Regularities of ground states of quantum field models

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Abstract

Regularities and higher order regularities of ground states of quantum field models are investigated through the fact that asymptotic annihilation operators vanish ground states. Moreover a sufficient condition for the absence of a ground state is given.

1 Introduction

A basic object in a quantum field model is a ground state of it which is defined to be an eigenvector of the Hamiltonian H (a self-adjoint operator on a Hilbert space) of the model with eigenvalue equal to the infimum $E(H)$ of the spectrum of H, provided that $E(H)$ is an eigenvalue of H. In this paper we investigate regularities of ground states of quantum field models. Here we mean by regularities properties of what class of subspaces ground states belong to, including absence of ground states in certain subspaces.

As is well known, existence of a ground state of a quantum field model may depend on properties of objects contained in its Hamiltonian such as a one-particle energy function and cutoff functions. In particular, it is subtle in the case where the quantum field is massless ([5, 2, 9, 10, 11, 12, 18] and references therein), being related to the so-called infrared divergence [20]. From this point of view, it is important to characterize regularities of ground states, in particular, absence of them, in terms of objects contained in the Hamiltonian of the model under consideration, as generally as possible. This is the main motivation of this work. Preliminary work concerning this theme was done in a previous paper [5], where the absence of ground states of an abstract and general model, called the *generalized spin boson* (GSB) model, was

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discussed. In the present work we extend results obtained in [5] to a more general class of quantum field models, establishing new criteria for regularities of ground states.

This paper is organized as follows. In Section 2 we define the quantum field model to be considered. We prove general results on regularities of ground states of the model. Section 3 is concerned with absence of ground states of the model. In Section 4 we consider higher order regularities of ground states, where higher order regularities means properties that ground states belong to smaller subspaces indexed by powers of a nonnegative self-adjoint operator. In Section 5 we apply the general results established in the previous sections to the GSB model and obtain results which extend those in [5]. In the last section we give some remarks on other quantum field models in view of the present work.

2 Regularities of ground states: general aspects

2.1 Fock spaces and second quantizations

Let $\mathcal{K}$ be a separable Hilbert space over complex field $\mathbb{C}$, and $\otimes_s^n\mathcal{K}$ denote the n-fold symmetric tensor product of $\mathcal{K}$ with $\otimes_s^0\mathcal{K} := \mathbb{C}$. The norm and the scalar product on Hilbert space $\mathcal{X}$ are denoted by $\|f\|_{\mathcal{X}}$ and $(f,g)_{\mathcal{X}}$, $f, g \in \mathcal{X}$, respectively, where $(f,g)_{\mathcal{X}}$ is anti-linear in f and linear in g. The norm of bounded operator from $\mathcal{X}$ to a Hilbert space $\mathcal{Y}$ is denoted by $\|X\|_{\mathcal{X}\to\mathcal{Y}}$ and the domain of unbounded operator Y is by $D(Y)$. The Boson Fock space over $\mathcal{K}$ is defined by

$$\mathcal{F}_{\mathrm{b}}(\mathcal{K}) := \bigoplus_{n=0}^{\infty}[\otimes_s^n\mathcal{K}] = \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \|\Psi\|^2_{\mathcal{F}_{\mathrm{b}}(\mathcal{K})} := \sum_{n=0}^{\infty}\|\Psi^{(n)}\|^2_{\otimes_s^n\mathcal{K}} < \infty\}.$$

The Fock vacuum is defined by

$$\Omega := \{1, 0, 0, ...\} \in \mathcal{F}_{\mathrm{b}}(\mathcal{K})$$

and the finite particle subspace of $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ by

$$\mathcal{F}_{\mathrm{b},0}(\mathcal{K}) := \left\{\{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}}(\mathcal{K}) | \Psi^{(n)} = 0 \text{ for all } n \geq n_0 \text{ with some } n_0\right\}.$$

It is known that $\mathcal{F}_{\mathrm{b},0}(\mathcal{K})$ is dense in $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$. The annihilation operator $a(f)$ with $f \in \mathcal{K}$ is defined to be a densely defined closed operator on $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ whose adjoint is given by

$$(a(f)^*\Psi)^{(n)} := \sqrt{n}S_n(f \otimes \Psi^{(n-1)}), \quad \Psi \in D(a(f)^*),$$

where S_n denotes the symmetrization operator on $\otimes^n\mathcal{K}$, i.e., $S_n(\otimes^n\mathcal{K}) = \otimes_s^n\mathcal{K}$. We note that $a(f)$ is anti-linear in f and $a^*(g)$ linear in g. The opertors $a(f)$ and $a^*(f)$

leave $\mathcal{F}_{\mathrm{b},0}(\mathcal{K})$ invariant and satisfy canonical commutation relations on $\mathcal{F}_{\mathrm{b},0}(\mathcal{K})$:

$$[a(f), a^*(g)] = (f, g)_{\mathcal{K}}, \tag{2.1}$$
$$[a(f), a(g)] = 0, \tag{2.2}$$
$$[a^*(f), a^*(g)] = 0. \tag{2.3}$$

Since $a(f)$ and $a^*(f)$ are closable operators, their closed extensions are denoted by the same symbols, respectively. Define $\mathcal{F}^{\mathcal{D}}_{\mathrm{b},0}(\mathcal{K})$ with subspace $\mathcal{D} \subset \mathcal{K}$ by the finite linear hull of

$$\{a^*(f_1)\cdots a^*(f_n)\Omega, \Omega | f_j \in D, j = 1, ..., n, n \geq 1\}.$$

It is known that $\mathcal{F}^{D}_{\mathrm{b},0}(\mathcal{K})$ is dense in $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ if D is dense in $\mathcal{K}$. Let C be a closed operator on $\mathcal{K}$. Define $d\Gamma_n(C) : \otimes^n_s \mathcal{K} \to \otimes^n_s \mathcal{K}$ by

$$d\Gamma_n(C) := \sum_{j=1}^{n} \underbrace{1 \otimes 1 \cdots 1 \otimes \overset{j}{\breve{C}} \otimes 1 \cdots 1 \otimes 1}_{n}, \quad n \geq 1,$$

and $d\Gamma_0(C) : \mathbb{C} \to \mathbb{C}$ by

$$d\Gamma_0(C)z = 0, \quad z \in \mathbb{C}.$$

The second quantization of C is the operator defined by

$$d\Gamma(C) := \bigoplus_{n=0}^{\infty} d\Gamma_n(C)$$

with $D(d\Gamma(C)) := \mathcal{F}^{D(C)}_{\mathrm{b},0}(\mathcal{K})$. Note that

$$\begin{aligned} &d\Gamma(C)\Omega = 0 \\ &d\Gamma(C)a^*(f_1)\cdots a^*(f_n)\Omega = \sum_{j=1}^{n} a^*(f_1)\cdots a^*(Cf_j)\cdots a^*(f_n)\Omega, \\ &\qquad\qquad\qquad\qquad\qquad\qquad f_j \in D(C), \quad j = 1, ..., n. \end{aligned}$$

For a nonnegative self-adjoint operator K, $d\Gamma(K)$ is a nonnegative essentially self-adjoint operator. We denote its self-adjoint extension by the same symbol, $d\Gamma(K)$. $d\Gamma(1)$ is refered to as the number operator, which is denoted by

$$N := d\Gamma(1).$$

It is known that, for all $\Psi \in D(d\Gamma(K)^{1/2})$ and $f \in D(K^{-1/2})$,

$$\|a(f)\Psi\|^2_{\mathcal{F}_{\mathrm{b}}(\mathcal{K})} \leq \|K^{-1/2}f\|^2_{\mathcal{K}} \|d\Gamma(K)^{1/2}\Psi\|^2_{\mathcal{F}_{\mathrm{b}}(\mathcal{K})}, \tag{2.4}$$
$$\|a^*(f)\Psi\|^2_{\mathcal{F}_{\mathrm{b}}(\mathcal{K})} \leq \|K^{-1/2}f\|^2_{\mathcal{K}} \|d\Gamma(K)^{1/2}\Psi\|^2_{\mathcal{F}_{\mathrm{b}}(\mathcal{K})} + \|f\|^2_{\mathcal{K}} \|\Psi\|^2_{\mathcal{F}_{\mathrm{b}}(\mathcal{K})}. \tag{2.5}$$

Lemma 2.1 *Let $0 \leq \epsilon < 1$ and n be a non-negative integer. Let $f \in D(K^{-1/2}) \cap D(K^{n+1})$. Then $a^\sharp(f)$ maps $D(d\Gamma(K)^{n+\epsilon+1/2})$ into $D(d\Gamma(K)^{n+\epsilon})$. In particular, it follows that for $\Psi \in D(d\Gamma(K)^{n/2})$ and $f_j \in D(K^{-1/2}) \cap D(K^{[n/2]})$, $j = 1, ..., n$,*

$$\Psi \in D(a^\sharp(f_1) \cdots a^\sharp(f_n)),$$

where $a^\sharp$ denotes a or a^*, and $[n/2]$ the integer part of $n/2$. Moreove it follows that

$$[d\Gamma(K), a(f)]\Psi = -a(Kf), \quad \Psi \in D(d\Gamma(K)^{3/2}), \quad f \in D(K),$$
$$[d\Gamma(K), a^*(f)]\Psi = a^*(Kf), \quad \Psi \in D(d\Gamma(K)^{3/2}), \quad f \in D(K).$$

Proof: See [1, Lemmas 2.3 and 2.5]. □

2.2 Total Hamiltonians

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. Then one can make the Hilbert space

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\rm b}(\mathcal{K}).$$

Let A be a self-adjoint operator bounded from below on $\mathcal{H}$ and S a nonnegative self-adjoint operator on $\mathcal{H}$. The decoupled Hamiltonian

$$H_0 := A \otimes 1 + 1 \otimes d\Gamma(S)$$

is self-adjoint on

$$D(H_0) := D(A \otimes 1) \cap D(1 \otimes d\Gamma(S))$$

and bounded from below. Let $H_{\rm I}$ be a symmetric operator on $\mathcal{F}$ such that $D(H_0) \subset D(H_{\rm I})$. The total Hamiltonian under consideration is the symmetric operator

$$H := H_0 + gH_{\rm I}$$

on $\mathcal{F}$, where $g \in \mathbb{R}$ is a coupling constant. Assumption (A.1) is as follows.

(A.1) There exist constants $a \geq 0$ and $b \geq 0$ such that

$$\|H_{\rm I}\Psi\| \leq a\|H_0\Psi\| + b\|\Psi\|, \quad \Psi \in D(H_0).$$

Proposition 2.2 *Suppose (A.1). Then for g with $|g| < 1/a$, H is self-adjoint on $D(H_0)$ and bounded from below. Moreover it is essentially self-adjoint on any core of H_0.*

Proof: It follows from the Kato-Rellich theorem [22]. □

It is known that there exist a finite measure space $\langle M, \mu\rangle$, a nonnegative measurable function S on $\langle M, \mu\rangle$ and a unitary operator $U : \mathcal{K} \to L^2(M) := L^2(M, d\mu)$ such that

(1) $\Psi \in D(S)$ if and only if $S(\cdot)(U\Psi)(\cdot) \in L^2(M)$,

(2) $(USU^{-1}\Psi)(k) = S(k)\Psi(k)$ for $\Psi \in UD(S)$ for almost every $k \in M$.

Hence, without loss of generality, we can take $\mathcal{K}$ to be an L^2-space. Thus, in what follows, we set

$$\mathcal{K} = L^2(M, d\mu),$$

where we do not assume that μ is a finite measure, but σ-finite measure, and take S to be a multiplication operator on $\mathcal{K}$ by a non-negative function $S(k)$ such that

$$0 < S(k) < \infty, \quad \mu - a.e.k.$$

We set

$$\mathcal{M}_m := D(S^m).$$

2.3 Regularities of ground states

We denote $\inf \sigma(K)$ by $E(K)$ for a self-adjoint operator K. Let us assume that a ground state $\varphi_{\rm g}$ of H exists. Then

$$H\varphi_{\rm g} = E(H)\varphi_{\rm g}.$$

We fix a normalized ground state $\varphi_{\rm g}$ of H, i.e., $\|\varphi_{\rm g}\|_{\mathcal{F}} = 1$. Let B and C be operators on a Hilbert space $\mathcal{W}$. We define a quadratic form $[B, C]_W^{\mathcal{D}}$ with form domain $\mathcal{D} \times \mathcal{D}$ such that

$$\mathcal{D} \subset D(B^*) \cap D(B) \cap D(C^*) \cap D(C),$$

by

$$[B, C]_W^{\mathcal{D}}(\Psi, \Phi) := (B^*\Psi, C\Phi)_{\mathcal{W}} - (C^*\Psi, B\Phi)_{\mathcal{W}}, \quad \Psi, \Phi \in \mathcal{D}.$$

Assumption (A.2) is as follows.

(A.2) There exists an operator

$$T(k) : \mathcal{F} \to \mathcal{F}, \quad a.e.\ k \in M,$$

such that

(1) $D(H) \subset D(T(k))$ for almost every $k \in M$,

(2) $T(k)\Phi$, $\Phi \in D(H)$, is weakly measurable in k,

(3) $[1 \otimes a(f), H_{\mathrm{I}}]_W^{D(H)}(\Psi, \Phi) = \int_M \overline{f(k)}(\Psi, T(k)\Phi)_{\mathcal{F}} d\mu(k)$ for $\Psi, \Phi \in D(H)$.

Define

$$a_t(f) := e^{-itH} e^{itH_0}(1 \otimes a(f)) e^{-itH_0} e^{itH} = e^{-itH}(1 \otimes a(e^{itS} f)) e^{itH}.$$

We want to investigate $a_t(f)$ as $t \to \infty$ strongly. Let $\sigma(S)$, $\sigma_{\mathrm{p}}(S)$ and $\sigma_{\mathrm{ac}}(S)$ be the spectrum, the point spectrum, and the absolutely continuous spectrum of S, respectively. Assumptions (A.3)-(A.5) are as follows.

(A.3) The operator S is purely absolutely continuous.

(A.4) There exists a dense subspace $\mathcal{C}_0 \subset L^2(M)$ such that

(1) $\mathcal{C}_0 \subset \mathcal{M}_{-1/2}$, and for any $f \in \mathcal{M}_{-1/2} \cap \mathcal{M}_0$, there exists a sequence $\{f_m\}_m \subset \mathcal{C}_0$ such that $\text{s-lim}_{m\to\infty} f_m = f$ and $\text{s-lim}_{m\to\infty} f_m/\sqrt{S} = f/\sqrt{S}$,

(2) for $f \in \mathcal{C}_0$ and $\Psi \in D(H)$,

$$\int_M \overline{f(k)}(\Psi, e^{-is(H-E(H)+S(k))} T(k)\varphi_{\mathrm{g}})_{\mathcal{F}} d\mu(k) \in L^1([0,\infty), ds).$$

(A.5) $\|T(\cdot)\varphi_{\mathrm{g}}\|_{\mathcal{F}} \in \mathcal{M}_{-1/2} \cap \mathcal{M}_0$.

Lemma 2.3 *Suppose (A.3). Then for any $a \in \mathbb{R}$,*

$$\mu(\{k \in M | S(k) = a\}) = 0.$$

Proof: Let $\mathcal{N} := \{k \in M | S(k) = a\}$ and suppose that $0 < \mu(\mathcal{N}) \le \infty$. Since μ is a σ-finite measure, $M = \cup_{n=1}^\infty M_n$ with $\mu(M_n) < \infty$ for all n. Then there exists m such that $\mu(M_m \cap \mathcal{N}) > 0$. Let $1_{M_m \cap \mathcal{N}}$ be the characteristic function on $M_m \cap \mathcal{N}$. Then $1_{M_m \cap \mathcal{N}} \in L^2(M)$ and $S 1_{M_m \cap \mathcal{N}} = a 1_{M_m \cap \mathcal{N}}$, which implies that $a \in \sigma_{\mathrm{p}}(S)$. It contradicts with (A.3). Thus $\mu(\mathcal{N}) = 0$. $\square$

Lemma 2.4 *Assume (A.1)-(A.5). Then for $f \in \mathcal{M}_{-1/2} \cap \mathcal{M}_0$,*

$$\int_M \|\overline{f(k)}(H - E(H) + S(k))^{-1} T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}} d\mu(k) < \infty$$

and

$$(1 \otimes a(f))\varphi_{\mathrm{g}} = -g \int_M \overline{f(k)}(H - E(H) + S(k))^{-1} T(k)\varphi_{\mathrm{g}} d\mu(k),$$

where the integral on the right-hand side above is in the strong sense.

Remark 2.5 **(1)** *Lemma 2.4 is a generalization of [11, 12].*

(2) *By Lemma 2.3, $S(k) \neq 0$ for almost every $k \in M$. Thus $(H - E(H) + S(k))^{-1}$ is a bounded operator for almost every $k \in M$.*

(3) *$T(k)\Phi$, $\Phi \in D(H)$, is strongly measurable since $T(k)\Phi$ is weakly measurable [21, Theorem IV.22].*

Proof: Note that

$$\begin{aligned} &\int_M \|\overline{f(k)}(H - E(H) + S(k))^{-1} T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}} d\mu(k) \\ &\leq \|f/\sqrt{S}\|_{L^2(M)} \left(\int_M \|T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2 / S(k) d\mu(k) \right)^{1/2} < \infty, \end{aligned} \tag{2.6}$$

and $\varphi_{\mathrm{g}} \in D(1 \otimes a(f))$ follows from the fact that $\varphi_{\mathrm{g}} \in D(1 \otimes d\Gamma(S))$ and Lemma 2.1. We see that for $\Psi, \Phi \in D(H)$ and for $f \in \mathcal{C}_0$,

$$\begin{aligned} \frac{d}{dt}(\Psi, a_t(f)\Phi)_{\mathcal{F}} &= ig[1 \otimes a(e^{itS} f), H_{\mathrm{I}}]_W^{D(H)} (e^{itH}\Psi, e^{itH}\Phi) \\ &= ig \int_M \overline{f(k)} e^{-itS(k)} (\Psi, e^{-itH} T(k) e^{itH} \Phi)_{\mathcal{F}} d\mu(k). \end{aligned}$$

Then

$$\begin{aligned} &(\Psi, a_t(f)\Phi)_{\mathcal{F}} \\ &= (\Psi, (1 \otimes a(f))\Phi)_{\mathcal{F}} + ig \int_0^t ds \left(\int_M \overline{f(k)} e^{-isS(k)} (\Psi, e^{-isH} T(k) e^{isH} \Phi)_{\mathcal{F}} d\mu(k) \right). \end{aligned} \tag{2.7}$$

Since, by (A.2),

$$\text{s-}\lim_{t \to \infty} a_t(f)\varphi_{\mathrm{g}} = 0, \quad f \in L^2(M),$$

it follows from (2.7) and (A.4) that

$$(\Psi, (1 \otimes a(f))\varphi_{\mathrm{g}})_{\mathcal{F}} = -ig \int_0^\infty ds \left(\int_M \overline{f(k)} (\Psi, e^{-is(H - E(H) + S(k))} T(k)\varphi_{\mathrm{g}})_{\mathcal{F}} d\mu(k) \right).$$

We have

$$\begin{aligned} &(\Psi, (1 \otimes a(f))\varphi_{\mathrm{g}})_{\mathcal{F}} \\ &= -ig \lim_{\epsilon \to 0} \int_0^\infty ds e^{-\epsilon s} \left(\int_M \overline{f(k)} (\Psi, e^{-is(H - E(H) + S(k))} T(k)\varphi_{\mathrm{g}})_{\mathcal{F}} d\mu(k) \right) \\ &= -g \int_M (\Psi, \overline{f(k)} (H - E(H) + S(k))^{-1} T(k)\varphi_{\mathrm{g}})_{\mathcal{F}} d\mu(k). \end{aligned}$$

Here on the first equality we used the Lebesgue dominated convergence theorem and (A.4)-(2), on the second equality, Fubini's theorem and (A.5). Hence by (2.6), for $f \in \mathcal{C}_0$,

$$(\Psi, (1 \otimes a(f))\varphi_{\rm g})_{\mathcal{F}} = (\Psi, -g \int_M \overline{f(k)}(H - E(H) + S(k))^{-1} T(k)\varphi_{\rm g} d\mu(k))_{\mathcal{F}}$$

and then

$$(1 \otimes a(f))\varphi_{\rm g} = -g \int_M \overline{f(k)}(H - E(H) + S(k))^{-1} T(k)\varphi_{\rm g}. \tag{2.8}$$

We can take a sequence $\{f_m\} \subset \mathcal{C}_0$ such that s-$\lim_{m\to\infty} f_m = f$ and s-$\lim_{m\to\infty} f_m/\sqrt{S} = f/\sqrt{S}$ for $f \in \mathcal{M}_{-1/2} \cap \mathcal{M}_0$. We see that

$$\text{s-}\lim_{m\to\infty} (1 \otimes a(f_m))\varphi_{\rm g} = (1 \otimes a(f))\varphi_{\rm g} \tag{2.9}$$

by

$$\|a(f)\Psi\|_{\mathcal{F}} \le \|S^{-1/2} f\|_{L^2(M)} \|d\Gamma(S)^{1/2}\Psi\|_{\mathcal{F}}, \quad \Psi \in D(d\Gamma(S)^{1/2}).$$

Moreover from

$$\begin{aligned} &\| \int_M (\overline{f_m}(k) - \overline{f}(k))(H - E(H) + S(k))^{-1} T(k)\varphi_{\rm g} d\mu(k)\| \\ &\le \|(\overline{f_m}(k) - \overline{f}(k))/\sqrt{S}\|_{L^2(M)} \left(\int_M \|T(k)\varphi_{\rm g}\|^2_{\mathcal{F}}/S(k) d\mu(k)\right)^{1/2}, \end{aligned}$$

it follows that

$$\begin{aligned} &\text{s-}\lim_{m\to\infty} \int_M \overline{f_m(k)}(H - E(H) + S(k))^{-1} T(k)\varphi_{\rm g} d\mu(k) \\ &= \int_M \overline{f(k)}(H - E(H) + S(k))^{-1} T(k)\varphi_{\rm g} d\mu(k). \end{aligned} \tag{2.10}$$

By (2.9) and (2.10), (2.8) can be extended for $f \in \mathcal{M}_{-1/2} \cap \mathcal{M}_0$. Thus the lemma is proven. □

We want to find a necessary and sufficient conditions for $\varphi_{\rm g} \in D(1 \otimes d\Gamma(G)^{1/2})$ with a nonnegative multiplication operator G on $L^2(M)$. We define $\kappa_G(k) \in \mathcal{F}$ by

$$\kappa_G(k) := \sqrt{G(k)}(H - E(H) + S(k))^{-1} T(k)\varphi_{\rm g}, \quad \textit{a.e. } k \in M,$$

and $T_G : L^2(M) \to \mathcal{F}$ by

$$T_G f := \int_M f(k)\kappa_G(k) d\mu(k), \quad f \in L^2(M).$$

Then by Lemma 2.4, for $f \in \mathcal{M}_{-1/2} \cap \mathcal{M}_0$,

$$(1 \otimes a(\sqrt{G} f))\varphi_{\rm g} = -g T_G \bar{f}.$$

Lemma 2.6 *(1) T_G is a Hilbert-Schmidt operator if and only if*

$$\int_M \|\kappa_G(k)\|_{\mathcal{F}}^2 d\mu(k) < \infty. \tag{2.11}$$

(2) Suppose that T_G is a Hilbert-Schmidt operator. Then

$$\mathrm{Tr}(T_G^* T_G) = \int_M \|\kappa_G(k)\|_{\mathcal{F}}^2 d\mu(k).$$

Proof: It is enough to show (1) and (2) for adjoint operator $T_G^* : \mathcal{F} \to L^2(M)$ instead of T_G. T_G^* is refered to as a Carleman operator with kernel κ_G, i.e.,

$$T_G^* \Phi(\cdot) = (\kappa_G(\cdot), \Phi)_{\mathcal{F}}, \quad \Phi \in \mathcal{F}.$$

Then it is established in e.g., [24, p.141] that T_G^* is a Hilbert-Schmidt operator if and only if (2.11) holds, and

$$\mathrm{Tr}(T_G T_G^*) = \int_M \|\kappa_G(k)\|_{\mathcal{F}}^2 d\mu(k).$$

Hence the lemma is proven. □

Lemma 2.7 *Let K be a bounded operator on $\mathcal{K}$, and $\{e_m\}_{m=1}^\infty$ a complete orthonormal system in $\mathcal{K}$ such that $e_m \in \mathcal{M}_{-1/2} \cap \mathcal{M}_0$. Then (1) and (2) are equivalent.*

(1) $\Psi \in D(d\Gamma(K^*K)^{1/2})$.

