# ANISOTROPIC SOBOLEV SPACES AND DYNAMICAL TRANSFER OPERATORS: $C^\infty$ FOLIATIONS

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ABSTRACT. We consider a $C^\infty$ Anosov diffeomorphism $T$ with a $C^\infty$ stable dynamical foliation. We show upper bounds on the essential spectral radius of its transfer operator acting on anisotropic Sobolev spaces. (Such bounds are related to the essential decorrelation rate for the SRB measure.) We compare our results to the estimates of Kitaev on the domain of holomorphy of dynamical determinants for differentiable dynamics.

## 1. INTRODUCTION

Let $T$ be an Anosov diffeomorphism on a $d$-dimensional compact $C^\infty$ Riemann manifold $\mathcal{X}$ (i.e., $T\mathcal{X} = E^u \oplus E^s$ and there are $C > 0$, $\gamma > 1$, with $|DT^n|E^s| \le C\gamma^{-n}$, $|DT^{-n}|E^u| \le C\gamma^{-n}$ for all $n \ge 1$). Denote the Jacobian of $T$ with respect to Lebesgue by $|\det DT|$. To construct SRB measures and to analyse their speed of mixing, it is natural to consider the following operators, defined initially on $C^\infty$ functions:

$$\mathcal{M}\varphi = \frac{\varphi \circ T^{-1}}{|\det DT| \circ T^{-1}}, \qquad \mathcal{L}\varphi = \varphi \circ T. \tag{1.1}$$

The operator $\mathcal{L}$ fixes the constant functions, while $\mathcal{M}$ fixes the constant functions if and only if $\det DT$ is constant (i.e., if $T$ is volume preserving). The dual of $\mathcal{M}$ restricted to elements of the dual of $\mathcal{C}^\infty$ which are finite complex measures, absolutely continuous with respect to Lebesgue with a $\mathcal{C}^\infty$ density, coincides with $\mathcal{L}$ viewed as acting on the density and vice-versa. Alternatively, the dual of $\mathcal{M}$ acting on $L^1(\mathcal{X}, \mathrm{Leb})$ is $\mathcal{L}$ acting on $L^\infty(\mathcal{X}, \mathrm{Leb})$.

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For $w \in \mathcal{X}$, and $\widetilde{T}$ an Anosov diffeomorphism on $\mathcal{X}$, introduce local hyperbolicity exponents ($|\cdot|$ denotes euclidean norm)

$$\lambda_w(\widetilde{T})^{-1} = \sup_{v \in E^u(\widetilde{T}(w)), |v|=1} |D_{T(w)} \widetilde{T}^{-1}(v)| \, ,$$
$$\nu_w(\widetilde{T}) = \sup_{v \in E^s(w), |v|=1} |D_w \widetilde{T}(v)| \, .$$

Assume $T$ is $C^{r+1}$ for some $r > 0$. Kitaev [14] proved that the following "dynamical Fredholm determinant"

$$d(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n(x)=x} \frac{1}{|\det(T^n(x) - \mathrm{Id}\,)|}$$

extends to a holomorphic function in each disc $\{z \mid |z| \cdot \rho_1^{(p,s)}(T) < 1\}$, where $p \in (-r, 0)$, $s = r + p$, and

$$\rho_1^{(p,s)}(T) = \lim_{n\to\infty} \left( \int_{\mathcal{X}} \max\bigl((\lambda_w(T^n))^p, (\nu_w(T^n))^s\bigr) \, dLeb(w) \right)^{1/n} < 1 \, .$$

One may take $s = -p = r/2$: Kitaev's result is then reminiscent of the "loss of one half of the Hölder exponent" which occurs when going from two-sided subshifts to one-sided subshifts in symbolic dynamics [7], since one easily sees that $\rho_1^{(-r/2,r/2)}(T) \le \gamma^{-r/2}$.

In view of the results of Ruelle [16] for smooth expanding maps, it is natural to look for Banach spaces $\mathcal{B}_{p,s,\mathcal{L}}$, respectively $\mathcal{B}_{p,s,\mathcal{M}}$, on which the essential spectral radius of $\mathcal{L}$, respectively $\mathcal{M}$, is $\le \rho_1^{(p,s)}$. Set

$$\rho_\infty^{(p,s)}(T) = \lim_{n\to\infty} \left( \sup_{\mathcal{X}} \max\bigl((\lambda_w(T^n))^p, (\nu_w(T^n))^s\bigr) \right)^{1/n} < 1 \, .$$

Clearly $\rho_\infty^{(p,s)}(T) \ge \rho_1^{(p,s)}(T)$ and, e.g., $\rho_\infty^{(-r/2,r/2)}(T) \le \gamma^{-r/2}$.

We shall assume that $T$ is $C^\infty$ and the stable foliation of $T$ is $C^\infty$. (This is a very strong assumption, and the corresponding case should be viewed as a "toy model" in which the features of our symbolic calculus approach are completely transparent: The heart of the proof is contained in a half page, between (2.7) and (2.8) below.) We introduce in Subsection 2.3, for all $p$, $s$ in $\mathbb{R}$ and all $1 \le t \le \infty$, a Banach space $W^{p,s-p,t}(\mathcal{X}) = W^{p,s-p,t}(\mathcal{X}, T)$ of distributions, based on $L^t(\mathrm{Leb})$. [1]

Our main result (Theorem 2.8) is that, whenever the stable foliation is $C^\infty$, the essential spectral radius $\rho_{ess}$ of $\mathcal{L}$ on $W^{p,s-p,\infty}(\mathcal{X})$ is at most $\rho_\infty^{(p,s)}(T)$ for all $p < 0$, $s > 0$. If $T$ is in addition volume preserving, then the same bound holds for $\mathcal{L}$ on $W^{p,s-p,t}(\mathcal{X})$ for all $t > 1$. In

[1] Controlling the spectrum on a scale of Sobolev spaces may be useful: see [5].

general, $\limsup_{t\to\infty} \rho_{ess}(\mathcal{L}|_{W^{p,s-p,t}(\mathcal{X})}) \le \rho_\infty^{(p,s)}(T)$. If the unstable foliation is $C^\infty$ (Theorem 2.11), the essential spectral radius of $\mathcal{M}$ on $W^{p,s-p,1}(\mathcal{X},T^{-1})$ is at most $\rho_\infty^{(-s,-p)}(T)$ for all $p<0$, $s>0$.

Propositions 2.10 and 2.12 give upper bounds related to $\rho_1^{(p,s)}(T)$: They imply

$$\rho_{ess}(\mathcal{L}|_{W^{p,s-p,\infty}}) \le \lim_{n\to\infty} \| \det DT^n|_{E^u}\|^{1/n}_{L^\infty(\mathrm{Leb})} \rho_1^{(p,s)}(T)\,,$$

$$\limsup_{t\to 1} \rho_{ess}(\mathcal{L}|_{W^{p,s-p,t}}) \le \lim_{n\to\infty} \|(\det DT^n|_{E^s})^{-1}\|^{1/n}_{L^\infty(\mathrm{Leb})} \rho_1^{(p,s)}(T)\,.$$

Finally, we study in the appendix the essential spectral radii of

$$\mathcal{L}_t\varphi = |\det DT|^{1/t}\cdot(\varphi\circ T)\,, \qquad \mathcal{M}_t\varphi = \frac{\varphi\circ T^{-1}}{|\det DT|^{1-1/t}\circ T^{-1}}\,.$$

The case when $T$ is $C^{r+1}$ (for some $r>0$) and neither of the dynamical foliations is $C^\infty$, but at least one of them is $C^{1+\epsilon}$ (for $\epsilon>0$) will be treated in a forthcoming work [4], using spaces due to Alinhac [1]. We hope that the (general) $C^\alpha$ foliation case will be amenable to the present approach. Gouëzel and Liverani [10] have independently obtained non trivial, but weaker, bounds for the essential spectral radius of $\mathcal{M}$, on a different Banach space, in this general case.

