Triviality of Hierarchical Models with Small Negative ϕ^4 Model in Four Dimensions

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1 Introduction

In this paper, we investigate trajectories of the Kadanoff-Wilson renormalization group (RG) acting on a Gawędzki-Kupiainen's hierarchical spin model in 4 dimensions with the following initial single spin potential;

$$v_0(\phi_x) = \mu_0\phi_x^2 - \lambda_0\phi_x^4 + \rho_0\phi_x^6. \tag{1}$$

Gawędzki-Kupiainen's hierarchical spin model is an equilibrium statistical mechanical system [4], [5], [6] which is almost massless Gaussian lattice field ϕ approximated by independent identically distributed random variables;

$$\phi_x \quad = \quad \sum_{n=0}^{\infty} L^{-n(d-2)/2} A_{[L^{-n}x]} Z^n_{[L^{-n-1}x]}. \tag{2}$$

where [ ] denotes the integral part. We take $L \geq 10$ even, $A_x = \pm 1$, and the average of A over the blocks of size L^d centered at $L\mathbf{Z}^d = \{Lx; x \in \mathbf{Z}^d\}$ equal to zero. Z^n's are assumed to be ultralocal; all Z^n_x's are indepent identically distributed random variables according to standard normal distribution $N_{0,1}$. Purpose of introducing hierarchical model is to caluculate RG easily. In fact, we have only to study following nonlinear transformation $\mathcal{R}$:

$$\exp[-\mathcal{R}v(\phi_x)] = \frac{\int \exp[-\frac{1}{2}L^d[v(L^{-(d-2)/2}\phi_x + z) + v(L^{-(d-2)/2}\phi_x - z)]]d\nu(z)}{\int \exp[-L^4 v(z)]d\nu(z)},\ x \in \mathbf{Z}^d, \tag{3}$$

where

$$\nu(z) \quad = \quad \frac{1}{(2\pi)^{1/2}} \exp\big(-\frac{1}{2}z^2\big)dz.$$

The connection between this hierarchical model and (3) is as follows (seeing also [6]). Let $V(\phi)$ denote the potential:

$$V(\phi) \quad = \quad \sum_x v(\phi_x). \tag{4}$$

Let also $\langle\, \cdot \,\rangle_{V,G_h}$ denote the following expectation

$$\langle F\rangle_{V,G_h} = \frac{1}{\mathcal{N}}\int F(\phi)\exp[-V(\phi)]d\nu_{G_h}(\phi), \tag{5}$$

where $d\nu_{G_h}$ is hierarchical massless Gaussian measure corresponding to (2), $\mathcal{N}$ is normalization, and $F = F(\phi)$ is a measurable function of ϕ's. Define the block spins ϕ' by

$$\phi'_x = L^{-(d+2)/2}\sum_{y\in B_L(x)} \phi_y, \tag{6}$$

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where

$$B_L(x) = \{y = (y_1, \cdots, y_d) \in \mathbf{Z}^d : -L/2 < Lx_i - y_i \leq L/2, 1 \leq i \leq d\}.$$

Notice that any ϕ_x can be written as

$$\phi_x = L^{-(d-2)/2}\phi'_{[L^{-1}x]} + A_x Z^0_{[L^{-1}x]}, \tag{7}$$

If a function $F(\phi)$ depends on ϕ through ϕ' only, namely, if there is a function $F'(\phi')$ onthe block spins such that

$$F(\phi) = F'(\phi'), \tag{8}$$

then it holds that

$$\langle F \rangle_{V,G_h} = \langle F' \rangle_{\mathcal{R}V,G_h}, \tag{9}$$

where

$$\begin{aligned} \exp[-\mathcal{R}V(\phi')] &= \frac{1}{\mathcal{N}} \prod_{x\in\mathbf{Z}^d} \exp[-\mathcal{R}v(\phi'_x)] \\ &= \prod_{x\in\mathbf{Z}^d} \frac{\int_{-\infty}^{\infty} \exp[-\sum_{[y/L]=x} v(L^{-(d-2)/2}\phi'_x + A_y z)]d\nu(z)}{\int_{-\infty}^{\infty} \exp[-\sum_{[y/L]=x} v(A_y z)]d\nu(z)}. \end{aligned} \tag{10}$$

With the use of $A_y = \pm 1, \sum_{[y/L]=x} A_y = 0$ and assumption that v is even, (10) is reduced to (3).

Notice that $V(\phi) \equiv 0$ is a fixed point of the transformation $\mathcal{R}$, we call this the "Gaussian fixed point". Our aim is to study the transformation $\mathcal{R}$ and to show that its iteration drives a wider class of v's to a "Gaussian fixed point" compared with Gawędzki and Kupiainen [6]. More precisely, we wish to show that $\mathcal{R}^n[v - \sum_{i=1}^2 \frac{1}{2}c_{2i}(v)\phi^{2i}] \to 0 (n \to \infty)$ for a suitable "counter terms" $\sum_{i=1}^2 \frac{1}{2}c_{2i}(v)\phi^{2i}$. Moreover we wish to esatblish a sufficiently strong convergence so that the Gibbs states will converge. In weak coupling ϕ^4 case, that is $v_0(\phi) - \frac{1}{2}c_2(v)\phi^2, v_0(\phi) = \mu_0\phi^2 + \lambda_0\phi^4$ and these coefficients are sufficiently small, triviality of the model in 4 dimensions by K. Gawędzki and A. Kupiainen in [6]. Our potential (1) is not included in the class they studied, since the coefficients of ϕ^4 can be negative. T. Hara, T. Hattori, and H. Watanabe proved the triviality of Dyson's Hierarchical Ising model in 4 dimensions [8]. This model is defined by the Ising spin measure density parameterized by $s \geq 0$:

$$\begin{aligned} h_{I,s}(\phi) &= \frac{1}{2}\big(\delta(\phi - s) + \delta(\phi + s)\big) && (11) \\ &= \lim_{\lambda\to\infty} \text{const.} \exp(-\lambda s\phi^2 - \lambda\phi^4). && (12) \end{aligned}$$

Their method, using Newman's inequalities on truncated correlations has the power to analyze in the strong coupling regime. So, we expect to apply it to analyzing not only Hierarchical Ising model, but also very general class of initial single spin potentials. However, (1) is still out of the range of their method. Because these initial single spin potentials doesn't satisfy Lee-Yang property $\mathcal{L}$ (seeing also [10]), so truncated correlations of these potentials dissatisfy Newman's inequalities that are the key estimations to perform their method (seeing also [8] and [11]). Nevertheless, we will prove that under a suitable conditions on μ_0, λ_0, ρ_0 and for suitable counter terms there exists a trajectory which leads to a Gaussian fixed point using Gawędzki and Kupiainen's method. Notice that triviality of the hierarchical model with the potential $\mu\phi^2 + \lambda\phi^4 + \rho\phi^6$ in 3 dimensions was already studied by Müller and Schiemann [9]. So, in principle it should be clear that trivialiy of $d \geq 4$ case. However they didn't mention it in their paper, so we believe it is worth while to confirm that it is true exactly. Remark also that Gawędzki and Kupiainen succeeded the construction of non-trivial Eucledean ϕ^4_4 theory with complex coupling constant with negative real part [7]. However parameter region of coupling constants which we studied is not included in their studied region. Our strategy is quite simple. Namely, we will show that the parameters will enter the region where the theorem of Gawędzki and Kupiainen [6] can be applied (we call this region "G-K region"), after some iteration (finite time iteration) of the RG.

First of all, we state results precisely. We shall use as following notation:

$$\begin{aligned} v_n(\phi) &= \mathcal{R}^n v(\phi), && (13) \\ (v_n)_k(\phi) &= \frac{1}{k!}\frac{d^k(v_n)(0)}{d\phi^k}\phi^k, && (14) \\ (v_n)_{\geq k}(\phi) &= v_n(\phi) - \sum_{l<k}(v_n)_l(\phi). && (15) \end{aligned}$$

Given a single spin potential

$$v(\phi) = \mu\phi^2 + \lambda\phi^4 + \rho\phi^6 + v_{\geq 8}(\phi), \tag{16}$$

let us define $\mu(v) \in \mathbf{R}$ by

$$v(\phi) - \frac{1}{2}\sum_{i=1}^{2} c_{2i}(v)\phi^{2i} = (\mu(v) - \frac{1}{2}c_2(v))\phi^2 + (\lambda - \frac{1}{2}c_4(v))\phi^4 + \rho_n\phi^6 + v_{\geq 8}(\phi), \tag{17}$$

where

$$c_2(v) = \frac{12\lambda}{1-L^{-2}} - \frac{180\rho}{(1-L^{-2})(1-L^{-4})} + \frac{90\rho}{1-L^{-4}}, \tag{18}$$

$$c_4(v) = \frac{30\rho}{1-L^{-2}}, \tag{19}$$

are the counter terms originating from Wick ordering. Now let us define a class of initial single spin potentials $\mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ satisfying the following conditions for constants L, D, C_1, n_0, and ρ_0.

Ta for $|\mathbf{Im}\phi| < C_1((L^{-4}\rho_0^{-1})^{1/6} \wedge n_0^{1/4})$, $\exp[-v_0(\phi)]$ is analytic, positive for real ϕ even, and

$$|e^{-(v_0)(\phi)}| \leq \exp[D - (\lambda_0^{1/2} + \rho_0^{1/3})|\phi|^2 + 20\lambda_0(\mathbf{Im}\phi)^4 + 2004\rho_0(\mathbf{Im}\phi)^6], \tag{20}$$

Tb for $|\phi| < C_1((L^{-4}\rho_0^{-1})^{1/6} \wedge n_0^{1/4})$, $(v_0)_{\geq 4}(\phi)$ is analytic,

$$(v_0)_{\geq 4}(\phi) = \big(\lambda_0 - \frac{15\rho_0}{1-L^{-2}}\big)\phi^4 + \rho_0\phi^6 + (v_0)_{\geq 8}(\phi), \tag{21}$$

with

$$\frac{C_{--}L^{-4}}{n_0} \leq \lambda_0 \leq \frac{C_{++}L^{-4}}{n_0}, \quad C_{--} = \frac{1}{42}, \quad C_{++} = \frac{1}{28}, \tag{22}$$

$$|(v_0)_{\geq 8}(\phi)| \leq \rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}. \tag{23}$$

Remark that the coefficient of ϕ^4 is represented as $\lambda_0 - \frac{15\rho_0}{1-L^{-2}}$. If we take value of ρ_0 suitably large, then the coefficient of ϕ^4 will be negative. So, the class $\mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ includes small negative ϕ^4 case. This case is the main object of our study in this paper. Of course, this class includes potentials which Gawędzki and Kupiainen studied. In fact it is easy to see that $\mathcal{V}_0(L, D, C_1, n_0, 0)$ is a class which are investigated in [6]. We will prove the following for our class.

Theorem 1 *There exist positive constants D, $\bar{C_1}(L, D) \geq L$, $\bar{n_0}(L, D, C_1) \geq L^{48}$, such that the following holds. Let $C_1 \geq C_1(L, D)$, $n_0 \geq \bar{n}_0(L, D, C_1)$, and*

$$0 \leq \rho_0 \leq L^{-4}n_0^{-1}. \tag{24}$$

Define the RG as follows,

$$e^{-v_{n+1}(\phi)} = \int \exp[-\frac{1}{2}L^4 \sum_{\pm} v_n(L^{-1}\phi \pm z)]d\nu(z)/(\phi = 0). \tag{25}$$

Then there exists $\mu_{crit} \in \mathbf{R}$ such that the iterates v_n of the recursion converge to zero uniformly on compacts in $\mathbf{C^1}$, if we start from $v_0 \in \mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ with $\mu(v_0) = \mu_{crit}$.

The proof goes along the following line. In the beginning, we are in the regions where $(v_n)_{\geq 6}(\phi)$ is dominant. For properly chosen initial data, $(v_n)_{\geq 6}(\phi)$ decreases rapidly, and we then go into the region where ϕ^4 term of $v_n(\phi)$ is not negligible compared to $(v_n)_{\geq 6}(\phi)$. As the recursion proceeds, the ϕ^4 term gradually becomes dominant but small and finally we can show that $v_n(\phi)$ falls into the G-K region. We will prove this by means of two lemmas. First, for $n \geq 0$, let $\mathcal{V}_n^1(L, D, C_1, n_0, \rho_0)$ be the class of potentials v_n satisfying,

L1a for $|\mathbf{Im}\phi| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, $\exp[-v_0(\phi)]$ is analytic, positive for real ϕ, even, and

$$|e^{-v_n(\phi)}| \leq \exp[D - (\lambda_n^{1/2} + \rho_n^{1/3})|\phi|^2 + 20\lambda_n(\mathbf{Im}\phi)^4 + 2004\rho_n(\mathbf{Im}\phi)^6], \tag{26}$$

L1b for $|\phi| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, $(v_n)_{\geq 4}(\phi)$ is analytic, and

$$(v_n)_{\geq 4}(\phi) = \big(\lambda_n - \frac{15\rho_n}{1-L^{-2}}\big)\phi^4 + \rho_n\phi^6 + (v_n)_{\geq 8}(\phi), \tag{27}$$

with

$$|\lambda_n - \lambda_0| \leq n n_0^{-7/4}, \tag{28}$$

$$|\rho_n - L^{-2n}\rho_0| \leq nL^{-2n}n_0^{-7/4}, \tag{29}$$

$$|(v_n)_{\geq 8}(\phi)| \leq (\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{-n}, \tag{30}$$

Remark that $\mathcal{V}_0(L, D, C_1, n_0, \rho_0) = \mathcal{V}_0^1(L, D, C_1, n_0, \rho_0)$, if L, D, C_1, n_0, and ρ_0 are same constants.

Lemma 1 *There exist constants D, $\bar{C}_1(L, D) \geq L$, $\bar{n_0}(L, D, C_1) \geq L^{48}$ such that the following holds. Let $C_1 \geq \bar{C}_1(L, D)$, $n_0 \geq \bar{n_0}(L, D, C_1)$and $n \geq 0$ satisfies the inequality*

$$(n_0 + n)^{1/4} \geq (L^{2n-4}\rho_0^{-1})^{1/6}. \tag{31}$$

Suppose also that $v_0(\phi) \in \mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ with

$$L^{-4}n_0^{-3/2} \leq \rho_0 \leq L^{-4}n_0^{-1}, \tag{32}$$

and $v_n(\phi) \in \mathcal{V}_n^1(L, D, C_1, n_0, \rho_0)$. Then, there exists a closed interval $J_n \subset I_n = [-(n_0+n)^{-3/2}, (n_0+n)^{-3/2}]$ such that for μ_n running through J_n, $v_{n+1} \in \mathcal{V}_{n+1}^1(L, D, C_1, n_0, \rho_0)$. Further, the map $\mu_n \to \mu_{n+1}$ sweeps I_{n+1} continuously. Here, μ_n, μ_{n+1}, are $\mu(v_n)$ and $\mu(v_{n+1})$.

Iterating Lemma 1, each time we can choose a subinterval $J_{n+1} \subset J_n \subset I_0$ of the initial mass squared values such that μ_n sweeps I_{n+1}. The effect of relatively large coefficient of ϕ^6 provides us with a positive coefficient of ϕ^4 in the next step, and we can get rid of negative ϕ^4 term. However, we can't iterate Lemma 1 arbitary times because of assumption (31). In other words, $(v_n)_{\geq 6}(\phi)$ will not be dominant any more compared with ϕ^4 term after some iteration. After applying Lemma 1 as much as possible, we are still not in the G-K region, and we have to trace the trajectory carefully for some more steps. So we must prepare new assumptions.

Put

$$n_1 = \max\{n \in \mathbf{N} | (L^{2n-4}\rho_0^{-1})^{1/6} \leq (n_0+n)^{1/4}\} + 1. \tag{33}$$

Notice that this number is the first n for which Lemma 1 cannot be applied. Obviously, we have

$$n_1 \leq \frac{1}{4}\log_L n_0. \tag{34}$$

Let us define a class of single spin potentials $\mathcal{V}_{n_1+n}^2(L, D, C_1, n_0, \rho_0)$ satisfying,

L2a for $|\mathbf{Im}\phi| < C_1(n_0 + n_1 + n)^{1/4}$, $\exp[-v_{n_1+n}]$ is analytic and positive for real ϕ, even, and

$$|e^{-v_{n_1+n}(\phi)}| \leq \exp[D - (\lambda_{n_1+n}^{1/2} + \rho_{n_1+n}^{1/3})|\phi|^2 + 20\lambda_{n_1+n}(\mathbf{Im}\phi)^4 + 2004\rho_{n_1+n}(\mathbf{Im}\phi)^6], \tag{35}$$

L2b for $|\phi| < C_1(n_0 + n_1 + n)^{1/4}$, $(v_n)_{\geq 4}(\phi)$ is analytic,

$$(v_{n_1+n})_{\geq 4}(\phi) = \big(\lambda_{n_1+1} - \frac{15\rho_{n_1+n}}{1-L^{-2}}\big)\phi^4 + \rho_{n_1+n}\phi^6 + (v_{n_1+n})_{\geq 8}, \tag{36}$$

with

$$|\lambda_{n_1+n} - \lambda_0| \leq (n_1+n)n_0^{-7/4}, \tag{37}$$

$$|\rho_{n_1+n} - L^{-2(n_1+n)}\rho_0| \leq (n_1+n)L^{-2(n_1+n-1)}n_0^{-7/4}, \tag{38}$$

$$|(v_{n_1+n})_{\geq 8}(\phi)| \leq L^{-3n-n_1}\big(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}\big). \tag{39}$$

Remark that $\mathcal{V}_{n_1}^1(L, D, C_1, n_0, \rho_0) = \mathcal{V}_0^2(L, D, C_1, n_0, \rho_0)$, if L, D, C_1, n_0, and ρ_0 are same constants.