(2) $\Psi \in \bigcap_{m=1}^\infty D(a(K^* e_m))$ *and* $\sum_{m=1}^\infty \|a(K^* e_m)\Psi\|_{\mathcal{F}_{\mathrm{b}}(\mathcal{K})}^2 < \infty$.

Furthermore suppose that (1) or (2) is fulfilled. Then

$$\|d\Gamma(K^*K)^{1/2}\Psi\|_{\mathcal{F}_{\mathrm{b}}(\mathcal{K})}^2 = \sum_{m=1}^\infty \|a(K^* e_m)\Psi\|_{\mathcal{F}_{\mathrm{b}}(\mathcal{K})}^2.$$

Proof: See Appendix. □

Theorem 2.8 *Assume (A.1)-(A.5). Let G be a nonnegative multiplication operator on $L^2(M)$. Then (1) and (2) are equivalent.*

(1) $\varphi_{\mathrm{g}} \in D(1 \otimes d\Gamma(G)^{1/2})$.

(2) $\int_M G(k)\|(H - E(H) + S(k))^{-1} T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2 d\mu(k) < \infty$.

Furthermore suppose that (1) or (2) holds. Then it follows that

$$\|(1\otimes d\Gamma(G)^{1/2})\varphi_{\rm g}\|^2_{\mathcal F}=g^2\int_M G(k)\|(H-E(H)+S(k))^{-1}T(k)\varphi_{\rm g}\|^2_{\mathcal F}d\mu(k). \quad (2.12)$$

Proof: We divide a proof into two steps.
(Step I) The case where G is bounded.
Let $\{e_m\}_{m=1}^\infty$ be a orthonormal system of $\mathcal K$ such that $e_m\in\mathcal M_{-1/2}\cap\mathcal M_0$, $m\geq 1$. By Lemma 2.7, $\varphi_{\rm g}\in D(1\otimes d\Gamma(G)^{1/2})$ is equivalent to

$$\sum_{m=1}^\infty\|(1\otimes a(\sqrt{G}e_m))\varphi_{\rm g}\|^2_{\mathcal F}<\infty, \quad (2.13)$$

and, if (2.13) holds, the left-hand side of (2.13) is identical with $\|(1\otimes d\Gamma(G)^{1/2})\varphi_{\rm g}\|^2_{\mathcal F}$. By the definition of T_G, (2.13) can be rewritten as

$$g^2\sum_{m=1}^\infty\|T_G\overline{e_m}\|^2_{\mathcal F}<\infty.$$

Then T_G is a Hilbert-Schmidt operator. Hence by Lemma 2.6, (1) is equivalent to

$$\int_M\|\kappa_G(k)\|^2_{\mathcal F}d\mu(k)<\infty. \quad (2.14)$$

Namely (1) is equivalent to (2).
(Step II) The case where G is unbounded.
Let $\varphi_{\rm g}=\{\varphi_{\rm g}^{(n)}\}_{n=0}^\infty$ and

$$G_\Lambda(k):=\begin{cases} G(k), & G(k)<\Lambda,\\ \Lambda, & G(k)\geq\Lambda.\end{cases}$$

Proof of (1) $\Longrightarrow$ (2). Note that

$$\begin{aligned}&\|(1\otimes d\Gamma_n(G_\Lambda)^{1/2})\varphi_{\rm g}^{(n)}\|^2_{\mathcal F_n}\\ &=\int_{M^n}\left(\sum_{j=1}^n G_\Lambda(k_j)\right)\|\varphi_{\rm g}^{(n)}(k_1,\cdots,k_n)\|^2_{\mathcal H}d\mu(k_1)\cdots d\mu(k_n).\end{aligned}$$

Hence we see that

$$\|(1\otimes d\Gamma(G_\Lambda)^{1/2})\varphi_{\rm g}\|^2_{\mathcal F}=\sum_{n=0}^\infty\|(1\otimes d\Gamma_n(G_\Lambda)^{1/2})\varphi_{\rm g}^{(n)}\|^2_{\mathcal F_n}$$

is monotonously increasing in Λ. Then the monotone convergence theorem yields that

$$\begin{aligned}&\lim_{\Lambda\to\infty}\|(1\otimes d\Gamma(G_\Lambda)^{1/2})\varphi_{\rm g}\|^2_{\mathcal F}=\lim_{\Lambda\to\infty}\sum_{n=0}^\infty\|(1\otimes d\Gamma_n(G_\Lambda)^{1/2})\varphi_{\rm g}^{(n)}\|^2_{\mathcal F_n}\\ &=\sum_{n=0}^\infty\lim_{\Lambda\to\infty}\|(1\otimes d\Gamma_n(G_\Lambda)^{1/2})\varphi_{\rm g}^{(n)}\|^2_{\mathcal F_n}=\|(1\otimes d\Gamma(G)^{1/2})\varphi_{\rm g}\|^2_{\mathcal F}.\end{aligned}$$

Since $\varphi_{\rm g} \in D(1 \otimes d\Gamma(G_\Lambda)^{1/2})$, we have by (Step I),

$$\|(1 \otimes d\Gamma(G_\Lambda)^{1/2})\varphi_{\rm g}\|_{\mathcal{F}}^2 = \int_M G_\Lambda(k)\|(H - E(H) + S(k))^{-1}T(k)\varphi_{\rm g}\|_{\mathcal{F}}^2 d\mu(k).$$

Take $\Lambda \to \infty$ on the both sides above. By the monotone convergence theorem again, we obtain that

$$\infty > \|(1 \otimes d\Gamma(G)^{1/2})\varphi_{\rm g}\|_{\mathcal{F}}^2 = \int_M G(k)\|(H - E(H) + S(k))^{-1}T(k)\varphi_{\rm g}\|_{\mathcal{F}}^2 d\mu(k).$$

Thus (2) follows.

Proof of (1) $\Longleftarrow$ (2). From (2) it follows that

$$\infty > \int_M G_\Lambda(k)\|(H - E(H) + S(k))^{-1}T(k)\varphi_{\rm g}\|_{\mathcal{F}}^2 d\mu(k) = \sum_{n=0}^{\infty} \|(1 \otimes d\Gamma_n(G_\Lambda)^{1/2})\varphi_{\rm g}^{(n)}\|_{\mathcal{F}_n}^2.$$

Then by the monotone convergence theorem, as $\Lambda \to \infty$ on the both sides above we obtain that

$$\begin{aligned} \infty &> \int_M G(k)\|(H - E(H) + S(k))^{-1}T(k)\varphi_{\rm g}\|_{\mathcal{F}}^2 d\mu(k) \\ &= \sum_{n=0}^{\infty} \|(1 \otimes d\Gamma_n(G)^{1/2})\varphi_{\rm g}^{(n)}\|_{\mathcal{F}_n}^2 = \|(1 \otimes d\Gamma(G)^{1/2})\varphi_{\rm g}\|_{\mathcal{F}}^2. \end{aligned}$$

Thus (1) follows.

Finally (2.12) follows from the fact that

$$\begin{aligned} &\|(1 \otimes d\Gamma(G)^{1/2})\varphi_{\rm g}\|_{\mathcal{F}}^2 = g^2 \sum_{m=1}^{\infty} \|T_G\overline{e_m}\|_{\mathcal{F}}^2 = g^2 {\rm Tr}(T_G^* T_G) \\ &= g^2 \int_M G(k)\|(H - E(H) + S(k))^{-1}T(k)\varphi_{\rm g}\|_{\mathcal{F}}^2 d\mu(k). \end{aligned}$$

Thus the proof is complete. □

Corollary 2.9 *Let G be a nonnnegative multiplication operator on $L^2(M)$. In addition to (A.1)-(A.5), suppose that*

$$\sqrt{G}\|T(\cdot)\varphi_{\rm g}\|_{\mathcal{F}}/S \in L^2(M). \tag{2.15}$$

Then $\varphi_{\rm g} \in D(1 \otimes d\Gamma(G)^{1/2})$. In particular suppose that

$$\|T(\cdot)\varphi_{\rm g}\|_{\mathcal{F}}/S \in L^2(M). \tag{2.16}$$

Then $\varphi_{\rm g} \in D(1 \otimes N^{1/2})$.

Remark 2.10 *Condition (2.15) is a generalization of an IR regularity condition in [4].*

Proof: Since

$$\int_M G(k)\|(H-E(H)+S(k))^{-1}T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2 d\mu(k) \le \int_M \frac{G(k)}{S(k)^2}\|T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2 d\mu(k) < \infty,$$

Theorem 2.8 yields the corollary. □

3 Absence of ground states

Let P_{g} be the projection onto the one-dimensional subspace $\{z\varphi_{\mathrm{g}}|z \in \mathbb{C}\}$ of $\mathcal{F}$.

Theorem 3.1 *Assume (A.1)-(A.5). Let G be a nonnegative multiplication operator on $L^2(M)$. Then H has no ground state φ_{g} in $D(1 \otimes d\Gamma(G)^{1/2})$ such that*

$$\sqrt{G}(\varphi_{\mathrm{g}}, T(\cdot)\varphi_{\mathrm{g}})_{\mathcal{F}}/S \notin L^2(M). \tag{3.1}$$

Remark 3.2 *Condition (3.1) is a generalization of an IR singularity condition in [4].*

Proof: Suppose that there exists a ground state φ_{g} such as in (3.1). We have

$$\begin{aligned}&\int_M G(k)\|(H-E(H)+S(k))^{-1}T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2 d\mu(k)\\ &\ge \int_M G(k)\|(H-E(H)+S(k))^{-1}P_{\mathrm{g}}T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2 d\mu(k).\end{aligned}$$

Since

$$\begin{aligned}&\|(H-E(H)+S(k))^{-1}P_{\mathrm{g}}T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2\\ &= \frac{1}{S(k)^2}\|P_{\mathrm{g}}T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2 = \frac{1}{S(k)^2}|(\varphi_{\mathrm{g}}, T(k)\varphi_{\mathrm{g}})_{\mathcal{F}}|^2,\end{aligned}$$

it follows that

$$\int_M G(k)\|(H-E(H)+S(k))^{-1}T(k)\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2 d\mu(k) \ge \int_M \left(\frac{\sqrt{G(k)}}{S(k)}|(\varphi_{\mathrm{g}}, T(k)\varphi_{\mathrm{g}})_{\mathcal{F}}|\right)^2 d\mu(k).$$

Since the right-hand side above diverges by (3.1), Lemma 2.6 yields that

$$\varphi_{\mathrm{g}} \notin D(1 \otimes d\Gamma(G)^{1/2}).$$

Thus the desired result is obtained. □

Setting $G = 1$ in Theorem 3.1, we have a corollary.

Corollary 3.3 *Assume (A.1)-(A.5). Then H has no ground state φ_{g} in $D(1 \otimes N^{1/2})$ such that*

$$\frac{1}{S}(\varphi_{\mathrm{g}}, T(\cdot)\varphi_{\mathrm{g}})_{\mathcal{F}} \notin L^2(M). \tag{3.2}$$

4 Higher order regularities of ground states

Through this section we fix a natural number n and assume (A.1)-(A.5). We shall consider an asymptotic field for $1 \otimes a(f_1) \cdots a(f_n)$. Namely we investigate

$$\text{s-}\lim_{t\to\infty} e^{-itH}(1 \otimes a(e^{itS} f_1) \cdots a(e^{itS} f_n))e^{itH}.$$

We introduce assumptions.

(H.1) $T(k)$ satisfies that

$$D(H) \subset D(T(k)(1 \otimes a(f))) \cap D((1 \otimes a(f))T(k)) \quad a.e. k \in M,$$

and

$$[T(k), 1 \otimes a(f)]\Psi = 0, \qquad \Psi \in D(H), \quad a.e.\ k \in M. \tag{4.1}$$

(H.2) The operator

$$(1 \otimes d\Gamma(S)^{(n+1)/2})(H + z)^{-m}$$

is a bounded operator for some m and $z \in \rho(H) \cap \mathbb{R}$, where $\rho(H)$ denotes the resolvent of H.

Lemma 4.1 *Suppose (H.2). Then for $f_j \in \mathcal{M}_{-1/2} \cap \mathcal{M}_{[n/2]}$, $j = 1, ..., n$,*

$$\varphi_{\rm g} \in D(1 \otimes a(f_1) \cdots a(f_n)) \tag{4.2}$$

and

$$(1 \otimes a(f_1) \cdots a(f_m))\varphi_{\rm g} \in D(H), \quad m = 1, ..., n - 1. \tag{4.3}$$

Proof: From (H.2) it follows that

$$\|(1 \otimes d\Gamma(S))^{(n+1)/2}\varphi_{\rm g}\|_{\mathcal{F}} \le \|(1 \otimes d\Gamma(S))^{(n+1)/2}(H + z)^{-m}\|_{\mathcal{F}\to\mathcal{F}} \,|E(H) + z|^m \,\|\varphi_{\rm g}\|_{\mathcal{F}}.$$

Hence

$$\varphi_{\rm g} \in D(1 \otimes d\Gamma(S)^{(n+1)/2}) \tag{4.4}$$

follows, from which we can see (4.2) by Lemma 2.1. From Lemma 2.1 it follows that

$$(1 \otimes a(f_1) \cdots a(f_n))\varphi_{\rm g} \in D(1 \otimes d\Gamma(S)).$$

Together with

$$(1 \otimes a(f_1) \cdots a(f_n))\varphi_{\rm g} \in D(A \otimes 1),$$

we obtain that

$$(1 \otimes a(f_1) \cdots a(f_n))\varphi_{\rm g} \in D(A \otimes 1) \cap D(1 \otimes d\Gamma(S)) = D(H).$$

Then (4.3) is proven. □

$\mathcal{P}_n$ denotes the set of all the permutations of degree n and we set for $\sigma \in \mathcal{P}_n$,

$$R_i^\sigma := \left(H - E(H) + \sum_{j=i}^n S(k_{\sigma(j)}) \right)^{-1},$$
$$\tilde{R}_i^\sigma := \left(H - E(H) + S(k_{\sigma(i)}) \right)^{-1}.$$

Remark 4.2 *Let $\otimes^m \mu$ denote the product measure on M^m. Then*

$$(\otimes^m \mu)(\{(k_1, ..., k_m) \in M^m | S(k_1) + \cdots + S(k_m) = 0\}) = 0$$

follows. This implies that R_i^σ and $\tilde{R}_i^\sigma$ are bounded operators on $\mathcal{F}$ for almost every $(k_{\sigma(i)}, ..., k_{\sigma(n)}) \in M^{n-i+1}$.

We set

$$\widehat{H} := H - E(H).$$

Assumption (H.3) and (H.4) are as follows.

(H.3) There exist dense subspaces $\mathcal{C}_n \subset L^2(M)$ and $\mathcal{E} \subset \mathcal{F}$ such that

- **(1)** $\mathcal{C}_n \subset \mathcal{M}_{-1/2} \cap \mathcal{M}_{[n/2]}$, and for $f \in \mathcal{M}_{-1/2} \cap \mathcal{M}_{[n/2]}$, there exists a sequence $\{f_m\}_m \subset \mathcal{C}_n$ such that $\text{s-lim}_{m\to\infty} f_m/\sqrt{S^k} = f/\sqrt{S^k}$ for $0 \le k \le n$,
- **(2)** $\Psi \subset \bigcap_{m=1}^n D(T(k_m)^* e^{it_m \widehat{H}} \cdots T(k_1)^* e^{it_1 \widehat{H}})$ for almost every $(k_1, ..., k_m) \in M^m$ and for $\Psi \in \mathcal{E}$,
- **(3)** for arbitrary $f_j \in \mathcal{C}_n$, $j = 1, ..., n$, and $\Psi \in \mathcal{E}$,

$$\int_M d\mu(k_m) e^{-iT_m S(k_m)} f_m(k_m)$$
$$(T(k_m)^* e^{it_m \widehat{H}} \cdots T(k_1)^* e^{it_1 \widehat{H}} \Psi, (1 \otimes a(e^{iT_m S} f_{m+1}) \cdots a(e^{iT_m S} f_n))\varphi_{\rm g})_{\mathcal{F}}$$

  is in $L^1([0,\infty); dt_m)$ for $m = 1, 2, ..., n-1$, where $T_m := t_1 + \cdots + t_m$,
- **(4)** for arbitrary $f_j \in \mathcal{C}_n$, $j = 1, ..., n$, and $\Psi \in \mathcal{E}$,

$$\int_M d\mu(k_{\sigma(m)}) \cdots \int_M d\mu(k_{\sigma(n)}) \prod_{j=m}^n f_{\sigma(k)}(k_{\sigma(j)})$$
$$(T(k_{\sigma(m)})^* e^{it_m(\widehat{H} + \sum_{j=m}^n S(k_{\sigma(j)}))} \cdots T(k_{\sigma(1)})^* e^{it_1(\widehat{H} + \sum_{j=1}^n S(k_{\sigma(j)}))} \Psi,$$
$$R_{m+1}^\sigma T(k_{\sigma(m+1)}) \cdots R_n^\sigma T(k_{\sigma(n)}) \varphi_{\rm g})_{\mathcal{F}}$$

  is in $L^1([0,\infty); dt_m)$ for $m = 1, 2, ..., n-1$.

