The Asymptotic Behaviour of the Fourier Transforms of Orthogonal Polynomials II: IFS measures and Quantum Mechanics

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Abstract

We study measures generated by systems of linear iterated functions, their Fourier transforms, and those of their orthogonal polynomials. We show that a Mellin analysis permits to fully characterize the asymptotic behaviours of these and other related quantities. Relevance of this analysis to quantum mechanics is briefly discussed.

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1 Introduction

This paper applies the general results of the companion paper I in the case of measures generated by Linear Iterated Functions Systems, that are perhaps the most manageable example of singular measures. In addition, further details and different proofs can be obtained for this class of measures. Specifically, we shall consider the Fourier Bessel functions $\mathcal{J}_n(\mu;t)$:

$$\mathcal{J}_n(\mu;t) := \int d\mu(s)\ p_n(\mu;s)\ e^{-its}, \tag{1}$$

where, as in paper I, $p_n(\mu;s)$ are the orthogonal polynomials of the measure μ, that are easily proven to exist. We shall study the asymptotic, for large argument t, of the Cesaro averages of $\mathcal{J}_n(\mu;t)$, and of the quadratic quantities $\mathcal{J}_n(\mu;t)\mathcal{J}^*_m(\mu;t)$. These averages will be indicated by $\bar{\mathcal{J}}_n(\mu;t)$ and $A_{nm}(\mu;t)$, respectively. We refer to paper I for further notations and general results. In particular, results from paper I will be referred to with the suffix -I.

In the quadratic case, when $n = m = 0$, this problem has already been treated in the literature, as remarked in Paper I. In our context, measures generated by L.I.F.S. are particularly simple, since they allow for the examination of the nature of the singularities determining the analyticity range of the Mellin transforms, as originally remarked by Turchetti et al. [1] and by Makarov [2]. Motivation of the extension to the general n, m case is dictated by the quantum mechanical applications of the formalism [3]. As a consequence of these applications, we shall sometimes refer to the argument t of the F-B. functions as to the "time". In fact, when the Jacobi matrix of a measure is considered as the Hamiltonian in the Schrödinger equation of the evolution in a separable Hilbert space, the components of the quantum motion are precisely the Fourier transforms of the orthogonal polynomials of the spectral measure [4].

We shall proceed as follows. In the next section we outline the relations between generalized Fourier-Bessel functions and quantum time evolution. In Sect. 3 we review the essentials of the formalism of Iterated Function Systems. We then assume a sort of separation condition for the I.F.S. measure, that leads in Sect. 4 to derive the local dimension of the measure μ at fixed points of the I.F.S. maps. We also obtain a theorem on the measure of the ball of radius ε centered at such points. The same direct techniques allow to derive the asymptotics of the F-B. functions, developed in Sect. 5. Graphical illustrations are also displayed. Starting from Sect. 6 we resume the theory of Paper I, by studying the Mellin transforms of F-B. functions, whose analytic continuation leads to the inversion theorems described in Sect. 7. We derive here a trigonometric series representation for the time-rescaled F-B. functions. In the following Sect. 8, we discuss how the same Mellin techniques can be applied to averaging procedures other than Cesaro, and as a by-product, we derive a Fourier series representation for the the measure of the ball of radius ε, when rescaled by the appropriate exponent. The asymptotic behaviour, in a strip of the complex plane, of the Mellin transforms is studied in Sect. 9, where we present quite general results, and graphical–numerical illustrations. We employ these results in Sect. 10 to further enhance the convergence properties of the trigonometric series derived in the previous sections. The same goal is achieved in Sect. 11 by introducing discrete Cesaro sums. These latter constitute an argument worth of investigation in themselves, that is briefly sketched. In Sect. 12 we then move to consider products of F-B. functions. The theory is now seen as an extension of the techniques of the preceding sections. To avoid repetitions, the conclusions in Sect. 13 are no more than a brief recap of the aims of this paper.

2 Fourier-Bessel Functions and Quantum Dynamics

The relevance of the generalized Fourier-Bessel functions $\mathcal{J}_n(\mu; t)$ in a quantum mechanical context can be seen as follows. Let $\mathcal{H}$ be the Hilbert space given by the closure in $L^2(\mathbf{R}, d\mu)$ of the vector subspace generated by $\{1, s, s^2, ..., s^n, ...\}$. In $\mathcal{H}$ the orthonormal polynomials $\{p_n(\mu; s)\}_{n \in \mathbf{N}}$ can be taken as basis set $\{e_n\}_{n \in \mathbf{N}}$ of $\mathcal{H}$. It is well known that these polynomials satisfy the recursion relation

$$s p_n(\mu; s) = a_{n+1} p_{n+1}(\mu; s) + b_n p_n(\mu; s) + a_n p_{n-1}(\mu; s), \tag{2}$$

where $a_n > 0$, and b_n are real numbers which obviously depend on μ (so that we feel no need to indicate this dependence explicitly). In $\mathcal{H}$ we consider the operator H defined by

$$(Hf)(s) = sf(s) \quad \text{for any } f \in \mathcal{D}(H) = \{f \in \mathcal{H} | sf(s) \in \mathcal{H}\}$$

$\mathcal{D}(H)$ is dense in $\mathcal{H}$ and H is self-adjoint on it. The recursion relation (2) can now be written as

$$Hp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}$$

and the matrix elements of H on $\{p_n\}_{n\in\mathbf{N}}$ form the *Jacobi Matrix* J of the system of orthogonal polynomials

$$J = ((p_n, Hp_m), n, m = 0, \ldots) = \begin{pmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathcal{H}$. Outside the tri-diagonals, the matrix elements are null. We can interpret H (equivalently, J) as the Hamiltonian of a tight-binding model on a lattice with canonical basis $\{e_n\}_{n\in\mathbf{N}}$.

Let us then consider the evolution generated in the separable Hilbert $\mathcal{H}$ space by Schrödinger equation,

$$H\psi(t) = i\frac{d}{dt}\psi(t). \tag{3}$$

In this equation, ψ is the *wave-function*. The physical amplitudes of the quantum motion are the square moduli of the projections of the wave-function on the basis states of Hilbert space, $\psi_n := (\psi(t), e_n)$:

$$A_{nn}(t) := |\psi_n(t)|^2 := |(\psi(t), e_n)|^2. \tag{4}$$

The initial state of the evolution, $\psi(0)$, can be chosen freely. Letting it coincide with the first basis state $e_0 = p_0(\mu; s) = 1$ leads to:

Proposition 1 *Let μ, H, J, $\psi(0) = e_0$ and $\psi(t)$ be defined as in the above. Then, $\psi_n(t)$, the projection of the time evolution on the n-th basis state, is $\mathcal{J}_n(\mu; t)$, the n-th generalized Fourier-Bessel function.*

Proof. We first observe that μ coincides with the spectral measure associated to p_0, that is, $\mu_{p_0} = (p_0|P_B p_0) = \int d\mu(s)\ \chi_B(s) = \int_B d\mu(s)$, where P_B is a projector in the spectral family of H associated to the Borel set B and χ_B is the characteristic function of B. Then, from $\psi(t) = e^{-itH}e_0 = e^{-its}p_0(\mu; s) = e^{-its}$, one has at once

$$\psi_n(t) = (e_n, \psi(t)) = \int d\mu(s)\ p_n(\mu; s)\ e^{-its} = \mathcal{J}_n(\mu; t). \quad \square \tag{5}$$

This proposition has been the basis of various investigations of the quantum dynamics of systems with singular continuous spectral measures [3, 4, 5, 6, 7, 8, 9, 10].

3 Linear Iterated Function Systems

Systems of hyperbolic linear iterated functions [11, 12, 13] are finite collections of real maps

$$\ell_i(s) := \delta_i s + \beta_i, \quad i = 1, \ldots, M, \tag{6}$$

where δ_i, β_i are real constants, and where the contraction rates δ_i have modulus larger than zero and less than one. For simplicity, we shall assume that these constant are positive. A weight, π_i, is associated with each map: $\pi_i > 0$, $\sum_i \pi_i = 1$. Employing these weights, a measure μ can be defined as:

Definition 1 *The balanced invariant I.F.S. measure μ is the unique measure that satisfies*

$$\int f \, d\mu \; = \sum_{i=1}^{M} \pi_i \int (f \circ \ell_i) \, d\mu, \tag{7}$$

for any continuous function f.

This measure is supported on A, the subset of $\mathbf{R}$ that solves the equation $A = \bigcup_{i=1,\ldots,M} \ell_i(A)$. The set A is invariant under the action of shrinking, and pasting. Because of this, the geometry of this set is typically fractal (except for special choices of the map parameters). In turn, the balance relation (7) is responsible for the multi-fractal properties of the measure μ.

We shall need in this paper a few results easily derived in IFS theory.

Lemma 1 *For any n,*

$$p_n(\ell_i(s)) = \sum_{l=0}^{n} \Gamma^n_{i,l} \, p_l(s), \quad i = 1, \ldots, M. \tag{8}$$

Proof. It is immediate, since $p_n(\ell_i(s))$ is an n-th degree polynomials, that can be expanded on the first n orthogonal polynomials; $\Gamma^n_{i,l}$, $l = 0, \ldots, n$, are the related coefficients. □

Lemma 2 *Let Γ^n_{jl} be the coefficients in eq. (8) for a linear I.F.S., as defined in eqs. (6, 7). Then, $\Gamma^n_{j,n} = {\delta_j}^n$.*

Proof. Obviously, $p_n(s) = a^n_n s^n + a^n_{n-1} s^{n-1} + \ldots + a^n_0$; then,

$$p_n(\ell_j(s)) = a^n_n(\delta_j s + \beta_j)^n + \ldots + a^n_0 = {\delta_j}^n a^n_n s^n + \ldots = {\delta_j}^n p_n(s) + Q_{n-1}(s),$$

where Q_{n-1} is a polynomial of degree $n-1$. Comparing with

$$p_n(\ell_j(s)) = \Gamma^n_{j,n} p_n(s) + \Gamma^n_{j,n-1} p_{n-1}(s) + \ldots + \Gamma^n_{j,0},$$

and using the orthogonality of the polynomials $\{p_n(\mu; s)\}$ gives the result. □

4 Local analysis of IFS measures

Let us now study in detail the local properties of the I.F.S. measures μ. For simplicity, we assume that zero is the fixed point of the first I.F.S. map, and that a kind of separability condition holds:

Assumption 1 *The I.F.S. maps are such that $\ell_1(s) = \delta_1 s$, and that the distance between zero and $\ell_j(A)$ is strictly positive, for any $j \neq 1$. Here A is the attractor of the I.F.S.*

Notice that this assumption is implied by the open set condition, but it is weaker than this latter. Indeed, it allows for overlaps of the map images. A realization of this hypothesis is given by an I.F.S. whose parameters satisfy $\beta_1 = 0$, $\beta_j > 0$ for $j \neq 1$, $0 < \delta_j < 1$ for all j.

The analysis extends with only notational complications to fixed points of finite combinations of I.F.S. maps, when the analogue hypothesis to 1 holds, and, via a suitable approximation argument, to the generic point in the support of the I.F.S. measure μ. The measure μ so defined enjoys distinctive properties. We first observe that

Proposition 2 *If assumption 1 holds, then $\mu([-\varepsilon, \varepsilon]) = \varepsilon^a A(\log \varepsilon)$, where $a = \log \pi_1 / \log \delta_1$, and A is a strictly positive periodic function, of period $\log \delta_1^{-1}$: $A(\zeta) = A(\zeta + \log \delta_1^{-1})$. Moreover, a is the local dimension of the IFS measure μ at zero: $\gamma_-(\mu; 0) = \gamma_+(\mu; 0) = d_0(\mu; 0) = \log \pi_1 / \log \delta_1$.*

Proof. Let $m(\varepsilon)$ be the measure of the ball of radius ε centered at zero:

$$m(\varepsilon) = \int d\mu(s) \chi_{(-\varepsilon,\varepsilon)}(s). \tag{9}$$

Applying the balance relation (7) one obtains

$$m(\varepsilon) = \sum_{i=1}^{M} \pi_i \int d\mu(s) \chi_{(-\varepsilon,\varepsilon)}(\delta_i s + \beta_i). \tag{10}$$

We now let $B(\varepsilon)$ denote the terms with $i \neq 1$ at r.h.s.:

$$B(\varepsilon) = \sum_{i \neq 1} \pi_i \int d\mu(s) \chi_{(-\varepsilon,\varepsilon)}(\delta_i s + \beta_i). \tag{11}$$

In the term $i = 1$ in eq. (10) we use the property $\chi_{(-\varepsilon,\varepsilon)}(\delta_1 s) = \chi_{(-\varepsilon/\delta_1, \varepsilon/\delta_1)}(s)$ of the characteristic function, to arrive at

$$m(\varepsilon) = \pi_1 m(\varepsilon/\delta_1) + B(\varepsilon). \tag{12}$$

Eq. (12) is a sort of renormalization equation for the function A. To proceed further, we need to study the function $B(\varepsilon)$. It is easy to see that $B(\varepsilon)$ is null for

