LINEAR RESPONSE FUNCTION FOR COUPLED HYPERBOLIC ATTRACTORS

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ABSTRACT

We prove that when we take the thermodynamic limit in the context of coupled hyperbolic attractors, Ruelle's derivative formula of the SRB measure with respect to the underlying dynamical system remains true if one of the terms is interpreted appropriately.

1. Introduction: Derivative formulas of the SRB measure and its entropy

For uniformly hyperbolic systems, Ruelle proved that the (generalized) SRB measure depends on the system differentiably and calculated its derivative formula [14, 16]. The derivative formula of the SRB state with respect to the underlying system is also called the linear response function and is used to develop the theory of non-equilibrium statistical mechanics [15]. Since Statistical Mechanics is mainly concerned with infinite systems and the relevant physical quantities are obtained through thermodynamic limits, it is of interest to study the thermodynamic limit of the linear response function. The differentiability of the thermodynamic limit of SRB measures was established in [7]. In this paper, we want to study the thermodynamic limit of the formulas for the derivative of the SRB measures.

For a finite dimensional hyperbolic attractor f, let $\varphi(x)$ be a differentiable function on M and δf be the vector field on M in which the derivative of the SRB measure μ_f is taken. Denote $X = \delta f \circ f^{-1}$. Ruelle's derivative formula states that

$$\delta\mu_f(\varphi) = \sum_{n=0}^{\infty} \int [< \mathrm{grad}(\varphi(f^n)), X^s > -\varphi(f^n)\mathrm{div}_{\sigma}^{u} X^u] d\mu_f, \tag{1.1}$$

where X^s and X^u are projections of X onto the stable and unstable subspaces, respectively, and $\mathrm{div}_{\sigma}^{u} X^u$ is the divergence of the vector field X^u on the unstable manifold with respect to the canonical metric σ induced by the SRB measure. Even though the vector field X^u is in general, not differentiable as observed by D. Dolgopyat, its divergence is well-defined and it is Hölder continuous on the hyperbolic attractor [16].

The formula (1.1) is expected to hold whenever the SRB measure exists and is differentiable with respect to the map f [17]. In this paper, combining the techniques developed in our proof [7] of the smooth dependence of the SRB measure for coupled hyperbolic systems and the strategy that Ruelle used to obtain the formula in finite dimensional systems [14, 16], we prove that an analogous formula for the derivative holds in the thermodynamic limit for coupled hyperbolic attractors. We just need to require that the function $\varphi(x)$ is sufficiently differentiable and that it satisfies certain decay property commonly needed for infinite-dimensional systems. We also need to give a proper interpretation to the term involving divergences: While the divergence $\mathrm{div}^u_\sigma X^u$ is infinite for most meaningful vector fields on the coupled system, the divergence of the projection of each component of X onto the unstable subspace X^u_i can be shown to be finite and

$$\sum_{n=0}^{\infty}\int \varphi(\Phi^n)\mathrm{div}^u_\sigma X^u d\mu$$

can be interpreted as

$$\sum_{n=0}^{\infty}\int \varphi(\Phi^n)\mathrm{div}^u_\sigma X^u d\mu = \sum_{n=0}^{\infty}\sum_{i\in\mathbb{Z}^d}\int \varphi(\Phi^n)\mathrm{div}^u_\sigma X^u_i d\mu.$$

The sum on the righthand side converges because of the exponential decay of correlation functions. Indeed, we will also show that the sum

$$\sum_{n=0}^{\infty}\int \varphi(\Phi^n)\mathrm{div}^u_\sigma X^u_V d\mu_V \tag{1.2}$$

converges as $V \to \mathbb{Z}^d$, where Φ_V, X_V denote the restrictions of the coupled map Φ and the vector filed X onto the finite product $M_V = \otimes_{i\in V} M$, $V \subset \mathbb{Z}^d$ is a finite set. As usual in statistical mechanics, we will need to understand that the limit $V \to \mathbb{Z}^d$ is reached in some sufficiently strong sense. For example, in [19], one can find discussions of limits in the sense van Hove or of Fisher. We will carry out the details only when V are cubes but indicate that there are some more general limits that could be considered. We denote by μ_V is the SRB measure of Φ_V. Thus, we can use the limit as $V \to \mathbb{Z}^d$ in (1.2) as the definition of the series

$$\sum_{n=0}^{\infty}\int \varphi(\Phi^n)\mathrm{div}^u_\sigma X^u d\mu.$$

We then, have the same derivative formula for coupled hyperbolic attractors

$$\delta\mu_\Phi(\psi) = \sum_{n=0}^{\infty} \int [< \mathrm{grad}(\varphi(\Phi^n)), X^s > -\varphi(\Phi^n)\mathrm{div}_\sigma^u X^u] d\mu_\Phi.$$

In the finite dimensional case, the derivative formula of the entropy with respect to the SRB measure is given by [16]

$$\delta h_{\mu_f}(f) = \sum_{n=-\infty}^{\infty} \int -\log J^u f(f^n(x))\mathrm{div}_\sigma^u X^u d\mu_g.$$

By an analogous argument, the derivative formula of the entropy in the thermodynamic limit is given by

$$\delta h_{\mu_\Phi}(\Phi) = \sum_{n=-\infty}^{\infty} \int [-\log Jf^u + \psi_\epsilon] \circ \Phi^n(\bar{x})\mathrm{div}_\sigma^u X^u d\mu_\Phi,$$

where $-\log Jf^u + \psi_\epsilon$ is the potential function for the SRB measure μ_Φ [10, 7] and the integral

$$\int [-\log Jf^u + \psi_\epsilon] \circ \Phi^n(\bar{x})\mathrm{div}_\sigma^u X^u d\mu_\Phi$$

is defined via the thermodynamic limit of finite dimensional approximation as $V \to \mathbb{Z}^d$.

For additional recent results in the area of smooth dependence and the differentiation of SRB measures, we refer readers to [1], [5], [12], and [18].

2. Notation and Formulation of the Main Result

First, we briefly introduce the notation. For more details, see [7].

1. Let M be a smooth compact Riemannian manifold and f a C^r ($r > 4$) diffeomorphism of M. Assume that f possesses a hyperbolic attractor Λ_f. The direct product space $\otimes_{i\in\mathbb{Z}^d} M_i$ over the integer lattice of dimension n is denoted by $\mathcal{M}$, where M_i is a copy of M. This direct product space is a Banach manifold under the usual supremum metric ρ

$$\rho(\bar{x}, \bar{y}) = \sup_{i\in\mathbb{Z}^d} \mathrm{d}(x_i, y_i),\ \bar{x} = (x_i), \bar{y} = (y_i) \in \mathcal{M}.$$

$\mathcal{M}$ is also equipped with the so called *"compact metric"* $\rho_q, 0 < q < 1$.

$$\rho_q(\bar{x}, \bar{y}) = \sup_{i\in\mathbb{Z}^d} q^{|i|}\mathrm{d}(x_i, y_i),\ \bar{x} = (x_i), \bar{y} = (y_i) \in \mathcal{M}, \tag{2.1}$$

where $i = (i_1, i_2, \cdots, i_d) \in \mathbb{Z}^d$ and $|i|$ is the lenght, $|i| = \sum_{k=1}^{d} |i_k|$. Note that, the supremun metric is invariant under translations of the indices i, but the compact metric is not.

The infinite dimensional system $F = \otimes_{i\in\mathbb{Z}^d} f$ over $\mathcal{M}$ is called the uncoupled system and it possesses a hyperbolic attractor $\Delta = \otimes_{i\in\mathbb{Z}^d}\Lambda_f$.

2. For the integer lattice $\mathbb{Z}^d$, given $\alpha_0 > d, \theta_0 \geq 0$, there exists $a > 0$ such that the function defined by

$$\Gamma(i) = \begin{cases} a|i|^{-\alpha_0}\exp(-\theta_0|i|) & i \neq 0, \\ a & i = 0, \end{cases} \tag{2.2}$$

satisfies the following properties:

(1) $\sum_{i\in\mathbb{Z}^d} \Gamma(i) < \infty$,
(2) $\sum_{j\in\mathbb{Z}^d} \Gamma(i-j)\Gamma(j-k) \leq \Gamma(i-k)$.

A function satisfying these two conditions is called a decay function. (See [7] Proposition 3.2).

In this paper, we will only consider a fixed decay function of the form (2.2) and with $\theta_0 > 0$. Let $V_n = \{i : i \in \mathbb{Z}^d, |i| \leq n\}$ and $\widehat{V}_n = \mathbb{Z}^d \setminus V_n$. The following properties can be easily verified for decay functions as in (2.2).

(1) For any number $\alpha > 0$, there exist some constants C and β depending only on the decay function Γ and α, such that
$$\sum_{i\in\widehat{V}_n} (\Gamma(i))^\alpha \leq C\exp(-\beta n).$$
(2) For any number $\alpha > 0$, there exist some constants C and β depending only on the decay function Γ and α, such that
$$\sum_{j\in\mathbb{Z}^d} (\Gamma(i-j)\Gamma(j-k))^\alpha \leq C\exp(-\beta|i-k|).$$

The inequalities above and (3.7) are the only places were we use that the V_n are cubes. It is clear that there are more general sequences of sets which converge to $\mathbb{Z}^d$ and which also satisfy the inequalities. These inequalities are slightly more restrictive than the sense of Van Hove considered in [19]. We do not attempt to investigate the optimal sense in which the limit can be reached. We note that, by considering cubes, we can obtain explicit estimates on the rate at which the limit is reached.

Throughout the paper, we will use C, α, and β to denote generic constants that will appear in various definitions and estimates. Instead of using different symbols for different constants, we will use these generic symbols with explicit explanations of their dependence.

3. A coupled map lattice Φ is defined to be a small perturbation of F of the form

$$\Phi = G \circ F,$$

where G is a $C^r, r > 4$ diffeomorphism of $\mathcal{M}$ close to the identity. By lifting $G(\bar{x})$ to the tangent space $T_{\bar{x}}\mathcal{M}$, we can identify G with a smooth section of the tangent bundle $T\mathcal{M}$.