Lemma 2 *There exist constants D, $\bar{C}_1(L,D) \geq L$, $\bar{n_0}(L,D,C_1) \geq L^{48}$ such that the following holds. Let $C_1 \geq \bar{C}_1(L,D)$, $n_0 \geq \bar{n_0}(L,D,C_1)$, $\log_L n_0 \geq n \geq 0$. $v_0(\phi) \in \mathcal{V}_0(L,D,C_1,n_0,\rho_0)$, and $v_{n_1+n}(\phi) \in \mathcal{V}^2_{n_1+n}(L,D,C_1,n_0,\rho_0)$. Then, there exists a closed interval $J_{n_1+n} \subset I_{n_1+n} = [-(n_0+n_1+n)^{-3/2}, (n_0+n_1+n)^{-3/2}]$ such that for μ_n running through J_n, $v_{n+1} \in \mathcal{V}^2_{n+1}$. Further, the map $\mu_n \to \mu_{n+1}$ sweeps I_{n_1+n+1} continuously.*

A proof of Lemma 2 is very close to that of Lemma 1. Difference of two proofs is that the condition that $|\phi| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$ is replaced by the condition that $|\phi| < C_1(n_0+n_1+n)^{1/4}$. In other words, ϕ^4 term becomes dominant compared with $(v_{n_1+n})_{\geq 6}(\phi)$. With Lemma 2 we can continue iterations, and we can make sure that after a finite number of iterations, this potential is in the G-K region. More precisely, Gawędzki and Kupiainen introduced the class of potentials $\mathcal{V}^{G-K}_n(L,D,C_1,n_0)$, which is defined to satisfy

G-Ka $e^{-(v_n)_{\geq 4}(\phi)}$ is analytic for $|\mathbf{Im}\phi| < C_1(n_0+n)^{1/4}$, positive for real ϕ, even and

$$|\exp[-(v_n)_{\geq 4}(\phi)]| \leq \exp[D - \lambda_n^{1/2}|\phi|^2 + 20\lambda_n(\mathbf{Im}\phi)^4], \tag{40}$$

G-Kb for $|\phi| < C_1(n_0+n)^{1/4}$, $(v_n)_{\geq 4}(\phi)$ is analytic,

$$(v_n)_{\geq 4}(\phi) = \lambda_n\phi^4 + (v_n)_{\geq 6}(\phi) \tag{41}$$

with

$$\frac{C_- L^{-4}}{n_0+n} \leq \lambda_n \leq \frac{C_+ L^{-4}}{n_0+n}, \; C_- = \frac{1}{48}, \; C_+ = \frac{1}{24}, \tag{42}$$

$$|(v_n)_{\geq 6}(\phi)| \leq (n_0+n)^{-3/4} \tag{43}$$

In this class $\mathcal{V}^{G-K}_n(L,D,C_1,n_0)$, Gawędzki and Kupiainen proved the following,

Theorem 2 *(Gawędzki and Kupiainen) There exist constants D, $\bar{C}_1(L,D)$, $\bar{n}_0(L,D,C_1)$ such that the following holds. Let $C_1 \geq \bar{C}_1(L,D)$, $n_0 \geq \bar{n}_0(L,D,C_1)$ and $n \geq 0$. Put*

$$v_n(\phi) = \mu_n - \frac{6\lambda_n}{1-L^{-2}}\phi^2 + (v_n)_{\geq 4}(\phi) \tag{44}$$

where $(v_n)_{\geq 4}(\phi) \in \mathcal{V}^{G-K}_n(L,D,C_1,n_0)$. Then, there exists a closed interval $J_n \subset I_n$ such that for μ_n running through J_n, $(v_{n+1})_{\geq 4}(\phi) = v_{n+1}(\phi) - \mu_{n+1}\phi^2 + \frac{6\lambda_{n+1}}{1-L^{-2}}\phi^2 \in \mathcal{V}^{G-K}_{n+1}(L,D,C_1,n_0)$. Futher, the map $\mu_n \to \mu_{n+1}$ sweeps I_{n+1} continuously.

Finally, let us explain differences between Gawędzki and Kupiainen [6] and this paper. First of all, we study contribution of the term of ϕ^6 to the term of ϕ^4 rigolously to control the term of ϕ^4 even with negative coefficient. Secondly, our class of single spin potential permits us to take $v_{\geq 6}$ in wider class than that Gawędzki and Kupiainen studied. However, we are still not in the G-K region, and we have to trace the trajectory carefully for some more steps. Finally, we would like to remark about some constants, like $2 \times 2\dot{1}3!! \times 606 \times 392 \times 2$ in this paper. These numbers themselves are not important, however it is important that these are absolute constants.

2 Proof of Lemma 1

We shall prove that $v'_n(\phi) = v_{n+1}(\phi)$ is in $\mathcal{V}^1_{n+1}(L,D,C_1,n_0,\rho_0)$,if μ_n is in I_n. This proof involves Small field region, and Large field region analyses corresponding to the cases either $|\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$, or $|\mathbf{Im}\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$ respedtively (see [6]). In the small field, we prove that $v'_n(\phi)$ satisfies **L1b**, n being replaced by $n+1$. We call it simply **L1b′**. On the other hand, we only investigate the global behavior of $v'_n(\phi)$ in the large field region, i.e. we confirm that $v'_n(\phi)$ satisfies with (30) of **L1a′**, n being replaced by $n+1$. Throughout the proof, we assume that $|\mu_n| \leq (n_0+n)^{-3/2}$.

2.1 Small field region

Let $v_n \in \mathcal{V}^1_n$. Write $\chi_1(z) = \chi(|z| < (L^{2n-4}\rho_0^{-1})^{1/6})$ and throughout this subsection, we assume that ϕ is in the region, $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$. Remark that we have to put C_1 to satisfy the inequality $|L^{-1}\phi \pm z| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$ for $|z| < (L^{2n-4}\rho_0^{-1})^{1/6}$ and $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$. Next, decompose $v_{n+1}(\phi)$ as follows,

$$v_{n+1}(\phi) = v'_n(\phi) = \tilde{v}'_n(\phi) + \approx{v}'_n(\phi), \tag{45}$$

$$e^{-\overset{\sim}{v'_n}(\phi)} = \int \exp[-\frac{1}{2}L^4 \sum_{\pm} v_n(L^{-1}\phi \pm z)]d\nu_1(z)/(\phi = 0)_{small}, \tag{46}$$

where

$$\begin{aligned} (\phi = 0)_{small} &= \int \exp[-L^4 v_n(z)]d\nu_1(z), & (47)\\ d\nu_1(z) &\equiv \chi_1(z)e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}. & (48) \end{aligned}$$

First of all , we estimate $\overset{\sim}{v'_n}(\phi)$ in 3.1.1, and then do $\overset{\approx}{v'_n}(\phi)$ in 2.1.2.

2.1.1 Estimation of $\overset{\sim}{v'_n}(\phi)$

By definition of μ_n, λ_n, and ρ_n, we have

$$\begin{aligned} e^{-\overset{\sim}{v'_n}(\phi)} &= \exp[-L^2\mu_n\phi^2 + (\frac{6L^2\lambda_n}{1-L^{-2}} - \frac{90L^2\rho_n}{(1-L^{-2})(1-L^{-4})} + \frac{45L^2\rho_n}{1-L^{-4}})\phi^2 - (\lambda_n - \frac{15\rho_n}{1-L^{-2}})\phi^4 - L^{-2}\rho_n\phi^6] \\ &\quad \times \int \exp[-w_\phi(z)]d\nu_1(z)/(\phi = 0)_{small}, \qquad (49) \end{aligned}$$

$$w_\phi(z) = w_0(z) + w_2(z)\phi^2 + w_4(z)\phi^4 + w_6(z)\phi^6 + w_{\geq 8}(\phi, z), \tag{50}$$

and

$$\begin{aligned} w_0(z) &= L^4 v_n(z) \\ &= L^4\big(\mu_n z^2 - (\frac{6\lambda_n}{1-L^{-2}} - \frac{90\rho_n}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho_n}{1-L^{-4}})z^2\big) \\ &\quad + L^4\big((\lambda_n - \frac{15\rho_n}{1-L^{-2}})z^4 + \rho_n z^6 + (v_n)_{\geq 8}(z)\big), & (51)\\ w_2(z) &= L^2\big(6(\lambda_n - \frac{15\rho_n}{1-L^{-2}})z^2 + 15\rho_n z^4 + \frac{1}{2}\frac{d^2}{dz^2}(v_n)_{\geq 8}(z)\big), & (52)\\ w_4(z) &= 15\rho_n z^2 + \frac{1}{4!}\frac{d^4}{dz^4}(v_n)_{\geq 8}(z), & (53)\\ w_6(z) &= L^{-2}\frac{1}{6!}\frac{d^6}{dz^6}(v_n)_{\geq 8}(z), & (54)\\ w_{\geq 8}(\phi, z) &= \frac{L^{-4}\phi^8}{7!}\int_0^1 dt(1-t)^7\frac{d^8}{dz^8}\{(v_n)_{\geq 8}(L^{-1}t\phi + z) + (v_n)_{\geq 8}(L^{-1}t\phi - z)\}. & (55) \end{aligned}$$

Notice that $v_n(\phi)$ is even and analytic from condition **L1a - L1b**. Using the Cauchy formula and (30) to estimate $\frac{d^8}{dz^8}(v_n)_{\geq 8}(\phi)$ on the support of $d\nu(z)$, we obtain

$$\begin{aligned} |(v_n)_{\geq 8}(z)| &\leq \frac{1}{7!}\int_0^1 dt(1-t)^7|z^8\frac{d^8}{dz^8}(v_n)_{\geq 8}(tz)| & (56)\\ &\leq \frac{C_1}{8!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{-11n/3}z^8. \end{aligned}$$

$$\begin{aligned} |\frac{d^2}{dz^2}(v_n)_{\geq 8}(z)| &\leq \frac{1}{5!}\int_0^1 dt(1-t)^5|z^6\frac{d^8}{dz^8}(v_n)_{\geq 8}(tz)| & (57)\\ &\leq \frac{C_1}{6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{-11n/3}z^6, \end{aligned}$$

$$\begin{aligned} |\frac{d^4}{dz^4}(v_n)_{\geq 8}(z)| &\leq \frac{1}{3!}\int_0^1 dt(1-t)^3|z^4\frac{d^8}{dz^8}(v_n)_{\geq 8}(tz)| & (58)\\ &\leq \frac{C_1}{4!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{-11n/3}z^4, \end{aligned}$$

$$
\begin{aligned}
|\frac{d^6}{dz^6}(v_n)_{\geq 8}(z)| &\leq \int_0^1 dt(1-t)|z^2\frac{d^8}{dz^8}(v_n)_{\geq 8}(tz)| \\
&\leq \frac{C_1}{2(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{-11n/3}z^2.
\end{aligned} \tag{59}
$$

Put

$$
w_0(z) = \alpha_2^0 z^2 + \alpha_4^0 z^4 + \alpha_6^0 z^6 + L^4(v_n)_{\geq 8}(z). \tag{60}
$$

To estimate α_2^0, we perform the following calculation. First we estimate μ_n, λ_n, and ρ_n. From (22), (23), (28) to (30) and (50) to (55), we have for a sufficiently large n_0,

$$
\begin{aligned}
|\mu_n(L^{2n}\rho_0^{-1})^s| &\leq 1 \text{ for } s \leq 1, & (61)\\
\lambda_n &\leq 2\lambda_0, \text{ for } 0 \leq n \leq n_1 & (62)\\
|\rho_n(L^{2n}\rho_0^{-1})^s| &\leq 2 \text{ for } s \leq 1, & (63)\\
|(v_n)_{\geq 8}(\phi)|n_0^{1/2} &\leq 1. & (64)
\end{aligned}
$$

These are valid for $n \leq (1/4)\log_L n_0$, $n_0 \geq L^{48}$. Using this and by the assumptions of the lemma on μ_n, ρ_0.

$$
\begin{aligned}
|\alpha_2^0| &\leq L^4|\mu_n| + \frac{6L^4|\lambda_n|}{1-L^{-2}} + \frac{90L^4|\rho_n|}{(1-L^{-2})(1-L^{-4})} + \frac{45L^4|\rho_n|}{(1-L^{-4})} \\
&\leq n_0^{-1} + \frac{n_0^{-1}}{2(1-L^{-2})} + \frac{180n_0^{-1}}{(1-L^{-2})(1-L^{-4})} + \frac{90n_0^{-1}}{1-L^{-4}} \\
&\leq 542n_0^{-1}
\end{aligned} \tag{65}
$$

In the same way, we have α_4^0, α_6^0, α_8^0,

$$
\begin{aligned}
|\alpha_4^0| &= L^4|\lambda_n| + \frac{15L^4|\rho_n|}{1-L^{-2}} \leq 61n_0^{-1}, & (66)\\
|\alpha_6^0| &\leq L^4|\rho_n| \leq 2L^{4-2n}\rho_0, & (67)\\
|L^4(v_n)_{\geq 8}(z)| &\leq \frac{C_1}{8!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{4-11n/3}z^8 = |\alpha_8^0|z^8. & (68)
\end{aligned}
$$

In the same way, we can estimate $|w_2(z)|$, $|w_4(z)|$, $|w_6(z)|$, $|w_{\geq 8}(\phi, z)|$ as follows,

$$
\begin{aligned}
|w_2(z)| &\leq |\alpha_2^2||z^2 + |\alpha_4^2|z^4 + |\alpha_6^2|z^6, & (69)\\
|w_4(z)| &\leq |\alpha_2^4|z^2 + |\alpha_4^4|z^4, & (70)\\
|w_6(z)| &\leq |\alpha_2^6|z^2, & (71)\\
|w_{\geq 8}(\phi, z)| &\leq |\alpha_0^8| + |\alpha_2^8|z^2 + |\alpha_4^8|z^4 + |\alpha_6^8|z^6 + |\alpha_8^8|z^8, & (72)
\end{aligned}
$$

where $\alpha_{2i}^{2j}, i, j = 0, 1, \cdots 4$ are

$$
\begin{aligned}
|\alpha_2^2| &\leq L^2 6\big(|\lambda_n| + \frac{15|\rho_n|}{1-L^{-2}}\big) \leq 361L^{-2}n_0^{-1}, & (73)\\
|\alpha_4^2| &\leq 15L^2|\rho_n| \leq 30L^{2-2n}\rho_0, & (74)\\
|\alpha_6^2| &\leq \frac{C_1}{6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{2-11n/3}, & (75)\\
|\alpha_2^4| &\leq 15|\rho_n| \leq 30L^{-2n}\rho_0, & (76)\\
|\alpha_4^4| &\leq \frac{C_1}{4!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{-11n/3}, & (77)\\
|\alpha_2^6| &\leq \frac{C_1}{2(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{-2-11n/3}, & (78)\\
|\alpha_0^8| &\leq L^{4-n}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}), & (79)\\
|\alpha_2^8| &\leq \frac{2C_1^7}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{1/3}L^{4-5n/3}, & (80)
\end{aligned}
$$

$$
\begin{aligned}
|\alpha_4^8| &\le \frac{6C_1^5}{3!\cdot 4!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})(L^4\rho_0)^{2/3}L^{4-7n/3}, && (81)\\
|\alpha_6^8| &\le \frac{2C_1^3 5!!}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})(L^4\rho_0)L^{4-3n}, && (82)\\
|\alpha_8^8| &\le \frac{2C_1 7!!}{8!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{4-11n/3}. && (83)
\end{aligned}
$$

Remark that $|w_{\ge 8}(\phi,z)|$ is estimated as follows. (30) of **L1b** implies

$$
|\frac{L^4}{2}\sum_{\pm}(v_n)_{\ge 8}(L^{-1}\phi\pm z)| \ \le\ (\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{4-n}. \tag{84}
$$

