# Universality of residence-time distributions in non-adiabatic stochastic resonance

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**Abstract.** – We present a mathematically rigorous expression for the residence-time distribution of a periodically forced Brownian particle in a bistable potential. For a broad range of forcing frequencies and amplitudes, the distribution is close to a periodically modulated exponential one. Remarkably, the periodic modulation is governed by a universal function, depending on a single parameter related to the forcing period. The behaviour of the distribution and its moments is analysed, in particular in the low- and high-frequency limits.

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The amplification by noise of a weak periodic signal acting on a multistable system is known as stochastic resonance (SR). A simple example of a system showing SR is an overdamped Brownian particle in a symmetric double-well potential, subjected to deterministic periodic forcing as well as white noise. Despite of the amplitude of the forcing being too small to enable the particle to switch from one potential well to the other, such transitions can be made possible by the additive noise. For sufficiently large noise intensity, depending on the forcing period, the transitions between potential wells can become close to periodic. This mechanism was originally proposed by Benzi *et al.* and Nicolis and Nicolis [1–3] in order to offer an explanation for the close-to-periodic occurence of the major Ice Ages. Since then, it has been observed in a large variety of physical and biological systems (for reviews see, *e.g.*, [4–7]).

Although much progress has been made in the quantitative description of the phenomenon of SR, many of its aspects are not yet fully understood. Mathematically rigorous results have

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so far been limited to the regimes of exponentially slow forcing [8, 9], or moderately slow forcing of close-to-threshold amplitude [10, 11].

One of the measures introduced in order to quantify SR is the residence-time distribution, that is, the distribution of the random time spans the Brownian particle spends in each potential well between transitions. SR is characterized by the fact that residence times are more likely to be close to multiples of the forcing period (plus a phase shift) than not. The residence-time distribution was first studied by Eckmann and Thomas for a two-level system [12]. For continuous systems, it has been estimated, in the case of adiabatic forcing, by averaging the escape rate for frozen potential over the distribution of jump phases [13, 14].

For larger forcing frequencies, however, the adiabatic approximation can no longer be used. An alternative approach is to consider time as an additional dynamic variable, which yields a two-dimensional problem. In the absence of noise, the system has two stable periodic orbits, one oscillating around each potential well, and one unstable periodic orbit, which oscillates around the saddle and separates the basins of attraction of the two stable orbits. Determining the residence-time distribution is equivalent to finding the distribution of first passages of the stochastic process through the unstable orbit. This problem was first investigated by Graham and Tél [15, 16] and Day [17–19], and later by Maier and Stein [20] and others (*e.g.*, [21, 22]).

At first glance, however, this two-dimensional approach seems to produce a paradoxical result. Indeed, it is known from the classical Wentzell–Freidlin theory [23–25] that the distribution of first-passage locations through a periodic orbit looks uniform on the level of exponential asymptotics [17]. This is due to the fact that translations along the periodic orbit do not contribute to the cost in terms of action functional. How can this fact be conciled with the quasistatic picture, which yields residence times concentrated near multiples of the forcing period? Obviously, the answer has to lie in the subexponential behaviour of the distribution of transitions.

In this Letter, we extend previous results of [19–21] to a mathematically rigorous expression for the residence-time distribution up to multiplicative errors in the subexponential prefactor, valid for a broad range of forcing periods [26, 27]. A particularly interesting aspect of the result is that the residence-time distribution is governed by a universal periodic function, depending only on the period of the unstable periodic orbit times its Lyapunov exponent. All the model-dependent properties of the distribution can be eliminated by a deterministic time change.

*Assumptions.* – We consider one-dimensional stochastic differential equations of the form

$$\mathrm{d}x_t = -\frac{\partial}{\partial x} V(x_t, t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{1}$$

where $W_t$ is a standard Wiener process, describing white noise, and the small parameter $\sigma$ measures the noise intensity (the diffusion constant being $D = \sigma^2/2$). The double-well potential $V(x,t)$ depends periodically on time, with period $T$. The simplest example is

$$V(x,t) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - A\sin(\omega t)x, \tag{2}$$

where the forcing has angular frequency $\omega = 2\pi/T$ and amplitude $|A| < \sqrt{4/27}$.