4. Now we give the definitions of Banach spaces for the objects we study.

(1) $C^0(\mathcal{M}, T\mathcal{M})$, the space of continuous sections of the tangent bundle:

$$\{\bar{v}(\bar{x}):\ \bar{x} \to v(\bar{x}) \in T_{\bar{x}}\mathcal{M} \text{ is continuous}, ||\bar{v}||_{C^0} \equiv \sup_{i\in\mathbb{Z}^d} \sup_{\bar{x}\in\mathcal{M}} |v_i(\bar{x})| < \infty\}$$

(2)$C^\alpha(\mathcal{M}, T\mathcal{M})$, the space of Hölder continuous sections of the tangent bundle:

For $0 < \alpha < 1$, we fix a trivialization of TM by a finite number of coordinate charts, $\{P_1, \ldots, P_l\}$

(2.3)

$$C^\alpha(\mathcal{M}, T\mathcal{M}) =$$

$$\{\bar{v}(\bar{x}) : ||\bar{v}||_{C^\alpha} \equiv \max(||\bar{v}||_{C^0}, \sup_{i\in\mathbb{Z}^d} \max_k \sup_{\bar{x}\neq\bar{y}\in\mathcal{M}, x_i, y_i\in P_k} \frac{|v_i(\bar{x}) - v_i(\bar{y})|}{d(\bar{x}, \bar{y})^\alpha} < \infty)\}$$

The Banach space of all Hölder continuous functions on $\mathcal{M}$ with Hölder exponent α will be denoted by $C^\alpha(\mathcal{M}, \mathbb{R})$.

(3) $C^\alpha_\Gamma(\mathcal{M}, T\mathcal{M})$, the space of Hölder continuous sections satisfying decay property specified by Γ:

Let $h : \mathcal{M} \to \mathbb{R}^n$ be a function. Define

$$\gamma_{\alpha,i}(h) = \sup_{(x_j)_{j\neq i}} \sup_{(z_j=y_j=x_j)_{j\neq i}, y_i\neq z_i} \frac{\|h(\bar{y}) - h(\bar{z})\|}{d^\alpha(y_i, z_i)},$$

where the first supremum is taken over all $\bar{x} = (x_j)$ without the ith variable and the second supremum is taken over all pairs of $\bar{y}$ and $\bar{z}$ with their variables equal to that of $\bar{x}$ except at the lattice site i.

We can identify v_j, the jth variable of $v = (v_j)$, with a function from $\mathcal{M}$ to $\mathbb{R}^n$. For $0 < \alpha <\leq 1$.

$$C^\alpha_\Gamma(\mathcal{M}, T\mathcal{M}) =$$

$$\big\{v \in C^0(\mathcal{M}, T\mathcal{M}) : \|v\|_{C^\alpha_\Gamma} \equiv \max\{\|v\|_{C^0}, \sup_{i,j\in\mathbb{Z}^d} \gamma_{\alpha,j}(v_i)\Gamma^{-1}(i-j)\} < \infty\big\}.$$

(4) $C^r_\Gamma(\mathcal{M}, T\mathcal{M})$, the space of C^r-sections satisfying decay property specified by Γ: For each $r \in \mathbb{N}$, we denote

$$
\begin{aligned}
&C^r_\Gamma(\mathcal{M}, T\mathcal{M}) = \big\{ v \in C^r(\mathcal{M}, T\mathcal{M}) \cap C^1_\Gamma(\mathcal{M}, T\mathcal{M}) :\\
\|v\|_{C^r_\Gamma} \equiv &\max_{0\le k\le r} \sup_{i_1,\cdots,i_k\in\mathbb{Z}^d} \sup_{i\in\mathbb{Z}^d} \Big|\Big| \frac{\partial^k}{\partial x_{i_1}\cdots \partial x_{i_k}} v_i(\bar{x}) \Big|\Big|_{C^0} \max\{\Gamma^{-1}(i-i_1), \cdots \Gamma^{-1}(i-i_k)\}\\
&< \infty \big\}.
\end{aligned}
$$

The space of C^r functions on $\mathcal{M}$ satisfying decay property specified by Γ will be denoted by $C^r_\Gamma(\mathcal{M}, \mathbb{R})$. For any function $\varphi \in C^r_\Gamma(\mathcal{M}, \mathbb{R})$, we have

$$
\left| \frac{\partial \varphi}{\partial x_i} \right| \le \Gamma(i) \|\varphi\|_{C^r_\Gamma}.
$$

For convenience, we also introduce a semi-norm for each coupled map

$$
\|\Phi\|_{C^r_\Gamma} = \max_{0\le k\le r} \sup_{i_1,\cdots,i_k\in\mathbb{Z}^d} \sup_{i\in\mathbb{Z}^d} \Big|\Big| \frac{\partial^k}{\partial x_{i_1}\cdots \partial x_{i_k}} \Phi_i(\bar{x}) \Big|\Big|_{C^0} \max\{\Gamma^{-1}(i-i_1), \cdots, \Gamma^{-1}(i-i_k)\}
$$

This norm is finite as long as the C^r_Γ-norm of the corresponding perturbation function G is finite.

The following results on the existence, uniqueness, and the differentiability of the SRB measure for the coupled map lattice Φ are contained in [7, 10].

Given a decay function Γ, a smooth Riemannian manifold M, and a C^r $(r > 4)$ system f with a topologically mixing hyperbolic attractor Λ, there exists a C^r_Γ neighborhood $\mathcal{U}$ of the uncoupled system $F = \otimes_{i\in\mathbb{Z}^d} f$ such that for all $\Phi \in \mathcal{U}$, the following statements are true.

1) *There is a topologically mixing hyperbolic attractor set Δ_Φ close to Δ_F.*
2) *There is a map $h_\Phi : \Delta_F \to \Delta_\Phi$ such that*

$$
\Phi \circ h_\Phi = h_\Phi \circ F.
$$

   The map h_Φ is unique among those that are C^0_Γ close to the identity. It is C^α_Γ for some $\alpha > 0$.
3) *The map $\Phi \to h_\Phi$ is C^{r-2} considered as a map from C^r_Γ to C^α_Γ.*
4) *There exists a measure μ_Φ on Δ_Φ (called SRB measure), invariant under Φ, such that it is the limit of the SRB measures of finite approximations of the coupled lattice map.*

5) *Given any function* $\varphi \in C_\Gamma^{r-1}(\mathcal{M}, \mathbb{R})$, *the function* $\mathfrak{A}_\varphi : \mathcal{U} \to \mathbb{R}$ *defined by*

$$\mathfrak{A}_\varphi(\Phi) = \int \varphi \, d\mu_\Phi$$

is C^{r-3} *when* $\mathcal{U}$ *is given the* C_Γ^r *topology.*

In this article, we obtain the derivative formula for the function

$$\mathfrak{A}_\varphi(\Phi) = \int \varphi \, d\mu_\Phi$$

under the conditions of the previous theorem.

Theorem 1. *Let* $\delta f \in C_\Gamma^r(\mathcal{M}, T\mathcal{M})$ *be a* C^r*-vector* $(r > 4)$ *field on* $\mathcal{M}$. *Assume that* $\Phi = G \circ F \in \mathcal{U}$ *with* $G \in C_\Gamma^r(\mathcal{M}, T\mathcal{M})$ *and* μ_Φ *is its SRB measure. Let* $X = \delta f(\Phi^{-1})$. *Then, for any function* $\varphi(\bar{x}) \in C_\Gamma^{r-1}(\mathcal{M}, \mathbb{R})$, *the functional*

$$\varphi \to \int \varphi d\mu_\Phi$$

is differentiable with respect to $\Phi = G \circ F, G \in C_\Gamma^r(\mathcal{M}, T\mathcal{M})$ *in the direction of* δf. *The derivative (linear response function) is given by*

$$< \delta\mu_\Phi, \varphi >= \sum_{n=0}^{\infty} \int [< \mathrm{grad}(\varphi(\Phi^n)), X^s > - \varphi(\Phi^n) \mathrm{div}_\sigma^u X^u] d\mu_\Phi,$$

where the integral $\int \varphi(\Phi^n) \mathrm{div}_\sigma^u X^u$ *is defined by taking the limit in the finite dimensional approximation as* $V \to \mathbb{Z}^d$.

3. The proof

3.1. **The strategy of the proof.**

We use the following standard approach to obtain the derivative:

Proposition 2. *Assume that* $g_V(\epsilon)$ *is a sequence of real-valued functions and converges to* $g(\epsilon)$ *as* $V \to \mathbb{Z}^d$ *for every* $\epsilon \in (-\epsilon_0, \epsilon_0)$. *Assume that* $g_V(\epsilon)$ *is* C^1 *in* ϵ *and* $g_V'(\epsilon)$ *is bounded uniformly in* V *and that* $\lim_{V \to \mathbb{Z}^d} g_V'(\epsilon) = \tilde{g}(\epsilon)$ *in* $(-\epsilon_0, \epsilon_0)$. *Then,* $g(\epsilon)$ *is* C^1 *and* $g'(\epsilon) = \tilde{g}(\epsilon)$.

The differentiability of the SRB measure with respect to Φ was proved in [7] using symbolic representation of the system. Here, in the process of determining the derivative formula, we obtain another proof based on Ruelle's derivative formula of the SRB measures for finite dimensional systems. This proof gives the derivative formula in the infinite dimensional setting as the limit of the derivative formulas of finite dimensional approximation. The proof in [7] established existence of the derivative for hyperbolic sets but did not provide a formula. The present paper establishes the formula for attractors and also proves the convergence of the finite dimensional approximations for the derivative.

3.2. **Finite dimensional approximations.** In the rest of the section, we make precise the notions of the finite approximations to the coupled map lattice.

Pick any point $\bar{x}^* \in \Delta_\Phi$, which we will use as the boundary conditions at infinity. It will be a byproduct of our work that one obtains the same results no matter which point we choose.

Given any set $V \subset \mathbb{Z}^d$ – it suffices for later applications consider the case that V is a cube –, we define the map $\Phi_V : x_V \to \Phi_V(x_V)$ on $M_V = \otimes_{i\in V} M_i$ is the restriction of Φ on M_V.

$$\Phi_V(x_V) = \Phi(x_V, x^*_{\widehat{V}})\Big|_V.$$

The coordinatewise description is given by

$$\big(\Phi_V(x_V)\big)_i = \big(\Phi(x_V, x^*_{\widehat{V}})\big)_i, \quad i \in V,$$

where the point $(x_V, x^*_{\widehat{V}})$ denotes an element in $\mathcal{M}$ whose restrictions to V and its complement $\widehat{V} \equiv \mathbb{Z}^d - V$, are x_V and $x^*_{\widehat{V}}$, respectively.