We end this introduction with three open problems:

*Remark* 1.1 (Links with SRB measures). With the techniques of Blank–Keller–Liverani [6], one should obtain that the spectral radius of $\mathcal{L}$ on each $W^{p,q,\infty}(\mathcal{X})$ is one, that 1 is is a semi-simple eigenvalue, and that the corresponding eigenvector (in the dual of $W^{p,q,\infty}(\mathcal{X})$) for the dual of $\mathcal{L}$ is an invariant probability measure $\mu$ with ergodic basin of full Lebesgue measure. Furthermore, the multiplicity of the eigenvalue 1 is equal to the number of ergodic components of $\mu$, and each ergodic component is an SRB measure. Also, if 1 is a simple eigenvalue then it is the only eigenvalue on the unit circle: this corresponds to exponential decay of correlations for smooth observables. If the unstable foliation is $C^\infty$, the SRB measure(s) of $T$ can alternatively be constructed with the fixed point of $\mathcal{M}$ in $W^{p,q,1}(\mathcal{X},T^{-1})$. [2]

*Remark* 1.2 (Spectral stability). The perturbation techniques of [6], should imply stability of the spectrum of $\mathcal{L}$ (including eigenprojectors) outside a disc of radius $\rho$, under stochastic and $C^{r+1}$ deterministic perturbations of $T$, perhaps up to taking $\rho > \rho_\infty^{(p,s)}$. For deterministic perturbations $\widetilde{T}$ of $T$, the Banach spaces $W^{p,q,t}(\mathcal{X},T)$ and

[2]The operators $\mathcal{L}_t$, $\mathcal{M}_t$ "interpolate" between the SRB measures of $T$, $T^{-1}$.

$\widetilde{W}^{p,q,t} = W^{p,q,t}(\mathcal{X}, \widetilde{T})$ are different. "Stability of the eigenprojector" $\Pi$ of $T$ associated to an eigenvalue $\tau$ of large enough modulus means the following (assume $\tau$ is simple): Let $\widetilde{\mathcal{L}}$ denote the transfer operator of $\widetilde{T}$; then, if $\widetilde{T}$ is close enough to $T$, there are a Banach space $W_\epsilon$ contained in the intersection of $\widetilde{W}^{p,q,t}$ and $W^{p,q,t}$, a rank-one projector $\Pi_\epsilon$ (on $W_\epsilon$, $\widetilde{W}^{p,q,t}$, and $W^{p,q,t}$), and a simple eigenvalue $(\tilde{\tau}, \widetilde{\Pi})$ for $\widetilde{\mathcal{L}}$ on $\widetilde{W}^{p,q,t}$, so that both $\|\Pi_\epsilon - \Pi\|_W$ and $\|\Pi_\epsilon - \widetilde{\Pi}\|_{\widetilde{W}}$ are small.

*Remark* 1.3 (Essential decorrelation radius). For $C^{r+1}$ expanding circle endomorphisms $F$, the essential spectral radius $\rho_{ess}(\mathcal{M}_F|_{C^r})$ of $\mathcal{M}_F\varphi(x) = \sum_{F(y)=x} \varphi(y)/|\det DF(y)|$ acting on $C^r$ functions (see [9] and references therein) is equal to

$$\lim_{n\to\infty} \Big(\int |(F^n)'(x)|^{-r}\, dLeb(x)\Big)^{1/n} = \lim_{n\to\infty} \Big(\int |(F^n)'(x)|^{-r}\, d\mu_{SRB}(x)\Big)^{1/n} .$$

However, for $C^{r+1}$ expanding maps in arbitrary dimension [11]

(1.2)

$$\begin{aligned}\rho_{ess}(\mathcal{M}_F|_{C^r}) &= \exp(\sup_\mu \{h_\mu - \int \log|\det DF|\, d\mu - r\cdot \chi_\mu\}) \\ &\le \lim_{n\to\infty} \Big(\int \sup_{|v|=1} |D_x(F^n)(v)|^{-r}\, dLeb(x)\Big)^{1/n} ,\end{aligned}$$

where $\mu$ ranges over ergodic $F$-invariant probability measures, $h_\mu$ is the entropy of $\mu$, and $\chi_\mu$ denotes the smallest (positive) Lyapunov exponent of $DF$. The inequality in (1.2) can be strict. In the other direction, note that $\rho_{ess}(\mathcal{M}_F|_{C^r}) \ge \exp(-r\chi_{\mu_{SRB}})$, and the inequality can be strict [9], even in dimension one. The results of Avila et al. [3], indicate that in dimension at least two there may be Banach spaces containing all $C^r$ functions on which the essential spectral radius of $\mathcal{M}_F$ is strictly smaller than $\rho_{ess}(\mathcal{M}_F|_{C^r})$. (This would imply [9] that $\rho_{ess}(\mathcal{M}_F|_{C^r})$ may be strictly larger than the essential decorrelation radius of $F$ for $C^r$ observables and thus $\rho_{point-ess}(\mathcal{M}_F|_{C^r}) < \rho_{ess}(\mathcal{M}_F|_{C^r})$.)

Let $T$ be a transitive $C^\infty$ Anosov diffeomorphism with both foliations $C^\infty$. Let $\rho^+_{ess}(p,s,t)$ and $\rho^-_{ess}(p,s,t)$ be the essential spectral radii of $\mathcal{L}$ acting on $W^{p,s-p,t}(\mathcal{X})$ and $W^{-p,-s+p,t}(\mathcal{X}, T^{-1})$, respectively, and set

$$\rho(r) := \min\Big( \inf_{\substack{t,p\in(-r,0)\\ s\in(0,r+p)}} \rho^+_{ess}(p,s,t), \inf_{\substack{t,p\in(-r,0)\\ s\in(0,r+p)}} \rho^-_{ess}(p,s,t)\Big) .$$

We expect that $\inf_{\mathcal{B}_r} \rho_{ess}(\mathcal{L}|_{\mathcal{B}_r})$, where $\mathcal{B}_r$ spans all Banach spaces of distributions of order $\le r$, containing all $C^r$ functions, and on which $\mathcal{L}$ acts boundedly, coincides with the essential decorrelation radius $\hat\rho(r)$ of $T$ for $C^r$ functions, and that $\hat\rho(r) < \rho(r)$ can occur.

## 2. Bounding the essential spectral radius

2.1. **Preliminaries.** From now on and until the end of Subsection 2.6, $T$ is Anosov and $C^\infty$, with a $C^\infty$ stable foliation $\mathcal{F}^s$. Write $I = (-1, 1)$, and let $d_s$ be the dimension of $\mathcal{F}^s$. We work with $C^\infty$ foliated charts $\kappa$, $V$: let $\cup_{i \in \mathcal{I}} V_i$ be a finite covering of $\mathcal{X}$ by small open sets, and let $U_i = I^d = I^{d_s} \times I^{d-d_s}$ be $\#\mathcal{I}$ disjoint copies of $I^d$, viewed as subsets of disjoint copies $\mathbb{R}^d_i$ of $\mathbb{R}^d$. Write $U = \cup_i U_i$. Let $\kappa_i : V_i \to U_i$ be $C^\infty$ diffeomorphisms so that $\kappa_i^{-1}$ of each horizontal segment is the intersection of a leaf of $\mathcal{F}^s$ with $V_i$. In addition, we require that $\kappa_i^{-1}(\{(0, y)\})$ is the unstable leaf of $\kappa_i^{-1}(0, 0)$ intersected with $V_i$ (this is a way to require closeness of the vertical foliation in $I^d$ and the image of leaves of the unstable foliation $\mathcal{F}^u$).

