\forall n < 0$, which implies $t_n^0 = 0$, for each $n < 0$, and the result follows from the genericity of $\{\alpha_n\}$, like in the proof of Proposition 3.4.
Moreover F and G are analytic in a neighborhood of $p = 0$ (see Lemma 3.1), so that

$$q_0 = F' + \sqrt{p}\ G'$$

where F' and G' are analytic and

$$G' \equiv \frac{2\sqrt{-2\pi i}(F+1) + (1+ip)G}{(1+ip)(F^2+ip)}$$

□

4.3 Complete ionization in the generic case

Like in section 3 we can now state the main result:

Theorem 4.1 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$q(t) \sim A\, t^{-\frac{3}{2}} + R(t) \tag{4.7}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.

Proof: See the proof of Theorem 3.1.

□

Corollary 4.1 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3) with respect to $\mathcal{T}$, the system shows asymptotic complete ionization,*

$$\lim_{t\to\infty} \theta(t) = 0$$

Proof: See the proof of Corollary 3.1.

□

Theorem 4.2 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3) with respect to $\mathcal{T}$, every $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state of $H_{\alpha(t)}$, i.e.*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \left\| F(|\vec{x}| \leq R) U(\tau, 0)\Psi \right\|^2 = 0$$

Moreover the discrete spectrum of the Floquet operator is empty.

Proof: See the proof of Proposition 3.6 and Theorem 3.2.

□

5 CASE III: $\alpha_0 > 0$

To complete the analysis of the problem, we shall consider the case of mean greater than 0: taking the normalization (3.1) and the initial condition (2.7), (2.11) assumes the form (3.4):

$$\tilde{q}(p) = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{-ip}} \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \alpha_k \, \tilde{q}(p + i\omega k) - \frac{2\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{-ip}} \frac{1-\sqrt{-ip}}{1+ip} \tag{5.1}$$

Following the discussion contained in section 3 and setting $q_n(p) \equiv \tilde{q}(p + i\omega n)$, $\Im(p) \in [0, \omega)$, the equation assumes the form (3.6): analyticity on the open right half plane is then a consequence of Proposition 3.2.
Let us now consider the behavior on the imaginary axis: singularities for $\Re(p) = 0$ are associated to zeros of $4\pi\alpha_0 + \sqrt{\omega n + s}$, $s \in [0, \omega)$, but, since $\alpha_0 > 0$, it is clear that the expression can not have zeros on the imaginary axis. Hence the proof of Proposition 3.2 can be extended to the closed right half plane except the origin:

Proposition 5.1 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition with respect to $\mathcal{T}$ (3.3), the solution $\tilde{q}(p)$ of (5.1) exists and is analytic for $\Re(p) \geq 0$, $p \neq i\omega n$, $n \in \mathbb{Z}$.*

Proof: See the proof of Proposition 3.2, Proposition 3.1 and the previous discussion.

□

Moreover the behavior at the origin is described by the following

Proposition 5.2 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition with respect to $\mathcal{T}$ (3.3), then, in an imaginary neighborhood of $p = i\omega n$, $n \in \mathbb{Z}$, the solution of equation (5.1) has the form $\tilde{q}(p) = c_n(p) + d_n(p)\sqrt{p - i\omega n}$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = i\omega n$.*

Proof: The proof of Proposition 3.4 still applies with only one difference: since, independently on ω, the solution can not have a pole on the imaginary axis, we need not to distinguish between the resonant case and the non-resonant one.

□

We can now prove asymptotic complete ionization of the system:

Theorem 5.1 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$q(t) \sim A\, t^{-\frac{3}{2}} + R(t) \tag{5.2}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$. Moreover the system shows asymptotic complete ionization,

$$\lim_{t \to \infty} \theta(t) = 0$$

Proof: See the proof of Theorem 3.1 and Corollary 3.1.

□

Theorem 5.2 *If $\{\alpha_n\}$ satisfies (2.9) and the genericity condition (3.3) with respect to $\mathcal{T}$, every $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state of $H_{\alpha(t)}$, i.e.*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t d\tau \, \left\| F(|\vec{x}| \leq R) U(\tau, 0) \Psi \right\|^2 = 0$$

Moreover the discrete spectrum of the Floquet operator is empty.

Proof: See the proof of Proposition 3.6 and Theorem 3.2.

□

Remark: If $\alpha(t) \geq 0$, $\forall t \in \mathbb{R}^+$, Proposition 5.1 and 5.2 hold without the genericity condition on the Fourier coefficients of $\alpha(t)$. Let us consider for example the proof of Proposition 5.1: the genericity condition enters in the analysis of solutions of the homogeneous equation on the imaginary axis. In particular it is necessary to prove absence of non zero solutions of

$$q_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n + s}} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k \, q_{n+k}$$

where $s \in (0, \omega)$. Let us suppose that there exists a non zero solution $\{Q_n\} \in \ell_2(\mathbb{Z})$. Multiplying both sides of the equation by Q_n^*, one has

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(4\pi\alpha_0 + \sqrt{\omega n + s}\right) |Q_n|^2 = -4\pi \sum_{\substack{n,k \in \mathbb{Z} \\ n,k \neq 0}} Q_n^* \alpha_{k-n} Q_k$$

or

$$\sum_{n \in \mathbb{Z}} \sqrt{\omega n + s}\, |Q_n|^2 = -4\pi \sum_{n,k \in \mathbb{Z}} Q_n^* \alpha_{k-n} Q_k$$

Since the right hand side is real, $Q_n = 0$, $\forall n < 0$. Moreover

$$-4\pi \sum_{n,k \in \mathbb{Z}} Q_n^* \alpha_{k-n} Q_k = -4\pi \Big(Q(t), \alpha(t) Q(t)\Big)_{L^2([0,T])} \leq 0$$

because $\alpha(t) \geq 0$, $\forall t \in [0, T]$, but the left hand side is positive and then $Q_n = 0$, $\forall n \in \mathbb{Z}$.