Our results apply to a general class of $T$-periodic double-well potentials. We assume that for each fixed $t$, $V(x,t)$ has two minima at $X^{\mathrm{s}}_{1,2}(t)$ and a saddle at $X^{\mathrm{u}}(t)$, such that

$$X^{\mathrm{s}}_1(t) < c_1 < X^{\mathrm{u}}(t) < c_2 < X^{\mathrm{s}}_2(t) \qquad \forall t \tag{3}$$

for two constants $c_1, c_2$ (in the particular case of the potential (2), one can take $c_2 = -c_1 = 1/\sqrt{3}$). Then it is straightforward to show, using Poincaré maps, that in the absence of noise the system (1) has exactly three periodic orbits, one of them unstable and staying between $c_1$ and $c_2$, which we denote by $x^{\mathrm{per}}(t)$. We introduce the notations

$$a(t) = -\frac{\partial^2}{\partial x^2} V(x^{\mathrm{per}}(t), t) \tag{4}$$

for the curvature of the potential at $x^{\mathrm{per}}(t)$, and

$$\lambda = \frac{1}{T} \int_0^T a(t)\, \mathrm{d}t \tag{5}$$

for the Lyapunov exponent of the unstable orbit. We assume that $\lambda$ is of order 1, but $T$ can become comparable to Kramers' time.

Finally, we need a non-degeneracy assumption for the system, which assures that the action functional is minimized on a discrete set of paths, and excludes symmetries other than time-periodicity [26]. In particular, it should not be possible to transform the equation into an autonomous one by a time-periodic change of variables. In the special case of the potential (2), this condition is met when $A \neq 0$. In addition, we will assume that $|A|$ is of order 1, while $\sigma^2 \ll |A|$.

*Results.* – Assume the system starts at time 0 in a given initial point at or near the bottom of the left-hand potential well. We shall call *residence time* the (random) first time $\tau$ at which $x_t$ reaches the unstable periodic orbit $x^{\mathrm{per}}(t)$. One should note that once $x^{\mathrm{per}}(t)$ has been reached, the process still needs some time to relax to one of the potential wells; this time is short compared to the period in the adiabatic case, but can be significant for larger forcing frequencies.

Our main result states that the probability distribution of $\tau$ is governed by the following function, in a sense made precise in Theorem 1 below. Let

$$p(t) = \frac{1}{N} Q_{\lambda T}\big(\theta(t) - |\ln\sigma|\big) \frac{\theta'(t)}{\lambda T_{\mathrm{K}}(\sigma)}\, \mathrm{e}^{-\theta(t)/\lambda T_{\mathrm{K}}(\sigma)}\, f_{\mathrm{trans}}(t) \tag{6}$$

where we use the following notations:

- $T_{\mathrm{K}}(\sigma)$ is the analogue of Kramers' time in the autonomous case; it has the form

  $$T_{\mathrm{K}}(\sigma) = \frac{C}{\sigma}\, \mathrm{e}^{\overline{V}/\sigma^2}, \tag{7}$$

  where $\overline{V}$ is the constant value of the quasipotential on $x^{\mathrm{per}}(t)$. $\overline{V}$ can be computed by a variational method, as the minimum of the action functional over all paths connecting the bottom of the left-hand potential well to $x^{\mathrm{per}}(t)$ (see [25]). In the limit of small forcing amplitude, $\overline{V}$ reduces to twice the potential barrier height. The prefactor has order $\sigma^{-1}$ rather than 1, due to the fact that most paths reach $x^{\mathrm{per}}(t)$ through a bottleneck of width $\sigma$ (the width would be larger if $|A|$ were not of order 1 [28]).

- $Q_{\lambda T}(y)$ is the announced universal periodic function, of period $\lambda T$; it has the explicit expression

$$Q_{\lambda T}(y) = 2\lambda T \sum_{k=-\infty}^{\infty} A(y - k\lambda T) \qquad \text{with} \qquad A(z) = \frac{1}{2}\, \mathrm{e}^{-2z} \exp\Big\{-\frac{1}{2}\, \mathrm{e}^{-2z}\Big\}, \tag{8}$$

and thus consists of a superposition of identical asymmetric peaks, shifted by a distance $\lambda T$. The average of $Q_{\lambda T}(y)$ over one period is equal to 1.