Choose a $C^\infty$ partition of the unity $\{\psi_i\}$ on $\mathcal{X}$, compatible with $V = \{V_i\}$, i.e., each $\psi_i$ is supported in $V_i$. Then, for each $n \ge 1$

$$\mathcal{L}^n \varphi(w) = \sum_{i,j} \psi_j(T^n(w)) \psi_i(w) \cdot \varphi(T^n(w)) \, .$$

If $V_{i,j} = V_{i,j,n} := T^{-n}(V_j) \cap V_i \neq \emptyset$, setting $U_{i,j,n} := \kappa_i(T^{-n}(V_j) \cap V_i) \subset U_i$, the map $T^n_{ij} : U_{i,j,n} \to U_j$ has a derivative in block form:

$$\begin{pmatrix} A^{tr}_{ij,n}(x,y) & B^{tr}_{ij,n}(x,y) \\ 0 & D^{tr}_{ij,n}(x,y) \end{pmatrix} , \ (x,y) \in (I^{d_s}, I^{d-d_s}) \, ,$$

with $A_{ij,n}$ a $d_s \times d_s$ matrix, $D_{ij,n}$ a $(d-d_s) \times (d-d_s)$ matrix, and

$$\begin{aligned} &|A_{ij,n}(x,y)| \le \nu_{i,j}(T^n) := \sup_{w \in V_{i,j}} \nu_w(T^n) < 1 \, , \\ &|D_{ij,n}(x,y)^{-1}| \le \lambda_{i,j}(T^n)^{-1} := \sup_{w \in V_{i,j}} (\lambda_{T^n(w)}(T^n))^{-1} < 1 \ . \end{aligned} \tag{2.1}$$

Furthermore, for each $\epsilon$, there exists $\delta$ so that if $\operatorname{diam} V < \delta$ then

$$|B_{ij,n}(x,y)v| \le \epsilon |D_{ij,n}(x,y)v| \, , \forall n \ge 1 \, , \ \forall v \in \mathbb{R}^{d-d_s} \, . \tag{2.2}$$

2.2. **Local spaces $W^{p,q,t}(I^d)$.** Let $p$ and $q$ be real numbers. We introduce the "symbol" $a_{p,q}(x, y, \xi, \eta)$, for $(\xi, \eta) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d-d_s}$ and $(x, y) \in I^{d_s} \times I^{d-d_s}$:

$$a_{p,q}(x, y, \xi, \eta) = (1 + |\xi|^2 + |\eta|^2)^{p/2} (1 + |\xi|^2)^{q/2} \, .$$

The corresponding linear operator $a^{Op}_{p,q}$ acts on $C^\infty(I^d)$ by

$$a^{Op}_{p,q}(f)(x,y) = (2\pi)^{-d} \int \int e^{ix\xi} e^{iy\eta} a(x, y, \xi, \eta) \hat{f}(\xi, \eta) \, d\xi \, d\eta \, ,$$

where the Fourier transform is $\hat{f}(\xi, \eta) = \int \int e^{-ix\xi} e^{-iy\eta} f(x, y) \, dx \, dy$.

**Definition 2.1** (Anisotropic Sobolev spaces)**.** For $1 \le t \le \infty$, let $W^{p,q,t}(I^d)$ be the closure of $\{f \in C^\infty(I^d)\}$, for the $L^t(I^d, \mathrm{Leb})$ norm of $a_{p,q}^{Op}(f)$, with induced norm, denoted $\|\cdot\|_{p,q,t,I^d}$.

By construction, $a_{p,q}^{Op}$ extends to a bounded invertible operator from $W^{p,q,t}(I^d)$ to $L^t(I^d)$. Clearly, $H^{p,q}(I^d) = W^{p,q,2}(I^d)$ is a Hilbert space.

**Lemma 2.2** (Boundedness/compactness of embedding)**.** *Assume that $1 < t < \infty$. Denote by $W_C^{p',q',t}(I^d)$ those $f \in W^{p',q',t}(I^d)$ supported in a compact subset of $I^d$. If $q' \ge q$ and $p' \ge p$ then the natural injection $W_C^{p',q',t}(I^d) \subset W^{p,q,t}(I^d)$ is bounded. This injection is compact if $q' \ge q$ and $p' > p$ .*

*Proof.* The proofs of Theorems 2.5.2 and 2.5.3 in [13] adapt readily. □

*Remark* 2.3. More generally, we may introduce classes of (symbols) of pseudodifferential operators: Let $p$ and $q$ be real numbers. We say that $b \in C^\infty(I^d \times \mathbb{R}^d, \mathbb{R})$, belongs to $S^{p,q}$ if for any multi-indices[3] $\alpha = (\alpha', \alpha'')$ and $\beta = (\beta', \beta'')$ in $\mathbb{Z}_+^{d_s + (d-d_s)}$, there exists $C_{\alpha,\beta}$ so that

$$\sup \left| \partial_\xi^{\alpha'} \partial_\eta^{\alpha''} \partial_x^{\beta'} \partial_y^{\beta''} b(x,y,\xi,\eta) \right| \le C_{\alpha,\beta} (1 + |\xi| + |\eta|)^{p - |\alpha''|} (1 + |\xi|)^{q - |\alpha'|} .$$

The 1963 edition of Hörmander's book [13, II.2.5] contains a treatment of a special case of the spaces $S^{p,q}$ when $d_s = 1$. See also Sablé-Tougeron [17] for applications of these special cases.

2.3. **Banach spaces $W^{p,q,t}(\mathcal{X})$ and Leibniz formula.** Let $\kappa$, $V$ be a chart and $\psi$ a compatible partition of unity as in Subsection 2.1.

**Definition 2.4.** Let $p$, $q$ be real numbers, and let $1 \le t \le \infty$. $W^{p,q,t}(\mathcal{X}, \kappa, V, \psi)$ is $\{\varphi \in \mathcal{D}'(\mathcal{X}) \mid (\psi_i \cdot \varphi) \circ \kappa_i^{-1} \in W^{p,q,t}(U_i), \forall i \in \mathcal{I}\}$, normed by

$$\|\varphi\|_{p,q,t} = \sum_{i \in \mathcal{I}} \|(\psi_i \cdot \varphi) \circ \kappa_i^{-1}\|_{p,q,t,U_i} .$$

*Remark* 2.5. $(L^t(\mathrm{Leb}))^* = L^{1+1/(t-1)}(\mathrm{Leb})$ for $1 \le t < \infty$, and the operator $a_{p,q}^{Op}$ is self-adjoint. Therefore, for all $1 \le t < \infty$ the dual of $W^{p,q,t}(\mathcal{X}, \kappa, V, \psi)$ is $W^{-p,-q,1+1/(t-1)}(\mathcal{X}, \kappa, V, \psi)$.

The Banach spaces $W^{p,q,t}(\mathcal{X}, \kappa, V, \psi)$ are independent of the charts $(\kappa, V)$ and of the partition of unity $\psi$: A version of the change of variables theorem for pseudodifferential operators, see e.g. [2, I.7.1],

[3] We decompose multi-indices $\gamma = (\gamma', \gamma'')$ in this way tacitly from now on.

shows that the norms corresponding to different $(\kappa, V, \psi)$ are equivalent. (See Lemmas 2.7 and 2.9 below if $1 < t < \infty$; if $t = \infty$, recall that

$$f_\infty := \sup_p \|f\|_{L^p(I^d)} < \infty \Longrightarrow f \in L^\infty(I^d) \text{ and } \|f\|_{L^\infty} = f_\infty\,; \tag{2.3}$$

if $t = 1$ work with the dual space and use the above argument.) We may thus write $W^{p,q,t}(\mathcal{X})$. ($H^{p,q}(\mathcal{X}) = W^{p,q,2}(\mathcal{X})$ is a Hilbert space.)