(H.4) The closure of $(H - E(H) + S(k))^{-1}T(k)$, $[(H - E(H) + S(k))^{-1}T(k)]^{\bar{}}$, is a bounded opeator and

$$\|[(H - E(H) + S(\cdot))^{-1}T(\cdot)]^{\bar{}}\|_{\mathcal{F}\to\mathcal{F}} \in \mathcal{M}_{1/2}.$$

Lemma 4.3 *Assume (H.1)-(H.4). Then for* $f_\ell \in \mathcal{M}_{-1/2} \cap \mathcal{M}_{[n/2]}$, $\ell = 1, ..., n$,

$$\int_{M^n} \prod_{j=1}^n d\mu(k_j) \left(\prod_{j=1}^n |f_j(k_j)| \right) \left\| R_1^\sigma T(k_{\sigma(1)}) \cdots R_n^\sigma T(k_{\sigma(n)}) \varphi_{\rm g} \right\|_{\mathcal{F}} < \infty \tag{4.5}$$

and

$$\begin{aligned} &(1 \otimes a(f_1) \cdots a(f_n))\varphi_{\rm g} \\ &= (-g)^n \sum_{\sigma \in \mathcal{P}_n} \int_{M^n} \prod_{j=1}^n d\mu(k_j) \left(\prod_{j=1}^n \overline{f_j(k_j)} \right) R_1^\sigma T(k_{\sigma(1)}) \cdots R_n^\sigma T(k_{\sigma(n)}) \varphi_{\rm g}. \end{aligned} \tag{4.6}$$

Proof: Note that

$$\|R_j^\sigma\|_{\mathcal{F}\to\mathcal{F}} \le \|\tilde{R}_j^\sigma\|_{\mathcal{F}\to\mathcal{F}}, \quad j = 1, ..., n.$$

From (H.4) it follows that

$$f(k_{\sigma(j)}) \|[\tilde{R}_j^\sigma T(k_{\sigma(j)})]^{\bar{}}\|_{\mathcal{F}\to\mathcal{F}} \in L^1(M) \tag{4.7}$$

for $f \in \mathcal{M}_{-1/2} \cap \mathcal{M}_0$ and $\sigma \in \mathcal{P}_n$. We have, by (4.7),

$$\begin{aligned} &\int_{M^n} \prod_{j=1}^n d\mu(k_j) \left(\prod_{j=1}^n |f_j(k_j)| \right) \left\| R_1^\sigma T(k_{\sigma(1)}) \cdots R_n^\sigma T(k_{\sigma(n)}) \varphi_{\rm g} \right\|_{\mathcal{F}} \\ &\le \|\varphi_{\rm g}\|_{\mathcal{F}} \prod_{j=1}^n \int_M |f_{\sigma(j)}(k_{\sigma(j)})| \|[\tilde{R}_j^\sigma T(k_{\sigma(j)})]^{\bar{}}\|_{\mathcal{F}\to\mathcal{F}} d\mu(k_{\sigma(j)}) < \infty. \end{aligned} \tag{4.8}$$

Then (4.5) follows. Lemma 2.1 yields that the left-hand side of (4.6) is well defined. Let $f_j \in \mathcal{C}_n$, $j = 1, ..., n$. Note that by (4.3),

$$(1 \otimes a(f_1) \cdots a(f_m))\varphi_{\rm g} \in D(H), \qquad m = 1, ..., n-1.$$

From this and (H.1) it follows that

$$[T(k), 1 \otimes a(g)](1 \otimes a(f_1) \cdots a(f_m))\varphi_{\rm g} = 0. \tag{4.9}$$

By (4.9) we see that for $\Psi \in \mathcal{E}$ and $f_j \in \mathcal{C}_n$, $j = 1, ..., n$,

$$\frac{d}{dt}(\Psi, e^{-itH}(1 \otimes a(e^{itS} f_1) \cdots a(e^{itS} f_n)) e^{itH} \varphi_{\rm g})_{\mathcal{F}}$$

$$
\begin{aligned}
&= ig\sum_{j=1}^{n}\int_M d\mu(k_j)e^{-itS(k_j)}f_j(k_j)\\
&(e^{it\widehat{H}}\Psi,(1\otimes a(e^{itS}f_1)\cdots a(e^{itS}f_{j-1}))T(k_j)(1\otimes a(e^{itS}f_{j+1})\cdots a(e^{itS}f_n))\varphi_{\rm g})_{\mathcal F}\\
&= ig\sum_{j=1}^{n}\int_M d\mu(k_j)e^{-itS(k_j)}f_j(k_j)\\
&\qquad (e^{it\widehat{H}}\Psi,T(k_j)(1\otimes a(e^{itS}f_1)\cdots\widehat{a(e^{itS}f_j)}\cdots a(e^{itS}f_n))\varphi_{\rm g})_{\mathcal F}.
\end{aligned}
$$

Then by (H.3)-(3) and the fact that

$$
\text{s-}\lim_{t\to\infty}(\Psi,e^{-itH}(1\otimes a(e^{itS}f_1)\cdots a(e^{itS}f_n))e^{itE(H)}\varphi_{\rm g})=0,
$$

we have

$$
\begin{aligned}
&(\Psi,(1\otimes a(f_1)\cdots a(f_n))\varphi_{\rm g})_{\mathcal F}\\
&= -ig\sum_{j=1}^{n}\int_0^\infty dt\int_M d\mu(k_j)e^{-itS(k_j)}\overline{f_j(k_j)}\\
&\qquad\qquad (T(k_j)^*e^{it\widehat{H}}\Psi,(1\otimes a(e^{itS}f_1)\cdots\widehat{a(e^{itS}f_j)}\cdots a(e^{itS}f_n))\varphi_{\rm g})_{\mathcal F}.
\end{aligned}
\tag{4.10}
$$

Using (4.10) again, we have by (H.3)-(3),

$$
\begin{aligned}
&(T(k_j)^*e^{it\widehat{H}}\Psi,(1\otimes a(e^{itS}f_1)\cdots\widehat{a(e^{itS}f_j)}\cdots a(e^{itS}f_n))\varphi_{\rm g})_{\mathcal F}\\
&= -ig\sum_{j'=1,j'\neq j}^{n}\int_0^\infty dt'\int_M d\mu(k_{j'})e^{-i(t+t')S(k_{j'})}\overline{f_{j'}(k_{j'})}(T(k_{j'})^*e^{it'\widehat{H}}T(k_j)^*e^{it\widehat{H}}\Psi,\\
&\qquad (1\otimes a(e^{i(t+t')S}f_1)\cdots\widehat{a(e^{i(t+t')S}f_j)}\cdots\widehat{a(e^{i(t+t')S}f_{j'})}\cdots a(e^{i(t+t')S}f_n))\varphi_{\rm g})_{\mathcal F}.
\end{aligned}
$$

We can inductively obtain that

$$
\begin{aligned}
&(\Psi,(1\otimes a(f_1)\cdots a(f_n))\varphi_{\rm g})_{\mathcal F}\\
&= (-ig)^n\sum_{\sigma\in\mathcal P_n}\int_0^\infty dt_1\int_M d\mu(k_{\sigma(1)})\cdots\int_0^\infty dt_n\int_M d\mu(k_{\sigma(n)})\left(\prod_{j=1}^n\overline{f_{\sigma(j)}(k_{\sigma(j)})}\right)\\
&\qquad\qquad e^{-it_1\sum_{j=1}^n S(k_{\sigma(j)})}e^{-it_2\sum_{j=2}^n S(k_{\sigma(j)})}\cdots e^{-it_nS(k_{\sigma(n)})}\\
&\qquad\qquad\qquad (T(k_n)^*e^{it_n\widehat{H}}\cdots T(k_1)^*e^{it_1\widehat{H}}\Psi,\varphi_{\rm g})_{\mathcal F}\\
&= (-ig)^n\sum_{\sigma\in\mathcal P_n}\int_0^\infty dt_1\int_M d\mu(k_{\sigma(1)})\cdots\int_0^\infty dt_n\int_M d\mu(k_{\sigma(n)})\left(\prod_{j=1}^n\overline{f_{\sigma(j)}(k_{\sigma(j)})}\right)\\
&\qquad (T(k_{\sigma(n)})^*e^{it_n(\widehat{H}+\sum_{j=n}^n S(k_{\sigma(j)}))}\cdots T(k_{\sigma(1)})^*e^{it_1(\widehat{H}+\sum_{j=1}^n S(k_{\sigma(j)}))}\Psi,\varphi_{\rm g})_{\mathcal F}.
\end{aligned}
$$

Since by (H.3)-(4),

$$
\begin{aligned}
&\int_M d\mu(k_{\sigma(n)})\prod_{j=1}^n\overline{f_{\sigma(j)}(k_{\sigma(j)})}\\
&\qquad (T(k_{\sigma(n)})^*e^{it_n(\widehat{H}+\sum_{j=n}^n S(k_{\sigma(j)}))}\cdots T(k_{\sigma(1)})^*e^{it_1(\widehat{H}+\sum_{j=1}^n S(k_{\sigma(j)}))}\Psi,\varphi_{\rm g})_{\mathcal F}
\end{aligned}
$$

is in $L^1([0,\infty), dt_n)$, we have

$$\begin{aligned}
&\int_0^\infty dt_n \int_M d\mu(k_{\sigma(n)}) \left(\prod_{j=1}^n \overline{f_{\sigma(j)}(k_{\sigma(j)})}\right) \\
&\qquad (T(k_{\sigma(n-1)})^* e^{it_{n-1}(\widehat{H}+\sum_{j=n-1}^n S(k_{\sigma(j)}))} \cdots T(k_{\sigma(1)})^* e^{it_1(\widehat{H}+\sum_{j=1}^n S(k_{\sigma(j)}))}\Psi, \\
&\qquad\qquad e^{-it_n(\widehat{H}+S(k_{\sigma(n)}))} T(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}} \\
&= \lim_{\epsilon\to 0}\int_0^\infty dt_n e^{-\epsilon t_n} \int_M d\mu(k_{\sigma(n)}) \left(\prod_{j=1}^n \overline{f_{\sigma(j)}(k_{\sigma(j)})}\right) \\
&\qquad (T(k_{\sigma(n-1)})^* e^{it_{n-1}(\widehat{H}+\sum_{j=n-1}^n S(k_{\sigma(j)}))} \cdots T(k_{\sigma(1)})^* e^{it_1(\widehat{H}+\sum_{j=1}^n S(k_{\sigma(j)}))}\Psi, \\
&\qquad\qquad e^{-it_n(\widehat{H}+S(k_{\sigma(n)}))} T(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}} \\
&= (-i)\int_M d\mu(k_{\sigma(n)}) \left(\prod_{j=1}^n \overline{f_{\sigma(j)}(k_{\sigma(j)})}\right) \\
&\qquad (T(k_{\sigma(n-1)})^* e^{it_{n-1}(\widehat{H}+\sum_{j=n-1}^n S(k_{\sigma(j)}))} \cdots T(k_{\sigma(1)})^* e^{it_1(\widehat{H}+\sum_{j=1}^n S(k_{\sigma(j)}))}\Psi, \\
&\qquad\qquad R_n^\sigma T(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}}.
\end{aligned}$$

Here we used the Lebesgue dominated convergence theorem and Fubini's lemma. Similarly using (H.3)-(4) we can obtain that

$$\begin{aligned}
&\int_0^\infty dt_{n-1} \int_M d\mu(k_{\sigma(n-1)}) \int_0^\infty dt_n \int_M d\mu(k_{\sigma(n)}) \left(\prod_{j=1}^n \overline{f_{\sigma(j)}(k_{\sigma(j)})}\right) \\
&\qquad (T(k_{\sigma(n)})^* e^{it_n(\widehat{H}+\sum_{j=n}^n S(k_{\sigma(j)}))} \cdots T(k_{\sigma(1)})^* e^{it_1(\widehat{H}+\sum_{j=1}^n S(k_{\sigma(j)}))}\Psi, \varphi_{\mathrm{g}})_{\mathcal{F}} \\
&= (-i)\int_0^\infty dt_{n-1} \int_M d\mu(k_{\sigma(n-1)}) \int_M d\mu(k_{\sigma(n)}) \left(\prod_{j=1}^n \overline{f_{\sigma(j)}(k_{\sigma(j)})}\right) \\
&\qquad (T(k_{\sigma(n-1)})^* e^{it_{n-1}(\widehat{H}+\sum_{j=n-1}^n S(k_{\sigma(j)}))} \cdots T(k_{\sigma(1)})^* e^{it_1(\widehat{H}+\sum_{j=1}^n S(k_{\sigma(j)}))}\Psi, \\
&\qquad\qquad R_n^\sigma T(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}} \\
&= (-i)^2 \int_M d\mu(k_{\sigma(n-1)}) \int_M d\mu(k_{\sigma(n)}) \left(\prod_{j=1}^n \overline{f_{\sigma(j)}(k_{\sigma(j)})}\right) \\
&\qquad (T(k_{\sigma(n-2)})^* e^{it_{n-2}(\widehat{H}+\sum_{j=n-2}^n S(k_{\sigma(j)}))} \cdots T(k_{\sigma(1)})^* e^{it_1(\widehat{H}+\sum_{j=1}^n S(k_{\sigma(j)}))}\Psi, \\
&\qquad\qquad R_{n-1}^\sigma T(k_{\sigma(n-1)}) R_n^\sigma T(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}}.
\end{aligned}$$

Thus inductively we can see that

$$\begin{aligned}
&(\Psi, (1\otimes a(f_1)\cdots a(f_n))\varphi_{\mathrm{g}})_{\mathcal{F}} \\
&= (-g)^n \sum_{\sigma\in\mathcal{P}_n} \int_{M^n} \prod_{j=1}^n d\mu(k_{\sigma(j)}) \prod_{j=1}^n \overline{f_{\sigma(j)}(k_{\sigma(j)})} (\Psi, R_1^\sigma T(k_{\sigma(1)})\cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}}.
\end{aligned}$$

By (4.8) and the fact that $\mathcal{E}$ is dense, we obtain that for $f_j \in \mathcal{C}_n$, $j = 1, ..., n$,

$$
\begin{aligned}
&(1 \otimes a(f_1) \cdots a(f_n))\varphi_{\rm g} \\
&= (-g)^n \sum_{\sigma\in\mathcal{P}_n} \int_{M^n} \prod_{j=1}^n d\mu(k_{\sigma(j)}) \left(\prod_{j=1}^n \overline{f_{\sigma(j)}(k_{\sigma(j)})} \right) R_1^\sigma T(k_{\sigma(1)}) \cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\rm g} \\
&= (-g)^n \sum_{\sigma\in\mathcal{P}_n} \int_{M^n} \prod_{j=1}^n d\mu(k_j) \left(\prod_{j=1}^n \overline{f_j(k_j)} \right) R_1^\sigma T(k_{\sigma(1)}) \cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\rm g}. \qquad (4.11)
\end{aligned}
$$

Let $f_j \in \mathcal{M}_{-1/2} \cap \mathcal{M}_{[n/2]}$, $j = 1, ..., n$. Take sequences $\{f_{jm}\}_m$, $j = 1, ..., n$, in $\mathcal{C}_n$ such that s-$\lim_{m\to\infty} f_{jm}/\sqrt{S^k} = f_j/\sqrt{S^k}$, $j = 1, ..., n$, for $0 \le k \le n$. Then

$$
\text{s-}\lim_{m\to\infty} (1 \otimes a(f_{1m}) \cdots a(f_{nm}))\varphi_{\rm g} = (1 \otimes a(f_1) \cdots a(f_n))\varphi_{\rm g}. \qquad (4.12)
$$

Moreover by (4.8) it follows that

$$
\begin{aligned}
&\text{s-}\lim_{m\to\infty} \int_{M^n} \prod_{j=1}^n d\mu(k_j) \left(\prod_{j=1}^n \overline{f_{jm}(k_j)} \right) R_1^\sigma T(k_{\sigma(1)}) \cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\rm g} \\
&= \int_{M^n} \prod_{j=1}^n d\mu(k_j) \left(\prod_{j=1}^n \overline{f_j(k_j)} \right) R_1^\sigma T(k_{\sigma(1)}) \cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\rm g}. \qquad (4.13)
\end{aligned}
$$

Hence (4.11) can be extended for $f_j \in \mathcal{M}_{-1/2} \cap \mathcal{M}_{[n/2]}$, $j = 1, ..., n$, by (4.12) and (4.13). Then the lemma follows. □

Lemma 4.4 *Let $\{e_j\}_{j=1}^\infty$ be a complete orthonormal system in $\mathcal{K}$. Then (1) and (2) are equivalent.*

(1) $\Psi \in \bigcap_{i_1,...,i_n=1}^\infty D(a(e_{i_1}) \cdots a(e_{i_n}))$ *and* $\sum_{i_1,...,i_n=1}^\infty \|a(e_{i_1}) \cdots a(e_{i_n})\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} < \infty$.

(2) $\Psi \in D(\prod_{j=1}^n (N - j + 1)^{1/2})$.

Suppose that (1) or (2) holds. Then

$$
\sum_{i_1,...,i_n=1}^\infty \|a(e_{i_1}) \cdots a(e_{i_n})\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} = \| \prod_{j=1}^n (N - j + 1)^{1/2}\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})}.
$$

Proof: See Appendix. □

Theorem 4.5 *Assume (H.1)-(H.4). Then (1) and (2) are equivalent.*

(1) $\displaystyle\int_{M^n}\prod_{j=1}^n d\mu(k_j)\|\sum_{\sigma\in\mathcal{P}_n}R_1^\sigma T(k_{\sigma(1)})\cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\rm g}\|_{\mathcal{F}}^2<\infty.$

(2) $\displaystyle\varphi_{\rm g}\in D(1\otimes\prod_{j=1}^n(N-j+1)^{1/2}).$

Furthermore suppose that (1) or (2) holds. Then

$$\begin{aligned}&\|(1\otimes\prod_{j=1}^n(N-j+1)^{1/2})\varphi_{\rm g}\|_{\mathcal{F}}^2\\&=g^{2n}\int_{M^n}\prod_{j=1}^n d\mu(k_j)\|\sum_{\sigma\in\mathcal{P}_n}R_1^\sigma T(k_{\sigma(1)})\cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\rm g}\|_{\mathcal{F}}^2.\end{aligned}\tag{4.14}$$

Proof: Let us define

$$\kappa(k_1\cdots k_n):=\sum_{\sigma\in\mathcal{P}_n}R_1^\sigma T(k_{\sigma(1)})\cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\rm g},\quad a.e.\ (k_1,...,k_n)\in M^n,$$

and $T:\otimes^n L^2(M)\to\mathcal{F}$ by

$$T(f_1\otimes\cdots\otimes f_n):=\int_{M^n}\prod_{j=1}^n d\mu(k_j)f(k_1)\cdots f(k_n)\kappa(k_1\cdots k_n).$$

Then

$$(1\otimes a(f_1)\cdots a(f_n))\varphi_{\rm g}=(-g)^nT(\overline{f_1}\otimes\cdots\otimes\overline{f_n})$$

for $f_j\in\mathcal{M}_{-1/2}\cap\mathcal{M}_{[n/2]}$, $j=1,...,n$. Adjoint operator $T^*:\mathcal{F}\to\otimes^nL^2(M)\cong L^2(M^n)$ is given by

$$(T^*\Phi)(k_1,...,k_n)=(\kappa(k_1,...,k_n),\Phi)_{\mathcal{F}}$$

for almost ever $(k_1,...,k_n)\in M^n$. Then T^* is a Carleman operator [24] with kernel κ. Thus T^* is a Hilbert-Schmidt operator if and only if

$$\int_{M^n}\prod_{j=1}^n d\mu(k_j)\|\kappa(k_1\cdots k_n)\|_{\mathcal{F}}^2<\infty.$$

Namely T is a Hilbert-Schmidt operator if and only if

$$\int_{M^n}\prod_{j=1}^n d\mu(k_j)\|\sum_{\sigma\in\mathcal{P}_n}R_1^\sigma T(k_{\sigma(1)})\cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\rm g}\|_{\mathcal{F}}^2<\infty.\tag{4.15}$$

Let $\{e_j\}_{j=1}^\infty$ be a complete orthonormal system in $L^2(M)$ such that $e_j\in\mathcal{M}_{-1/2}\cap\mathcal{M}_{[n/2]}$, $j\ge1$. If T is a Hilbert-Schmidt operator, then

$$\infty>g^{2n}\int_{M^n}\prod_{j=1}^n d\mu(k_j)\|\sum_{\sigma\in\mathcal{P}_n}R_1^\sigma T(k_{\sigma(1)})\cdots R_n^\sigma T(k_{\sigma(n)})\varphi_{\rm g}\|_{\mathcal{F}}^2$$

$$= g^{2n}\int_{M^n}\prod_{j=1}^n d\mu(k_j)\|\kappa(k_1\cdots k_n)\|_{\mathcal{F}}^2 = g^{2n}\mathrm{Tr}(T^*T) = g^{2n}\sum_{i_1,\ldots,i_n}^\infty \|T(\overline{e_{i_1}}\otimes\cdots\otimes\overline{e_{i_n}})\|_{\mathcal{F}}^2$$

$$= \sum_{i_1,\ldots,i_n}^\infty \|(1\otimes a(e_{i_1})\cdots a(e_{i_n}))\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2 = \|(1\otimes\prod_{j=1}^n(N-j+1)^{1/2})\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2. \quad (4.16)$$

Thus (4.15) is equivalent to $\varphi_{\mathrm{g}} \in D(1\otimes\prod_{j=1}^n(N-j+1)^{1/2})$. Moreover (4.14) follows from (4.16). The proof is complete. □

Assumption (H.5) is as follows.