The map Φ_V is a diffeomorphism of M_V when the perturbation G is sufficiently close to the identity and it is C^1-close to the diffeomorphism $F_V = \otimes_{i\in V} f$. By the structural stability theorem, Φ_V possesses a hyperbolic attractor Δ_{Φ_V} since F_V has a hyperbolic attractor $\Delta_{F_V} = \otimes_{i\in V}\Lambda$. There exists a conjugating homeomorphism $h_V : \Delta_{F_V} \to \Delta_{\Phi_V}$,

$$\Phi_V \circ h_V = h_V \circ F_V.$$

We note that, the hyperbolicity constants are uniform in V. Hence, we can apply the structural stability theorem for all V. See [7] for more details.

For any smooth function $\varphi \in C^{r-1}_\Gamma(\mathcal{M}, \mathbb{R})$, we can similarly define its restriction to the finite product M_V:

$$\varphi_V(x_V) = \varphi(x_V, x^*_{\widehat{V}}).$$

It is clearly a C^{r-1}-function on M_V with respect to the differentiable structure induced by the direct product of Riemannian metric.

We fix a coupled map $\Phi = G \circ F$ in $\mathcal{U}$. Let $\delta f \in C^r_\Gamma(\mathcal{M}, T\mathcal{M})$ be a translation invariant vector field. We describe the ϵ-perturbation of Φ in the direction of δf by the map

$$\Phi_\epsilon(\bar{x}) = \mathcal{A}^{-1}_{\Phi(\bar{x})}\epsilon\delta f(\Phi(\bar{x})),$$

where ϵ belongs to certain open interval $(-\epsilon_0, \epsilon_0)$ and $\mathcal{A}_{\Phi(\bar{x})}$ is the exponential map at $\Phi(\bar{x})$.

We need to truncate the map Φ_ϵ over the finite volume V:

$$\Phi_{V,\epsilon}(x_V) = \Phi_\epsilon(x_V, x^*_{\widehat{V}})\Big|_V.$$

Note that

$$\Phi_\epsilon(x_V, x^*_{\widehat{V}})\Big|_V = [\mathcal{A}^{-1}_{\Phi(x_V, x^*_{\widehat{V}})}\epsilon\delta f(\Phi(x_V, x^*_{\widehat{V}}))]\Big|_V.$$

So, it is an ϵ-perturbation of the map Φ_V in the direction of the vector field

$$\delta f_V = \delta f(x_V, x^{**}_{\widehat{V}})\Big|_V,$$

where $x^{**} = \Phi(x_V, x^*_{\widehat{V}})$.

With V fixed, we apply the Ruelle's derivative formula (1.1) to the finite dimensional system $(\Phi_{V,\epsilon}, M_V)$ in the direction of δf_V.

The function

$$g_V(\epsilon) : \epsilon \to \int_{\Delta_{\Phi_{V,\epsilon}}} \varphi_V d\mu_{\Phi_{V,\epsilon}}$$

is C^1 in ϵ and the derivative is given by

$$\frac{dg_V(\epsilon)}{d\epsilon} = \sum_{n=0}^{\infty} \int_{\Delta_{\Phi_{V,\epsilon}}} [< \mathrm{grad}(\varphi_V(\Phi^n_{V,\epsilon})), X^s_V > -\varphi(\Phi^n_{V,\epsilon})\mathrm{div}^u X^u_V] d\mu_{\Phi_{V,\epsilon}},$$

where X^s_V and X^u_V are projections of the vector field $X_V = \delta f(\Phi^{-1}_{V,\epsilon})$ onto the stable and unstable subspaces of $D\Phi_{V,\epsilon}$ and $\Delta_{\Phi_{V,\epsilon}}$ is the hyperbolic attractor for $\Phi_{V,\epsilon}$.

Since we already know that

$$\lim_{V\to\mathbb{Z}^d} g_V(\epsilon) = \lim_{V\to\mathbb{Z}^d} \int_{\Delta_{\Phi_{V,\epsilon}}} \varphi_V d\mu_{\Phi_{V,\epsilon}} = g(\epsilon) =: \int_{\Delta_{\Phi_\epsilon}} \varphi d\mu_{\Phi_\epsilon},$$

in order to obtain the derivative formula for the SRB measure of Φ, we only need to prove the following statements on the derivative of $g_V(\epsilon)$:

Proposition 3. *(1) As functions of ϵ, both infinite series*

$$\sum_{n=0}^{\infty} \int < \mathrm{grad}(\varphi_V(\Phi^n_{V,\epsilon})), X^s_V > d\mu_{\Phi_{V,\epsilon}}$$

and

$$\sum_{n=0}^{\infty} \int (\varphi \circ \Phi^n_{V,\epsilon}) \mathrm{div}^u X^u_V d\mu_{\Phi_{V,\epsilon}}$$

are uniformly bounded in V.

(2) For each $\epsilon \in (-\epsilon_0, \epsilon_0)$, the following limits exist.
(a)

$$\lim_{V\to\mathbb{Z}^d} \sum_{n=0}^{\infty} \int < \mathrm{grad}(\varphi_V(\Phi^n_{V,\epsilon})), X^s_V > d\mu_{\Phi_{V,\epsilon}} = \sum_{n=0}^{\infty} \int < \mathrm{grad}(\varphi(\Phi^n_\epsilon)), X^s > d\mu_{\Phi_\epsilon}.$$

(b)

$$\lim_{V\to\mathbb{Z}^d} \sum_{n=0}^{\infty} \int \varphi(\Phi^n_{V,\epsilon}) \mathrm{div}^u X^u_V d\mu_{\Phi_{V,\epsilon}} = \sum_{n=0}^{\infty} \int \varphi(\Phi^n_\epsilon) \mathrm{div}^u X^u d\mu_{\Phi_\epsilon}.$$

These limits should be considered as the definitions of the righthand sides.

3.3. **Proof of Proposition 3: Convergence of the first term.** In this section, we consider the convergence of:

$$\sum_{n=0}^{\infty} \int < \mathrm{grad}(\varphi_V(\Phi^n_{V,\epsilon})), X^s_V > d\mu_{\Phi_{V,\epsilon}}. \tag{3.1}$$

3.3.1. *Uniform boundedness in V.* We first estimate the integral

$$\left| \int < \mathrm{grad}\varphi_V(\Phi^n_{V,\epsilon}), X^s_V > d\mu_{\Phi V,\epsilon} \right|.$$

For any function $\varphi \in C^{r-1}_\Gamma(\mathcal{M}, \mathbb{R})$, its gradient is well defined. Assume that $X \in C^0(\mathcal{M}, T\mathcal{M})$ is a bounded vector field on $\mathcal{M}$. We have the following useful estimates. The proofs are omitted as the estimates follow directly from the decay property of φ. The norm $\|X\|$ denotes the C^0-norm of the vector field X and $\|\varphi\|$ denotes the C^{r-1}_Γ norm of φ.

Lemma 1. *For any $\varphi \in C^{r-1}_\Gamma(\mathcal{M}, \mathbb{R})$ and $X \in C^0(\mathcal{M}, T\mathcal{M})$, we have*
(1)

$$| < \mathrm{grad}\varphi, X > | \le \|\varphi\| \|X\| \sum_{i\in\mathbb{Z}^d} \Gamma(i).$$

(2) *If $X = \oplus_{i\in\mathbb{Z}^d} X_i, X_i \in TM_i$ and $|X_i| \le C\Gamma(i-j)$ for some constant C, then*

$$| < \mathrm{grad}\varphi, X > | \le C\|\varphi\|\|X\|\Gamma(j).$$

We also notice that

$$< \mathrm{grad}\varphi_V(\Phi^n_{V,\epsilon}), X^s_V > = < \mathrm{grad}\varphi_V, D\Phi^n_{V,\epsilon} X^s_V >$$

and by the uniform contraction condition

$$\|D\Phi^n_{V,\epsilon} X^s_V\| \le C' \lambda^n \|X^s_V\|,$$

where $C' > 0$ is a constant and $0 < \lambda < 1$ is the contracting constant along the stable direction. Note that as long as our ϵ_0 is chosen small, the contracting constant λ can be so chosen that it is independent of ϵ and V. Thus, by the decay property of φ_V and Lemma 1 (1) we have

$$|<\mathrm{grad}\varphi_V, D\Phi^n_{V,\epsilon}X^s_V>| \le CC'\lambda^n\|\varphi\|\|X^s_V\|\sum_{i\in\mathbb{Z}^d}\Gamma(i).$$

So, as functions of ϵ,

$$\sum_{n=0}^{\infty}\int <\mathrm{grad}(\varphi_V(\Phi^n_{V,\epsilon})), X^s_V> d\mu_{\Phi_{V,\epsilon}} \le \frac{C}{1-\lambda}\|\varphi\|\|X\|\sum_{i\in\mathbb{Z}^d}\Gamma(i)$$

is uniformly bounded in V.