$\varepsilon < \varrho_{\min}$, where $\varrho_{\min}$ is the minimum distance of the images $\ell_j(A)$ from zero, for $j \neq 1$. Also, $m(\varepsilon)$ is a continuous, monotone non-decreasing function, bounded from below by zero and from above by one. It can also be noted that $m(\varrho_{\min})$ is strictly larger than zero, since $m(\varepsilon)$ is such for any $\varepsilon > 0$, being zero the fixed point of one of the I.F.S. maps. Let now ε_0 belong to $I_0 = (\varrho_{\min}, \varrho_{\min}/\delta_1)$, and let $\varepsilon_k = \delta_1^k \varepsilon_0$. Then, applying eq. (12) once, we have

$$m(\varepsilon_1) = \pi_1 m(\varepsilon_0) + B(\varepsilon_1). \tag{13}$$

Since $B(\varepsilon_k) = 0$ for $k \geq 1$, the above equation extends to

$$m(\varepsilon_k) = \pi_1^k m(\varepsilon_0), \quad k = 1, 2, \ldots. \tag{14}$$

Let now $a = \log \pi_1 / \log \delta_1$, $\varsigma = \log(\varepsilon)$, and $\varsigma_k = \log(\varepsilon_k) = \varsigma_0 + k \log \delta_1$. Let also $A(\varsigma) = e^{-a\varsigma} m(e^{\varsigma})$: in the new variables, eq. (14) becomes

$$A(\varsigma_k) = A(\varsigma_0). \tag{15}$$

Therefore, $A(\varsigma)$ is a continuous, periodic function. It is strictly positive, because $m(\varepsilon)$ is such in I_0. As a consequence, we obtain the first part of the thesis: $m(\varepsilon) = \varepsilon^a A(\log \varepsilon)$. Recall now the definition of local dimension, Def. 7-I, Paper I, and Thm. 1-I, paper I, that permit to conclude. □

We shall come back to the function $A(\varsigma)$ at the end of Sect. 8. It is evident that we can construct I.F.S. measures with arbitrary local dimension $d_0(\mu)$ in $(0, \infty)$ [14]. This fact will be helpful to investigate the import of the value $d_0(\mu)$ on the asymptotic of F-B. functions.

5 Asymptotics of F-B. functions for I.F.S. measures

Recall the definition of the asymptotic exponents $\alpha_n(\mu)$ from paper I and the property:

$$\alpha_n(\mu) = \min\{d_n(\mu), 1\}. \tag{16}$$

These quantities provide us with the asymptotic decay of the generalized F-B. functions, via Thm. 5-I. A further application of the balance relation of I.F.S. measures is a renormalization argument that leads to a different proof, and to further detail:

Theorem 1 *For an I.F.S. in the conditions of assumption 1, when $d_0 = \alpha_0(\mu) < 1$ and when $d_0 > \alpha_0(\mu) = 1$, the function*

$$j_n(\mu; t) := t^{\alpha_0(\mu)} \bar{\mathcal{J}}_n(\mu; t) \tag{17}$$

is bounded. In the first instance, moreover, $j_n(\mu; t)$ is an asymptotically log periodic function of period $\log \delta_1^{-1}$:

$$j_n(\mu; \delta_1^{-k} t) \to u_n(t),$$

as $k \to \infty$, for all $t \in \mathbf{R}$. The function $u_n(t)$ is continuous, and log-periodic: $u_n(\delta_1^{-1} t) = u_n(t)$ for all $t \in \mathbf{R}$.

Proof. Recall that $d_0(\mu) = \log \pi_1 / \log \delta_1$. In the following, we drop for simplicity the reference to the measure μ in the arguments of j_n, p_n, and elsewhere. Let

$$\theta_n(t) := t^{d_0} \bar{\mathcal{J}}_n(\mu;t) = t^{d_0} \int d\mu(s) \frac{\sin(ts)}{ts} p_n(s). \tag{18}$$

Apply eqs. (7) and (8), to obtain

$$\theta_n(t) = t^{d_0-1} \sum_{i=1}^{M} \pi_i \int d\mu(s) \; \frac{\sin(t(\delta_i s + \beta_i))}{\delta_i s + \beta_i} p_n(\delta_i s + \beta_i), \tag{19}$$

$$\theta_n(t) = t^{d_0-1} \sum_{i=1}^{M} \pi_i \sum_{l=0}^{n} \Gamma^n_{i,l} \int d\mu(s) \; \frac{\sin(t(\delta_i s + \beta_i))}{\delta_i s + \beta_i} p_l(s). \tag{20}$$

Observe that the index i labels the summation over all I.F.S. maps. Let us now part the terms with $i = 1$ from the rest:

$$\theta_n(t) = \frac{\pi_1}{\delta_1} t^{d_0-1} \sum_{l=0}^{n} \Gamma^n_{1,l} \int d\mu(s) p_l(s) \frac{\sin t\delta_1 s}{s} + t^{d_0-1}\phi_n(t), \tag{21}$$

where we have put

$$\phi_n(t) := \sum_{i=2}^{M} \pi_i \int d\mu(s) \; \frac{\sin(t(\delta_i s + \beta_i))}{\delta_i s + \beta_i} p_n(\delta_i s + \beta_i). \tag{22}$$

Working out the first term in eq. (21) we finally obtain

$$\theta_n(t) = \sum_{l=0}^{n} \Gamma^n_{1,l} \theta_l(\delta_1 t) + t^{d_0-1}\phi_n(t), \tag{23}$$

where the definition of d_0 has been employed, to simplify $\pi_1 \delta_1^{-d_0} = 1$.

Observe that when assumption 1 holds, $\phi_n(t)$ are bounded functions: $|\phi_n(t)| \leq C_n$. In fact, take any integral appearing in eq. (22), that contributes to $\phi_n(t)$: on the support of μ, the denominators $\delta_i s + \beta_i$ are larger than, or equal to $\varrho_{\min}$. Then,

$$\left| \int d\mu(s) \; \frac{\sin(t(\delta_i s + \beta_i))}{\delta_i s + \beta_i} p_n(\delta_i s + \beta_i) \right| \leq \frac{\|p_n\|_\infty}{\varrho_{\min}},$$

where we have introduced the infinity norm of p_n over the bounded support of μ, $\|p_n\|_\infty$.

Let us now first consider the case when $n = 0$. The relation (23) becomes:

$$\theta_0(t) = \theta_0(\delta_1 t) + t^{d_0-1}\phi_0(t). \tag{24}$$

Take now an arbitrary $t = t_0 \in \mathbf{R}$ and set $t_k := \delta_1^{-k} t_0$. Eq. (24) becomes

$$\theta_0(t_k) - \theta_0(t_{k-1}) = t_k^{d_0-1}\phi_0(t_k), \tag{25}$$

and

$$\sum_k |\theta_0(t_k) - \theta_0(t_{k-1})| \leq C \sum_k t_k^{d_0-1} = C t_0^{d_0-1} \sum_k \delta_1^{-k(d_0-1)}. \tag{26}$$

When $d_0 = \alpha < 1$ the function $\theta_0(t)$ coincides with $j_0(t)$. In addition, the last series is convergent, and therefore the sequence $j_0(t_k)$ converges to a limit, that we call $u_0(t)$. Quite obviously, $u_0(\delta_1^{-1}t) = u_0(t)$ for all $t \in \mathbf{R}$. Take now the interval $I = [t_0, \delta_1^{-1}t_0]$, and the sequence of functions $u_0^k(t) := j_0(\delta_1^{-k}t)$ defined over I. These functions are clearly continuous. Because of the estimate (26) this sequence is uniformly convergent, and therefore $u_0 = \lim_{k\to\infty} u_0^k$ is a continuous function. This proves the full theorem for $n = 0$ and $d_0 < 1$.

The case of $n > 0$, $d_0 = \alpha < 1$, can now be proven by induction. In fact, the relation $\Gamma^n_{1,n} = \delta_1^n$ allows us to re-write eq. (23) as follows:

$$\theta_n(t) = \delta_1^n \theta_n(\delta_1 t) + \sum_{l=0}^{n-1} \Gamma^n_{1,l} \theta_l(\delta_1 t) + t^{d_0-1}\phi_n(t), \tag{27}$$

and therefore

$$\begin{aligned} j_n(t_k) - j_n(t_{k-1}) &= \delta_1^n (j_n(t_{k-1}) - j_n(t_{k-2})) + \\ + \sum_{l=0}^{n-1} \Gamma^n_{1,l}(j_l(t_{k-1}) - j_l(t_{k-2})) &+ t_k^{d_0-1}(\phi_n(t_k) - \delta_1^{d_0-1}\phi_n(t_{k-1})). \end{aligned} \tag{28}$$

Take now the modulus, bound the r.h.s., and sum over k from two to infinity:

$$\begin{aligned} \sum_{k=2}^{\infty} |j_n(t_k) - j_n(t_{k-1})| &\leq \delta_1^n \sum_{k=2}^{\infty} |(j_n(t_{k-1}) - j_n(t_{k-2}))| + \\ + \sum_{l=0}^{n-1} |\Gamma^n_{1,l}| \sum_{k=2}^{\infty} |(j_l(t_{k-1}) - j_l(t_{k-2}))| &+ \sum_{k=2}^{\infty} t_k^{d_0-1} |(\phi_n(t_k) - \delta_1^{d_0-1}\phi_n(t_{k-1}))|. \end{aligned} \tag{29}$$

Because of the difference in the indices, the first line of the above inequality can be modified as:

$$(1 - \delta_1^n) \sum_{k=2}^{\infty} |j_n(t_k) - j_n(t_{k-1})| \leq \delta_1^n |(j_n(t_1) - j_n(t_0))| + \ldots$$

without affecting the rest. In the second line, the terms $|(\phi_n(t_k) - \delta_1^{d_0-1}\phi_n(t_{k-1}))|$ are bounded; when $d_0 < 1$ the series $\sum_{k=2}^{\infty} t_k^{d_0-1}$ was proven to converge in eq. (26) above. Therefore, the induction hypothesis that $\sum_{k=2}^{\infty} |(j_l(t_{k-1}) - j_l(t_{k-2}))|$

is convergent for $l = 0, \ldots, n-1$ guarantees that this is the case also for $l = n$. The induction seed for $l = 0$ has been proven in in eq. (26), and in conclusion

$$\sum_{k=1}^{\infty} |(j_n(t_k) - j_n(t_{k-1}))| < \infty \tag{30}$$

for any value of n. The proof can then be completed as in the case $n = 0$.

Observe now that a convenient representation of the function $\theta_0(t)$ can be obtained by applying iteratively eq. (24) k times:

$$\theta_0(t) = \theta_0(\delta_1^k t) + t^{d_0-1} \sum_{j=0}^{k-1} (\frac{\pi_1}{\delta_1})^j \phi_0(\delta_1^j t). \tag{31}$$

Because of the definition (18), and because $d_0 > 0$, $\lim_{t\to 0} \theta_n(t) = 0$; hence, $\delta_1 < 1$ implies that $\theta_0(\delta_1^k t) \to 0$ as $k \to \infty$. Taking this limit in eq. (31) we get

$$\theta_0(t) = t^{d_0-1} \sum_{j=0}^{\infty} (\frac{\pi_1}{\delta_1})^j \phi_0(\delta_1^j t). \tag{32}$$

Remark that our proof shows that this representation is valid even in the absence of the full assumption 1: it only requires zero to be the fixed point of one of the I.F.S. maps.

To further exploit this point, let now $\psi_0(t) := \phi_0(t)/t$. Eq. (32) can now be written in terms of the F-B. function and of ψ_0:

$$\bar{\mathcal{J}}_0(\mu; t) = \frac{1}{t} \sum_{j=0}^{\infty} (\frac{\pi_1}{\delta_1})^j \phi_0(\delta_1^j t) = \sum_{j=0}^{\infty} {\pi_1}^j \psi_0(\delta_1^j t). \tag{33}$$

Observe that, in force of the definition (22) $\psi_0(t)$ is always a bounded function, so that the last series in the above is totally convergent.