(H.5) The closure of $(H-E(H)+S(k))^{-1}T(k)$, $[(H-E(H)+S(k))^{-1}T(k)\bar{]}$, is a bounded operator for almsot every $k\in M$, and

$$\|[(H-E(H)+S(\cdot))^{-1}T(\cdot)\bar{]}\|_{\mathcal{F}\to\mathcal{F}} \in \mathcal{M}_0.$$

Theorem 4.6 *In addition to (H.1)-(H.4), we assume (H.5). Then*

$$\varphi_{\mathrm{g}} \in D(1\otimes N^{n/2}).$$

Proof: Note that

$$D(N^{n/2}) = \bigcap_{\ell=1}^n D(\prod_{j=1}^\ell(N-j+1)^{1/2}). \quad (4.17)$$

See e.g., [15, Lemma 3.2]. We see that for all $1\le\ell\le n$,

$$\begin{aligned}
&\|(1\otimes\prod_{j=1}^\ell(N-j+1)^{1/2})\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2\\
&= g^{2\ell}\int_{M^\ell}\prod_{j=1}^\ell d\mu(k_j)\|\sum_{\sigma\in\mathcal{P}_\ell}R_1^\sigma T(k_{\sigma(1)})\cdots R_\ell^\sigma T(k_{\sigma(\ell)})\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2\\
&\le \ell(\ell!)^2 g^{2\ell}\|\varphi_{\mathrm{g}}\|_{\mathcal{F}}^2\left(\int_M\|(H-E(H)+S(k))^{-1}T(k)\|_{\mathcal{F}\to\mathcal{F}}^2 d\mu(k)\right)^\ell < \infty.
\end{aligned}$$

Then

$$\varphi_{\mathrm{g}} \in \bigcap_{\ell=1}^n D(\prod_{j=1}^\ell(N-j+1)^{1/2}).$$

By (4.17) the theorem follows. □

5 GSB models

5.1 Regularities of ground states

The Hilbert space for the GSB model is defined by

$$\mathcal{F}_{\mathrm{SB}} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^\nu)).$$

Let $a(f)$ and $a^*(f)$, $f \in L^2(\mathbb{R}^\nu)$, be the annihilation operator and the creation operator of $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^\nu))$, respectively. Set

$$\phi(\lambda) := \frac{1}{\sqrt{2}} \left(a^*(\lambda) + a(\lambda)\right), \quad \lambda \in L^2(\mathbb{R}^\nu).$$

Hamiltonians of GSB models are defined by

$$H_{\mathrm{SB}} := H_{\mathrm{SB},0} + \alpha H_{\mathrm{SB,I}},$$

where α is a coupling constant,

$$H_{\mathrm{SB},0} := A \otimes 1 + 1 \otimes d\Gamma(\omega)$$

and

$$H_{\mathrm{SB,I}} := \overline{\sum_{j=1}^{J} B_j \otimes \phi(\lambda_j)}.$$

Here ω is a nonnegative multiplication operator on $L^2(\mathbb{R}^\nu)$, B_j, $j = 1, ..., J$, closed symmetric operators on $\mathcal{H}$, and $\overline{\sum_{j=1}^{J} B_j \otimes \phi(\lambda_j)}$ denotes the closure of $\sum_{j=1}^{J} B_j \otimes \phi(\lambda_j)$. Using fundamental inequalities (2.4) and (2.5) we have

$$\|\phi(\lambda)\Psi\|_{\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^\nu))} \leq \frac{1}{\sqrt{2}}(\|\lambda/\sqrt{\omega}\|_{L^2(\mathbb{R}^\nu)} + 2\|\lambda\|_{L^2(\mathbb{R}^\nu)})\|(d\Gamma(\omega)+1)^{1/2}\Psi\|_{\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^\nu))} \quad (5.1)$$

for $\Psi \in D(d\Gamma(\omega)^{1/2})$. Assumptions are as follows.

(B.1) A is self-adjoint and bounded from below.

(B.2) $\lambda_j, \lambda_j/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$, $j = 1, ..., J$.

(B.3) $D(\hat{A}^{1/2}) \subset \bigcap_{j=1}^{J} D(B_j)$, where $\hat{A} := A - E(A)$, and there exist constants a_j and b_j such that

$$\|B_j f\|_{\mathcal{H}} \leq a_j \|\hat{A}^{1/2} f\|_{\mathcal{H}} + b_j \|f\|_{\mathcal{H}}, \quad j = 1, ..., J, \quad f \in D(\hat{A}^{1/2}).$$

Moreover

$$|\alpha| \leq \left(\sum_{j=1}^{J} a_j \|\lambda_j/\sqrt{\omega}\|^2_{L^2(\mathbb{R}^\nu)}\right)^{-1}.$$

Proposition 5.1 *Assume (B.1)-(B.3). Then H_{SB} is self-adjoint and bounded from below on $D(H_{\mathrm{SB},0}) = D(A \otimes 1) \cap (1 \otimes d\Gamma(\omega))$. Moreover it is essentially self-adjoint on any core of $H_{\mathrm{SB},0}$.*

Proof: It is easily seen that for $\Psi \in D(H_{\mathrm{SB},0})$,

$$\|H_{\mathrm{SB,I}}\Phi\|_{\mathcal{F}_{\mathrm{SB}}} \leq \left(\sum_{j=1}^{J} a_j \|\lambda_j/\sqrt{\omega}\|^2_{L^2(\mathbb{R}^\nu)} \right) \|H_{\mathrm{SB},0}\Phi\|_{\mathcal{F}_{\mathrm{SB}}} + b\|\Phi\|_{\mathcal{F}_{\mathrm{SB}}} \tag{5.2}$$

with some constant $b > 0$. Then by the Kato-Rellich theorem, the proposition follows. □

To formulate additional assumptions we define

$$Y := \bigcup_{n=1}^{\nu} \{k = (k_1, ..., k_\nu) \in \mathbb{R}^\nu | k_n = 0\}.$$

We assume (B.4) and (B.5).

(B.4) $\lambda_j \in C^2(\mathbb{R}^\nu \setminus Y)$, $j = 1, ..., J$.

(B.5) **(1)** $\omega \in C^3(\mathbb{R}^\nu \setminus Y)$ and $\partial\omega(k)/\partial k_n \neq 0$ on $\mathbb{R}^\nu \setminus Y$, $n = 1, ..., \nu$,

(2) ω is purely absolutely continuous.

Remark 5.2 *In [3] the existence of ground states of H_{SB} is proven under some conditions on ω for α with $|\alpha|$ sufficiently small. Moreover for massive cases the multiplicity of ground states is studied in [3]. For massless cases, an upper bound of the multiplicity of ground states is shown in [16].*

We see that, for $\Psi, \Phi \in D(H_{\mathrm{SB},0})$,

$$[1 \otimes a(f), H_{\mathrm{SB,I}}]_W^{D(H_{\mathrm{SB},0})}(\Psi, \Phi) = \int_{\mathbb{R}^\nu} \overline{f(k)} (\Psi, T_{\mathrm{SB}}(k)\Phi)_{\mathcal{F}_{\mathrm{SB}}} dk, \tag{5.3}$$

where

$$T_{\mathrm{SB}}(k) := \left(\sum_{j=1}^{J} \lambda_j(k) B_j \right) \otimes 1.$$

Theorem 5.3 *Assume that (B.1)-(B.5). Let G be a nonnegative multiplication operator on $L^2(\mathbb{R}^\nu)$. Then*

$$\varphi_{\mathrm{g}} \in D(1 \otimes d\Gamma(G)^{1/2})$$

if and only if

$$\int_{\mathbb{R}^\nu} G(k) \|(H_{\mathrm{SB}} - E(H_{\mathrm{SB}}) + \omega(k))^{-1} T_{\mathrm{SB}}(k)\varphi_{\mathrm{g}}\|^2_{\mathcal{F}_{\mathrm{SB}}} dk < \infty.$$

Furthermore suppose $\varphi_{\rm g} \in D(1 \otimes d\Gamma(G)^{1/2})$. *Then*

$$\|(1 \otimes d\Gamma(G)^{1/2})\varphi_{\rm g}\|^2_{\mathcal{F}_{\rm SB}} = \int_{\mathbb{R}^\nu} G(k)\|(H_{\rm SB} - E(H_{\rm SB}) + \omega(k))^{-1} T_{\rm SB}(k)\varphi_{\rm g}\|^2_{\mathcal{F}_{\rm SB}} dk.$$

Proof: Under the identifications:

$$H = H_{\rm SB}, \quad H_{\rm I} = H_{\rm SB,I}, \quad S(k) = \omega(k), \quad T(k) = T_{\rm SB}(k), \quad \mathcal{C}_0 = C_0^2(\mathbb{R}^\nu \setminus Y), \tag{5.4}$$

it is enough to check (A.1)-(A.5) by Theorem 2.8. (A.1), (A.2) and (A.3) follow from (5.2), (5.3) and (B.5)-(2), respectively. Since

$$\|T_{\rm SB}(k)\varphi_{\rm g}\|_{\mathcal{F}_{\rm SB}} \le \sum_{j=1}^J \lambda_j(k)\|(B_j \otimes 1)\varphi_{\rm g}\|_{\mathcal{F}_{\rm SB}},$$

it is seen that

$$\|T_{\rm SB}(\cdot)\varphi_{\rm g}\|_{\mathcal{F}_{\rm SB}}, \|T_{\rm SB}(\cdot)\varphi_{\rm g}\|_{\mathcal{F}_{\rm SB}}/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$$

by (B.2). Thus (A.5) follows. We shall check (A.4). (A.4)-(1) is trivial. It is seen that for $k \in \mathbb{R}^\nu \setminus Y$,

$$e^{-is\omega(k)} = -\frac{1}{s^2}\left(\frac{\partial\omega(k)}{\partial k_\mu}\right)^{-1} \frac{\partial}{\partial k_\mu}\left(\left(\frac{\partial\omega(k)}{\partial k_\mu}\right)^{-1} \frac{\partial}{\partial k_\mu} e^{-is\omega(k)}\right), \quad \mu = 1, ..., \nu. \tag{5.5}$$

Then by the integration by parts formula,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^\nu} \overline{f(k)}(\Psi, e^{-is(H_{\rm SB} - E(H_{\rm SB}) + \omega(k))} T_{\rm SB}(k)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}} dk\right| \\
&\le \frac{1}{s^2}\sum_{j=1}^J \int_{\mathbb{R}^\nu} dk \left|\frac{\partial}{\partial k_\mu}\left(\frac{\partial\omega(k)}{\partial k_\mu}\right)^{-1} \frac{\partial}{\partial k_\mu}\left(\frac{\partial\omega(k)}{\partial k_\mu}\right)^{-1} \overline{f(k)}\lambda_j(k)\right| \\
&\qquad\qquad \left|(\Psi, e^{-is(H_{\rm SB} - E(H_{\rm SB}))}(B_j \otimes 1)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}}\right| \\
&\le \frac{1}{s^2}\sum_{j=1}^J \int_{\mathbb{R}^\nu} dk \left|\frac{\partial}{\partial k_\mu}\left(\frac{\partial\omega(k)}{\partial k_\mu}\right)^{-1} \frac{\partial}{\partial k_\mu}\left(\frac{\partial\omega(k)}{\partial k_\mu}\right)^{-1} \overline{f(k)}\lambda_j(k)\right| \\
&\qquad\qquad \|\Psi\|_{\mathcal{F}_{\rm SB}}\|(B_j \otimes 1)\varphi_{\rm g}\|_{\mathcal{F}_{\rm SB}}.
\end{aligned}$$

Since the integrand of the right-hand side above is integrable for $f \in C_0^2(\mathbb{R}^\nu \setminus Y)$, we obtain that

$$\int_{\mathbb{R}^\nu} \overline{f(k)}(\Psi, e^{-is(H_{\rm SB} - E(H_{\rm SB}) + \omega(k))} T_{\rm SB}(k)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}} dk \in L^1([0, \infty), ds).$$

Thus (A.4)-(2) follows and the proof is complete. □

Corollary 5.4 *Let G be a nonnegative multiplication operator on $L^2(\mathbb{R}^\nu)$. In addition to (B.1)-(B.5), suppose that*

$$\sqrt{G}\lambda_j/\omega \in L^2(\mathbb{R}^\nu), \quad j=1,...,J. \tag{5.6}$$

Then $\varphi_{\rm g} \in D(1\otimes d\Gamma(G)^{1/2})$. In particular suppose that

$$\lambda_j/\omega \in L^2(\mathbb{R}^\nu), \quad j=1,...,J.$$

Then $\varphi_{\rm g} \in D(1\otimes N^{1/2})$

Proof: From (5.6) it follows that

$$\int_{\mathbb{R}^\nu} G(k)\|(H_{\rm SB}-E(H_{\rm SB})+\omega(k))^{-1}T_{\rm SB}(k)\varphi_{\rm g}\|^2_{\mathcal{F}_{\rm SB}}dk$$
$$<\sum_{j=1}^J \int_{\mathbb{R}^\nu} G(k)|\lambda_j(k)/\omega(k)|^2 dk\|(B_j\otimes 1)\varphi_{\rm g}\|^2_{\mathcal{F}_{\rm SB}}<\infty.$$

Thus the corollary follows from Theorem 5.3. □

Example 5.5 *Typical examples of λ_j, $j=1,...,J$, and ω are*

$$\omega(k)=|k|, \qquad \lambda_j=\rho_j/\sqrt{\omega}, \quad j=1,...,J,$$

with some nonnegative functions ρ_j such that $\rho_j\in C^2(\mathbb{R}^\nu)$, $\rho_j/\sqrt{\omega}\in L^2(\mathbb{R}^\nu)$ and $\rho_j/\omega \in L^2(\mathbb{R}^\nu)$, $j=1,...,J$. Let $\gamma\ge 0$. In addition to (B.1) and (B.3), suppose that

$$\rho_j\omega^\gamma/\omega^{\frac{3}{2}} \in L^2(\mathbb{R}^\nu), \quad j=1,...,J.$$

Then $\varphi_{\rm g}\in D(1\otimes d\Gamma(\omega^\gamma)^{1/2})$.

5.2 Absence of ground states

Theorem 5.6 *Assume (B.1)-(B.5). Let G be a nonnegative multiplication operator on $L^2(\mathbb{R}^\nu)$. Then $H_{\rm SB}$ has no ground state $\varphi_{\rm g}$ in $D(1\otimes d\Gamma(G)^{1/2})$ such that*

$$\frac{\sqrt{G}}{\omega}\sum_{j=1}^J(\varphi_{\rm g},(B_j\otimes 1)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}}\lambda_j \notin L^2(\mathbb{R}^\nu).$$

Proof: By Theorems 5.3 and 3.1 under identification (5.4), there exists no ground state $\varphi_{\rm g}$ in $D(1\otimes d\Gamma(G)^{1/2})$ such that

$$\frac{\sqrt{G}}{\omega}(\varphi_{\rm g},T_{\rm SB}(\cdot)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}}\notin L^2(\mathbb{R}^\nu).$$

Since

$$(\varphi_{\rm g}, T_{\rm SB}(k)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}} = \sum_{j=1}^{J}(\varphi_{\rm g}, (B_j\otimes 1)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}}\lambda_j(k),$$

the theorem follows. □

Corollary 5.7 *Assume (B.1)-(B.5) and for some ℓ,*

$$\lambda_\ell/\omega \notin L^2(\mathbb{R}^\nu).$$

Then $H_{\rm SB}$ has no ground state $\varphi_{\rm g}$ in $D(1\otimes N^{1/2})$ such that $(\varphi_{\rm g}, (B_\ell\otimes 1)\varphi_{\rm g}) > 0$ and $(\varphi_{\rm g}, (B_j\otimes 1)\varphi_{\rm g})\lambda_j \geq 0$ for all j but $j \neq \ell$.

Proof: Since $\sum_{j=1}^{J}(\varphi_{\rm g}, (B_j\otimes 1)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}}\lambda_j \geq (\varphi_{\rm g}, B_\ell\varphi_{\rm g})\lambda_\ell$, we have

$$\frac{1}{\omega}\sum_{j=1}^{J}(\varphi_{\rm g}, (B_j\otimes 1)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}}\lambda_j \notin L^2(\mathbb{R}^\nu).$$

Thus by Theorem 5.6 with $G = 1$, the corollary follows. □

Example 5.8 *Let $\omega(k) = |k|$ and $\lambda_j = \rho_j/\sqrt{\omega}$ with some nonnegative functions ρ_j such that $\rho_j \in C^2(\mathbb{R}^\nu)$, $\rho_j/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$ and $\rho_j/\omega \in L^2(\mathbb{R}^\nu)$, $j = 1, ..., J$. Suppose (B.1), (B.3) and*

$$\rho_\ell\omega^\gamma/\omega^{3/2} \notin L^2(\mathbb{R}^\nu)$$

for some $\gamma \geq 0$. Then $H_{\rm SB}$ has no ground state $\varphi_{\rm g}$ in $D(1\otimes d\Gamma(\omega^\gamma)^{1/2})$ such that $(\varphi_{\rm g}, (B_\ell\otimes 1)\varphi_{\rm g}) > 0$ and $(\varphi_{\rm g}, (B_j\otimes 1)\varphi_{\rm g})\lambda_j \geq 0$ for all j but $j \neq \ell$.

5.3 Higher order regularities

In this subsection, we fix a natural number n and consider cores of $(H_{\rm SB}+1)^n$.

5.3.1 Cores of $(H_{\rm SB}+1)^n$

We define ${\rm ad}_A^k(B)$ by ${\rm ad}_A^0(B) := B$ and ${\rm ad}_A^k(B) := [A, {\rm ad}_A^{k-1}(B)]$ for $k \geq 1$. If D is an invariant subspace of A and B, we have for all $\Psi \in D$,

$$[A^k, B]\Psi = \sum_{\ell=1}^{k}\binom{k}{\ell}{\rm ad}_A^\ell(B)A^{k-\ell}\Psi, \quad {\rm ad}_A^k(BC)\Psi = \sum_{\ell=0}^{k}\binom{k}{\ell}{\rm ad}_A^\ell(B){\rm ad}_A^{k-\ell}(C)\Psi. \quad (5.7)$$

(B.6) There exists a dense subspace $\mathcal{D}_\infty \subset \mathcal{H}$ such that

(1) $\mathcal{D}_\infty \subset D(A) \cap \left[\cap_{j=1}^{J} D(B_j)\right]$,

(2) $A\mathcal{D}_\infty \subset \mathcal{D}_\infty$ and $B_j\mathcal{D}_\infty \subset \mathcal{D}_\infty$, $j = 1, ..., J$,

(3) $A^n\lceil_{\mathcal{D}_\infty}$ is essentially self-adjoint,

(4) there exist constants a_k and b_k such that for all $\Psi \in \mathcal{D}_\infty$ and $j = 1, ..., J$,

$$\|\mathrm{ad}_A^k(B_j)\Psi\|_{L^2(\mathbb{R}^\nu)} \leq a_k \|\hat{A}^{(k+1)/2}\Psi\|_{L^2(\mathbb{R}^\nu)} + b_k\|\Psi\|_{L^2(\mathbb{R}^\nu)}, \quad 0 \leq k \leq n.$$

(B.7) $\omega^k\lambda_j/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$ and $\omega^k\lambda_j \in L^2(\mathbb{R}^\nu)$ for $0 \leq k \leq n$ and $j = 1, ..., J$.

Example 5.9 *Let A and B_j, $j = 1, ..., J$, be bounded. Then (B.6) are satisfied with $\mathcal{D}_\infty = \mathcal{H}$.*

Example 5.10 *Let $\mathcal{S}(\mathbb{R}^\nu)$ be the Schwartz space of rapidly dicreasing C^∞ functions on $\mathbb{R}^\nu$ and $V \in \mathcal{S}(\mathbb{R}^\nu)$ be real-valued. Let $A = -\Delta + \beta V$ and $B_j = -i\nabla_j$, $j = 1, ..., \nu$. Then (B.6) is satisfied with $\mathcal{D}_\infty = \mathcal{S}(\mathbb{R}^\nu)$ for β with $|\beta|$ sufficiently small. See Appendix for details.*

Let us define a Hamiltonian K by

$$K := K_0 + \alpha H_{\mathrm{SB,I}},$$

where

$$K_0 := \hat{A} \otimes 1 + 1 \otimes d\Gamma(\omega).$$

The self-adjoint operator $(K+1)^n$ is defined through the spectral theorem, i.e.,

$$(K+1)^n = \int_{[E(H_{\mathrm{SB}}),\infty)} (\lambda - E(A) + 1)^n dE(\lambda),$$

where $E(\lambda)$ is the spectral projection associated with H_{SB}. Let

$$\mathcal{F}_\infty := \mathcal{D}_\infty \otimes_{\mathrm{alg}} \mathcal{F}_{\mathrm{b},0}^{C^\infty(\omega)},$$

where $C^\infty(\omega) := \cap_{n=1}^\infty D(\omega^n)$ and $\otimes_{\mathrm{alg}}$ denotes the algebraic tensor product. Since H_{SB} leaves $\mathcal{F}_\infty$ invariant, it follows that

$$\mathcal{F}_\infty \subset \cap_{n=1}^\infty D(H_{\mathrm{SB}}^n),$$

and canonical commutation relations for $a(f)$ and $a^*(g)$ hold on $\mathcal{F}_\infty$.

Theorem 5.11 *Suppose (B.1), (B.6) and (B.7). Then there exists $\alpha_* > 0$ such that for α with $|\alpha| < \alpha_*$, $(K+1)^n$ is self-adjoint on $D((K_0+1)^n)$ and essentially self-adjoint on any core of $(K_0+1)^n$. In particular it is essentially self-adjoint on $\mathcal{F}_\infty$.*

To prove Theorem 5.11 we prepare some lemmas.