Therefore, we have from (56) to (59) and (84)

$$
\begin{aligned}
|w_{\ge 8}(\phi,z)| &\le |\frac{L^4}{2}\sum_{\pm}(v_n)_{\ge 8}(L^{-1}\phi\pm z)| + |L^4(v_n)_{\ge 8}(z)| + |\frac{1}{2}L^2\phi^2\frac{d^2}{dz^2}(v_n)_{\ge 8}(z)|\\
&\quad +|\frac{1}{4!}\phi^4\frac{d^4}{dz^4}(v_n)_{\ge 8}(z)| + |\frac{1}{6!}L^{-2}\phi^6\frac{d^6}{dz^6}(v_n)_{\ge 8}(z)|\\
&\le (\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{4-n} + \frac{2C_1 7!!}{8!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{4-11n/3}z^8\\
&\quad +\frac{2C_1^3 5!!}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})(L^4\rho_0)L^{4-3n}z^6\\
&\quad +\frac{6C_1^5}{3!\cdot 4!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})(L^4\rho_0)^{2/3}L^{4-7n/3}z^4\\
&\quad +\frac{2C_1^7}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})(L^4\rho_0)^{1/3}L^{4-5n/3}z^2,
\end{aligned} \tag{85}
$$

which gives (79) - (83). Taking logarithm of (49), we get

$$
\begin{aligned}
\tilde{v'_n}(\phi) &= L^2\mu_n\phi^2 - \Big(\frac{6L^2\lambda_n}{1-L^{-2}} - \frac{90L^2\rho_n}{(1-L^{-2})(1-L^{-4})} + \frac{45L^2\rho_n}{1-L^{-4}}\Big)\phi^2 + \Big(\lambda_n - \frac{15\rho_n}{1-L^{-2}}\Big)\phi^4\\
&\quad +L^{-2}\rho_n\phi^6 - \log\int e^{-w_\phi(z)}d\nu_1(z) + \log(\phi=0)_{small}.
\end{aligned} \tag{86}
$$

Notice that $e^{-\tilde{v'_n}(\phi)}\ne 0$ if ϕ is in the small field region i.e. $|\phi|\le(10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$ and n_0 is sufficiently large. Because

$$
\begin{aligned}
|\int e^{-w_\phi(z)}d\nu_1(z) - \int e^{-w_0(z)}d\nu_1(z)| &\le e^{||w_0(z)||_{\chi_1}}\int |e^{-w_\phi(z)+w_0(z)}-1|d\nu_z(z)\\
&\le 65^2 e^{126}L^8C_1^8 n_0^{-1/3},
\end{aligned} \tag{87}
$$

where

$$
||f(z)||_{\chi_1} = \sup_{|z|<(L^{2n-4}\rho_0)^{1/6}}|f(z)|. \tag{88}
$$

Here, we used (65) to (68). Remark that $\int e^{-w_0(z)}d\nu_1(z)$ is real positive which does not depend on n_0. From the pertubation expansion :

$$
-\log\int e^{-w_\phi(z)}d\nu_1(z) = -\log\int d\nu_1(z) + \langle w_\phi(z)\rangle_0 - \int_0^1 dt(1-t)\langle w_\phi(z);w_\phi(z)\rangle_t. \tag{89}
$$

Where

$$
\langle\cdots\rangle_t \ \equiv\ \int\cdots e^{-tw_\phi(z)}d\nu_1(z)\Big/\int e^{-tw_\phi(z)}d\nu_1(z). \tag{90}
$$

Now, we shall estimate each part of (89).

Estimation of $\langle w_\phi(z)\rangle_0$

By (50) - (55),

$$\begin{aligned}\langle w_\phi(z)\rangle_0 &= \langle w_0(z)\rangle_0 + L^2\phi^2\langle w_2(z)\rangle_0 + \phi^4\langle w_4(z)\rangle_0 + L^{-2}\phi^6\langle w_6(z)\rangle_0 + \langle w_{\geq 8}(\phi,z)\rangle_0 \\ &= L^4\langle v_n(z)\rangle_0 + 6L^2\phi^2\big(\lambda_n - \frac{15\rho_n}{1-L^{-2}}\big)\langle z^2\rangle_0 + 15L^2\rho_n\phi^2\langle z^4\rangle_0 + L^2\phi^2\langle\frac{d^2}{2dz^2}(v_n)_{\geq 8}(z)\rangle_0 \\ &\quad +15\rho_n\phi^4\langle z^2\rangle_0 + \phi^4\langle\frac{d^4}{4!dz^4}(v_n)_{\geq 8}(z)\rangle_0 + L^{-2}\phi^6\langle\frac{d^6}{6!dz^6}(v_n)_{\geq 8}(z)\rangle_0 + \langle w_{\geq 8}(\phi,z)\rangle_0. \qquad (91)\end{aligned}$$

Note, that for $m = 0, 1, \cdots, 8$

$$|\langle z^{2m}\rangle_0 - (2m-1)!!| \leq \frac{2\{(2m-1)!!\check{\Phi}((L^{2n-4}\rho_0)^{1/6}) + M\sqrt{2}\check{\Phi}((L^{2n-4}\rho_0)^{1/6}/\sqrt{2})\}}{\sqrt{2\pi} - 2\check{\Phi}((L^{2n-4}\rho_0)^{1/6})}. \qquad (92)$$

where

$$\begin{aligned}\check{\Phi}(x) &= \int_x^\infty \frac{\exp[-y^2/2]}{\sqrt{2\pi}}dy \leq \sqrt{2}\exp[-x^2/4] \\ M &= \max_{0\leq m\leq 8, z\in\mathbf{R}} z^{2m}\exp[-z^2/4] = \max_{0\leq m\leq 8}(4m)^m\exp[-m] = \big(\frac{32}{e}\big)^8\end{aligned}$$

We get

$$\begin{aligned}\langle w_\phi(z)\rangle_0 &= L^4\langle v_n(z)\rangle_0 + 6L^2\big(\lambda_n - \frac{15\rho_n}{1-L^{-2}}\big)\phi^2 + 45L^2\rho_n\phi^2 + \overset{\sim 0,0}{R_2}(L,n_0,\rho_0,n)\phi^2 \\ &\quad +15\rho_n\phi^4 + \overset{\sim 0,0}{R_4}(L,n_0,\rho_0,n)\phi^4 + \overset{\sim 0,0}{R_6}(L,n_0,\rho_0,n)\phi^6 + \langle w_{\geq 8}(\phi,z)\rangle_0, \qquad (93)\end{aligned}$$

where, the terms $\overset{\sim 0,0}{R_{2i}}(L,n_0,\rho_0,n), i = 1,\cdots,3$ satisfy

$$|\overset{\sim 0,0}{R_{2i}}(L,n_0,\rho_0,n)| \leq (\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{-11n/3}, \qquad (94)$$

because of (57) - (59), and (92). Finally, let us estimate $\langle w_{\geq 8}(\phi,z)\rangle_0$. From (72), (79) to (83), we have

$$\begin{aligned}|\langle w_{\geq 8}(\phi,z)\rangle_0| &\leq \alpha_0^8 + \alpha_2^8\langle z^2\rangle_0 + \alpha_4^8\langle z^4\rangle_0 + \alpha_6^8\langle z^6\rangle_0 + \alpha_8^8\langle z^8\rangle_0 \\ &\leq L^{4-n}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}) + \frac{2C_1 7!!}{8!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3}L^{4-11n/3} \\ &\quad + \frac{2C_1^3 5!!}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)L^{4-3n} \\ &\quad + \frac{6C_1^5}{3!\cdot 4!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{2/3}L^{4-7n/3} \\ &\quad + \frac{2C_1^7}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{1/3}L^{4-5n/3} \\ &\leq L^{4-n}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}) + (\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{1/3}L^{4-5n/3}. \qquad (95)\end{aligned}$$

Here, we take $C_1 \geq 6$.

Estimation of $\int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t$

Let us estimate

$$\begin{aligned}\int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t &= \int_0^1 dt(1-t)\sum_{i,j}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t \\ &= \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t + \int_0^1 dt(1-t)\sum_{i,j\neq 0}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t, \qquad (96)\end{aligned}$$

where

$$\tilde{w}_{2i} = \begin{cases} w_{2i}(z)\phi^{2i} & i = 0,1,2,3 \\ w_{\geq 8}(\phi,z) & i = 4.\end{cases}$$

By definition, we have

$$\langle \tilde{w}_{2i}; \tilde{w}_{2j} \rangle_t = \langle e^{-tw_\phi(z)} \rangle_0^{-1} \langle \tilde{w}_{2i}^2 e^{-tw_\phi(z)} \rangle_0 - \langle e^{-tw_\phi(z)} \rangle_0^{-2} \langle \tilde{w}_{2i} e^{-tw_\phi(z)} \rangle_0^2. \tag{97}$$

Note that the support of $d\nu_1(z)$ is $|z| < (L^{2n-4}\rho_0^{-1})^{1/6}$ and by the assumption of the lemma we have $(L^{-2n}\rho_0)^{1/6} \geq (n_0+n)^{-1/4}$. We get uniform estimate for $|z| < (L^{2n-4}\rho_0^{-1})^{1/6}$ and $|\phi| < \frac{10LC_1}{11}(L^{2n-4}\rho_0^{-1})^{1/6}$,

$$\begin{aligned} |w_\phi(z)| &\leq |w_0(z)| + |w_2(z)||\phi|^2 + |w_4(z)||\phi|^4 + |w_6(z)||\phi|^6 + |w_{\geq 8}(\phi, z)| \\ &\leq 126 + 753C_1^2 + 31C_1^4 + 1 + 1 \\ &\leq 912C_1^4. \end{aligned} \tag{98}$$

We have

$$|\langle \tilde{w}_{2i}; \tilde{w}_{2j} \rangle_t| \quad \leq \quad \exp[3648C_1^4]\big(\langle |\tilde{w}_{2i}||\tilde{w}_{2j}| \rangle_0 + \langle |\tilde{w}_{2i}| \rangle_0 \langle |\tilde{w}_{2j}| \rangle_0\big). \tag{99}$$

Estimating these terms similarly as in (93), we obtain

$$\begin{aligned} |\int_0^1 dt(1-t) \sum_{(i,j)\neq(0,0)} \langle \tilde{w}_{2i}; \tilde{w}_{2j} \rangle_t| &\leq \exp(3648C_1^4) \sum_{0\leq i,j\leq 4, i+j>0} \langle |\tilde{w}_{2i}||\tilde{w}_{2j}| \rangle_0 + \langle |\tilde{w}_{2i}| \rangle_0 \langle |\tilde{w}_{2j}| \rangle_0 \\ &\leq \exp[(3648C_1^4)] \sum_{1\leq i+j\leq 3} \sum_{m,n} |\alpha_{2m}^{2i}||\alpha_{2n}^{2j}|\big(\langle z^{2(m+n)} \rangle_0 + \langle z^{2m} \rangle_0 \langle z^{2n} \rangle_0\big)|\phi|^{2(i+j)} \\ &\quad +\text{higher order terms} \\ &\leq \exp[(3648C_1^4)] \sum_{i,j} \sum_{m,n} |\alpha_{2m}^{2i}||\alpha_{2n}^{2j}| \cdot 4 \cdot (2(m+n)-1)!!|\phi|^{2(i+j)} \\ &\quad +\text{higher order terms} \\ &\leq \exp[(3648C_1^4)] \sum_{i,j} 4 \max_{m,n}(2(m+n)-1)!!\big(\sum_m |\alpha_{2m}^{2i}|\big)\big(\sum_n |\alpha_{2n}^{2j}|\big)|\phi|^{2(i+j)} \\ &\quad +\text{higher order terms} \\ &\leq 2 \times 2 \cdot 13!! \times 606 \times 392 \times 2\exp(3648C_1^4)L^{-2}n_0^{-2}|\phi|^2 \\ &\quad +\{2 \times 606 \times 31 + 392^2\} \times 2 \cdot 11!! \times 2\exp(3648C_1^4)L^{-4}n_0^{-2}|\phi|^4 \\ &\quad +2 \times (606 \times 1 + 392 \times 31) \times 2 \cdot 9!! \times 2\exp(3648C_1^4)L^{-2n-6}n_0^{-2}|\phi|^6 \\ &\quad +\text{higher order terms}. \end{aligned} \tag{100}$$

Higher terms are written as follows,

$$\begin{aligned} \text{higher order terms} &= \exp[3648C_1^4] \sum_{2i+2j\geq 8} \langle |\tilde{w}_{2i}||\tilde{w}_{2j}| \rangle_0 + \langle |\tilde{w}_{2i}| \rangle_0 \langle |\tilde{w}_{2j}| \rangle_0 \\ &= \exp[3648C_1^4] \sum_{8\leq i+j} \sum_{k,l} \alpha_{2k}^{2i}\alpha_{2j}^{2l} 2\langle z^{2(k+l)} \rangle_0 |\phi|^{2(i+j)} \\ &\quad + \exp[3648C_1^4]\langle (2|w_0(z)| + 2|w_2(z)\phi^2| + 2|w_4(z)\phi^4| + 2|w_6(z)\phi^6| + |w_{\geq 8}(\phi,z)|)|w_{\geq 8}(\phi,z)| \rangle_0 \\ &\quad + \exp[3648C_1^4]\langle (2|w_0(z)| + 2|w_2(z)\phi^2| + 2|w_4(z)\phi^4| + 2|w_6(z)\phi^6| + |w_{\geq 8}(\phi,z)|) \rangle_0 \langle |w_{\geq 8}(\phi,z)| \rangle_0 \\ &\leq 30\exp[3648C_1^4] \cdot 961 \cdot L^{12-n}C_1^{12}n_0^{-1/8}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})2 \cdot 15!! \end{aligned} \tag{101}$$

Next, we estimate $\int_0^1 dt(1-t)\langle w_0(z); w_0(z) \rangle_t$. From taylor expantion of ϕ, we give

$$\begin{aligned} \int_0^1 dt(1-t)\langle w_0(z); w_0(z) \rangle_t &= \int_0^1 dt(1-t)\langle w_0(z); w_0(z) \rangle_t|_{\phi=0} \\ &\quad + \int_0^1 dt(1-t)\phi^2 \int_0^1 ds(1-s)\frac{d^2}{d^2\phi}\langle w_0(z); w_0(z) \rangle_t|_{\phi=s\phi}. \end{aligned} \tag{102}$$

Notice that $\langle w_0(z); w_0(z) \rangle_t$ is analytic in $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$, using the Cauchy formula, we give

$$\begin{aligned} |\int_0^1 ds(1-s)\frac{d^2}{d\phi^2}\langle w_0(z); w_0(z) \rangle_t|_{\phi=s\phi}| &= |\int_0^1 ds(1-s)\frac{1}{2\pi i}\oint_{|\Phi-s\phi|=(L^{2n-4}\rho_0)^{1/6}} d\Phi \frac{\langle w_0(z); w_0(z) \rangle_t|_{\phi=\Phi}}{(\Phi-s\phi)^3}| \\ &\leq \frac{\exp(3648C_1^4)}{2\pi(L^{2n-4}\rho_0^{-1})^{1/2}} \oint_{|\Phi-s\phi|=(L^{2n-4}\rho_0)^{1/6}} d\Phi \langle |w_0(z)|^2 \rangle_0 \\ &\leq \frac{\exp(3648C_1^4)}{(L^{2n-4}\rho_0^{-1})^{1/3}} 2 \cdot 15!! \times (606)^2 L^{-2}n_0^{-2}. \end{aligned} \tag{103}$$

So we have,

$$
\begin{aligned}
|\int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t \quad &- \quad \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0}| \\
&\leq 2\times 2\times 2\cdot 13!!\times 606\times 392\times 2\exp(3648C_1^4)L^{-2}n_0^{-2}|\phi|^2 \\
&+\{2\times 606\times 31+392^2\}\times 2\cdot 11!!\times 2\exp(3648C_1^4)L^{-4}n_0^{-2}|\phi|^4 \\
&+2\times(606\times 1+392\times 31)\times 2\cdot 9!!\times 2\exp(3648C_1^4)L^{-2n-6}n_0^{-2}|\phi|^6 \\
&+\text{higher order terms}, \\
|\text{higher order terms}| \quad &\leq \quad 30\exp[3648C_1^4]\cdot 961\cdot L^{12-n}C_1^{12}n_0^{-1/8}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})2\cdot 15!! \qquad (104)
\end{aligned}
$$

These coefficients are large , but not terrible, because we can take n_0 sufficiently large. In the following, we put $n_0^{1/8}\geq 30\cdot 961\times 2\cdot 15!!C_1^{12}L^{12}\exp(3648C_1^4)$. From these estimation, we infer that

$$
\begin{aligned}
\tilde{v}_n'(\phi) \quad &= \quad L^2\mu_n\phi^2 - L^2\Big(\frac{6\lambda_n}{1-L^{-2}} - \frac{90\rho_n}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho_n}{1-L^{-4}}\Big)\phi^2 + 6L^2\Big(\lambda_n - \frac{15\rho_n}{1-L^{-2}}\Big)\phi^2 \\
&+45L^2\rho_n\phi^2 + \tilde{R}_2(L,n_0,\rho_0,n)\phi^2 + \Big(\lambda_n - \frac{15\rho_n}{1-L^{-2}}\Big)\phi^4 + 15\rho_n\phi^4 + \tilde{R}_4(L,n_0,\rho_0,n)\phi^4 \\
&+L^{-2}\rho_n\phi^6 + \tilde{R}_6(L,n_0,\rho_0,n)\phi^6 + (\tilde{v}_n)'_{\geq 8}(\phi), \qquad (105)
\end{aligned}
$$

where, the terms $\tilde{R}_{2i}(L,n_0,\rho_0,n)$, $i=1,2,3$ satisfy

$$
\begin{aligned}
|\tilde{R}_2(L,n_0,\rho_0,n)| \quad &\leq \quad |\tilde{R}_2^{0,0}(L,n_0,\rho_0,n)| + L^{-12}n_0^{-2+1/8}, \qquad (106) \\
|\tilde{R}_4(L,n_0,\rho_0,n)| \quad &\leq \quad |\tilde{R}_4^{0,0}(L,n_0,\rho_0,n)| + L^{-16}n_0^{-2+1/8}, \qquad (107) \\
|\tilde{R}_6(L,n_0,\rho_0,n)| \quad &\leq \quad |\tilde{R}_6^{0,0}(L,n_0,\rho_0,n)| + L^{-2n-18}n_0^{-2+1/8}, \qquad (108)
\end{aligned}
$$

and $(\tilde{v}_n)'_{\geq 8}(\phi)$ satisfy

$$
|(\tilde{v}_n)'_{\geq 8}(\phi)| \leq L^{4-n}(1+L^{-2n/3}(L^4\rho_0)^{1/3}+L^{-4})(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4}), \qquad (109)
$$

for $|\phi| < (10/11)LC_1(\rho_0^{-1}L^{2n})^{1/6}$.