- $\theta(t)$ contains the model-dependent part of the distribution; it is an increasing function of $t$, satisfying $\theta(t+T) = \theta(t) + \lambda T$, and is given by

$$\theta(t) = \text{const} + \int_0^t a(s)\,\mathrm{d}s - \frac{1}{2}\ln\frac{v(t)}{v(0)}, \tag{9}$$

where $v(t)$ is the unique periodic solution of the differential equation $\dot{v}(t) = 2a(t)v(t)+1$. It is related to the variance of Eq. (1) linearized around $x^{\text{per}}(t)$, and has the expression

$$v(t) = \frac{1}{\mathrm{e}^{2\lambda T} - 1}\int_t^{t+T} \exp\biggl\{\int_s^{t+T} 2a(u)\,\mathrm{d}u\biggr\}\,\mathrm{d}s. \tag{10}$$

- $f_{\text{trans}}(t)$ accounts for the initial transient behaviour of the system; it is an increasing function satisfying

$$f_{\text{trans}}(t) = \begin{cases} \mathcal{O}\biggl(\exp\biggl\{-\dfrac{L}{\sigma^2}\dfrac{\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-2\lambda t}}\biggr\}\biggr) & \text{for } \lambda t < 2|\ln\sigma| \\ 1 - \mathcal{O}\biggl(\dfrac{\mathrm{e}^{-\lambda t}}{\sigma^2}\biggr) & \text{for } \lambda t \geqslant 2|\ln\sigma| \end{cases} \tag{11}$$

for some constant $L$, and thus behaves roughly like $\exp\{-L\,\mathrm{e}^{-\theta(t)}/\sigma^2(1-\mathrm{e}^{-2\theta(t)})\}$. However, $f_{\text{trans}}(t)$ can be different when starting with an initial distribution that is not concentrated in a single point near the bottom of the potential well.

- $N$ is the normalization, which we compute below.

The precise formulation of our result is the following:

**Theorem 1.** *For any $\Delta \geqslant \sqrt{\sigma}$, and all times $t \geqslant 0$,*

$$\mathbb{P}\bigl\{\tau \in [t, t+\Delta]\bigr\} = \int_t^{t+\Delta} p(s)\,\mathrm{d}s\,[1 + r(\sigma)], \tag{12}$$

*where $r(\sigma) = \mathcal{O}(\sqrt{\sigma})$.*

If it were not for the limitation on $\Delta$, which is due to technical reasons, this result would show that the probability density of $\tau$ is given by $p(t)[1+r(\sigma)]$. We expect the remainder to be of order $\sigma$ rather than $\sqrt{\sigma}$. This result has been derived in [26] in the simplified setting of a piecewise quadratic potential, with explicit values for $\overline{V}$ and $C$, $r(\sigma) = \sigma$, and no restriction on $\Delta$. A full proof for the general case will be given in [27].

The main idea behind the proof is that sample paths reaching $x^{\text{per}}(t)$, say, during a time interval $[t, t+\Delta] \subset [nT, (n+1)T]$, are concentrated in a neighbourhood of $n$ deterministic paths, the most probable exit paths, or MPEPs. Each of these paths contributes to the probability (12). The $k$th term of the sum (8) is the contribution of a MPEP remaining inside the left-hand well for $n-k$ periods, and then idling along $x^{\text{per}}(t)$ during the remaining $k$ periods (extending the sum from $k \in \{0, \dots, n-1\}$ to $\mathbb{Z}$ only results in an error of order $\sigma$). The special form of the sum (8), involving double-exponentials, has been previously noted in [18,20] and [21].