Lemma 5.12 *Suppose (B.1), (B.6) and (B.7). Then there exist constants C_ℓ, $\ell = 1, ..., m$, such that, for $\Psi \in \mathcal{F}_\infty$,*

$$\|[(K_0+1)^m, H_{\mathrm{SB,I}}]\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \le \sum_{\ell=1}^m \binom{m}{\ell} C_\ell \|(K_0+1)^{m+1-(\ell/2)}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}.$$

Proof: We see that, for $\Psi \in \mathcal{F}_\infty$,

$$[(K_0+1)^m, H_{\mathrm{SB,I}}]\Psi = \sum_{\ell=1}^m \binom{m}{\ell} \mathrm{ad}^\ell_{K_0+1}(H_{\mathrm{SB,I}})(K_0+1)^{m-\ell}\Psi. \tag{5.8}$$

Using formula (5.7) we have

$$\begin{aligned}
\mathrm{ad}^\ell_{K_0+1}(H_{\mathrm{SB,I}})\Psi &= \sum_{j=1}^J \mathrm{ad}^\ell_{K_0}((B_j\otimes 1)(1\otimes\phi(\lambda_j))\Psi \\
&= \sum_{j=1}^J\sum_{k=0}^\ell \binom{\ell}{k} \mathrm{ad}^k_{K_0}(B_j\otimes 1)\mathrm{ad}^{\ell-k}_{K_0}(1\otimes\phi(\lambda_j))\Psi \\
&= \sum_{j=1}^J\sum_{k=0}^\ell \binom{\ell}{k} \mathrm{ad}^k_{\hat{A}}(B_j)\otimes \mathrm{ad}^{\ell-k}_{d\Gamma(\omega)}(\phi(\lambda_j))\Psi \\
&= \sum_{j=1}^J\sum_{k=0}^\ell \binom{\ell}{k} \mathrm{ad}^k_{\hat{A}}(B_j)\otimes \phi((-i)^{\ell-k}\omega^{\ell-k}\lambda_j)i^{\ell-k}\Psi.
\end{aligned}$$

From (B.6)-(4), it follows that

$$\begin{aligned}
&\|\mathrm{ad}^k_{\hat{A}}(B_j)\otimes\phi((-i)^{\ell-k}\omega^{\ell-k}\lambda_j)i^{\ell-k}\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \\
&\le a_k\|(\hat{A}^{(k+1)/2}\otimes\phi((-i)^{\ell-k}\omega^{\ell-k}\lambda_j))\Psi\|_{\mathcal{F}_{\mathrm{SB}}} + b_k\|(1\otimes\phi((-i)^{\ell-k}\omega^{\ell-k}\lambda_j))\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \\
&\le \xi_{\ell-k,j}\left\{a_k\|(\hat{A}^{(k+1)/2}\otimes(d\Gamma(\omega)+1)^{1/2})\Psi\|_{\mathcal{F}_{\mathrm{SB}}} + b_k\|(1\otimes(d\Gamma(\omega)+1)^{1/2})\Psi\|_{\mathcal{F}_{\mathrm{SB}}}\right\},
\end{aligned}$$

where

$$\xi_{m,j} := (\|\omega^m\lambda_j/\sqrt{\omega}\|_{L^2(\mathbb{R}^\nu)} + 2\|\omega^m\lambda_j\|_{L^2(\mathbb{R}^\nu)})/\sqrt{2}.$$

Note that

$$\|(\hat{A}^{(k+1)/2}\otimes(d\Gamma(\omega)+1)^{1/2})\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \le \|(K_0+1)^{(k+2)/2}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}, \quad k\ge 0,$$

and

$$\|(1\otimes(d\Gamma(\omega)+1)^{1/2})\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \le \|(K_0+1)^{1/2}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}.$$

Hence we have

$$\begin{aligned}
&\|\mathrm{ad}^k_{\hat{A}}(B_j)\otimes\phi((-i)^{\ell-k}\omega^{\ell-k}\lambda_j))i^{k-\ell}\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \\
&\le \xi_{\ell-k,j}\left(a_k\|(K_0+1)^{(k+2)/2}\Psi\|_{\mathcal{F}_{\mathrm{SB}}} + b_k\|(K_0+1)^{1/2}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}\right).
\end{aligned}$$

From this it follows that

$$\begin{aligned}
&\|\mathrm{ad}^{\ell}_{K_0+1}(H_{\mathrm{SB,I}})\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \\
&\le \sum_{j=1}^{J}\sum_{k=0}^{\ell}\binom{\ell}{k}\xi_{\ell-k,j}\left(a_k\|(K_0+1)^{(k+2)/2}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}+b_k\|(K_0+1)^{1/2}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}\right)\\
&\le \left(\sum_{k=0}^{\ell}\binom{\ell}{k}\sum_{j=1}^{J}\xi_{\ell-k,j}(a_k+b_k)\right)\|(K_0+1)^{(\ell+2)/2}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}. \qquad (5.9)
\end{aligned}$$

Hence from (5.9) and (5.8) the lemma follows. □

Lemma 5.13 *Suppose (B.1), (B.6) and (B.7). Then there exist constants c_k, $k = 1, ..., n$, such that*

$$\|(K_0+1)^k H_{\mathrm{SB,I}}\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \le c_{k+1}\|(K_0+1)^{k+1}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}, \quad \Psi\in\mathcal{F}_\infty, \quad 0\le k\le n-1. \quad (5.10)$$

Proof: We prove the lemma by induction with respect to k. For $k = 0$, (5.10) holds true. Assume that (5.10) is satisfied for $k = 0, 1, ..., m-1$. We see that for $\Psi \in \mathcal{F}_\infty$,

$$(K_0+1)^m H_{\mathrm{SB,I}}\Psi = H_{\mathrm{SB,I}}(K_0+1)^m\Psi + [(K_0+1)^m, H_{\mathrm{SB,I}}]\Psi.$$

By Lemma 5.12 we have

$$\begin{aligned}
&\|(K_0+1)^m H_{\mathrm{SB,I}}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}\\
&\le \|H_{\mathrm{SB,I}}(K_0+1)^m\Psi\|_{\mathcal{F}_{\mathrm{SB}}} + \sum_{\ell=1}^{m}\binom{m}{\ell}C_\ell\|(K_0+1)^{m+1-(\ell/2)}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}\\
&\le c_0\|(K_0+1)^{m+1}\Psi\|_{\mathcal{F}_{\mathrm{SB}}} + \sum_{\ell=1}^{m}\binom{m}{\ell}C_\ell\|(K_0+1)^{m+1-(\ell/2)}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}\\
&\le (c_0+\sum_{\ell=1}^{m}\binom{m}{\ell}C_\ell)\|(K_0+1)^{m+1}\Psi\|_{\mathcal{F}_{\mathrm{SB}}}
\end{aligned}$$

with some constant c_0. Thus the lemma follows with $c_{m+1} = c_0 + \sum_{\ell=1}^{m}\binom{m}{\ell}C_\ell$, $m \ge 1$. □

Proof of Theorem 5.11
We have on $\mathcal{F}_\infty$,

$$(K+1)^n = (K_0+1)^n + \alpha H_{\mathrm{I}}(n),$$

where

$$H_{\mathrm{I}}(n) := H^{(1)}_{\mathrm{I}} + \alpha H^{(2)}_{\mathrm{I}} + \cdots + \alpha^{n-1}H^{(n)}_{\mathrm{I}},$$

and

$$
\begin{aligned}
H_{\mathrm{I}}^{(1)} &:= \sum_{i=1}^{n} \underbrace{(K_0+1)\cdots \overset{i}{H_{\mathrm{SB,I}}} \cdots (K_0+1)}_{n}, \\
H_{\mathrm{I}}^{(2)} &:= \sum_{i_1<i_2}^{n} \underbrace{(K_0+1)\cdots \overset{i_1}{H_{\mathrm{SB,I}}} \cdots \overset{i_2}{H_{\mathrm{SB,I}}} \cdots (K_0+1)}_{n}, \\
H_{\mathrm{I}}^{(3)} &:= \sum_{i_1<i_2<i_3}^{n} \underbrace{(K_0+1)\cdots \overset{i_1}{H_{\mathrm{SB,I}}} \cdots \overset{i_2}{H_{\mathrm{SB,I}}} \cdots \overset{i_3}{H_{\mathrm{SB,I}}} \cdots (K_0+1)}_{n}, \\
&\ \vdots \\
H_{\mathrm{I}}^{(n)} &:= H_{\mathrm{SB,I}}^{n}.
\end{aligned}
$$

We see that

$$
(K_0+1)^n\Psi = \sum_{k=0}^{n}\sum_{\ell=0}^{k} \binom{n}{k}\binom{k}{\ell} \hat{A}^{\ell}\otimes d\Gamma(\omega)^{k-\ell}\Psi, \quad \Psi\in\mathcal{F}_\infty,
$$

and $\sum_{k=0}^{n}\sum_{\ell=0}^{k} \binom{n}{k}\binom{k}{\ell} \hat{A}^{\ell}\otimes d\Gamma(\omega)^{k-\ell}$ is essentially self-adjoint on

$$
C(\hat{A}^n)\otimes_{\mathrm{alg}} C(d\Gamma(\omega)^n),
$$

where $C(\hat{A}^n)$ and $C(d\Gamma(\omega)^n)$ are any cores of A^n and $d\Gamma(\omega)^n$, respectively. In particular $\mathcal{F}_\infty$ is a core of $(K_0+1)^n$. From Lemma 5.13 and the definition of $H_{\mathrm{I}}^{(j)}$, $j=1,...,n$, we can see that for $\Psi\in\mathcal{F}_\infty$,

$$
\|H_{\mathrm{I}}^{(j)}\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \le d_j\|(K_0+1)^n\Psi\|_{\mathcal{F}_{\mathrm{SB}}}, \quad j=1,...,n,
$$

with some constant d_j, which implies that

$$
\|H_{\mathrm{I}}(n)\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \le (d_1+|\alpha|d_2+\cdots+|\alpha|^{n-1}d_n)\|(K_0+1)^n\Psi\|_{\mathcal{F}_{\mathrm{SB}}}.
$$

Since $\mathcal{F}_\infty$ is a core of $(K_0+1)^n$, we can see that $D((K_0+1)^n)\subset D(\overline{H_{\mathrm{I}}(n)})$ and

$$
\|\overline{H_{\mathrm{I}}(n)}\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \le (d_1+|\alpha|d_2+\cdots+|\alpha|^{n-1}d_n)\|(K_0+1)^n\Psi\|_{\mathcal{F}_{\mathrm{SB}}}, \quad \Psi\in D((K_0+1)^n),
$$

where $\overline{H_{\mathrm{I}}(n)}$ denotes the closure of $H_{\mathrm{I}}(n)\lceil_{\mathcal{F}_\infty}$. For α such that

$$
|\alpha|(d_1+|\alpha|d_2+\cdots+|\alpha|^{n-1}d_n)<1,
$$

by the Kato-Rellich theorem

$$
K_n := (K_0+1)^n + \alpha\overline{H_{\mathrm{I}}(n)}
$$

is self-adjoint on $D((K_0+1)^n)$ and bounded from below. Moreover it is essentially self-adjoint on any core of $D((K_0+1)^n)$. In particular $K_n\lceil_{\mathcal{F}_\infty}$ is essentially self-adjoint. Let

$$\alpha_* := \max\left\{|\alpha| \,\Big|\, |\alpha|(d_1+|\alpha|d_2+\cdots+|\alpha|^{n-1}d_n)\le 1\right\}.$$

Since

$$(K+1)^n\lceil_{\mathcal{F}_\infty} = K_n\lceil_{\mathcal{F}_\infty} \subset K_n\lceil_{D((K_0+1)^n)}$$

and $K_n\lceil_{D((K_0+1)^n)}$ is self-adjoint for α with $|\alpha|<\alpha_*$, we conclude that

$$(K+1)^n = K_n\lceil_{D((K_0+1)^n)},$$

i.e., $(K+1)^n$ is self-adjoint on $D((K_0+1)^n)$ and essentially self-adjoint on any core of $(K_0+1)^n$ for α with $|\alpha|<\alpha_*$. Thus we get the desired results. □

5.3.2 Higher order regularities of ground states

Lemma 5.14 *Suppose (B.1), (B.6) and (B.7). Then there exist constants $\alpha_{**}>0$ and ξ_k, $k=1,...,n$, such that for α with $|\alpha|<\alpha_{**}$,*

$$\|(K_0+1)^k\Psi\|_{\mathcal{F}_{\rm SB}} \le \xi_k\|(K+1)^k\Psi\|_{\mathcal{F}_{\rm SB}}, \quad 1\le k\le n, \quad \Psi\in D((K+1)^k).$$

In particular $-1\in\rho(K)$ and $(K_0+1)^k(K+1)^{-k}$, $k=1,...,n$, is a bounded operator with

$$\|(K_0+1)^k(K+1)^{-k}\|_{\mathcal{F}_{\rm SB}\to\mathcal{F}_{\rm SB}} \le \xi_k.$$

Proof: We prove the lemma by induction with respect to k. For $k=1$, we have

$$\begin{aligned}\|(K_0+1)\Psi\|_{\mathcal{F}_{\rm SB}} &\le \|(K+1)\Psi\|_{\mathcal{F}_{\rm SB}} + |\alpha|\|H_{\rm SB,I}\Psi\|_{\mathcal{F}_{\rm SB}}\\ &\le \|(K+1)\Psi\|_{\mathcal{F}_{\rm SB}} + |\alpha|c_0\|(K_0+1)\Psi\|_{\mathcal{F}_{\rm SB}}\end{aligned}$$

with some constant c_0. Thus

$$\|(K_0+1)\Psi\|_{\mathcal{F}_{\rm SB}} \le \xi_1\|(K+1)\Psi\|_{\mathcal{F}_{\rm SB}}, \quad \Psi\in\mathcal{F}_\infty, \tag{5.11}$$

with $\xi_1 = 1/(1-|\alpha|c_0)$ follows for α with $|\alpha|<1/c_0$. Since $\mathcal{F}_\infty$ is a core of $K+1$, (5.11) can be extended for $\Psi\in D(K+1)$. Thus the lemma follows for $k=1$. Suppose that the lemma holds for $k=m<n$. Note that for $\Psi\in\mathcal{F}_\infty$,

$$\begin{aligned}(K_0+1)^{m+1}\Psi = {}&(K_0+1)(K+1)^{-1}(K_0+1)^m(K+1)\Psi\\ &+(K_0+1)(K+1)^{-1}[K+1,(K_0+1)^m]\Psi.\end{aligned}$$

We have

$$\|(K_0+1)(K+1)^{-1}(K_0+1)^m(K+1)\Psi\|_{\mathcal{F}_{\rm SB}} \le \xi_1\xi_m\|(K+1)^{m+1}\Psi\|_{\mathcal{F}_{\rm SB}} \tag{5.12}$$

and by Lemma 5.12,

$$\begin{aligned}
&\|(K_0+1)(K+1)^{-1}[K+1,(K_0+1)^m]\Psi\|_{\mathcal{F}_{\rm SB}}\\
&\le \xi_1|\alpha|\|[H_{\rm SB,I},(K_0+1)^m]\Psi\|_{\mathcal{F}_{\rm SB}}\\
&\le \xi_1|\alpha|\sum_{\ell=1}^m \binom{m}{\ell} C_\ell\|(K_0+1)^{m+1-(\ell/2)}\Psi\|_{\mathcal{F}_{\rm SB}}\\
&\le \xi_1|\alpha|\left(\sum_{\ell=2}^m \binom{m}{\ell} C_\ell\|(K_0+1)^{m+1-(\ell/2)}\Psi\|_{\mathcal{F}_{\rm SB}} + mC_1\|(K_0+1)^{m+(1/2)}\Psi\|_{\mathcal{F}_{\rm SB}}\right)\\
&\le \xi_1|\alpha|\left(\sum_{\ell=2}^m \binom{m}{\ell} C_\ell\|(K_0+1)^{m}\Psi\|_{\mathcal{F}_{\rm SB}} + mC_1\|(K_0+1)^{m+1}\Psi\|_{\mathcal{F}_{\rm SB}}\right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\|(K_0+1)^{m+1}\Psi\|_{\mathcal{F}_{\rm SB}}\\
&\le \frac{\xi_1}{1-\xi_1|\alpha|mC_1}\left(\xi_m\|(K+1)^{m+1}\Psi\|_{\mathcal{F}_{\rm SB}} + |\alpha|\sum_{\ell=2}^m \binom{m}{\ell} C_\ell\|(K_0+1)^{m}\Psi\|_{\mathcal{F}_{\rm SB}}\right)\\
&\le \frac{\xi_1}{1-\xi_1|\alpha|mC_1}(\xi_m+|\alpha|\sum_{\ell=2}^m \binom{m}{\ell} C_\ell\xi_m)\|(K+1)^{m+1}\Psi\|_{\mathcal{F}_{\rm SB}}, \quad \Psi\in\mathcal{F}_\infty,
\end{aligned} \tag{5.13}$$

for α with $|\alpha|<1/\xi_1 mC_1$. Since $\mathcal{F}_\infty$ is a core of $(K+1)^{m+1}$, (5.13) can be extended for $\Psi\in D((K+1)^{m+1})$. Thus the lemma follows with $\alpha_{**}:=1/(n\xi_1C_1)$. □

Let

$$\epsilon_0 := E(A)-1.$$

Corollary 5.15 *Suppose (B.1), (B.6) and (B.7). Then $\epsilon_0\in\rho(H_{\rm SB})$ and for α with $|\alpha|<\min\{\alpha_*,\alpha_{**}\}$, operator $(1\otimes d\Gamma(\omega)^m)(H_{\rm SB}-\epsilon_0)^{-n}$ for $m\le n$ is a bounded operator.*

Proof: We have

$$\|(1\otimes d\Gamma(\omega)^m)\Psi\|_{\mathcal{F}_{\rm SB}} \le \|(K_0+1)^m\Psi\|_{\mathcal{F}_{\rm SB}} \le \|(K_0+1)^n\Psi\|_{\mathcal{F}_{\rm SB}} \tag{5.14}$$

for $\Psi\in\mathcal{F}_\infty$. Since $\mathcal{F}_\infty$ is a core of $(K_0+1)^n$, (5.14) can be extended for $\Psi\in D((K_0+1)^n)$. Thus $(1\otimes d\Gamma(\omega)^m)(K_0+1)^{-n}$ is a bounded operator with

$$\|(1\otimes d\Gamma(\omega)^m)(K_0+1)^{-n}\|_{\mathcal{F}_{\rm SB}\to\mathcal{F}_{\rm SB}}\le 1.$$

Hence Lemma 5.14 yields that

$$\begin{aligned}
&\|(1\otimes d\Gamma(\omega)^m)(H_{\rm SB}-\epsilon_0)^{-n}\Psi\|_{\mathcal{F}_{\rm SB}}\\
&\leq \|(1\otimes d\Gamma(\omega)^m)(K_0+1)^{-n}(K_0+1)^n(K+1)^{-n}\Psi\|_{\mathcal{F}_{\rm SB}}\\
&\leq \xi_n\|\Psi\|_{\mathcal{F}_{\rm SB}}.
\end{aligned}$$

Thus the corollary follows. □

Lemma 5.16 *Suppose (B.1), (B.6) and (B.7). Then for α with $|\alpha|<\alpha_*$,*

$$T_{\rm SB}(k):D((H_{\rm SB}-\epsilon_0)^m)\longrightarrow D((H_{\rm SB}-\epsilon_0)^{m-1}),\quad 1\leq m\leq n. \tag{5.15}$$

In particular

$$D((H_{\rm SB}-\epsilon_0)^n)\subset D(T_{\rm SB}(k_m)^*e^{it_m\widehat{H}_{\rm SB}}\cdots T_{\rm SB}(k_1)^*e^{it_1\widehat{H}_{\rm SB}}),\quad 1\leq m\leq n, \tag{5.16}$$

where $\widehat{H}_{\rm SB}:=H_{\rm SB}-E(H_{\rm SB})$.

