Symmetry and Reduction for Coordinated Rigid Body Networks

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Abstract

Motivated by interest in the collective behavior of autonomous agents, we study networks of rigid bodies and the problem of coordinated orientation and position across the group. Our main result is the reduction of the networked system in the case that individuals are coupled by control inputs that depend only on relative configuration. We use reduction theory based on semi-direct products; this yields flat Poisson spaces which enable efficient formulation of control laws. In the second part of the paper, we apply the reduction results to particular choices of kinetic energy and prove stability of coordinated behaviors.

1 Introduction

The study of control laws for groups of autonomous vehicles has emerged as a challenging new research topic in recent years. There are currently few examples of, and yet many possible applications for, groups of highly autonomous agents that exhibit complex collective behavior in the engineered world. Researchers in this area are finding much inspiration from biological examples. Animal aggregations, such as schools of fish, are believed to use simple, local traffic rules at the individual level but exhibit remarkable capabilities at the group level. These include rapid maneuverability, the ability to quickly process data and a markedly intelligent decision-making ability [16].

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Numerous researchers have attempted to model animal aggregation with each agent considered as a point mass (see, for example, [2, 15, 20]). Likewise in engineering applications, group control problems are often formulated with point-mass vehicle models (e.g. [4, 5, 8, 21]). Less often, the individuals are modelled as rigid bodies in three-dimensional physical space. Here, we consider a group in which individuals are modelled as rigid bodies from the outset. We note that a close biological analogue to this system, that of a school of fish, is believed to maintain group cohesion through each individual's desire to match the speed and direction of nearby fish, and to be surrounded by a certain amount of open space (see [17]).

There is a natural connection between coordinated rigid body networks and multi-body dynamics problems; the former can be made to resemble the latter by imposing control inputs that artificially couple the individuals so that the network functions as one large multi-body system. We aim to realize this connection and develop a framework for analysis and design of coordinated rigid body networks in which we can make best use of tools and results from mechanics.

In this paper we focus on symmetry and reduction in coordinated rigid body networks for which control inputs provide coupling that depends only on the *relative* configuration of the individuals in the network (relative orientations and possibly relative positions). These control inputs can be interpreted as breaking certain symmetries, namely those associated with the relative configurations. We neglect interaction with the environment and disregard the mission one wants the group as a whole to achieve, and so the absolute position and orientation of the group remains a symmetry. In the reduction, we abstract from the explicit position and orientation of each individual rigid body in space and only retain information on the relative positions and orientations within the group.

There are a number of advantages for control design and analysis that come from the reduction derived here. These include the reduced dimensionality, and thus complexity, of the system, the simplification in the equations of motion for a given Hamiltonian and, importantly, the derivation of a Hamiltonian structure for the reduced system complete with Casimir functions which can be used readily in the study of stability and stabilization of the coordinated network. Later, the interaction with the environment and prescription for a group objective can be re-interpreted as breaking the symmetries we use in this paper to reduce the phase space.

In our general problem, we consider N rigid bodies, each with configuration space $SE(3)$ (or some subgroup of $SE(3)$) so that the configuration space for the network is $SE(3)^N = SE(3) \times \cdots \times SE(3)$. $SE(3)$ is the Euclidean group (or group of rigid motions) which globally describes rigid body positions and orientations. The phase space for the network can be described by the cotangent bundle $T^*SE(3)^N$, and the dynamics are Hamiltonian dynamics given a Hamiltonian function $H:\ T^*SE(3)^N \to \mathbb{R}$.

We examine, in depth, the problem of $N = 2$ and discuss at the end of the paper scalability and extensions to $N > 2$. For $N = 2$ we consider two rigid bodies labelled A and B. An element in $SE(3) \times SE(3)$ is given by (R_A, b_A, R_B, b_B), where $R_k \in SO(3)$ is a rotation matrix that maps body coordinates into inertial coordinates, and describes the orientation of the vehicle k. Here b_k is the vector from the origin of the inertial coordinate frame to the origin of the body frame of vehicle k and describes the position of this vehicle.

We consider in this paper three general settings motivated by applications. In case the control depends on the relative orientations *and* the relative positions of the rigid bodies, the remaining symmetry will be $G = SE(3)$, interpreted as the invariance of the system dynamics to the global position and orientation of the group as a whole. The way to encode this 'information loss' is to reduce the (left) diagonal action of $SE(3)$ on the phase space $T^*SE(3) \times \ldots \times T^*SE(3)$. In the case $N = 2$, dividing out the 6-dimensional symmetry group $G = SE(3)$ from the 24-dimensional phase space $T^*SE(3) \times T^*SE(3)$, leads to the 18-dimensional reduced phase space

$$T^*SE(3) \times T^*SE(3) \big/ G \;\cong\; (\mathfrak{se}(3)^* \times \mathfrak{se}(3)^*) \times SE(3) \tag{1}$$

where $\mathfrak{se}(3)^*$ is the dual of the Lie algebra $\mathfrak{se}(3)$ of $SE(3)$. The isomorphism and the direct product on the right hand side refer to the manifold character of the sets involved and not to the group structure; in fact, the left hand side does not even come equipped with a group structure. We call this *Case C*.

Our second setting is the case studied in [19] in which we not only abstract from the absolute positions of the two rigid bodies, but also from their relative position, only retaining information on their relative orientation. In other words, we consider control inputs that depend only on the relative orientation of the individual bodies, e.g., for orientation alignment of the two individuals but *not* relative position alignment. To this end we again reduce the left action of G, but instead of the semi-direct product $SE(3)$ of $SO(3)$ with $\mathbb{R}^3$, as in Case C, this group G now becomes the 9-dimensional semi-direct product of $SO(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$ (see [6] for more on this semi-direct product). The additional copy of $\mathbb{R}^3$ in the symmetry group, as compared with Case C, comes from the invariance of the system to the relative position of the bodies. Correspondingly, we do not end up with (1), but with the 15-dimensional reduced phase space

$$\mathfrak{t}^*_- \;=\; (\mathfrak{se}(3)^* \times \mathfrak{se}(3)^*) \times SO(3) \tag{2}$$

(again concentrating on the manifold character). We call this *Case B*.

Our final setting which we call *Case A* is even simpler. Here we ignore the translational part of the problem altogether and only treat the rotational aspect. The phase space is the 12-dimensional tangent bundle $T^*SO(3) \times T^*SO(3)$, and we assume control inputs that depend on relative orientation. The 3-dimensional symmetry group $G = SO(3)$ corresponds to the invariance of the controlled system with respect to the orientation of the group of rigid bodies as a whole. The 9-dimensional reduced phase space is

$$T^*SO(3) \times T^*SO(3) \big/ SO(3) \;\cong\; (\mathfrak{so}(3)^* \times \mathfrak{so}(3)^*) \times SO(3)\,. \tag{3}$$

where $\mathfrak{so}(3)^*$ is the dual of the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$.

In the first part of this paper we describe the reduction for Cases A-C. We do this *without* assuming the form of the kinetic energy of the rigid bodies of interest; we only assume the dependency of the coupling potentials that are realized by the control inputs.

The reduction can therefore be seen as providing a framework for a class of problems. For example, Cases B and C which concern rigid bodies, each on $SE(3)$, can be applied to a group of stratospheric balloons, underwater vehicles, helicopters, etc. Case A, which concerns rigid bodies on $SO(3)$, is applicable to satellite cluster coordination, for example. The presentation of the reduction is organized as follows. In §2 we obtain the quotient spaces provided by the abstract theory. In §3 we review the reduction theory by means of semi-direct products and apply this to Cases A-C. This yields flat Poisson spaces $\mathfrak{s}^*_- \cong \mathbb{R}^\nu$ which we treat in their own right in §4; we write down Casimirs, study how the abstract quotients of §2 are realized as submanifolds and examine the singular (lower-dimensional) symplectic leaves. These flat Poisson spaces described in §3 and §4 are preferable to the abstract reduced spaces described in §2 since they make it possible for us to efficiently formulate control laws.

In the second part of the paper we apply the reduction results to particular examples, i.e., we study the controlled dynamics for particular choices of kinetic energy. In §5 the dynamics are given by Hamiltonians that are obtained by adding a potential energy incorporating our control law to the (free) kinetic energy. The (relative) equilibria of these controlled dynamics are described in §6, while the stability properties of these are analysed in §7. Extensions to $N > 2$ are presented in §8 and final remarks in §9.

2 Abstract reduction theory

According to Noether's theorem every continuous symmetry leads to a conserved quantity [9]. Hamiltonian mechanics are well suited for a concise formulation of this result and its implications.

Let the Hamiltonian system (vector field) X_H on the phase space P be invariant under the proper Poisson action of some Lie group G . We call G a symmetry group. Then the quotient space

$$P/G \;=\; \{\; \{\, g(p) = p \mid g \in G \,\} \;\mid\; p \in P \;\}$$

is again a Poisson space, even a Poisson manifold if the action of G is free [9]. The conserved quantities predicted by Noether's theorem are *Casimir functions* on this quotient space. The system X_H canonically projects to a Hamiltonian system on the quotient space, the so-called reduced system. Like every Hamiltonian system on P/G , it leaves the values of the Casimir functions fixed. Therefore, the reduced dynamics are restricted to a submanifold of the quotient space, defined by the Casimir functions.

A second way of doing reduction swaps the order of the steps and fixes the values of conserved quantities *before* passing to the quotient. Because of the intimate relation between the two procedures, going through the steps of each helps to efficiently work with the reduced dynamics in applications. For this reason we present here the latter, more abstract, reduction, often called *regular reduction* in the literature.

