A Product Formula Related to Quantum Zeno Dynamics

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We prove a product formula which involves the unitary group generated by a semibounded self-adjoint operator and an orthogonal projection P on a separable Hilbert space $\mathcal{H}$, with the convergence in $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{H})$. It gives a partial answer to the question about existence of the limit which describes quantum Zeno dynamics in the subspace $\mathrm{Ran}\, P$. The convergence in $\mathcal{H}$ is demonstrated in the case of a finite-dimensional P. The main result is illustrated in the example where the projection corresponds to a domain in $\mathbb{R}^d$ and the unitary group is the free Schrödinger evolution.

1 Introduction

The fact that the decay of an unstable system can be slowed down, or even fully stopped in the ideal case, by frequently repeated measurements checking whether the system is still undecayed was noticed first by Beskow and Nilsson [BN]. It was only decade later, however, when Misra and Sudarshan [MS] caught the imagination of the community by linking the effect to the well-known Zeno aporia about a flying arrow. While at first the subject was rather academical, in recent years the possibility of observing Zeno-type effects experimentally has become real and at present there are scores of physical papers discussing this topic.

On the mathematical side, the first discussion of the continuous observation appeared in [Fr]. Two important questions, however, namely the existence of Zeno dynamics and the form of its effective Hamiltonian have been left open both in this paper and later in [MS]. The second problem is particularly important when the subspace into which the state of the system is repeatedly reduced has dimension larger than one. A partial answer was given in [Ex, Sec. 2.4] where it was shown that the results of Chernoff [Ch1, Ch2] allow to determine the generator of the Zeno time evolution naturally through the appropriate quadratic form.

Our interest to the problem was rekindled by a recent paper by Facchi et al. [FPS] who studied the important special case when the presence of a particle in a domain of $\Omega \subset \mathbb{R}^d$ is repeatedly ascertained. Using the method of stationary phase the authors showed that the Zeno dynamics describes in this case the free particle confined to Ω, with the hard-wall (Dirichlet) condition at the boundary of the domain. The result cannot be regarded as fully rigorous, because detailed properties of the convergence are not worked out, but the idea is sound without any doubt.

In the present paper we combine the results of [Ch1, Ch2] with that of Kato [Ka2] to address this question in a general setting. We show that if the natural effective Hamiltonian mentioned above is densely defined — which is a nontrivial assumption — then the Zeno dynamics exists and the said operator is its generator in a topology which includes an averaging over the time variable – cf. Theorem 2.1 for exact statement (a part of the present result given in Corollary 2.3 was announced in [EI]). Our conclusion cannot be thus regarded as fully satisfactory from the mathematical point of view, because the natural topology to be used here is given by the norm of the Hilbert space, and in this respect an important part of the problem remains open. We demonstrate, however, the strong convergence in $\mathcal{H}$ for the particular case when the projections involved are finite-dimensional – cf. Theorem 2.4. On the other hand, from the physical point of view the result given in Theorem 2.1 is quite plausible taking into account that any real measurement is burdened with errors – see Remark 2.5 below.

We will formulate the theorems together with their corollaries in the next section. Theorem 2.1 will be then proven in Sections 3 and 4, Theorem 2.4

in Section 5. As an example we discuss in the concluding section reduction of a free dynamics to a domain in $\mathbb{R}^d$ by permanent observation. We will establish that the Zeno generator mentioned above is in this case the Dirichlet Laplacian, obtaining thus in a different way the result of the paper [FPS].

2 The main result

Throughout the paper H will be a nonnegative self-adjoint operator in a *separable* Hilbert space $\mathcal{H}$, and P will be an orthogonal projection. The nonnegativity assumption is made for convenience; our main result extends easily to any self-adjoint operator H bounded from below as well as one bounded from above, i.e. to each semi-bounded self-adjoint operator in $\mathcal{H}$.

Consider the quadratic form $u \mapsto \|H^{1/2}Pu\|^2$ with form domain $D[H^{1/2}P]$. Note that $H^{1/2}P$ involved here is a closed operator and HP has the same property. Let $H_P := (H^{1/2}P)^*(H^{1/2}P)$ be the self-adjoint operator associated with this quadratic form. In general, H_P may not be densely defined in which case it is a self-adjoint operator in a closed subspace of $\mathcal{H}$. More specifically, it is obviously defined and acts nontrivially in a closed subspace $\operatorname{Ran} P$ of the closure of the form domain $D[H^{1/2}P]$, while in the orthogonal complement $(\operatorname{Ran} P)^\perp$ it acts as zero.

The quadratic form $u \mapsto \|H^{1/2}Pu\|^2$ defined on $D[H^{1/2}P]$ is a closed extension of the form $u \mapsto \langle Pu, HPu\rangle$ defined on $D[HP]$, but the former is not in general the closure of the latter. Indeed, if H is unbounded, $D[H]$ is a proper subspace of $D[H^{1/2}]$. Take $u_0 \in D[H^{1/2}]\backslash D[H]$ such that the vector $H^{1/2}u_0$ is nonzero, and set P to be the orthogonal projection onto the one-dimensional subspace spanned by u_0. Taking into account that $D[HP] = \{u \in \mathcal{H};\ Pu \in D[H]\}$ which $u_0 = Pu_0$ does not belong to, we find $HPu = 0$ for $u \in D[HP]$, while $H^{1/2}Pu_0 = H^{1/2}u_0 \neq 0$ by assumption.

To describe our results, we denote by $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H}) = L^2_{\mathrm{loc}}([0,\infty)) \otimes \mathcal{H}$ the Fréchet space of the $\mathcal{H}$-valued strongly measurable functions $v(\cdot)$ on $[0,\infty)$ such that $\|v(\cdot)\|$ is locally square integrable there, with the topology induced by the semi-norms $v \mapsto (\int_0^{T_\ell} \|v(t)\|^2 dt)^{1/2}$ for a countable set $\{T_\ell\}_{\ell=1}^\infty$ of increasing positive numbers accumulating at infinity, $\lim_{\ell\to\infty} T_\ell = \infty$. In a similar way one defines the Fréchet space $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{H}) = L^2_{\mathrm{loc}}(\mathbb{R}) \otimes \mathcal{H}$.

Our main result can be stated as follows:

Theorem 2.1 *Let H be a nonnegative self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and P an orthogonal projection. Let $t \mapsto P(t)$ be a strongly continuous function whose values are orthogonal projections in $\mathcal{H}$, defined in some neighborhood of zero, with $P(0) =: P$. Moreover, suppose that $D[H^{1/2}P(t)] \supset D[H^{1/2}P]$ and $\lim_{t\to 0} \|H^{1/2}P(t)v\| = \|H^{1/2}Pv\|$ holds for $v \in D[H^{1/2}P]$. If the operator H_P specified above is densely defined in the whole Hilbert space $\mathcal{H}$, then for every $f \in \mathcal{H}$ and $\varepsilon = \pm 1$ it holds that*

$$[P(1/n)\exp(-i\varepsilon tH/n)P(1/n)]^n f \longrightarrow \exp(-i\varepsilon tH_P)\, Pf\,, \qquad (2.1)$$

$$[P(1/n)\, \exp(-i\varepsilon tH/n)]^n f \longrightarrow \exp(-i\varepsilon tH_P)\, Pf\,, \qquad (2.2)$$

$$[\,\exp(-i\varepsilon tH/n)\, P(1/n)]^n f \longrightarrow \exp(-i\varepsilon tH_P)\, Pf\,, \qquad (2.3)$$

in the topology of $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{H})$ as $n \to \infty$.

Note that H_P differs in general from the operator PHP, which may not be self-adjoint in $\mathcal{H}$, nor even closed, because PH is not necessarily closed, though HP is. H_P is a self-adjoint extension of PHP. The requirement of the theorem that H_P is densely defined in $\mathcal{H}$ means nothing else but that the domain $D[H^{1/2}P]$ of the quadratic form in question is dense in $\mathcal{H}$.

Note also that for $\varepsilon = 1$, the theorem concerns a nonnegative self-adjoint operator $\varepsilon H = H$, while for $\varepsilon = -1$, we get product formulae for the nonpositive self-adjoint operator $\varepsilon H = -H$. Moreover, the result is preserved when H is replaced with a shifted operator $H + cI$, i.e. for any semi-bounded self-adjoint operator in a separable Hilbert space.

An important particular case, most often met in the applications, concerns the situation when the projection-valued function is constant.