Proof: Let $\Psi\in\mathcal{F}_\infty$. We have

$$\begin{aligned}
&(K_0+1)^{m-1}(B_j\otimes 1)\Psi\\
&=(B_j\otimes 1)(K_0+1)^{m-1}\Psi+[(K_0+1)^{m-1},B_j\otimes 1]\Psi\\
&=(B_j\otimes 1)(K_0+1)^{m-1}\Psi+\sum_{\ell=1}^{m-1}\binom{m-1}{\ell}\mathrm{ad}_{K_0}^\ell(B_j\otimes 1)(K_0+1)^{m-1-\ell}\Psi.
\end{aligned}$$

By (B.6)-(4), we have

$$\begin{aligned}
&\|\mathrm{ad}_{K_0}^\ell(B_j\otimes 1)\Psi\|_{\mathcal{F}_{\rm SB}}=\|(\mathrm{ad}_A^\ell(B_j)\otimes 1)\Psi\|_{\mathcal{F}_{\rm SB}}\\
&\leq a_\ell\|\left(\hat{A}^{(\ell+1)/2}\otimes 1\right)\Psi\|_{\mathcal{F}_{\rm SB}}+b_\ell\|\Psi\|_{\mathcal{F}_{\rm SB}}\leq c_\ell\|(K_0+1)^{(\ell+1)/2}\Psi\|_{\mathcal{F}_{\rm SB}}
\end{aligned}$$

with some constant c_ℓ. Hence it follows that

$$\begin{aligned}
&\|(K_0+1)^{m-1}(B_j\otimes 1)\Psi\|_{\mathcal{F}_{\rm SB}}\\
&\leq c_0\|(K_0+1)^{m-(1/2)}\Psi\|_{\mathcal{F}_{\rm SB}}+\sum_{\ell=1}^{m-1}\binom{m-1}{\ell}c_\ell\|(K_0+1)^{m-(\ell+1)/2}\Psi\|_{\mathcal{F}_{\rm SB}}\\
&\leq C\|(K_0+1)^m\Psi\|_{\mathcal{F}_{\rm SB}}
\end{aligned}$$

with come constant C. Thus for $\Psi,\Phi\in\mathcal{F}_\infty$, it follows that

$$|((B_j\otimes 1)\Psi,(K_0+1)^{m-1}\Phi)_{\mathcal{F}_{\rm SB}}|\leq C\|(K_0+1)^m\Psi\|_{\mathcal{F}_{\rm SB}}\|\Phi\|_{\mathcal{F}_{\rm SB}}. \tag{5.17}$$

It is seen that

$$\|(B_j \otimes 1)\Psi\|_{\mathcal{F}_{\rm SB}} \le C\|(K_0+1)^{1/2}\Psi\|_{\mathcal{F}_{\rm SB}}, \quad \Psi \in D((K_0+1)^{1/2})).$$

From this it follows that

$$\|(B_j \otimes 1)\Psi\|_{\mathcal{F}_{\rm SB}} \le C\|(K_0+1)^{m}\Psi\|_{\mathcal{F}_{\rm SB}}, \quad \Psi \in D((K_0+1)^{m})). \tag{5.18}$$

Since $\mathcal{F}_\infty$ is a core of $(K_0+1)^m$ and B_j is a closed operator, using (5.18) we can extend (5.17) for $\Psi \in D((K_0+1)^m)$ and $\Phi \in D((K_0+1)^{m-1})$. Set

$$Q(\Psi,\Phi) := ((B_j \otimes 1)\Psi, (K_0+1)^{m-1}\Phi)_{\mathcal{F}_{\rm SB}}, \quad \Psi \in D((K_0+1)^m), \Phi \in D((K_0+1)^{m-1}).$$

For each fixed $\Psi \in D((K_0+1)^m)$, $Q(\Psi,\Phi)$ can be extended for all $\Phi \in \mathcal{F}_{\rm SB}$ by (5.17) as a linear bounded functional, which is denoted by $\overline{Q}(\Psi,\Phi)$. Thus, by the Riesz representation theorem, there exists a unique $F_\Psi \in \mathcal{F}_{\rm SB}$ such that

$$\overline{Q}(\Psi,\Phi) = (F_\Psi,\Phi)_{\mathcal{F}_{\rm SB}}, \quad \Psi \in D((K_0+1)^m), \Phi \in \mathcal{F}_{\rm SB}. \tag{5.19}$$

In particular

$$((B_j \otimes 1)\Psi, (K_0+1)^{m-1}\Phi)_{\mathcal{F}_{\rm SB}} = (F_\Psi,\Phi)_{\mathcal{F}_{\rm SB}}, \quad \Psi \in D((K_0+1)^m), \Phi \in D((K_0+1)^{m-1}).$$

This implies that $(B_j \otimes 1)\Psi \in D((K_0+1)^{m-1})$ for $\Psi \in D((K_0+1)^m)$, i.e.,

$$B_j \otimes 1 : D((K_0+1)^m) \to D((K_0+1)^{m-1}).$$

Since $T_{\rm SB}(k) = (\sum_{j=1}^J \lambda_j(k)B_j) \otimes 1$, we have

$$T_{\rm SB}(k) : D((K_0+1)^m) \to D((K_0+1)^{m-1}).$$

(5.15) follows from the fact that $D((K_0+1)^m) = D((K+1)^m) = D((H_{\rm SB}-\epsilon_0)^m)$. Noting that

$$e^{it\widehat{H}_{\rm SB}} : D((H_{\rm SB}-\epsilon_0)^m) \to D((H_{\rm SB}-\epsilon_0)^m),$$

we can conclude (5.16) and the proof is complete. □

Theorem 5.17 *Suppose (B.1), (B.4)–(B.7) and*

$$\lambda_j/\omega \in L^2(\mathbb{R}^\nu), \quad j = 1,..,J. \tag{5.20}$$

Then for α with $|\alpha| < \min\{\alpha_*, \alpha_{**}\}$,

$$\varphi_{\rm g} \in D(1 \otimes N^{n/2}).$$

Proof: By Theorem 4.6 it is enough to check (H.1)-(H.5) under identifcation (5.4) and

$$\mathcal{E} = D((H_{\rm SB} - \epsilon_0)^n), \quad \mathcal{C}_n = C_0^2(\mathbb{R}^\nu \setminus Y).$$

Since $T_{\rm SB}(k) = \left(\sum_{j=1}^J \lambda_j(k) B_j\right) \otimes 1$, (H.1) is satisfied. In Corollary 5.15 we checked that $(1 \otimes d\Gamma(\omega)^m)(H_{\rm SB} - \epsilon_0)^{-n}$, $m \le n$, is a bounded operator. This implies (H.2). Let

$$\beta := \sum_{j=1}^J a_j \|\lambda_j/\sqrt{\omega}\|^2_{L^2(\mathbb{R}^\nu)}.$$

Then

$$\begin{aligned}
\|(\hat{A}^{1/2} \otimes 1)\Psi\|^2_{\mathcal{F}_{\rm SB}} &\le (\Psi, H_{\rm SB,0}\Psi)_{\mathcal{F}_{\rm SB}} + E(A)\|\Psi\|^2_{\mathcal{F}_{\rm SB}} \\
&\le (\frac{1}{2} + |E(A)|)\|\Psi\|^2_{\mathcal{F}_{\rm SB}} + \frac{1}{2}\|H_{\rm SB,0}\Psi\|^2_{\mathcal{F}_{\rm SB}} \\
&\le (\frac{1}{2} + |E(A)|)\|\Psi\|^2_{\mathcal{F}_{\rm SB}} + \frac{1}{2(1-|\alpha|\beta)^2}\|H_{\rm SB}\Psi\|^2_{\mathcal{F}_{\rm SB}},
\end{aligned}$$

and

$$\|(\hat{A}^{1/2} \otimes 1)\Psi\|_{\mathcal{F}_{\rm SB}} \le \frac{1}{\sqrt{2}(1-|\alpha|\beta)}\|H_{\rm SB}\Psi\|_{\mathcal{F}_{\rm SB}} + \left(\frac{1}{2} + |E(A)|\right)^{1/2}\|\Psi\|_{\mathcal{F}_{\rm SB}}.$$

Hence we have

$$\begin{aligned}
&\|T_{\rm SB}(k)\Psi\|_{\mathcal{F}_{\rm SB}} \le \sum_{j=1}^J \|\lambda_j(k)(B_j \otimes 1)\Psi\|_{\mathcal{F}_{\rm SB}} \\
&\le \sum_{j=1}^J |\lambda_j(k)| \left(a_j\|(\hat{A}^{1/2} \otimes 1)\Psi\|_{\mathcal{F}_{\rm SB}} + b_j\|\Psi\|_{\mathcal{F}}\right) \\
&\le \sum_{j=1}^J |\lambda_j(k)| \left\{\frac{a_j/\sqrt{2}}{1-|\alpha|\beta}\|H_{\rm SB}\Psi\|_{\mathcal{F}_{\rm SB}} + \left(b_j + a_j\left(\frac{1}{2} + |E(A)|\right)^{1/2}\right)\|\Psi\|_{\mathcal{F}_{\rm SB}}\right\} \\
&\le \sum_{j=1}^J |\lambda_j(k)| \left(d_j\|\widehat{H}_{\rm SB}\Psi\| + d'_j\|\Psi\|\right),
\end{aligned}$$

where

$$d_j = \frac{a_j/\sqrt{2}}{1-|\alpha|\beta}, \quad d'_j = b_j + a_j\left(\frac{1}{2} + |E(A)|\right)^{1/2} + \frac{a_j|E(H_{\rm SB})|/\sqrt{2}}{1-|\alpha|\beta}.$$

Thus we can obtain that

$$\|T_{\rm SB}(k)(\widehat{H}_{\rm SB} + \omega(k))^{-1}\Psi\|_{\mathcal{F}_{\rm SB}} \le \sum_{j=1}^J |\lambda_j(k)| \left(d_j + d'_j\frac{1}{\omega(k)}\right)\|\Psi\|_{\mathcal{F}_{\rm SB}},$$

from which it follows that

$$\|[(\widehat{H}_{\rm SB}+\omega(k))^{-1}T_{\rm SB}(k)]\Psi\|_{\mathcal{F}_{\rm SB}} \le \sum_{j=1}^J |\lambda_j(k)| \left(d_j + d_j' \frac{1}{\omega(k)}\right) \|\Psi\|_{\mathcal{F}_{\rm SB}}.$$

From (B.2) and (5.20), it follows that

$$\sqrt{\omega}\|[(\widehat{H}_{\rm SB}+\omega(\cdot))^{-1}T_{\rm SB}(\cdot)]\Psi\|_{\mathcal{F}_{\rm SB}}, \|[(\widehat{H}_{\rm SB}+\omega(\cdot))^{-1}T_{\rm SB}(\cdot)]\Psi\|_{\mathcal{F}_{\rm SB}} \in L^2(\mathbb{R}^\nu).$$

Thus (H.4) and (H.5) follow. Finally we shall check from (H.3)-(1) to (H.3)-(4). (H.3)-(1) is trivial. In Lemma 5.16 we obtained that

$$D((H_{\rm SB}-\epsilon_0)^n) \subset D(T_{\rm SB}(k_m)^* e^{it_m \widehat{H}_{\rm SB}} \cdots T_{\rm SB}(k_1)^* e^{it_1\widehat{H}_{\rm SB}}), \quad 1 \le m \le n,$$

which implies (H.3)-(2). Note that

$$\begin{aligned} &T_{\rm SB}(k_m)^* e^{it_m \widehat{H}_{\rm SB}} \cdots T_{\rm SB}(k_1)^* e^{it_1\widehat{H}_{\rm SB}} \\ &= \sum_{\ell_1,\dots,\ell_m}^J \overline{\lambda_{\ell_m}(k_m)} \cdots \overline{\lambda_{\ell_1}(k_1)} (B_{\ell_m}\otimes 1) e^{it_m \widehat{H}_{\rm SB}} \cdots (B_{\ell_1}\otimes 1) e^{it_1\widehat{H}_{\rm SB}}, \end{aligned}$$

Using (5.5) and the integration by parts formula, we obtain that for $\Psi \in D((H_{\rm SB}-\epsilon_0)^n)$, $f_j \in C_0^2(\mathbb{R}^\nu \setminus Y)$, $j=1,...,n$, and $T_m = t_1+\cdots+t_m$,

$$\begin{aligned} &\Big| \int_{\mathbb{R}^\nu} dk_m e^{-iT_m\omega(k_m)} \overline{f_m(k_m)} \\ &(T_{\rm SB}(k_m)^* e^{it_m \widehat{H}_{\rm SB}} \cdots T_{\rm SB}(k_1)^* e^{it_1\widehat{H}_{\rm SB}}\Psi, (1\otimes a(e^{iT_m\omega} f_{m+1})\cdots a(e^{iT_m\omega} f_n))\varphi_{\rm g})_{\mathcal{F}_{\rm SB}} \Big| \\ &\le \frac{1}{|T_m|^2} \sum_{j=1}^J \int_{\mathbb{R}^\nu} dk_m \Big| F_j(k_m) (T_{\rm SB}(k_{m-1})^* e^{it_{m-1}\widehat{H}_{\rm SB}} \cdots T_{\rm SB}(k_1)^* e^{it_1 \widehat{H}_{\rm SB}}\Psi, \\ &\qquad e^{-it_m\widehat{H}_{\rm SB}} B_j \otimes a(e^{iT_m\omega} f_{m+1}) \cdots a(e^{iT_m\omega} f_n)\varphi_{\rm g})_{\mathcal{F}_{\rm SB}} \Big| \\ &\le \frac{1}{|T_m|^2} \sum_{j=1}^J \int_{\mathbb{R}^\nu} dk_m |F_j(k_m)| \left\| T_{\rm SB}(k_{m-1})^* e^{it_{m-1}\widehat{H}_{\rm SB}} \cdots T_{\rm SB}(k_1)^* e^{it_1\widehat{H}_{\rm SB}}\Psi \right\|_{\mathcal{F}_{\rm SB}} \\ &\qquad \left\| B_j \otimes a(e^{iT_m\omega} f_{m+1}) \cdots a(e^{iT_m\omega} f_n)\varphi_{\rm g}\right\|_{\mathcal{F}_{\rm SB}}, \end{aligned} \tag{5.21}$$

where

$$F_j(k_m) = \frac{\partial}{\partial k_{m\mu}} \left(\frac{\partial \omega(k_m)}{\partial k_{m\mu}}\right)^{-1} \frac{\partial}{\partial k_{m\mu}} \left(\frac{\partial \omega(k_m)}{\partial k_{m\mu}}\right)^{-1} \overline{f_m(k_m)} \lambda_j(k_m).$$

Since

$$\left\| [B_j \otimes a(e^{iT_m\omega} f_{m+1}) \cdots a(e^{iT_m\omega} f_n)]\varphi_{\rm g}\right\|_{\mathcal{F}_{\rm SB}} \le C \left\| (B_j \otimes d\Gamma(\omega)^{(n-m)/2})\varphi_{\rm g}\right\|_{\mathcal{F}_{\rm SB}}$$

with some C independent of T_m. Then the integrand of the right-hand side of (5.21) is independent of T_m and integrable by (B.4) and (B.5). Thus the right-hand side of (5.21) is in $L^1([0,\infty);dt_m)$. Hence (H.3)-(3) follows. Let

$$R_k^\sigma = (\widehat{H}_{\mathrm{SB}} + \sum_{\ell=k}^n \omega(k_{\sigma(\ell)}))^{-1}, \quad \sigma \in \mathcal{P}_n.$$

Using (5.5) and the integration by parts formula again, we see that for $\Psi \in D((H_{\mathrm{SB}} - \epsilon_0)^n)$, $f_j \in C_0^2(\mathbb{R}^\nu \setminus Y)$, $j = 1, ..., n$,

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^\nu} dk_{\sigma(m)} \cdots \int_{\mathbb{R}^\nu} dk_{\sigma(n)} \prod_{j=m}^n f_{\sigma(j)}(k_{\sigma(j)}) \\
&\qquad (T_{\mathrm{SB}}(k_{\sigma(m)})^* e^{it_m(\widehat{H}_{\mathrm{SB}} + \sum_{j=m}^n \omega(k_{\sigma(j)}))} \cdots T_{\mathrm{SB}}(k_{\sigma(1)})^* e^{it_1(\widehat{H}_{\mathrm{SB}} + \sum_{j=1}^n \omega(k_{\sigma(j)}))}\Psi, \\
&\qquad\qquad R_{m+1}^\sigma T_{\mathrm{SB}}(k_{\sigma(m+1)}) \cdots R_n^\sigma T_{\mathrm{SB}}(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}_{\mathrm{SB}}} \Big| \\
&= \Big| \int_{\mathbb{R}^\nu} dk_{\sigma(m)} \cdots \int_{\mathbb{R}^\nu} dk_{\sigma(n)} \prod_{j=m}^n f_{\sigma(j)}(k_{\sigma(j)}) \\
&\qquad e^{it_1\omega(k_{\sigma(1)})} e^{i(t_1+t_2)\omega(k_{\sigma(2)})} \cdots e^{i(t_1+\cdots+t_{m-1})\omega(k_{\sigma(m-1)})} e^{i(t_1+\cdots+t_m)(\omega(k_{\sigma(m)})+\cdots+\omega(k_{\sigma(n)}))} \\
&\qquad (T_{\mathrm{SB}}(k_{\sigma(m)})^* e^{it_m\widehat{H}_{\mathrm{SB}}} \cdots T_{\mathrm{SB}}(k_{\sigma(1)}^* e^{it_1\widehat{H}_{\mathrm{SB}}}\Psi, \\
&\qquad\qquad R_{m+1}^\sigma T_{\mathrm{SB}}(k_{\sigma(m+1)}) \cdots R_n^\sigma T_{\mathrm{SB}}(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}_{\mathrm{SB}}} \Big| \\
&= \frac{1}{t_1+\cdots+t_m} \Big| \sum_{j=1}^J \int_{\mathbb{R}^\nu} dk_{\sigma(m)} \cdots \int_{\mathbb{R}^\nu} dk_{\sigma(n)} \prod_{j=m+1}^n f_{\sigma(j)}(k_{\sigma(j)}) F_j(k_{\sigma(m)}) \\
&\qquad e^{it_1\omega(k_{\sigma(1)})} e^{i(t_1+t_2)\omega(k_{\sigma(2)})} \cdots e^{i(t_1+\cdots+t_{m-1})\omega(k_{\sigma(m-1)})} e^{i(t_1+\cdots+t_m)(\omega(k_{\sigma(m+1)})+\cdots+\omega(k_{\sigma(n)}))} \\
&\qquad (T_{\mathrm{SB}}(k_{s(m-1)})^* e^{it_{m-1}\widehat{H}_{\mathrm{SB}}} \cdots T_{\mathrm{SB}}(k_{\sigma(1)})^* e^{it_1\widehat{H}_{\mathrm{SB}}}\Psi, \\
&\qquad\qquad e^{-it_m\widehat{H}_{\mathrm{SB}}}(B_j \otimes 1) R_{m+1}^\sigma T_{\mathrm{SB}}(k_{\sigma(m+1)}) \cdots R_n^\sigma T_{\mathrm{SB}}(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}_{\mathrm{SB}}} \Big| \\
&\le \frac{1}{t_1+\cdots+t_m} \sum_{j=1}^J \int_{\mathbb{R}^\nu} dk_{\sigma(m)} \cdots \int_{\mathbb{R}^\nu} dk_{\sigma(n)} \Big| \prod_{j=m+1}^n f_{\sigma(j)}(k_{\sigma(j)}) \Big| |F_j(k_{\sigma(m)})| \\
&\qquad \|T_{\mathrm{SB}}(k_{s(m-1)})^* e^{it_{m-1}\widehat{H}_{\mathrm{SB}}} \cdots T_{\mathrm{SB}}(k_{\sigma(1)})^* e^{it_1\widehat{H}_{\mathrm{SB}}}\Psi\|_{\mathcal{F}_{\mathrm{SB}}} \\
&\qquad \|(B_j \otimes 1) R_{m+1}^\sigma T_{\mathrm{SB}}(k_{\sigma(m+1)}) \cdots R_n^\sigma T_{\mathrm{SB}}(k_{\sigma(n)})\varphi_{\mathrm{g}})_{\mathcal{F}_{\mathrm{SB}}}\|_{\mathcal{F}_{\mathrm{SB}}},
\end{aligned} \tag{5.22}$$

where

$$F_j(k_{\sigma(m)}) = \frac{\partial}{\partial k_{\sigma(m)}{}_\mu} \left(\frac{\partial \omega(k_{\sigma(m)})}{\partial k_{\sigma(m)}{}_\mu} \right)^{-1} \frac{\partial}{\partial k_{\sigma(m)}{}_\mu} \left(\frac{\partial \omega(k_{\sigma(m)})}{\partial k_{\sigma(m)}{}_\mu} \right)^{-1} \overline{f_{\sigma(m)}(k_{\sigma(m)})} \lambda_j(k_{\sigma(m)}).$$

Then the right-hand side of (5.22) is in $L^1([0,\infty);dt_m)$. Hence (H.3)-(4) follows. Thus the proof is complete. □

6 Concluding Remarks

In this section we give some remarks on ground states of the Nelson model and the Pauli-Fierz model. Through this section we assume that ω is the multiplication operator on $L^2(\mathbb{R}^\nu)$ by

$$\omega(k) := |k|. \tag{6.1}$$

6.1 The Nelson models

The so-called Nelson model was introduced by Nelson [19], which describes an iteraction between nonrelativistic particles and a scalar quantum field. Here we consider only the case where one nonrelativistic particle interacts with a scalar quantum field in $\mathbb{R}^\nu$. Then the Hilbert space for the Nelson model is defined by

$$\mathcal{F}_{\mathrm{N}} := L^2(\mathbb{R}^\nu) \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^\nu)) \cong \int_{\mathbb{R}^\nu}^{\oplus} \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^\nu)) dx,$$

where $\int_{\mathbb{R}^\nu}^{\oplus} \cdots dx$ denotes a constant fiber direct integral [23]. The Nelson Hmailtonian, H_{N}, is defined by

$$H_{\mathrm{N}} := H_{\mathrm{N},0} + gH_{\mathrm{N,I}},$$

where $g \in \mathbb{R}$ is a coupling constant,

$$H_{\mathrm{N},0} := \left(-\frac{1}{2}\Delta + V\right) \otimes 1 + 1 \otimes d\Gamma(\omega)$$

with $V : \mathbb{R}^\nu \to \mathbb{R}$ an external potential, and

$$H_{\mathrm{N,I}} := \int_{\mathbb{R}^\nu}^{\oplus} \phi(x) dx$$

with

$$\phi(x) := \frac{1}{\sqrt{2}} \left\{ a(f^x_{\omega\lambda}) + a^*(f^x_{\omega\lambda}) \right\}.$$

Here for each $x \in \mathbb{R}^\nu$ we define $f^x_{\omega\lambda} \in L^2(\mathbb{R}^\nu)$ by

$$f^x_{\omega\lambda}(k) := \lambda(k) e^{-ikx} / \sqrt{\omega(k)}.$$

Proposition 6.1 *Assume that $\lambda/\sqrt{\omega}, \lambda/\omega \in L^2(\mathbb{R}^\nu)$ and that V is relatively bounded with respect to $-\frac{1}{2}\Delta$ with a relative bound strictly less than one. Then for all $g \in \mathbb{R}$, H_{N} is self-adjoint on $D(H_{\mathrm{N},0})$ and bounded from below. Moreover it is essentially self-adjoint on any core of $H_{\mathrm{N},0}$.*

Proof: See [19]. □

We see that

$$[1\otimes a(f),H_{\mathrm{N,I}}]_W^{D(H_{\mathrm{N}})}(\Psi,\Phi)=\int_{\mathbb{R}^\nu}\bar{f}(k)(\Psi,T_{\mathrm{N}}(k)\Phi)_{\mathcal{F}_{\mathrm{N}}}dk,$$

where

$$T_{\mathrm{N}}(k):=\frac{\lambda(k)}{\sqrt{2\omega(k)}}\int_{\mathbb{R}^\nu}^{\oplus}e^{-ikx}dx.$$

Under the following identifications

$$H=H_{\mathrm{N}},\quad H_{\mathrm{I}}=H_{\mathrm{N,I}},\quad S(k)=\omega(k),\quad T(k)=T_{\mathrm{N}}(k),\quad \mathcal{C}_0=C_0^2(\mathbb{R}^\nu\setminus Y),$$

We can check that H_{N} satisfies assumptions (A.1)-(A.5). We introduce an assumption.