Assembling the conserved quantities in the unreduced space, P, leads to the momentum mapping

$$\mathbf{J} \;:\; P \;\longrightarrow\; \mathfrak{g}^*$$

with values in the dual of the Lie algebra $\mathfrak{g}$ of the symmetry group G. Given the value $\mu \in \mathfrak{g}^*$ of $\mathbf{J}$, the dynamics are restricted to the subset $\mathbf{J}^{-1}(\mu) \subseteq P$. In the important case that μ is a *regular* value of $\mathbf{J}$ the subset $\mathbf{J}^{-1}(\mu)$ is a (smooth) submanifold. Regular reduction aims to divide out the remaining symmetry.

The action of the whole group G would in general alter the value of the momentum mapping. An important case is when $\mathbf{J}$ is equivariant with respect to the given action of G on P, Φ_g, and the *coadjoint* action of G on $\mathfrak{g}^*$, $\mathrm{Ad}^*_{g^{-1}}$, i.e.,

$$\mathrm{Ad}^*_{g^{-1}} \circ \mathbf{J} \;=\; \mathbf{J} \circ \Phi_g \qquad \text{for all } g \in G.$$

In this case, we have that the *isotropy subgroup*

$$G_\mu \;=\; \left\{ \, g \in G \;\middle|\; \mathrm{Ad}^*_{g^{-1}}(\mu) = \mu \, \right\}$$

does leave $\mathbf{J}^{-1}(\mu)$ invariant and hence defines a free action, i.e. one that does not have any fixed points. (This is true provided that the action of G on P is free and μ is a regular value of the momentum mapping). If P is a symplectic manifold, then the quotient

$$P_\mu \;=\; \mathbf{J}^{-1}(\mu) \big/ G_\mu$$

is (in a canonical way) again a symplectic manifold. The two ways of reduction are related through

$$P \big/ G \;=\; \bigcup_{\mu \in \mathfrak{g}^*} P_\mu \quad ; \tag{4}$$

in fact the P_μ constitute the *symplectic leaves* of this Poisson manifold.

In the important case that the phase space P is given by the cotangent bundle T^*G of the symmetry group G (acting by left translation) – for $G = SO(3)$ this is the situation of the free rigid body and for $G = SE(3)$ that of a free underwater vehicle – equation (4) expresses that $\mathfrak{g}^*$ is foliated by the coadjoint orbits, i.e. the orbits of the coadjoint action $\mathrm{Ad}^*_{g^{-1}}$ [9]. We now examine the abstract reduction theory in the three cases of interest in this paper.

2.1 Case A

The phase space of two rigid bodies in three-dimensional space,

$$P \;=\; T^*(SO(3) \times SO(3)) \;\cong\; T^*SO(3) \times T^*SO(3) \quad ,$$

is a symplectic manifold, and we define the action of the group $G = SO(3)$ on this space through the lift of the diagonal action

$$\begin{array}{ccc} G \times (SO(3) \times SO(3)) & \longrightarrow & SO(3) \times SO(3) \\ (R, R_A, R_B) & \mapsto & (RR_A, RR_B) \end{array} \quad . \tag{5}$$

This corresponds to rotating bodies A and B, whose orientation with respect to an inertial frame is specified by R_A and R_B, respectively, by the same (fixed) rotation R. A fixed value $\mu \in \mathfrak{so}(3)^* \cong \mathbb{R}^3$ of the momentum mapping $\mathbf{J} : P \longrightarrow \mathbb{R}^3$ corresponds to the sum

$$\mu \ = \ R_A\pi_A \ + \ R_B\pi_B \tag{6}$$

of the two angular momenta of the two bodies, measured in the inertial frame, where π_A and π_B are the individual angular momenta in the respective body frames of the two vehicles. Hence, the isotropy subgroup G_μ is isomorphic to S^1 and consists of all rotations about μ (provided this vector does not vanish). Since $\mathbf{J}^{-1}(\mu)$ is 9-dimensional and $\dim S^1 = 1$, we obtain 8-dimensional symplectic leaves P_μ. Correspondingly, the squared length

$$\mu^2 \ = \ \pi_A^2 \ + \ 2\pi_B^T(R_B^TR_A)\pi_A \ + \ \pi_B^2 \tag{7}$$

is the Casimir function on the 9-dimensional Poisson manifold (3). The rank of the reduced Poisson bracket is 8; we have reduced from 6 to 4 degrees of freedom.

2.2 Case B

The phase space of two rigid bodies with translational dynamics in three-dimensional space,

$$P \ = \ T^*(SE(3) \times SE(3)) \ \cong \ T^*SE(3) \times T^*SE(3),$$

is again a symplectic manifold, while the symmetry group G of interest in this case, as described in §1, is given by the semi-direct product of $SO(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$. The action of G we choose is the lift of

$$\begin{array}{rcc} G \times (SE(3) \times SE(3)) & \longrightarrow & SE(3) \times SE(3) \\ (R, a, b, R_A, b_A, R_B, b_B) & \mapsto & (RR_A, Rb_A + a, RR_B, Rb_B + b) \end{array} \tag{8}$$

and corresponds to rotating bodies A and B by the same rotation R while translating them with two (possibly different) vectors a and b. A fixed value $\mu \in \mathfrak{g}^* \cong \mathbb{R}^9$ of the momentum mapping $\mathbf{J} : P \longrightarrow \mathbb{R}^9$ corresponds to the nine components of the three vectors

$$R_Ap_A \ , \quad R_Bp_B \ , \quad R_A\pi_A \ + \ b_A \times R_Ap_A \ + \ R_B\pi_B \ + \ b_B \times R_Bp_B$$

where p_A, p_B denote the linear momenta of the vehicles A, B, respectively, measured in the respective body frames. For regular values μ, the isotropy subgroup G_μ is isomorphic to $S^1 \times S^1 \times S^1$ and consists of all rotations about these three vectors. Since $\mathbf{J}^{-1}(\mu)$ is 15-dimensional and $\dim(S^1 \times S^1 \times S^1) = 3$, we obtain 12-dimensional symplectic leaves. Correspondingly, we have on the 15-dimensional Poisson manifold (2) the three Casimir functions

$$p_A^2 \ , \quad p_B^2 \quad \text{and} \quad p_B^T(R_B^TR_A)p_A \quad . \tag{9}$$

The rank of the reduced Poisson bracket is 12 ; we have reduced from 12 to 6 degrees of freedom.

2.3 Case C

The phase space is again

$$P \;=\; T^*(SE(3)\times SE(3)) \;\cong\; T^*SE(3)\times T^*SE(3)$$

and we choose the symmetry group G, the semi-direct product $SE(3)$ of $SO(3)$ with $\mathbb{R}^3$, to act through the lift of the diagonal action

$$\begin{array}{ccc} SE(3)\times(SE(3)\times SE(3)) & \longrightarrow & SE(3)\times SE(3) \\ (R,b,R_A,b_A,R_B,b_B) & \mapsto & (RR_A, Rb_A+b, RR_B, Rb_B+b) \end{array} \quad . \tag{10}$$

This corresponds to rotating bodies A and B by the same rotation R while translating them with the same vector b . A fixed value $\mu\in\mathfrak{se}(3)^*\cong\mathbb{R}^6$ of the momentum mapping corresponds to the six components of the two vectors

$$R_A\pi_A \;+\; b_A\times R_Ap_A \;+\; R_B\pi_B \;+\; b_B\times R_Bp_B \quad\text{and}\quad R_Ap_A \;+\; R_Bp_B \quad .$$

Hence, the isotropy subgroup G_μ is isomorphic to $S^1\times S^1$ and consists of all rotations about these two vectors. Since $\mathbf{J}^{-1}(\mu)$ is 18-dimensional and $\dim(S^1\times S^1)=2$, we obtain 16-dimensional symplectic leaves. On the 18-dimensional Poisson manifold (1), we have the two Casimir functions

$$\begin{array}{l} \pi_A\cdot p_A \;+\; p_B^T(R_B^TR_A)\pi_A \;+\; \pi_B^T(R_B^TR_A)p_A \\ +\;\pi_B\cdot p_B \;+\; R_B^T(b_B-b_A)\cdot(\pi_B\times(R_B^TR_A)p_A) \\ \text{and}\quad p_A^2 \;+\; 2p_B^T(R_B^TR_A)p_A \;+\; p_B^2 \quad . \end{array} \tag{11}$$

Fixing the values of these two Casimirs yields the 16-dimensional symplectic leaves of (1). Hence, the rank of the reduced Poisson bracket is 16; we have reduced from 12 to 8 degrees of freedom.

3 Semi-direct product reduction

While regular reduction yields symplectic manifolds, it does not provide an algorithm to obtain a realization of the quotients $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ as submanifolds of some flat Poisson space $\mathbb{R}^\nu$, a result that would contribute to efficient formulation of control laws. Finding a set of ν generators of the G-invariant functions on P would provide a Poisson-embedding of P/G in $\mathbb{R}^\nu$; the image of this embedding being defined by the various relations between the ν generators. However, this is not the approach we take.