*Remark* 2.6. $W^{p,q,t}(\mathcal{X})$ is the Banach space of distributions $f$ on $\mathcal{X}$ so that $(1+\Delta_s)^{q/2}(1+\Delta)^{p/2} f \in L^t(\mathcal{X})$, with the induced $L^t(\mathcal{X})$ norm, where $\Delta$ is the Laplacian and $\Delta_s$ is the stable foliated Laplacian. In particular, if $p \in 0$ and $0 \le q \le r$, it contains all $C^r$ functions.

**Lemma 2.7** (Leibniz formula)**.** *Let $1 < t < \infty$, let $p$, $q$ be real numbers and let $h$ be a $C^\infty$ function on $I^d$. Then there is $C(h) > 0$, and there exists $C_x(h)$, depending only on $\sup_{|\beta'|\in\{1,2\},(x,y)\in I^d} |\partial_x^{\beta'} h(x,y)|$, so that for every $f \in W^{p,q,t}(I^d)$*

$$a^{Op}_{p,q}(h \cdot f) = h \cdot a^{Op}_{p,q}(f) + g_1 + g_2\,, \tag{2.4}$$

*with $\|g_1\|_{L^t} \le C_x(h)\|f\|_{p,q-1,t}$ and $\|g_2\|_{L^t} \le C(h)\|f\|_{p-1,q,t}$.*

*Proof.* Multiplication by $h$ is a pseudodifferential operator. Composing it with $a^{Op}_{p,q}$, we get a new operator $b^{Op}$. Using a Taylor series of order one (see e.g. [2, Théorème I.4.1 and §I.8.2]), we find $b(x,y,\xi,\eta) =$

$$a_{p,q}(\xi,\eta)\cdot h(x,y) + \frac{2}{(2\pi)^d} \sum_{|\alpha|+|\beta|=2} \frac{(-1)^{|\alpha|+|\beta|}}{\alpha!\beta!} \int_0^1 (1-s)\cdot$$
$$\cdot \int e^{-i(u,v)(\omega,\theta)} \omega^{\beta'}\theta^{\beta''} u^{\alpha'} v^{\alpha''} \cdot \partial_\omega^{\beta'}\partial_\theta^{\beta''} a_{p,q}(\xi - s\omega, \eta - s\theta)\, d\omega\, d\theta$$
$$\cdot \partial_u^{\alpha'}\partial_v^{\alpha''} h(x-su, y-sv)\, du\, dv\, ds\,.$$

The symbol $a_{p,q}(\xi,\eta)\cdot h(x,y)$ gives rise to the first term in the right-hand-side of (2.4). For the remainder term, the usual integrations by parts [2, §I.8.2, p.56] yield a linear combination of terms

$$b^{\gamma,j}(x,y,\xi,\eta) := \int_0^1 (1-s)s^j \cdot \int e^{-i(u,v)(\omega,\theta)} \cdot \partial_\omega^{\gamma'}\partial_\theta^{\gamma''} a_{p,q}(\xi - s\omega, \eta - s\theta)$$
$$\cdot \partial_u^{\gamma'}\partial_v^{\gamma''} h(x-su, y-sv)\, d\omega d\theta\, dudv\, ds\,,$$

where $j \in \{0,1,2\}$ and $|\gamma'|+|\gamma''| \in \{1,2\}$ (the number of terms and the coefficients in the linear combination are independent of $h$ and $a_{p,q}$). If $|\gamma''| = 0$ then $|\gamma'| \in \{1,2\}$, and this gives $g_1$, as we explain next. Define a symbol $\tilde b = b^{\gamma,j}(a_{p,q-1})^{-1}$. By [8, Théorème 9] it suffices to show that there is $C_x(h)$ so that $\sup |\partial^\alpha_{x,y}\partial^\beta_{\xi,\eta}\tilde b(x,y,\xi,\eta)| \le C_x(h)$, for all $|\alpha| \le 1$ and all $\beta$. This can be shown by a straightforward (although tedious) implementation of the standard oscillatory integral argument [2, §I.8.2, p.56]. Finally, $|\gamma''| \ge 1$ then the term corresponding to $b^{\gamma,j}$ may be included in $g_2$, working with $b^{\gamma,j}(a_{p-1,q})^{-1}$. □

2.4. **Bounding the essential spectral radius of $\mathcal{L}$.**

**Theorem 2.8.** *Let $T$ be a $C^\infty$ Anosov diffeomorphism on a compact manifold, with a $C^\infty$ stable foliation. For any $p<0$, $s>0$:*
*(0) The essential spectral radius of $\mathcal{L}$ on $W^{p,s-p,\infty}(\mathcal{X})$ is $\le \rho_\infty^{(p,s)}(T)$.*
*(1) The essential spectral radius of $\mathcal{L}$ on $W^{p,s-p,t}(\mathcal{X})$ is not larger than $\lim_{n\to\infty}\sup_{\mathcal{X}} |\det DT^n|^{-1/tn} \cdot \rho_\infty^{(p,s)}(T)$, for each $t \in (1,\infty)$.*

Here are two immediate consequences of the above theorem: The essential spectral radius of the dual of $\mathcal{L}$ (i.e., $\mathcal{M}$) extended to the dual of $W^{p,s-p,\infty}(\mathcal{X})$ is $\le \rho_\infty^{(p,s)}(T)$. Also, if the unstable foliation is $C^\infty$, then $\varphi \mapsto \varphi\circ T^{-1}$ on $W^{p,s-p,\infty}(\mathcal{X}, T^{-1})$ has essential spectral radius $\le \rho_\infty^{(p,s)}(T^{-1}) = \rho_\infty^{(-s,-p)}(T)$.

It is convenient to extend each $T^n_{ij} = \kappa_j \circ T^n \circ \kappa_i^{-1} : U_{i,j,n} \to U_j$ to a $C^\infty$ diffeomorphism from $U_i$ onto its image in $\mathbb{R}^d_j$, still denoted by $T^n_{ij}$, in such a way that the extended map preserves the horizontal foliation and that its derivative in block form satisfies (2.1,2.2) for the same constants $\lambda_{i,j}$, $\nu_{i,j}$, $\epsilon$. The theorem will be a consequence of the following lemma, proved in §2.6:

**Lemma 2.9** (Lasota-Yorke inequality)**.** *There exist $\delta_0>0$ and $C_0$, so that for each cover $V$ with $\operatorname{diam} V < \delta_0$, and for each $n \ge 1$ there exists $C(n)>1$, so that for every $f \in W^{p,q,t}_C(U_j)$ and each $i$*

$$\begin{aligned}
&\|f\circ T^n_{ij}\|_{p,q,t,U_i} \cdot \inf_{V_{i,j}} |\det DT^n|^{1/t} \\
&\qquad \le C_0 \cdot \max((\lambda_{i,j}(T^n))^p, (\nu_{i,j}(T^n))^{q+p}) \|\chi_{U_j \cap T^n_{ij}(U_i)} \cdot a^{Op}_{p,q}(f)\|_{L^t(U_j)} \\
&\qquad\qquad + C(n)\|f\|_{p-1/2,q,t,U_j}\,, \ \forall p \le 0\,,\ q \ge -p\,,\ 1<t<\infty\,.
\end{aligned}$$

*Proof of Theorem 2.8 using Lemma 2.9.* We show (1). The proof, combined with (2.3), gives (0).