Let us now re-assume validity of hypothesis 1, that amounts to have $\phi_0(t)$ bounded. Consider the case $n = 0$, and $d_0 > \alpha = 1$, that implies $\frac{\pi_1}{\delta_1} < 1$. Then, we have total convergence of the first series in eq. (33):

$$\sum_{j=0}^{\infty} (\frac{\pi_1}{\delta_1})^j |\phi_0(\delta_1^j t)| \leq C_0 \frac{\delta_1}{\delta_1 - \pi_1},$$

whose sum defines the bounded function $\Phi_0(t)$. This permits to write

$$\bar{\mathcal{J}}_0(\mu; t) = \frac{1}{t} \Phi_0(t). \tag{34}$$

By utilizing again eq. (27) we can prove iteratively that there exist bounded functions $\Phi_n(t)$ such that $\bar{\mathcal{J}}_n(\mu; t) = \frac{1}{t}\Phi_n(t)$, for all n. In fact, when written in terms of $j_n(\mu;t) = t\bar{\mathcal{J}}_n(\mu;t)$ (recall that now $\alpha = 1$), eq. (27) becomes:

$$j_n(\mu;t) = \delta_1^{n+d_0-1} j_n(\mu;\delta_1 t) + \sum_{l=0}^{n-1} \Gamma_{1,l}^n \delta_1^{d_0-1} j_l(\mu;\delta_1 t) + \phi_n(t). \tag{35}$$

Suppose now that $j_l(\mu;t)$ are bounded functions for $l = 0, \ldots, n-1$. Then, the last two terms in eq. (35) define a bounded function $B_n(t)$:

$$j_n(\mu;t) = \delta_1^{n+d_0-1} j_n(\mu;\delta_1 t) + B_n(t). \tag{36}$$

Let us iterate this equation, to get

$$j_n(\mu;t) = \delta_1^{k(n+d_0-1)} j_n(\mu;\delta_1^k t) + \sum_{j=0}^{k} \delta_1^{k(n+d_0-1)} B_n(\delta_1^k t). \tag{37}$$

Again, if $d_0 > 1$ the first term at r.h.s. vanishes for $k \to \infty$, and

$$|j_n(\mu;t)| \leq \sum_{j=0}^{k} \delta_1^{k(n+d_0-1)} |B_n(\delta_1^k t)| \leq \frac{C}{1-\delta_1^{n+d_0-1}}. \quad \square \tag{38}$$

Remark 1 Equation (33), or better (24), are a convenient means for numerical experiments. In fact, the functions at r.h.s. can be efficiently computed using the techniques of [15, 16, 17]. Figures (1) and (2) obtained in this way provide an illustration of the content of this Theorem.

Remark 2 The asymptotically log-periodic character of the functions $j_n(\mu;t)$, with their convergence to the fractal curve $u_n(t)$, is a remarkable result, born out of the technique of Cesaro averaging. This is particularly evident in comparison with the parallel situation in the absence of averaging [18].

6 Mellin Analysis of F-B. functions: Singularities

In the previous section, the exact asymptotic behaviour of the Cesaro averaged F-B. functions have been derived. Compare in fact the statement of Thm. 1 with that of Thm. 5-I of paper I. The unifying power of the analysis presented in paper I permits to understand why this achievement has been possible: the nature of the singularities of the Mellin transform for I.F.S. can be fully mastered.

Recall the definitions: we let $M_n(\mu;z)$ be the Mellin transform of the function $\bar{\mathcal{J}}_n(\mu;t)$:

$$M_n(\mu;z) = \int_0^\infty dt\; t^{z-1} \bar{\mathcal{J}}_n(\mu;t). \tag{39}$$

This can be written as

$$M_n(\mu;z) = H(z)\; G_n(\mu;z), \tag{40}$$

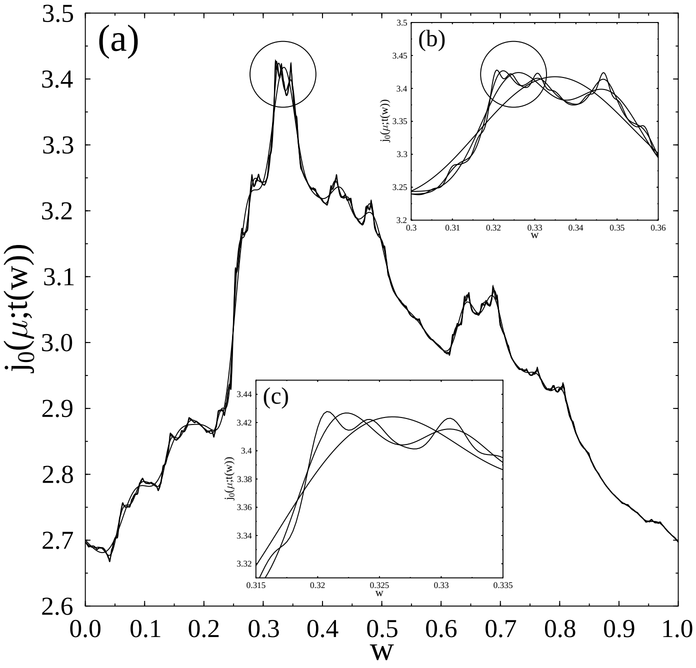

Figure 1: Convergence of $j_0(\mu;t)$ to the log-periodic limit for the L.I.F.S. with $\ell_1(s) = \frac{2}{5}s$, $\ell_2(s) = \frac{2}{5}s + \frac{4}{5}$, and $\pi_1 = \frac{3}{5}$, $\pi_2 = \frac{2}{5}$. Here, $d_0(\mu) \simeq 0.5574929506...$ is a number between one half and one. Time is logarithmically scaled via the variable $w \in [0,1]$: $t(w) = t_0 \delta_1^{-n-w}$, with $t_0 = 100$. The vertical scale has been divided by δ_1. In (a) four different periods ($n = 0, \ldots, 3$) are shown. The region around $w = .33$ (circled) is magnified in the inset (b): a single peak for $n = 0$ splits into a twin structure. The same pattern is then repeated (inset (c), where only $n = 1, 2, 3$ are plotted). In the limit, the graph of the function $j_0(t(w))$ is a fractal curve.

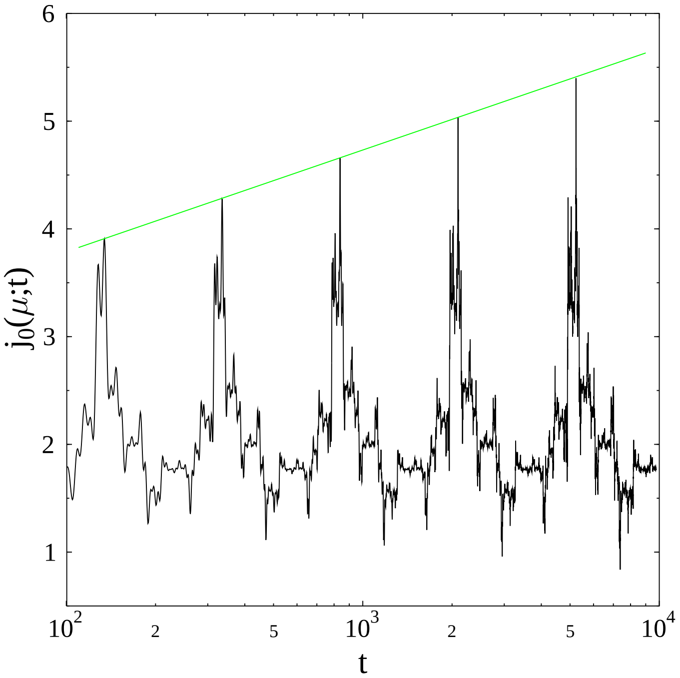

Figure 2: Function $j_0(\mu;t)$ for the L.I.F.S. with $\ell_1(s) = \frac{2}{5}s$, $\ell_2(s) = \frac{2}{5}s + \frac{4}{5}$, and $\pi_1 = \frac{2}{5}$, $\pi_2 = \frac{3}{5}$. Here, $d_0(\mu) = 1$, and $\bar{\mathcal{J}}_0(\mu;t)$ is infinitesimal with respect to t^{-x}, for all $x < 1$. In this borderline case we are not assured that $j_0(\mu;t)$ is a bounded function; yet its divergence—if any—should be slower than any power-law. The numerical data suggest that a logarithmic divergence is taking place.

with $H(z) = \Gamma(z-1)\sin[\frac{\pi}{2}(z-1)]$, and

$$G_n(\mu; z) = \int d\mu(s) \frac{p_n(s)}{|s|^z}. \tag{41}$$

It is now easy to find the singularity picture of $M_n(\mu; z)$, in the family of I.F.S. measures we are treating.

Proposition 3 *$G_n(\mu, z)$ has simple poles at $z_m + k$, $m = -\infty, \ldots, \infty$, $k = 0, \ldots, n$. We have set $z_m := d_0(\mu) + im\omega$, with $d_0(\mu) = \frac{\log \pi_1}{\log |\delta_1|}$, and $\omega := -\frac{2\pi}{\log |\delta_1|}$.*

Proof: Combining the I.F.S. balance relation and eq. (8) in the integral defining $G_n(\mu, z)$, one obtains

$$G_n(\mu, z) = \sum_{i=1}^{M} \sum_{l=0}^{n} \Gamma^n_{i,l} \pi_i \int d\mu(s)\; p_l(s) |\delta_i s + \beta_i|^{-z}. \tag{42}$$

Under assumption 1, terms with $i \neq 1$ at r.h.s. are analytic functions of z (because $\delta_i s + \beta_i$ is never null, when s belongs to the support of μ), and will be collectively denoted by $\Phi_n(\mu; z)$:

$$\Phi_n(\mu; z) = \sum_{i=2}^{M} \sum_{l=0}^{n} \Gamma^n_{i,l} \pi_i \int d\mu(s)\; p_l(s) |\delta_i s + \beta_i|^{-z}. \tag{43}$$

In particular,

$$\Phi_0(\mu; z) = \sum_{i=2}^{M} \pi_i \int d\mu(s)\; |\delta_i s + \beta_i|^{-z}. \tag{44}$$

Then, we also split off the term with $i = 1$, and $l = n$. We use the relation $\Gamma^n_{i,n} = \delta^n_i$, proven in section 3, to get:

$$E(z-n) G_n(\mu, z) = \Phi_n(\mu; z) + \pi_1 |\delta_1|^{-z} \sum_{l=0}^{n-1} \Gamma^n_{1,l} G_l(\mu, z), \tag{45}$$

where the function $E(z)$ is defined by:

$$E(z) := 1 - \pi_1 |\delta_1|^{-z}. \tag{46}$$

The set of equations (45) can be iteratively solved: the first gives

$$G_0(\mu, z) = \frac{\Phi_0(\mu; z)}{E(z)}, \tag{47}$$

which has simple poles at $z_m := d_0(\mu) + im\omega$, with $d_0(\mu) = \frac{\log \pi_1}{\log |\delta_1|}$, and $\omega := -\frac{2\pi}{\log |\delta_1|}$. This proves the lemma for $n = 0$. The general case follows by iteration of eq. (45), with the aid of eqs. (46),(47). $\square$

We can now apply the theory of paper I to obtain the the asymptotic decay of the Cesaro averages $\bar{\mathcal{J}}_n(\mu;t)$, as in the first part of Theorem 1. We so prove that $\bar{\mathcal{J}}_n(\mu;t) = o(t^{-x})$, for any $x < \alpha_0(\mu)$. Yet, the Mellin technique can lead us to obtain stronger results.

7 Analytic Continuation and Inversion Thms.

The full power of the Mellin transform approach is appreciated when effecting the analytic continuation of $M_n(\mu;z)$. Because of Prop. 3 just one of the vertical lines of singularity can occur in the strip $0 < \Re(z) < 1$. We now study what happens in this case. We therefore assume for starters that $\alpha_0(\mu) = d_0(\mu) < 1$. Otherwise, we can apply Thms. 6-I, 7-I, and 8-I of paper I. The results obtained in this fashion are two theorems:

Theorem 2 *Let* $0 < d_0(\mu) = \alpha_0(\mu) < \frac{1}{2}$. *One can write*

$$\bar{\mathcal{J}}_n(\mu;t) = t^{-\alpha_0(\mu)}\Psi_n(\log t) + N_n(t), \tag{48}$$

where Ψ_n *is a periodic function, and* $t^{\bar{x}}N_n(t)$ *is infinitesimal, when* $t \to \infty$, *for any* $\alpha_0(\mu) < \bar{x} < \frac{1}{2}$.

Theorem 3 *Let* $\frac{1}{2} \leq d_0(\mu) = \alpha_0(\mu) < 1$. *One can write*

$$\bar{\mathcal{J}}_n(\mu;t) = t^{-\alpha_0(\mu)}\Psi_n(\log t) + N_n(t), \tag{49}$$

where Ψ_n *is a periodic function, and* $t^{\bar{x}}N_n(t)$ *belongs to* $L^2((1,\infty),\frac{dt}{t})$ *for all* $\alpha_0(\mu) < \bar{x} < 1$.