In the present case, where the phase space P is the cotangent bundle of some Lie group F, and the symmetry group G is a subgroup of F, the quotient P/G can be embedded in the flat Poisson space

$$\mathfrak{s}_-^* \;=\; \mathfrak{f}_-^* \times_{\rho'} V_-^*$$

where $\mathfrak{f}^*$ is the dual of the Lie algebra of F and V is a vector space. To define this semi-direct product one needs a dummy dependent variable $a \in V_+^*$ for the system and a left representation

$$\rho \; : \; F \; \longrightarrow \; \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ is the general linear group. This representation defines the semi-direct product $S = F \times_\rho V$ in such a way that the symmetry group G can be identified with the isotropy subgroup

$$F_a \;=\; \left\{ \, f \in F \;\middle|\; \rho^*(f)a = a \, \right\}$$

where ρ^* is the associated right representation of F on V^*. Further, ρ' is the induced Lie algebra representation. The vector space V acts on itself by vector addition. This allows us to use regular reduction on T^*S. The reduction of the S-symmetry on T^*S provided by the lift of the (left) action of S on itself is the well-known Lie-Poisson reduction leading to

$$T^*S\big/S \;=\; \mathfrak{s}^*_-$$

with lower sign choice in the Poisson bracket on $\mathfrak{s}^*_-$

$$\begin{aligned} \{W, Q\}_\pm(\ell, \alpha) \;&=\; \pm\left\langle \ell, \left[\frac{\delta W}{\delta \ell}, \frac{\delta Q}{\delta \ell}\right]\right\rangle \\ &\pm \left\langle \alpha, \rho'\left(\frac{\delta W}{\delta \ell}\right)\cdot\frac{\delta Q}{\delta \alpha} - \rho'\left(\frac{\delta Q}{\delta \ell}\right)\cdot\frac{\delta W}{\delta \alpha}\right\rangle \quad . \end{aligned} \tag{12}$$

To obtain the upper sign one would have to reduce the right action of S on itself. The symplectic leaves of this Poisson manifold are the quotients

$$\mathbf{J}^{-1}(\mu, a)\big/S_{(\mu,a)} \;=\; \mathbf{J}^{-1}(\mu) \times V \times \{a\}\big/F_{(\mu,a)} \times V \quad ,$$

where

$$F_{(\mu,a)} \;=\; \left\{ \, g \in F_a \;\middle|\; \mathrm{Ad}^*_{g^{-1}}\mu = \mu \, \right\} \;=\; G_\mu \quad .$$

The vector space V cancels and we have embedded

$$T^*F\big/G \;=\; \bigcup_{\mu \in \mathfrak{f}^*_+} \mathbf{J}^{-1}(\mu)\big/G_\mu$$

in the flat Poisson space $\mathfrak{s}^*_-$ by choosing for the dummy variable $a \in V_+^*$ some fixed value. This is called semi-direct product reduction and is summarized in the following theorem.

Theorem 3.1 (Marsden et al [10]) *Let $H_a : T^*F \to \mathbb{R}$ be a Hamiltonian depending smoothly on a parameter $a \in V^*$, and left invariant under the action on T^*F of the stabiliser F_a, defined as:*

$$F_a \;=\; \left\{ \, f \in F \;\middle|\; \rho^*(f)a = a \, \right\} \tag{13}$$

where ρ is a left representation of F on the vector space V, and ρ^* is the associated right representation of F on V^*. The family of Hamiltonians $\{H_a \mid a \in V^*\}$ induces a Hamiltonian function H on the space s^*_-, defined by $H((T_eL_f)^*\alpha_f, \rho^*(f)a) = H_a(\alpha_f)$, thus yielding Lie-Poisson equations on $\mathfrak{s}^*_-$. Let $c_a(t) \in T^*F$ denote an integral curve of H_a and let $f_a(t)$ be its projection onto F. Then the reduced variable describes the evolution of the parameter $a \in V^*$, where this evolution is given by $t \to \rho^*(f_a(t))a$.

Again we explicitly formulate the implications of this procedure for the cases of interest in this paper.

3.1 Case A

Here $F = SO(3) \times SO(3)$ and we shall consider the symmetry group $G = SO(3)$. Define

$$F_K \;=\; \left\{ \, (\bar{R}_A, \bar{R}_B) \in F \;\middle|\; \bar{R}_A^T K \bar{R}_B = K \, \right\} \quad , \tag{14}$$

where $K \in \mathbb{R}^{3\times 3}$ is a dummy variable. We shall eventually fix $K = I_3$, the 3×3 identity matrix, whence F_{I_3} becomes $\left\{ (R,R) \;\middle|\; R \in SO(3) \right\}$, and we recover the diagonal action (5) of the symmetry group $G = SO(3)$. As we shall see in §5.1, this choice of G may arise, for example, in the addition of a potential function which is added to stabilise the equilibrium $R_A = KR_B$; the choice $K = I_3$ corresponds to the two vehicles, A and B, having orientations in alignment.

We may now define a representation $\rho : SO(3) \times SO(3) \longrightarrow \mathrm{GL}(\mathbb{R}^{3\times 3})$ by $\rho(R_A, R_B)K = R_A K (R_B)^T$ where $\mathrm{GL}(\mathbb{R}^{3\times 3})$ is the general linear group of the vector space $\mathbb{R}^{3\times 3}$ of three by three matrices. Given two matrices $Y \in \mathbb{R}^{n_1\times m_1}$, $Z \in \mathbb{R}^{n_2\times m_2}$, we define their *Kronecker Product* [18], denoted by $Y \otimes Z \in \mathbb{R}^{n_1n_2\times m_1m_2}$, as

$$Y \otimes Z = \begin{bmatrix} y_{11}Z & \cdots & y_{1m_1}Z \\ \vdots & \ddots & \vdots \\ y_{n_11}Z & \cdots & y_{n_1m_1}Z \end{bmatrix}. \tag{15}$$

With this definition it is easy to check that there is a direct correspondence between the elements of $R_A K R_B^T \in \mathbb{R}^{3\times 3}$ and those of $(R_A \otimes R_B)\widetilde{K} \in \mathbb{R}^9$, where $\widetilde{K} = (e_1^T K, e_2^T K, e_3^T K)^T \in \mathbb{R}^9$. We can thus equivalently define $\rho : SO(3) \times SO(3) \longrightarrow \mathrm{GL}(\mathbb{R}^9)$ by $\rho(R_A, R_B)\widetilde{K} = (R_A \otimes R_B)\widetilde{K}$, where $\mathrm{GL}(\mathbb{R}^9)$ denotes the set of invertible linear mappings on $\mathbb{R}^9$.

Given this parameter space $V = \mathbb{R}^{3\times 3} \cong \mathbb{R}^9$, we may write the entire 15-dimensional space $S = F \times_\rho V = (SO(3) \times SO(3)) \times_\rho \mathbb{R}^9$ as a semi-direct product where group multiplication is given by

$$\begin{aligned} ((\bar{R}_A, \bar{R}_B), \bar{\widetilde{K}})((R_A, R_B), \widetilde{K}) &= ((\bar{R}_A, \bar{R}_B)(R_A, R_B), \rho(\bar{R}_A, \bar{R}_B)\widetilde{K} + \bar{\widetilde{K}}) \\ &= ((\bar{R}_A R_A, \bar{R}_B R_B, (\bar{R}_A \otimes \bar{R}_B)\widetilde{K} + \bar{\widetilde{K}}). \end{aligned}$$

Using this correspondence, an element in the Lie group S can be represented by a 16×16 matrix of the form

$$\begin{bmatrix} R_A & 0 & 0 & 0 \\ 0 & R_B & 0 & 0 \\ 0 & 0 & R_A\otimes R_B & \tilde{K} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the group action is given simply by matrix multiplication.

Let ρ^* be the associated right representation of $SO(3)\times SO(3)$ on $\mathbb{R}^9$, given by

$$\rho^*(R_A,R_B)\widetilde{K} = (R_A^T\otimes R_B^T)\widetilde{K}. \tag{16}$$

With this definition, G may be identified with the isotropy subgroup

$$F_{\widetilde{K}} \;=\; \left\{(\bar{R}_A,\bar{R}_B)\;\in F \;\;\middle|\;\; \rho^*(\bar{R}_A,\bar{R}_B)\widetilde{K}=\widetilde{K}\right\}\quad . \tag{17}$$

By Theorem 3.1 the Hamiltonian dynamics on $T^*(SO(3)\times SO(3))$ can be reduced to a Lie-Poisson system on the fifteen-dimensional space $\mathfrak{s}^*_-$, the dual of the Lie algebra $\mathfrak{s}$ of S. In fact, $\mathfrak{s}=(\mathfrak{so}(3)\times\mathfrak{so}(3))\times_{\rho'}\mathbb{R}^{3\times 3}$ where ρ' is the induced Lie algebra representation, given by

$$\rho'(\widehat{\alpha},\widehat{\beta})\nu = \widehat{\alpha}\nu-\nu\widehat{\beta}. \tag{18}$$

Here $(\widehat{\alpha},\widehat{\beta})\in\mathfrak{so}(3)\times\mathfrak{so}(3)$ and $\nu\in\mathbb{R}^{3\times 3}$. This implies that the Lie bracket of two elements of the Lie algebra $\mathfrak{s}$ is given by

$$\begin{aligned}[((\widehat{\alpha},\widehat{\beta},\nu),((\widehat{\alpha}',\widehat{\beta}'),\nu')] &= ([(\widehat{\alpha},\widehat{\beta}),(\widehat{\alpha}',\widehat{\beta}')],\;\rho'(\widehat{\alpha},\widehat{\beta})\nu'-\rho'(\widehat{\alpha}',\widehat{\beta}')\nu) \\ &= (\widehat{\alpha}\widehat{\alpha}'-\widehat{\alpha}'\widehat{\alpha},\widehat{\beta}\widehat{\beta}'-\widehat{\beta}'\widehat{\beta},\widehat{\alpha}\nu'-\nu'\widehat{\beta}-\widehat{\alpha}'\nu+\nu\widehat{\beta}').\end{aligned}$$

Again using the definition for the Kronecker product, elements in the Lie algebra $\mathfrak{s}$ can be represented by 16×16 matrices of the form

$$\begin{bmatrix} \widehat{\alpha} & 0 & 0 & 0 \\ 0 & \widehat{\beta} & 0 & 0 \\ 0 & 0 & \widehat{\alpha}\otimes I+I\otimes\widehat{\beta} & \tilde{\nu} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where, as before, $\widetilde{\nu}=(e_1^T\nu,e_2^T\nu,e_3^T\nu)^T$. The Lie bracket of two Lie algebra elements in this form is given simply by the matrix commutator.