3.3.2. *Convergence as $V \to \mathbb{Z}^d$.* Because of the uniform convergence of the series

$$\sum_{n=0}^{\infty}\int <\mathrm{grad}(\varphi_V(\Phi^n_{V,\epsilon})), X^s_V> d\mu_{\Phi_{V,\epsilon}}$$

in V, to prove that its limit as $V \to \mathbb{Z}^d$ is

$$\sum_{n=0}^{\infty}\int <\mathrm{grad}(\varphi(\Phi^n_{\epsilon})), X^s> d\mu_{\Phi_\epsilon},$$

we just need to show that for each fixed n,

$$\lim_{V\to\mathbb{Z}^d}\int <\mathrm{grad}(\varphi_V(\Phi^n_{V,\epsilon})), X^s_V> d\mu_{\Phi_{V,\epsilon}} = \int <\mathrm{grad}(\varphi(\Phi^n_{\epsilon})), X^s> d\mu_{\Phi_\epsilon}.$$

Let $X^s_{V,i}, X^s_i$ be the ith coordinate of the vector fields X^s_V, X^s, respectively. We have

$$X^s_V = \sum_{i\in V} X^s_{V,i},\ \ X^s = \sum_{i\in\mathbb{Z}^d} X^s_i.$$

By Lemma 1, the term $<\mathrm{grad}\varphi(\Phi^n_\epsilon), X^s>$ is well-defined and

$$<\mathrm{grad}\varphi(\Phi^n_\epsilon), X^s> = \sum_{i\in\mathbb{Z}^d} <\mathrm{grad}\varphi(\Phi^n_\epsilon), X^s_i> .$$

Since

$$<\mathrm{grad}\varphi_V(\Phi^n_{V,\epsilon}), X^s_V> = \sum_{i\in V} <\mathrm{grad}\varphi_V(\Phi^n_{V,\epsilon}), X^s_{V,i}>$$

and the measure μ_{Φ_ϵ} converges to μ_{Φ_ϵ} weakly, it suffices to show that for each fixed n and all V

$$| < \text{grad}\varphi_V(\Phi^n_{V,\epsilon}), X^s_{V,i} > | \leq a_i, \text{ and } \sum_{i\in\mathbb{Z}^d} a_i < \infty$$

and for each i

$$\lim_{V\to\mathbb{Z}^d} < \text{grad}\varphi_V(\Phi^n_{V,\epsilon}), X^s_{V,i} >=< \text{grad}\varphi(\Phi^n_\epsilon), X^s_i >$$

uniformly in $\bar{x} \in \mathcal{M}$.

We first prove several properties needed in the proof of the convergence. For typographical simplicity, we drop the subscript C^r_Γ in the norm $\|\Phi\|_{C^r_\Gamma}$: $\|\Phi\| = \|\Phi\|_{C^r_\Gamma}$.

Lemma 2.
(1) *For each n, there exists a constant $C(n)$ depending only on the norm $\|\Phi\|$ and the size of the manifold M such that the ith coordinates of $\Phi^n_{V,\epsilon}$ and Φ^n_ϵ satisfy*

$$d(\big(\Phi^n_{V,\epsilon}(x_V)\big)_i, \big(\Phi^n_\epsilon(\bar{x})\big)_i) \leq C(n) \sum_{j\in\widehat{V}} \Gamma(i-j)$$

for every $i \in V$ and $\bar{x} \in \mathcal{M}$.
(2) *For each n, the truncated function $\varphi_V(\Phi^n_{V,\epsilon}(x_V))$ converges to $\varphi(\Phi^n_\epsilon(\bar{x}))$ uniformly in $\bar{x}$ and ϵ. Indeed, we have*

$$|\varphi_V(\Phi^n_{V,\epsilon}(x_V)) - \varphi(\Phi^n_\epsilon(\bar{x}))| \leq (C(n) + \|\Phi\|)\|\varphi\| \sum_{j\in\widehat{V}} \Gamma(j)$$

(3) *For each $i \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, $\frac{\partial}{\partial x_i}\varphi_V(\Phi^n_{V,\epsilon}(x_V))$ converges to $\frac{\partial}{\partial x_i}\varphi(\Phi^n_\epsilon)$ exponentially as $V \to \mathbb{Z}^d$.*
(4) *For each fixed n, we have*

$$< \text{grad}\varphi(\Phi^n_V), X_i > | \leq \|X\|\|\varphi\|\|\Phi\|^n\Gamma(i).$$

Proof. (1) Notice that for the ith coordinate, $i \in V$, we have

$$\begin{aligned} d(\big(\Phi_{\epsilon,V}(x_V)\big)_i, \big(\Phi_\epsilon(\bar{x})\big)_i) &= d(\big(\Phi_\epsilon(x_V, x^*_{\widehat{V}})\big)_i, \big(\Phi_\epsilon(\bar{x})\big)_i) \\ &\leq \sum_{j\in\widehat{V}} |\frac{\partial\Phi_{\epsilon,i}}{\partial x_j}| d(x^*_j, x_j), \end{aligned}$$

where x^*_j denotes the coordinate of x^* at j. Let L be the maximum distance between two points on the compact manifold M. Note that

$$|\frac{\partial\Phi_{\epsilon,i}}{\partial x_j}| \leq \|\Phi\|\Gamma(i-j).$$

We have

$$d(\big(\Phi_{V,\epsilon}(x_V)\big)_i,\big(\Phi_\epsilon(\bar{x})\big)_i)\le L\|\Phi\|\sum_{j\in\widehat{V}}\Gamma(i-j).$$

In general, for the ith coordinate, $i\in V$ and $\bar{y}=(y_i)_{i\in\mathbb{Z}^d}=(y_V,x^*_{\widehat{V}})$, we have

$$\begin{aligned}d(\big(\Phi_{V,\epsilon}(y_V)\big)_i,\big(\Phi_\epsilon(\bar{x})\big)_i)&=d(\big(\Phi_\epsilon(y_V,x^*_{\widehat{V}})\big)_i,\big(\Phi_\epsilon(\bar{x})\big)_i)\\&\le\sum_{j\in\mathbb{Z}^d}|\frac{\partial\Phi_{\epsilon,i}}{\partial x_j}|d(y_j,x_j)\\&\le L\|\Phi\|\sum_{j\in\mathbb{Z}^d}\Gamma(i-j)d(y_j,x_j).\end{aligned}$$

Thus,

$$d(\big(\Phi^2_{V,\epsilon}(x_V)\big)_i,\big(\Phi^2_\epsilon(\bar{x})\big)_i)=d(\big(\Phi_{V,\epsilon}(y_V)\big)_i,\big(\Phi_\epsilon(\bar{z})\big)_i)\le\sum_{j\in\mathbb{Z}^d}\frac{\partial\Phi_{\epsilon,i}}{\partial x_j}d(y_j,z_j),$$

where $\bar{y}=(\Phi_{V,\epsilon}(x_V),x^*_{\widehat{V}})$ and $\bar{z}=(z_j)_{j\in\mathbb{Z}^d}=\Phi_\epsilon(\bar{x})$. Using the previous estimation, we have:

$$\begin{aligned}&d(\big(\Phi^2_{V,\epsilon}(x_V)\big)_i,\big(\Phi^2_\epsilon(\bar{x})\big)_i)\\&\quad\le L\|\Phi\|\sum_{j\in V}\Gamma(i-j)d(y_j,z_j)+L\|\Phi\|\sum_{j\in\widehat{V}}\Gamma(i-j)d(x^*_j,z_j)\\&\quad\le L\|\Phi\|\sum_{j\in V}\Gamma(i-j)L\|\Phi\|\sum_{k\in\widehat{V}}\Gamma(j-k)+L^2\|\Phi\|\sum_{j\in\widehat{V}}\Gamma(i-j)\\&\quad\le L^2\|\Phi\|^2\sum_{k\in\widehat{V}}\sum_{j\in V}\Gamma(i-j)\Gamma(j-k)+L^2\|\Phi\|\sum_{j\in\widehat{V}}\Gamma(i-j)\\&\quad\le L^2\|\Phi\|^2\sum_{k\in\widehat{V}}\Gamma(i-k)+L^2\|\Phi\|\sum_{j\in\widehat{V}}\Gamma(i-j)\\&\quad\le L^2(\|\Phi\|^2+\|\Phi\|)\sum_{j\in\widehat{V}}\Gamma(i-j).\end{aligned}$$

By induction in n, we have, for $x_V=\bar{x}|_V$,

$$|\big(\Phi^n_{V,\epsilon}(x_V)\big)_i-\big(\Phi^n_\epsilon(\bar{x})\big)_i|\le C(n)\sum_{j\in\widehat{V}}\Gamma(i-j),$$

where $C(n)$ is a constant depending on $L,\|\Phi\|$ and n.

The claim (2) is now a direct consequence of (1).

$$
\begin{aligned}
&|\varphi(\Phi^n_{V,\epsilon}(x_V), x^*_{\widehat{V}}) - \varphi(\Phi^n_\epsilon(\bar{x}))| \\
&\qquad \leq \sum_{i\in\mathbb{Z}^d} |\frac{\partial\varphi}{\partial x_i}| d\big(\big(\Phi^n_{V,\epsilon}(x_V)\big)_i, \big(\Phi^n_\epsilon(\bar{x})\big)_i\big) \\
&\qquad \leq C(n)\|\varphi\| \sum_{i\in V}\Gamma(i)\sum_{j\in\widehat{V}}\Gamma(i-j) + L\|\varphi\|\sum_{i\in\widehat{V}}\Gamma(i) \\
&\qquad \leq \|\varphi\|(C(n)+L)\sum_{i\in\widehat{V}}\Gamma(i).
\end{aligned}
$$

(3) With $i \in V$ and n fixed, we wish to estimate

$$
\left|\frac{\partial}{\partial x_i}\varphi_V(\Phi^n_{V,\epsilon}(x_V)) - \frac{\partial}{\partial x_i}\varphi(\Phi^n_\epsilon(\bar{x}))\right| = \left|\frac{\partial}{\partial x_i}\varphi(\Phi^n_{V,\epsilon}(x_V), x^*_{\widehat{V}}) - \frac{\partial}{\partial x_i}\varphi(\Phi^n_\epsilon(\bar{x}))\right|.
$$

We may drop the symbol ϵ in our calculation since the conditions involved are always independent of ϵ.

Using the chain rule, we have:

$$
\begin{aligned}
&\Big|\frac{\partial}{\partial x_i}\varphi(\Phi^n_V(x_V), x^*_{\widehat{V}}) - \frac{\partial}{\partial x_i}\varphi(\Phi^n(\bar{x}))\Big| \\
&\qquad \leq \sum_{j\in\mathbb{Z}^d}\Big|\frac{\partial\varphi}{\partial x_j}\Big|_{(\Phi^n_V(x_V), x^*_{\widehat{V}})}\frac{\partial(\Phi^n_V)_j}{\partial x_i} - \frac{\partial\varphi}{\partial x_j}\Big|_{\Phi^n(\bar{x})}\frac{\partial(\Phi^n)_j}{\partial x_i}\Big|,
\end{aligned}
\tag{3.2}
$$

where $(\Phi^n_V)_j$ and $(\Phi^n)_j$ denote the jth coordinate of maps Φ^n_V and Φ^n, respectively. We note that since the manifold M is multidimensional the products in the formula are actually matrix products.