(IR) On a neighboorhood of $\{0\}$, λ is continuous and $\lambda(k)\sim|k|^p$, where $2p\le 3-\nu$.

Suppose (IR). Then

$$\lambda/\omega^{3/2}\notin L^2(\mathbb{R}^\nu), \tag{6.2}$$

and if a ground state φ_{g} of H_{N} exists, then

$$\frac{1}{\omega}(\varphi_{\mathrm{g}},T_{\mathrm{N}}(\cdot)\varphi_{\mathrm{g}})_{\mathcal{F}_{\mathrm{N}}}=(\varphi_{\mathrm{g}},e^{-i\cdot x}\varphi_{\mathrm{g}})_{\mathcal{F}_{\mathrm{N}}}\frac{\lambda}{\sqrt{2}\omega^{3/2}}\notin L^2(\mathbb{R}^3). \tag{6.3}$$

Thus it follows from Theorem 3.1 that, if (6.2) holds, then H_{N} has no ground state φ_{g} in $D(1\otimes N^{1/2})$. Actually the absence of ground states of H_{N} under condition (6.2) has been established. See [10, 18].

6.2 Infrared regular representation of the Nelson models

The Nelson model in a non-Fock representation is introduced and investigated in [2]. The Nelson Hamiltonian in a non-Fock representation is given as a self-adjoint operator on $\mathcal{F}_{\mathrm{N}}$ by

$$H_{\mathrm{N}}^{\mathrm{reg}}:=H_{\mathrm{N},0}+gH_{\mathrm{N,I}}^{\mathrm{reg}},$$

where $H_{\mathrm{N,I}}^{\mathrm{reg}}:=\tilde{H}_{\mathrm{N,I}}^{\mathrm{reg}}-gW\otimes 1+gc$, and

$$W(x):=\int_{\mathbb{R}^\nu}\frac{\lambda(k)^2}{\omega(k)^2}e^{-ikx}dk,\quad c:=\frac{1}{2}\|\lambda/\omega\|_{L^2(\mathbb{R}^\nu)}^2,\quad \tilde{H}_{\mathrm{N,I}}^{\mathrm{reg}}:=\int_{\mathbb{R}^\nu}^{\oplus}\phi_{\mathrm{reg}}(x)dx.$$

Here

$$\phi_{\mathrm{reg}}(x):=\frac{1}{\sqrt{2}}\left\{a^*(f_{\omega\lambda}^x-f_{\omega\lambda^0})+a(f_{\omega\lambda}^x-f_{\omega\lambda^0})\right\}.$$

It is known [2] that in the case of $\lambda/\omega^{3/2} \in L^2(\mathbb{R}^\nu)$, there exists a unitary operator $\mathcal{U}$ on $\mathcal{F}_{\mathrm{N}}$ such that

$$\mathcal{U} H_{\mathrm{N}} \mathcal{U}^{-1} = H_{\mathrm{N}}^{\mathrm{reg}}. \tag{6.4}$$

However, in the case of $\lambda/\omega^{3/2} \notin L^2(\mathbb{R}^\nu)$, H_{N} and $H_{\mathrm{N}}^{\mathrm{reg}}$ are *not* unitarily equivalent. We see that

$$[1 \otimes a(f), H_{\mathrm{N,I}}^{\mathrm{reg}}]_W^{D(H_{\mathrm{N}}^{\mathrm{reg}})}(\Psi, \Phi) = \int_{\mathbb{R}^\nu} \overline{f}(k)(\Psi, T_{\mathrm{N}}^{\mathrm{reg}}(k)\Phi)_{\mathcal{F}_{\mathrm{N}}} dk,$$

where

$$T_{\mathrm{N}}^{\mathrm{reg}}(k) := \frac{\lambda(k)}{\sqrt{2\omega(k)}} \int_{\mathbb{R}^\nu}^{\oplus} (e^{-ikx} - 1) dx.$$

Suppose that a ground state φ_{g} of $H_{\mathrm{N}}^{\mathrm{reg}}$ exists and

$$\|(|x| \otimes 1)\varphi_{\mathrm{g}}\|_{\mathcal{F}_{\mathrm{N}}} < \infty. \tag{6.5}$$

Actually for some V, e.g., $V(x) = -1/|x|, |x|^2$, (6.5) has been established. See, e.g., [7, 8, 16]. Then

$$\begin{aligned} \frac{1}{\omega(k)} |(\varphi_{\mathrm{g}}, T_{\mathrm{N}}^{\mathrm{reg}}(k)\varphi_{\mathrm{g}})_{\mathcal{F}_{\mathrm{N}}}| &\leq \frac{\lambda(k)|k|}{\sqrt{2}\omega(k)^{3/2}} \|\varphi_{\mathrm{g}}\|_{\mathcal{F}_{\mathrm{N}}} \|(|x| \otimes 1)\varphi_{\mathrm{g}}\|_{\mathcal{F}_{\mathrm{N}}} \\ &= \frac{\lambda(k)}{\sqrt{2\omega(k)}} \|\varphi_{\mathrm{g}}\|_{\mathcal{F}_{\mathrm{N}}} \|(|x| \otimes 1)\varphi_{\mathrm{g}}\|_{\mathcal{F}_{\mathrm{N}}}. \end{aligned}$$

Hence

$$\frac{1}{\omega}(\varphi_{\mathrm{g}}, T_{\mathrm{N}}^{\mathrm{reg}}(\cdot)\varphi_{\mathrm{g}})_{\mathcal{F}_{\mathrm{N}}} \in L^2(\mathbb{R}^\nu). \tag{6.6}$$

Remark 6.2 *We do not assume $\lambda/\omega^{3/2} \in L^2(\mathbb{R}^\nu)$ in (6.6).*

In [2], the existence of a ground state φ_{g} of $H_{\mathrm{N}}^{\mathrm{reg}}$ such that $\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2})$ is established without assuming $\lambda/\omega^{3/2} \in L^2(\mathbb{R}^\nu)$.

6.3 The Pauli-Fierz models

The Pauli-Fierz model [20] describes an interaction between nonrelativistic particles and a quantum radiation field. The Hilbert space of the Pauli-Fierz model is given by

$$\mathcal{F}_{\mathrm{PF}} := L^2(\mathbb{R}^\nu) \otimes \mathcal{F}_{\mathrm{b}}(\oplus^{\nu-1} L^2(\mathbb{R}^\nu)) \cong \int_{\mathbb{R}^\nu}^{\oplus} \mathcal{F}_{\mathrm{b}}(\oplus^{\nu-1} L^2(\mathbb{R}^\nu)) dx.$$

The creation operator and the annihilation operator are denoted by $a^*(f_1 \oplus \cdots \oplus f_{\nu-1})$ and $a(f_1 \oplus \cdots \oplus f_{\nu-1})$, respectively. The Pauli-Fierz Hamiltonian is defined by

$$H_{\mathrm{PF}} := H_{\mathrm{PF},0} + eH_{\mathrm{PF,I}},$$

where

$$H_{\mathrm{PF},0} := \Big(-\frac{1}{2}\Delta + V\Big) \otimes 1 + 1 \otimes d\Gamma(\oplus^{\nu-1}\omega),$$
$$H_{\mathrm{PF,I}} := -(p \otimes 1) \cdot A + \frac{e}{2} A \cdot A.$$

Here $p = (-i\partial/\partial x_1, ..., -i\partial/\partial x_n)$ denotes the set of generalized partial differential operator, $e \in \mathbb{R}$ a coupling constant, V an external potential and $A = (A_1, ..., A_\nu)$ denotes a quantum radition field defined by

$$A_\mu := \int_{\mathbb{R}^\nu}^{\oplus} A_\mu(x)dx, \quad \mu = 1, ..., \nu,$$

where

$$A_\mu(x) := \frac{1}{\sqrt{2}} \Big\{ a^*(\oplus_j^{\nu-1} e_\mu^j f_{\omega\lambda}^x) + a(\oplus_j^{\nu-1} e_\mu^j f_{\omega\lambda}^x) \Big\}$$

and $e^j(k) = (e_1^j(k), ..., e_\nu^j(k))$, $j = 1, .., \nu - 1$, denote ν-dimensional polarization vectors such that

$$e^j(k) \cdot e^{j'}(k) = \delta_{jj'} 1, \quad k \cdot e^j(k) = 0, \quad j, j' = 1, .., \nu - 1.$$

We can take $e^1, e^2, ..., e^{\nu-1}$ such that e_μ^j $(\mu = 1, ..., \nu, j = 1, ..., \nu - 1)$ is continuous on $\mathbb{R}^\nu \setminus Z$ for some $Z \subset \mathbb{R}^\nu$ with Lebesgue measure $|Z| = 0$.

Proposition 6.3 *Suppose that $\sqrt{\omega}\lambda, \lambda, \lambda/\sqrt{\omega}, \lambda/\omega \in L^2(\mathbb{R}^\nu)$ and that external potential V is relatively bounded with respect to $-\frac{1}{2}\Delta$ with a relative bound strictly less than one. Then H_{PF} is self-adjoint on $D(H_{\mathrm{PF},0})$ for all $e \in \mathbb{R}$.*

Proof: See [13, 14]. □

We see that

$$[1 \otimes a(f_1 \oplus \cdots \oplus f_{\nu-1}), H_{\mathrm{PF,I}}]_W^{D(H_{\mathrm{PF}})} = \sum_{j=1}^{\nu-1} \int_{\mathbb{R}^\nu} \overline{f_j}(k) (\Psi, T_{\mathrm{PF}j}(k)\Phi)_{\mathcal{F}_{\mathrm{PF}}} dk,$$

where

$$T_{\mathrm{PF}j}(k) := -\frac{\lambda(k)}{\sqrt{2\omega(k)}} u_k e^j(k) \cdot (p \otimes 1 - eA)$$

and $u_k : \mathcal{F}_{\mathrm{PF}} \to \mathcal{F}_{\mathrm{PF}}$ is the unitary operator defined by $u_k = \int_{\mathbb{R}^\nu}^{\oplus} e^{-ikx} dx$. Under the following identifications

$$H = H_{\mathrm{PF}}, \quad H_{\mathrm{I}} = H_{\mathrm{PF,I}}, \quad S(k) = \oplus^{\nu-1}\omega(k),$$
$$T(k) = \oplus_{j=1}^{\nu-1} T_{\mathrm{PF}j}(k), \quad \mathcal{C}_0 = C_0^2(\mathbb{R}^\nu \setminus (Z \cup Y)),$$

we can check that $H_{\rm PF}$ satisfies assumptions (A.1)-(A.5). Suppose that there exists a ground state $\varphi_{\rm g}$ of $H_{\rm PF}$ such that $\|(|x|\otimes 1)\varphi_{\rm g}\|_{\mathcal{F}_{\rm PF}}<\infty$. Then we obtain that

$$T_{{\rm PF}j}(k)\varphi_{\rm g} = i\frac{\lambda(k)}{\sqrt{2\omega(k)}}\left\{(H_{\rm PF}-E(H_{\rm PF}))+(p\otimes 1-eA)\cdot k+\frac{1}{2}|k|^2\right\}u_k e^j(k)\cdot(x\otimes 1)\varphi_{\rm g}.$$

Then

$$\begin{aligned}&\sum_{j=1}^{\nu-1}(\varphi_{\rm g}, T_{{\rm PF}j}(k)\varphi_{\rm g})_{\mathcal{F}_{\rm PF}}\\ &=\sum_{j=1}^{\nu-1} i\frac{\lambda(k)}{\sqrt{2\omega(k)}}(\varphi_{\rm g}, \left\{(p\otimes 1-eA)\cdot k+\frac{1}{2}|k|^2\right\}u_k e^j(k)\cdot(x\otimes 1)\varphi_{\rm g})_{\mathcal{F}_{\rm PF}}. \quad (6.7)\end{aligned}$$

By (6.7) we can obtain that

$$|\sum_{j=1}^{\nu-1}(\varphi_{\rm g}, T_{{\rm PF}j}(k)\varphi_{\rm g})_{\mathcal{F}_{\rm PF}}| \leq (c_1|k|+c_2|k|^2)\|\varphi_{\rm g}\|_{\mathcal{F}_{\rm PF}}\|(|x|\otimes 1)\varphi_{\rm g}\|_{\mathcal{F}_{\rm PF}} \quad (6.8)$$

with some constants c_1 and c_2. See [16] for details. From this we can coclude that

$$\frac{1}{\omega}\sum_{j=1}^{\nu-1}(\varphi_{\rm g}, T_{{\rm PF}j}(k)\varphi_{\rm g})_{\mathcal{F}_{\rm PF}} \in L^2(\mathbb{R}^\nu). \quad (6.9)$$

Remark 6.4 *We do not assume* $\lambda/\omega^{3/2}\in L^2(\mathbb{R}^\nu)$ *in (6.9).*

In [6] the existence of a ground state $\varphi_{\rm g}$ of $H_{\rm PF}$ such that $\varphi_{\rm g}\in D(1\otimes N^{1/2})$ is actually established without assuming $\lambda/\omega^{3/2}\in L^2(\mathbb{R}^\nu)$.

Remark 6.5 *(6.9) holds for the dipole approximation of* $H_{\rm PF}$*, too. We omit the details. See [9].*

7 Appendix

7.1 Proof of Lemma 2.7

Proof of (1) $\Longrightarrow$ (2).
Let $\Psi=\{\Psi^{(n)}\}_{n=0}^\infty\in\mathcal{F}_{\rm b,0}(\mathcal{K})$ be such that $\Psi^{(n)}=a^*(f_1^{(n)})\cdots a^*(f_n^{(n)})\Omega$. Then, using canonical commutation relations (2.1)–(2.3), we can show that

$$\text{s-}\lim_{M\to\infty}\sum_{m=1}^{M} a^*(K^*e_m)a(K^*e_m)\Psi^{(n)}$$

$$
\begin{aligned}
&= \text{s-}\lim_{M\to\infty} \sum_{j=1}^{n} a^*(f_1^{(n)}) \cdots a^*(K^* \sum_{m=1}^{M} (e_m, K f_j^{(n)}) e_m) \cdots a^*(f_n^{(n)})\Omega \\
&= \sum_{j=1}^{n} a^*(f_1^{(n)}) \cdots a^*(K^* K f_j^{(n)}) \cdots a^*(f_n^{(n)})\Omega \\
&= d\Gamma(K^*K)\Psi^{(n)}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\sum_{m=1}^{\infty} \|a(K^* e_m)\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} = \sum_{n=0}^{\text{finite}} (\Psi^{(n)}, \sum_{m=1}^{\infty} a^*(K^* e_m) a(K^* e_m)\Psi^{(n)})_{\otimes_s^n \mathcal{K}} \\
&= (\Psi, d\Gamma(K^*K)\Psi)_{\mathcal{F}_{\rm b}(\mathcal{K})} = \|d\Gamma(T^*K)^{1/2}\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})}.
\end{aligned}
$$

Since the finite linear hull of such Ψ's, say $\mathcal{D}$, is a core of $d\Gamma(K^*K)^{1/2}$, we can choose $\Psi_\epsilon \in \mathcal{D}$ for $\Psi \in D(d\Gamma(K^*K)^{1/2})$ such that $\Psi_\epsilon \to \Psi$ and $d\Gamma(K^*K)^{1/2}\Psi_\epsilon \to d\Gamma(K^*K)^{1/2}\Psi$ as $\epsilon \to 0$ strongly. From the facts that $a(f)$ is a closed operator and that by (2.4),

$$
\|a(K^* e_m)\Psi\|_{\mathcal{F}_{\rm b}(\mathcal{K})} \le \|d\Gamma(K^*K)^{1/2}\Psi\|_{\mathcal{F}_{\rm b}(\mathcal{K})},
$$

it follows that

$$
\lim_{\epsilon\to 0} \|a(K^* e_m)\Psi_\epsilon\|_{\mathcal{F}_{\rm b}(\mathcal{K})} = \|a(K^* e_m)\Psi\|_{\mathcal{F}_{\rm b}(\mathcal{K})}.
$$

Then we obtain that

$$
\sum_{m=1}^{M} \|a(K^* e_m)\Psi_\epsilon\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} \le \sum_{m=1}^{\infty} \|a(K^* e_m)\Psi_\epsilon\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} = \|d\Gamma(K^*K)^{1/2}\Psi_\epsilon\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})},
$$

and as $\epsilon \to 0$ on the both sides above,

$$
\sum_{m=1}^{M} \|a(K^* e_m)\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} \le \|d\Gamma(K^*K)^{1/2}\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})}.
$$

Hence, taking $M \to \infty$ on the both sides above, we can conclude that

$$
\sum_{m=1}^{\infty} \|a(K^* e_m)\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} < \infty.
$$

Proof of (1) $\Longleftarrow$ (2).
Let $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty}$. We have

$$
\begin{aligned}
&\sum_{m=1}^{\infty} \|a(K^* e_m)\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \|a(K^* e_m)\Psi^{(n)}\|^2_{\otimes_s^{n-1}\mathcal{K}} \\
&= \lim_{M\to\infty} \sum_{n=1}^{\infty}\sum_{m=1}^{M} \|a(K^* e_m)\Psi^{(n)}\|^2_{\otimes_s^{n-1}\mathcal{K}} = \sum_{n=1}^{\infty} \lim_{M\to\infty} \sum_{m=1}^{M} \|a(K^* e_m)\Psi^{(n)}\|^2_{\otimes_s^{n-1}\mathcal{K}}. \quad (7.1)
\end{aligned}
$$

Here on the third equality we used the monotone convergence theorem. The restriction $A_M := \sum_{m=1}^M a^*(K^*e_m)a(K^*e_m)\lceil_{\otimes_s^n\mathcal{K}}$ is a bounded operator, and

$$\|A_M\|_{\otimes_s^n\mathcal{K}\to\otimes_s^n\mathcal{K}} \leq \|d\Gamma_n(K^*K)^{1/2}\|_{\otimes_s^n\mathcal{K}\to\otimes_s^n\mathcal{K}}.$$

Then

$$\text{s-}\lim_{M\to\infty} A_M = d\Gamma_n(K^*K)$$

on $\otimes_s^n\mathcal{K}$. Hence we have by (7.1),

$$\begin{aligned}
&\infty > \sum_{m=1}^\infty \|a(K^*e_m)\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} = \sum_{n=1}^\infty \lim_{M\to\infty}(\Psi^{(n)}, A_M\Psi^{(n)})_{\otimes_s^n\mathcal{K}} \\
&= \sum_{n=1}^\infty (\Psi^{(n)}, d\Gamma_n(K^*K)\Psi^{(n)})_{\otimes_s^n\mathcal{K}} = \sum_{n=0}^\infty \|d\Gamma_n(K^*K)^{1/2}\Psi^{(n)}\|^2_{\otimes_s^n\mathcal{K}} \\
&= \|d\Gamma(K^*K)^{1/2}\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})}.
\end{aligned}$$

Thus the lemma is proven. □

7.2 Proof of Lemma 4.4

It is seen that for $\Psi = a^*(f_1)\cdots a^*(f_m)\Omega$,

$$\lim_{m_1,\ldots,m_n\to\infty} \sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} \|a(e_{i_1})\cdots a(e_{i_n})\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} = m(m-1)\cdots(m-n+1)\|\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})}.$$

Hence by a limiting argument we have for $\Psi \in \mathcal{F}_{{\rm b},0}(\mathcal{K})$,

$$\lim_{m_1,\ldots,m_n\to\infty} \sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} \|a(e_{i_1})\cdots a(e_{i_n})\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} = \|\prod_{j=1}^n (N-j+1)\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})}.$$

Proof of (1) $\Longrightarrow$ (2)

Let $\Psi_M \in \mathcal{F}_{{\rm b},0}(\mathcal{K})$ be a truncated vector for Ψ defined by $\Psi_M^{(m)} := \begin{cases} \Psi^{(m)}, & m \leq M, \\ 0, & m > M. \end{cases}$

Then

$$\begin{aligned}
\sum_{i_1,\ldots,i_n=1}^\infty \|a(e_{i_1})\cdots a(e_{i_n})\Psi_M\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} &= \|\prod_{j=1}^n (N-j+1)\Psi_M\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} \\
&= \sum_{m=0}^M \|\prod_{j=1}^n (N-j+1)\Psi^{(m)}\|^2_{\otimes_s^m\mathcal{K}}.
\end{aligned}$$

Take $M \to \infty$ on the both sides above. Then we have by the monotone convergence theorem,

$$\infty > \sum_{i_1,...,i_n=1}^{\infty} \|a(e_{i_1}) \cdots a(e_{i_n})\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} = \sum_{m=0}^{\infty} \| \prod_{j=1}^{n} (N-j+1)\Psi^{(m)}\|^2_{\otimes_s^m \mathcal{K}}$$
$$= \| \prod_{j=1}^{n} (N-j+1)\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})}.$$

Thus (2) follows.
Proof of (2) $\Longrightarrow$ (1)

$$\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \|a(e_{i_1}) \cdots a(e_{i_n})\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})} \le \| \prod_{j=1}^{n} (N-j+1)\Psi\|^2_{\mathcal{F}_{\rm b}(\mathcal{K})}.$$

Take $m_1, ..., m_n \to \infty$ on the both sides above. Thus (1) follows. $\square$

7.3 Example 5.10

In this section we shall prove that $A = -\Delta + \beta V$, $V \in \mathcal{S}(\mathbb{R}^\nu)$, and $B_j = -i\nabla_j (= p_j)$, $j = 1, ..., \nu$, satisfy (B.6) with $\mathcal{D}_\infty = \mathcal{S}(\mathbb{R}^\nu)$ for β with $|\beta|$ sufficiently small.