Let $\delta_0$ be as in Lemma 2.9. For $\delta<\delta_0$ let $(\kappa, V)$ be a foliated chart of diameter at most $\delta$ and let $\psi$ be an adapted partition of unity. Let $K_V$ be the maximum number of sets $V_i$ which contain a point $x \in \mathcal{X}$. Set $f_j|_{U_j} = (\psi_j \cdot \varphi)\circ \kappa_j^{-1}$. By definition, for all $n \ge 1$,

$$\|\mathcal{L}^n \varphi\|_{p,q,t} \le \sum_i \sum_{j: T^n(V_i)\cap V_j \ne \emptyset} \|(\psi_i \circ \kappa_i^{-1}) \cdot (f_j \circ T^n_{ij})\|_{p,q,t,U_i}\,.$$

By Lemma 2.7, $\|(\psi_i \circ \kappa_i^{-1}) \cdot (f_j \circ T^n_{ij})\|_{p,q,t,U_i}$ is bounded by

$$C(\psi) \cdot \|(f_j \circ T^n_{ij})\|_{p,q,t,U_i} + C(\psi) \cdot \|(f_j \circ T^n_{ij})\|_{p-1,q,t,U_i} .$$

Set $\rho(p,s,n) = \max_{i,j} \max((\lambda_{i,j}(T^n))^p, (\nu_{i,j}(T^n))^s))$. By Lemma 2.9

$$\begin{aligned}\|\mathcal{L}^n \varphi\|_{p,s-p,t} \cdot \inf |\det DT^n|^{1/t} &\le C_0 C(\psi) \rho(p,s,n) \\ &\quad \cdot \sum_{i,j} \|\chi_{U_j \cap T^n_{ij} U_i} \cdot a^{Op}_{p,s-p}(\psi_j \cdot \varphi) \circ \kappa_j^{-1}\|_{L^t(U_j)} \\ &\quad + C(n,\psi) \sum_{i,j} \|(\psi_j \cdot \varphi) \circ \kappa_j^{-1}\|_{p-1/2,s-p,t,U_j} \\ &\le C_0 K_V C(\psi) \rho(p,s,n) \cdot \sum_j \|(\psi_j \cdot \varphi) \circ \kappa_j^{-1}\|_{p,s-p,t,U_j} \\ &\quad + \#\mathcal{I} \cdot C(n,\psi) \sum_j \|(\psi_j \cdot \varphi) \circ \kappa_j^{-1}\|_{p-1/2,s-p,t,U_j} \\ &\le C_1 \rho(p,s,n) \cdot \|\varphi\|_{p,s-p,t} + C_2(n) \|\varphi\|_{p-1/2,s-p,t} .\end{aligned}$$

By Lemma 2.2, we can apply Hennion's theorem [12]. [4] □

### 2.5. **Bounds involving averaged hyperbolicity exponents.**

**Proposition 2.10.** *Let $T$ be a $C^\infty$ Anosov diffeomorphism on a compact manifold, with a $C^\infty$ stable foliation. For any $p < 0$, $s > 0$, and $1 < t \le \infty$, the essential spectral radius of $\mathcal{L}$ on $W^{p,s-p,t}(\mathcal{X})$ is*

$$\begin{aligned}&\le \lim_{n\to\infty} \Big( \int_{\mathcal{X}} \max\big((\lambda_w(T^n))^p, (\nu_w(T^n))^s\big) \\ &\qquad\qquad \cdot |\det DT^n|_{E^u}| \cdot |\det DT^n|^{-1/t} \, dLeb(w) \Big)^{1/n} \\ &= \lim_{n\to\infty} \Big( \int_{\mathcal{X}} \max\big((\lambda_w(T^n))^p, (\nu_w(T^n))^s\big) \\ &\qquad\qquad \cdot |\det DT^n|_{E^s}|^{-1} \cdot |\det DT^n|^{1-1/t} \, dLeb(w) \Big)^{1/n} .\end{aligned}$$

*Proof.* The reader is invited to check that there is $C_3 > 1$ (depending on $T$) so that for all $n \ge 1$, each cover $V$, all $i$, $j$, all $p \le 0$

$$\max_{w \in \overline{V}_{i,j,n}} (\lambda_w(T^n))^p - \min_{w \in \overline{V}_{i,j,n}} (\lambda_w(T^n))^p \le n C_3 (\lambda_{i,j}(T^n))^p \mathrm{diam}\, V ,$$

[4] We sketch an alternative proof: For fixed $V$, $\psi$ and all $n$, set $\mathcal{Q}_n \varphi = \sum_{i,j} \psi_i \max(\lambda_{i,j}(T^n)^p, \nu_{i,j}(T^n)^{q+p})(\varphi\psi_j) \circ T^n$. There is $C_0$ and for each $n \ge 1$ there is a compact operator $\mathcal{M}'_n$ on $L^t(\mathrm{Leb})$ so that $\mathcal{M}''_n := a^{Op}_{p,q} \circ \mathcal{M}^n \circ a^{Op}_{-p,-q} - \mathcal{M}'_n$ (viewing $a^{Op}$ as an operator on $\mathcal{X}$) satisfies $\|\mathcal{M}''_n\|_{L^t} \le C_0 \|\mathcal{Q}_n\|_{L^t}$.

and similarly for the $\nu_w$ (this is a bounded distortion argument). If $\ell_{i,j}(n) := \max((\lambda_{i,j}(T^n))^p, (\nu_{ij}(T^n))^s) = (\lambda_{i,j}(T^n))^p$ (the other case is similar) then

$$\max_{w\in\overline{V}_{i,j}} (\ell_{i,j}(n) - \max((\lambda_w(T^n))^p, (\nu_w(T^n))^s)) \le \ell_{i,j}(n) - \min_{w\in\overline{V}_{i,j}} (\lambda_w(T^n))^p$$
$$\le \max_{w\in\overline{V}_{i,j}} (\lambda_w(T^n))^p - \min_{w\in\overline{V}_{i,j}} (\lambda_w(T^n))^p\,.$$

Choose a partition $\mathcal{X} = \cup_{i\in\mathcal{I}} W_i$ with $W_i \subset V_i$, and write $W_{i,j} = W_i \cap T^{-n} W_j$. Then

$$\sum_{i,j} \mathrm{Leb}(W_{i,j})\ell_{i,j}(n) - \int_{\mathcal{X}} \max((\lambda_w(T^n))^p, (\nu_w(T^n))^s)\, dLeb$$
$$\le \sum_{i,j} \mathrm{Leb}(W_{i,j})\Big(\ell_{i,j}(n) - \min_{W_{i,j}} \max((\lambda_w(T^n))^p, (\nu_w(T^n))^s)\Big)$$
$$\le C_3 n \cdot \mathrm{diam}\, V \sum_{i,j} \mathrm{Leb}(W_{i,j})\ell_{i,j}(n)\,.$$

Therefore, fixing $\delta \in (0,1)$, if $V(n)$ satisfies $\mathrm{diam}\, V(n) = \delta/(C_3 n)$, then, recalling the notation $K_V$ from the proof of Theorem 2.8,

$$\sum_{i,j} \mathrm{Leb}(V_{i,j,n})\ell_{i,j}(n) \le \frac{K_{V(n)}}{(1-\delta)} \int_{\mathcal{X}} \max((\lambda_w(T^n))^p, (\nu_w(T^n))^s)\, dLeb\,.$$