We first estimate the following difference between partial derivatives of φ. We use the fact that the decay function Γ is of the form (2.2) and the elementary inequality

$\min\{a,b\} \leq (ab)^{\frac{1}{2}}$ for $a, b \geq 0$.

$$
\begin{aligned}
&\Big| \frac{\partial \varphi}{\partial x_j}\Big|_{(\Phi_V^n(x_V), x^*_{\widehat{V}})} - \frac{\partial \varphi}{\partial x_j}\Big|_{\Phi^n(\bar{x})}\Big| \\
&\qquad \leq \sum_{k\in\mathbb{Z}^d} \left|\frac{\partial^2 \varphi}{\partial x_j \partial x_k}\right| d((\Phi_V^n(x_V), x^*_{\widehat{V}}), \Phi^n(\bar{x})) \\
&\qquad \leq \sum_{k\in\mathbb{Z}^d} \|\varphi\| \min\{\Gamma(j), \Gamma(k)\} C(n) \sum_{j\in\widehat{V}} \Gamma(k-j) \\
&\qquad \leq \|\varphi\| C(n) \Gamma(j)^{\frac{1}{2}} \sum_{k\in\mathbb{Z}^d} \Gamma(k)^{\frac{1}{2}} \sum_{j\in\widehat{V}} \Gamma(k-j) \\
&\qquad \leq \|\varphi\| C(n) \Gamma(j)^{\frac{1}{2}} \sum_{k\in\mathbb{Z}^d} \sum_{j\in\widehat{V}} \Gamma(k)^{\frac{1}{2}} \Gamma(k-j) \\
&\qquad \leq \|\varphi\| C(n) \Gamma(j)^{\frac{1}{2}} e^{-\beta l(V)},
\end{aligned}
\tag{3.3}
$$

where β is some constant and $l(V)$ is the linear size of V. Note that for all $j \in \mathbb{Z}^d$, we also have

$$
\left|\frac{\partial \varphi}{\partial x_j}\right| \leq \|\varphi\| \Gamma(j). \tag{3.4}
$$

Now, we estimate

$$
\frac{\partial (\Phi_V^n)_j}{\partial x_i}, \frac{\partial (\Phi^n)_j}{\partial x_i}, \text{ and } \left| \frac{\partial (\Phi_V^n)_j}{\partial x_i} - \frac{\partial (\Phi^n)_j}{\partial x_i} \right|.
$$

When $n = 1$,

$$
\left|\frac{\partial (\Phi_V)_j}{\partial x_i}\right| = \left|\frac{\partial \Phi_j(x_V, x^*_{\widehat{V}})}{\partial x_i}\right| \leq \|\Phi\| \Gamma(i-j).
$$

The same estimation holds for $|\frac{\partial \Phi_j}{\partial x_i}|$.

When $n = 2$,

$$
\begin{aligned}
\left|\frac{\partial (\Phi_V^2)_j}{\partial x_i}\right| &= \left|\frac{\partial (\Phi_V)_j(\Phi_V(x_V))}{\partial x_i}\right| = \left|\sum_{k\in V} \frac{\partial (\Phi_V)_j}{\partial x_k} \frac{\partial (\Phi_V)_k}{\partial x_i}\right| \\
&\leq \sum_{k\in V} \|\Phi\|^2 \Gamma(j-k)\Gamma(k-i) \leq \|\Phi\|^2 \Gamma(i-j)
\end{aligned}
$$

Inductively, we have

$$\left|\frac{\partial(\Phi_V^n)_j}{\partial x_i}\right| \le \|\Phi\|^n \Gamma(i-j). \tag{3.5}$$

The same estimation holds for $\frac{\partial(\Phi^n)_j}{\partial x_i}$.

The difference

$$\frac{\partial(\Phi_V^n)_j}{\partial x_i} - \frac{\partial(\Phi^n)_j}{\partial x_i}$$

can also be estimated inductively.

When $n=1$, we have

$$\begin{aligned}
\left|\frac{\partial\Phi_j(x_V, x^*_{\widehat{V}})}{\partial x_i} - \frac{\partial\Phi_j(\bar{x})}{\partial x_i}\right| &\le \sum_{k\in\mathbb{Z}^d} |\frac{\partial^2\Phi_j}{\partial x_i \partial x_k}| d((x_V, x^*_{\widehat{V}})_k, x_k) \\
&\le L \sum_{k\in\widehat{V}} |\frac{\partial^2\Phi_j}{\partial x_i \partial x_k}| \le L \sum_{k\in\widehat{V}} \|\Phi\| \min\{\Gamma(i-j), \Gamma(k-j)\} \\
&\le C_1 \exp(-\alpha_1|i-j|) \sum_{k\in\widehat{V}} \exp(-\beta_1|i-k|) \\
&\le C_1 \exp(-\alpha_1|i-j|) \exp(-\beta_1|i-\partial V|),
\end{aligned}$$

for some positive constants C_1, α_1, and β_1. The term $|i-\partial V|$ denotes the minimal distance between i and ∂V, the boundary of V.

Assume that this estimation holds for $n-1$:

$$\left|\frac{\partial(\Phi_V^{n-1})_j}{\partial x_i} - \frac{\partial(\Phi^{n-1})_j}{\partial x_i}\right| \le C_{n-1} \exp(-\alpha_{n-1}|i-j|) \exp(-\beta_{n-1}|i-\partial V|),$$

for some positive constants C_{n-1}, α_{n-1}, and β_{n-1} depending on n, $\|\Phi\|$, and L.

$$\begin{aligned}
\left|\frac{\partial(\Phi_V^n)_j}{\partial x_i} - \frac{\partial(\Phi^n)_j}{\partial x_i}\right| &= |\frac{\partial}{\partial x_i}\Phi_j(\Phi_V^{n-1}(x_V), x^*_{\widehat{V}}) - \frac{\partial}{\partial x_i}\Phi_j(\Phi^{n-1}(\bar{x}))| \\
&\le \sum_{k\in V} \left| \frac{\partial\Phi_j}{\partial x_k}\Big|_{(\Phi_V^{n-1}(x_V), x^*_{\widehat{V}})} \frac{\partial(\Phi_V^{n-1})_k}{\partial x_i} - \frac{\partial\Phi_j}{\partial x_k}\Big|_{\Phi^{n-1}(\bar{x})} \frac{\partial\Phi_k^{n-1}}{\partial x_i}\right| \\
&\quad + \sum_{k\in\widehat{V}} \left| \frac{\partial\Phi_j}{\partial x_k}\Big|_{\Phi^{n-1}(\bar{x})} \frac{\partial\Phi_k^{n-1}}{\partial x_i}\right|.
\end{aligned}$$

We estimate these sums separately. We estimate the second sum first.

$$\sum_{k\in\widehat{V}}\left|\frac{\partial\Phi_j}{\partial x_k}\Big|_{\Phi^{n-1}(\bar{x})}\frac{\partial\Phi_k^{n-1}}{\partial x_i}\right|\le\sum_{k\in\widehat{V}}\|\Phi\|\Gamma(k-j)\|\Phi\|^{n-1}\Gamma(i-k)$$
$$\le\|\Phi\|^n\exp(-\alpha|i-j|)\sum_{k\in\widehat{V}}\Gamma^{\frac{1}{2}}(i-k).$$

For the first sum, we have

$$\begin{aligned}
&\sum_{k\in V}\left|\frac{\partial\Phi_j}{\partial x_k}\Big|_{(\Phi_V^{n-1}(x_V),x^*_{\widehat{V}})}\frac{\partial(\Phi_V^{n-1})_k}{\partial x_i}-\frac{\partial\Phi_j}{\partial x_k}\Big|_{\Phi^{n-1}(\bar{x})}\frac{\partial\Phi_k^{n-1}}{\partial x_i}\right|\\
&\le\sum_{k\in V}\left|\frac{\partial\Phi_j}{\partial x_k}|_{(\Phi_V^{n-1}(x_V),x^*_{\widehat{V}})}-\frac{\partial\Phi_j}{\partial x_k}|_{\Phi^{n-1}(\bar{x})}\right|\cdot\left|\frac{\partial(\Phi_V^{n-1})_k}{\partial x_i}\right|\\
&\quad+\left|\frac{\partial\Phi_j}{\partial x_k}|_{\Phi^{n-1}(\bar{x})}\right|\cdot\left|\frac{\partial(\Phi_V^{n-1})_k}{\partial x_i}-\frac{\partial\Phi_k^{n-1}}{\partial x_i}\right|.
\end{aligned}\tag{3.6}$$

The second part of the sum (3.6) can be estimated as follows:

$$\begin{aligned}
&\sum_{k\in V}\left|\frac{\partial\Phi_j}{\partial x_k}\Big|_{\Phi^{n-1}(\bar{x})}\right|\left|\frac{\partial(\Phi_V^{n-1})_k}{\partial x_i}-\frac{\partial\Phi_k^{n-1}}{\partial x_i}\right|\\
&\le\sum_{k\in V}\|\Phi\|\Gamma(j-k)C_{n-1}\exp(-\alpha_{n-1}|i-k|)\exp(-\beta_{n-1}|i-\partial V|)\\
&\le\|\Phi\|C_{n-1}\exp(-\beta_{n-1}|i-\partial V|)\sum_{k\in V}\Gamma(j-k)\exp(-\alpha_{n-1}|i-k|)\\
&\le C_n\exp(-\beta_{n-1}|i-\partial V|)\exp(-\alpha_n|i-j|).
\end{aligned}$$

The first part of the sum (3.6) needs to be estimated more carefully.

$$\begin{aligned}
&\sum_{k\in V}\left|\frac{\partial\Phi_j}{\partial x_k}|_{(\Phi_V^{n-1}(x_V),x^*_{\widehat{V}})}-\frac{\partial\Phi_j}{\partial x_k}|_{\Phi^{n-1}(\bar{x})}\right|\left|\frac{\partial(\Phi_V^{n-1})_k}{\partial x_i}\right|\\
&=\sum_{k\in V}\sum_{l\in\mathbb{Z}^d}\left|\frac{\partial^2\Phi_j}{\partial x_k\partial x_l}\right|d((\Phi_V^{n-1}(x_V),x^*_{\widehat{V}})_l,(\Phi^{n-1}(\bar{x}))_l)\|\Phi\|^{n-1}\Gamma(i-k)\\
&=\sum_{k\in V}\sum_{l\in V}\|\Phi\|\min\{\Gamma(j-k),\Gamma(j-l)\}C(n-1)\sum_{j'\in\widehat{V}}\Gamma(l-j')\|\Phi\|^{n-1}\Gamma(i-k)\\
&\quad+\sum_{k\in V}\sum_{l\in\widehat{V}}\|\Phi\|\min\{\Gamma(j-k),\Gamma(j-l)\}L\|\Phi\|^{n-1}\Gamma(i-k).
\end{aligned}$$

Both sums in the previous expression are bounded by

$$C\exp(-\alpha|i-j|)\exp(-\beta|i-\partial V|),$$

for some constants C, α, and β depending on n, $\|\Phi\|$, and L. Thus, we have

$$\left|\frac{\partial(\Phi_V^n)_j}{\partial x_i} - \frac{\partial(\Phi^n)_j}{\partial x_i}\right| \le C_n \exp(-\alpha_n|i-j|)\exp(-\beta_n|i-\partial V|), \tag{3.7}$$

for some positive constants C_n, α_n, and β_n depending on n, $\|\Phi\|$, and L. The statement (3) follows from the estimates (3.2), (3.3) (3.4) (3.5),and (3.7).