Notice that $(\phi=0)_{small} = \log\int d\nu_1(z) + \int\langle w_0(z)\rangle_0 + \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t$, so we can check that constant term $(\phi=0)_{small}$ vanishes. The estimate (109) is a little weaker than what we want (see (30)). So, we need a stronger estimate. Since $\tilde{v}_n'(\phi)$ is analytic for $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$, $\phi^{-8}(\tilde{v}_n)'_{\geq 8}(\phi)$ is also analytic for $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$. We obtain from the maximum principle

$$
|(\tilde{v}_n)'_{\geq 8}(\phi)| \quad \leq \quad \Big(\frac{|\phi|}{(10L/11)C_1(L^{2n-4}\rho_0^{-1})^{1/6}}\Big)^8\big(L^{4-n}(1+L^{-2n/3}(L^4\rho_0)^{1/3}+L^{-4})(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})\big), \qquad (110)
$$

so that for $|\phi| < C_1(L^{2(n+1)-4}\rho_0)^{1/6}$,

$$
|(\tilde{v}_n)'_{\geq 8}(\phi)| \quad \leq \quad \Big(\frac{11}{10}\Big)^8 L^{-16/3}\big(L^{4-n}(1+L^{-2n/3}(L^4\rho_0)^{1/3}+L^{-4})(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})\big). \qquad (111)
$$

2.1.2 Estimation of $\approx{v}_n'(\phi)$ for $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$

Represent (45) as

$$
\begin{aligned}
\overset{\approx}{v}_n'(\phi) \quad &= \quad \log\Big(1 + \int\exp[-\frac{1}{2}L^4\sum_{\pm}v_n(L^{-1}\phi\pm z) - \frac{1}{2}z^2](1-\chi_1(z))\frac{dz}{\sqrt{2\pi}}/e^{-\tilde{v}_n'(\phi)}(\phi=0)_{small}\Big) \qquad (112) \\
&+ \quad \log(\phi=0)_{small} - \log(\phi=0).
\end{aligned}
$$

We want to prove that $\overset{\approx}{v}_n'(\phi)$ is analytic in $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$ and sufficiently smaller than $\tilde{v}_n'(\phi)$. To prove these properties, we have only to prove that

$$
\int\exp[-\frac{1}{2}L^4\sum_{\pm}v_n(L^{-1}\phi\pm z) - \frac{1}{2}z^2](1-\chi_1(z))\frac{dz}{\sqrt{2\pi}}/e^{-\tilde{v}_n'(\phi)}(\phi=0)_{small} \qquad (113)
$$

is analytic and sufficiently small in $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$. First of all, we estimate the denominator part of (113). We will show that the denominator is bounded from below by a constant which depends on L and C_1, but not on n_0. From (105) together with the condition that $(n_0+n)^{1/4} \geq (L^{2n-4}\rho_0^{-1})^{1/6}$, we have

$$\begin{aligned}
|\text{denominator}| &= |\exp[-\overset{\sim}{v'_n}(\phi)](\phi=0)_{small}| \\
&= |\exp[\log(\phi=0)_{small} - L^2\mu_n\phi^2 + (\frac{6L^2\lambda_n}{1-L^{-2}} - \frac{90L^2\rho_n}{(1-L^{-2})(1-L^{-4})} + \frac{45L^2\rho_n}{1-L^{-4}})\phi^2] \\
&\quad \times|\exp[-6L^2(\lambda_n - \frac{15\rho_n}{1-L^{-2}})\phi^2 - 45L^2\rho_n\phi^2 + \tilde{R}_2(L,n_0,\rho_0,n)\phi^2| \\
&\quad \times|\exp[-(\lambda_n - \frac{15\rho_n}{1-L^{-2}})\phi^4 - 15\rho_n\phi^4 + \tilde{R}_4(L,n_0,\rho_0,n)\phi^4]| \\
&\quad \times|\exp[-L^{-2}\rho_n\phi^6 + \tilde{R}_6(L,n_0,\rho_0,n)\phi^6 - (\overset{\sim}{v_n})'_{\geq 8}(\phi)]|
\end{aligned} \tag{114}$$

We estimate $\log(\phi=0)_{small}$ as follows

$$\begin{aligned}
\log(\phi=0)_{small} &= \log\int\exp[-w_\phi(z)]d\nu_1(z)|_{\phi=0} \\
&\geq \log\exp[-\|w_\phi(z)\|_{\chi_1}]\int d\nu_1(z)|_{\phi=0} \\
&\geq -\|w_0(z)\|_{\chi_1} + \log\int d\nu_1(z) \geq -126-1 = -127.
\end{aligned} \tag{115}$$

where

$$\|f(z)\|_{\chi_1} = \sup_{|z|<(L^{2n-4}\rho_0^{-1})^{1/6}} |f(z)|. \tag{116}$$

By using of (22), (23), (28) to (30), and (61) to (64), we can estimate other terms.

$$\begin{aligned}
|\text{denominator}| &\geq \exp[-127 - 4C_1^2 - C_1^4 - 3C_1^6 - 1] \\
&= \exp[-136C_1^6].
\end{aligned} \tag{117}$$

Next, we estimate the numerator part of (113),

$$\begin{aligned}
|\text{numerator}| &\leq \int|(1-\chi_1(z))\exp[-\frac{L^4}{2}\sum_{\pm}v_n(-L^{-1}\phi\pm z) - \frac{1}{2}z^2]|\frac{dz}{\sqrt{2\pi}} \\
&= \int(1-\chi_1(z))\prod_{\pm}|\exp[-v_n(L^{-1}\phi\pm z)]|^{L^4/2}\exp[-\frac{1}{2}z^2]\frac{dz}{\sqrt{2\pi}},
\end{aligned} \tag{118}$$

Using (26) of **L1a** for $|L^{-1}\phi \pm z| < C_1(L^{2n-4}\rho_0)^{1/6}$, we have

$$\begin{aligned}
|\text{numerator}| &\leq \exp[L^4D - L^2(\lambda_n^{1/2}+\rho_n^{1/3})|\phi|^2 + 20\lambda_n(\mathbf{Im}\phi)^4 + 2004L^{-2}\rho_n(\mathbf{Im}\phi)^6] \qquad (119)\\
&\quad \times\int(1-\chi_1(z))\exp[(-L^4\lambda_n^{1/2} - L^4\rho_n^{1/3} - \frac{1}{2})z^2]\frac{dz}{\sqrt{2\pi}} \\
&\leq \exp[3/2\log 2 + L^4D + 20C_1^4 + 4008C_1^6]\times\check{\Phi}((L^{2n-4}\rho_0^{-1})^{1/6}). \qquad (120)
\end{aligned}$$

Since

$$\check{\Phi}(x) \leq \sqrt{2}\exp[-x^2/4], \tag{121}$$

we obtain

$$|\text{numerator}| \leq \exp[3/2\log 2 + L^4D + 20C_1^4 + 4008C_1^6 - \frac{1}{4}(L^{2n-4}\rho_0^{-1})^{1/3}]. \tag{122}$$

So,

$$\begin{aligned}
&\int(1-\chi_1(z))\exp[-\frac{L^4}{2}\sum_{\pm}v_n(L^{-1}\pm z)]d\nu(z)/(\phi=0)_{small}\exp[-\overset{\sim}{v'_n}(\phi)] \\
&\leq \exp[136C_1^6 + 3/2\log 2 + L^4D + 20C_1^4 + 4008C_1^6 - \frac{1}{4}(L^{2n-4}\rho_0^{-1})^{1/3}].
\end{aligned} \tag{123}$$

For given L, D and C_1, we can take n_0 large enough to obtain

$$\text{RHS of (123)} \le \exp[-\frac{1}{8}(L^{2n-4}\rho_0^{-1})^{1/3}]. \tag{124}$$

This estimate is also valid for $\log(\phi = 0) - \log(\phi = 0)_{small}$. According to (124), we can show that $\stackrel{\approx}{v'_n}(\phi)$ is analytic and

$$|\stackrel{\approx}{v'_n}(\phi)| \le 2e^{-1/8(L^{2n-4}\rho_0^{-1})^{1/3}}. \tag{125}$$

2.1.3 Estimation of coefficients

Now, we assume that $|\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$ i.e. ϕ is in the small field region of $v'_n(\phi)$. Notice that the small field region is in the region $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$, so we can use the result in 2.1.2. Thus, $\stackrel{\approx}{v'_n}(\phi)$ is analytic for small field region, and we can obtain power series expansion of $\stackrel{\approx}{v'_n}(\phi)$. With the use of Cauchy estimate, we see that coefficients of ϕ^2, ϕ^4 and ϕ^6 satisfy,

$$\begin{aligned}
|\frac{1}{2}\frac{d^2}{d\phi^2}\stackrel{\approx}{v'_n}(0)| &\le e^{-1/8(L^{2n-4}\rho_0^{-1})^{1/3}}, & (126)\\
|\frac{1}{4!}\frac{d^4}{d\phi^4}\stackrel{\approx}{v'_n}(0)| &\le e^{-1/8(L^{2n-4}\rho_0^{-1})^{1/3}}, & (127)\\
|\frac{1}{6!}\frac{d^6}{d\phi^6}\stackrel{\approx}{v'_n}(0)| &\le e^{-1/8(L^{2n-4}\rho_0^{-1})^{1/3}}. & (128)
\end{aligned}$$

Using the bounded convergence theorem, we see that $\frac{1}{2}\frac{d^2}{d\phi^2}\stackrel{\approx}{v'_n}(0)$, $\frac{1}{4!}\frac{d^4}{d\phi^4}\stackrel{\approx}{v'_n}(0)$ and $\frac{1}{6!}\frac{d^6}{d\phi^6}\stackrel{\approx}{v'_n}(0)$ are continuous functions of μ_n on $I_n = [-(n_0+n)^{-3/2}, (n_0+n)^{-3/2}]$.

From (111) and (125), if n_0 is sufficiently large, then we have

$$|(v_n)'_{\ge 8}(\phi)| \le L^{-(n+1)}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}), \text{ for } |\phi| < C_1(L^{2(n+1)-4}\rho_0)^{1/6}. \tag{129}$$

From (105), (108), and (128), we know that

$$\begin{aligned}
|\rho'_n - L^{-2}\rho_n| &= |R_6(L, n_0, \rho_0, n) + \frac{1}{6!}\frac{d^6}{d\phi^6}\stackrel{\approx}{v'_n}(0)| \\
&\le L^{-2n}n_0^{-15/8}. & (130)
\end{aligned}$$

Thus, if n_0 is sufficiently large, we have

$$|\rho'_n - L^{-2(n+1)}\rho_0| \quad < \quad (n+1)L^{-2n}n_0^{-7/4} \tag{131}$$

which proves $(30)'$ of **L1b$'$**. From (105), (107), we know

$$\begin{aligned}
\lambda'_n - \frac{15\rho'_n}{1-L^{-2}} &= (\lambda_n - \frac{15\rho_n}{1-L^{-2}}) + 15\rho_n + R_4(L, n_0, \rho_0, n) + \frac{1}{4!}\frac{d^4}{d\phi^4} & (132)\\
&= \lambda_n - \frac{15L^{-2}\rho_n}{1-L^{-2}} + R_4(L, n_0, \rho_0, n) + \frac{1}{4!}\frac{d^4}{d\phi^4}\stackrel{\approx}{v'_n}(0) \\
|\lambda'_n - \lambda_n| &= |R_4(L, n_0, \rho_0, n) + \frac{1}{4!}\frac{d^4}{d\phi^4}\stackrel{\approx}{v'_n}(0) + \frac{15}{1-L^{-2}}(\rho'_n - L^{-2}\rho_n)| \\
&\le n_0^{-15/8} & (133)
\end{aligned}$$

Thus, we have

$$|\lambda'_n - \lambda_0| < (n+1)n_0^{-7/4}, \tag{134}$$

which completes the proof of **L1b$'$**. Consider the quartic part $(v'_n)_2(\phi)$

$$\begin{aligned}
(v'_n)_2(\phi) &= \big(\mu_{n+1} - (\frac{6\lambda_{n+1}}{1-L^{-2}} - \frac{90\rho_{n+1}}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho_{n+1}}{1-L^{-4}})\big)\phi^2 \\
&= L^2\mu_n\phi^2 - L^2(\frac{6\lambda_n}{1-L^{-2}} - \frac{90\rho_n}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho_n}{1-L^{-4}})\phi^2 + 6L^2(\lambda_n - \frac{15\rho_n}{1-L^{-2}})\phi^2 \\
&\quad +45L^2\rho_n\phi^2 + R_2(L, n_0, \rho_0, n)\phi^2 + \frac{1}{2}\frac{d^2}{d\phi^2}\stackrel{\approx}{v'_n}(0)\phi^2. & (135)
\end{aligned}$$

Comparing these coefficients , we get

$$\begin{aligned}|\mu_n' - L^2\mu_n| &\leq |(\frac{6(\lambda_n - \lambda_n')}{1-L^{-2}} - \frac{90(L^{-2}\rho_n - \rho_n')}{(1-L^{-2})(1-L^{-4})} + \frac{45(L^{-2}\rho_n - \rho_n')}{1-L^{-4}})| \\ &\quad + |R_2(L, n_0, \rho_0, n) + \frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v}_n'(0)| \\ &\leq 284 n_0^{-15/8}. \end{aligned} \tag{136}$$

We know that map $R : \mu \to \mu'$ is continuous ,and image $R(I_n)$ include I_{n+1}. So that we can take for J_{n+1} a conected component of this inverse image $R^{-1}(I_{n+1}) \subset I_n$. This ends the analysis of the small field properties.

2.2 Large field region analysis

Next, we prove that $e^{-(v_n)'(\phi)}$ satisfy the condition **L1a$'$**. First, we prove it in the case where $|\mathbf{Re}\phi| > C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$. Next, we prove it in $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$ i.e. this region includes the small field region of $v'(\phi)$.