Corollary 2.2 *Let H be a self-adjoint operator bounded from below in a separable Hilbert space $\mathcal{H}$ and P an orthogonal projection. If the operator H_P specified above is densely defined, then for every $f \in \mathcal{H}$ and $\varepsilon = \pm 1$ we have in the topology of $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{H})$ the limiting relation*

$$[P\exp(-i\varepsilon tH/n)P]^n f \longrightarrow \exp(-i\varepsilon tH_P)\, Pf \qquad (2.4)$$

for $n \to \infty$ as well as its nonsymmetric counterparts obtained by setting $P(1/n) = P$ in (2.2) and (2.3).

¿From the viewpoint of quantum Zeno effect described in the introduction the optimal result would be a strong convergence on $\mathcal{H}$ for a fixed value of the time variable, moreover uniformly on each compact interval in t. Our Theorem 2.1 implies the following weaker result on pointwise convergence.

Corollary 2.3 *Under the same hypotheses as in Theorem 2.1, there exist a set $M \subset \mathbb{R}$ of Lebesgue measure zero and a strictly increasing sequence $\{n'\}$ of positive integers along which we have*

$$[P(1/n')\exp(-i\varepsilon tH/n')P(1/n')]^{n'}f \longrightarrow \exp(-i\varepsilon tH_P)\,Pf\,, \quad (2.5)$$

$$[P(1/n')\,\exp(-i\varepsilon tH/n')]^{n'}f \longrightarrow \exp(-i\varepsilon tH_P)\,Pf\,, \quad (2.6)$$

$$[\,\exp(-i\varepsilon tH/n')\,P(1/n')]^{n'}f \longrightarrow \exp(-i\varepsilon tH_P)\,Pf\,, \quad (2.7)$$

for every $f \in \mathcal{H}$, strongly in $\mathcal{H}$ for all $t \in \mathbb{R} \setminus M$.

As we have indicated above, one need not resort to subsequences in the particular case when the projections involved are finite-dimensional.

Theorem 2.4 *In addition to the hypotheses of Theorem 2.1, assume that the orthogonal projection P as well as $P(t)$ is of finite dimension. Then*
(i) the formulae (2.1)–(2.3) hold in the norm of $\mathcal{H}$ as $n \to \infty$, uniformly on each compact interval of the variable t in $\mathbb{R} \setminus \{0\}$,
(ii) it also holds for $\varepsilon = \pm 1$ that as $n \to \infty$,

$$[P(t/n)\exp(-i\varepsilon tH/n)P(t/n)]^{n} \longrightarrow \exp(-i\varepsilon tH_P)\,P\,,$$

$$[P(t/n)\,\exp(-i\varepsilon tH/n)]^{n} \longrightarrow \exp(-i\varepsilon tH_P)\,P\,,$$

$$[\,\exp(-i\varepsilon tH/n)\,P(t/n)]^{n} \longrightarrow \exp(-i\varepsilon tH_P)\,P\,,$$

strongly on $\mathcal{H}$, uniformly on each compact interval in the variable $t \in \mathbb{R}$.

Before proving Theorems 2.1 and 2.4 and Corollary 2.3 let us comment briefly on several other aspects of the result.

Remark 2.5 While the necessity to pick a subsequence makes the pointwise convergence result weaker than desired, let us notice that from the physical point of view the convergence in $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{H})$ can be regarded as satisfactory. The point is that any actual measurement, in particular that of time, is burdened with errors. Suppose thus we perform the Zeno experiment on numerous copies of the system. The time value in the results will be characterized by a probability distribution $\phi\,:\,\mathbb{R}_+ \to \mathbb{R}_+$, which is typically a bounded, compactly supported function – in a precisely posed experiment it is sharply peaked, of course. Corollary 2.2 then gives

$$\int \phi(t)\;\|\,[P\exp(-i\varepsilon tH/n)P]^n f - \exp(-i\varepsilon tH_P)\,Pf\|^2\;dt \to 0 \qquad (2.8)$$

as $n \to \infty$, in other words, the Zeno dynamics limit is valid after averaging over experimental errors, however small they are.

Remark 2.6 While the proof of strong convergence in $\mathcal{H}$ in Theorem 2.1 and Corollaries 2.2 and 2.3 remains elusive without the finite-dimension assumption, such a claim can be easily established in the orthogonal complement of the subspace $P\mathcal{H}$. Indeed, taking $f \in Q\mathcal{H}$, where $Q := I - P$, we have

$$\begin{aligned} (P(1/n)e^{-i\varepsilon tH/n}P(1/n))^n f &= (P(1/n)e^{-i\varepsilon tH/n}P(1/n))^n P(1/n)Qf\,, \\ (e^{-i\varepsilon tH/n}P(1/n))^n f &= (e^{-i\varepsilon tH/n}P(1/n))^n P(1/n)Qf\,, \end{aligned}$$

which converge to zero, uniformly on each compact t-interval in $\mathbb{R}$, as $n \to \infty$, because $P(\tau) \xrightarrow{s} P$ as $\tau \to 0$. This gives the result for (2.5) and (2.7), while for (2.6) one has to employ in addition the relation (3.11) below.

Remark 2.7 The fact that the product formulae require H_P to be densely defined is nontrivial. Recall the example of [Ex, Rem. 2.4.9] in which H is the multiplication operator, $(H\psi)(x) = x\psi(x)$ on $L^2(\mathbb{R}_+)$, and P is the one-dimensional projection onto the subspace spanned by the vector ψ_0 : $\psi_0(x) = [(\pi/2)(1+x^2)]^{-1/2}$. In this case obviously H_P is the zero operator on the domain $D[H_P] = \{\psi_0\}^\perp$. On the other hand, $P\,e^{-itH}P$ acts on $\operatorname{Ran} P$ as multiplication by the function

$$v(t) := e^{-t} - \frac{i}{\pi}\left[e^{-t}\,\overline{\mathrm{Ei}}(t) - e^t\,\mathrm{Ei}(-t)\right] = 1 + \frac{2i}{\pi}\,t\ln t + \mathcal{O}(t),$$

where $E_i(-t)$ and $\overline{E}_i(t)$ are exponential integrals [AS]; due to the rapid oscillations of the imaginary part as $t \downarrow 0$ a pointwise limit of $v(t/n)^n$ for $n \to \infty$ does not exist. Notice also that different limits may be obtained in this example along suitably chosen subsequences $\{n'\}$.

Remark 2.8 In their recent study of Trotter-type formulae involving projections Matolcsi and Shvidkoy [MaS] presented two examples in which expressions of the type $[\exp(-iH/n)P]^n$ do not converge strongly. This result does not answer the question, however, whether the product expressions considered here converge in the strong topology of $\mathcal{H}$ or not, because our assumptions are not satisfied there. In the first example of [MaS] the analogue of the operator H_P is not densely defined, in the second one H is not semi-bounded.

3 Proof of Theorem 2.1

We present the argument for $\varepsilon = 1$, the case $\varepsilon = -1$ can be treated similarly. We first prove (2.1) in (a), and next (2.2), (2.3) in (b).

(a) Let us begin with the symmetric product case and prove the formula (2.1) with $\varepsilon = 1$. We will check the convergence in (2.1) on an arbitrary compact t-interval in the closed right half-line $[0, \infty)$. The proof for t-intervals in the closed left half-line $(-\infty, 0]$ is analogous, and in addition, it can be included in the case $\varepsilon = -1$ with the convergence in (2.1) on compact t-intervals of the closed right half-line $[0, \infty)$.