The last statement (4) of the lemma follows immediately from our previous estimates.

$$\begin{aligned}\left|\frac{\partial}{\partial x_i}\varphi(\Phi_V^n)\right| &= \left|\sum_{k\in V}\frac{\partial\varphi}{\partial x_k}\frac{\partial(\Phi_V^n)_k}{\partial x_i}\right| \\ &\le \sum_{k\in V}\|\varphi\|\Gamma(k)\|\Phi\|^n\Gamma(i-k) \le \|\varphi\|\|\Phi\|^n\Gamma(i).\end{aligned}$$

□

Now, the only thing left to show is that for each i

$$\lim_{V\to\mathbb{Z}^d} X^s_{V,i} = X^s_i$$

uniformly in $\bar{x}$. We will need the closeness of stable and unstable subspaces for $\Phi_{V,\epsilon}$ and Φ in compact metric (2.1) ρ_q for some $0 < q < 1$. Using the Grassmanian distance on subspaces of the manifold M, we introduce the corresponding ρ_q metric for Hölder continuous sections of the tangent bundles close to $\otimes_{i\in\mathbb{Z}^d}E_i^s$ and $\otimes_{i\in\mathbb{Z}^d}E_i^u$. We formulate these properties in the following lemma but omit their proofs as they are proved using standard techniques (the fixed point theorem of contracting maps in the compact metric ρ_q, see for example, Lemma 2 of [9]). We take $V = V_m$, the hypercubes in $\mathbb{Z}^d$ for convenience. $E_V^s, E_V^u, \mathcal{E}^s, \mathcal{E}^u$ denote stable and unstable subspaces of Φ_V and Φ, respectively. The map Φ_V is extended to the entire $\mathcal{M}$ by adding copies of f to it outside the hypercube V_m.

Lemma 3.

(1) *There exist constants* $C > 0$, $0 < \lambda < 1$ *such that*

$$\rho_q(h_{\Phi_{V_n}}, h_\Phi) \le C\lambda^n.$$

(2) *There exist constants* $C > 0$, $0 < \lambda < 1$ *such that*

$$\rho_q(E^s_{V_n}, \mathcal{E}^s) \le C\lambda^n, \quad \rho_q(E^u_{V_n}, \mathcal{E}^u) \le C\lambda^n.$$

(3) $\lim_{V_n\to\mathbb{Z}^d} X^s_{V_n,i}(x_{V_n}) = X^s_i(\bar{x})$ *uniformly in* $\bar{x}$.

Proof of the convergence. By Lemmas 2 and 3, we have

$$\lim_{V\to\mathbb{Z}^d} < \mathrm{grad}(\varphi_V(\Phi^n_{V,\epsilon})), X^s_V >= \lim_{V\to\mathbb{Z}^d} \sum_{i\in V} \frac{\partial\varphi_V(\Phi^n_{V,\epsilon})}{\partial x_i} X^s_{V,i}$$

$$= \sum_{i\in\mathbb{Z}^d} \frac{\partial\varphi(\Phi^n_\epsilon)}{\partial x_i} X^s_i =< \mathrm{grad}(\varphi_V(\Phi^n_\epsilon)), X^s >,$$

i.e.,

$$\lim_{V\in\mathbb{Z}^d} \int < \mathrm{grad}(\varphi_V(\Phi^n_{V,\epsilon})), X^s_V > d\mu_{\Phi_{V,\epsilon}} = \int < \mathrm{grad}(\varphi_V(\Phi^n_\epsilon)), X^s > d\mu_{\Phi_\epsilon}.$$

3.4. **Proof of Proposition 3: Convergence of the second term.** To show the convergence of the second series

$$\sum_{n=0}^\infty \int \varphi(\Phi^n_{V,\epsilon}) \mathrm{div}^u_\sigma X^u_V d\mu_{\Phi_{V,\epsilon}} \to \sum_{n=0}^\infty \int \varphi(\Phi^n_\epsilon) \mathrm{div}^u_\sigma X^u d\mu_{\Phi\epsilon},$$

the termwise convergence as $V \to \mathbb{Z}^d$ is proved exactly in the same way since we have the divergence formula [14]

$$\int < \mathrm{grad}\varphi(\Phi^n_{V,\epsilon}), X^u_V > d\mu_{\Phi_{V,\epsilon}} = -\int \varphi(\Phi^n_{V,\epsilon}) \mathrm{div}^u_\sigma X^u_V d\mu_{\Phi_{V,\epsilon}}. \tag{3.8}$$

But the convergence of the series uniform in the volume V requires a careful analysis.

We need to estimate the term

$$\int \varphi(\Phi^n_{V,\epsilon}) \mathrm{div}^u_\sigma X^u_V d\mu_{\Phi_{V,\epsilon}}$$

using the exponential decay of correlation functions for SRB measures $\mu_{\Phi_{V,\epsilon}}$. The meaning of the divergence $\mathrm{div}^u_\sigma X^u_V$ is given by Ruelle in [16] and we will also briefly discuss it later.

We decompose the vector field in a different way. We project each component of X_V onto the unstable subspace first and then, add them up:

$$X^u_V = \left(\sum_{i\in V} X_{V,i}\right)^u = \sum_{i\in V} X^u_{V,i},$$

i.e., $X^u_{V,i}$ denotes the projection of the ith coordinate of X_V onto the unstable subspace E^u_V. Thus,

$$\int \varphi(\Phi^n_{V,\epsilon})\mathrm{div}^u_\sigma X^u_V d\mu_{\Phi_{V,\epsilon}} = \sum_{i\in V}\int \varphi(\Phi^n_{V,\epsilon})\mathrm{div}^u_\sigma X^u_{V,i} d\mu_{\Phi_{V,\epsilon}}.$$

Now it suffices to prove the following theorem.

Theorem 4. *There exist constants $C > 0$, $0 < \alpha, \beta < 1$ that depend on the map Φ, the vector field X, and the decay function Γ but are independent of V such that*

$$\left|\int \varphi(\Phi^n_{V,\epsilon})\mathrm{div}^u_\sigma X^u_{V,i} d\mu_{\Phi_{V,\epsilon}}\right| \leq Ce^{-\alpha n-\beta|i|}.$$

This theorem follows from results of Bricmont and Kupiainen in [3, 4] once we show that the function $\mathrm{div}^u_\sigma X^u_{V,i}(x_V)$ is Hölder continuous in x_V with the Hölder constant and exponent independent of the volume V with respect to the (translated) metric $\rho_{q,i}$:

$$\rho_{q,i}(x_V, y_V) = \sup_{j\in V} q^{|i-j|}d(x_j, y_j).$$

Since the vector field X is assumed to be translation invariant, we have

$$\mathrm{div}^u_\sigma X^u_{V,i} = \mathrm{div}^u_\sigma X^u_{V-i,0},$$

where $V - i$ denotes the volume V translated by $-i$. So, we just need to prove the following theorem.

Theorem 5. *There exist constant $C > 0$ and $0 < \alpha < 1$ independent of the volume V such that*

$$|\mathrm{div}^u_\sigma X^u_{V,0}(x_V) - \mathrm{div}^u_\sigma X^u_{V,0}(y_V)| \leq C[\rho_q(x_V, y_V)]^\alpha.$$

Now we focus on the proof of this theorem. We first need to recall the definition of $\mathrm{div}^u_\sigma X^u_{V,0}$. Note that $X^u_{V,0}$ is usually not differentiable even though $X_{V,0}$, the coordinate of X_V at $0 \in \mathbb{Z}^d$ is. However, the divergence, which measures the change of the volume form under the flow determined by the vector field, exists for non-differentiable vector fields satisfying certain conditions. We will see that since the stable foliation is absolutely continuous, the divergence of any smooth vector field projected onto the unstable manifold can be properly defined and calculated with the help of the holonomy map defined by the stable foliation. Ruelle showed in [16] that using a specially adapted local coordinate system, the divergence $\mathrm{div}^u_\sigma X^u_{V,0}$ can be expressed as the derivative of the Jacobian of the holonomy map of the stable foliation. The choice of the adapted coordinate system depends on the vector field $X_{V,0}$ and is not convenient in our case since we need to take the limit $V \to \mathbb{Z}^d$. We use Ruelle's idea of calculation of the divergence to obtain a more general relation between the divergence $\mathrm{div}^u_\sigma X^u_{V,0}$ and the Jacobian of the holonomy map. Since the divergence is independent of the choice of the coordinate system, we identify a neighborhood on the Riemannian manifold with a neighborhood of $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$ equipped with Riemannian metric and the unstable manifolds are identified with $\mathbb{R}^u$.