In body coordinates, which describe the isomorphism $SO(3)\times\mathfrak{so}(3)^*_-\cong T^*SO(3)$, the reduction mapping is given by

$$\begin{array}{ccccc} SO(3)\times\mathfrak{so}(3)^*_-\times SO(3)\times\mathfrak{so}(3)^*_- & \longrightarrow & s^*_- & = & \mathfrak{so}(3)^*_-\times\mathfrak{so}(3)^*_-\times V^*_- \\ (R_A,\pi_A,R_B,\pi_B) & \mapsto & & & (\pi_A,\pi_B,\Delta,\Sigma,\Gamma) \end{array} ,$$

where we denote the three columns of $(R_A^T K R_B)^T$ by Δ, Σ and Γ , respectively. In the particular case $K = I_3$ the embedded reduced phase space is defined by the equations

$$\begin{array}{lll} \Delta^2 = 1 , & \Sigma^2 = 1 , & \Gamma^2 = 1 , \\ \Sigma \cdot \Gamma = 0 , & \Gamma \cdot \Delta = 0 , & \Delta \cdot \Sigma = 0 \end{array} \tag{19}$$

where $(\Delta, \Sigma, \Gamma) \in SO(3)$ measures the relative orientation $R_B^T R_A$ of the two bodies.

Since we have performed reduction by means of the left action, the reduced phase space is $\mathfrak{s}^*_-$ (with lower signs in (12)). Using the formula (12), and defining the pairing between matrices in $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{3\times 3^*}$ to be $\langle Y, Z\rangle = \mathrm{Tr}(Y^T Z)$ yields the following expression for the Poisson bracket on $\mathfrak{s}^*_-$:

$$\begin{aligned} \{W, Q\}(\Pi_A, \Pi_B, X) &= -\Pi^A \cdot (\nabla_{\Pi_A} W \times \nabla_{\Pi_A} Q) - \Pi_B \cdot (\nabla_{\Pi_B} W \times \nabla_{\Pi_B} Q) \\ &\quad - \mathrm{Tr}\left[X^T \left[\widehat{\frac{\delta W}{\delta \Pi_A}} \frac{\delta Q}{\delta X} - \frac{\delta Q}{\delta X} \widehat{\frac{\delta W}{\delta \Pi_B}} - \widehat{\frac{\delta Q}{\delta \Pi_A}} \frac{\delta W}{\delta X} + \frac{\delta W}{\delta X} \widehat{\frac{\delta Q}{\delta \Pi_B}} \right]\right] . \end{aligned}$$

The Poisson bracket (12) has the structure matrix

$$\Lambda(\pi_A, \pi_B, \Delta, \Sigma, \Gamma) = \begin{bmatrix} \widehat{\pi}_A & 0 & a & b & c \\ 0 & \widehat{\pi}_B & \widehat{\Delta} & \widehat{\Sigma} & \widehat{\Gamma} \\ -a^T & \widehat{\Delta} & 0 & 0 & 0 \\ -b^T & \widehat{\Sigma} & 0 & 0 & 0 \\ -c^T & \widehat{\Gamma} & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

where the 3×3 matrices a, b and c are given by

$$a = \begin{bmatrix} 0 \\ \Gamma^T \\ -\Sigma^T \end{bmatrix} , \quad b = \begin{bmatrix} -\Gamma^T \\ 0 \\ \Delta^T \end{bmatrix} , \quad c = \begin{bmatrix} \Sigma^T \\ -\Delta^T \\ 0 \end{bmatrix} .$$

For a given Hamiltonian, $H(\pi_A, \pi_B, \Delta, \Sigma, \Gamma)$, expressed in terms of the reduced variables, we may now write the system dynamics as

$$(\dot{\pi}_A, \dot{\pi}_B, \dot{\Delta}, \dot{\Sigma}, \dot{\Gamma})^T = \Lambda \nabla H(\pi_A, \pi_B, \Delta, \Sigma, \Gamma) \quad . \tag{21}$$

3.2 Case B

Here $F = SE(3) \times SE(3)$ and the symmetry group G, the semi-direct product of $SO(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$, is embedded as

$$F_K = \left\{ (R_A, b_A, R_B, b_B) \in F \;\middle|\; R_A^T K R_B = K \right\}$$

with the same dummy variable $K \in \mathbb{R}^{3\times 3}$ as in Case A. Analogous to Case A, we recover the action (8) for the fixed value $K = I_3$ of interest. The reduction proceeds via Theorem 3.1 as before, with the reduction mapping now reading

$$\begin{array}{ccc} SE(3) \times \mathfrak{se}(3)^*_- \times SE(3) \times \mathfrak{se}(3)^*_- & \longrightarrow & s^*_- = \mathfrak{se}(3)^*_- \times \mathfrak{se}(3)^*_- \times V^*_- \\ (R_A, b_A, \pi_A, p_A, R_B, b_B, \pi_B, p_B) & \mapsto & (\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma) \end{array}$$

with the same convention for Δ, Σ, Γ as in Case A. The variables p_A, p_B are the linear momenta of the two bodies (measured in their respective body frames).

Elements (R, b) in the Lie groups are in one-to-one correspondence with matrices $\left(\begin{smallmatrix} R & b \\ 0 & 1 \end{smallmatrix}\right)$ and computations similar to those of Case A yield the structure matrix

$$\Lambda(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma) = \begin{bmatrix} \widehat{\pi}_A & \widehat{p}_A & 0 & 0 & a & b & c \\ \widehat{p}_A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\pi}_B & \widehat{p}_B & \widehat{\Delta} & \widehat{\Sigma} & \widehat{\Gamma} \\ 0 & 0 & \widehat{p}_B & 0 & 0 & 0 & 0 \\ -a^T & 0 & \widehat{\Delta} & 0 & 0 & 0 & 0 \\ -b^T & 0 & \widehat{\Sigma} & 0 & 0 & 0 & 0 \\ -c^T & 0 & \widehat{\Gamma} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

of the Poisson bracket.

For a given Hamiltonian, $H(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma)$, expressed in terms of the reduced variables, we may now write the system dynamics as

$$(\dot{\pi}_A, \dot{p}_A, \dot{\pi}_B, \dot{p}_B, \dot{\Delta}, \dot{\Sigma}, \dot{\Gamma})^T = \Lambda \nabla H(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma) \quad . \tag{23}$$

3.3 Case C

Again $F = SE(3) \times SE(3)$, while the symmetry group $G = SE(3)$ is now embedded as

$$F_{(K,k)} = \left\{ (R_A, b_A, R_B, b_B) \in F \;\middle|\; R_A^T K R_B = K \,,\; R_A k + b_A - K b_B = k \right\}, \tag{24}$$

where the dummy variable has become $(K, k) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^3$. For the fixed value $(K, k) = (I_3, 0)$ we recover the diagonal action (10). One may easily check that F_K may also be written

$$F_K = \left\{ (R_A, b_A, R_B, b_B) \in F \;\middle|\; \begin{bmatrix} R_A & 0 \\ -b_A^T R_A & 1 \end{bmatrix}^T \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_B & b_B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} \right\}. \tag{25}$$

We may now define a representation $\rho : SE(3) \times SE(3) \longrightarrow \mathrm{GL}(\mathbb{R}^{3\times 3} \times \mathbb{R}^3)$ by

$$\rho(R_A, b_A, R_B, b_B) \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_A & 0 \\ -b_A^T R_A & 1 \end{bmatrix}^T \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_B & b_B \\ 0 & 1 \end{bmatrix}.$$

One may use the definition (15) to demonstrate that there is a direct correspondence between the elements of

$$\begin{bmatrix} R_A & 0 \\ -b_A^T R_A & 1 \end{bmatrix}^T \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_B & b_B \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

and those of

$$\begin{bmatrix} R_A^T & -R_A^T b_A \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} R_B^T & 0 \\ b_B^T & 1 \end{bmatrix} \widehat{K} \in \mathbb{R}^{16}$$

where $\widehat{K} = (e_1^T K, k_1, e_2^T K, k_2, e_3^T K, k_3, 0, 0, 0, 1)^T$. We can thus equivalently define $\rho : SE(3) \times SE(3) \longrightarrow \mathrm{GL}(\mathbb{R}^{16})$ by

$$\rho(R_A, b_A, R_B, b_B)\widehat{K} = \begin{bmatrix} R_A^T & -R_A^T b_A \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} R_B^T & 0 \\ b_B^T & 1 \end{bmatrix} \widehat{K}.$$