Put $Q(t) := I - P(t)$ and $Q := Q(0) = I - P(0) = I - P$, where I is the identity operator on $\mathcal{H}$. Since H is nonnegative by assumption, there exists a spectral measure $E(d\lambda)$ on the nonnegative real line such that $H = \int_{0-}^{\infty} \lambda\, E(d\lambda)$. For $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta \geq 0$ and $\tau > 0$, we put

$$F(\zeta, \tau) = P(\tau)\, e^{-\zeta\tau H} P(\tau)\,, \tag{3.1}$$

which is a contraction, and

$$S(\zeta, \tau) = \tau^{-1}[I - F(\zeta, \tau)] = \tau^{-1}[I - P(\tau)\, e^{-\zeta\tau H} P(\tau)], \tag{3.2}$$

which exists as a bounded operator on $\mathcal{H}$ with $\operatorname{Re}\langle f, S(\zeta, \tau) f\rangle \geq 0$ for every $f \in \mathcal{H}$. For definiteness we use here and in the following the physicist convention about the inner product supposing that it is antilinear in the first argument. For a non-zero $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta \geq 0$, we put also

$$H(\zeta) := \zeta^{-1}[I - e^{-\zeta H}]\,. \tag{3.3}$$

Each element $v(\cdot)$ in $L^2_{\rm loc}([0, \infty); \mathcal{H})$ is an equivalence class such that any two representatives of it are equal a.e. on $[0, \infty)$. However, at some places we will not avoid an abuse of notation using for a particular representative of such an element the same symbol $v(\cdot)$. At the same time, in the following the convergence of a family of vectors $v(\cdot, \tau)$ to $v(\cdot)$ in the topology of the space $L^2_{\rm loc}([0, \infty); \mathcal{H}) = L^2_{\rm loc}([0, \infty)) \otimes \mathcal{H}$ as τ tends to zero will be often written as $v(t, \tau) \longrightarrow v(t)$; this will be the case when writing $v(\cdot, \tau) \longrightarrow v(\cdot)$ would require to introduce a separate symbol for this $v(t, \tau)$ the meaning of which is clear from the context.

The key ingredient of the proof is the following lemma.

Lemma 3.1 *$(I+S(it,\tau))^{-1}$ converges to $(I+itH_P)^{-1}P$ as $\tau \to 0$ strongly in $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$, in other words, for all $f \in \mathcal{H}$ and every finite $T > 0$ we have*

$$\int_0^T \|(I+S(it,\tau))^{-1}f - (I+itH_P)^{-1}Pf\|^2\,dt \to 0\,, \quad \tau \to 0\,. \tag{3.4}$$

We postpone the proof of Lemma 3.1 to the next section. For the moment we will accept its claim and use it to show that it implies the symmetric case (2.1) of the product formula in Theorem 2.1.

To this end, let $\{m_n\}$ be a strictly increasing sequence of positive integers, i.e. a subsequence of the sequence of all positive integers. We have only to show that there exists a subsequence $\{n'\}$ in any such sequence $\{m_n\}$ along which (2.1) holds. Then by a standard argument we can conclude that (2.1) actually holds along the sequence of all positive integers n. For if this were not the case, there would exist a subsequence $\{n'\}$ of strictly increasing positive integers along which (2.1) does not converge. However, we see that there is a subsequence $\{n''\}$ of $\{n'\}$ along which the convergence takes place to the same limit, which is a contradiction.

Fix $\{m_n\}$ and $f \in \mathcal{H}$. Lemma 3.1 holds, in particular, along the sequence $\{\tau_n\}$ with $\tau_n := 1/m_n$, and since L^2 convergence implies pointwise convergence a.e. along a subsequence, there exist a subset M_f of Lebesgue measure zero of the variable t in $[0,\infty)$ and a subsequence $\{\tau_{f,n}\}$ of $\{\tau_n\}$, both dependent on f, such that

$$(I+S(it,\tau_{f,n}))^{-1}f \longrightarrow (I+itH_P)^{-1}Pf$$

holds strongly in $\mathcal{H}$ for $t \in [0,\infty)\setminus M_f$. Since $\mathcal{H}$ is separable by assumption, we can choose a countable dense subset $\mathcal{D} = \{f_\ell\}_{\ell=1}^\infty$ in $\mathcal{H}$. Then we infer that for $f_1 \in \mathcal{D}$ there exist a set $M_1 := M_{f_1}$ of Lebesgue measure zero and a subsequence $\{\tau_{1,n}\}$ of $\{\tau_n\}$ along which $(I+S(it,\tau_{1,n}))^{-1}f$ converges to $(I+itH_P)^{-1}Pf$ for every $t \notin M_1$. Next, for $f_2 \in \mathcal{D}$ there exist a set $M_2 := M_{f_2}$ of Lebesgue measure zero and a subsequence $\{\tau_{2,n}\}$ of $\{\tau_{1,n}\}$ along which $(I+S(it,\tau_{2,n}))^{-1}f$ converges to $(I+itH_P)^{-1}Pf$ for every $t \notin M_2$. Proceeding in this way, we associate in the ℓ-th step with $f_\ell \in \mathcal{D}$ a set $M_\ell := M_{f_\ell}$ of Lebesgue measure zero and a subsequence $\{\tau_{\ell,n}\}$ of $\{\tau_{\ell-1,n}\}$ along which $(I+S(it,\tau_{\ell,n}))^{-1}f$ converges to $(I+itH_P)^{-1}Pf$ for every $t \notin M_\ell$.

Now we put $\tau'_n := \tau_{n,n}$ and $n' := 1/\tau'_n$, so that $\{n'\}$ is a subsequence of the strictly increasing sequence $\{m_n\}$ of positive integers from which we have started. Then it follows that for every $t \in [0,\infty)\setminus\cup_{\ell=1}^{\infty} M_\ell$, the sequence $\{(I+S(it,\tau'_n))^{-1}f\}$ converges to $(I+itH_P)^{-1}Pf$ strongly in $\mathcal{H}$ as $\tau'_n \to 0$ for every $f \in \mathcal{D}$, and therefore also in $\mathcal{H}$, because both $(I+S(it,\tau_{\ell,n}))^{-1}$ and $(I+itH_P)^{-1}P$ are bounded operators on $\mathcal{H}$ with the norms not exceeding one. We denote $M := \cup_{\ell=1}^{\infty} M_\ell$, which is, of course, again a set of Lebesgue measure zero. In this way we have found a subsequence $\{\tau'_n\}$ of $\{\tau_n = 1/m_n\}$ and an exceptional subset M of $[0,\infty)$ such that

$$(I+S(it,\tau'_n))^{-1}f = (I+S(it,1/n'))^{-1}f \longrightarrow (I+itH_P)^{-1}Pf \tag{3.5}$$

strongly in $\mathcal{H}$ as $\tau'_n \to 0$ or $n' \to \infty$ for every $f \in \mathcal{H}$ and for each fixed $t \notin M$; it is important that M is independent of f.

Lemma 3.2 *For the sequence $\{n'\}$ specified above and every $f \in \mathcal{H}$ we have*

$$[P(1/n')\exp(-itH/n')P(1/n')]^{n'}f \longrightarrow e^{-itH_P}Pf \tag{3.6}$$

as $n' \to \infty$ strongly in $\mathcal{H}$ provided $t \notin M$.

Notice that this claim is in fact the "symmetric" part of Corollary 2.3.

Proof of Lemma 3.2: We use arguments analogous to those employed in derivation of Chernoff's theorem – see [Ch2, Theorem 1.1], [Ch1] and [Ka1, Thm IX.3.6]. We divide the proof into two steps referring to f belonging to $P\mathcal{H}$ and to its orthogonal complement.

Suppose first that $f \in P\mathcal{H}$. For $t \notin M$ and τ fixed, the operator $S(it,\tau)$ generates a strongly continuous semigroup $\{\, e^{-\theta S(it,\tau)} : \ \theta \ge 0 \,\}$ on $\mathcal{H}$, and the resolvent convergence (3.5) implies the convergence of the corresponding semigroups [Ka1, Thm IX.2.16], so we have

$$e^{-\theta S(it,1/n')}f \xrightarrow{s} e^{-i\theta tH_P}f$$

in $\mathcal{H}$ as $n' \to \infty$ for $t \notin M$, uniformly on each compact interval of the variable $\theta \ge 0$. In particular, choosing $\theta = 1$ we get for each $t \in [0,\infty)\setminus M$

$$e^{-S(it,1/n')}f \xrightarrow{s} e^{-itH_P}f\,, \quad n' \to \infty\,. \tag{3.7}$$

The same equivalence implies for any $\lambda \ge 0$ and $t \in [0,\infty)\setminus M$ that

$$(I+\lambda S(it,1/n'))^{-1}f \xrightarrow{s} (I+i\lambda tH_P)^{-1}Pf\,,$$

in particular, using the diagonal trick we obtain

$$\Big(I + \frac{1}{\sqrt{n'}} S(it, 1/n')\Big)^{-1} f \xrightarrow{\ s\ } Pf \quad \text{as } n' \to \infty, \tag{3.8}$$

for every $t \in [0, \infty) \setminus M$. Next we use [Ch1, Lemma 2] which gives for any $g \in \mathcal{H}$ the inequality

$$\Big\| F(it, 1/n')^{n'} g - e^{-n'(I - F(it,1/n'))} g \Big\| \le \sqrt{n'}\, \|(I - F(it, 1/n'))g\| \,.$$