Proposition 7.1 *Suppose that $|\beta|$ is sufficiently small. Then (1) A^n is self-adjoint on*

$$D(A^n) = D((-\Delta)^n)$$

and essentially self-adjoint on any core of $(-\Delta)^n$. In particular A^n is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^\nu)$, (2) there exist constants a_k and b_k such that for all $\Psi \in \mathcal{S}(\mathbb{R}^\nu)$ and $j = 1, ..., \nu$,

$$\|{\rm ad}_A^k(p_j)\Psi\|_{L^2(\mathbb{R}^\nu)} \le a_k \|\hat{A}^{(k+1)/2}\Psi\|_{L^2(\mathbb{R}^\nu)} + b_k\|\Psi\|_{L^2(\mathbb{R}^\nu)}, \quad k \ge 0.$$

Before going to a proof of Proposition 7.1 we prepare some lemmas. In this section we write $\|\cdot\|$ for $\|\cdot\|_{L^2(\mathbb{R}^\nu)}$ for simplicity. Note that

$$\|p_{j_1} \cdots p_{j_m}\Phi\| \le \|(-\Delta)^{m/2}\Phi\|, \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu), \quad 1 \le j_1, ..., j_m \le \nu,$$

and for $k \le l$,

$$\|(-\Delta)^{k/2}\Phi\| \le C_{k,\ell}(\|(-\Delta)^{l/2}\Phi\| + \|\Phi\|), \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu), \tag{7.2}$$

with some constant $C_{k,\ell}$. The operators A and p_j leave $\mathcal{S}(\mathbb{R}^\nu)$ invariant and we see that

$$A^n\lceil_{\mathcal{S}(\mathbb{R}^\nu)} = ((-\Delta)^n + \beta K_{\rm I})\lceil_{\mathcal{S}(\mathbb{R}^\nu)},$$

where

$$K_{\mathrm{I}} := \sum_j \underbrace{(-\Delta)\cdots \overset{j}{V} \cdots(-\Delta)}_{n} + \beta \sum_{j_1<j_2} \underbrace{(-\Delta)\cdots \overset{j_1}{V} \cdots \overset{j_2}{V} \cdots(-\Delta)}_{n}$$
$$+\beta^2 \sum_{j_1<j_2<j_3} \underbrace{(-\Delta)\cdots \overset{j_1}{V} \cdots \overset{j_2}{V} \cdots \overset{j_3}{V} \cdots(-\Delta)}_{n} + \cdots + \beta^{n-1}V^n.$$

Lemma 7.2 *There exists a constant* C_m *such that*

$$\|[p_{j_1}p_{j_2}\cdots p_{j_m}, V]\Phi\| \le C_m(\|(-\Delta)^{(m-1)/2}\Phi\| + \|\Phi\|), \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu), \quad 1 \le j_1, ..., j_m \le \nu.$$

Proof: Since on $\mathcal{S}(\mathbb{R}^\nu)$,

$$[p_{j_1}p_{j_2}\cdots p_{j_m}, V] = \sum_{\ell=1}^m \sum_{\{i_1,...,i_\ell\}\subset\{j_1,...,j_m\}} V^{i_1,...,i_\ell} p_{j_1}\cdots \widehat{p_{i_1}} \cdots \widehat{p_{i_\ell}} \cdots p_{j_m},$$

where

$$V^{i_1,...,i_\ell} = (-i)^\ell \frac{\partial^\ell V}{\partial x_{i_1}\cdots\partial x_{i_\ell}},$$

we have

$$\|[p_{j_1}p_{j_2}\cdots p_{j_m}, V]\Phi\| \le \sum_{\ell=1}^m \sum_{\{i_1,...,i_\ell\}\subset\{j_1,...,j_m\}} \|V^{i_1,...,i_\ell}\|_\infty \|(-\Delta)^{(m-\ell)/2}\Phi\|. \tag{7.3}$$

Here $\|f\|_\infty := \mathrm{ess.sup}_{k\in\mathbb{R}^\nu}|f(k)|$. From (7.2) it follows that

$$\|(-\Delta)^{(m-\ell)/2}\Phi\| \le C_{(m-\ell)/2,(m-1)/2}(\|(-\Delta)^{(m-1)/2}\Phi\| + \|\Phi\|). \tag{7.4}$$

Then the lemma follows from (7.3) and (7.4). □

Lemma 7.3 *There exists a constant* C_ℓ *such that*

$$\|(-\Delta)^\ell V\Phi\| \le C_\ell(\|(-\Delta)^\ell\Phi\| + \|\Phi\|), \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu). \tag{7.5}$$

Proof: Note that for $\Phi \in \mathcal{S}(\mathbb{R}^\nu)$,

$$\|(-\Delta)^\ell V\Phi|| \le \|V(-\Delta)^\ell\Phi\| + \|[(-\Delta)^\ell, V]\Phi\|.$$

Since on $\mathcal{S}(\mathbb{R}^\nu)$,

$$[(-\Delta)^\ell, V] = \sum_{j_1,...,j_\ell=1}^\nu [p_{j_1}^2\cdots p_{j_\ell}^2, V],$$

by Lemma 7.2

$$\|[(-\Delta)^\ell, V]\Phi\| \le C(\|(-\Delta)^{(2l-1)/2}\Phi\| + \|\Phi\|) \le C'(\|(-\Delta)^\ell\Phi\| + \|\Phi\|) \tag{7.6}$$

with some constants C and C'. Moreover

$$\|V(-\Delta)^\ell \Phi\| \leq \|V\|_\infty \|(-\Delta)^\ell \Phi\|. \tag{7.7}$$

From (7.5), (7.6) and (7.7), it follows that

$$\|(-\Delta)^\ell V\Phi\| \leq (C' + \|V\|_\infty)(\|(-\Delta)^\ell \Phi\| + \|\Phi\|).$$

Then the proof is complete. □

Lemma 7.4 *There exists a constant $C_{i_1,...,i_m}$ such that for $1 \leq i_1 < \cdots < i_m \leq n$,*

$$\| \underbrace{(-\Delta)\cdots \overset{i_1}{V} \cdots \overset{i_2}{V} \overset{\dots}{\cdots} \overset{i_m}{V} \cdots (-\Delta)}_{n} \Phi\| \leq C_{i_1,...,i_m}(\|(-\Delta)^{n-m}\Phi\| + \|\Phi\|), \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu).$$

Proof: It inductively follows from Lemma 7.3. □

Proof of Proposition 7.1 (1)
From the definition of K_{I}, (7.2) and Lemma 7.4, it follows that

$$\begin{aligned} \|K_{\mathrm{I}}\Phi\| &\leq \sum_{\ell=1}^{n} |\beta|^{\ell-1} C_{\mathrm{I},\ell}(\|(-\Delta)^{n-\ell}\Phi\| + \|\Phi\|) \\ &\leq C_{\mathrm{I}}(\|(-\Delta)^n \Phi\| + \|\Phi\|), \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu), \end{aligned} \tag{7.8}$$

with some constants C_{I} and $C_{\mathrm{I},\ell}$, $\ell = 1, ..., n$. Since $\mathcal{S}(\mathbb{R}^\nu)$ is a core of $(-\Delta)^n$, we can extend (7.8) for $\Phi \in D((-\Delta)^n)$ with

$$\|\overline{K_{\mathrm{I}}}\Phi\| \leq C_{\mathrm{I}}(\|(-\Delta)^n \Phi\| + \|\Phi\|), \tag{7.9}$$

where $\overline{K_{\mathrm{I}}}$ denotes the closure of $K_{\mathrm{I}}\lceil_{\mathcal{S}(\mathbb{R}^\nu)}$. Then for β with $|\beta| < 1/C_{\mathrm{I}}$, the Kato-Rellich theorem yields that $(-\Delta)^n + \beta\overline{K_{\mathrm{I}}}$ is self-adjoint on $D((-\Delta)^n)$ and bounded from below. Moreover it is essentially self-adjoint on any core of $(-\Delta)^n$. In particular $(-\Delta)^n + \beta\overline{K_{\mathrm{I}}}$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^\nu)$. Since

$$A^n\lceil_{\mathcal{S}(\mathbb{R}^\nu)} = ((-\Delta)^n + \beta\overline{K_{\mathrm{I}}})\lceil_{\mathcal{S}(\mathbb{R}^\nu)} \subset ((-\Delta)^n + \beta\overline{K_{\mathrm{I}}})\lceil_{D((-\Delta)^n)},$$

we obtain that

$$A^n = ((-\Delta)^n + \beta\overline{K_{\mathrm{I}}})\lceil_{D((-\Delta)^n)}.$$

Hence for β with $|\beta| < 1/C_{\mathrm{I}}$, A^n is self-adjoint on $D(-\Delta)^n$ and essentially self-adjoint on any core of $(-\Delta)^n$. Thus Proposition 7.1 (1) follows. □

Lemma 7.5 *Let $g \in \mathcal{S}(\mathbb{R}^\nu)$ and $m \geq 1$. Then there exist $g^{(1)}, ..., g^{(m-1)}$, $g^{(k)}_{j_1 \cdots j_\ell} \in \mathcal{S}(\mathbb{R}^\nu)$, $j_1, ..., j_\ell = 1, ..., \nu$, $\ell = 1, ..., m$, such that*

$$\mathrm{ad}_A^m(g) = \sum_{\ell=0}^{m-1} \mathrm{ad}_A^\ell(g^{(\ell)}) + \sum_{\ell=1}^{m} \sum_{j_1,..,j_\ell=1}^{\nu} g^{(k)}_{j_1 \cdots j_\ell} p_{j_1} \cdots p_{j_\ell}. \tag{7.10}$$

Proof: We prove the lemma by induction with respect to m. Let $m = 1$. Then

$$\mathrm{ad}_A(g) = -g'' + 2\sum_{j=1}^{\nu} g'_j p_j = \mathrm{ad}_A^0(-g'') + 2\sum_{j=1}^{\nu} g'_j p_j, \tag{7.11}$$

where $g'' = \Delta g$ and $g'_j = -i\partial g/\partial x_j$. Thus (7.10) follows for $m = 1$. Suppose (7.10) holds for $m = 0, 1, ..., k$. Then we have

$$\mathrm{ad}_A^{k+1}(g) = \mathrm{ad}_A \mathrm{ad}_A^k(g) = \sum_{\ell=0}^{k-1} \mathrm{ad}_A^{\ell+1}(g^{(\ell)}) + \sum_{\ell=1}^{k} \mathrm{ad}_A(g^{(k)}_{j_1 \cdots j_\ell} p_{j_1} \cdots p_{j_\ell}). \tag{7.12}$$

Directly we can see by (7.11) that

$$\begin{aligned}
&\mathrm{ad}_A(g^{(k)}_{j_1 \cdots j_\ell} p_{j_1} \cdots p_{j_\ell}) = \mathrm{ad}_A(g^{(k)}_{j_1 \cdots j_\ell}) p_{j_1} \cdots p_{j_\ell} + g^{(k)}_{j_1 \cdots j_\ell} \mathrm{ad}_A(p_{j_1} \cdots p_{j_\ell}) \\
&= (-(g^{(k)}_{j_1 \cdots j_\ell})'' + 2\sum_{j=1}^{\nu} (g^{(k)}_{j_1 \cdots j_\ell})'_j p_j) p_{j_1} \cdots p_{j_\ell} + g^{(k)}_{j_1 \cdots j_\ell} [V, p_{j_1} \cdots p_{j_\ell}] \\
&= g^{(k)}_{j_1 \cdots j_\ell} \sum_{m=1}^{\ell-1} \sum_{\{i_1,...,i_\ell\} \subset \{j_1,...,j_m\}} V^{i_1,...,i_\ell} \underbrace{p_{j_1} \cdots \cancel{p_{i_1}} \cdots \cancel{p_{i_\ell}} \cdots p_{j_m}}_{\leq \ell-1} \\
&-(g^{(k)}_{j_1 \cdots j_\ell})'' \underbrace{p_{j_1} \cdots p_{j_\ell}}_{\ell} + 2\sum_{j=1}^{\nu} (g^{(k)}_{j_1 \cdots j_\ell})'_j \underbrace{p_j p_{j_1} \cdots p_{j_\ell}}_{\ell+1} + \underbrace{g^{(k)}_{j_1 \cdots j_\ell} V^{j_1,...,j_\ell}}_{=\mathrm{ad}_A^0(g^{(k)}_{j_1 \cdots j_\ell} V^{j_1,...,j_\ell})} .
\end{aligned} \tag{7.13}$$

Substituting (7.13) to (7.12) and rearranging, we can see that

$$\mathrm{ad}_A^{k+1}(g) = \sum_{\ell=0}^{k} \mathrm{ad}_A^\ell(\tilde{g}^{(\ell)}) + \sum_{\ell=1}^{k+1} \sum_{j_1,..,j_\ell=1}^{\nu} \tilde{g}^{(k)}_{j_1 \dots j_\ell} p_{j_1} \cdots p_{j_\ell}$$

with some $g^{(\ell)}$, $\tilde{g}^{(k)}_{j_1 \dots j_\ell} \in \mathcal{S}(\mathbb{R}^\nu)$, $j_1, ..., j_\ell = 1, .., \nu$, $\ell = 1, ..., k+1$. Thus the lemma follows. $\square$

Lemma 7.6 *Let $g \in \mathcal{S}(\mathbb{R}^\nu)$. Then there exists a constant $C_{g,m}$ such that*

$$\|\mathrm{ad}_A^m(g)\Phi\| \leq C_{g,m}(\|(-\Delta)^{m/2}\Phi\| + \|\Phi\|), \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu), \quad m \geq 0. \tag{7.14}$$

Proof: We prove the lemma by induction with respect to m. For $m = 0$, (7.14) follows. Assume that (7.14) holds for $m = 0, 1, .., k$. Then by Lemma 7.5 we see that

$$\begin{aligned}\|\mathrm{ad}_A^{k+1}(g)\Phi\| &\leq \sum_{\ell=0}^{k} \|\mathrm{ad}_A^{\ell}(g^{(\ell)})\Phi\| + \sum_{\ell=1}^{k+1} \sum_{j_1,..,j_\ell=1}^{\nu} \|g^{(k)}_{j_1\cdots j_\ell} p_{j_1}\cdots p_{j_\ell}\Phi\| \\ &\leq \sum_{\ell=0}^{k} \|\mathrm{ad}_A^{\ell}(g^{(\ell)})\Phi\| + C\sum_{\ell=1}^{k+1} \|(-\Delta)^{\ell/2}\Phi\|\end{aligned}$$

with some constant C. By the assumption of the induction and (7.2) we have

$$\begin{aligned}\|\mathrm{ad}_A^{k+1}(g)\Phi\| &\leq C'\sum_{\ell=0}^{k+1}(\|(-\Delta)^{\ell/2}\Phi\| + \|\Phi\|) \\ &\leq C''(\|(-\Delta)^{(k+1)/2}\Phi\| + \|\Phi\|)\end{aligned}$$

with some constants C' and C''. Then the lemma follows. □

Lemma 7.7 *We have* $\mathrm{ad}_A^k(p_j) = \beta\mathrm{ad}_A^{k-1}(i\partial V/\partial x_j)$ *on* $\mathcal{S}(\mathbb{R}^\nu)$.

Proof: We see that on $\mathcal{S}(\mathbb{R}^\nu)$,

$$\begin{aligned}\mathrm{ad}_A^k(p_j) &= \mathrm{ad}^k_{(-\Delta)+\beta V}(p_j) \\ &= \mathrm{ad}^k_{(-\Delta)}(p_j) \\ &\quad +\beta\sum_j \underbrace{\mathrm{ad}_{(-\Delta)}\cdots \overset{j}{\mathrm{ad}_V}\cdots \mathrm{ad}_{(-\Delta)}}_{k}(p_j) \\ &\quad +\beta^2\sum_{j_1<j_2} \underbrace{\mathrm{ad}_{(-\Delta)}\cdots \overset{j_1}{\mathrm{ad}_V}\cdots \overset{j_2}{\mathrm{ad}_V}\cdots \mathrm{ad}_{(-\Delta)}}_{k}(p_j) \\ &\quad +\beta^3\sum_{j_1<j_2<j_3} \underbrace{\mathrm{ad}_{(-\Delta)}\cdots \overset{j_1}{\mathrm{ad}_V}\cdots \overset{j_2}{\mathrm{ad}_V}\cdots \overset{j_3}{\mathrm{ad}_V}\cdots \mathrm{ad}_{(-\Delta)}}_{k}(p_j) \\ &\quad \vdots \\ &\quad +\beta^k\mathrm{ad}_V^k(p_j).\end{aligned}$$

Since $\mathrm{ad}_{(-\Delta)}(p_j) = 0$, we have

$$\begin{aligned}\mathrm{ad}_A^k(p_j) &= \beta\underbrace{\mathrm{ad}_{(-\Delta)}\cdots\mathrm{ad}_{(-\Delta)}}_{k-1}\mathrm{ad}_V(p_j) \\ &\quad +\beta^2\sum_j \underbrace{\mathrm{ad}_{(-\Delta)}\cdots \overset{j}{\mathrm{ad}_V}\cdots \mathrm{ad}_{(-\Delta)}\mathrm{ad}_V}_{k-1}(p_j)\end{aligned}$$

$$
\begin{aligned}
& +\beta^3 \sum_{j_1<j_2} \underbrace{\mathrm{ad}_{(-\Delta)} \cdots \overset{j_1}{\mathrm{ad}_V} \cdots \overset{j_2}{\mathrm{ad}_V} \cdots \mathrm{ad}_{(-\Delta)}\mathrm{ad}_V}_{k-1}(p_j) \\
& \vdots \\
& +\beta^k \mathrm{ad}_V^{k-1}\mathrm{ad}_V(p_j) \\
= \; & \beta \mathrm{ad}_A^{k-1}\mathrm{ad}_V(p_j) \\
= \; & \beta \mathrm{ad}_A^{k-1}(i\partial V/\partial x_j).
\end{aligned}
$$

Thus the lemma follows. □

Lemma 7.8 *Let $k \geq 1$. Then there exists a constant $C_{V,k}$ such that*

$$\|\mathrm{ad}_A^k(p_j)\Phi\| \leq |\beta| C_{V,k}(\|(-\Delta)^{(k-1)/2}\Phi\| + \|\Phi\|), \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu).$$

Proof: By Lemmas 7.6 and 7.7 we have

$$\|\mathrm{ad}_A^k(p_j)\Phi\| = |\beta| \|\mathrm{ad}_A^{k-1}(i\partial V/\partial x_j)\Phi\| \leq |\beta| C(\|(-\Delta)^{(k-1)/2}\Phi\| + \|\Phi\|)$$

with some constant C. Then the lemma follows. □

Proof of Proposition 7.1 (2)
Let $\Phi \in \mathcal{S}(\mathbb{R}^\nu)$. We have

$$
\begin{aligned}
\|(-\Delta)^n\Phi\| &= \|((-\Delta)^n + \beta K_{\mathrm{I}} - \beta K_{\mathrm{I}})\Phi\| \leq \|A^n\Phi\| + |\beta| \|K_{\mathrm{I}}\Phi\| \\
&\leq \|A^n\Phi\| + |\beta| C(\|(-\Delta)^n\Phi\| + \|\Phi\|)
\end{aligned}
$$

with some constant C. Hence it follows that

$$\|(-\Delta)^n\Phi\| \leq \frac{1}{1-|\beta|C}(\|A^n\Phi\| + \|\Phi\|)$$

for β with $|\beta| < 1/C$. Moreover

$$\|A^n\Phi\| \leq a\|\hat{A}^n\Phi\| + b\|\Phi\|, \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu),$$

with some constants a and b. Then

$$\|(-\Delta)^n\Phi\| \leq C'(\|\hat{A}^n\Phi\| + \|\Phi\|), \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu), \tag{7.15}$$

follows with some constant C'. Since $\mathcal{S}(\mathbb{R}^\nu)$ is a core of $\hat{A}^n$, one can see that (7.15) can be extended to $\Phi \in D(\hat{A}^n)$. Then

$$\|(-\Delta)^{n/2}\Phi\| \leq C''(\|\hat{A}^{n/2}\Phi\| + \|\Phi\|), \quad \Phi \in D(\hat{A}^{n/2}), \tag{7.16}$$

with some constant C''. By Lemma 7.8 and (7.16) it follows that

$$\|\mathrm{ad}_A^k(p_j)\Phi\| \leq |\beta|C'''(\|\hat{A}^{(k-1)/2}\Phi\| + \|\Phi\|), \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu),$$

with some constant C'''. In particular

$$\|\mathrm{ad}_A^k(p_j)\Phi\| \leq |\beta|C(\|\hat{A}^{(k+1)/2}\Phi\| + \|\Phi\|) \quad \Phi \in \mathcal{S}(\mathbb{R}^\nu),$$

follows. Then the Proposition 7.1 (2) follows. □

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