Proof: We now compute a contour integral similar to that of Thms 7-I of paper I, but where the rightmost vertical component is to the right of the first line of singularities: Consider the sequence of rectangular paths γ_N, composed of the straight line $x+iy$, with $x < \frac{1}{2}$, $y \in [-(N+\frac{1}{2})\omega, (N+\frac{1}{2})\omega]$, the horizontal line from $x + i(N+\frac{1}{2})\omega$ to $\bar{x} + i(N+\frac{1}{2})\omega$, with $\alpha_0(\mu) < \bar{x} < 1$ and the two remaining segments needed to form a rectangle in the complex plane. As in paper I, consider the function $m_n(z) := M_n(\mu;z)e^{-\tau z}$, which is now meromorphic in the the strip, and let $I_N(n,\tau)$ be its contour integral. It can be parted according to the contour components, as

$$I_N(n,\tau) = I_N(x,n,\tau) + H_N(x,\bar{x},n,\tau) - I_N(\bar{x},n,\tau), \tag{50}$$

where $I_N(x,n,\tau)$ denotes the integral over the vertical component with $\mathrm{Re}(z) = x$, and the terms over the horizontal segments have been collected in the function H_N. We now interrupt for a moment the proof of Thms. 2, 3 to introduce a series of technical lemmas needed in the evaluation of the terms in eq. (50). $\triangle$

Lemma 3 *The functions $\Phi_n(\mu;z)$ are uniformly bounded in the strip $x < \Re(z) < \bar{x}$, with arbitrary $0 < x < \bar{x}$.*

Proof. The integrals in eq. (43) can be bounded by:

$$\|p_l\|_\infty \int d\mu(s)\ |\delta_i s + \beta_i|^{-\Re(z)},$$

for $i = 2, \ldots$, and for these maps the minimum of $|\delta_i s + \beta_i|$ over the support of μ, $\varrho_{\min}$, is strictly positive, in force of the assumption 1. Therefore,

$$|\Phi_n(\mu;z)| \leq \max\{1, \varrho_{\min}^{-\bar{x}}\} \sum_{i=2}^{M} \pi_i \sum_{l=0}^{n} |\Gamma_{i,l}^n| \|p_l\|_\infty. \tag{51}$$

The r.h.s. is then a constant that depends only on n. □

Lemma 4 *Let $\kappa_N = \{z \in \mathbf{C}\ \ s.t\ z = u \pm i(N + \frac{1}{2})\omega, u \in [x, \bar{x}]\}$ with arbitrary $0 < x < \bar{x}$. Then, there exist positive constants g_n depending on n, but not on N, such that $|G_n(\mu;z)| \leq g_n$, for all $z \in \kappa_N$.*

Proof. First of all, the functions $\Phi_n(\mu;z)$ are uniformly bounded on the horizontal segments κ_N, in force of the previous lemma 3. Secondly, the functions $E(z-n)$ are real, and uniformly inferiorly bounded on κ_N: since $|\delta_1|^{\pm i(N+\frac{1}{2})\omega} = -1$,

$$E(z-n) = 1 - \pi_1 |\delta_1|^{-u-n\pm i(N+\frac{1}{2})\omega} = 1 + \pi_1 |\delta_1|^{-u-n} > 1\ \ \forall z \in \kappa_N. \tag{52}$$

For the same reason, $\pi_1|\delta_1|^{-z} = -\pi_1|\delta_1|^{-u}$. Finally, eqs. (51,52) imply that $|G_0(\mu;z)| \leq C_0$ for all $z \in \kappa_N$, and the lemma follows by induction using eq. (43), together with eqs. (51,52). □

Lemma 5 *For any $x, \bar{x}$ in $(0, \frac{3}{2})$, $\tau \geq 0$, $H_N(x, \bar{x}, n, \tau) \to 0$ uniformly in τ, as N tends to infinity.*

Proof. Clearly,

$$H_N(x, \bar{x}, n, \tau) = \int_{\kappa_N} e^{-\tau z} H(z) G_n(\mu;z)\ dz.$$

We can then use Lemma 4, and the asymptotic information

$$H(z_m) \simeq |m|^{-\frac{3}{2}+\alpha}, \tag{53}$$

to show that

$$H_N(x, \bar{x}, n, \tau) \leq C_n e^{-\tau x} |(N + \frac{1}{2})\omega|^{-\frac{3}{2}+\bar{x}} |\bar{x} - x|,$$

from which the lemma follows. □

Lemma 6 *The residuals of $M_n(\mu; z)$ at the poles $z_m := d_0(\mu) + im\omega$, have the form*

$$\rho_m^n = H(z_m)\Phi_0(z_m)q_n, \tag{54}$$

where the coefficients q_n are independent of m.

Proof. Let ρ_m^n be the residuals of $M_n(\mu; z)$. We have that

$$\rho_m^n = \eta_m^n H(z_m), \tag{55}$$

where, obviously, the coefficients η_m^n are the residuals of $G_n(\mu; z)$ inside the prescribed strip. These latter can be obtained taking limits in eq. (45):

$$\eta_m^n = \lim_{z \to z_m} (z - z_m) G_n(\mu; z).$$

After reduction of the functions $E(z-n)$ to their explicit form, eq (46), we obtain the recursion relation:

$$\eta_m^n = \frac{1}{1 - |\delta_1|^n} \sum_{l=0}^{n-1} \Gamma_{1,l}^n \, \eta_m^l, \tag{56}$$

initialized by

$$\eta_m^0 = \frac{\Phi_0(z_m)}{E'(z_m)} = \frac{\Phi_0(z_m)}{\log|\delta_1|}. \tag{57}$$

This can be simplified, somehow, by noticing that the term $\Phi_0(z_m)$ is a factor of everything. In so doing, we arrive at the form (54), where q_n can be obtained from the relation

$$q_n = \frac{1}{1 - |\delta_1|^n} \sum_{l=0}^{n-1} \Gamma_{1,l}^n \, q_l, \tag{58}$$

initialized by

$$q_0 = \frac{1}{\log|\delta_1|}. \quad \square \tag{59}$$

We now need an estimate of $|\rho_m^n|$ for large $|m|$. This is contained in

Lemma 7 *For large m, there exist a constant C_n such that*

$$|\rho_m^n| \leq C_n |m|^{-\frac{3}{2} + d_0(\mu)}. \tag{60}$$

Proof. In force of the estimates in the proof of Lemma 4, $|\Phi_0(z_m)| < C$, with C a suitable constant. The asymptotic information (53) then completes the proof of the Lemma. $\square$

Continuation of the proof of Thms. 2, 3. We can now consider the evaluation of the integral $I_N(n, \tau)$. Clearly, $I_N(n, \tau)$ is equal to $2\pi i$ times the sum of residuals of $M_n(\mu; z)e^{-\tau z}$ inside the contour γ_N. Using the information just derived we can write:

$$I_N(x, n, \tau) = I_N(\bar{x}, n, \tau) - H_N(x, \bar{x}, n, \tau) + 2\pi i \sum_{m=-N}^{N} \rho_m^n e^{-\tau z_m}. \tag{61}$$

Since $z_m = d_0(\mu) + im\omega$, a trigonometric polynomial in τ appears at r.h.s. We let N go to infinity in eq. (61). The horizontal contribution H_N vanishes, because of lemma 5. The l.h.s. tends to the limit $i\bar{\mathcal{J}}_n(e^\tau)$, as in Theorem 7-I of paper I. Moreover, when $\alpha_0(\mu) < \frac{1}{2}$, we can also take $\bar{x} < \frac{1}{2}$, so that $\lim_{N\to\infty} I_N(\bar{x}, n, \tau)$ is a Fourier transform of a L^1 function, and therefore

$$I_\infty(\bar{x}, n, \tau) = 2\pi i e^{-\bar{x}\tau} u_n(\bar{x}, \tau) \tag{62}$$

with $u_n(\bar{x}, \tau)$ a C_∞ function of τ. The Fourier series in (61),

$$\sum_{m=-\infty}^{\infty} \rho_m^n e^{-i\tau\omega m} = \sum_{m=-\infty}^{\infty} H(z_m)\Phi_0(z_m)q_n e^{-i\tau\omega m} \tag{63}$$

converges uniformly, because of the boundedness of $\Phi_n(z_m)$, of Lemma 3, and of the estimate (53) on the growth of $H(z_m)$, giving rise to a periodic function of τ. Collecting this information, we can write

$$\bar{\mathcal{J}}_n(e^\tau) = e^{-\alpha_0(\mu)\tau} \sum_{m=-\infty}^{\infty} \rho_m^n e^{-im\omega\tau} + e^{-\bar{x}\tau} u_n(\bar{x}, \tau). \tag{64}$$

Returning to linear time, $t = e^\tau$, we obtain the thesis of Thm. 2

If $\alpha_0(\mu) > \frac{1}{2}$, we can still perform the limit $N \to \infty$ in eq. (61), but convergence of the individual terms at r.h.s. of (60) is in L^2 sense: precisely, we have again equation (64), now with square summable terms: $\sum_m |\rho_m^n|^2 < \infty$, and $u_n(\bar{x}, \tau) \in L^2(\mathbf{R}, d\tau)$. Finally, it is evident from lemma 5 that $e^{-\bar{x}\tau} u_n(\bar{x}, \tau)$ does not depend on $\bar{x}$, as far as $\bar{x} > \alpha$. This completes the proof of Thm. 3 □

The above theorems complete the asymptotic analysis of $\bar{\mathcal{J}}_n(t)$, showing the presence of log-periodic oscillations superimposed to a power-law decay given by the Hölder exponent of the spectral measure at zero. This behaviour has a correction which decays faster, on the average, at infinity. A strengthening of these theorems will be obtained later on, after we derive an enhanced version of Lemma 3.

8 Different forms of Averaging

Compare eqs. (9) and (18). It is apparent that the kernels $\chi_{(-\varepsilon,\varepsilon)}(s)$ and *sinc* (ts) are sampling the measure μ in the neighbourhood of zero, when $\varepsilon \to 0$, or $t \to \infty$. One also easily realizes that the two kernels are Fourier conjugated pairs: in fact, to the kernel *sinc* (ts) in the integration over the measure μ corresponds a Cesaro average in the time integration, rendered by the kernel $\chi_{(-1/t,1/t)}$. Quite symmetrically, the kernel $\chi_{(-\varepsilon,\varepsilon)}(s)$ in eq. (9) corresponds to *sinc* averaging of F-B. functions:

$$\overline{\mathcal{J}}_n^S(\mu;\frac{1}{\varepsilon}) = \frac{\varepsilon}{2}\int_{-\infty}^{\infty} \frac{\sin\varepsilon t}{\varepsilon t}\mathcal{J}_n(\mu;t)dt. \tag{65}$$

A theory quite parallel to that of paper I can then be performed also with this different averaging, and with any other time averaging kernel K, leading to the formula

$$M_n(\mu;z) = H^K(z)\; G_n(\mu;z), \tag{66}$$

where $H^K(z)$ originates from the Mellin transform of the Fourier conjugate kernel $\hat{K}$. In the case of sinc averaging, we have $H^K(z) = \frac{1}{z}$.

In the same vein, it is possible to recast Prop. 2 in terms of the Mellin transform of the function $m(\varepsilon)$:

$$\mathcal{M}(m;z) = -\frac{1}{z}G_0(\mu;-z), \tag{67}$$

a relation valid in the strip $-d_0(\mu) < \Re(z) < 0$. In so doing, the *left* divergence abscissa of the Mellin transform governs the small ε behavior of $m(\varepsilon)$. The right divergence abscissa, zero, if of course related to the constant behaviour of $m(\varepsilon)$ when ε is larger than the diameter of the support of μ. Again developing a suitable path integral, in the case of L.I.F.S. measures fulfilling hypothesis 1, the function $A(\tau)$ of Prop. 2 can be associated to the Fourier series

$$\lim_{N\to\infty}\sum_{m=-N}^{N}\frac{\Phi_0(z_m)}{z_m\log|\delta_1|}e^{-i\tau\omega m}, \tag{68}$$

where $\tau := \log\varepsilon$. This analysis has been performed in [15]. Figure 3 shows the graph of this function for the I.F.S. described in Fig. 1.