Choosing $g = \Big(I + \frac{1}{\sqrt{n'}} S(it, 1/n')\Big)^{-1} f$ we infer that

$$\begin{aligned} &\left\| \Big[F(it, 1/n')^{n'} - e^{-S(it,1/n')}\Big] \Big(1 + \frac{1}{\sqrt{n'}} S(it, 1/n')\Big)^{-1} f \right\| \\ &\quad \le \left\| \Big(I + \frac{1}{\sqrt{n'}} S(it, 1/n')\Big)^{-1} f - f \right\|, \end{aligned}$$

where the right-hand side tends to zero as $n' \to \infty$ by (3.8). Using (3.8) once again we get

$$\Big\| F(it, 1/n')^{n'} f - e^{-S(it,1/n')} f \Big\| \longrightarrow 0\,. \tag{3.9}$$

The sought relation (3.6) immediately follows from (3.7) and (3.9), since by (3.1) we have $F(it, 1/n')^{n'} = [P(1/n') \exp(-itH/n') P(1/n')]^{n'}$.

The case $f \in Q\mathcal{H}$ is easier being independent of the arguments preceding Lemma 3.2. We have, along the sequence of *all* positive integers n,

$$[P(1/n) \exp(-itH/n) P(1/n)]^n f \to 0$$

strongly in $\mathcal{H}$ and for each $t \in [0, \infty)$, since $P(1/n)f = P(1/n)Qf$ converges by assumption to $PQf = 0$ as $n \to \infty$, while $\exp(-itH_P)Pf = 0$. ∎

This yields the sought result because $\{[P(1/n') \exp(-itH/n') P(1/n')]^{n'}\}$ is a bounded sequence for any $t \ge 0$ and by Lebesgue dominated-convergence theorem it tends to the expected limit in $L^2_{\mathrm{loc}}([0, \infty); \mathcal{H})$. Using the standard "subsequence" trick mentioned above we have thus shown that Lemma 3.1 implies the symmetric product formula (2.1) of Theorem 2.1.

(b) Let us turn to the non-symmetric product-formula cases, i.e. to prove that (2.1) implies (2.2) and (2.3).

Proof of (2.2): We employ the standard notation, $[U, S] = US - SU$, for the commutator of bounded operators U and S. First we observe the following fact.

Lemma 3.3 *It holds that* $[\,e^{-it\tau H}, P(\tau)\,] \xrightarrow{\ s\ } 0$ *as* $\tau \to 0$*, uniformly on each compact* t*-interval in* $\mathbb{R}$*.*

Proof: By (3.3) with $\zeta = it\tau$ we have

$$[\,e^{-it\tau H}, P(\tau)\,] = i\Big(P(\tau)t\tau H(it\tau) - t\tau H(it\tau)P(\tau)\Big),$$

and hence for any $v \in \mathcal{H}$ we can estimate

$$\|[\,e^{-it\tau H}, P(\tau)\,]v\| \le \|t\tau H(it\tau)v\| + \|t\tau H(it\tau)P(\tau)v\|\,.$$

We rewrite (3.3) with $\zeta = it\tau$ as

$$iH(it\tau) = \frac{I - \cos t\tau H}{t\tau} + i\,\frac{\sin t\tau H}{t\tau} =: B(t\tau) + iA(t\tau)\,, \tag{3.10}$$

where $B(t\tau)$ and $A(t\tau)$ are obviously bounded self-adjoint operators on $\mathcal{H}$, and $B(t\tau)$ is in addition nonnegative. The definition makes sense if $t \neq 0$ but we need not exclude this case because what we really need is the operator $t\tau H(it\tau)$. For any $w \in \mathcal{H}$ we get

$$\begin{aligned}
\|t\tau H(it\tau)w\|^2 &= \|\,[t\tau B(t\tau) + it\tau A(t\tau)]w\|^2 \\
&= \|\,[(I - \cos t\tau H) + i \sin t\tau H]w\|^2 \\
&= \langle w, [(I - \cos t\tau H)^2 + \sin^2(t\tau H)]w\rangle \\
&= 4\|\sin(t\tau H/2)w\|^2 \to 0\,,
\end{aligned}$$

uniformly on compact t-intervals in $\mathbb{R}$. In this way we have proved the claim, noting that $P(\tau) \xrightarrow{\ s\ } P$ holds uniformly on each compact t-interval in $\mathbb{R}$ as $\tau \to 0$. ∎

Now we employ the following identity,

$$\begin{aligned}
&(P(1/n)\,e^{-itH/n})^n v - (P(1/n)\,e^{-itH/n}P(1/n))^n v \\
&= -(P(1/n)e^{-itH/n}P(1/n))^{n-1}[\,e^{-itH/n}, P(1/n)\,]v\,,
\end{aligned} \tag{3.11}$$

the right-hand side of which converges by Lemma 3.3 to zero for all $t \neq 0$ and any $v \in \mathcal{H}$, because $(P(1/n)e^{-itH/n}P(1/n))^{n-1}$ is a contraction on $\mathcal{H}$, and hence also in $L^2_{\mathrm{loc}}([0,\infty); \mathcal{H})$. This yields the formula (2.2).

Proof of (2.3): In view of the already proven formula (2.1) we have for every $f \in \mathcal{H}$ and $T > 0$ the following chain of relations

$$\begin{aligned}
T\|Pf\|^2 &\geq \limsup \int_0^T \|(e^{-itH/n}P(1/n))^n f\|^2\,dt \\
&= \limsup \Big[\int_0^T \|P(1/n)(e^{-itH/n}P(1/n))^n f\|^2\,dt \\
&\qquad + \int_0^T \|Q(1/n)(e^{-itH/n}P(1/n))^n f\|^2\,dt \Big] \\
&\geq \limsup \int_0^T \|(P(1/n)e^{-itH/n}P(1/n))^n f\|^2\,dt \\
&= \int_0^T \|e^{-itH_P}Pf\|^2\,dt = T\|Pf\|^2,
\end{aligned}$$

with the lim sup taken along $n \to \infty$, because $I = P(1/n) + Q(1/n)$ and $P(\tau) \xrightarrow{s} P$ as $\tau \to 0$. It follows that $\int_0^T \|Q(1/n)(e^{-itH/n}P(1/n))^n f\|^2\,dt \longrightarrow 0$ as $n \to \infty$. Thus for any $v(\cdot) \in L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$ and every $T > 0$ we have, again by (2.1),

$$\begin{aligned}
&\int_0^T \langle v(t), (e^{-itH/n}P(1/n))^n f\rangle\,dt \\
= &\int_0^T \langle P(1/n)v(t), (P(1/n)e^{-itH/n}P(1/n))^n f\rangle\,dt \\
&\qquad + \int_0^T \langle Q(1/n)v(t), Q(1/n)(e^{-itH/n}P(1/n))^n f\rangle\,dt \\
&\longrightarrow \int_0^T \langle v(t), e^{-itH_P}Pf\rangle\,dt
\end{aligned}$$

as $n \to \infty$. It means that $\{(e^{-itH/n}P(1/n))^n f\}$ converges to $e^{-itH_P}Pf$ weakly in $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$ together with all the seminorms, and therefore the convergence is strong in $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$. This yields the formula (2.3).

It remains to prove Lemma 3.1 on which the above arguments were based.

4 Proof of Lemma 3.1

To demonstrate (3.4), we shall use the Vitali theorem – see, e.g., [HP] – for holomorphic functions and employ arguments analogous to those used in

Kato's paper [Ka2] for the self-adjoint Trotter product formula with the form sum of a pair of nonnegative self-adjoint operators. We do it in three steps.

I. In the first step we will show the following lemma.

Lemma 4.1 *For a fixed $\zeta = t > 0$,*

$$(I + S(t,\tau))^{-1} \xrightarrow{s} (I + tH_P)^{-1}P \quad \text{as} \quad \tau \to 0\,. \tag{4.1}$$

Proof: The argument will be analogous to that in [Ka2], and indeed, validity of the result in the particular case when our projection-valued function is constant is remarked in [Ka2, Eq. (5.2), p. 194].