Note that $\rho(R_A, b_A, R_B, b_B)$ fixes the 12-dimensional affine subspace V of $\mathbb{R}^{16}$ that consists of vectors of the above form $\widehat{K}$, i.e. vectors that have as last four components $0, 0, 0, 1$. By "not adding up the last vector component fixed at 1," we can give V the structure of a vector space and end up with the desired $\rho : SE(3) \times SE(3) \longrightarrow \mathrm{GL}(V)$. Given this parameter space $V \cong \mathbb{R}^{12}$, we may write the entire 24-dimensional space $S = F \times_\rho V = (SE(3) \times SE(3)) \times_\rho V$ as a semi-direct product and proceed as in Case A. In particular, G may be identified with the isotropy subgroup

$$F_{\widehat{K}} \;=\; \left\{ (\bar{R}_A, \bar{R}_B, \bar{b}_A, \bar{b}_B) \in F \;\middle|\; \rho^*(\bar{R}_A, \bar{b}_A, \bar{R}_B, \bar{b}_B)\widehat{K} = \widehat{K} \right\} \quad ,$$

where the associated right representation of the symmetry group G, is

$$\rho^*(R_A, b_A, R_B, b_B) \;=\; \begin{bmatrix} R_A^T & -R_A^T b_A \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} R_B^T & 0 \\ b_B^T & 1 \end{bmatrix} \quad . \tag{26}$$

By Theorem 3.1 the Hamiltonian dynamics on $T^*(SE(3) \times SE(3))$ can be reduced to a Lie-Poisson system on the 24-dimensional space $\mathfrak{s}^*_-$, the dual of the Lie algebra $\mathfrak{s}$ of S. In fact, $\mathfrak{s} = (\mathfrak{se}(3) \times \mathfrak{se}(3)) \times_{\rho'} V$ where ρ' is the induced Lie algebra representation

$$\rho'(\widehat{\alpha}, \nu_A, \widehat{\beta}, \nu_B)\nu \;=\; \begin{bmatrix} \widehat{\alpha} & 0 \\ -\nu_A^T & 0 \end{bmatrix} \otimes I_4 \;+\; I_4 \otimes \begin{bmatrix} \widehat{\beta} & \nu_B \\ 0 & 0 \end{bmatrix} \quad . \tag{27}$$

The reduction mapping is given by

$$\begin{array}{ccc} SE(3) \times \mathfrak{se}(3)^*_- \times SE(3) \times \mathfrak{se}(3)^*_- & \longrightarrow & \mathfrak{s}^*_- \;=\; \mathfrak{se}(3)^*_- \times \mathfrak{se}(3)^*_- \times V^*_- \\ (R_A, b_A, \pi_A, p_A, R_B, b_B, \pi_B, p_B) & \mapsto & (\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma, \beta) \end{array}$$

where in addition to the three columns Δ, Σ, Γ of $(R_A^T K R_B)^T$ we now have $\beta = R_A^T(k - b_A + K b_B)$. In the particular case $(K, k) = (I_3, 0)$ the embedded reduced phase space is still defined by (19) and the new variable β measures the relative position of the two bodies (in the body frame of the A underwater vehicle).

The Poisson bracket (12) has the structure matrix

$$\Lambda \;=\; \begin{bmatrix} \widehat{\pi}_A & \widehat{p}_A & 0 & 0 & a & b & c & \widehat{\beta} \\ \widehat{p}_A & 0 & 0 & 0 & 0 & 0 & 0 & I_3 \\ 0 & 0 & \widehat{\pi}_B & \widehat{p}_B & \widehat{\Delta} & \widehat{\Sigma} & \widehat{\Gamma} & 0 \\ 0 & 0 & \widehat{p}_B & 0 & 0 & 0 & 0 & -(\Delta, \Sigma, \Gamma) \\ -a^T & 0 & \widehat{\Delta} & 0 & 0 & 0 & 0 & 0 \\ -b^T & 0 & \widehat{\Sigma} & 0 & 0 & 0 & 0 & 0 \\ -c^T & 0 & \widehat{\Gamma} & 0 & 0 & 0 & 0 & 0 \\ \widehat{\beta} & -I_3 & 0 & (\Delta, \Sigma, \Gamma)^T & 0 & 0 & 0 & 0 \end{bmatrix} \quad . \tag{28}$$

For a given Hamiltonian, $H(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma, \beta)$, expressed in terms of the reduced variables, we may now write the system dynamics as

$$(\dot{\pi}_A, \dot{p}_A, \dot{\pi}_B, \dot{p}_B, \dot{\Delta}, \dot{\Sigma}, \dot{\Gamma}, \dot{\beta})^T = \Lambda \nabla H(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma, \beta) \quad . \tag{29}$$

4 Casimirs in flat spaces

In the previous section we were able to realize the abstract reduced spaces of §2 as submanifolds of (concrete) flat Poisson spaces. We now study these Poisson spaces in their own right, determining the (maximal) rank of the bracket and looking for the Casimir functions. These Casimirs play a central role in proving stability for the coordinated rigid body networks. While we do not lose sight of our particular case(s) $K = I_3$ (and $k = 0$), letting K (and k) vary allows to decompose the the whole flat space(s) into various reduced manifolds. One may think of $\mathfrak{s}^*_-$ as collecting all possible reductions of F_K-actions for all possible matrices $K \in \mathbb{R}^{3\times3}$ (and $k \in \mathbb{R}^3$).

4.1 Case A

On the whole 15-dimensional space $\mathfrak{s}^*_-$ the rank of the Poisson structure (20) cannot be more than 12; this follows because of the identically vanishing lower right 9×9-block. We also know from §2.1 that the symplectic leaves of the reduced phase space $\mathfrak{t}^*_- \subseteq \mathfrak{s}^*_-$ defined by (19), i.e., when $K = I_3$, have dimension 8, which then must also be the rank of (20) for points in $\mathfrak{t}^*_-$. However, regular reduction of the F_K-action with

$$K \quad = \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{30}$$

does indeed lead to 12-dimensional quotient manifolds. Correspondingly, on the whole space $\mathfrak{s}^*_-$ there are only three Casimir functions

$$\begin{aligned} C_1 &= \Delta^2 + \Sigma^2 + \Gamma^2 \\ C_2 &= \|\Sigma \times \Gamma\|^2 + \|\Gamma \times \Delta\|^2 + \|\Delta \times \Sigma\|^2 \\ C_3 &= (\Delta \times \Sigma) \cdot \Gamma \quad . \end{aligned} \tag{31}$$

The functions $c_1 = \Delta^2$, $c_2 = \Sigma^2$, $c_3 = \Gamma^2$, $c_4 = \Sigma \cdot \Gamma$, $c_4 = \Gamma \cdot \Delta$, $c_6 = \Delta \cdot \Sigma$ themselves are not Casimirs. We may, however, use these functions to define a 10-dimensional *Poisson submanifold*,

$$\mathfrak{r}^*_- \quad = \quad \left\{ \, \Delta^2 = \Sigma^2 = \Gamma^2 \, , \; \Sigma \cdot \Gamma = \Gamma \cdot \Delta = \Delta \cdot \Sigma = 0 \, \right\} \quad , \tag{32}$$

on which the three Casimirs (31) are constrained by the relations

$$3C_2 = C_1^2 \quad \text{and} \quad 27C_3^2 = C_1^3 \quad .$$

Thus on $\mathfrak{r}^*_-$ we have, effectively, only one Casimir remaining from (31). Since the rank of the Poisson bracket on $\mathfrak{r}^*_-$ is 8, there must be a further Casimir on this subspace; it is given by

$$C_0 \;=\; \frac{3(\Delta\times\Sigma)\cdot\Gamma}{\Delta^2+\Sigma^2+\Gamma^2}\left((\pi_A)^2 \;+\; (\pi_B)^2\right) \;+\; 2\pi_B^T(\Delta,\Sigma,\Gamma)\pi_A \quad . \tag{33}$$

Thus, in the particular case $K = \gamma R$ with nonzero scalar γ and $R \in SO(3)$, regular reduction of the F_K-action leads to 8-dimensional quotients. We may go on to identify the hypersurface $\mathfrak{t}^*_- \subseteq \mathfrak{r}^*_-$ as given by $C_3 \equiv 1$ (whence $C_1 \equiv 3$ and $C_2 \equiv 3$), confirming that $\mathfrak{t}^*_-$ is a Poisson submanifold of $\mathfrak{s}^*_-$ as well. Here C_0 equals the total angular momentum (7) of the system. The rank of the Poisson structure on $\mathfrak{r}^*_-$ drops from 8 to 6 when $\pi_A = \pi_B$, but there is no further drop when both vanish. The Poisson submanifold $\mathfrak{r}^*_-$ of $\mathfrak{s}^*_-$ also contains the non-generic points $C_3 \equiv 0$ (whence $C_1 \equiv 0$ and $C_2 \equiv 0$) where the rank of the Poisson structure drops from 8 to 4 for π_A and π_B in general position. Here $F_K = F_0 = SO(3)\times SO(3)$ and there are two sub-Casimirs π_A^2 and π_B^2 ; reducing by this large symmetry group abstracts from all inter-body information and leads to two uncoupled free rigid bodies.

4.2 Case B

On the whole 21-dimensional space $\mathfrak{s}^*_-$ the rank of the Poisson structure (22) cannot be more than 12 since the rows associated with the momentum variables can be linearly combined from the last 9 rows for a proper choice of Δ,Σ,Γ; this is possible if *e.g.* K is given by (30). We also know from §2.2 that the (maximal) symplectic leaves of the reduced phase space $\mathfrak{t}^*_- \subseteq \mathfrak{s}^*_-$ defined by (19) are 12-dimensional as well. Thus, there is no drop in the rank of the Poisson bracket on $\mathfrak{s}^*_-$ when $K = I_3$. The parametrization by invertible $K \in \mathbb{R}^{3\times 3}$ provides a foliation into symplectic leaves that all have the same dimension. The extreme choice $K = 0$ still leads to $F_0 = F$, though. In addition to the three Casimirs in (31) we found the Casimirs

$$\begin{aligned} C_4 &= (p_A)^2 \\ C_5 &= (p_B)^2 \\ C_6 &= p_B^T(\Delta,\Sigma,\Gamma)p_A \end{aligned} \tag{34}$$

which equal (9) on $\mathfrak{t}^*_-$. We conjecture that there are three further Casimirs but these have not yet been found.