Definition 6. *Let X be a continuous vector field defined in an open set U on a Riemannian manifold with a volume form dw. Let $\mathcal{G}$ denote the family of C^∞ functions in U with compact support. Assume that there is an integrable function $h(x)$ such that*

$$\int < \mathrm{grad} g, X > dw = -\int g(x)h(x)dw,$$

for all $g \in \mathcal{G}$. Then, we call $h(x)$ the divergence of X and denote it by $\mathrm{div}X$.

It is clear that the divergence of a vector field is unique up to a measure zero set. When X is C^1, the definition is consistent with the common definition if we choose $h(x)$ to be continuous. The following proposition can be directly verified and it gives the geometric meaning of the divergence.

Proposition 7. *Let X be a continuous vector field on $\mathbb{R}^n$ with a measure m determined by a volume form $d\omega$. Let $t \in (-1,1)$ and A_t be a map on $\mathbb{R}^n$ defined by*

$$A_t(x) = x + tX(x).$$

Assume that the induced measure $A_t^* m$ is absolutely continuous with respect to m with a Radon-Nikodym derivative $\rho(x,t)$ continuous in x. Assume that $\frac{1}{t}(\rho(x,t) - \rho(x,0)) = \frac{1}{t}(\rho(x,t) - 1)$ is bounded in x for all t sufficiently small. Then, the divergence of $X(x)$ is equal to the derivative $\mathrm{div}X = \frac{d}{dt}\rho(x,t)\big|_{t=0}$ when it exists.

When X is differentiable, the Radon-Nikodym derivative becomes the Jacobian of A_t. Using the definition of the divergence, one can easily see that linearity holds, i.e.,

$$\mathrm{div}(aX_1 + bX_2) = a\mathrm{div}X_1 + b\mathrm{div}X_2.$$

The Gauss divergence theorem can also be verified.

We prove the following lemma.

Lemma 4. *Let X be a C^1- vector field on $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$ and $E^s(x)$ denote the distribution tangent to an absolutely continuous foliation W with smooth leaves transversal to the manifold $\mathbb{R}^u$. Let $X = X^u + X^s$ denote the projections of X onto $\mathbb{R}^u$ and $E^s(x)$, respectively. Then,* $\mathrm{div}^u X^u$*, the divergence of X^u on the manifold $\mathbb{R}^u$ with respect to the volume form restricted to the manifold is well defined. Moreover, when the Jacobian*

$J\xi(x^u,t)$ of the the holonomy map $\xi(x^u,t)$ from $\mathbb{R}^u$ to $\mathbb{R}^u+tX$ defined by the foliation is differentiable in t and continuous in x^u, we have

$$\mathrm{div}^u X^u = \mathrm{div}^u X^u_c - \frac{d}{dt} J\xi(x^u,t)|_{t=0}, x^u \in \mathbb{R}^u$$

where X^u_c is the projection of the vector field X onto the coordinate subspaces $\mathbb{R}^u$ and $\mathbb{R}^s$: $X = X^u_c + X^s_c$.

Proof. Note that since the foliation W has differentiable leaves, the derivative $\frac{d}{dt}J\xi(x^u,t)|_{t=0}$ is uniquely determined by its first order linear approximation, the distribution $E^s(x^u)$, $x^u \in \mathbb{R}^u$. Thus, in calculating the Jacobian of the holonomy map at $t=0$, we may consider $\xi(x^u,t)$ as the intersection of hyperplanes $E^s(x^u)$ and $\mathbb{R}^u + tX$. For each fixed $x^u \in \mathbb{R}^u$, $\xi(x^u,t)$ is linear in t. Indeed, since any point in $E^s(x^u)$ can be expressed as $(x^u + B(x^u)x^s, x^s)$ where $B(x^u)$ is a linear map from $\mathbb{R}^s$ to $\mathbb{R}^u$ and $\mathbb{R}^u + tX = \mathbb{R}^u + tX^s_c$, we have, in the coordinate system of $\mathbb{R}^u \times \mathbb{R}^s$,

$$\xi(x^u,t) = (x^u + tB(x^u)X^s_c, X^s_c).$$

Thus, the holonomy map maps x^u to $x^u + tB(x^u)X^s_c$, and by Proposition 7,

$$\frac{d}{dt} J\xi(x^u,t)|_{t=0} = \mathrm{div}\big(B(x^u)X^s_c\big).$$

We can now determine X^u in terms of X^u_c, X^s_c and $B(x^u)$. We look for coefficients a and b such that $X = ae^u + bf^s$, where e^u, f^s denotes the basis of $\mathbb{R}^u$ and $E^s(x^u)$: $f^s = (B(x^u)e^s, e^s)$. We have equations:

$$X^u_c = a + bB(x^u);\ \ X^s_c = b.$$

Thus,

$$X^u = a = X^u_c - B(x^u)X^s_c.$$

Note that X^u_c is C^1. We have

$$\mathrm{div}X^u = \mathrm{div}(X^u_c - B(x^u)X^s_c) = \mathrm{div}X^u_c - \frac{d}{dt}J\xi(x^u,t)|_{t=0}.$$

□

Remark. We have assumed in the proof that the volume forms on $\mathbb{R}^u$ and $\mathbb{R}^u + tX$ are the same. This is not true in general. But this will not affect the formula as we are taking limit $t \to 0$. Note also that $\mathrm{div}X^u_c$ is not zero in general, even when X^u_c is constant since the volume form on $\mathbb{R}^u$ is not necessarily constant. But it is possible to choose a coordinate system such that $\mathrm{div}X^u_c = 0$ locally.

Proof. We now prove Theorem 5.

In the context of our coupled hyperbolic attractors, we need to show that $\mathrm{div}^u_\sigma X^u_{V,0}$ depends on x_V Hölder continuously in the metric ρ_q, where σ is the canonical metric on the unstable manifold. We note that the density function s_V of this metric satisfies the equation

$$\frac{s_V(x_V)}{s_V(y_V)} = \prod_{k=1}^{\infty} \Big[\frac{J^u\Phi_V(\Phi_V^{-k}x_V)}{J^u\Phi_V(\Phi_V^{-k}y_V)}\Big]^{-1}.$$

We also have the identity

$$\mathrm{div}^u_\sigma X^u_{V,0} = \mathrm{div}^u_\omega X^u_{V,0} + < \mathrm{grad}\log s_V, X^u_{V,0} >,$$

where w is the induced volume form on the unstable manifold. We will drop w for simplicity.

Note that the gradient

$$\mathrm{grad}\log s_V = \mathrm{grad}\log \prod_{k=1}^{\infty} J^u\Phi_V(\Phi_V^{-k}x_V)$$

is taken in the unstable direction $X^u_{V,0}$. We can prove its Hölder continuity with straightforward estimations as we did in Part I. For $\mathrm{div}^u X^u_{V,0}$, we use the representation from Lemma 4

$$\mathrm{div}^u X^u_{V,0} = \mathrm{div}^u X^u_{V,0,c} - \frac{d}{dt} J\xi_V(x_V, t),$$

where $\xi_V(x_V, t)$ denotes the corresponding holonomy map defined on the unstable manifold of Φ_V. Note that the first term $X^u_{V,0,c}$ is differentiable in x_V and its partial derivatives with respect to x_i approach zero exponentially as $|i| \to \infty$. This guarantees the Hölder continuity of this divergence and the Hölder constant and exponent can be chosen independent of V. For the second term, we need an explicit representation of the Jacobian.

1. *Calculating the derivative of the Jacobian with respect to t.*

We identify a small neighborhood of the manifold with $\mathbb{R}^u \times \mathbb{R}^s$. We shall now consider the Jacobian whose formula is given [11, p. 190] by

$$J\xi_V(x_V, t) = \prod_{n=0}^{\infty} \frac{|\det(D_{p_n}\Phi_V\Big|_{\mathcal{E}_n})|}{|\det(D_{p'_n}\Phi_V\Big|_{\mathcal{E}'_n})|},$$

where $p_n = \Phi^n_V(x_V), p'_n = \Phi^n_V(\xi(x_V, t)), \mathcal{E}_n = D_{p_n}\Phi^n_V E^u_V(x_V)$, and $\mathcal{E}'_n = D_{p'_n}\Phi^n_V E^u_{V,t}$, where $E^u_{V,t} = E^u_V(x_V) + tX_{V,0}(x_V)$.

It is clear that this Jacobian is bounded away from zero since it is equal to 1 when $t = 0$. We prove the Hölder continuity of

$$\frac{d}{dt} \log J\xi_V(x_V, t)\Big|_{t=0}$$

in terms of x_V with respect to the metric ρ_q with Hölder exponent and constant independent of V.

Note that the numerator $\det(D_{p_n}\Phi_V\Big|_{\mathcal{E}_n})$ is independent of t. So, we have

$$\frac{d}{dt} \log J_V(x, t)\Big|_{t=0} = -\sum_{n=0}^{\infty} \frac{d}{dt} \log |\det(D_{p'_n}\Phi_V\Big|_{\mathcal{E}'_n})|. \tag{3.9}$$

We now consider $\frac{d}{dt} \log \det(D_{p'_n}\Phi_V\Big|_{\mathcal{E}'_n})$ with n and x_V fixed. Note that when t varies, $p'_n = \Phi_V^n(\xi(\bar{x}, t))$ is moving along the stable manifold. The subspace $D_{p'_n}\Phi_V E^u_{V,t}(\bar{x})$ varies smoothly in t.

We denote $\psi(y, E) = \log |\det(D_y\Phi_V\Big|_E)|$, where $D_y\Phi_V\Big|_E$ is the differential of Φ_V along the subspace E at the point y in a neighborhood $\mathcal{U}$ of p_n, and E is any subspace in a C^α_Γ-neighborhood $\mathcal{V}$ (using the Grassmannian distance) of the direct sum $\otimes_{i\in V} E^u_i(y)$. We take $\mathcal{V}$ large enough to include all subspaces $E^u_{V,t}$ for $0 \le t \le t_0$ (for some $t_0 > 0$).

Thus, we have a smooth function $\psi(y, E)$ on the product space $\mathcal{U} \times \mathcal{V}$ with differentiable manifold structure.