For $\zeta = t\tau > 0$ we have from (3.3) $H(t\tau) = (t\tau)^{-1}[I - e^{-t\tau H}]$, which is a bounded, nonnegative and self-adjoint operator on $\mathcal{H}$. It allows us to rewrite

$$\begin{aligned} S(t,\tau) &= \tau^{-1}[I - P(\tau)(I - t\tau H(t\tau))P(\tau)] \\ &= \tau^{-1}Q(\tau) + tP(\tau)H(t\tau)P(\tau)\,, \end{aligned}$$

which is in this case also a bounded and nonnegative self-adjoint operator. To prove (4.1) take any $f \in \mathcal{H}$ and put $\hat{u}(t,\tau) := (I + S(t,\tau))^{-1}f$, so that

$$f = (I + S(t,\tau))\hat{u}(t,\tau) = [I + \tau^{-1}Q(\tau) + tP(\tau)H(t\tau)P(\tau)]\hat{u}(t,\tau)\,. \tag{4.2}$$

Then we have

$$\langle \hat{u}(t,\tau), f\rangle = \|\hat{u}(t,\tau)\|^2 + \tau^{-1}\|Q(\tau)\hat{u}(t,\tau)\|^2 + t\|H(t\tau)^{1/2}P(\tau)\hat{u}(t,\tau)\|^2. \tag{4.3}$$

Thus the families $\{\hat{u}(t,\tau)\}$, $\{\tau^{-1/2}Q(\tau)\hat{u}(t,\tau)\}$ and $\{t^{1/2}H(t\tau)^{1/2}P(\tau)\hat{u}(t,\tau)\}$ are all bounded by $\|f\|$ for all $t > 0$, uniformly as $\tau \to 0$, and therefore they are weakly compact in $\mathcal{H}$. It follows that for each fixed $t > 0$ there exists a sequence $\{\tau_n(t)\}$ with $\tau_n(t) \to 0$ as $n \to \infty$, in general dependent on t, along which these vectors converge weakly in $\mathcal{H}$,

$$\begin{aligned} &\hat{u}(t,\tau) \xrightarrow{w} \hat{u}(t)\,, \quad \tau^{-1/2}Q(\tau)\hat{u}(t,\tau) \xrightarrow{w} g_0(t)\,, \\ &t^{1/2}H(t\tau)^{1/2}P(\tau)\hat{u}(t,\tau) \xrightarrow{w} h(t)\,, \end{aligned} \tag{4.4}$$

for some vectors $\hat{u}(t)$, $g_0(t)$ and $h(t)$ in $\mathcal{H}$. Note that the sequence $\{\tau_n(t)\}_{n=1}^{\infty}$ can be chosen the same for all three families.

¿From this result we see first that $Q(\tau)\hat{u}(t,\tau) \xrightarrow{s} 0$ uniformly in $t > 0$ as $\tau \to 0$, so that we have $Q\hat{u}(t) = 0$ or $\hat{u}(t) = P\hat{u}(t) \in P\mathcal{H}$. For every $v \in D[H^{1/2}]$ we have, with the limit taken along $\{\tau_n(t)\}$,

$$\begin{aligned}\langle v, h(t)\rangle &= \lim \langle v, t^{1/2}H(t\tau)^{1/2}P(\tau)\hat{u}(t,\tau)\rangle \\ &= t^{1/2} \lim \langle H(t\tau)^{1/2}v, P(\tau)\hat{u}(t,\tau)\rangle = t^{1/2}\langle H^{1/2}v, P\hat{u}(t)\rangle\,,\end{aligned}$$

because $H(t\tau)^{1/2}v \xrightarrow{s} H^{1/2}v$ as $\tau \to 0$. Hence $\hat{u}(t) = P\hat{u}(t)$ belongs to $D[H^{1/2}]$ and $h(t) = t^{1/2}H^{1/2}P\hat{u}(t)$ because $D[H^{1/2}]$ is dense by assumption. Furthermore, multiplying (4.2) by $\tau^{1/2}$ and taking the weak limit along the sequence $\{\tau_n(t)\}$ we get $g_0(t) = 0$. Similarly, multiplying (4.2) by $P(\tau)$ we have for every $v \in D[H^{1/2}P]$

$$\langle v, P(\tau)f\rangle = \langle v, P(\tau)\hat{u}(t,\tau)\rangle + \langle t^{1/2}H(t\tau)^{1/2}P(\tau)v, t^{1/2}H(t\tau)^{1/2}P(\tau)\hat{u}(t,\tau)\rangle.$$

Then taking the limit along the sequence $\{\tau_n(t)\}$ we get

$$\langle v, Pf\rangle = \langle v, P\hat{u}(t)\rangle + \langle t^{1/2}H^{1/2}Pv, h(t)\rangle,$$

because by spectral theorem

$$\begin{aligned}\|H(t\tau)^{1/2}(P(\tau)-P)v\| &= \|H(t\tau)^{1/2}(I+H)^{-1/2}(I+H)^{1/2}(P(\tau)-P)v\| \\ &\le \|(I+H)^{1/2}(P(\tau)-P)v\|\,,\end{aligned}$$

which tends to zero since $P(\tau) \xrightarrow{s} P$ as $\tau \to 0$, $D[H^{1/2}P(\tau)] \supset D[H^{1/2}P]$ and $\|H^{1/2}P(\tau)v\| \to \|H^{1/2}Pv\|$ for $v \in D[H^{1/2}P]$ by assumption[1]. Hence $H^{1/2}P\hat{u}(t) \in D[H^{1/2}P]$ and

$$\begin{aligned}Pf &= P\hat{u}(t) + t^{1/2}(H^{1/2}P)^*h(t) = \hat{u}(t) + t(H^{1/2}P)^*(H^{1/2}P)\hat{u}(t) \\ &= \hat{u}(t) + tH_P\hat{u}(t)\,,\end{aligned} \tag{4.5}$$

because $D[H^{1/2}P]$ is supposed to be dense. Applying once again the standard argument mentioned after Lemma 3.1 to all the three families we conclude that the weak convergence in (4.4) takes place independently of a sequence $\{\tau_n(t)\}$ chosen.

[1]This part of the proof shows that the hypotheses of Theorem 2.1 can be slightly weakened, because we need in fact only that $s-\lim_{\tau\to 0} H(t\tau)^{1/2}P(\tau)v = H^{1/2}Pv$ holds for any $v \in D[H^{1/2}P]$.

On the other hand, we infer from (4.3) that

$$\begin{aligned}\langle \hat{u}(t), f\rangle &\geq \liminf \|\hat{u}(t,\tau)\|^2 + \liminf \tau^{-1}\|Q(\tau)\hat{u}(t,\tau)\|^2 \\ &\qquad + \liminf \|t^{1/2}H(t\tau)^{1/2}P(\tau)\hat{u}(t,\tau)\|^2 \\ &\geq \|\hat{u}(t)\|^2 + \|g_0(t)\|^2 + \|h(t)\|^2 \\ &= \|\hat{u}(t)\|^2 + \|t^{1/2}H^{1/2}P\hat{u}(t)\|^2 \\ &= \|\hat{u}(t)\|^2 + t\|H_P^{1/2}\hat{u}(t)\|^2\end{aligned}$$

with $\liminf$ taken along $\tau \to 0$. Since by (4.5) the left-hand side of the above inequality is equal to

$$\langle \hat{u}(t), f\rangle = \langle \hat{u}(t), Pf\rangle = \|\hat{u}(t)\|^2 + t\langle \hat{u}(t), H_P\hat{u}(t)\rangle = \|\hat{u}(t)\|^2 + t\|H_P^{1/2}\hat{u}(t)\|^2,$$

we see that the norms of these vectors converge to the norms of their limit vectors. It allows us to conclude that the $\mathcal{H}$-valued families in question, $\{\hat{u}(t,\tau)\}$, $\{\tau^{-1/2}Q(\tau)\hat{u}(t,\tau)\}$ and $\{t^{1/2}H(t\tau)^{1/2}P(\tau)\hat{u}(t,\tau)\}$ converge to $\hat{u}(t)$, 0 and $t^{1/2}H^{1/2}P\hat{u}(t)$ strongly in $\mathcal{H}$, respectively, as $\tau \to 0$. In particular, we have shown that $Pf = (I + tH_P)\hat{u}(t)$ and $\hat{u}(t,\tau) \xrightarrow{s} \hat{u}(t) = (I + tH_P)^{-1}Pf$, or (4.1). This proves Lemma 4.1. ∎

II. Next, for a fixed $\tau > 0$, the function $\zeta \mapsto F(\zeta,\tau)$ is holomorphic in the open right half-plane $\mathrm{Re}\,\zeta > 0$ and uniformly bounded in norm by one. This makes it possible to mimick the argument of Feldman [Fe], which is reproduced in Chernoff's book [Ch2, p. 90], see also [Fr], to conclude by means of the Vitali theorem (see, e.g., [HP, Thm 3.14.1]) that for $\mathrm{Re}\,\zeta > 0$

$$(I + S(\zeta,\tau))^{-1} \xrightarrow{s} (I + \zeta H_P)^{-1}P \quad \text{as} \quad \tau \to 0 \tag{4.6}$$

holds uniformly on compact subsets of $\mathrm{Re}\,\zeta > 0$.