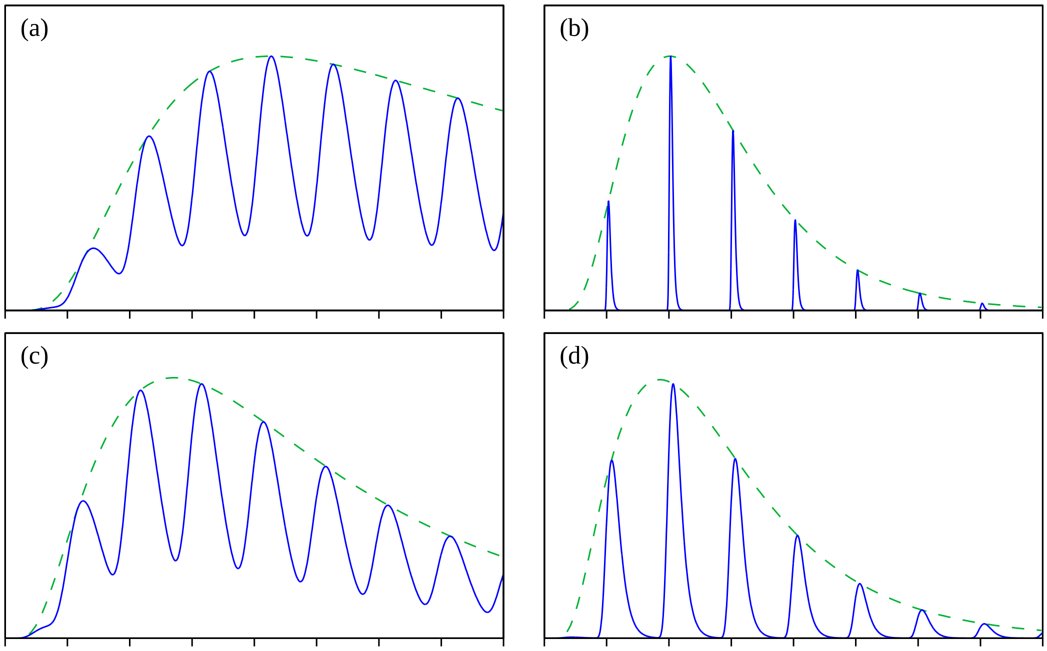


Fig. 1 – Plots of the residence-time distribution $p(t)$ (full curve) for various parameter values. The broken curve is proportional to the average density (14), but scaled to match the peak height in order to guide the eye. The $x$-axis comprises 8 periods on each plot; the vertical scale is not respected between plots. Parameter values are $\overline{V} = 0.5$, $\lambda = 1$ and (a) $\sigma = 0.4$ (*i.e.*, $D = 0.08$), $T = 2$, (b) $\sigma = 0.4$, $T = 20$, (c) $\sigma = 0.5$ (*i.e.*, $D = 0.125$), $T = 2$ and (d) $\sigma = 0.5$, $T = 5$.

*Discussion.* – Let us now analyse the expression (6) in more detail.

Taking $\theta(t)/\lambda$ as new time variable eliminates the factor $\theta'(t)/\lambda$ in the density. Thus $\theta(t)/\lambda$ can be considered as a natural parametrization of time, in which one has to measure the residence-time distribution in order to reveal its universal character. We may thus henceforth assume that $\theta(t) = \lambda t$.

The universal periodic function $Q_{\lambda T}$ depends only on the single parameter $\lambda T$. For large $\lambda T$, it consists of well-separated asymmetric peaks, while for decreasing $\lambda T$ these peaks overlap more and more and $Q_{\lambda T}(y)$ becomes flatter. In fact, one can easily compute the Fourier series of $Q_{\lambda T}$, which reads

$$Q_{\lambda T}(y) = \sum_{q\in\mathbb{Z}} 2^{\pi \mathrm{i} q/\lambda T}\,\Gamma\left(1 + \frac{\pi \mathrm{i} q}{\lambda T}\right) \mathrm{e}^{2\pi \mathrm{i} q y/\lambda T} \,. \qquad (13)$$

Since the Euler Gamma function $\Gamma$ decreases exponentially fast as a function of the imaginary part of its argument, $Q_{\lambda T}(y)$ is close, for small $\lambda T$, to a sinusoïd of mean value 1 and amplitude exponentially small in $1/2\lambda T$.

The remarkable fact that $|\ln \sigma|$ enters in the argument of $Q_{\lambda T}$ has been discovered, to our best knowledge, by Day, who termed it *cycling* [18, 19]. It means that as $\sigma$ decreases, the peaks of the residence-time distribution are translated along the time-axis, proportionally to $|\ln \sigma|$. See also [20] for an interpretation of this phenomenon in terms of MPEPs.

The remaining, non-periodic time-dependence of (6) corresponds to an averaged density, and behaves roughly like

$$\exp\biggl\{-\frac{L}{\sigma^2}\frac{\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-2\lambda t}}-\frac{t}{T_{\mathrm{K}}(\sigma)}\biggr\}. \tag{14}$$

This function grows from 0 to almost 1 in a time of order $2|\ln\sigma|/\lambda$, and then slowly decays on the scale of the Kramers time $T_{\mathrm{K}}(\sigma)$. It is maximal for $\lambda t \simeq \overline{V}/\sigma^2$.

The residence-time distribution is thus in effect controlled by two parameters: the quantity $\lambda T$, measuring the instability of the saddle, which determines the shape of the distribution within a period; and the Kramers time, which governs the decay of the average density (14).