We may choose $V(n)$ and $\psi(n)$ with $K_V \le \#V = O(n)$ and also $(\min_i \mathrm{Leb}(V_i))^{-1} = O(n^d)$, ensuring that the derivatives of the $\psi_i$ from Lemma 2.7 satisfy $O(n^Q)$ bounds; for some $Q \ge 1$. Finally, there is $C_4 \ge 1$ so that

$$\frac{1}{\mathrm{Leb}(V_{i,j,n})} \le \frac{C_4}{\min_i \mathrm{Leb}(V_i))} \inf_{V_{i,j,n}} |\det DT^n|_{E^u}|\,.$$

for all $n$ and all covers $V$. Lemma 2.9 allows to conclude, by a straightforward adaptation of the proof of Theorem 2.8. □

## 2.6. **Proof of the Lasota-Yorke inequality.**

*Proof of Lemma 2.9.* We replace $T$ by $T^n$ (the reader should keep in mind that $A_{ij}$, $B_{ij}$, $D_{ij}$, $\lambda_{i,j}$, and $\nu_{i,j}$ depend on $n$) and drop the indices $i$, $j$. We study the action of the composition by $T$ on our symbol $a_{p,q}(\xi,\eta)$ (see e.g. [2, Chapter I.7, Proposition 7.1, and Chapter I.8, Théorème 3]). Taking a Taylor series of order zero (i.e., $k = 1$ in the proof of [2, I.8, Lemme 4]), we find that $(a_{p,q}(\xi,\eta)^{Op}(f\circ T))\circ T^{-1}$ decomposes as

(2.5)
$$(a_{p,q}((DT)^{tr}_{(x,y)}(\xi,\eta))^{Op}(f) + r_1(x,y,\xi,\eta)^{Op}(f) + r_2(x,y,\xi,\eta)^{Op}(f)\,,$$

where $r_1$ and $r_2$ are described next. The symbol $r_1(x, y, \xi, \eta)$ is a universal finite linear combination of

(2.6)

$$\int_{I^d \cup -I^d} dudv \int_{\mathbb{R}^d} d\omega d\theta \, e^{-i(u,v)(\omega,\theta)} \int_0^1 ds \, (1-s)s^j$$
$$\cdot \, \partial_{u_\ell}\Big(e^{i(R_{(x,y)}(x+su,y+sv))^{tr}(\xi,\eta)}\Big)$$
$$\cdot \, (1+|s\omega + A_{(x,y)}\xi|^2 + |s\theta + B_{(x,y)}\xi + D_{(x,y)}\eta|^2)^{p/2}$$
$$\cdot \, \partial_{\omega_\ell}\big(1+|s\omega + A_{(x,y)}\xi|^2)^{q/2}\chi\bigg(\frac{(s\omega + A_{(x,y)}\xi, s\theta + B_{(x,y)}\xi + D_{(x,y)}\eta)}{1+|(A_{(0,0)}\xi, B_{(0,0)}\xi + D_{(0,0)}\eta)|}\bigg) \, ,$$

where $j \in \{0, 1\}$, $1 \le \ell \le d_s$, the function $\chi : \mathbb{R}^d \to [0, 1]$ is $C^\infty$ and compactly supported in a suitable annulus, and

$$R_{(x,y)}(u, v) = T(u, v) - T(x, y) - DT_{(x,y)}(u - x, v - y) \, .$$

To describe $r_2$, set $\tilde r_2 = r_2 \cdot (a_{p-1/2,q})^{-1}$, so that $r_2^{Op} = \tilde r_2^{Op} a^{Op}_{p-1/2,q}$. (In the proof we shall use the notation $\tilde r = r \cdot (a_{p-1/2,q})^{-1}$ several times.) We claim that $\tilde r_2^{Op}$ is a bounded operator on each $L^t(I^d, \mathrm{Leb})$, so that

$$\begin{aligned}\|(r_2^{Op} f)\circ T\|_{L^t(U_i)} &\le C \sup_{U_i} |\det DT|^{-1/t} \cdot \bigg(\int_{T(U_i)\cap U_j} |r_2^{Op}(f)|^t \, dLeb\bigg)^{1/t} \\ &\le C_2(n) \|\chi_{U_j \cap T(U_i)} f\|_{p-1/2,q,t,U_j} \, .\end{aligned}$$

By [8, Théorème 9] it suffices to show that $\sup |\partial^\alpha_{x,y}\partial^\beta_{\xi,\eta} \tilde r_2(x, y, \xi, \eta)| < \infty$, for all $|\alpha| \le 1$ and all $\beta$. This can be seen by observing that $r_2$ is made on the one hand with contributions due to $1 - \chi$, which have rapid decay in $1+|(\omega, \theta)|+|(A_{(0,0)}\xi, B_{(0,0)}\xi + D_{(0,0)}\eta)|$ (by a small modification of [2, p.58], using bounded distortion for $DT^n$). The other terms forming $r_2$ correspond to a $\partial_{\theta_\ell}$ derivative, or to a $\partial_{\omega_\ell}$, but acting on a factor $(1 + |s\omega + A_{(x,y)}\xi|^2 + |s\theta + B_{(x,y)}\xi + D_{(x,y)}\eta|^2)^{p/2}$. (Details are left to the reader, see [2, p.60].)

We may thus concentrate on the first two terms in (2.5). The first one is called the *principal symbol.*

We get $a_{p,q}((DT_{(x,y)})^{tr}(\xi, \eta))^{Op} = b^{Op} \circ a^{Op}_{p,q}$ by setting $b(x, y, \xi, \eta) = a_{p,q}((DT_{(x,y)})^{tr}(\xi, \eta))/a_{p,q}(\xi, \eta)$. Again by [8, Théorème 9] it suffices to show that, up to replacing $b$ by $b - r_3$, with $\tilde r_3^{Op}$ bounded on each $L^t$, we have $\sup |\partial^\alpha_{x,y}\partial^\beta_{\xi,\eta} b| \le (C_\delta/2)\max(\lambda^p, \nu^{q+p})$, for all $|\alpha| \le 1$ and all $\beta$. Of course, we must also prove the same bounds for $r_1 \cdot (a_{p,q})^{-1}$ (modulo $r_4 + r_5$, with $\tilde r_4^{Op}$ and $\tilde r_5^{Op}$ bounded on each $L^t(I^d)$).

Consider first $\alpha = \beta = 0$ and the principal symbol, i.e., the bound for $\sup |b|$. For $\xi \ne 0$ write $\nu_\xi = \sup_{x,y} |A_{(x,y)}\xi|/|\xi|$. Then, setting

$\Lambda_1 = \max_\xi((1-2\nu_\xi)/\nu_\xi)$, we get for any $|\xi| \ge \Lambda_1$

$$(1+|A_{(x,y)}\xi|^2)^{q/2} \le (1+\nu_\xi^2|\xi|^2)^{q/2} \le 2\nu^q_{\xi/|\xi|}(1+|\xi|^2)^{q/2}\,. \tag{2.7}$$

For $|\xi| < \Lambda_1$ we always have $(1+|A_{(x,y)}\xi|^2)^{q/2} \le (1+|\xi|^2)^{q/2}$.