At the boundary $\mathrm{Re}\,\zeta = 0$, or $\zeta = it$ with t real, $(I + S(\zeta,\tau))^{-1}$ still converges as $\tau \to 0$ but in a weaker sense only. Using the argument of [Fe] based on the Poisson kernel, we can check that for each pair of f, $g \in \mathcal{H}$ and all $\phi \in L^1(\mathbb{R})$ the following relation is valid,

$$s-\lim_{\tau\to 0}\int_{\mathbb{R}} \phi(t)\langle g, (I+S(it,\tau))^{-1}f\rangle\, dt = \int_{\mathbb{R}} \phi(t)\langle g, (I+itH_P)^{-1}Pf\rangle\, dt\,. \tag{4.7}$$

This says that for each pair of f, $g \in \mathcal{H}$ the family $\{\langle g, (I + S(it,\tau))^{-1}f\rangle\}$ of functions of t in $L^\infty(\mathbb{R})$ converges to $\langle g, (I + itH_P)^{-1}Pf\rangle$ as $\tau \to 0$ weakly*, or equivalently, in the weak topology defined by the dual pairing between $L^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$ – see, e.g., [Kö].

III. Now we shall show the family of the bounded operators $\{(I+S(it,\tau))^{-1}\}$ is weakly convergent in $L^2_{\mathrm{loc}}([0,\infty);\,\mathcal{H})$, and in fact, strongly convergent there too. To do so, we will employ an argument analogous to that used in the proof of Lemma 4.1 on the Hilbert space $\mathcal{H}$, however, this time on the Fréchet space $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$.

Using the decomposition (3.10) with $t \neq 0$, we find (cf. [Ich])

$$\begin{aligned} S(it,\tau) &= \tau^{-1}[I - P(\tau)(I - it\tau H(it\tau))P(\tau)] \\ &= \tau^{-1}Q(\tau) + tP(\tau)(B(t\tau) + iA(t\tau))P(\tau)\,. \end{aligned}$$

To prove (3.4), take any $f \in \mathcal{H}$ and put $u(t,\tau) := (I + S(it,\tau))^{-1}f$. Note that this $u(t,\tau)$ represents an element in $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$ as well as its unique representative in $(0,\infty)$, because $u(t,\tau)$ is strongly continuous at this interval as a function of t. Then

$$\begin{aligned} f &= (I + S(it,\tau))u(t,\tau) \\ &= [I + \tau^{-1}Q(\tau) + tP(\tau)(B(t\tau) + iA(t\tau))P(\tau)]u(t,\tau)\,, \end{aligned} \tag{4.8}$$

so we have

$$\begin{aligned} \langle u(t,\tau), f\rangle &= \langle u(t,\tau), (I + S(it,\tau))u(t,\tau)\rangle \\ &= \|u(t,\tau)\|^2 + \tau^{-1}\|Q(\tau)u(t,\tau)\|^2 + t\|B(t\tau)^{1/2}P(\tau)u(t,\tau)\|^2 \\ &\quad + it\langle P(\tau)u(t,\tau), A(t\tau)P(\tau)u(t,\tau)\rangle\,. \end{aligned} \tag{4.9}$$

Observing the real part of (4.9) we see that for τ small enough, each of the $\mathcal{H}$-valued families $\{u(t,\tau)\}$, $\{\tau^{-1/2}Q(\tau)u(t,\tau)\}$ and $\{t^{1/2}B(t\tau)^{1/2}P(\tau)u(t,\tau)\}$ is bounded by $\|f\|$ for all $t > 0$. Moreover, they are strongly continuous in t for fixed $\tau > 0$, and locally bounded as $\mathcal{H}$-valued functions of t in $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$, uniformly as $\tau \to 0$.

Hence we infer first of all that $Q(\tau)u(t,\tau) \xrightarrow{s} 0$, uniformly in $t \in (0,\infty)$, as $\tau \to 0$. Next, since $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$ is reflexive [GV, Chap. 1, Sec. 3.1, pp. 57-62], any bounded set in it is weakly compact [Kö, Sec. 23.5, pp. 302-304]. Consequently, there is a sequence $\{\tau_n\}_{n=1}^\infty$ with $\tau_n \to 0$ as $n \to \infty$ along which the above families are weakly convergent in $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$:

$$\begin{aligned} &u(t,\tau) \xrightarrow{w} u(t)\,, \qquad \tau^{-1/2}Q(\tau)u(t,\tau) \xrightarrow{w} f_0(t)\,, \\ &t^{1/2}B(t,\tau)^{1/2}P(\tau)u(t,\tau) \xrightarrow{w} z(t)\,, \end{aligned} \tag{4.10}$$

with some vectors $u(\cdot)$, $f_0(\cdot)$ and $z(\cdot) \in L^2_{\rm loc}([0,\infty);\mathcal{H})$. Note that as before the sequence $\{\tau_n\}_{n=1}^\infty$ can be chosen the same for all three families.

Lemma 4.2 *These above mentioned vectors have the following properties,*

$$u(t) = Pu(t) \in P\mathcal{H} \text{ for a.e. } t\,, \quad z(\cdot) = 0\,, \quad f_0(\cdot) = 0\,.$$

Proof: For $B(t\tau)$ and $A(t\tau)$ in (3.10), the spectral theorem gives

$$\begin{aligned} \|(B(t\tau)^{1/2}v\|^2 &= \int_{0-}^\infty \left|\frac{1-\cos t\tau\lambda}{t\tau\lambda}\right| \|E(d\lambda)H^{1/2}v\|^2 \to 0\,, \;\; v \in D[H^{1/2}]\,; \\ \|(B(t\tau)v\|^2 &= \int_{0-}^\infty \left|\frac{1-\cos t\tau\lambda}{t\tau\lambda}\right|^2 \|E(d\lambda)Hv\|^2 \to 0\,, \;\; v \in D[H] \end{aligned} \tag{4.11}$$

as $\tau \to 0$ by the Lebesgue dominated-convergence theorem.

Since $Q(\tau)u(t,\tau) \xrightarrow{s} 0$ uniformly in $t \in (0,\infty)$ and $Q(\tau) \xrightarrow{s} Q$ as $\tau \to 0$, we have $Qu(t) = 0$, or in other words $u(t) = Pu(t) \in P\mathcal{H}$ for a.e. t. Moreover, by (4.11) we infer that

$$\begin{aligned} \int_0^\infty \langle \phi(t)v, z(t)\rangle\, dt &= \lim \int_0^\infty \langle \phi(t)v, t^{1/2}B(t\tau)^{1/2}P(\tau)u(t,\tau)\rangle\, dt \\ &= \lim \int_0^\infty \bar\phi(t)\langle t^{1/2}B(t\tau)^{1/2}v, P(\tau)u(t,\tau)\rangle\, dt \\ &= \int_0^\infty \bar\phi(t)\langle 0, Pu(t)\rangle\, dt = 0 \end{aligned}$$

holds for every $\phi \in C_0^\infty([0,\infty))$ and $v \in D[H^{1/2}]$, hence $z(t) = 0$ a.e. because $D[H^{1/2}]$ is dense in $\mathcal{H}$, so that $z(\cdot)$ is the zero element of $L^2_{\rm loc}([0,\infty);\mathcal{H})$. Finally, the relation $f_0(\cdot) = 0$ follows from (4.8) which implies $\tau^{1/2}Q(\tau)f = \tau^{1/2}(1+\tau^{-1})Q(\tau)u(t,\tau)$, yielding the result; this concludes the proof. ∎