6 Conclusions and Perspectives

In sections 3,4 and 5 we have proved that, under the genericity condition on $\alpha(t)$, the system defined in section 2 shows asymptotic complete ionization, independently on its frequency.
If $\inf(\alpha(t)) < 0$, the genericity condition may be a necessary condition to have complete ionization: for example, in one dimension, it is possible to exhibit (see [4]) explicit functions $\alpha(t)$ for which the genericity condition fails[7] and the ionization is not complete. On the other hand, also in one dimension, it is not known whether the condition is necessary. It would be interesting to check if non generic $\alpha(t)$ gives rise to asymptotic partial ionization in three dimensions. A possible way to investigate this problem is the analysis of the discrete spectrum of the Floquet operator. If one can find an explicit relation between existence of eigenvalues of the Floquet operator and the genericity condition, it would be probably easy to check if the condition is truly necessary.
On the other hand, as we expected, if $\alpha(t)$ is positive at any time, no further condition on $\alpha(t)$ is required to prove complete ionization.
Two interesting future applications of these methods can be the problem of complete ionization for moving point interactions and for N time-dependent point interactions. Indeed there are simple examples in which asymptotic complete ionization occurs also for moving sources (see [3]).

[7] A simple example of $\alpha(t)$, for which the genericity condition is not satisfied is the geometric series, $\alpha_n = \lambda^{|n|}$ for some $\lambda < 1$.

References

[1] S.A. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, *Solvable Models in Quantum Mechanics*, Springer-Verlag, New York, 1988.

[2] F.A. Berezin, L.D. Faddeev, A Remark on Schrödinger Equation with a Singular Potential, *Sov. Math. Dokl.* **2**, 372-375, 1961.

[3] M. Correggi, G.F. Dell'Antonio, Rotating Singular Perturbations of the Laplacian, preprint *math-ph/0307056*, 2003.

[4] O. Costin, R.D. Costin, J.L. Lebowitz, A. Rokhlenko, Evolution of a Model Quantum System under Time Periodic Forcing: Conditions for Complete Ionization, *Comm. Math. Phys.* **221**, no.1 1-26, 2001.

[5] O. Costin, J.L. Lebowitz, A. Rokhlenko, Decay versus Survival of a Localized State Subjected to Harmonic Forcing: Exact Results, *J. Phys. A: Math. Gen.* **35**, 8943-8951, 2002.

[6] O. Costin, J.L. Lebowitz, A. Rokhlenko, Exact Results for the Ionization of a Model Quantum System, *J. Phys. A: Math. Gen.* **33**, 6311-6319, 2000.

[7] O. Costin, R.D. Costin, J.L. Lebowitz, Transition to the Continuum of a Particle in Time-Periodic Potentials, in Advances in Differential Equations and Mathematical Physics, Birmingham, AL, 2002, *Contemp. Math.* **327**, 75-86, AMS, Providence, 2003.

[8] O. Costin, J.L. Lebowitz, A. Rokhlenko, Ionization of a Model Atom: Exact Results and Connection with Experiment, preprint *physics/9905038*, 1999.

[9] G.F. Dell'Antonio, R. Figari, A. Teta, Schrödinger Equation with Moving Point Interactions in Three Dimensions, in Stochastic Processes, Physics and Geometry: New Interplays, Leipzig, 1999, *CMS Conference Proceedings* **28**, 99-113, AMS, Providence, 2000.

[10] G.F. Dell'Antonio, Point Interactions, in Mathematical Physics in Mathematics and Physics, Siena, 2000, *Fields Institute Communications* **30**, 139-150, AMS, Providence, 2001.

[11] V. Enss, K. Veselic, Bound States and Propagating States for Time-dependent Hamiltonians, *Ann. Inst. H. Poincaré A* **39**, 159-191, 1983.

[12] R. Figari, Time Dependent and Non Linear Point Interactions, in *Proceedings of Mathematical Physics and Stochastic Analysis, Lisbon, 1998*, 184-197, World Scientific Publisher, New York, 2000.

[13] J.S. Howland, Stationary Scattering Theory for Time-dependent Hamiltonians, *Math. Ann.* **207**, 315-335, 1974.

[14] J.S. HOWLAND, Scattering Theory for Hamiltonians Periodic in Time, *Indiana Univ. Math. J.* **28** No.3, 471-494, 1979.

[15] M. REED, B. SIMON, *Methods of Modern Mathematical Physics, Vol.I: Functional Analysis*, Academic Press, San Diego, 1975.

[16] M.R. SAYAPOVA, D.R. YAFAEV, The Evolution Operator for Time-dependent Potentials of Zero Radius, *Proc. Stek. Inst. Math.* **2**, 173-180, 1984.

[17] D.R. YAFAEV, Scattering Theory for Time-dependent Zero-range Potentials, *Ann. Inst. H. Poincaré A* **40**, 343-359, 1984.

[18] K. YAJIMA, H. KITADA, Bound States and Scattering States for Time Periodic Hamiltonians, *Ann. Inst. H. Poincaré A* **39**, 145-157, 1983.