If $|\xi| \ge \max(\Lambda_1, |\eta|)$ then [5]

$$(1+|A_{(x,y)}\xi|^2+|B_{(x,y)}\xi+D_{(x,y)}\eta|^2)^{p/2} \le 2\nu^p_{\xi/|\xi|}(1+|\xi|^2+|\eta|^2)^{p/2}\,.$$

For $\eta \ne 0$ write $\lambda_\eta = \inf_{x,y}|D_{(x,y)}\eta|/|\eta|$. Fix $\Lambda_2 = \max_\eta \lambda_\eta$. If $\epsilon < 1/4$ and $|\eta| \ge \max(\Lambda_2, |\xi|)$ then [6]

$$(1+|A_{(x,y)}\xi|^2+|B_{(x,y)}\xi+D_{(x,y)}\eta|^2)^{p/2} \le 3\lambda^p(1+|\xi|^2+|\eta|^2)^{p/2}\,.$$

Finally, if $|\xi| \le \Lambda_1$, and $|\eta| \le \Lambda_2$, there is $C_{\Lambda_1,\Lambda_2}(n)$ with

$$(1+|A_{(x,y)}\xi|^2+|B_{(x,y)}\xi+D_{(x,y)}\eta|^2)^{p/2} \le C_{\Lambda_1,\Lambda_2}(1+|\xi|^2+|\eta|^2)^{p/2-1}\,. \tag{2.8}$$

We include the above contribution in $r_3$.

To bound $\sup_{x,y,\xi,\eta}|r_1\cdot(a_{p,q})^{-1}|$, multiply the integrand of (2.6) by

$$\left(1-\tilde\chi\Big(\frac{s\omega+A_{(x,y)}\xi}{1+|A_{(0,0)}\xi|}\Big)\right)+\tilde\chi\Big(\frac{s\omega+A_{(x,y)}\xi}{1+|A_{(0,0)}\xi|}\Big)\,,$$

where $\tilde\chi:\mathbb{R}^{d_s}\to[0,1]$ is $C^\infty$ and compactly supported in an annulus. We consider separately the two terms in this decomposition:

The term containing $\chi\cdot(1-\tilde\chi)$ enjoys $C_k(n)(1+|A_{(0,0)}\xi|+|\omega|)^{-k}$ rapid decay (adapting [2, p.58]). By choosing first $k$ and then $\Lambda_3$ we get a bound $(C_\delta/4)\max(\lambda^p,\nu^{q+p})$ for $|\xi|\ge\Lambda_3$. If $|\xi|\le\Lambda_3$, we use that if $|\eta| > \max(|\xi|,\Lambda_4)$ then

$$\begin{aligned}\sup_{s,(\omega,\theta),(u,v),(x,y)} &(1+|s\omega+A_{(x,y)}\xi|^2+|s\theta+B_{(x,y)}\xi+D_{(x,y)}\eta|^2)^{p/2}\\ &\cdot\chi\left(\frac{(s\omega+A_{(x,y)}\xi,\, s\theta+B_{(x,y)}\xi+D_{(x,y)}\eta)}{1+|(A_{(0,0)}\xi, B_{(0,0)}\xi+D_{(0,0)}\eta)|}\right)\\ &\le 2\lambda^p(1+|\xi|^2+|\eta|^2)^{p/2}\,.\end{aligned} \tag{2.9}$$

The compact set $\{|\xi|\le\Lambda_3, |\eta|\le\Lambda_4\}$ gives rise to a term $r_4$.

For the $\chi\cdot\tilde\chi$ term, use the ideas exploited for the principal symbol (see also – again – [2, p.60]) to get a bound $(C_\delta/4)\max(\lambda^p,\nu^{q+p})$, up to

[5] Here we pay the price of $p<0$.

[6] If $|\xi|$ is small but $|\eta|$ is large we need $p<0$ to get a contraction here.

a perturbation $r_5$. In particular, if $|\xi|$ is large with respect to $|\eta|$ then

$$\begin{aligned}(2.10)\qquad \sup_{s,(\omega,\theta),(x,y)} &(1+|s\omega+A_{(x,y)}\xi|^2+|s\theta+B_{(x,y)}\xi+D_{(x,y)}\eta|^2)^{p/2}\\ &\cdot\partial_{\omega_\ell}\big((1+|s\omega+A_{(x,y)}\xi|^2)^{q/2}\chi(\cdots)\cdot\tilde\chi(\cdots)\big)\end{aligned}$$

is bounded by $C(1+|\xi|+|\eta|)^{p-1}(1+|\xi|)^q$ (giving a contribution $r_5$); while if $|\eta|$ is large then (2.10) is bounded by $2\lambda^p(1+|\xi|^2+|\eta|^2)^{p/2}(1+|\xi|^2)^{q/2}$.

The control of the derivatives of $b$ and $r_1\cdot(a_{p,q})^{-1}$, i.e., the case of nonzero $|\alpha|+|\beta|$, is straightforward although rather tedious. □

2.7. **The essential spectral radius of $\mathcal{M}$.** Let $T$ be a $C^\infty$ Anosov diffeomorphism on a compact manifold, with a $C^\infty$ unstable foliation. Let $W^{p,q,t}(\mathcal{X},T^{-1})$ denote the Banach space in Definition 2.4, but using now the stable foliation of $T^{-1}$, i.e. the unstable foliation of $T$. (In other words, $W^{p,q,t}(\mathcal{X},T^{-1})=(1+\Delta_u)^{-q/2}(1+\Delta)^{-p/2}(L^t(\mathcal{X}))$.)

**Theorem 2.11.** *For any $p<0$ and $s>0$:*
*(0) The essential spectral radius of $\mathcal{M}$ on $W^{p,s-p,1}(\mathcal{X},T^{-1})$ is not larger than $\rho_\infty^{(-s,-p)}(T)$.*
*(1) The essential spectral radius of $\mathcal{M}$ on $W^{p,s-p,t}(\mathcal{X},T^{-1})$ is not larger than $\lim_{n\to\infty}\sup_{\mathcal{X}}|\det DT^n|^{-(t-1)/tn}\cdot\rho_\infty^{(-s,-p)}(T)$ for $t\in(1,\infty)$.*

**Proposition 2.12.** *For any $p<0$, $s>0$, and $1\le t<\infty$, the essential spectral radius of $\mathcal{M}$ on $W^{p,s-p,t}(\mathcal{X},T^{-1})$ is*

$$\begin{aligned}(2.11)\qquad &\le \lim_{n\to\infty}\bigg(\int_{\mathcal{X}}\max\big((\lambda_w(T^n))^{-s},(\nu_w(T^n))^{-p}\big)\\ &\qquad\cdot|\det DT^n|_{E^u}|\cdot|\det DT^n|^{-(t-1)/t}\,dLeb(w)\bigg)^{1/n}.\end{aligned}$$

*Proof of the theorem and the proposition.* Adapt the proofs of Theorem 2.8 and Proposition 2.10, for $\mathcal{M}^*=\mathcal{L}$ on $W^{-p,p-s,1+1/(t-1)}(\mathcal{X},T^{-1})$. □

## Appendix A. Operators $\mathcal{M}_t$ and $\mathcal{L}_t$

**Theorem A.1.**
*(1) If the stable foliation of $T$ is $C^\infty$ then for any $p<0$ and $s>0$ with $q=s-p$ integer, there exists $t_1(q)>1$ so that the essential spectral radius of $\mathcal{L}_t$ on $W^{p,s-p,t}(\mathcal{X})$ is $\le\rho_\infty^{(p,s)}(T)$, for each $1<t<t_1$.*
*(2) If the unstable foliation of $T$ is $C^\infty$ then for any $p<0$ and $s>0$ with $q=s-p$ integer there exists $t_2(q)\ge 1$ so that the essential*

*spectral radius of $\mathcal{M}_t$ on $W^{p,s-p,t}(\mathcal{X},T^{-1})$ is $\le \rho_\infty^{(-s,-p)}(T)$, for each $t_2 < t < \infty$.*

For any integer $q \ge 1$, there exist $C \ge 1$ and $t_1(q) > 1$ so that for every $\gamma'$ with $1 \le |\gamma'| \le q$, all $1 \le t < t_1$ and all $n \ge 1$, setting $h(x,y) = |\det DT^n|^{1/t} \circ \kappa_i^{-1}(x,y)$, then $|\partial_x^{\gamma'} h(x,y)| \le C h(x,y)$. (If $q = 1$ we may take $t_1 = \infty$.) Theorem A.1 is therefore a consequence of the following corollary of the proof of Lemma 2.9 combined with a refinement of the Leibniz formula for $a_{p,q}$ if $q \in \mathbb{Z}_+$, Lemma A.3.