Note that

$$E^u_{V,t} = tX_{V,0}(x_V) + E^u_V(x_V) = tX^s_{V,0}(x_V) + E^u_V(x_V)$$

and that

$$D_{p'_n}\Phi_V^n E^u_{V,t} = tD_{p'_n}\Phi^n_V X^s_{V,0}(x_V) + D_{p'_n}\Phi^n_V E^u_V(\bar{x}).$$

Taking the derivative of $\log |\det(D_{p'_n}\Phi_V\Big|_{\mathcal{E}'_n})|$ with respect to t, we have

$$\frac{d}{dt}|\det(D_{p'_n}\Phi_V\Big|_{\mathcal{E}'_n})| = \frac{\partial}{\partial y}\psi(y, E)D\Phi^n_V \frac{d}{dt}\xi(\bar{x}, t) + \frac{\partial}{\partial E}\psi(y, E)D_{p'_n}\Phi^n_V X^s_{V,0}(\bar{x}),$$

where the differentials $\frac{\partial}{\partial y}\psi(y, E)$ and $\frac{\partial}{\partial E}\psi(y, E)$ are partial differential operators of $\psi(y, E)$ taken with respect to y and E, respectively.

At the point $t = 0$, since the stable subspace E^s_V is tangent to the leaf of the stable foliation passing the point $\bar{x}$, we simply have

$$\frac{d}{dt}\xi_V(x_V, 0) = X^s_{V,0}$$

and therefore,

$$\frac{\partial}{\partial y}\psi(y,E)D\Phi_V^n\frac{d}{dt}\xi_V(x_V,t)=\frac{\partial}{\partial y}\psi(y,E)D\Phi_V^n X_{V,0}^s.$$

Evaluating the derivative at $t=0$, we have

$$\frac{d}{dt}\log|\det(D_{p_n'}\Phi_V\Big|_{\mathcal{E}_n'})|\Big|_{t=0}=\frac{\partial}{\partial y}\psi(y,E)\Big|_{y=p_n}D\Phi_V^n X_{V,0}^s(\bar{x})+\frac{\partial}{\partial E}\psi(y,E)\Big|_{\mathcal{E}_n}D\Phi_V^n X_{V,0}^s(\bar{x}).$$

2.*Proof of the Hölder continuity of the derivative*

We need a standard lemma.

Lemma 5. *Let $\sum_{n=1}^{\infty}\phi_n(x)$ be a series of Hölder continuous functions with a common Hölder exponent α and Hölder constants c_n satisfying the condition $c_n<L^n$ for some constant $L>0$. Assume that $|\psi_n(x)|<C\lambda^n$ for another constants $C>0, 0<\lambda<1$. Then, the sum is a Hölder continuous with Hölder exponent and constant dependent on constants α, L, C and λ.*

Since $X_{V,0}^s$ is in the stable subspace and $\Phi\in C_\Gamma^r(\mathcal{M},\mathcal{M})$, we have

$$\|D\Phi_V^n X_{V,i}^s(x_V)\|\le C\lambda^n\|X_{V,0}^s\|$$

for some constants $C>0, 0<\lambda<1$.

We now need to consider partial derivatives of the function

$$\psi(y,E)=\log|\det(D_y\Phi_V\Big|_E)|.$$

Using the basis of the tangent space, we represent the differential $D_y\Phi_V\Big|_E$ as a matrix.

Note that y is changing along a stable manifold of Φ_V. So, we can use the coordinate system provided by the stable and unstable subspaces $\oplus_{i\in V}E_i^s$, $\oplus_{i\in V}E_i^u$ of the uncoupled map F_V at the corresponding conjugating point $h_V^{-1}y$. Let $\{\mathbf{e}_i^u, i\in V\}$ and $\{\mathbf{e}_i^s, i\in V\}$ denote the orthonormal bases of the direct sums $\oplus_{i\in V}E_i^u$ and $\oplus_{i\in V}E_i^s$. We take an orthonormal basis of E, $\{\tilde{\mathbf{e}}_i^u\}$. Since E is in the C_Γ^α- neighborhood $\mathcal{V}$ of $\oplus_{i\in V}E_i^u$, $\{\tilde{\mathbf{e}}_i^u\}$ can be uniquely represented as a linear combination of $\{\mathbf{e}_i^u, i\in V\}$ and $\{\mathbf{e}_i^s, i\in V\}$:

$$\tilde{\mathbf{e}}_i^u=\sum_{j\in V}a_{ij}\mathbf{e}_i^u+b_{ij}\mathbf{e}_i^s, \tag{3.10}$$

where a_{ij} and b_{ij} satisfy the conditions

$$\|a_{ij}-\delta_{ij}\|\le C\Gamma(i-j),\ \|b_{ij}\|\le C\epsilon\Gamma(i-j).$$

The constants C and ϵ are valid for the entire neighborhood of $\oplus_{i\in V} E_i^u$.

The matrix representation of $D_y\Phi_V|_E$ is now given by the images of the basis vectors under the map $D\Phi_V$, i.e.,

$$(D\Phi_V \tilde{\mathbf{e}}_i^u), i \in V.$$

If we use the same letter E to denote the coefficient matrix in (3.10), then, the matrix representation is simply

$$D\Phi_V E.$$

If the unstable subspace E_i^u is of dimension u and the manifold M is of dimension n, then $D\Phi_V E$ is a $n|V| \times u|V|$ matrix. We now have

$$[\det(D_y\Phi_V|_E)]^2 = \det(D\Phi_V E)^T D\Phi_V E = \det E^T (D\Phi_V)^T D\Phi_V E,$$

where $D\Phi_V$ is evaluated at the point y.

This determinant as a function of the point y is well defined in an open neighborhood of the hyperbolic attractor Δ_F. For each point y, it is a function of E in the C_Γ^α-neighborhood $\mathcal{V}$ of the unstable subspace at $h_V^{-1}y$. We prove the following estimates on its partial derivatives. These estimates allow us to verify the conditions of Lemma 5 and thus, obtain the desired Hölder continuity.

Lemma 6. *Assume that the map $\Phi \in C_\Gamma^r(\mathcal{M},\mathcal{M})$, $r > 4$. Then, there exist constants C and β independent of V such that*

$$\|\frac{\partial^2\psi(y,E)}{\partial y_k \partial y_j}\| \le Ce^{-\beta|k-j|},\ k,j \in V \tag{3.11}$$

and

$$\|\frac{\partial^2\psi(y,E)}{\partial y_k \partial E}\| \le Ce^{-\beta|k|},\ k \in V. \tag{3.12}$$

Proof. We rewrite the matrix B^TB in the form

$$B^TB = E^T(D\Phi_V)^T D\Phi_V E = (DF)(I + A),$$

where (DF) is a diagonal matrix dependent on F_V and E, but independent of the perturbation.

By the Banach algebra property of the decay function Γ, we know that the off-diagonal entries of the matrix A satisfies the same decay property as those of $D\Phi_V$ and E.

We now calculate the partial derivatives of $\log\det(DF)(I + A)$.

$$\log\det(DF)(I + A) = \log\det(DF) + \log\det(I + A).$$

Clearly, when $k \neq j$,

$$\frac{\partial^2 \log\det(DF)}{\partial x_k \partial x_j} = 0.$$

We expand the function $\log\det(I+A)$:

$$\log\det(I + A_V) = \operatorname{trace}(\log(I + A_V)) = -\sum_{i\in V} w_i(x_V),$$

where

$$w_i(x_V) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} a_{ii}^{(n)}(x_V),$$

where $\{a_{ii}^{(n)}(x_V),\ i \in V\}$ are the entries on the main diagonal of the matrix $(A_V)^n$.

It suffices now to show that there exist constants C, β_1, β_2 such that

$$\| \frac{\partial^2 w_i}{\partial x_k \partial x_l} \| \leq C e^{-(\beta_1|i-k|+\beta_2|i-l|)}. \tag{3.13}$$

We denote the entries of $(A_V)^n$ by $a_{ij}^{(n)}(x_V)$. The following lemma is similar to what were proved in earlier papers [7, 9] about the same matrix and the estimation (3.13) follows immediately.

Sublemma 1. *The entries of the matrix $(A_V)^n$ satisfy the following properties.*

(1) $|a_{ij}^{(n)}(x_V)| \leq (c\epsilon)^n C_1 \exp(-\beta_1|i-j|)$.

(2) $|\frac{\partial a_{ij}^{(n)}}{\partial x_k}(x_V)| \leq (c\epsilon)^n C_2 \exp(-\beta_2(|i-j|+|j-k|+|i-k|))$.

(3) $|\frac{\partial^2 a_{ij}^{(n)}}{\partial x_k \partial x_l}(x_V)| \leq (c\epsilon)^n C_3 \exp(-\beta_3(|i-j|+|i-k|+|i-l|+|j-k|+|j-l|+|k-l|))$.

We leave out the proof since it involves only straightforward estimations.

We are left to calculate the partial derivative of

$$\psi(\bar{y}, E) = \frac{1}{2}\log\det E^T (D\Phi_V)^T D\Phi_V E$$

with respect to E. For convenience, we denote the matrix $(D\Phi_V)^T D\Phi_V$ by B again. We have:

$$\begin{aligned}
\log\det(E+tE')^T B(E+tE') &= \log\det(E^TBE + t(E'^TBE + E^TBE') + t^2E'^TBE') \\
&= \log\det E^TBE \\
&\quad + \log\det(I + t(E'^TBE(E^TBE)^{-1} + (E^TBE)^{-1}E^TBE') + t^2E'^TBE'(E^TBE)^{-1})
\end{aligned}$$

Thus, the derivative of $2\psi(\bar{y}, E)$ along the direction E' is equal to

$$\operatorname{trace} E'^TBE(E^TBE)^{-1} + (E^TBE)^{-1}E^TBE'.$$

With E, E' fixed, $\frac{\partial^2 \psi(y,E)}{\partial y_k \partial E}$ satisfies the estimate (3.12) in the lemma as claimed since $\frac{\partial B}{\partial y_k}$ satisfies the same estimate and is independent of the volume V.

$\square$

By Lemmas 5 and 6, we conclude that the Jacobian $\frac{d}{dt} J\xi_V(x_V, t)\Big|_{t=0}$ and thus,$\mathrm{div}^u X^u_{V,0}(x_V)$ is a Hölder continuous function with respect to a compact metric ρ_q with the Hölder constant and exponent independent of the volume V. This completes our proof of the formula for the linear response function.

$\square$

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