For small $T/T_{\mathrm{K}}(\sigma)$, the residence-time distribution consists of many peaks whose height decreases only slowly on the time scale $T_{\mathrm{K}}(\sigma)$. Fig. 1 shows residence-time distributions for relatively large noise intensities, in order to make the decay more apparent. Increasing the period for constant noise intensity has two effects (Fig. 1 (a) and (b)): the peaks become narrower relatively to the period, while their height decreases faster. When $T$ increases beyond $T_{\mathrm{K}}(\sigma)$, the distribution becomes dominated by a single peak, and one enters the synchronization regime, with the particle switching wells twice per period. Increasing the noise intensity for constant period (Fig. 1 (a) and (c)) also produces a faster decay of the peaks, while at the same time the peak's location is shifted due to the cycling phenomenon. Fig. 1 (b) and (d) correspond to the same value of $T/T_{\mathrm{K}}(\sigma)$, but in (d) a larger noise intensity is responsible for broader peaks.

Moments of the residence-time distribution can easily be computed up to a correction stemming from $r(\sigma)$ (the correction due to $f_{\mathrm{trans}}(t)$ is of smaller order). Using the Fourier series (13), one finds (for $\theta(t)=\lambda t$)

$$\mathbb{E}\{\tau^n\} = \frac{1}{N}n!\,T_{\mathrm{K}}(\sigma)^n\biggl[1+2\,\mathrm{Re}\sum_{q\geqslant 1}\frac{(2\sigma^2)^{\pi\mathrm{i}q/\lambda T}}{(1-2\pi\mathrm{i}qT_{\mathrm{K}}(\sigma)/T)^{n+1}}\Gamma\biggl(1+\frac{\pi\mathrm{i}q}{\lambda T}\biggr)\biggr]\bigl[1+\mathcal{O}(r(\sigma))\bigr]. \tag{15}$$

In particular, taking $n=0$ yields the normalization $N$. One easily sees that

$$\lim_{\sigma\to 0}\sigma^2\log\mathbb{E}\{\tau\} = \overline{V}, \tag{16}$$

in accordance with the classical Wentzell–Freidlin theory.

Two other limits are of particular interest. For $\lambda T \ll 1$, the decay properties of $\Gamma(1+\mathrm{i}x)$ imply

$$\mathbb{E}\{\tau^n\} = n!\,T_{\mathrm{K}}(\sigma)^n\bigl[1+\mathcal{O}(\mathrm{e}^{-\pi^2/2\lambda T})+\mathcal{O}(r(\sigma))\bigr], \tag{17}$$

which is close to the moments of an exponential distribution with expectation $T_{\mathrm{K}}(\sigma)$. This is natural since the periodic modulation becomes flat in this limit. However, it is also true that for $T \ll T_{\mathrm{K}}(\sigma)$,

$$\mathbb{E}\{\tau^n\} = n!\,T_{\mathrm{K}}(\sigma)^n\bigl[1+\mathcal{O}(T/T_{\mathrm{K}}(\sigma))+\mathcal{O}(r(\sigma))\bigr], \tag{18}$$

independently of the value of $\lambda T$. This is due to the fact that the period of modulation is short with respect to the scale of exponential decay. The moments of the residence-time distribution can thus differ significantly from those of an exponential distribution only when $T$ is not too small compared to both $\lambda^{-1}$ and $T_{\mathrm{K}}(\sigma)$.

A third limit in which the residence-time distribution should approach an exponential one is the limit of vanishing forcing amplitude $A$ (for the particular case of the potential (2)). However, the expression (6) does not hold in cases where $A$ is not large compared to $\sigma^2$, because it makes use of the saddle-point method in the vicinity of MPEPs. An asymptotic expression for $A \ll \sigma^2$ has been proposed in [13, 14].

*Conclusion.* – The most important aspect of our rigorous expression for the residence-time distribution is the fact that it is governed essentially by two dimensionless parameters, $\lambda T$ and $T/T_{\mathrm{K}}(\sigma)$, which can be modified independently. The ratio $T/T_{\mathrm{K}}(\sigma)$ between period and Kramers time appears in most quantitative measures of SR, which indicate an optimal amplification when $T$ is close to $2T_{\mathrm{K}}(\sigma)$. In this regime, the probability of transitions between potential wells becomes significant during each period. The parameter $\lambda T$, by contrast, controls the concentration of residence times within each period, *i.e.*, the phase of transition times. Large values of $\lambda T$ yield a sharply peaked residence-time distribution, regardless of the peak's relative height.

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