9 Asymptotics of the Mellin Transform of F-B. functions in the strip of analyticity

In line with the general theory presented in Paper I, let us now consider the asymptotic behaviour, in the strip of analyticity, of the Mellin transforms $G_n(\mu;x+iy)$. The properties of L.I.F.S. measures permits to carry the analysis to a much larger extent than in the general case. Recall that in this study a crucial role is played by the measure ν_x appearing in Sect. 14 of paper I, and defined via

$$\int d\nu_x(\tau)f(\tau) = \int d\mu(s)|s|^{-x}f(\log(|s|)), \tag{69}$$

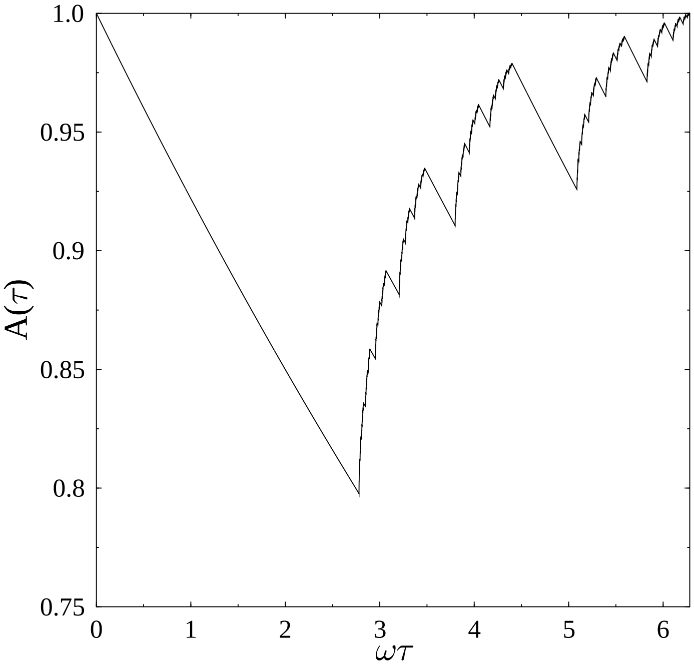

Figure 3: Function $A(\tau)$ versus $\omega\tau$, with $\omega = 2\pi/\log\delta_1^{-1}$, for the L.I.F.S. of Fig. 1. The curve has been obtained by the partial sum of eq. (68) with $N = 40,000$.

that is required to hold for any continuous function f which vanishes at infinity. Thm. 13-I of Paper I asserts that the asymptotic decay of the Cesaro average of $G_n(\mu; x+iy)$ is determined by the local dimension, at zero, of ν_x. Its value can be easily characterized:

Proposition 4 *The local dimensions of ν_x at zero equal the local dimensions of μ at one, for any $x < d_0(\mu)$.*

Proof: Let us compute the ν_x-measure of the ball of radius ε at zero:

$$\nu_x(B_\varepsilon(0)) = \int d\nu_x(\tau)\chi_{[-\varepsilon,\varepsilon]}(\tau). \tag{70}$$

Because of eq. (69), we obtain

$$\nu_x(B_\varepsilon(0)) = \int_{e^{-\varepsilon}}^{e^\varepsilon} d\mu(s)\frac{1}{s^x} \sim \int_{1-\varepsilon}^{1+\varepsilon} d\mu(s)\frac{1}{s^x} \sim \mu(B_\varepsilon(1)), \tag{71}$$

and therefore the upper and lower dimensions of ν_x at zero are the same as the local dimensions of μ at one.

The case of the electrostatic dimension is similar: the generalized electrostatic potential of ν at zero can be written:

$$\mathcal{G}(\nu; 0, w) := \int d\nu(\tau)\frac{1}{|\tau|^w} = \int \frac{d\mu(r)}{|r|^x}\frac{1}{|\log|r||^w}.$$

We now part the integral with respect to $d\mu$ in two parts: one for $|r-1| < 1/2$, and the complementary. The former defines an analytic function in w. In the domain of the second, the logarithm can be bounded as $(|r|-1) \geq \log|r| \geq c_1(|r|-1)$ for $|r| \geq 1$, and $c_2(1-|r|) \leq \log|r| \leq (1-|r|)$ for $|r| \leq 1$, with c_1 and c_2 suitable constants. Using these constraints into the argument of the integrand we can bound the divergence abscissa of $\mathcal{G}(\nu; 0, w)$ by that of $\mathcal{G}(\mu; 1, w)$ on both sides, so that the thesis follows. □
Thm 13-I, paper I, can now be applied, to the effect that:

Proposition 5 *When assumption 1 holds, and $x < d_0(\mu)$, the Mellin transform $G_n(\mu; x+iy)$ decays in Cesaro average as $o(t^{-a})$, for any $a < \min\{1, d_0(\mu; 1)\}$.*□

This Proposition is illustrated in Figure 4. Because the support of the L.I.F.S. considered in the figure is contained in the interval $[0, 1]$, and one is the fixed point of the second map, the local dimension of ν_x at one can be computed explicitly.

Let us now turn to the asymptotic behaviour of the Cesaro average of the square modulus of $G_n(\mu; x+iy)$: $\frac{1}{2T}\int_{-T}^{T} |G_n(\mu; x+iy)|^2 dy$. To study this latter, we need to compute the correlation dimension of ν_x. A remarkable extension of Theorem 14-I of paper I can be obtained for linear I.F.S. measures:

Theorem 4 *For an I.F.S. verifying assumption 1, for all $x < d_0(\mu)$, the divergence abscissa of the electrostatic energy of the measure ν_x does not depend on x and coincides with that of the balanced invariant measure μ: $D_2(\nu_x) = D_2(\mu)$.*

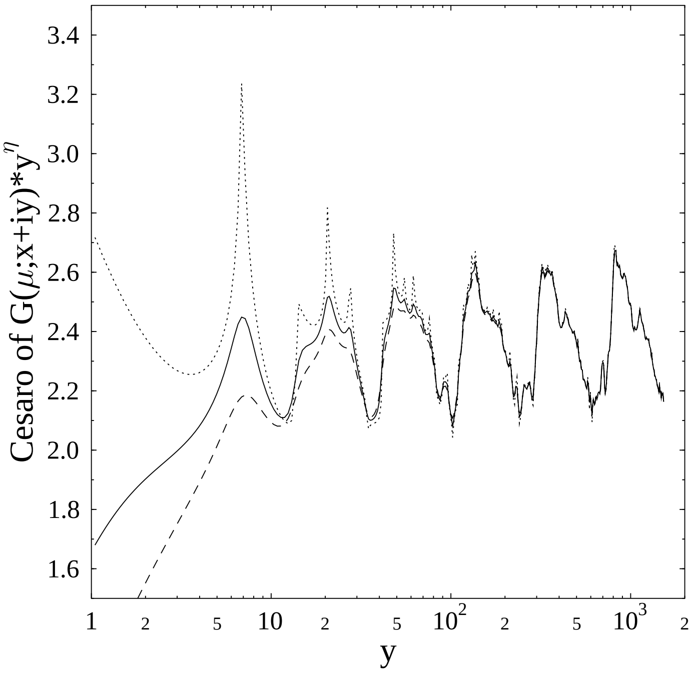

Figure 4: Cesaro average of $G(\mu; x + iy)$ multiplied by y^η, with $\eta = 0.55749295065024..$ the local dimension of μ at one, for the L.I.F.S. with $\ell_1(s) = \frac{2}{5}s$, $\ell_2(s) = \frac{2}{5}s + \frac{4}{5}$, and $\pi_1 = \frac{2}{5}$, $\pi_2 = \frac{3}{5}$. The three lines are drawn for $x = -.9, 0.$ and $x = .9$. Their difference fades asymptotically.

Proof. The electrostatic energy $\mathcal{E}(\nu_x;z)$ of the measure ν_x is defined by the double integral

$$\mathcal{E}(\nu_x;z)=\iint\frac{d\nu_x(r)d\nu_x(s)}{|r-s|^z}=\iint d\mu(r)d\mu(s)\frac{1}{|r|^x|s|^x\;|\log|r|-\log|s||^z} \tag{72}$$

The second equality in the above equation is a consequence of the definition of ν_x. We use the I.F.S. balance relation, eq. (7), in eq. (72) to obtain $\mathcal{E}(\nu_x;z)$ in the form of a sum of various complex functions of the variable z:

$$\mathcal{E}(\nu_x;z)=\sum_{i,j=1}^{M}m_{i,j}(z). \tag{73}$$

The indices i,j label the maps of the I.F.S. Explicitly, the functions $m_{i,j}(z)$ are given by

$$m_{i,j}(z)=\pi_i\pi_j\iint\frac{d\mu(r)d\mu(s)}{|\delta_i r+\beta_i|^x|\delta_j s+\beta_j|^x\;|\log|\delta_i r+\beta_i|-\log|\delta_j s+\beta_j||^z}. \tag{74}$$

We now part the diagonal from the outdiagonal terms in the sum in eq. (73), and in the former the term corresponding to the first map:

$$\mathcal{E}(\nu_x;z)=m_{11}(z)+\sum_{i=j\neq 1}m_{ij}(z)+\sum_{i\neq j}m_{ij}(z). \tag{75}$$

We easily recognize that the first term at r.h.s. can be written

$$m_{11}(z)=\pi_1^2|\delta_1|^{-2x}\iint\frac{d\mu(r)d\mu(s)}{|r|^x|s|^x\;|\log|r|-\log|s||^z}, \tag{76}$$

and therefore $m_{11}(z)$ is just $\pi_1^2|\delta_1|^{-2x}\mathcal{E}(\nu;z)$, that allows to write

$$(1-\pi_1^2|\delta_1|^{-2x})\mathcal{E}(\nu_x;z)=\sum_{j>1}m_{jj}(z)+\sum_{i\neq j}m_{ij}(z). \tag{77}$$

We must now use the geometric properties of the support of μ. Assumption 1 on the I.F.S. implies that

$$\mathrm{diam}(A)\geq|\delta_j s+\beta_j|\geq\varrho_{\min}>0,\ \text{ for } j\neq 1, \tag{78}$$

where the first term is the diameter of A, the support of μ, and the chain of inequalities holds for any s in this set. Moreover, since $|r-s|/b\leq|\log(r)-\log(s)|\leq|r-s|/a$ for any r,s in a finite interval $[a,b]\subset\mathbf{R}_+$, this also implies that the logarithmic difference in the denominator of $m_{ij}(z)$ can be bounded as

$$\eta_1||\delta_i r+\beta_i|-|\delta_j s+\beta_j||\leq|\log|\delta_i r+\beta_i|-\log|\delta_j s+\beta_j||\leq\eta_2||\delta_i r+\beta_i|-|\delta_j s+\beta_j||, \tag{79}$$

with strictly positive constants η_1, η_2, for any $i \neq 1, j \neq 1$ and for any r, s in the support of μ.

Let us now consider the terms m_{jj} with $j > 1$. Inequalities (78) provide finite upper and lower bounds for the first two terms in the denominator in eq. (74). In addition, since the sign of $\delta_j r + \beta_j$ is the same for all r in the support of μ, inequalities (79) become

$$\eta_1 |\delta_j||r-s| \leq |\log|\delta_j r + \beta_j| - \log|\delta_j s + \beta_j|| \leq \eta_2 |\delta_j||r-s|. \tag{80}$$

Therefore, under assumption 1, the divergence abscissa of $m_{jj}(z)$, $j \neq 1$ is the same as that of the electrostatic energy integral of the measure μ, that we have called $D_2(\mu)$.

We are then left with the terms $m_{ij}(z)$ with $i \neq j$. If we were to assume disconnectedness of the I.F.S., it would be immediate to show that they are analytic functions. In fact, in this case $\delta_j s + \beta_j$ would belong to a branch of the hierarchical structure of the support of μ, separated by a finite distance from the others. Therefore, in this case the thesis would follow.

Yet, we can obtain the result under the more general assumption 1. First, let us assume that both i and j are different from 1. Then, the estimates (78),(79) still apply, so that the divergence abscissa of $m_{ij}(z)$ is the same as that of the integral

$$l_{i,j}(z) = \iint \frac{d\mu(r)d\mu(s)}{||\delta_i r + \beta_i| - |\delta_j s + \beta_j||^z}. \tag{81}$$

We shall back to these functions momentarily. Prior to that, we consider the case of $i = 1$, $j > 1$. We divide the integral $m_{1j}(z)$ in two parts, according to the integration being extended to $|\delta_1 r| \leq 1/2|\delta_j s + \beta_j|$, or to its complementary. In the former case,

$$|\log|\delta_1 r| - \log|\delta_j s + \beta_j|| \geq \log 2,$$

and therefore the related integral is an analytic function. Notice in fact that the integral

$$\iint \frac{d\mu(r)d\mu(s)}{|\delta_1 r|^x |\delta_j s + \beta_j|^x} = \frac{1}{|\delta_1|^x} \int \frac{d\mu(r)}{|r|^x} \int \frac{d\mu(s)}{|\delta_j s + \beta_j|^x}$$

is convergent, for $x < d_0(\mu)$. In the second case region, since $|\delta_j s + \beta_j| \geq \varrho_{\min}$, one has $|\delta_1 r| \geq \varrho_{\min}/2$. Because of these inequalities, $|\delta_1 r|^x |\delta_j s + \beta_j|^x$ is bounded between two finite constants, and moreover the linear inequalities (79) still apply: as a consequence, convergence of the integral over this region is implied by that of the integral

$$\iint \frac{d\mu(r)d\mu(s)}{||\delta_1 r| - |\delta_j s + \beta_j||^z},$$

where the integration is extended to the full domain. This integral has been denoted $l_{1j}(z)$ in eq. (81), so that convergence of $m_{ij}(z)$ is implied by convergence of $l_{ij}(z)$, for all pairs i, j with $i \neq j$.