Our next aim is to show that the weak limits in (4.10) do not depend upon a sequence chosen. The $u(\cdot,\tau_n) = (I+S(it,\tau_n))^{-1}f$ converge to $u = u(\cdot)$ weakly in $L^2_{\rm loc}([0,\infty);\mathcal{H})$ as $n \to \infty$. It obviously implies that for all $\phi \in C_0^\infty([0,\infty))$ and for every $g \in \mathcal{H}$ we have

$$\int_0^\infty \phi(t)\langle g, (I+S(it,\tau))^{-1}f\rangle\, dt \longrightarrow \int_0^\infty \phi(t)\langle g, u(t)\rangle\, dt\,,$$

again along the sequence $\{\tau_n\}$. It follows from (4.7) that

$$u(t) = (I + itH_P)^{-1}Pf, \quad \text{for a.e. } t \text{ in } [0,\infty), \tag{4.12}$$

because the set of all such $\bar{\phi}(\cdot)g$ is total in $L^2_{\rm loc}([0,\infty);\mathcal{H})$. This shows that for every $f \in \mathcal{H}$, $(I + S(it,\tau_n))^{-1}f$ converges to $(I + itH_P)^{-1}Pf$ weakly in $L^2_{\rm loc}([0,\infty);\mathcal{H})$ as $n \to \infty$. Together with the fact that $z(\cdot) = 0$, $f_0(\cdot) = 0$ in view of Lemma 4.2, this yields the desired property, namely that the weak limits of (4.10) are independent of the particular subsequence $\{\tau_n\}$ chosen. The standard argument sketched below Lemma 3.1 shows that (4.10) holds as $\tau \to 0$ without any restriction on subsequences.

Finally, we are going to check the strong convergence $u(\cdot,\tau) \xrightarrow{s} u(\cdot)$ in $L^2_{\rm loc}([0,\infty);\mathcal{H})$ as $\tau \to 0$. In fact, we will prove two other limiting relations at the same time.

Lemma 4.3 *In the topology of $L^2_{\rm loc}([0,\infty);\mathcal{H})$, the family $\{u(\cdot,\tau)\}$ converges to the vector $u = u(\cdot)$ as $\tau \to 0$, and moreover,*

$$\begin{aligned}
&\tau^{-1/2}Q(\tau)u(t,\tau) \longrightarrow f_0(t) = 0\,, \\
&t^{1/2}B(t,\tau)^{1/2}P(\tau)u(t,\tau) \longrightarrow z(t) = 0\,.
\end{aligned}$$

Proof: In the above reasoning we have already checked the weak convergence in (4.10) as $\tau \to 0$. Integrating the real part of (4.9) in t over the interval $[0,T]$ for any fixed $T > 0$ and taking $\liminf$ as $\tau \to 0$, we get by Lemma 4.2

$$\begin{aligned}
\mathrm{Re}\int_0^T \langle u(t), f\rangle\,dt \;\;&\geq\;\; \liminf \int_0^T \|u(t,\tau)\|^2\,dt \\
&\qquad + \liminf \int_0^T \tau^{-1}\|Q(\tau)u(t,\tau)\|^2\,dt \\
&\qquad + \liminf \int_0^T \|t^{1/2}B(t\tau)^{1/2}P(\tau)u(t,\tau)\|^2\,dt \\
&\geq\;\; \int_0^T \|u(t)\|^2\,dt + \int_0^T \|f_0(t)\|^2\,dt + \int_0^T \|z(t)\|^2\,dt \\
&\geq\;\; \int_0^T \|u(t)\|^2\,dt\,.
\end{aligned}$$

On the other hand, the left-hand side of the above inequality is by (4.12) equal to

$$\mathrm{Re}\int_0^T \langle u(t), f\rangle\, dt = \mathrm{Re}\int_0^T \langle u(t), (I + itH_P)u(t)\rangle\, dt = \int_0^T \|u(t)\|^2\, dt.$$

Hence we conclude that all the Fréchet-space semi-norms of the vectors $u(t,\tau)$, $\tau^{-1/2}Q(\tau)u(t,\tau)$ and $t^{1/2}B(t,\tau)^{1/2}P(\tau)u(t,\tau)$ converge to the semi-norms of the weak-limit vectors $u(t)$, 0 and 0, respectively, as $\tau \to 0$. Thus the convergence is strong with respect to each semi-norm, and since their family induces the topology in $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$ the lemma is proved. ∎

This completes the proof of Lemma 3.1, and by that the verification of our main result, Theorem 2.1.

5 The finite-dimensional case

In this section, we will prove Theorem 2.4 in which we assume that P and $P(t)$ are finite-dimensional orthogonal projections. Since the closed operator $H^{1/2}P$ is supposed to be densely defined, the domain $D[H^{1/2}P]$ of $H^{1/2}P$ becomes the whole space $\mathcal{H}$, for the restriction $H^{1/2}P|_{P\mathcal{H}}$ of the operator $H^{1/2}P$ to the finite-dimensional subspace $P\mathcal{H}$ is densely defined, so its domain must coincide with $P\mathcal{H}$, and it acts as zero on $Q\mathcal{H}$. The same is valid for $H^{1/2}P(t)$ when $P(t)$ is of a finite dimension. As a result, $H^{1/2}P$ and $H_P = (H^{1/2}P)^*(H^{1/2}P)$ as well as $H^{1/2}P(t)$ are bounded operators on $\mathcal{H}$ by the closed-graph theorem. By the assumptions common with Theorem 2.1, for each fixed $f \in D[H^{1/2}P] = \mathcal{H}$ the family $\{H^{1/2}P(t)f\}$ converges to $H^{1/2}Pf$ as $t \to 0$, and hence is uniformly bounded with respect to t near to zero, say, for $-1 \le t \le 1$. Then by the uniform boundedness principle we can conclude that $\sup_{|t|\le 1}\|H^{1/2}P(t)\| < \infty$.

To prove the assertions (i) and (ii) of Theorem 2.4 simultaneously, take a fixed $a \in \mathbb{R}$ and consider instead of $F(\zeta,\tau)$, $S(\zeta,\tau)$ defined by (3.1) and (3.2), respectively, the following operators

$$F_a(\zeta,\tau) := P(a\tau)\exp(-\zeta\tau H)P(a\tau), \quad S_a(\zeta,\tau) := \tau^{-1}[I - F_a(\zeta,\tau)].$$

In fact, we shall employ $F_a(it,\tau)$, $S_a(it,\tau)$ instead of $F(it,\tau)$, $S(it,\tau)$ in the proof of Lemma 3.2 and Lemma 3.1. Similarly $u(t,\tau)$ used above will be replaced by $u_a(t,\tau) = (I + S_a(it,\tau))^{-1}f$ corresponding to a given $f \in \mathcal{H}$.

Lemma 5.1 *For any $t, t' \geq 0$ and $0 < \tau \leq 1$ we have*

$$\|u_a(t,\tau) - u_a(t',\tau)\| \leq C(a)|t-t'|\,\|f\|$$

with a positive $C(a)$ independent of t, t', which is uniformly bounded as a function of a on each compact interval of $\mathbb{R}$.

Proof: By the resolvent equation we have

$$\begin{aligned}
&u_a(t,\tau) - u_a(t',\tau)\\
&\quad = (I + S_a(it,\tau))^{-1}f - (I + S_a(it',\tau))^{-1}f\\
&\quad = (I + S_a(it,\tau))^{-1}P(a\tau)\tau^{-1}[e^{-it\tau H} - e^{-it'\tau H}]P(a\tau)(I + S_a(it',\tau))^{-1}f\\
&\quad = (I + S_a(it,\tau))^{-1}P(a\tau)\,\frac{1}{\tau}\int_{t'}^{t}\frac{d}{ds}\,e^{-is\tau H}ds\;P(a\tau)(I + S_a(it',\tau))^{-1}f\\
&\quad = -i(I + S_a(it,\tau))^{-1}P(a\tau)\int_{t'}^{t}He^{-is\tau H}ds\;P(a\tau)(I + S_a(it',\tau))^{-1}f\\
&\quad = -i(I + S_a(it,\tau))^{-1}(H^{1/2}P(a\tau))^{*}\\
&\qquad\quad \times\int_{t'}^{t}e^{-is\tau H}ds\;(H^{1/2}P(a\tau))(I + S_a(it',\tau))^{-1}f\,.
\end{aligned}$$