**Corollary A.2** (More on Lasota-Yorke)**.** *There exist $\delta_0 > 0$ and $C_0$ so that, for all $V$ with $\operatorname{diam} V < \delta_0$ and $n \ge 1$, there exists $C(n) > 1$ so that for any multi-index $\gamma'$ with $|\gamma'| \le q$ and any $f \in W_C^{p,q,t}(U_j)$*

$$\begin{aligned}\|| \det DT^n \circ \kappa_i^{-1}|^{1/t} \cdot a_{p,0}\, \partial_x^{\gamma'}(f \circ T_{ij}^n)\|_{L^t(U_i)} \\ \le C_0 \max((\lambda_{i,j}(T^n))^p, (\nu_{i,j}(T^n))^{q+p}) \|\chi_{U_j \cap T_{ij}^n(U_i)} \cdot a_{p,q}^{Op}(f)\|_{L^t(U_j)} \\ + C(n)\|f\|_{p-1/2,q,t,U_j}\,,\ \forall p \le 0\,,\ q \ge -p\,,\ 1 < t < \infty\,.\end{aligned}$$

**Lemma A.3** (Leibniz formula for integer derivatives)**.** *Let $1 < t < \infty$. Let $q \in \mathbb{Z}_+$ and let $p \in \mathbb{R}$. There exists $C \ge 1$, and for every $h \in C^\infty(I^d)$ there exists $C(h) > 0$ so that for each $f \in W^{p,q,t}(I^d)$ we have $a_{p,q}^{Op}(h \cdot f) = h \cdot a_{p,q}^{Op}(f) + g_1 + g_2$ with*

$$\|g_1\|_{L^t} \le C \sum_{\substack{|\gamma'|=q-1 \\ \gamma_1'+\gamma_2'=\gamma'}} \|\partial_x^{\gamma_1'} h \cdot a_{p,0}^{Op}(\partial_x^{\gamma_2'} f)\|_{L^t}\,,\ \|g_2\|_{L^t} \le C(h)\|f\|_{p-1,q,t}\,.$$

*Proof.* Decompose $a_{p,q}^{Op} = a_{p,0}^{Op} \circ a_{0,q}^{Op}$. The proof of Lemma 2.7 gives that if $\tilde h \in C^\infty(I^d)$, then there is $C(\tilde h) \ge$ so that for all $\tilde f \in W^{p,0,t}(I^d)$ we have $a_{p,0}^{Op}(\tilde h \cdot \tilde f) = \tilde h \cdot a_{p,0}^{Op}(\tilde f) + \tilde g$ with $\|\tilde g\|_{L^t} \le C(\tilde h)\|\tilde f\|_{p-1,0,t}$.

Since $q$ is an integer, $a_{0,q}^{Op}(h \cdot f)$ decomposes as:

$$a_{0,q}^{Op}(h \cdot f) = \begin{cases} h \cdot f + \sum_{j=1}^{\ell} \binom{\ell}{j} \left(\partial_{x_1}^2 + \cdots + \partial_{x_{d_s}}^2\right)^j (h \cdot f) \\ \qquad\qquad\qquad\qquad \text{if } q = 2\ell \text{ is even,} \\ \mu_1 * a_{0,q-1}^{Op}(h \cdot f) + \mu_2 * \left(\sum_{j=1}^{d_s} R_{x_j}(\partial_{x_j} a_{0,q-1}^{Op}(h \cdot f)\right) \\ \qquad\qquad\qquad\qquad \text{if } q = 2\ell+1 \text{ is odd,} \end{cases}$$

where $\mu_1$ and $\mu_2$ are finite measures on $I^d$ (which do not depend on $h$ or $f$) and $*$ denotes convolution. Indeed, $q = 2\ell$ is even, just recall that $a_{0,q}^{Op} = (1+\Delta_s)^{q/2} = (1 + \sum_{j=1}^{d_s} \partial_{x_j}^2)^\ell$. If $q = 2\ell + 1$, recall [18, V.3.2–V.3.4] that $(1+\Delta_s)^{1/2}(\varphi) = \mu_1 * \varphi + \mu_2 * \left(\sum_{j=1}^{d_s} R_{x_j}(\partial_{x_j}\varphi)\right)$, where $R_{x_j}$ is the Riesz transform [18, III.1]. Finally, use the ordinary Leibniz formula for partial derivatives, that $a_{p,0}^{Op}$ commutes with each

$R_{x_j}$, that $R_{x_j}$ is bounded on $L^t$, and that if $\mu$ is a measure, with total mass $|\mu|$, then $a_{p,0}(\mu * f) = \mu * a_{p,0}(f)$ and $\|\mu * f\|_{L^t} \le |\mu| \cdot \|f\|_{L^t}$. $\square$

## References


1. S. Alinhac, *Intéraction d'ondes simples pour des équations complètement non-linéaires,* Ann. Scient. Éc. Norm Sup. 21 (1988) 91–132.
2. S. Alinhac and P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser,* Interéditions, CNRS, 1991.
3. A. Avila, S. Gouëzel, and M. Tsujii, *Smoothness of fat solenoidal attractors,* in preparation (2004).
4. V. Baladi, *Anisotropic Sobolev spaces and dynamical transfer operators: $C^{1+\alpha}$ foliations,* in preparation.
5. V. Baladi and M. Baillif, *Kneading determinants and spectra of transfer operators in higher dimensions, the isotropic case,* Preprint (2003).
6. M. Blank, G. Keller, and C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps,* Nonlinearity 15 (2002) 1905–1973.
7. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Springer Lecture Notes in Mathematics Vol 470 (1975).
8. R.R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudo-différentiels,* Astérisque 57 (1978).
9. P. Collet and J.-P. Eckmann, *Liapunov multipliers and decay of correlations in dynamical systems,* J. Stat. Phys. 115 (2004) 217–254.
10. S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems,* preprint (2004).
11. M. Gundlach and Y. Latushkin, *A sharp formula for the essential spectral radius of the Ruelle transfer operator on smooth and Hölder spaces,* Ergodic Theory Dynam. Systems 23 (2003) 175-191.
12. H. Hennion, *Sur un théorème spectral et son application aux noyaux lipschitziens,* Proc. Amer. Math. Soc. 118 (1993) 627–634.
13. L. Hörmander, *Linear Partial Differential Operators,* Springer (1963).
14. A.Yu. Kitaev, *Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness,* Nonlinearity 12 (1999) 141–179. *Corrigendum:* Nonlinearity 12 (1999) 1717–1719.
15. Y.A. Kordyukov, *Functional calculus for tangentially elliptic operators on foliated manifolds,* in Analysis and Geometry in Foliated Manifolds: international conference on differ. geom., Santiago de Compostela, 1994, World Scientific.
16. D. Ruelle, *An extension of the theory of Fredholm determinants,* Inst. Hautes Etudes Sci. Publ. Math. 72 (1990) 175–193.
17. M. Sablé-Tougeron, *Régularité microlocale pour des problèmes aux limites non linéaires,* Ann. Inst. Fourier 36 (1986) 39–82.
18. E.M. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton University Press, Princeton, NJ, 1970.


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