Now, let us apply the balance relation eq. (7) to the electrostatic energy $\mathcal{E}(\mu_e; z)$ of the measure $\mu_e(s) := \mu(s) + \mu(-s)$ defined in paper I:

$$\mathcal{E}(\mu_e; z) = \iint \frac{d\mu(r) d\mu(s)}{||r| - |s||^z} = \sum_{i,j} \pi_i \pi_j \iint \frac{d\mu(r) d\mu(s)}{||\delta_i r + \beta_i| - |\delta_j s - \beta_j||^z}. \tag{82}$$

By definition, the left hand side has divergence abscissa $D_2(\mu_e) = D_2(\mu)$ (see Lemma 8-I of paper I). We recognize at r.h.s. the integrals $l_{ij}(z)$. Therefore, these are convergent at least as far as $D_2(\mu)$, and at least one of them diverges at that value. Collecting all this information in eq. (77) provides the thesis. □

If we now recall the analysis of Sect. 14 and 15 of paper I, we can draw two sorts of conclusions. First,

Theorem 5 *When assumption 1 is verified for an I.F.S. balanced measure, the square modulus of the Mellin transform $|G_n(\mu; x+iy)|^2$ decays in Cesaro average, on lines parallel to the imaginary axis with $x < d_0(\mu)$, as*

$$\frac{1}{T} \int_{-T}^{T} |G_n(\mu; x+iy)|^2 dy = o(T^{-a}),$$

for all $a < D_2(\mu)$

Proof follows from Thm. 13-I of paper I, and Thm. 4. □
This result is noteworthy: for I.F.S. verifying assumption 1, the electrostatic correlation dimension also governs the asymptotic decay, for large imaginary argument, of the function $|G_n(\mu; x+iy)|^2$. A further consequence of this Theorem is the fact that we can take $\delta(x) = D_2(\mu)/2$ for any $x < d_0(\mu)$ in the analysis of Sect. 15 of paper I:

Proposition 6 *When assumption 1 is verified for an I.F.S. balanced measure, one has that $M_n(\mu; x+iy) \in L^1(\mathbf{R}, dy)$ for any $x < \min\{d_0(\mu), \frac{1+D_2(\mu)}{2}\}$, and consequently $\bar{\mathcal{J}}_n(\mu; t) = t^{-x} o(t)$, when $t \to \infty$.*

Proof. This is Thm. 15-I of paper I. □

10 More on the Fourier Series for F-B. Functions

A useful product of I.F.S. techniques is the following extension of Lemma 3.

Lemma 8 *The Cesaro average of the function $|\Phi_n(\mu; x+iy)|^2$ in the variable y decay as as $o(t^{-D_2(\mu)+\varepsilon})$, for any $\varepsilon > 0$, for all $x \neq d_0(\mu)$. In addition, for all x such that $d_0(\mu) \neq x < \frac{1+D_2(\mu)}{2}$, the function $M_n(\mu; x+iy)$ belongs to $L^1(\mathbf{R}, dy)$.*

Proof. Let $n = 0$. The function $\Phi_0(\mu; z)$ is given explicitly in eq. (44). It can be re-written as a Fourier transform:

$$\Phi_0(\mu; x+iy) = \int d\varsigma_x(u) e^{-iyu}, \tag{83}$$

where the measure ς_x is defined via the usual technique by:

$$\int d\varsigma_x(u)f(u) = \sum_{i=2}^{M} \pi_i \int d\mu(s)|\delta_i s + \beta_i|^{-x} f(\log|\delta_i s + \beta_i|). \tag{84}$$

It follows immediately from this definition and from hypothesis 1 that the measure ς_x is finite, and can be normalized. Its electrostatic energy $\mathcal{E}(\varsigma_x; w)$ can therefore be written as:

$$\mathcal{E}(\varsigma_x; w) = \sum_{i,j=2}^{M} \iint d\mu(r)d\mu(s) \frac{\pi_i \pi_j |\delta_i r + \beta_i|^{-x} |\delta_j s + \beta_j|^{-x}}{|\log|\delta_i r + \beta_i| - \log|\delta_j s + \beta_j||^w} = \sum_{i,j=2}^{M} m_{i,j}(w), \tag{85}$$

where the functions $m_{i,j}(w)$ have been defined in eq. (74). Repeating the proof of Thm. 4 one shows that $D_2(\varsigma_x) = D_2(\mu)$, now for *any* value of x. This entails the decay of the Cesaro average, in the variable y, of the square modulus of $\Phi_0(\mu; x + iy)$, and proves the first part of the thesis.

Next, eq. (47) gives $M_0(\mu; x+iy) = \Phi_0(\mu; x+iy)H(\mu; x+iy)/E(x+iy)$. If $x \neq d_0(\mu)$ the function $|E(x+iy)|$ of the variable y is superiorly bounded, and inferiorly bounded away from zero. Then, the ratio $\frac{H(\mu;x+iy)}{E(x+iy)}$ features the same asymptotic decay as that of the numerator. A similar analysis to that of Sect. 15 of paper I can then be carried over, to show that $M_n(\mu; x+iy) \in L^1(\mathbf{R}, dy)$, for all x such that $d_0(\mu) \neq x < \frac{1+D_2(\mu)}{2}$. This ends the proof for $n = 0$. The case of generic n can now be treated as in Remark 11 of paper I, thanks to the observation that $p_n(\mu; x)$ are bounded functions on the support of μ. □

Observe that $B := \frac{1+D_2(\mu)}{2}$ is necessarily smaller than 1, since $D_2(\mu) \leq D_0(\mu) \leq 1$. When B is larger than $\alpha_0(\mu) = \min\{1, d_0(\mu)\}$, we can return to Theorem 3 of Sect. 7, that can be strengthened as follows:

Theorem 6 *When assumption 1 is verified for an I.F.S., and $d_0(\mu) = \alpha_0(\mu) < \frac{1+D_2(\mu)}{2}$, one can write*

$$\bar{\mathcal{J}}_n(\mu; t) = t^{-\alpha_0(\mu)} \Psi_n(\log t) + N_n(t), \tag{86}$$

where Ψ_n is a periodic function, and $t^{\bar{x}} N_n(t)$ is infinitesimal, as $t \to \infty$, for any $\alpha_0(\mu) < \bar{x} < \frac{1+D_2(\mu)}{2}$.

Proof: develop the same path integral as in the proof of Thms. 2,3, to obtain eq. (61). Of course, the horizontal contribution H_N still vanishes, as N tends to infinity, and the l.h.s. $I_N(x, n, \tau)$ tends uniformly to the limit $i\bar{\mathcal{J}}_n(e^\tau)$. Observe that, in force of Lemma 8, Eq. (62) holds for $\bar{x} < \frac{1+D_2(\mu)}{2}$, and the Fourier series (63) is then uniformly convergent. □

Remark 3 Observe that the uniform convergence of the Fourier series (63) in the region $\alpha_0(\mu) > \frac{1}{2}$ has been obtained without exerting control of the coefficients

$H(z_m)\Phi_0(z_m)$. Indeed, since the latter factor is the pointwise result of the Fourier transform of a measure, it is possible to master its decay as $m \to \infty$. We shall do this in the next section. See also Fig. 5.

11 Discrete Cesaro Averages of F-B. Functions

The Fourier sums appearing in the previous section bring to our attention the decay of discrete Cesaro averages of F-B. functions. In fact, thanks to eq. (83), $\Phi_0(z_m)$ can be written as the Fourier transform of the measure ς_x, with $x = d_0(\mu)$, at the point $m\omega$:

$$\Phi_0(\mu; d_0(\mu) + im\omega) = \int d\varsigma_{d_0(\mu)}(u) e^{-im\omega u}. \tag{87}$$

Consider now the discrete Cesaro averages of the square moduli of these coefficients:

$$\widetilde{|\Phi_0|^2}(N) := \frac{1}{2N+1} \sum_{m=-N}^{N} |\Phi_0(\mu; d_0(\mu) + im\omega)|^2. \tag{88}$$

These are particular instances of a general problem: let μ be a positive measure, and $\mathcal{J}_n(\mu; t)$ the associated F-B. functions. Define their discrete averages as:

$$\widetilde{\mathcal{J}}_n(\mu; N, T) := \frac{1}{2N+1} \sum_{m=-N}^{N} \mathcal{J}_n(\mu; mT). \tag{89}$$

It is possible to prove a discrete analogue of Thm.s 5-I, 9-I of paper I:

Theorem 7 *Let $\alpha_n(\mu)$ be the divergence abscissa associated with $\mathcal{J}_n(\mu; t)$. For all x such that $x < \alpha_n(\mu)$, for all $T > 0$, one has $\widetilde{\mathcal{J}}_n(\mu; N, T) = o(N^{-x})$ when $N \to \infty$.*

Proof. Computing the discrete average, we find

$$\widetilde{\mathcal{J}}_n(\mu; N, T) = \int d\mu(s) p_n(\mu; s) \frac{\sin(N + \frac{1}{2})Ts}{(2N+1)\sin\frac{Ts}{2}}. \tag{90}$$

Therefore,

$$N^x \widetilde{\mathcal{J}}_n(\mu; N, T) = \int \frac{d\mu(s)}{|s|^x} p_n(\mu; s) |Ns|^x \frac{\sin(N + \frac{1}{2})Ts}{(2N+1)\sin\frac{Ts}{2}}.$$

Let $x \in [0, 1]$. Then for any $T > 0$ the kernel $|Ns|^x \frac{\sin(N+\frac{1}{2})Ts}{(2N+1)\sin\frac{Ts}{2}}$ is bounded by a constant that does not depend on N or s. Since $x < \alpha_n(\mu)$, the function $\frac{p_n(\mu;s)}{|s|^x}$ is

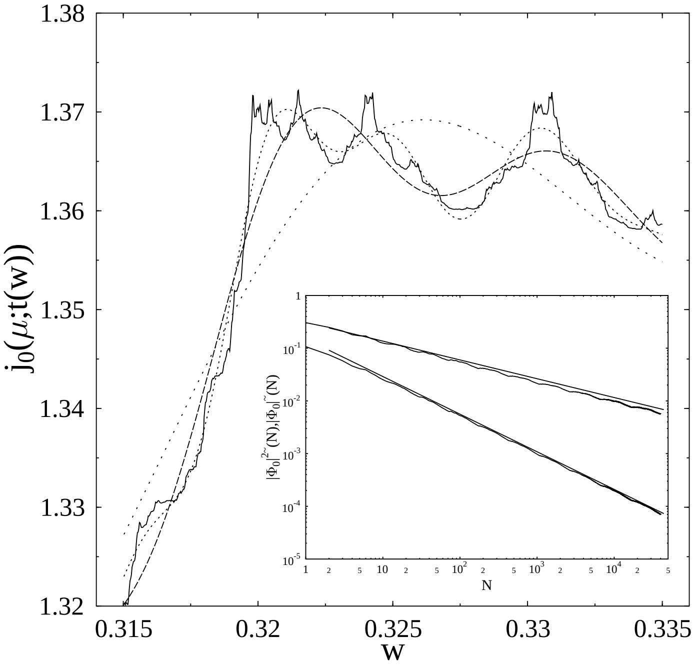

Figure 5: Partial sums of the Fourier series of $j_0(\mu;t)$ for the L.I.F.S. of Fig. 1. The horizontal axis is expressed in the same scaled variable w employed in Fig. 1, and covers the range of the inset (c) of Figure 1. Different curves are for $N = 64$ (dots), $N = 128$ (thick dots), $N = 256$ (dashes) and $N = 40,000$ (continuous line). Since $1/2 < d_0(\mu) < (D_2(\mu)+1)/2$ uniform convergence of the Fourier series is justified by Thm.s 6 and 9. In addition, the inset shows that the discrete Cesaro averages $\widetilde{|\Phi_0|^2}(N)$ (lower curve–accompanying line) and $\widetilde{|\Phi_0|}(N)$ (upper curve–accompanying line) decay as $N^{-D_2(\mu)}$ and $N^{-D_2(\mu)/2}$, respectively.

integrable, and the dominated convergence theorem permits to take the limit for infinite N inside the integral sign. □

Consider now the quadratic functions $\mathcal{J}_n(\mu;t)\mathcal{J}^*_m(\mu;t)$ that were defined in paper I and that we shall study extensively in the next section. Their discrete Cesaro averages can be treated by the same formalism, to obtain

Theorem 8 *Let $\alpha_{n,m}(\mu)$ be the divergence abscissa associated with $\mathcal{J}_n(\mu;t)\mathcal{J}^*_m(\mu;t)$. For all x such that $x < \alpha_{n,m}(\mu)$, for all $T > 0$, one has $\widetilde{\mathcal{J}_n\mathcal{J}^*_m}(\mu;N,T) = o(N^{-x})$ when $N \to \infty$.*

Proof. It is analogous to that of the previous theorem. □
Notice than when $n = m = 0$, $\alpha_{0,0}(\mu) = D_2(\mu)$, and the previous theorem is fully equivalent to Thm. 7 expressed in terms of the correlation measure Ω (see Remark 1 of paper I). We can therefore determine the asymptotic decay of the coefficients $\widetilde{|\Phi|^2}(N)$, and its effect on the convergence of the Fourier series (63).