At the beginning of this section we have argued that the operators $H^{1/2}P(a\tau)$ are uniformly bounded on $\mathcal{H}$ for $0 < \tau \leq 1$. It follows that

$$\|u_a(t,\tau) - u_a(t',\tau)\| \leq C(a)|t-t'|\,\|f\|$$

with $C(a) := \sup_{|a\tau|\leq 1}\|H^{1/2}P(a\tau)\|^2$. By the argument preceding the lemma the function $C(\cdot)$ is uniformly bounded on each compact a-interval in $\mathbb{R}$; this yields the claim. ∎

Proof of Theorem 2.4: It follows from the lemma that the vector family $\{u_a(t,\tau)\}$, continuous in $\mathcal{H}$, is uniformly bounded and equicontinuous. Hence we may infer by the Ascoli–Arzelà theorem that the sequence $\{\tau_n\}$ used in part III of the proof of Lemma 3.1 can chosen to have an additional property, namely that the sequence $\{u_a(t,\tau_n)\}$ converges strongly to $u(t)$ also pointwise, uniformly on $[0,\infty)$. Then the limit $u(t)$ becomes strongly continuous in $t \geq 0$, and coincides with $(I + itH_P)^{-1}f$ for all $t \geq 0$. Thus we have instead of Lemma 3.1 the following claim:

$$(I + S_a(it,\tau))^{-1} \longrightarrow (I + itH_P)^{-1}P \tag{5.1}$$

as $\tau \to 0$, strongly on $\mathcal{H}$ and uniformly on each compact interval of the variable t in $[0,\infty)$.

Next we will modify the reasoning of Sec. 3 based on [Ch2, Theorem 1.1] with the aim to show the symmetric product case,

$$[P(at/n)\exp(-itH/n)P(at/n)]^n \xrightarrow{\ s\ } \exp(-itH_P)P, \quad n\to\infty. \tag{5.2}$$

Let $f\in\mathcal{H}$. The resolvent convergence (5.1) with $t=1$ implies the convergence of the corresponding semigroups, so we have

$$e^{-\theta S_a(i,\tau)}f \xrightarrow{\ s\ } e^{-i\theta H_P}f \tag{5.3}$$

in $\mathcal{H}$ as $\tau\to 0$, uniformly on each compact interval of the variable $\theta\ge 0$. Using this equivalence once more we get for any $\lambda\ge 0$ the relation

$$(I+\lambda S_a(i,\tau))^{-1}f \xrightarrow{\ s\ } (I+i\lambda H_P)^{-1}Pf\,, \quad \tau\to 0.$$

In particular, taking $\tau=\theta/n$ and using the diagonal trick, we infer that

$$(I+\frac{\theta}{\sqrt{n}}S_a(i,\theta/n))^{-1}f \xrightarrow{\ s\ } Pf\,, \quad n\to\infty\,, \tag{5.4}$$

holds uniformly on each compact θ-interval in $[0,\infty)$. Then the mentioned lemma from [Ch1] yields

$$\left\|F_a(i,\theta/n)^n g - e^{-n(I-F_a(i,\theta/n))}g\right\| \le \sqrt{n}\,\|(I-F_a(i,\theta/n))g\|\,.$$

Choosing again $g=\left(I+\frac{\theta}{\sqrt{n}}S_a(i,\theta/n)\right)^{-1}f$ we find that

$$\begin{aligned}&\left\|\left[F_a(i,\theta/n)^n - e^{-\theta S_a(i,\theta/n)}\right]\left(1+\frac{\theta}{\sqrt{n}}S_a(i,\theta/n)\right)^{-1}f\right\|\\ &\quad\le \left\|\left(I+\frac{\theta}{\sqrt{n}}S_a(i,\theta/n)\right)^{-1}f-f\right\|\,,\end{aligned}$$

where the right-hand side tends to zero as $n\to\infty$ by (5.4). Using the last named convergence once more we get

$$\left\|F_a(i,\theta/n)^n f - e^{-\theta S_a(i,\theta/n)}f\right\| \longrightarrow 0 \tag{5.5}$$

uniformly on each compact θ-interval in $[0, \infty)$. Choosing now $\theta = t$ we see that the validity of (5.2) on $P\mathcal{H}$ follows immediately from (5.3) and (5.5). Consequently, on the subspace $P\mathcal{H}$ the assertion (i) is obtained by taking $a = 1/t$ for any t belonging to a compact interval in $\mathbb{R} \setminus \{0\}$ and (ii) by choosing simply $a = 1$.

The case $f \in Q\mathcal{H}$ can be treated as in the proof of Lemma 3.2; together this yields the relation (5.2) on $\mathcal{H}$, i.e. the symmetric product case. The non-symmetric product cases can also be checked with the help of Lemma 3.3 – cf. part (b) of the proof of Theorem 2.1 in Section 3. This concludes the proof of Theorem 2.4. ▮

6 An example

As we have said, our investigation was motivated by the result by Facchi et al. [FPS] mentioned in the introduction. Let us thus look how the result looks in this case. To see this, consider an open domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary, and denote by P the orthogonal projection on $L^2(\mathbb{R}^d)$ defined as the multiplication operator by the indicator function χ_Ω of the set Ω. Consider further the free quantum Hamiltonian $H := -\Delta$, i.e. the Laplacian in $\mathbb{R}^d$ which is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^d)$, and the Dirichlet Laplacian $-\Delta_\Omega$ in $L^2(\Omega)$ defined in the usual way [RS, Sec. XIII.15] as the Friedrichs extension of the appropriate quadratic form.

We consider the Zeno dynamics in the subspace $L^2(\Omega)$ corresponding to a permanent reduction of the wavefunction to the region Ω, which may be identified with the volume of a detector. In the sense of the $L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^d))$ topology, which is physically plausible as explained in Remark 2.5, we then claim that the generator of the dynamics in $L^2(\Omega)$ is just the appropriate Dirichlet Laplacian,

$$(Pe^{-it(-\Delta/n)}P)^n \to e^{-it(-\Delta_\Omega)}P \tag{6.1}$$

as $n \to \infty$, or in other words:

Proposition 6.1 *The self-adjoint operator*

$$-\Delta_P = ((-\Delta)^{1/2}P)^*((-\Delta)^{1/2}P) \tag{6.2}$$

is densely defined in $L^2(\mathbb{R}^d)$ and its restriction to the subspace $L^2(\Omega)$ is nothing but the Dirichlet Laplacian $-\Delta_\Omega$ of the region Ω, with the domain $D[-\Delta_\Omega] = W^1_0(\Omega) \cap W^2(\Omega)$.

Proof: Let $u \in D[-\Delta_P]$, so that u and $-\Delta_P u$ belong to $L^2(\mathbb{R}^d)$. We have

$$\langle -\Delta_P u, \varphi \rangle = \langle u, -\Delta\varphi \rangle = \langle -\Delta u, \varphi \rangle,$$

for any $\varphi \in C_0^\infty(\Omega)$ because φ has a compact support in Ω. Thus $-\Delta_P u = -\Delta u$ holds in Ω in the sense of distributions, which means that $\Delta u|_\Omega \in L^2(\Omega)$. On the other hand, since $(-\Delta)^{1/2} P u \in L^2(\mathbb{R}^d)$, we have $\chi_\Omega u \in W^1(\mathbb{R}^d)$. Since we have

$$\nabla(\chi_\Omega u) = \nabla((\chi_\Omega)^2 u) = (\nabla\chi_\Omega)\chi_\Omega u(x) + \chi_\Omega \nabla(\chi_\Omega u),$$

in order to belong to $L^2(\mathbb{R}^d)$ the function $\nabla(\chi_\Omega u)$ must not contain the δ-type singular term, which requires $u(\cdot) = 0$ on the boundary of Ω. This combined with the fact that $u|_\Omega$, $\Delta u|_\Omega \in L^2(\Omega)$ – see, e.g., [LM, Thm 5.4] – implies that $u|_\Omega$ belongs to $W^2(\Omega)$ and $W_0^1(\Omega)$.

Thus we have shown that $u|_\Omega \in D[-\Delta_\Omega]$ and $(-\Delta_P u)|_\Omega = -\Delta_\Omega(u|_\Omega)$ or $-\Delta_\Omega \supset -\Delta_P|_{L^2(\Omega)}$, but both operators are self-adjoint, so they coincide. ∎

In this sense therefore our result given in Theorem 2.1 provides one possible abstract version of the result by Facchi et al. [FPS].

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