2.2.1 Proof of condition L1a$'$ in the case where $|\mathbf{Re}\phi| > C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$

Note that the definition of renormalization (25) has the following expression

$$e^{-v_n'(\phi)} = \int \prod_{\pm} \exp[-v(L^{-1}\phi \pm z)]^{L^4/2} \exp[-\frac{1}{2}z^2]\frac{dz}{\sqrt{2\pi}}/(\phi = 0). \tag{137}$$

Therefore $|\mathbf{Im}(L^{-1}\phi \pm z)| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, if $|\mathbf{Im}\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$. From condition **L1a**,

$$\begin{aligned}|e^{-(v_n)'(\phi)}| &\leq \exp[L^4D - L^2(\lambda_n^{1/2} + \rho_n^{1/3})|\phi|^2 + 20\lambda_n(\mathbf{Im}\phi)^4 + 2004L^{-2}\rho_n(\mathbf{Im}\phi)^6] \\ &\quad \times \int_{-\infty}^{\infty} \exp[-L^4\lambda_n^{1/2}z^2 - L^4\rho_n^{1/3}z^2 - \frac{1}{2}z^2]\frac{dz}{\sqrt{2\pi}}/(\phi = 0). \end{aligned} \tag{138}$$

Note that, and hence λ_n and ρ_n are positive

$$\begin{aligned}\text{RHS of (138)} &\leq \exp[L^4D - L^2(\lambda_n^{1/2} + \rho_n^{1/3})|\phi|^2 + 20\lambda_n(\mathbf{Im}\phi)^4] \\ &\quad \times \exp[2004L^{-2}\rho_n(\mathbf{Im}\phi)^6 + 128]. \end{aligned} \tag{139}$$

Decompose (139) into the part of **L1a$'$** and the rest

$$\begin{aligned}&\leq \exp[D - ({\lambda_n'}^{1/2} + {\rho_n'}^{1/3})|\phi|^2 + 20\lambda_n'(\mathbf{Im}\phi)^4 + 2004\rho_n'(\mathbf{Im}\phi)^6] \\ &\quad \times \exp[(L^4-1)D - \{L^2(\lambda_n^{1/2} + \rho_n^{1/3}) - ({\lambda_n'}^{1/2} + {\rho_n'}^{1/3})\}|\phi|^2 + 20(\lambda_n' - \lambda_n)(\mathbf{Im}\phi)^4] \\ &\quad \times \exp[2004(L^{-2}\rho_n - \rho_n')(\mathbf{Im}\phi)^6 + 128] \end{aligned} \tag{140}$$

It is sufficient to prove that the remainder part of (140) is less than 1. Note that

$$L^2(\lambda_n^{1/2} + \rho_n^{1/3}) - ({\lambda_n'}^{1/2} + {\rho_n'}^{1/3}) \geq \sqrt{C_{--}}(1 - 2L^{-2})n_0^{1/2} + (L^{4/3} - 2L^{-4/3})(L^{2(n+1)-4}\rho_0^{-1})^{-1/3}.$$

From this, $|\mathbf{Re}\phi| > C_1(L^{2(n+1)}\rho_0)^{1/6}$ and $|\mathbf{Im}\phi| < C_1(L^{2(n+1)}\rho_0)^{1/6} < C_1(n_0 + n + 1)^{1/4} < C_1(2n_0)^{1/4}$,

$$\begin{aligned}\text{Remainder of RHS of (140)} &\leq \exp[(L^4-1)D - \sqrt{C_{--}}(1 - 3L^{-2})n_0^{-1/2}C_1^2(L^{2(n+1)-4}\rho_0)^{1/3}] \\ &\quad \times \exp[-(L^{4/3} - 2L^{-4/3})C_1^2 + 40n_0^{-15/8+1}C_1^4] \\ &\quad \times \exp[8016L^{-2n}n_0^{-15/8+3/2}C_1^6 + 128] \\ &\leq \exp[(L^4-1)D - (L^{4/3} - 2L^{-4/3})C_1^2 + 40C_1^4n_0^{-7/8}] \\ &\quad \times \exp[+8016L^{-4}C_1^6n_0^{-3/8} + 128], \end{aligned} \tag{141}$$

If D and L are given, we take C_1 sufficiently large and then we take n_0 suffiently large, so that the RHS of (141) is smaller than 1. Thus, we obtain

$$|\exp(-v'(\phi))| < \exp[D - ({\lambda_n'}^{1/2} + {\rho_n'}^{1/3})|\phi|^2 + 20\lambda'(\mathbf{Im}\phi)^4 + 2004\rho_n'(\mathbf{Im}\phi)^6], \tag{142}$$

for

$$|\mathbf{Im}\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}, |\mathbf{Re}\phi| > C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}.$$

2.2.2 Proof of condition L1a$'$ in the case where $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0)^{1/6}$

Let $(n_0+n)^{1/4} \geq (L^{2n-4}\rho_0^{-1})^{1/6}$, $\mu_n \in I_n$, and $|\phi| < (10/11)LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$. (109), (131), (134), and (136),

$$\begin{aligned} |e^{-(v_n)'(\phi)}| &= |\exp[-\mu_n'\phi^2 - (\frac{6\lambda_n'}{1-L^{-4}} - \frac{90\rho_n'}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho_n'}{1-L^{-4}})\phi^2 - (\lambda_n' - \frac{15\rho_n'}{1-L^{-2}})\phi^4 - \rho_n'\phi^6 - v_{\geq 8}'(\phi)]| \\ &\leq \exp[721L^{-2}C_1^2 n_0^{-1/2} + 60C_1^4 L^{4/3}(L^{-2(n+1)}\rho_0)^{1/3} - \lambda_n'(\mathbf{Re}\phi^4) - \rho_n'(\mathbf{Re}\phi^6) + L^4 n_0^{-1/2}]. \end{aligned} \tag{143}$$

$$\begin{aligned} \rho_n'(\mathbf{Re}\phi^6) &= \rho_n'\{(\mathbf{Re}\phi)^6 - 15(\mathbf{Re}\phi)^4(\mathbf{Im}\phi)^2 + 15(\mathbf{Re}\phi)^4(\mathbf{Im}\phi)^2 - (\mathbf{Im}\phi)^6\} \\ &\geq \rho_n'(\frac{1}{4}(\mathbf{Re}\phi)^6 - 2001(\mathbf{Im}\phi)^6) \\ &\geq -\frac{1}{2}D_6 + 2(\rho_n')^{1/3}|\phi|^2 - 2004\rho_n'(\mathbf{Im}\phi)^6. \end{aligned} \tag{144}$$

Notice that D_6 does not depend on C_1, n_0 or n. Similarly, we can estimate, $\lambda_n'\mathbf{Re}\phi^4 \geq -\frac{1}{2}D_4 + 2(\lambda_n')^{1/2}|\phi|^2 - 20\lambda_n'(\mathbf{Im}\phi)^4$. Notice that D_4 does not deped on C_1, n_0 or n, either. Put $D = D_4 + D_6$. From (143) to (144),

$$\begin{aligned} |e^{-(v_n)'(\phi)}| &\leq \exp[\frac{1}{2}D - ((\lambda_n')^{1/2} + (\rho_n')^{1/3})|\phi|^2 + 20(\lambda_n')(\mathbf{Im}\phi)^4 + 2004(\rho_n')(\mathbf{Im}\phi)^6] \\ &\quad \times \exp[361L^{-2}C_1^2 n_0^{-1/2} + 60L^{4/3}C_1^4(L^{-2(n+1)}\rho_0)^{1/3} + L^4 n_0^{-1/2}] \\ &\leq \exp[D - ((\lambda_n')^{1/2} + (\rho_n')^{1/3})|\phi|^2 + 20(\lambda_n')(\mathbf{Im}\phi)^4 + 2004(\rho_n')(\mathbf{Im}\phi)^6] \\ &\quad \times \exp[-\frac{1}{2}D + 361L^{-2}C_1^2 n_0^{-1/2} + 60L^{4/3}C_1^4(L^{-2(n+1)}\rho_0)^{1/3} + L^4 n_0^{-1/2}], \end{aligned} \tag{145}$$

which is smaller than

$$\exp[D - ((\lambda_n')^{1/2} + (\rho_n')^{1/3})|\phi|^2 + 20(\lambda_n')(\mathbf{Im}\phi)^4 + 2004(\rho_n')(\mathbf{Im}\phi)^6], \tag{146}$$

if n_0 is sufficiently large. Proof of Lemma 1 is completed.

3 Proof of Lemma 2

3.1 Small field region

Let $v_{n_1+n} \in \mathcal{V}^2_{n_1+n}$. Put $\chi_2(z) = \chi(|z| < (n_0+n_1+n)^{1/4})$ and throughout this subsection, we assume that ϕ is in the region, $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$. Remark that $|L^{-1}\phi \pm z| < C_1(n_0+n_1+n)^{1/4}$ in $|z| < (n_0+n_1+n)^{1/4}$ and $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$ if $C_1 > 11$. Let us decompose $v_{n+1}(\phi)$ as follows,

$$v_{n+1}(\phi) = v_n'(\phi) = \tilde{v_n'}(\phi) + \approx{v_n'}(\phi), \tag{147}$$

$$e^{-\tilde{v_n'}(\phi)} = \int \exp[-\frac{1}{2}L^4 \sum_{\pm} v_n(L^{-1}\phi \pm z)]d\nu_2(z)/[\phi = 0]_{small}, \tag{148}$$

where

$$\begin{aligned} [\phi = 0]_{small} &= \int \exp[-L^4 v_n(z)]d\nu_2(z), \end{aligned} \tag{149}$$

$$\begin{aligned} d\nu_2(z) &\equiv \chi_2(z)e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}. \end{aligned} \tag{150}$$

We estimate $\tilde{v_n'}(\phi)$ in 3.1.1, and we estimate $\overset{\approx}{v_n'}(\phi)$ in 3.1.2 later.

3.1.1 Estimation of $\tilde{v'}_{n_1+n}(\phi)$

By definition of the RG, we know

$$\begin{aligned} e^{-\tilde{v'}_{n_1+n}(\phi)} &= \exp[-L^2\mu_{n_1+n}\phi^2 + (\frac{6L^2\lambda_{n_1+n}}{1-L^{-2}} - \frac{90L^2\rho_{n_1+n}}{(1-L^{-2})(1-L^{-4})} + \frac{45L^2\rho_{n_1+n}}{1-L^{-4}})\phi^2] \\ &\quad \times \exp[-(\lambda_{n_1+n} - \frac{15\rho_{n_1+n}}{1-L^{-2}})\phi^4 - L^{-2}\rho_{n_1+n}\phi^6] \\ &\quad \times \int \exp[-w_\phi(z)]d\nu_2(z)/[\phi = 0]_{small}, \end{aligned} \tag{151}$$

$$w_\phi(z) = w_0(z) + w_2(z)\phi^2 + w_4(z)\phi^4 + w_6(z)\phi^6 + w_{\geq 8}(\phi, z), \tag{152}$$

and

$$\begin{aligned}
w_0(z) &= L^4 v_{n_1+n}(z) \\
&= L^4\big(\mu_{n_1+n}z^2 - (\frac{6\lambda_{n_1+n}}{1-L^{-2}} - \frac{90\rho_{n_1+n}}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho_{n_1+n}}{1-L^{-4}})z^2\big) \\
&\quad + L^4\big((\lambda_{n_1+n} - \frac{15\rho_{n_1+n}}{1-L^{-2}})z^4 + \rho_{n_1+n}z^6 + (v_{n_1+n})_{\geq 8}(z)\big), &(153) \\
w_2(z) &= L^2\big(6(\lambda_{n_1+n} - \frac{15\rho_{n_1+n}}{1-L^{-2}})z^2 + 15\rho_{n_1+n}z^4 + \frac{1}{2}\frac{d^2}{dz^2}(v_{n_1+n})_{\geq 8}(z)\big), &(154) \\
w_4(z) &= 15\rho_{n_1+n}z^2 + \frac{1}{4!}\frac{d^4}{dz^4}(v_{n_1+n})_{\geq 8}(z), &(155) \\
w_6(z) &= L^{-2}\frac{1}{6!}\frac{d^6}{dz^6}(v_{n_1+n})_{\geq 8}(z), &(156) \\
w_{\geq 8}(\phi, z) &= \frac{L^{-4}\phi^8}{7!}\int_0^1 dt(1-t)^7\frac{d^8}{dz^8}\{(v_{n_1+n})_{\geq 8}(L^{-1}t\phi + z) + (v_{n_1+n})_{\geq 8}(L^{-1}t\phi - z)\}. &(157)
\end{aligned}$$

We note that $v_{n_1+n}(\phi)$ is even and analytic from conditions **L2a - L2b**. Using the Cauchy formula and (39) to estimate $\frac{d^8}{dz^8}(v_{n_1+n})_{\geq 8}(\phi)$ on the support of $d\nu_2(z)$, we obtain

$$\begin{aligned}
|(v_{n_1+n})_{\geq 8}(z)| &\leq \frac{1}{7!}\int_0^1 dt(1-t)^7|z^8\frac{d^8}{dz^8}(v_{n_1+n})_{\geq 8}(tz)| &(158) \\
&\leq \frac{C_1}{8!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{-n_1-3n}(n_0+n_1+n)^{-2}z^8.
\end{aligned}$$

$$\begin{aligned}
|\frac{d^2}{dz^2}(v_{n_1+n})_{\geq 8}(z)| &\leq \frac{1}{5!}\int_0^1 dt(1-t)^5|z^6\frac{d^8}{dz^8}(v_{n_1+n})_{\geq 8}(tz)| &(159) \\
&\leq \frac{C_1}{6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{-n_1-3n}(n_0+n_1+n)^{-2}z^6,
\end{aligned}$$

$$\begin{aligned}
|\frac{d^4}{dz^4}(v_{n_1+n})_{\geq 8}(z)| &\leq \frac{1}{3!}\int_0^1 dt(1-t)^3|z^4\frac{d^8}{dz^8}(v_{n_1+n})_{\geq 8}(tz)| &(160) \\
&\leq \frac{C_1}{4!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{-n_1-3n}(n_0+n_1+n)^{-2}z^4,
\end{aligned}$$

$$\begin{aligned}
|\frac{d^6}{dz^6}(v_{n_1+n})_{\geq 8}(z)| &\leq \int_0^1 dt(1-t)|z^2\frac{d^8}{dz^8}(v_{n_1+n})_{\geq 8}(tz)| &(161) \\
&\leq \frac{C_1}{2(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{-n_1-3n}(n_0+n_1+n)^{-2}z^2.
\end{aligned}$$

Utilizing (158) - (161), we can estimate $|w_0(z)|$ in the following way,

$$w_0(z) = \alpha_2^0 z^2 + \alpha_4^0 z^4 + \alpha_6^0 z^6 + L^4(v_{n_1+n})_{\geq 8}(z). \tag{162}$$

To calculate (162), Let us prepare following estimates for μ_{n_1+n}, λ_{n_1+n}, ρ_{n_1+n}. From (22) to (23), (37) to (39) and (152) to (157),

$$\begin{aligned}
|\mu_{n_1+n}(n_0+n_1+n)| &\leq L^{-4} &(163) \\
|\lambda_{n_1+n}||(n_0+n_1+n)|^s &\leq L^{-4}, \text{for } s \leq 1 &(164) \\
|\lambda_{n_1+n}| &\leq 2\lambda_0, \text{for } n \geq 0 &(165) \\
|\rho_{n_1+n}(L^{2(n_1+n)}\rho_0^{-1})^s| &\leq 2, \text{for } s \leq 1 &(166) \\
|(v_{n_1+n})_{\geq 8}(\phi)| &\leq L^{-n_1}n_0^{-1/2}. &(167)
\end{aligned}$$

These estimates are valid for $n \le (1/4)\log_L n_0$, $n_0 \ge L^{48}$. Hence, we have

$$\begin{aligned}
|\alpha_2^0| &\le L^4|\mu_{n_1+n}| + \frac{6L^4|\lambda_{n_1+n}|}{1-L^{-2}} + \frac{90L^4|\rho_{n_1+n}|}{(1-L^{-2})(1-L^{-4})} + \frac{45L^4|\rho_{n_1+n}|}{(1-L^{-4})} \\
&\le n_0^{-1} + \frac{n_0^{-1}}{2(1-L^{-2})} + \frac{180L^{-2n_1}n_0^{-1}}{(1-L^{-2})(1-L^{-4})} + \frac{90L^{-2n_1}n_0^{-1}}{1-L^{-4}} \\
&\le 4n_0^{-1}. \qquad (168)
\end{aligned}$$

Similarly, we can estimate α_4^0, α_6^0, α_8^0,

$$\begin{aligned}
|\alpha_4^0| &\le L^4|\lambda_{n_1+n}| + \frac{15L^4|\rho_{n_1+n}|}{1-L^{-2}} \le 2n_0^{-1}, \\
|\alpha_6^0| &\le L^4|\rho_{n_1+n}| \le 2L^{4-2(n_1+n)}\rho_0, \qquad (169)\\
|L^4(v_{n_1+n})_{\ge 8}(z)| &\le \frac{1}{7!}\int_0^1 dt(1-t)^7|\frac{d^8}{dz^8}(v_{n_1+n})_{\ge 8}(tz)| \\
&\le \frac{C_1}{8!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{4-n_1-3n}(n_0+n_1+n)^{-2}z^8 \\
&\equiv |\alpha_8^0|z^8. \qquad (170)\\
& \qquad (171)
\end{aligned}$$