We are not short of Casimirs on $\mathfrak{r}^*_-$ defined by (32) and indeed $C_{-1} \equiv 0$ on the 16-dimensional Poisson submanifold $\mathfrak{r}^*_-$. On $\mathfrak{r}^*_-$ the 12-dimensional symplectic leaves are the level sets of C_3, C_4, C_5, C_6 . The rank drops again for certain special configurations of the four vectors π_A, p_A, π_B and p_B or when $K = 0$.

4.3 Case C

Here the (maximal) symplectic leaves of the 18-dimensional reduced phase space $\mathfrak{t}^*_- \subseteq \mathfrak{s}^*_-$ (defined by (19)) have dimension 16, while the maximal rank of the Poisson structure

(28) on the whole 24-dimensional Poisson space $\mathfrak{s}^*_-$ is 18. Indeed, the extra rows in (28) yield, for *e.g.* K given by (30), 6 additional linear independent rows in comparison with (22). Correspondingly, we have the Casimir functions (31) and possibly the three others conjectured to exist for Case B.

On $\mathfrak{r}^*_-$ we still have $3C_2 = C_1^2$ and $27C_3^2 = C_1^3$, whence the symplectic leaves are given by the level sets of C_3 and the two Casimirs

$$\begin{aligned} C_{10} &= \frac{3(\Delta \times \Sigma) \cdot \Gamma}{\Delta^2 + \Sigma^2 + \Gamma^2} \left((p_A)^2 \ + \ (p_B)^2\right) \ + \ 2p_B^T(\Delta, \Sigma, \Gamma)p_A \\ C_{11} &\approx \pi_A \cdot p_A \ + \ p_B^T(\Delta, \Sigma, \Gamma)\pi_A \ + \ \pi_B^T(\Delta, \Sigma, \Gamma)p_A \\ &\quad + \ \pi_B \cdot p_B \ + \ ((\Delta, \Sigma, \Gamma)\beta) \cdot (\pi_B \times (\Delta, \Sigma, \Gamma)p_A) \quad . \end{aligned}$$

We stress that C_{10} and C_{11} are Casimirs only on the Poisson manifold $\mathfrak{r}^*_-$, but not on the whole space $\mathfrak{s}^*_-$. They simplify to (11) when $(K, k) = (I_3, 0)$.

5 Controlled equations of motion

The equations of motion are obtained taking the Poisson bracket with the Hamiltonian, which is the sum of the kinetic and the potential energy. As an example, the kinetic energy is simply that of two satellites (in Case A) or underwater vehicles (in Cases B and C). The potential energy incorporates our control, by defining it on the reduced phase space $\mathfrak{t}^*_-$, as the restriction of some formal potential energy on $\mathfrak{s}^*_-$, we ensure that only the relative orientation (and relative position in Case C) enters.

5.1 Case A

Here we consider the problem of aligning two satellites which have purely rotational dynamics. We model each satellite as a free rigid body with moment of inertia matrix J, and write Ω_k for the angular velocity vector of the body k with respect to its body-fixed frame.

In this case, the kinetic energy of the system is merely rotational. In the unreduced space, $SO(3) \times SO(3)$, we write it as

$$T = \frac{1}{2}(\Omega_A^T J \Omega_A + \Omega_B^T J \Omega_B). \tag{35}$$

To enforce orientation matching we use the control with potential energy

$$V = \sigma \sum_{i=1}^{3} e_i^T R_A^T K R_B e_i \tag{36}$$

where σ is a scalar control gain. For $K = I_3$ and negative σ, an aligned orientation $R_A = R_B$ then minimizes the potential energy, as can easily be seen by expressing V in terms of quaternions. Additionally, as one can easily check, the Hamiltonian of the system will be invariant to all (lifted) group actions F_K such that (14) is satisfied. The reduction story then proceeds as in §3.1.

We may then express the potential and kinetic energies in the reduced variables as

$$T = \frac{1}{2}\left[\pi_A^T J^{-1}\pi_A + \pi_B^T J^{-1}\pi_B\right]$$

and

$$V = \sigma\left(\Delta\cdot e_1 + \Sigma\cdot e_2 + \Gamma\cdot e_3\right) \quad , \tag{37}$$

respectively. Writing $H = T + V$ and substituting into (21), one finds that the reduced equations of motion are

$$\begin{aligned}
\dot{\pi}_A &= \pi_A \times J^{-1}\pi_A - \sigma\left(\Delta\times e_1 + \Sigma\times e_2 + \Gamma\times e_3\right)\\
\dot{\pi}_B &= \pi_B \times J^{-1}\pi_B + \sigma\left(\Delta\times e_1 + \Sigma\times e_2 + \Gamma\times e_3\right)\\
\dot{\Delta} &= \frac{\pi_A^3}{J_3}\Sigma - \frac{\pi_A^2}{J_2}\Gamma + \Delta\times J^{-1}\pi_B\\
\dot{\Sigma} &= \frac{\pi_A^1}{J_1}\Gamma - \frac{\pi_A^3}{J_3}\Delta + \Sigma\times J^{-1}\pi_B\\
\dot{\Gamma} &= \frac{\pi_A^2}{J_2}\Delta - \frac{\pi_A^1}{J_1}\Sigma + \Gamma\times J^{-1}\pi_B \quad .
\end{aligned} \tag{38}$$

The control torque applied to body A is thus computed to be

$$u_A = -\,\sigma\left(\Delta\times e_1 + \Sigma\times e_2 + \Gamma\times e_3\right)$$

and the control torque applied to body B is $u_B = -u_A$.

One can easily check that the Casimir functions, given in (31) and (33), are conserved under the flow of this dynamical system.

5.2 Case B

Here we consider the case of two underwater vehicles as an example of a pair of bodies with translational as well as rotational dynamics. We assume they are identical and model each as an ellipsoidal body of mass m, as in [6]. For each vehicle, the matrix J will now denote the sum of the body inertia and the *added inertia* from the potential flow model of the fluid. Similarly, let M denote the sum of the body mass m multiplied by the identity matrix and the *added mass matrix*. We assume that m is also the mass of the displaced fluid so that each vehicle is neutrally buoyant. If each ellipsoid has uniformly distributed mass, the center of buoyancy is coincident with the center of gravity and both J and M are diagonal in a coordinate system defined by the ellipsoid's principal axes. We shall denote the translational velocity of each vehicle in body coordinates as v_k, and write $M = diag(m_1, m_2, .m_3)$, assuming $m_3 \geq m_2 > m_1$.

In this case, the kinetic energy is now written as

$$T = \frac{1}{2}(\Omega_A^T J\Omega_A + \Omega_B^T J\Omega_B + v_A^T M v_A + v_B^T M v_B) \quad , \tag{39}$$

while we continue to write the control potential energy as (36). Accordingly the reduction story here proceeds as outlined in §3.2. We then express the kinetic energy, in terms of the reduced variables, as

$$T = \frac{1}{2}\left[\pi_A^T J^{-1}\pi_A + p_A^T M^{-1} p_A + \pi_B^T J^{-1}\pi_B + p_B^T M^{-1} p_B\right], \tag{40}$$

and we use again (37) as potential energy. Writing $H = T + V$ and substituting into (23), one finds that the reduced equations of motion become

$$\begin{aligned}
\dot{\pi}_A &= \pi_A \times J^{-1}\pi_A + p_A \times M^{-1} p_A - \sigma\left(\Delta \times e_1 + \Sigma \times e_2 + \Gamma \times e_3\right) \\
\dot{p}_A &= p_A \times J^{-1}\pi_A \\
\dot{\pi}_B &= \pi_B \times J^{-1}\pi_B + p_B \times M^{-1} p_B + \sigma\left(\Delta \times e_1 + \Sigma \times e_2 + \Gamma \times e_3\right) \\
\dot{p}_B &= p_B \times J^{-1}\pi_B \\
\dot{\Delta} &= \frac{\pi_A^3}{J_3}\Sigma - \frac{\pi_A^2}{J_2}\Gamma + \Delta \times J^{-1}\pi_B \\
\dot{\Sigma} &= \frac{\pi_A^1}{J_1}\Gamma - \frac{\pi_A^3}{J_3}\Delta + \Sigma \times J^{-1}\pi_B \\
\dot{\Gamma} &= \frac{\pi_A^2}{J_2}\Delta - \frac{\pi_A^1}{J_1}\Sigma + \Gamma \times J^{-1}\pi_B \quad .
\end{aligned} \tag{41}$$

The control inputs are the same torques as in Case A.

5.3 Case C

Here we focus attention on the case of the two underwater vehicles of Case B where we now seek to add a control to stabilise the configuration where both vehicles are not only aligned, but also separated by a prescribed relative position vector d_{AB}, directed from body A to body B.