Theorem 9 *For an I.F.S. verifying assumption 1, one has that $\widetilde{|\Phi|^2}(N) = o(N^{-x})$ when $N \to \infty$, for all $x < D_2(\mu)$. In addition,*

$$\widetilde{|\Phi_0|}(N) := \frac{1}{2N+1} \sum_{m=-N}^{N} |\Phi_0(\mu; d_0(\mu) + im\omega)| = o(N^{-x}) \tag{91}$$

when $N \to \infty$, for all $x < D_2(\mu)/2$. Therefore, the series in eq. (63),

$$\sum_{m=-\infty}^{\infty} \rho^n_m e^{-i\tau\omega m} = \sum_{m=-\infty}^{\infty} H(z_m)\Phi_0(z_m)q_n e^{-i\tau\omega m} \tag{92}$$

converges uniformly, whenever $d_0(\mu) = \alpha_0(\mu) < \frac{1+D_2(\mu)}{2}$.

Proof. The first statement follows from Thm. 8 and the previous remark, with the additional consideration of eq. (87) and of the fact that the correlation dimension of the measure ς_x is $D_2(\mu)$, proven in Lemma 8. The estimate on the decay of the sums of the moduli $|\Phi(z_m)|$ can be obtained as in the proof of Lemma 9 of paper I. Finally, the summability of the Fourier coefficients of eq. (92) follows upon adaptation of the proof of Thm. 15 of paper I. □

Before ending this section we remark that an analogous theory to that of paper I can be developed also for discrete Cesaro averages, by using discrete Mellin transforms. We shall not pursue this idea any further in this paper, because it is now time to change the object of investigation.

12 Quadratic amplitudes for I.F.S. Measures

Let us now consider the quadratic amplitudes $A_{nm}(\mu;t)$ analyzed in paper I:

$$A_{nm}(\mu;t) := \frac{1}{2t}\int_{-t}^{t} \mathcal{J}_n(\mu;t')\mathcal{J}^*_m(\mu;t')\, dt'. \tag{93}$$

In this section, we shall make the following:

Assumption 2 *The I.F.S. is disconnected. That is to say, the distance between $\ell_i(A)$ and $\ell_j(A)$ is strictly positive, for any pair $i \neq j$. Here A is the attractor of the I.F.S.*

Assumption 3 *The I.F.S. is harmonic: the quantities $\ln \delta_i$ are rationally related.*

We observe that hypothesis 3 is equivalent to the existence of a real number $\lambda > 0$, and of $2M$ prime positive integers p_i, q_i, such that

$$- \log \delta_i = \lambda \frac{p_i}{q_i}.$$

When the I.F.S. is disconnected, also hypothesis 1 holds. The analysis that follows is a generalization of that for linear quantities. As we have seen in paper I, the large time behaviour of the crossed amplitudes $A_{nm}(\mu;t)$ is determined by the range of analyticity of their Mellin transforms, $M_{mn}(\mu;z)$. The main result of this chapter are two analogues of Thm.s 2, and 3.

Theorem 10 *Let $D_2(\mu) < \frac{1}{2}$. There exist $2L$ positive numbers $\alpha_1, ..., \alpha_L$ and $\beta_1, ..., \beta_L$ with the property:*

$$D_2(\mu) = \alpha_1 < \alpha_2 \leq ... \leq \alpha_L < \frac{1}{2},$$

such that the quadratic amplitude can be written as

$$A_{nm}(t) = t^{-D_2(\mu)} \Psi_{nm}(\log t) + \sum_{l=2}^{L} t^{-\alpha_l} e^{-i\beta_l \log t} \Psi_{nm,l}(\log t) + N_{nm}(t) \qquad (94)$$

where $\Psi_{nm}(\zeta)$, $\Psi_{nm,l}(\zeta)$ are periodic functions of their argument ζ, and $t^{\bar{x}} N_{nm}(t)$ is infinitesimal, when $t \to \infty$, for any $\bar{x}$ such that $\alpha_L < \bar{x} < \frac{1}{2}$.

Theorem 11 *Let $\frac{1}{2} \leq D_2(\mu) < 1$. There exist $2L$ positive numbers $\alpha_1, ..., \alpha_L$ and $\beta_1, ..., \beta_L$ with the property:*

$$D_2(\mu) = \alpha_1 < \alpha_2 \leq ... \leq \alpha_L < 1,$$

such that the quadratic amplitude can be written as in (94), where $\Psi_{nm}(\zeta)$, $\Psi_{nm,l}(\zeta)$ are periodic functions of ζ, defined in $L^2(d\zeta)$. Moreover, $t^{\bar{x}} N_{nm}(t)$ belongs to $L^2([0,\infty), \frac{dt}{t})$, for any $\bar{x}$ such that $\alpha_L < \bar{x} < 1$.

The method to prove the above theorems is similar to that used for Thms. 2, 3. According to paper I, we write

$$M_{nm}(\mu;z) = G_{nm}(\mu;z) \; H(z). \qquad (95)$$

Let us now apply the balance equation (7) and the expansion (8) to the function $G_{nm}(\mu; z)$, as defined by the double integral

$$G_{nm}(\mu; z) := \iint d\mu(r) d\mu(s) \; \frac{p_n(r) p_m(s)}{|r-s|^z}. \tag{96}$$

We find

$$G_{nm}(\mu; z) = \sum_{\substack{j=1,..,M \\ l=0,..,n \\ i=0,..,m}} \pi_j^2 \delta_j^{-z} \Gamma^n_{jl} \Gamma^m_{ji} \; G_{li}(\mu; z) \; +$$

$$+ \sum_{\substack{j \neq k \\ l=0,..,n \\ i=0,..,m}} \pi_j \pi_k \Gamma^n_{jl} \Gamma^m_{ki} \; \iint d\mu(s) d\mu(r) \; \frac{p_l(s) p_i(r)}{|\ell_j(s) - \ell_k(r)|^z}, \tag{97}$$

where the summation on the indices j and k runs from 1 to M (the number of I.F.S. maps), and where we have separated the diagonal contribution $j = k$, which gives rise to the functions $G_{li}(\mu; z)$. The second grand summation in the above equation defines an analytic function of z, which we shall denote by $\Phi_{nm}(z)$. To prove that this function is analytical, observe that, thanks to the disconnectedness of the I.F.S., the real numbers $|\ell_j(s) - \ell_k(r)|$ are all strictly larger than zero.

Having so taken care of the second term at r.h.s. of eq. (97), we can focus on the first, and split off the $l = n$, $i = m$ term:

$$G_{nm}(\mu; z) = \sum_{j=1}^{M} \pi_j^2 \delta_j^{-z+n+m} \; G_{nm}(\mu; z) + \sum_{\substack{j=1,..,M \\ l+i<n+m}} \pi_j^2 \Gamma^n_{jl} \Gamma^m_{ji} \delta_j^{-z} G_{li}(\mu; z) + \Phi_{nm}(z).$$

To make this last relation stand out in a more transparent way, introduce the following functions:

$$F(z) := 1 - \sum_{j=1}^{M} \pi_j^2 \delta_j^{-z}; \quad \text{and} \quad g^{nm}_{li}(z) := \sum_{j=1}^{M} \pi_j^2 \Gamma^n_{jl} \Gamma^m_{ji} \delta_j^{-z}. \tag{98}$$

They are defined in terms of the map parameters π_j and δ_j, for $j = 1, \ldots, M$, and of the coefficients Γ^n_{jl}, which also depend on the map parameters, although in a highly non-trivial fashion. Then, the functions G_{nm} are linked by the relations

$$G_{nm}(\mu; z) = \frac{1}{F(z-n-m)} \Big[\sum_{\substack{l=0,..,n \\ i=0,..,m \\ l+i<n+m}} \Big(g^{nm}_{li}(z) G_{li}(\mu; z) \Big) \; + \Phi_{nm}(z) \Big], \tag{99}$$

which obviously carries over, via eq. (95), to $M_{nm}(\mu; z)$, and permit the recursive determination of all of them. This recursive determination is the basis of the following Lemma

Lemma 9 *The functions $G_{nm}(\mu;z)$ are meromorphic, and can be analytically continued in all the complex plane, with the exception of poles at the zeros of $F(z-k)$, with $0 \leq k \leq m+n$.*

Proof. As we have already remarked, $\Phi_{nm}(z)$ are analytic functions in all the complex plane. The same goes for all the functions $g^{nm}_{li}(z)$. Therefore, singularities can only occur via the repeated divisions by the denominators $F(z-n-m)$.□

The analogous of Proposition 3 specifies the location of the poles described in Lemma 9.

Lemma 10 *When assumption 2 holds, the poles of $G_{nm}(\mu;z)$ lie in the half plane $\Re(z) \geq D_2(\mu)+n+m$. When hypothesis 3 also holds, there exist three positive integers W, Q, S and a finite set of complex numbers u_l, $l=1,2,...,S$, $u_1 = 1 < |u_2| \leq ... \leq |u_S|$, such that the poles of $G_{nm}(\mu;z)$ are located at the points*

$$z = D_2(\mu)+n+m+\frac{Q}{W\lambda}(\ln u_l + 2\pi i k_l)$$

$$l=1,2,...,S, \qquad k_l = -\infty,...,\infty, \qquad 0 \leq \arg(u_l) < 2\pi$$

Proof. We look for the solutions of $F(z-n-m)=0$. We reproduce the solution of this problem following Makarov [2]. Let us define the frequencies $\omega_j = -\ln\delta_j > 0$. By multiplying and dividing by $\delta_j^{D_2(\mu)}$, we can rewrite our problem in the form:

$$\sum_{j=1}^{M}\left(\frac{\pi_j^2}{\delta_j^{D_2(\mu)}}\right)\, e^{\omega_j(z-D_2(\mu)-n-m)} = 1. \tag{100}$$

For disconnected IFS measures, the correlation dimension $D_2(\mu)$ satisfies the equation $\sum_j \pi_j^2 \delta_j^{-D_2(\mu)} = 1$; using this result, we have solutions of eq. (100) only if there exist j such that $|e^{\omega_j(z-D_2(\mu)-n-m)}| \geq 1$, for otherwise the summation in eq. (100) has modulus strictly less than 1. This implies that the solutions must lie in the half plane $\{\ z \mid \Re e z \geq D_2(\mu)+n+m\}$. In particular, there is always the solution $z = D_2(\mu)+n+m$. In fact, in this case $e^{\omega_j(z-D_2(\mu)-n-m)} = 1$ for all j.
We now give the general solution of problem (100), under the assumption 3. Let $Z = z - D_2(\mu) - n - m$, and $a_i = \pi_i^2\delta_i^{-D_2(\mu)}$: (100) becomes

$$\sum_{i=1}^{M} a_i e^{\omega_i Z} = 1, \qquad \sum_{i=1}^{M} a_i = 1. \tag{101}$$

Following assumption 3, we write $\omega_i = \frac{p_i}{q_i}\lambda$, with p_i, q_i prime integers, $i=1,\dots,M$. Let Q be the minimum common multiple of the denominators $q_1,\dots,q_M$, and let $\lambda Z = QZ'$. Eq. (101) becomes

$$1 = \sum_{i=1}^{M} a_i e^{Q\frac{p_i}{q_i}Z'} = \sum_{i=1}^{M} a_i e^{w_i p_i Z'}, \qquad w_i = \frac{Q}{q_i} \in \mathbf{Z}. \tag{102}$$

The maximum common divisor of the w_i's is 1. Let W be the MCD of p_i's, then, $p_i = WP_i, \quad P_i \in \mathbf{Z}$, for all i. Let $\zeta = WZ'$; eq. (102) then becomes an algebraic equation in the variable $u = e^\zeta = \exp\{\frac{W}{Q}\lambda(z - D_2(\mu) - n - m)\}$, of degree $\max\{w_i P_i\}$:

$$1 = \sum_{i=1}^{M} a_i e^{w_i P_i \zeta} = \sum_{i=1}^{M} a_i u^{w_i P_i}. \tag{103}$$

Since $\sum a_i = 1$, $a_i > 0$, the only positive real root is $u_1 = 1$. There is at least an odd exponent $w_i P_i$ so that $u = -1$ is not a root of the equation. All other roots satisfy

$$|u_l| > 1, \quad l = 2, ..., S; \quad S := \max\{w_i P_i\}.$$

The set of roots can be ordered as follows,

$$u_1 = 1 < |u_2| \leq ... \leq |u_S|$$

so that, going back to the original variable z, the roots of eq. (100) are

$$z = D_2(\mu) + n + m + \frac{Q}{W\lambda}(\ln u_l + 2\pi i\, k_l), \quad l = 1, \ldots, S, \quad k_l = -\infty, \ldots, +\infty \tag{104}$$

with the restriction $0 \leq \arg(u_l) < 2\pi$. $\square$

We choose a sequence of rectangular paths γ_N like in Thms. 2, 3, with vertical sides at $\Re(z) = x$ and $\Re(z) = \bar{x}$, where $0 < x < D_2(\mu)$ and $D_2(\mu) < \bar{x} < \frac{1}{2}$ or $D_2(\mu) < \bar{x} < 1$, depending on $D_2(\mu)$ being less than 1/2 or greater or equal to 1/2. The horizontal sides are $\Im(z) = \pm \frac{Q}{W\lambda} 2\pi \left(N + \frac{1}{2}\right)$. Let $\kappa_N = \{z \in \mathbf{C}$ s.t $z = \tilde{x} \pm \frac{Q}{W\lambda} 2\pi i \left(N + \frac{1}{2}\right), \tilde{x} \in [x, \bar{x}]\}$. We can derive a Lemma equivalent to Lemmas 3 4:

Lemma 11 *The functions $\Psi_{nm}(z)$ are uniformly bounded on κ_N.*

Proof. Since the I.F.S. is disconnected, there exists $\delta_{\min} > 0$ such that when r, s belong to the support of μ, and $i \neq j$, $|\delta_i r + \beta_i - \delta_j s - \beta_j| > \delta_{\min}$. Therefore, when $z \in \kappa_N$,

$$|\Phi_{nm}(\mu; z)| \leq \sum_{\substack{i \neq j \\ l = 0...n \\ h = 0..m}} \pi_i \pi_j\, |\Gamma^n_{il}|\, |\Gamma^m_{jh}|\, ||p_l||_\infty\, ||p_h||_\infty \iint d\mu(r) d\mu(s) |\delta_i r + \beta_i - \delta_j s - \beta_j|^{-\Re z}$$

$$\leq \max\{1, \delta_{\min}^{-\bar{x}}\} \sum_{\substack{i \neq j \\ l = 0...n \\ h = 0..m}} \pi_i \pi_j\, |\Gamma^n_{il}|\, |\Gamma^m_{jh}|\, ||p_l||_\infty\, ||p_h||_\infty \leq B_{nm},$$

where B_{nm} is independent of N and $\tilde{x}$. $\square$

Lemma 12 *There exist positive constants g_{nm} depending on n, m, but not on N, such that $|G_{nm}(\mu; z)| \leq g_{nm}$, for all $z \in \kappa_N$.*

Proof. Let $z \in \kappa_N$. Notice that eq. (104) excludes that any root of (100) may lie on κ_N. Moreover, compute $|F(z-n-m)| =$

$$= \left|1 - \sum_i a_i \exp\{\pm(w_i P_i)\ 2\pi i\ N\} \exp\{\pm(w_i P_i)\pi i\} \exp\left\{\frac{p_i}{q_i}\left(\tilde{x} - D_2(\mu) - n - m\right)\right\}\right|$$

$$= \left|1 - \sum_i \exp\{\pm(w_i P_i)\pi i\} a_i \exp\left\{\frac{p_i}{q_i}\left(\tilde{x} - D_2(\mu) - n - m\right)\right\}\right|. \tag{105}$$

This shows that $|F(z-n-m)|$ is a continuous function of $\tilde{x}$, independent of N, characterized by a finite minimum over the closed interval $[x, \bar{x}]$. Therefore $|F(z-n-m)| \geq C_{nm}$ for $z \in \kappa_N$. This fact and lemma 11 yield:

$$|G_{nm}(z)| \leq \frac{1}{|F(z-n-m)|}\Big(\sum_{\substack{i=1..M\\ l=0..n-1\\ h=0..m-1}} \pi_i^2 \delta_i^{-\bar{x}}\ |\Gamma^n_{il}|\ |\Gamma^m_{ih}|\ |G_{lh}(z)| + |\Phi_{nm}(\mu; z)|\Big) \leq$$

$$\leq \frac{1}{C_{nm}}\Big(\sum_{\substack{i=1..M\\ l=0..n-1\\ h=0..m-1}} \pi_i^2 \delta_i^{-\bar{x}}\ |\Gamma^n_{il}|\ |\Gamma^m_{ih}|\ |G_{lh}(z)| + B_{nm}\Big). \tag{106}$$

When $n = m = 0$, $|G_{00}(z)| \leq C_{00}^{-1} B_{00} := g_{00}$ on κ_N. Then, eq. (106) implies by induction that there exists a constant g_{nm} independent of N such that $|G_{nm}(z)| \leq g_{nm}$, for all $z \in \kappa_N$. $\square$

As a corollary, we have the following

Lemma 13 *The integral of $M_{nm}(z)e^{-z\ln t}$ on the horizontal sides of integration vanishes as $N \to \infty$.*

Proof. Recall that $|H(z_N)| \leq C\left|N + \frac{1}{2}\right|^{-\frac{3}{2}+\bar{x}}$ as $N \to \infty$. Therefore, along the horizontal paths we have:

$$\Big|\int M_{nm}(z)e^{-z\ln t}\frac{dz}{2\pi i}\Big| \leq C g_{nm}\ e^{-x\ln t}\ |\bar{x} - x|\ \left|N + \frac{1}{2}\right|^{-\frac{3}{2}+\bar{x}} \to 0 \text{ as } N \to \infty. \ \square$$

Our next step is the computation of the residues of $M_{nm}(\mu; z)$ at the poles:

$$z_{lk_l} = D_2(\mu) + \frac{Q}{W\lambda}\ln u_l + 2\pi i\frac{Q}{W\lambda}\ k_l, \qquad k_l = -\infty, \ldots, +\infty, \qquad \text{with } |z_{lk_l}| < 1.$$

For $l = 1$, the poles $z_{1k_1} = D_2(\mu) + 2\pi i\frac{Q}{W\lambda}\ k_1$, are simple, because in the recursion formula for $G_{nm}(\mu; z)$ they are due only to $F(z) = 0$ in $G_{00}(\mu; z)$. For $l \geq 2$, only simple poles give non-zero residues. Non simple poles occur if u_l is not a simple root of (103).

Lemma 14 *Let $\rho_{nm,lk_l} := Res_{z=z_{lk_l}} M_{nm}(\mu;z)$ be the residues of $M_{nm}(\mu;z)$ at the (simple) poles z_{lk_l}. They have the form*

$$\rho_{nm,lk_l} = H(z_{lk_l})\ q_{nm,l}\ \Phi_{00}(z_{lk_l})$$

where the coefficients $q_{nm,l}$ are independent of k_l and are recursively determined by the relation (107).

Proof: We denote by $\eta_{nm,lk_l} := Res_{z=z_{lk_l}} G_{nm}(\mu;z)$. We preliminary observe that $\delta_i^{-z_{lk_l}} = \delta_i^{-D_2(\mu)} u_l^{w_i P_i}$. Let's begin with

$$\eta_{00,lk_l} = \mathrm{Res}_{z=z_{lk_l}} G_{00}(\mu;z) = \lim_{z\to z_{lk_l}} (z - z_{lk_l}) \frac{\Phi_{00}(z)}{F(z)} = \Phi_{00}(z_{lk_l}) \lim_{z\to z_{lk_l}} \frac{z - z_{lk_l}}{F(z)}.$$

We compute the last limit:

$$\lim_{z\to z_{lk_l}} \frac{z - z_{lk_l}}{F(z)} = \lim_{z\to z_{lk_l}} \frac{\frac{d}{dz}(z - z_{lk_l})}{\frac{d}{dz}F(z)} = \frac{1}{\sum_i \pi_i^2 \ln\delta_i e^{-z_{lk_l}\ln\delta_i}} = \frac{1}{\sum_i \frac{\pi_i^2}{\delta_i^{D_2(\mu)}} \ln\delta_i\ u_l^{w_i P_i}} =: q_{00,l}$$

Therefore $\eta_{00,lk_l} = q_{00,l}\Phi_{00}(z_{lk_l})$. Now we turn to

$$\eta_{nm,lk_l} = \lim_{z\to z_{lk_l}} (z - z_{lk_l}) G_{nm}(\mu;z) = \frac{1}{F(z_{lk_l} - n - m)} \Big(\sum_{i,r,h} \pi_i^2 \delta_i^{-z_{lk_l}} \Gamma_{ir}^n \Gamma_{ih}^m \eta_{rh,lk_l} + 0 \Big)$$

Moreover

$$F(z_{lk_l} - n - m) = 1 - \sum_i \pi_i^2 \delta_i^{-z_{lk_l}} \delta_i^{n+m} = 1 - \sum_i \frac{\pi_i^2}{\delta_i^{D_2(\mu)}} \delta_i^{n+m}\ u_l^{w_i P_i}.$$

This leads to the recursive procedure

$$q_{00,l} = \frac{1}{\sum_{i=1}^M \frac{\pi_i^2}{\delta_i^{D_2(\mu)}} \ln\delta_i\ u_l^{w_i P_i}};$$

$$q_{nm,l} = \frac{1}{1 - \sum_{i=1}^M \frac{\pi_i^2}{\delta_i^{D_2(\mu)}} \delta_i^{n+m} u_l^{w_i P_i}} \sum_{i=1}^M \sum_{r=0}^{n-1} \sum_{h=0}^{m-1} \frac{\pi_i^2}{\delta_i^{D_2(\mu)}}\ \Gamma_{ir}^n \Gamma_{ih}^m\ u_l^{w_i P_i}\ q_{rh,l}, \quad (107)$$

that permits to determine all quantities $\eta_{nm,lk_l} = q_{nm,l}\Phi_{00}(z_{lk_l})$, as well as the residues of $\rho_{nm,lk_l} = H(z_{lk_l})\eta_{nm,lk_l}$. □

As a corollary, we have the following

Lemma 15 *The residues of $M_{nm}(\mu;z)$ decay, for large k_l, as follows:*

$$\rho_{nm,l\ k_l} \le |q_{nm,l}|\ C\ |k_l|^{-\frac{3}{2}+D_2(\mu)},$$

where C is a costant, and

$$\sum_{k_l=-\infty}^{\infty} |\rho_{nm,lk_l}| < \infty \ \textit{for}\ D_2(\mu) < \frac{1}{2}$$

$$\sum_{k_l=-\infty}^{\infty} |\rho_{nm,lk_l}|^2 < \infty \ \textit{for}\ D_2(\mu) < 1.$$

Proof: This follows from the decay properties of $H(z)$ and from the uniform bound $|\Phi_{00}(z_{l\ k_l})| \leq B_{00}$ which was proven above. □

Proof of Thms. 10, 11: We proceed exactly as in the proof of theorems 2, 3. We integrate over the paths γ_N, and we let $N \to \infty$ while accounting for the residues of the poles getting into the integration contour. Note that we are considering the first L poles z_{lk_l} such that

$$D_2(\mu) = \Re z_{1k_1} < \Re z_{2k_2} \leq ... \leq \Re z_{Lk_L} < \frac{1}{2} \quad \text{if } D_2(\mu) < \frac{1}{2}$$

$$D_2(\mu) = \Re z_{1k_1} < \Re z_{2k_2} \leq ... \leq \Re z_{Lk_L} < 1 \quad \text{if } D_2(\mu) \geq \frac{1}{2}$$

We obtain the following result:

$$A_{nm}(t) = -t^{-D_2(\mu)} \sum_{k_1=-\infty}^{\infty} \rho_{nm,1k_1} e^{-i\left(2\pi \frac{Q}{W\lambda}\right)\ k_1\ \log t} +$$

$$-t^{-D_2(\mu)} \sum_{l=2}^{L} t^{-\frac{Q}{W\lambda}\log|u_l|}\ e^{-i\frac{Q}{W\lambda}\arg u_l\ \log t} \sum_{k_l=-\infty}^{\infty} \rho_{nm,lk_l}\ e^{-i(2\pi\frac{Q}{W\lambda})\ k_l\ \log t} +$$

$$+N_{nm}(t). \tag{108}$$

The series in the above are uniformly convergent if $D_2(\mu) < \frac{1}{2}$. They converge in $L^2(\mathbf{R}, d\log t) = L^2([0,\infty), \frac{dt}{t})$ if $\frac{1}{2} \leq D_2(\mu) < 1$. Moreover $N_{nm}(t) = t^{-\bar{x}}o(t)$ in the first case, or $t^{\bar{x}}N_{nm} \in L^2([0,\infty), \frac{dt}{t})$, in the second case (it is obtained as a Fourier-Plancherel transform by the limit $N \to \infty$ of the integral on the vertical line $\Re z = \bar{x}$). We note that $\log|u_l| > 0$, $l \geq 2$. The expansion (108) is eq. (94). □

In this fashion, we have obtained the Fourier series representation of the periodic functions Ψ_{nm} and $\Psi_{nm,l}$. These latter are generated by the periodic arrangement of poles in the complex plane. When the rationality hypothesis is violated, only a single pole is to be found with $\Re(z) = D_2(\mu)$, and Ψ_{nm} becomes a constant. The remaining poles in the complex plane give contributions that decay faster than $t^{-D_2(\mu)}$, as t tends to infinity. An analysis of both the rational and the irrational case for linear I.F.S. measures can be found also in [19, 20, 21, 22].

13 Conclusions

The properties of the Fourier transform of singular measures have been studied extensively both in the mathematical literature [2], [4], [22], [23, 24, 25, 26], [27] and in the physical [1] [28],[29], as described succinctly in paper I. This concept is generalized by that of Fourier–Bessel functions. In this paper we have examined in detail the asymptotic properties of the F-B. functions when the orthogonality measure is the invariant measure of an Iterated Functions System. We have shown that the analititicy structure of the Mellin transform of F-B. functions fully explains these properties.

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