In the same way, we can estimate $|w_2(z)|$, $|w_4(z)|$, $|w_6(z)|$, $|w_{\ge 8}(\phi,z)|$ as follows,

$$\begin{aligned}
|w_2(z)| &\le |\alpha_2^2|z^2 + |\alpha_4^2|z^4 + |\alpha_6^2|z^6, \qquad (172)\\
|w_4(z)| &\le |\alpha_2^4|z^2 + |\alpha_4^4|z^4, \qquad (173)\\
|w_6(z)| &\le |\alpha_2^6|z^2, \qquad (174)\\
|w_{\ge 8}(\phi,z)| &\le |\alpha_0^8| + |\alpha_2^8|z^2 + |\alpha_4^8|z^4 + |\alpha_6^8|z^6 + |\alpha_8^8|z^8, \qquad (175)
\end{aligned}$$

where $\alpha_{2i}^{2j}, i,j = 0,1,\cdots 4$ are

$$\begin{aligned}
|\alpha_2^2| &\le L^2 6\big(|\lambda_{n_1+n}| + \frac{15|\rho_{n_1+n}|}{1-L^{-2}}\big) \le 2L^{-2}n_0^{-1}, \qquad (176)\\
|\alpha_4^2| &\le 15L^2|\rho_{n_1+n}| \le 30L^{2-2(n_1+n)}\rho_0, \qquad (177)\\
|\alpha_6^2| &\le \frac{C_1}{6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{2-n_1-3n}(n_0+n_1+n)^{-2}, \qquad (178)\\
|\alpha_2^4| &\le 15|\rho_{n_1+n}| \le 30L^{-2(n_1+n)}\rho_0, \qquad (179)\\
|\alpha_4^4| &\le \frac{C_1}{4!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{-n_1-3n}(n_0+n_1+n)^{-2}, \qquad (180)\\
|\alpha_2^6| &\le \frac{C_1}{2(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{-2-n_1-3n}(n_0+n_1+n)^{-2}, \qquad (181)\\
|\alpha_0^8| &\le L^{4-n_1-3n}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}), \qquad (182)\\
|\alpha_2^8| &\le \frac{2C_1^7}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{4-n_1-3n}(n_0+n_1+n)^{-1/2}, \qquad (183)\\
|\alpha_4^8| &\le \frac{6C_1^5}{3!\cdot 4!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{4-n_1-3n}(n_0+n_1+n)^{-1}, \qquad (184)\\
|\alpha_6^8| &\le \frac{2C_1^3 5!!}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{4-n_1-3n}(n_0+n_1+n)^{-3/2}, \qquad (185)\\
|\alpha_8^8| &\le \frac{2C_1 7!!}{8!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{4-n_1-3n}(n_0+n_1+n)^{-2}. \qquad (186)
\end{aligned}$$

Notice that $|w_{\ge 8}(\phi,z)|$ is estimated as follows. From (39) of **L2b**, we have

$$|\frac{L^4}{2}\sum_{\pm}(v_{n_1+n})_{\ge 8}(L^{-1}\phi \pm z)| \le (\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4})L^{4-n_1-3n}. \qquad (187)$$

So, we have from (158) to (161) and (187)

$$\begin{aligned}
|w_{\geq 8}(\phi,z)| &\leq |\frac{L^4}{2}\sum_{\pm}(v_{n_1+n})_{\geq 8}(L^{-1}\phi\pm z)|+|L^4(v_{n_1+n})_{\geq 8}(z)|+|\frac{1}{2}L^2\phi^2\frac{d^2}{dz^2}(v_{n_1+n})_{\geq 8}(z)| \\
&+|\frac{1}{4!}\phi^4\frac{d^4}{dz^4}(v_{n_1+n})_{\geq 8}(z)|+|\frac{1}{6!}L^{-2}\phi^6\frac{d^6}{dz^6}(v_{n_1+n})_{\geq 8}(z)| \\
&\leq (\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{4-n}+\frac{2C_1 7!!}{8!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{4-n_1-3n}(n_0+n_1+n)^{-2}z^8 \\
&+\frac{2C_1^3 5!!}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{4-n_1-3n}(n_0+n_1+n)^{-3/2}z^6 \\
&+\frac{6C_1^5}{3!\cdot 4!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{4-n_1-3n}(n_0+n_1+n)^{-1}z^4 \\
&+\frac{2C_1^7}{2\cdot 6!(C_1-1)^9}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{4-n_1-3n}(n_0+n_1+n)^{-1/2}z^2. \qquad (188)
\end{aligned}$$

Let us take logarithm of (151), we have

$$\begin{aligned}
\overset{\sim}{v'}_{n_1+n}(\phi) &= L^2\mu_{n_1+n}\phi^2-(\frac{6L^2\lambda_{n_1+n}}{1-L^{-2}}-\frac{90L^2\rho_{n_1+n}}{(1-L^{-2})(1-L^{-4})}+\frac{45L^2\rho_{n_1+n}}{1-L^{-4}})\phi^2+(\lambda_{n_1+n}-\frac{15\rho_{n_1+n}}{1-L^{-2}})\phi^4 \\
&+L^{-2}\rho_{n_1+n}\phi^6-\log\int e^{-w_\phi(z)}d\nu_2(z)+\log[\phi=0]_{small}. \qquad (189)
\end{aligned}$$

Remark that $e^{-\overset{\sim}{v'}_{n_1+n}(\phi)}\neq 0$ if ϕ is in the small field region i.e. $|\phi|\leq(10/11)LC_1(n_0+n_1+n)^{1/4}$ and n_0 is sufficiently large. From the pertubation expansion :

$$-\log\int e^{-w_\phi(z)}d\nu_2(z)=-\log\int d\nu_2(z)+\langle w_\phi(z)\rangle_0-\int_0^1 dt(1-t)\langle w_\phi(z);w_\phi(z)\rangle_t. \qquad (190)$$

Where

$$\langle\cdots\rangle_t \equiv \int\cdots e^{-tw_\phi(z)}d\nu_2(z)/\int e^{-tw_\phi(z)}d\nu_2(z). \qquad (191)$$

Now, we shall estimate each part of (190).

Estimation of $\langle w_\phi(z)\rangle_0$

By (152) - (157),

$$\begin{aligned}
\langle w_\phi(z)\rangle_0 &= L^4\langle v_{n_1+n}(z)\rangle_0+6L^2\phi^2(\lambda_{n_1+n}-\frac{15\rho_{n_1+n}}{1-L^{-2}})\langle z^2\rangle_0+15L^2\rho_{n_1+n}\phi^2\langle z^4\rangle_0 \\
&+L^2\phi^2\langle\frac{d^2}{2dz^2}(v_{n_1+n})_{\geq 8}(z)\rangle_0+15\rho_{n_1+n}\phi^4\langle z^2\rangle_0+\phi^4\langle\frac{d^4}{4!dz^4}(v_{n_1+n})_{\geq 8}(z)\rangle_0 \\
&+L^{-2}\phi^6\langle\frac{d^6}{6!dz^6}(v_{n_1+n})_{\geq 8}(z)\rangle_0+\langle w_{\geq 8}(\phi,z)\rangle_0. \qquad (192)
\end{aligned}$$

Note, that for $m=0,1,\cdots,8$

$$|\langle z^{2m}\rangle_0-(2m-1)!!|\leq\frac{2\{(2m-1)!!\check{\Phi}((n_0+n_1+n)^{1/4})+M\sqrt{2}\check{\Phi}((n_0+n_1+n)^{1/4})/\sqrt{2})\}}{\sqrt{2\pi}-2\check{\Phi}((n_0+n_1+n)^{1/4})}. \qquad (193)$$

We obtain

$$\begin{aligned}
\langle w_\phi(z)\rangle_0 &= L^4\langle v_{n_1+n}(z)\rangle_0+6L^2(\lambda_{n_1+n}-\frac{15\rho_{n_1+n}}{1-L^{-2}})\phi^2+45L^2\rho_{n_1+n}\phi^2+\overset{\sim 0,0}{R_2}(L,n_0,\rho_0,n)\phi^2 \\
&+15\rho_{n_1+n}\phi^4+\overset{\sim 0,0}{R_4}(L,n_0,\rho_0,n)\phi^4+\overset{\sim 0,0}{R_6}(L,n_0,\rho_0,n)\phi^6+\langle w_{\geq 8}(\phi,z)\rangle_0, \qquad (194)
\end{aligned}$$

where, these terms $\overset{\sim 0,0}{R_{2i}}(L,n_0,\rho_0,n), i=1,\cdots,3$ satisfy

$$|\overset{\sim 0,0}{R_{2i}}(L,n_0,\rho_0,n)|\leq(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{-n_1-3n}(n_0+n_1+n)^{-2} \qquad (195)$$

for sufficiently n_0, because of (159) - (161), and (193). Finally, let us estimate $\langle w_{\geq 8}(\phi,z)\rangle_0$. From (175), (182) to (186), we have

$$\begin{aligned} |\langle w_{\geq 8}(\phi,z)\rangle_0| &\leq \alpha_0^8+\alpha_2^8\langle z^2\rangle_0+\alpha_4^8\langle z^4\rangle_0+\alpha_6^8\langle z^6\rangle_0+\alpha_8^8\langle z^8\rangle_0 \\ &\leq (\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{4-3n-n_1}+(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})L^{4-3n-n_1}(n_0+n_1+n)^{-1/2}. \quad (196) \end{aligned}$$

Here, we take $C_1 \geq 6$.

Estimation of $\int_0^1 dt(1-t)\langle w_\phi(z);w_\phi(z)\rangle_t$

Now, we estimate

$$\begin{aligned} \int_0^1 dt(1-t)\langle w_\phi(z);w_\phi(z)\rangle_t &= \int_0^1 dt(1-t)\sum_{i,j}\langle \tilde{w}_{2i};\tilde{w}_{2j}\rangle_t \\ &= \int_0^1 dt(1-t)\langle w_0(z);w_0(z)\rangle_t+\int_0^1 dt(1-t)\sum_{i,j\neq 0}\langle \tilde{w}_{2i};\tilde{w}_{2j}\rangle_t, \quad (197) \end{aligned}$$

where

$$\tilde{w}_{2i}=\begin{cases} w_{2i}(z)\phi^{2i} & i=0,1,2,3 \\ w_{\geq 8}(\phi,z) & i=4. \end{cases}$$

By definition, we get

$$\langle \tilde{w}_{2i};\tilde{w}_{2j}\rangle_t=\langle e^{-tw_\phi(z)}\rangle_0^{-1}\langle \tilde{w}_{2i}\tilde{w}_{2j}e^{-tw_\phi(z)}\rangle_0-\langle e^{-tw_\phi(z)}\rangle_0^{-2}\langle \tilde{w}_{2i}e^{-tw_\phi(z)}\rangle_0\langle \tilde{w}_{2j}e^{-tw_\phi(z)}\rangle_0. \quad (198)$$

From (162) - (186), we get uniform estimate of $|w_\phi(z)|$ in $|z|<(n_0+n_1+n)^{1/4}$ and $|\phi|<\frac{10LC_1}{11}(n_0+n_1+n)^{1/4}$,

$$\begin{aligned} |w_\phi(z)| &\leq |w_0(z)|+|w_2(z)||\phi|^2+|w_4(z)||\phi|^4+|w_6(z)||\phi|^6+|w_{\geq 8}(\phi,z)| \\ &\leq 8+35C_1^2+31C_1^4+1+1 \\ &\leq 82C_1^4. \quad (199) \end{aligned}$$

Using this, we obtain

$$|\langle \tilde{w}_{2i};\tilde{w}_{2j}\rangle_t| \leq \exp[328C_1^4](\langle |\tilde{w}_{2i}||\tilde{w}_{2j}|\rangle_0+\langle |\tilde{w}_{2i}|\rangle_0\langle |\tilde{w}_{2j}|\rangle_0). \quad (200)$$

As in the estimation of $\langle w_\phi(z)\rangle_0$, we have

$$\begin{aligned} |\int_0^1 dt(1-t)\sum_{(i,j)\neq(0,0)}\langle \tilde{w}_{2i};\tilde{w}_{2j}\rangle_t| &\leq \exp(328C_1^4)\sum_{0\leq i,j\leq 4,i+j>0}(\langle |\tilde{w}_{2i}||\tilde{w}_{2j}|\rangle_0+\langle |\tilde{w}_{2i}|\rangle_0\langle |\tilde{w}_{2j}|\rangle_0) \\ &\leq \exp[(328C_1^4)]\sum_{0\leq i+j\leq 3}\sum_{m,n}|\alpha_{2m}^{2i}||\alpha_{2n}^{2j}|(\langle z^{2(m+n)}\rangle_0+\langle z^{2m}\rangle_0\langle z^{2n}\rangle_0)|\phi|^{2(i+j)} \\ &\leq 2\times 8\times 4\times 2\cdot 13!!\times 2\exp(328C_1^4)L^{-2}n_0^{-2}|\phi|^2 \\ &\quad +\{2\times 8\times 31+4^2\}2\cdot 11!!\times 2\exp(328C_1^4)L^{-4}n_0^{-2}|\phi|^4 \\ &\quad +2\times(8\times 1+4\times 31)2\cdot 9!!\times 2\exp(328C_1^4) \\ &\quad \times(L^{-2(n_1+n)}\rho_0\vee L^{-n_1-3n}n_0^{-1/2}(n_0+n_1+n)^{-2})n_0^{-1}|\phi|^6 \\ &\quad +\text{higher order terms}. \quad (201) \end{aligned}$$

Here, we used the following estimate

$$\langle z^{2n}\rangle_0 \leq 2\cdot(2n-1)!!. \quad (202)$$

Higher terms are,

$$\text{higher order terms} = \exp[328C_1^4]\sum_{2i+2j\geq 8,0\leq i,j\leq 4}\{\langle |\tilde{w}_{2i}||\tilde{w}_{2j}|\rangle_0+\langle |\tilde{w}_{2i}|\rangle_0\langle |\tilde{w}_{2j}|\rangle_{2j}\}$$

$$
\begin{aligned}
&= \exp[328C_1^4] \sum_{0\le i,j\le 3, i+j\ge 4} \sum_{m,n} |\alpha_{2m}^{2i}||\alpha_{2n}^{2j}| \big(\big\langle z^{2(m+n)}\big\rangle_0 + \big\langle z^{2m}\big\rangle_0 \big\langle z^{2n}\big\rangle_0\big)|\phi|^{2(i+j)} \\
&\quad + \exp[328C_1^4]\big\langle (2|w_0|+2|w_2\phi^2|+2|w_4\phi^4|+2|w_6\phi^6|+|w_{\ge 8}(\phi,z)|)|w_{\ge 8}(\phi,z)|\big\rangle_0 \\
&\quad + \exp[328C_1^4]\big\langle 2|w_0|+2|w_2\phi^2|+2|w_4\phi^4|+2|w_6\phi^6|+|w_{\ge 8}(\phi,z)|\big\rangle_0 \big\langle |w_{\ge 8}(\phi,z)|\big\rangle_0 \\
&\le 30\exp(328C_1^4)\cdot 31^2\cdot L^{12-n_1-3n} C_1^{12} n_0^{-1/8}(\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}) 2\cdot 15!!.
\end{aligned}
\tag{203}
$$

Here, we used the following estimate

$$
(n_0+n_1+n) \quad \le \quad 2(n_0+n_1) \le 2(L^{2n_1-4}\rho_0^{-1})^{2/3}. \tag{204}
$$

if $n \ge \log_L n_0$. We estimate $\int_0^1 dt(1-t)\big\langle w_0; w_0\big\rangle_t$ likewise. Notice that $\big\langle w_0(z); w_0(z)\big\rangle_t$ is analytic in $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$, using taylor expantion of ϕ, we know

$$
\begin{aligned}
\int_0^1 dt(1-t)\big\langle w_0(z); w_0(z)\big\rangle_t &= \int_0^1 dt(1-t)\big\langle w_0(z); w_0(z)\big\rangle_t|_{\phi=0} \\
&\quad + \int_0^1 dt(1-t)\phi^2 \int_0^1 ds(1-s)\frac{d^2}{d^2\phi}\big\langle w_0(z); w_0(z)\big\rangle_t|_{\phi=s\phi}.
\end{aligned}
\tag{205}
$$

Using the Cauchy formula, we give

$$
\begin{aligned}
\Big|\int_0^1 ds(1-s)\frac{d^2}{d\phi^2}\big\langle w_0(z); w_0(z)\big\rangle_t|_{\phi=s\phi}\Big| &= \Big|\int_0^1 ds(1-s)\frac{1}{2\pi i}\oint_{|\Phi-s\phi|=(n_0+n_1+n)^{1/4}} d\Phi \frac{\big\langle w_0(z); w_0(z)\big\rangle_t|_{\phi=\Phi}}{(\Phi-s\phi)^3}\Big| \\
&\le \frac{2\exp(328C_1^4)}{2\pi(n_0+n_1+n)^{3/4}} \oint_{|\Phi-s\phi|=(n_0+n_1+n)^{1/4}} d\Phi \big\langle |w_0(z)|^2\big\rangle_0 \\
&\le \frac{2\exp(328C_1^4)}{(n_0+n_1+n)^{1/2}} 2\cdot 15!! \times 8^2 n_0^{-2}.
\end{aligned}
\tag{206}
$$

So we have,

$$
\begin{aligned}
\Big|\int_0^1 dt(1-t)\big\langle w_\phi(z); w_\phi(z)\big\rangle_t &- \int_0^1 dt(1-t)\big\langle w_0(z); w_0(z)\big\rangle_t|_{\phi=0}\Big| \\
&\le \{1+2\times 8\times 4\times 2\cdot 13!!\times 2\}\exp(328C_1^4)L^{-2}n_0^{-2}|\phi|^2 \\
&\quad +\{2\times 8\times 31+4^2\}\times 2\cdot 11!!\times 2\exp(328C_1^4)L^{-4}n_0^{-2}|\phi|^4 \\
&\quad +2\times(8\times 1+4\times 31)\times 2\cdot 9!!\times 2\exp(328C_1^4) \\
&\quad \times(L^{-2(n_1+n)}\rho_0 \vee L^{-n_1-3n}n_0^{-1/2}(n_0+n_1+n)^{-2})n_0^{-1}|\phi|^6 \\
&\quad +\text{higher order terms}, \\
|\text{higher order terms}| &\le 30\exp[328C_1^4]\cdot 961\cdot L^{12-n_1-3n}C_1^{12}n_0^{-1/8}(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})2\cdot 15!!
\end{aligned}
\tag{207}
$$

In the following, we take n_0, such that $n_0^{1/8} \ge 30\cdot 31^2\times 2\cdot 15!! C_1^{12}L^{12}\exp(328C_1^4)$, as in Lemma 1.