In this case, the kinetic energy is again given by (39) and we now use the control with potential energy

$$\begin{aligned}
V &= \frac{1}{2}\left(R_A^T(R_A k + b_A - K b_B) + d_{AB}\right)^T C\left(R_A^T(R_A k + b_A - K b_B) + d_{AB}\right) \\
&\quad + \sigma \sum_{i=1}^{3} e_i^T R_A^T K R_B e_i \qquad ,
\end{aligned}$$

where we have made an arbitrary choice in expressing the inter-vehicle distance in terms of the body frame of vehicle A. We observe that for $(K, k) = (I_3, 0)$, positive definite matrix C and $\sigma < 0$, the aligned configuration $R_A = R_B$ with the vehicles having the desired relative position $d_{AB} = -R_A^T(b_A - K b_B)$ then minimizes the potential energy. One can easily check that the Hamiltonian of the system, given by the sum of $T + V$, will be invariant to all (lifted) group actions F_K such that (25) is satisfied. The reduction story then proceeds as in §3.3.

We express the reduced Hamiltonian as the sum of the kinetic energy (40) and the potential energy

$$V \;=\; \frac{1}{2}\,(\beta + d_{AB})^T\,C\,(\beta + d_{AB}) \;+\; \sigma\,(\Delta\cdot e_1 \,+\, \Sigma\cdot e_2 \,+\, \Gamma\cdot e_3) \quad , \tag{42}$$

and then use (29) to write the reduced equations of motion as

$$\begin{aligned}
\dot{\pi}_A &= \pi_A \times J^{-1}\pi_A \;+\; p_A \times M^{-1}p_A \;+\; \beta \times C(\beta + d_{AB}) \;-\; \sigma\,(\Delta\times e_1 \,+\, \Sigma\times e_2 \,+\, \Gamma\times e_3)\\
\dot{p}_A &= p_A \times J^{-1}\pi_A \;+\; C(\beta + d_{AB})\\
\dot{\pi}_B &= \pi_B \times J^{-1}\pi_B \;+\; p_B \times M^{-1}p_B \;+\; \sigma\,(\Delta\times e_1 \,+\, \Sigma\times e_2 \,+\, \Gamma\times e_3)\\
\dot{p}_B &= p_B \times J^{-1}\pi_B \;-\; (\Delta,\Sigma,\Gamma)\,C\,(\beta + d_{AB})\\
\dot{\Delta} &= \frac{\pi_A^3}{J_3}\Sigma \;-\; \frac{\pi_A^2}{J_2}\Gamma \;+\; \Delta \times J^{-1}\pi_B\\
\dot{\Sigma} &= \frac{\pi_A^1}{J_1}\Gamma \;-\; \frac{\pi_A^3}{J_3}\Delta \;+\; \Sigma \times J^{-1}\pi_B\\
\dot{\Gamma} &= \frac{\pi_A^2}{J_2}\Delta \;-\; \frac{\pi_A^1}{J_1}\Sigma \;+\; \Gamma \times J^{-1}\pi_B\\
\dot{\beta} &= \beta \times J^{-1}\pi_A \;-\; M^{-1}p_A \;+\; (\Delta,\Sigma,\Gamma)^T M^{-1}p_B \quad .
\end{aligned} \tag{43}$$

Here the control torque applied to body A is thus computed to be

$$u_A = \beta \times C(\beta + d_{AB}) - u_B,$$

where the control torque applied to Body B is, as in Cases A and B,

$$u_B = \;\sigma\,(\Delta\times e_1 \,+\, \Sigma\times e_2 \,+\, \Gamma\times e_3)\,.$$

Additionally, we find there are control forces

$$f_A = C(\beta + d_{AB}), \qquad f_B = -(\Delta,\Sigma,\Gamma)C(\beta + d_{AB}),$$

acting on bodies A and B, respectively.

6 Equilibria of the reduced system

The first step in analysing the closed-loop dynamical systems defined by (38), (41) and (43) consists in identifying the simplest possible motions: the (relative) equilibria. In the interest of avoiding a great deal of unwieldy algebra, we shall only exhaustively discuss the equilibria for Case A.

6.1 Case A

Careful consideration of the last three equations of (38) indicates that, as might be expected, all equilibria in this case are non-generic, *i.e.* they are contained in the lower-dimensional symplectic leaf $\pi_A = \pi_B$. Let us set $\pi_0 = \pi_A = \pi_B$.

Since the control torques of bodies A and B are equal and opposite, it is clear that for real equilibria to exist we require

$$\pi_0 \times J^{-1}\pi_0 \quad = \quad 0 \quad . \tag{44}$$

For bodies with neither spherical nor axial symmetry, elementary considerations reveal that no real solutions exist for this equation with all three components of π_0 different from zero. While real solutions to (44) exist with merely one component of π_0 equal to zero, it is not possible to simultaneously satisfy the equations $\dot{\Delta} = 0$, $\dot{\Sigma} = 0$, $\dot{\Gamma} = 0$ (since this requires two of Δ, Σ and Γ to be parallel when, by definition, they must be orthogonal). We must therefore have two components of π_0 equal to zero.

Additionally, an equilibrium in this system requires that

$$\Delta \times e_1 \,+\, \Sigma \times e_2 \,+\, \Gamma \times e_3 = \begin{pmatrix} \Gamma_2 - \Sigma_3 \\ \Delta_3 - \Gamma_1 \\ \Sigma_1 - \Delta_2 \end{pmatrix} = 0 \tag{45}$$

and, consequently, the matrix (Δ, Σ, Γ) must be symmetric.

We use the unit quaternion representation for the matrix $(\Delta, \Sigma, \Gamma) \in SO(3)$ to rewrite its columns as:

$$\begin{aligned}
\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ 2(q_1 q_2 - q_0 q_3) \\ 2(q_1 q_3 + q_0 q_2) \end{bmatrix} \\
\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{bmatrix} &= \begin{bmatrix} 2(q_1 q_2 + q_0 q_3) \\ q_0^2 - q_1^2 + q_2^2 - q_3^2 \\ 2(q_2 q_3 - q_0 q_1) \end{bmatrix} \\
\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} &= \begin{bmatrix} 2(q_1 q_3 - q_0 q_2) \\ 2(q_2 q_3 + q_0 q_1) \\ q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}
\end{aligned}$$

where the q_j are real numbers such that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

The condition that (Δ, Σ, Γ) is symmetric is thus equivalent to stating that either $q_1 = q_2 = q_3 = 0$, in which case (Δ, Σ, Γ) is the identity matrix and the bodies A and B are aligned or, $q_0 = 0$, in which case the bodies need not be aligned.

For the case where the bodies are aligned we may immediately write down the form of two equilibria. This simplest possible case consists of the two bodies being stationary and aligned,

$$(\pi_A, \pi_B, \Delta, \Sigma, \Gamma) \quad = \quad (0, 0, e_1, e_2, e_3), \tag{46}$$

while the next simplest consists of the two rigid bodies spinning synchronously about their ith axis of inertia, i.e.

$$(\pi_A, \pi_B, \Delta, \Sigma, \Gamma) \;=\; (\pi_0 e_i, \pi_0 e_i, e_1, e_2, e_3). \tag{47}$$

. For the non-aligned configurations we introduce spherical coordinates

$$\begin{aligned} q_1 &= \sin\alpha\,\cos\beta \\ q_2 &= \sin\alpha\,\sin\beta \\ q_3 &= \cos\alpha \end{aligned}$$

to write these equilibria as

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} \sin^2\alpha\,\cos(2\beta) - \cos^2\alpha \\ \sin^2\alpha\,\sin(2\beta) \\ \sin(2\alpha)\cos\beta \end{bmatrix} \tag{48}$$

$$\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{bmatrix} = \begin{bmatrix} \sin^2\alpha\,\sin(2\beta) \\ -\sin^2\alpha\,\cos(2\beta) - \cos^2\alpha \\ \sin(2\alpha)\sin\beta \end{bmatrix} \tag{49}$$

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} = \begin{bmatrix} \sin(2\alpha)\cos\beta \\ \sin(2\alpha)\sin\beta \\ \cos(2\alpha) \end{bmatrix} \tag{50}$$

For the case of two stationary rigid bodies in a non-aligned configuration, there are no additional restrictions on the columns of the relative rotational position matrix given in (48)-(50). In contrast, for two bodies rotating and non-aligned with, say, $\pi_A = \pi_B = \pi_0 e_i$, we see from (38) that $(\Delta, \Sigma, \Gamma)^T e_i$ must be parallel to e_i and, hence, from (48)-(50), the relative rotation matrix is of the form:

$$(\Delta, \Sigma, \Gamma) = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{51}$$

where one of the ϵ_i is positive and the other two are negative. This configuration corresponds to one axis of the two bodies A and B being in alignment, while the other two axes are out of alignment by 180 degrees. This equilibrium therefore corresponds to the two bodies 'pointed' in opposite directions.

6.2 Cases B & C

The principal equilibrium of interest for Case B is of the form

$$(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma) = (0, C_A e_i, 0, C_B e_j, e_1, e_2, e_3), \tag{52}$$

while that for Case C is of the form

$$(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma, \beta) = (0, C e_i, 0, C e_i, e_1, e_2, e_3, -d_{AB}), \tag{53}$$

We shall prove the stability of the equilibrium (52) in §7.2.

7 Stability of Equilibria

We now obtain results on the stability of the equilibria of the controlled network dynamics found in the previous section. Where possible, we use the energy-Casimir method [9] to prove stability for the aligned equilibria of Cases A and B. We note that the application of the energy-Casimir method does not require any knowledge of the Poisson structure other than (some of) the Casimirs. In particular, only the partial derivatives of these (and of the Hamiltonian) within the flat space $\mathfrak{s}^*_-$ are needed, while the precise way in which the reduced manifold $\mathfrak{t}^*_-$ lies within $\mathfrak{s}^*_-$ is relevant only to the extent that this determines the Casimirs.