From these estimation, we infer that

$$
\begin{aligned}
\tilde{v}'_{n_1+n}(\phi) &= L^2\mu_{n_1+n}\phi^2 + L^2\Big(\frac{6\lambda_{n_1+n}}{1-L^{-2}} - \frac{90\rho_{n_1+n}}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho_{n_1+n}}{1-L^{-4}}\Big)\phi^2 + 6L^2\Big(\lambda_{n_1+n} - \frac{15\rho_{n_1+n}}{1-L^{-2}}\Big)\phi^2 \\
&\quad +45L^2\rho_{n_1+n}\phi^2 + \tilde{R}_2(L,n_0,\rho_0,n)\phi^2 + \Big(\lambda_{n_1+n} - \frac{15\rho_{n_1+n}}{1-L^{-2}}\Big)\phi^4 + 15\rho_{n_1+n}\phi^4 + \tilde{R}_4(L,n_0,\rho_0,n)\phi^4 \\
&\quad +L^{-2}\rho_{n_1+n}\phi^6 + \tilde{R}_6(L,n_0,\rho_0,n)\phi^6 + (\tilde{v}'_{n_1+n})_{\ge 8}(\phi),
\end{aligned}
\tag{208}
$$

where, the terms $\tilde{R}_{2i}(L,n_0,\rho_0,n), i=1,2,3$ satisfy

$$
\begin{aligned}
|\tilde{R}_2(L,n_0,\rho_0,n)| &\le |\tilde{R}_2^{0,0}(L,n_0,\rho_0,n)| + L^{-14}n_0^{-2+1/8}, & (209)\\
|\tilde{R}_4(L,n_0,\rho_0,n)| &\le |\tilde{R}_4^{0,0}(L,n_0,\rho_0,n)| + L^{-16}n_0^{-2+1/8}, & (210)\\
|\tilde{R}_6(L,n_0,\rho_0,n)| &\le |\tilde{R}_6^{0,0}(L,n_0,\rho_0,n)| + L^{-2(n_1+n)-12}n_0^{-2+1/8}, & (211)
\end{aligned}
$$

and $(\widetilde{v}'_{n_1+n})_{\geq 8}(\phi)$ satisfy

$$|(\widetilde{v}'_{n_1+n})_{\geq 8}(\phi)| \leq L^{4-n_1-3n}(1+(n_0+n_1+n)^{-1/2}+L^{-4})(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4}), \tag{212}$$

for $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$.

This estimate (212) is not sufficient by the same reason as in Lemma 1. So, we try estimating (212) for $|\phi| < C_1(n_0+n_1+n+1)^{1/4}$. From analyticity of $\widetilde{v'}_{n_1+n}(\phi)$ for $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$, $\phi^{-8}\ (\widetilde{v_{n_1+n}})'_{\geq 8}(\phi)$ is analytic for $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$. The maximum principle implies

$$\begin{aligned}&|(\widetilde{v}'_{n_1+n})_{\geq 8}(\phi)| \\ &\leq \Big(\frac{|\phi|}{(10L/11)C_1(n_0+n_1+n)^{1/4}}\Big)^8\big(L^{4-3n-n_1}(1+(n_0+n_1+n)^{-1/2}+L^{-4})(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4})\big).\end{aligned} \tag{213}$$

For $|\phi| < C_1(n_0+n_1+n+1)^{1/4}$, we obtain

$$\begin{aligned}|\ (\widetilde{v_{n_1+n}})'_{\geq 8}(\phi)| \quad &\leq \quad (\frac{11}{10})^8(1+\frac{1}{n_0+n_1+n})^2L^{-4-3n-n_1}\big(1+(n_0+n_1+n)^{-1/2}+L^{-4}\big) \\ &\times\big(\rho_0^{2/3}n_0^{1/8}\vee n_0^{-3/4}\big).\end{aligned} \tag{214}$$

3.1.2 Estimation of $\overset{\approx}{v'_{n_1+n}}(\phi)$ for $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$

Represent (45) as

$$\begin{aligned}\overset{\approx}{v'_{n_1+n}}(\phi) \quad &= \quad \log\big(1+\int\exp[-\frac{1}{2}L^4\sum_{\pm}v_{n_1+n}(L^{-1}\phi\pm z)-\frac{1}{2}z^2](1-\chi_2(z))\frac{dz}{\sqrt{2\pi}}/e^{-\widetilde{v}'_{n_1+n}(\phi)}[\phi=0]_{small}\big) \\ &+ \quad \log[\phi=0]_{small}-\log(\phi=0).\end{aligned} \tag{215}$$

We show that $\overset{\approx}{v'_{n_1+n}}(\phi)$ is analytic for $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$ and sufficiently smaller than $\widetilde{v'_{n_1+n}}(\phi)$. To show these properties, we only check that

$$\int\exp[-\frac{1}{2}L^4\sum_{\pm}v_{n_1+n}(L^{-1}\phi\pm z)-\frac{1}{2}z^2](1-\chi_2(z))\frac{dz}{\sqrt{2\pi}}/e^{-\widetilde{v}'_{n_1+n}(\phi)}[\phi=0]_{small} \tag{216}$$

is analytic and sufficiently small for $|\phi| < (10/11)LC_1(n_0+n_1+n)$. First of all, we confirm that the denominator part of (216) is bounded from below by a constant which depends on L and C_1, but not on n_0. (208) implies

$$\begin{aligned}|\text{denominator}| \quad &= \quad |\exp[-\widetilde{v'_{n_1+n}}(\phi)][\phi=0]_{small}| \\ &= \quad |\exp[\log[\phi=0]_{small}-L^2\mu_{n_1+n}\phi^2-\big(\frac{6L^2\lambda_{n_1+n}}{1-L^{-2}}-\frac{90L^2\rho_{n_1+n}}{(1-L^{-2})(1-L^{-4})}+\frac{45L^2\rho_{n_1+n}}{1-L^{-4}}\big)\phi^2]| \\ &\quad\times|\exp[-6L^2\big(\lambda_{n_1+n}-\frac{15\rho_n}{1-L^{-2}}\big)\phi^2-45L^2\rho_{n_1+n}\phi^2+\widetilde{R}_2\ (L,n_0,\rho_0,n)\phi^2]| \\ &\quad\times|\exp[-(\lambda_{n_1+n}-\frac{15\rho_{n_1+n}}{1-L^{-2}})\phi^4-15\rho_{n_1+n}\phi^4+\widetilde{R}_4\ (L,n_0,\rho_0,n)\phi^4]| \\ &\quad\times|\exp[-L^{-2}\rho_{n_1+n}\phi^6+\widetilde{R}_6\ (L,n_0,\rho_0,n)\phi^6+(\widetilde{v}'_{n_1+n})_{\geq 8}(\phi)]|\end{aligned} \tag{217}$$

We estimate $\log[\phi=0]_{small}$ as follows

$$\begin{aligned}\log[\phi=0]_{small} \quad &= \quad \log\int\exp[-w_\phi(z)]d\nu_2(z)|_{\phi=0} \\ &\geq \quad \log\exp[-\|w_\phi(z)\|_{\chi_2}]\int d\nu_2(z)|_{\phi=0} \\ &\geq \quad -\|w_0(z)\|_{\chi_2}+\log\int d\nu_2(z)\geq -8-1=-9.\end{aligned} \tag{218}$$

where

$$\|f(z)\|_{\chi_2} = \sup_{|z|<(n_0+n_1+n)^{1/4}}|f(z)|. \tag{219}$$

By using of (22), (23), (37) to (39), and (163) to (167), we can estimate other terms.

$$\begin{aligned}
|\text{denominator}| &\geq \exp[-9 - C_1^2 - C_1^4 - 3C_1^6 - 1] \\
&= \exp[-15C_1^6]. \qquad (220)
\end{aligned}$$

Next, we estimate the numerator part of (216),

$$\begin{aligned}
|\text{numerator}| &\leq \int |(1-\chi_2(z)) \exp[-\frac{L^4}{2}\sum_{\pm} v_{n_1+n}(-L^{-1}\phi \pm z) - \frac{1}{2}z^2]| \frac{dz}{\sqrt{2\pi}} \\
&= \int (1-\chi_2(z)) \prod_{\pm} |\exp[-v_{n_1+n}(L^{-1}\phi \pm z)]|^{L^4/2} \exp[-\frac{1}{2}z^2] \frac{dz}{\sqrt{2\pi}}, \qquad (221)
\end{aligned}$$

Utilizing (35) of **L2a** for $|L^{-1}\phi \pm z|$, we obtain

$$\begin{aligned}
|\text{numerator}| &\leq \exp[L^4 D - L^2(\lambda_{n_1+n}^{1/2} + \rho_{n_1+n}^{1/3})|\phi|^2 + 20\lambda_{n_1+n}(\mathbf{Im}\phi)^4 + 2004L^{-2}\rho_{n_1+n}(\mathbf{Im}\phi)^6] \\
&\quad \times \int (1-\chi_2(z)) \exp[(-L^4\lambda_{n_1+n}^{1/2} - L^4\rho_{n_1+n}^{1/3} - \frac{1}{2})z^2] \frac{dz}{\sqrt{2\pi}} \\
&\leq \exp[\log 2 + L^4 D + 20C_1^4 + 4008C_1^6] \times \check{\Phi}((n_0+n_1+n)^{1/4}). \qquad (222)
\end{aligned}$$

From

$$\check{\Phi}(x) \leq \sqrt{2}\exp[-x^2/4], \qquad (223)$$

we obtain

$$|\text{numerator}| \leq \exp[3/2\log 2 + L^4 D + 20C_1^4 + 4008C_1^6 - \frac{1}{4}(n_0+n_1+n)^{1/2}]. \qquad (224)$$

So, we know that

$$\begin{aligned}
&\int (1-\chi_2(z)) \exp[-\frac{L^4}{2}\sum_{\pm} v_{n_1+n}(L^{-1} \pm z)] d\nu(z)/[\phi=0]_{small} \exp[-\overset{\sim}{v'}_{n_1+n}(\phi)] \\
&\leq \exp[15C_1^6 + 3/2\log 2 + L^4 D + 20C_1^4 + 4008C_1^6 - \frac{1}{4}(n_0+n_1+n)^{1/2}]. \qquad (225)
\end{aligned}$$

If constants L, D and C_1 are given, we are able to take n_0 large enough to obtain

$$\text{RHS of (225)} \leq \exp[-\frac{1}{8}(n_0+n_1+n)^{1/2}]. \qquad (226)$$

This estimate is also valid for $\log(\phi = 0) - \log[\phi = 0]_{small}$. According to (226), we can show that $\overset{\approx}{v'}_{n_1+n}(\phi)$ is analytic and

$$|\overset{\approx}{v'}_{n_1+n}(\phi)| \leq 2e^{-1/8(n_0+n_1+n)^{1/2}}. \qquad (227)$$

3.1.3 Estimation of coefficients

In this subsection, we assume that $|\phi| < C_1(n_0+n_1+n+1)^{1/4}$ i.e. ϕ is in the small field region of $v'_{n_1+n}(\phi)$. Notice that the small field region is in the region $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$, so we can apply the result in 3.1.2. Thus, $\overset{\approx}{v}'_{n_1+n}(\phi)$ is analytic for small field region, and we can obtain power series expansion of $\overset{\approx}{v}'_{n_1+n}(\phi)$. Using of Cauchy estimate, we see that coefficients of ϕ^2, ϕ^4 and ϕ^6 satisfy,

$$\begin{aligned}
|\frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v'}_{n_1+n}(0)| &\leq e^{-1/8(n_0+n_1+n)^{1/2}}, \qquad (228) \\
|\frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v'}_{n_1+n}(0)| &\leq e^{-1/8(n_0+n_1+n)^{1/2}}, \qquad (229) \\
|\frac{1}{6!}\frac{d^6}{d\phi^6}\overset{\approx}{v'}_{n_1+n}(0)| &\leq e^{-1/8(n_0+n_1+n)^{1/2}}. \qquad (230)
\end{aligned}$$

Using the dominated convergence theorem, we see that $\frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v'_{n_1+n}}(0)$, $\frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v'_{n_1+n}}(0)$ and $\frac{1}{6!}\frac{d^6}{d\phi^6}\overset{\approx}{v'_{n_1+n}}(0)$ are continuous functions of μ_{n_1+n} on $I_n = [-(n_0+n_1+n)^{-3/2}, (n_0+n_1+n)^{-3/2}]$.

From (214) and (227), if n_0 is sufficiently large, then we have

$$|(v_{n_1+n})'_{\geq 8}(\phi)| \leq L^{-3(n+1)-n_1}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}), \text{ for } |\phi| < C_1(n_0+n_1+n+1)^{1/4}. \tag{231}$$

From (208) and (211), and (230), we know that

$$\begin{aligned} |\rho'_{n_1+n} - L^{-2}\rho_{n_1+n}| &= |\tilde{R}_6(L, n_0, \rho_0, n) + \frac{1}{6!}\frac{d^6}{d\phi^6}\overset{\approx}{v'_{n_1+n}}(0)| \\ &\leq L^{-2(n_1+n)}n_0^{-15/8}. \end{aligned} \tag{232}$$

We obtain

$$|\rho'_{n_1+n} - L^{-2(n_1+n+1)}\rho_0| < (n_1+n+1)L^{-2(n_1+n)}n_0^{-7/4} \tag{233}$$

which proves $(39)'$ of $\mathbf{L2b}'$. (208) implies

$$\begin{aligned} \lambda'_{n_1+n} - \frac{15\rho'_{n_1+n}}{1-L^{-2}} &= (\lambda_{n_1+n} - \frac{15\rho_{n_1+n}}{1-L^{-2}}) + 15\rho_{n_1+n} + \tilde{R}_4(L, n_0, \rho_0, n) + \frac{1}{4!}\frac{d^4}{d\phi^4} \\ &= \lambda_{n_1+n} - \frac{15L^{-2}\rho_{n_1+n}}{1-L^{-2}} + \tilde{R}_4(L, n_0, \rho_0, n) + \frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v'_{n_1+n}}(0), \end{aligned} \tag{234}$$

$$\begin{aligned} |\lambda'_{n_1+n} - \lambda_{n_1+n}| &= |\tilde{R}_4(L, n_0, \rho_0, n) + \frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v'_{n_1+n}}(0) + \frac{15}{1-L^{-2}}(\rho'_{n_1+n} - L^{-2}\rho_{n_1+n})| \\ &\leq n_0^{-15/8}. \end{aligned} \tag{235}$$

So, we have

$$|\lambda'_{n_1+n} - \lambda_0| < (n_1+n+1)n_0^{-7/4}, \tag{236}$$

which completes the proof of $\mathbf{L2b}'$. Consider the quartic part $(v'_{n_1+n})_2(\phi)$

$$\begin{aligned} (v'_{n_1+n})_2(\phi) &= \big(\mu_{n_1+n+1} - (\frac{6\lambda_{n_1+n+1}}{1-L^{-2}} - \frac{90\rho_{n_1+n+1}}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho_{n_1+n+1}}{1-L^{-4}})\big)\phi^2 \\ &= L^2\mu_{n_1+n}\phi^2 - L^2(\frac{6\lambda_{n_1+n}}{1-L^{-2}} - \frac{90\rho_{n_1+n}}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho_{n_1+n}}{1-L^{-4}})\phi^2 + 6L^2\big(\lambda_{n_1+n} - \frac{15\rho_{n_1+n}}{1-L^{-2}}\big)\phi^2 \\ &\quad +45L^2\rho_{n_1+n}\phi^2 + \tilde{R}_2(L, n_0, \rho_0, n)\phi^2 + \frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v'_{n_1+n}}(0)\phi^2. \end{aligned} \tag{237}$$

Comparing these coefficients , we get

$$\begin{aligned} |\mu'_{n_1+n} - L^2\mu_{n_1+n}| &\leq |(\frac{6(\lambda_{n_1+n} - \lambda'_{n_1+n})}{1-L^{-2}} - \frac{90(L^{-2}\rho_{n_1+n} - \rho'_{n_1+n})}{(1-L^{-2})(1-L^{-4})} + \frac{45(L^{-2}\rho_{n_1+n} - \rho'_{n_1+n})}{1-L^{-4}})| \\ &\quad +|\tilde{R}_2(L, n_0, \rho_0, n) + \frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v'_{n_1+n}}(0)| \\ &\leq 16n_0^{-15/8}. \end{aligned} \tag{238}$$

Remark that map $R : \mu \to \mu'$ is continuous, and image $R(I_n)$ include I_{n+1}. Thus, we can take for J_{n+1} a conected component of this inverse image $R^{-1}(I_{n+1}) \subset I_n$. This ends the analysis of the small field properties.