7.1 Case A

We claim that the equilibria of the form (46) are stable in the sense of Lyapunov. To find a Lyapunov function, we make the ansatz

$$H_\Phi \;=\; H \;+\; \Phi(C_1, C_2, C_3) \quad ,$$

where have included only those Casimirs (31), which are valid on all of $\mathfrak{s}^*_-$, as arguments of Φ.

Let $\Phi^{(l)}$ be the derivative of Φ with respect to C_l at the equilibrium (46). The first variation of H_Φ will vanish at the equilibrium provided that $\Phi^{(1)} = -\sigma/2$ and $\Phi^{(2)} = \Phi^{(3)} = 0$. We may thus choose Φ to have the simple form

$$\Phi \;=\; -\frac{\sigma}{2}(\Delta^2 + \Sigma^2 + \Gamma^2) \quad .$$

We now check for definiteness of the second variation at the equilibrium point. The Hessian of H_Φ has an associated matrix given by

$$\begin{bmatrix} J^{-1} & & & & \\ & J^{-1} & & & \\ & & -\sigma I_3 & & \\ & & & -\sigma I_3 & \\ & & & & -\sigma I_3 \end{bmatrix}$$

The Hessian is thus positive definite provided $\sigma < 0$. Stability follows by the energy-Casimir method.

The other equilibria in Case A require $\pi_A = \pi_B$ which implies they are non-generic equilibria. The energy-Casimir method does not apply to non-generic equilibria. Stability results are proved for these cases using alternative energy methods in [13].

7.2 Case B

We claim that the equilibria of the form (52) with $i = j = 3$ are stable in the sense of Lyapunov. Here we make the ansatz

$$H_\Phi \;=\; H \;+\; \Phi(C_1, C_2, C_3, C_4, C_5, C_6) \quad ,$$

where we now include the Casimirs (34) , which are valid on all $\mathfrak{s}_-^*$, as arguments of Φ. The first variation of H_Φ will vanish at the equilibrium provided that

$$\begin{aligned} \frac{C_A}{m_3} \;+\; 2C_A\Phi^{(4)} \;+\; C_B\Phi^{(6)} &= 0 \\ \frac{C_B}{m_3} \;+\; 2C_B\Phi^{(5)} \;+\; C_A\Phi^{(6)} &= 0 \\ \sigma \;+\; 2\Phi^{(1)} \;+\; 4\Phi^{(2)} \;+\; \Phi^{(3)} &= 0 \\ \sigma \;+\; C_AC_B\Phi^{(6)} \;+\; 2\Phi^{(1)} \;+\; 4\Phi^{(2)} \;+\; \Phi^{(3)} &= 0 \quad . \end{aligned}$$

This implies $\Phi^{(6)} = 0$, and taking $\Phi^{(2)} = \Phi^{(3)} = 0$ we may choose Φ to have the simple form

$$\Phi \;=\; ((p_A)^2 - C_A^2)^2 \;-\; \frac{(p_A)^2}{2m_3} \;+\; ((p_B)^2 - C_B^2)^2 \;-\; \frac{(p_B)^2}{2m_3} \;-\; \frac{\sigma}{2}(\Delta^2 + \Sigma^2 + \Gamma^2) \quad . \tag{54}$$

Here the Hessian of H_Φ has an associated matrix given by

$$\begin{bmatrix} J^{-1} & & & & & & \\ & \widetilde{M}_A & & & & & \\ & & J^{-1} & & & & \\ & & & \widetilde{M}_B & & & \\ & & & & -\sigma I_3 & & \\ & & & & & -\sigma I_3 & \\ & & & & & & -\sigma I_3 \end{bmatrix} \tag{55}$$

where

$$\widetilde{M}_k = \begin{bmatrix} 1/m_1 - 1/m_3 & & \\ & 1/m_2 - 1/m_3 & \\ & & 8\,C_k^2 \end{bmatrix}$$

for $k = A, B$, and is thus again positive definite provided $\sigma < 0$. As before, stability follows by the energy-Casimir method.

8 Systems of N vehicles

The case of N underwater vehicles, when considered as $N(N-1)/2$ possible pairwise interactions, can be treated as a natural extension of the two-vehicle problem. We label each of our N vehicles with an index (i) for $i = 1, \dots N$, and now seek an appropriate artificial potential to stabilise the equilibrium $R_{(j)} = K_{(jk)}R_{(k)}$ for all pairs j, k satisfying

$$1 \leq j < k \leq N \tag{56}$$

where $K_{(jk)} \in \mathrm{SO}(3)$ is a (matrix) parameter relating the orientation of body (j) to that of body (k). For consistency, we must include the requirement that

$$K_{(ji)}K_{(ik)} = K_{(jk)} \quad \forall\; j, k \quad \text{and} \quad j < i < k.$$

This requirement will be trivially satisfied in the case

$$K_{(jk)} = I_3 \quad \forall\, j, k \quad \text{satisfying} \quad (56).$$

As in §3, this choice corresponds to our desire to stabilize the equilibrium consisting of all N vehicles having the same orientation. We note that it is not necessary to include potential functions for all possible $N(N-1)/2$ pairwise interactions in order to stabilise this equilibrium. In the language of Graph Theory, all possible pairwise would correspond to a complete graph, where each of the vehicles is represented by a node on the graph, and the presence of a potential function coupling the orientations of any two vehicles may be described as an *undirected* link in the graph. Rather than this complete graph of all $N(N-1)/2$ pairwise interactions, we merely need at least $N-1$ of them to ensure that the graph is *connected*.

The form of our artificial potential, V_p, is given by

$$V_p = \sum_{(j,k)\in\mathfrak{J}} \sigma_{jk} \sum_{i=1}^{3} e_i^T R_{(j)}^T K_{(jk)} R_{(k)} e_i, \tag{57}$$

where $\mathfrak{J}$ is a set of $\mathfrak{n}$ ordered pairs (j,k) with $N-1 < \mathfrak{n} < N(N-1)/2$, such that the associated graph is connected.

An analysis similar to that in §3 shows the Lagrangian and thus the Hamiltonian to be invariant provided

$$\bar{R}_{(j)}^T K_{(jk)} \bar{R}_{(k)} = K_{(jk)}$$

for all $(j,k) \in \mathfrak{J}$.

We define for each $K_{(jk)}$ an associated vector, $\widetilde{K}_{(jk)} = (e_1^T K_{(jk)}, e_2^T K_{(jk)}, e_3^T K_{(jk)})^T \in \mathbb{R}^9$. In order to proceed with the semi-direct product reduction we consider $\widetilde{K} \in \mathbb{R}^{9\mathfrak{n}}$ defined by

$$\widetilde{K} = (\widetilde{K}_{(hi)}^T, \ldots, \widetilde{K}_{(jk)}^T, \ldots, \widetilde{K}_{(lm)}^T), \tag{58}$$

where $(h,i,),(j,k)$, and $(l,m) \in \mathfrak{J}$, to be a "global" parameter for this problem.

We then define $\rho : \mathrm{SE}(3)^N \to \mathrm{GL}(\mathbb{R}^{9\mathfrak{n}})$ by

$$\rho(R_{(1)}, b_{(1)}, \ldots, R_{(N)}, b_{(N)})\widetilde{K} = (\widetilde{\kappa}_{(hi)}, \ldots, \widetilde{\kappa}_{(jk)}, \ldots, \widetilde{\kappa}_{(lm)}) \tag{59}$$

where $\widetilde{\kappa}_{(jk)} = (R_{(j)} \otimes R_{(k)})\widetilde{K}_{(jk)}$, again using (15).

It can then be checked that the Hamiltonian, consisting of the sum of the individual body kinetic energies and (57), will be invariant to all isotropy groups F_K analogous to (17).

With these definitions, the semi-direct product reduction follows through in much the same way as in §3. In particular, the resulting structure matrix will be block diagonal with each block of the form (22). Most importantly, one can use the reduced Hamiltonian to show, using the energy Casimir method, that the equilibrium consisting of all bodies aligned will be stable provided all the control gains σ_{jk} are positive. This proof is very simple since the Hessian of the augmented Hamiltonian, $H_\Phi = H + \Phi$, where Φ is a sum of terms of the form (54), is again block diagonal where each block is of the form (55). This inherent block diagonality ensures that our control law is *scalable* (with number of bodies N).

9 Final Remarks

We have presented reduction for coordinated rigid body networks in which control inputs depend only on relative configurations. The semi-direct product reduction provides flat spaces from which we can efficiently formulate control laws. We have applied the reduction results to several cases of interest including orientation alignment of a cluster of rigid satellites and coordination of a network of underwater vehicles.

We note that the control law presented in §5 for orientation matching is related to those of [1, 14] constructed for the asymptotic tracking of a desired attitude for satellites and helicopters. The control laws in these works consist of state feedback laws which required cancellation of the natural dynamics of those vehicles to achieve asymptotic stabilisation. In contrast, our control law is derived from the addition of a potential function designed to preserve the Hamiltonian nature of the system (and therefore the natural dynamics of the system); this Hamiltonian structure was used to prove stabilitiy of the aligned configuration in §7.

We also point out that the stabilization proof of §7.2 applies to two aligned underwater vehicles with each moving along its short (stable) axis. More work is necessary to stably coordinate two (or more) aligned underwater vehicles with each moving along its long (unstable) axis. The problem of stable coordination of a network of mechanical systems of a particular class, each with otherwise unstable dynamics, was addressed in [12]; stable coordination was proved for a controlled network of carts, each moving steadily while balancing an inverted pendulum.

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