3.2 Large field region analysis

In this subsection, we prove that $e^{-(v_{n_1+n})'(\phi)}$ satisfy the condition $\mathbf{L2a}'$. First, we prove it in the case where $|\mathbf{Re}\phi| > C_1(n_0+n_1+n+1)^{1/4}$. Next, we prove it in $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$ i.e. this region includes the small field region of $v'(\phi)$.

3.2.1 Proof of condition L2a′ in the case where $|\mathbf{Re}\phi| > C_1(n_0+n_1+n+1)^{1/4}$

Remark that the definition of the RG (25) has the following expression

$$e^{-v'_{n_1+n}(\phi)} = \int\prod_{\pm}\exp[-v(L^{-1}\phi\pm z)]^{L^4/2}\exp[-\frac{1}{2}z^2]\frac{dz}{\sqrt{2\pi}}/(\phi=0). \tag{239}$$

Since $|\mathbf{Im}(L^{-1}\phi\pm z)| < C_1(n_0+n_1+n)^{1/4}$ if $|\mathbf{Im}\phi| < C_1(n_0+n_1+n+1)^{1/4}$, From condition **L2a**,

$$\begin{aligned} |e^{-(v_{n_1+n})'(\phi)}| &\le \exp[L^4D - L^2(\lambda_{n_1+n}^{1/2}+\rho_{n_1+n}^{1/3})\phi^2 + 20\lambda_{n_1+n}(\mathbf{Im}\phi)^4 + 2004L^{-2}\rho_{n_1+n}(\mathbf{Im}\phi)^6] \\ &\quad\times\int_{-\infty}^{\infty}\exp[-L^4\lambda_{n_1+n}z^2 - L^4\rho_{n_1+n}^{1/3}z^2 - \frac{1}{2}z^2]\frac{dz}{\sqrt{2\pi}}/(\phi=0). \end{aligned}\tag{240}$$

Noting that λ_{n_1+n} and ρ_{n_1+n} are positive, and from (218) we have

$$\begin{aligned} \text{RHS of (240)} &\le \exp[L^4D - L^2(\lambda_{n_1+n}^{1/2}+\rho_{n_1+n}^{1/3})\phi^2 + 20\lambda_{n_1+n}(\mathbf{Im}\phi)^4] \\ &\quad\times\exp[2004L^{-2}\rho_{n_1+n}(\mathbf{Im}\phi)^6+9]. \end{aligned}\tag{241}$$

Decompose (241) into the part of **L2a′** and the rest

$$\begin{aligned} &\le \exp[D - ({\lambda'_{n_1+n}}^{1/2}+{\rho'_{n_1+n}}^{1/3})\phi^2 + 20\lambda'_{n_1+n}(\mathbf{Im}\phi)^4 + 2004\rho'_{n_1+n}(\mathbf{Im}\phi)^6] \\ &\quad\times\exp[(L^4-1)D - \{L^2(\lambda_{n_1+n}^{1/2}+\rho_{n_1+n}^{1/3}) - ({\lambda'_{n_1+n}}^{1/2}+{\rho'_{n_1+n}}^{1/3})\}\phi^2] \\ &\quad\times\exp[20(\lambda'_{n_1+n}-\lambda_{n_1+n})(\mathbf{Im}\phi)^4 + 2004(L^{-2}\rho_{n_1+n}-\rho'_{n_1+n})(\mathbf{Im}\phi)^6+9]. \end{aligned}\tag{242}$$

We prove that the remainder part of (242) is less than 1. Remark that

$$L^2(\lambda_{n_1+n}^{1/2}+\rho_{n_1+n}^{1/3}) - ({\lambda'_{n_1+n}}^{1/2}+{\rho'_{n_1+n}}^{1/3}) \ge \sqrt{C_{--}}(1-2L^{-2})n_0^{1/2} + (L^{8/3}-2)(L^{2(n_1+n+1)}\rho_0^{-1})^{-1/3}.$$

If, $|\mathbf{Re}\phi| > C_1(n_0+n_1+n+1)^{1/4}$ and $|\mathbf{Im}\phi| < C_1(n_0+n_1+n+1)^{1/4} < C_1(L^{2(n_1+n+1)-4}\rho_0^{-1})^{1/6}$, then by (232), (235)

$$\begin{aligned} \text{Remainder of RHS of (242)} &\le \exp[(L^4-1)D - \sqrt{C_{--}}(1-2L^{-2})C_1^2] \\ &\quad\times\exp[-(L^{8/3}-2)C_1^2(L^{-2(n_1+n+1)}\rho_0)^{1/3}(n_0+n_1+n+1)^{1/2}] \\ &\quad\times\exp[40n_0^{-15/8+1}C_1^4 + 8016L^{-2(n_1+n)}n_0^{-15/8+3/2}C_1^6+9] \\ &\le \exp[(L^4-1)D - \sqrt{C_{--}}(1-2L^{-2})C_1^2 + 40C_1^4n_0^{-7/8}] \\ &\quad\times\exp[+8016L^{-2(n_1+n)}C_1^6n_0^{-3/8}+9], \end{aligned}\tag{243}$$

If D and L are given , we take C_1 sufficiently large, then n_0 large. Thus, RHS of (243) is smaller than 1. So, we obtain

$$|\exp(-v'(\phi))| < \exp[D - ({\lambda'_{n_1+n}}^{1/2}+{\rho'_{n_1+n}}^{1/3})\phi^2 + 20\lambda'(\mathbf{Im}\phi)^4 + 2004\rho'_{n_1+n}(\mathbf{Im}\phi)^6], \tag{244}$$

for

$$|\mathbf{Im}\phi| < C_1(n_0+n_1+n+1)^{1/4}, |\mathbf{Re}\phi| > C_1(n_0+n_1+n+1)^{1/4}.$$

3.2.2 Proof of condition L2a′ in the case where $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$

Let $|\phi| < (10/11)LC_1(n_0+n_1+n)^{1/4}$. Notice that $(L^{2(n_1+n)-4}\rho_0^{-1})^{1/6} > (n_0+n_1+n)^{1/4}$. From (37), (38) of **L2b′**, (163) to (167), (227) to (230), (212) and (238),

$$\begin{aligned} |e^{-(v_{n_1+n})'(\phi)}| &= |\exp[-\mu'_{n_1+n}\phi^2 - (\frac{6\lambda'_{n_1+n}}{1-L^{-4}} - \frac{90\rho'_{n_1+n}}{(1-L^{-2})(1-L^{-4})} + \frac{45\rho'_{n_1+n}}{1-L^{-4}})\phi^2] \\ &\quad\times\exp[-(\lambda'_{n_1+n} - \frac{15\rho'_{n_1+n}}{1-L^{-2}})\phi^4 - \rho'_{n_1+n}\phi^6 - v'_{\ge 8}(\phi)]| \\ &\le \exp[4L^{-2}C_1^2n_0^{-1/2} + 60L^4C_1^4(L^{-2(n_1+n+1)}\rho_0)^{1/3}] \\ &\quad\times\exp[-\lambda'_{n_1+n}(\mathbf{Re}\phi^4) - \rho'_{n_1+n}(\mathbf{Re}\phi^6) + L^4n_0^{-1/2}]. \end{aligned}\tag{245}$$

$$\begin{aligned}\rho'_{n_1+n}(\mathbf{Re}\phi^6) &= \rho'_{n_1+n}\{(\mathbf{Re}\phi)^6 - 15(\mathbf{Re}\phi)^4(\mathbf{Im}\phi)^2 + 15(\mathbf{Re}\phi)^2(\mathbf{Im}\phi)^4 - (\mathbf{Im}\phi)^6\} \\ &\geq \rho'_{n_1+n}\big(\frac{1}{4}(\mathbf{Re}\phi)^6 - 2001(\mathbf{Im}\phi)^6\big) \\ &\geq -\frac{1}{2}D_6 + (\rho'_{n_1+n})^{1/3}|\phi|^2 - 2004\rho'_{n_1+n}(\mathbf{Im}\phi)^6 \qquad (246)\end{aligned}$$

Notice that D_4 does not depend on C_1, n_0 or n. Similarly, we can estimate, $\lambda'_{n_1+n}\mathbf{Re}\phi^4 \geq -\frac{1}{2}D_4 + (\lambda'_{n_1+n})^{1/2}|\phi|^2 - 20\lambda_p lan_1 + n'(\mathbf{Im}\phi)^4$. Notice that D_4 does not deped on C_1, n_0 or n either. Now, put $D = D_4 + D_6$, from (245) to (246)

$$\begin{aligned}|e^{-(v_{n_1+n})'_{\geq 4}(\phi)}| &\leq \exp[\frac{1}{2}D - ((\lambda'_{n_1+n})^{1/2} + (\rho'_{n_1+n})^{1/3})|\phi|^2 + 20(\lambda'_{n_1+n})(\mathbf{Im}\phi)^4 + 2004(\rho'_{n_1+n})(\mathbf{Im}\phi)^6] \\ &\quad \times \exp[4L^{-2}C_1^2 n_0^{-1/2} + 60C_1^4(L^{-2(n_1+n+1)}\rho_0)^{1/3} + L^4 n_0^{-1/2}] \\ &\leq \exp[D - ((\lambda'_{n_1+n})^{1/2} + (\rho'_{n_1+n})^{1/3})|\phi|^2 + 20(\lambda'_{n_1+n})(\mathbf{Im}\phi)^4 + 2004(\rho'_{n_1+n})(\mathbf{Im}\phi)^6] \\ &\quad \times \exp[-\frac{1}{2}D + 4L^{-2}C_1^2 n_0^{-1/2} + 60C_1^4(L^{-2(n_1+n+1)}\rho_0)^{1/3} + L^4 n_0^{-1/2}]. \qquad (247)\end{aligned}$$

Take n_0 sufficiently large, we obtain

$$\leq \exp[D - ((\lambda'_{n_1+n})^{1/2} + (\rho'_{n_1+n})^{1/3})|\phi|^2 + 20(\lambda'_{n_1+n})(\mathbf{Im}\phi)^4 + 2004(\rho'_{n_1+n})(\mathbf{Im}\phi)^6]. \qquad (248)$$

Proof of Lemma 2 is completed.

4 Proof of Theorem 1

Finally, we prove Theorem 1, using Lemma 1, Lemma 2 and Theorem 2. First of all, we notice that it is possible to take constants L, D, $C_1(L, D)$, $n_0(L, D, C_1)$ to satisfy both Lemma 1, Lemma 2, and Theorem 2. We can check that potential $v(\phi)$ can be iterated n_1 times if initial parameters satisfy conditions **Ta** to **Tc** because of Lemma 1. Notice that n_1 time iterated potential $v_{n_1}(\phi)$ satisfies conditions **L2a** to **L2c** with $n = 0$, and so Lemma 2 can be applied to this potential. We have to iterate using Lemma 2, sufficiently many times so that the iterated potentials satisfy G-K conditions. Put

$$n_2 = \min\{n \in \mathbf{N} : |\rho_{n_1+n}||\phi|^6 + |(v_{n_1+n})_{\geq 8}(\phi)| < (n_0 + n_1 + n)^{-3/4} \text{ for } |\phi| < C_1(n_0 + n_1 + n)^{1/4}\} + 1. \qquad (249)$$

Then,

$$\rho_{n_1+n_2-1} < (n_0 + n_1 + n_2)^{-9/4}. \qquad (250)$$

By explicit calculation, n_1 and n_2 can be estimated as follows,

$$\frac{1}{8}\log_L n_0 < n_1, n_2 < \log_L n_0.$$

Since, $\rho_{n_1+n_2} \geq 0$, and by (37)

$$\begin{aligned}\lambda_{n_1+n_2} - \frac{15\rho_{n_1+n_2}}{1-L^{-2}} &< \lambda_0 + (n_1+n_2)n_0^{-7/4} \\ &< \frac{C_{++}}{L^4}n_0^{-1} + 2(\log_L n_0)n_0^{-7/4} \\ &< \frac{C_+}{L^4}(n_0+n_1+n_2)^{-1}. \qquad (251)\end{aligned}$$

Similarly , by (250) we have

$$\lambda_{n_1+n_2} - \frac{15\rho_{n_1+n_2}}{1-L^{-2}} > \frac{C_-}{L^4}(n_0+n_1+n_2)^{-1}.$$

So, we checked condition **G-Kb** completely. Next, let us check condition **G-Ka**. Notice that analyticity, positivity for real ϕ, and even function of $v_{n_1+n_2}(\phi)$ are checked easily. Now, We check the bound of $v_{n_1+n_2}(\phi)$

$$\begin{aligned}|\exp[-v_{n_1+n_2}(\phi)]| &\leq \exp[D - (\lambda_{n_1+n_2}^{1/2} + \rho_{n_1+n_2}^{1/3})|\phi|^2 + 20\lambda_{n_1+n_2}(\mathbf{Im}\phi)^4 + 2004\rho_{n_1+n_2}(\mathbf{Im}\phi)^6] \\ &\leq \exp[D - \lambda_{n_1+n_2}^{1/2}|\phi|^2 + 20\lambda_{n_1+n_2}(\mathbf{Im}\phi)^4 + 2004\rho_{n_1+n_2}(\mathbf{Im}\phi)^6]. \qquad (252)\end{aligned}$$

Notice that, from definition of n_1 and n_2,

$$-\rho_{n_1+n_2}^{1/3}|\phi|^2 + 2004\rho_{n_1+n_2}(\mathbf{Im}\phi)^6 \leq 0$$

for $(\mathbf{Im}\phi) < C_1(n_0+n_1+n_2)^{1/4}$, so we have

$$|\exp[-v_{n_1+n_2}(\phi)]| \quad \leq \quad \exp[D - \lambda_{n_1+n_2}^{1/2}|\phi|^2 + 20\lambda_{n_1+n_2}(\mathbf{Im}\phi)^4]. \tag{253}$$

We have checked all of G-K conditions. Notice that $\rho_{n_1+n_2-1}$ is sufficiently small by (250), we know

$$\begin{aligned} &|\mu_{n_1+n_2} - L^2\big(\mu_{n_1+n_2-1} + \frac{90\rho_{n_1+n_2-1}}{(1-L^{-2})(1-L^{-4})} - \frac{45\rho_{n_1+n_2-1}}{1-L^{-4}}\big)| \\ &\leq 16n_0^{-15/8} + |\frac{15L^2\rho_{n_1+n_2-1}}{1-L^{-2}}| \leq 17n_0^{-15/8} \end{aligned} \tag{254}$$

As in the proof Lemma 1 and Lemma 2, we can take for $J_{n_1+n_2}$ a suitable conected component. So, we can adapt Theorem 2. Theorem 1 is now proved.

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