New product formulæ and quantum Zeno dynamics with generalized observables

Pavel Exner[a], Takashi Ichinose[b], Hagen Neidhardt[c], and Valentin A. Zagrebnov[d]

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a) Department of Theoretical Physics, NPI, Academy of Sciences, CZ-25068 Řež, and Doppler Institute, Czech Technical University, Břehová 7, CZ-11519 Prague, Czech Republic
b) Department of Mathematics, Faculty of Science, Kanazawa University, Kanazawa 920-1192, Japan
c) Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstr. 39, D-10117 Berlin, Germany
d) Départment de Physique, Université de la Méditerranée (Aix-Marseille II) and Centre de Physique Théorique, CNRS, Luminy Case 907, F-13288 Marseille Cedex 9, France
exner@ujf.cas.cz, ichinose@kappa.s.kanazawa-u.ac.jp, neidhard@wias-berlin.de, zagrebnov@cpt.univ-mrs.fr

Abstract

We demonstrate a pair of new product formulæ which combine a projection with a resolvent of a positive operator, or with an exponential function and spectral projections, respectively. The convergence is strong or even operator-norm under more restrictive assumptions. The second mentioned formula can be used to describe Zeno dynamics in the situation when the usual non-decay measurement is replaced by a particular generalized observable in the sense of Davies.

1 Introduction

Product formulæ are a traditional tool in various branches of mathematics; their use dates back to the time of Sophus Lie. In more recent times most attention was paid to Trotter formula combining two semigroups or unitary groups, and its various generalizations, motivated in part by the usefulness of this tool in functional integration, quantum statistical physics and other parts of physics – see, e.g., [Ex, Chap. V], [Za] and references therein. The last decade brought a progress, with the participation of some of the present authors, in understanding of the convergence properties of such formulæ – for a review of these results we refer to the monograph [Za].

In the last few years we have witnessed a surge of interest to another type of product formulæ in which a (semi)group is combined with a projection operator motivated by the "quantum Zeno effect" (QZE). This is also a venerable problem known already to Alan Turing and formulated in the usual decay context for the first time by Beskow and Nilsson [BN67]: frequent measurements can slow down a decay of an unstable system, or even fully stop it in the limit of infinite measurement frequency. The effect was analyzed mathematically by Friedman [Fr71] but became popular only after the authors of [MS77] invented the above stated name. Recent interest is motivated mainly by the fact that now the effect is within experimental reach; an up-to-date bibliography can be found, e.g., in [FMP04] or [Sch04].

At the same time there are still important unanswered questions. The central among them concerns the existence and properties of the Zeno dynamics. To explain it, recall the usual way in which quantum kinematics of decays is described [Ex, Chap. I]. The unstable system is characterized by a projection P to a subspace in the state Hilbert space $\mathfrak{H}$ of a larger, isolated system, the dynamics of which is governed by a self-adjoint Hamiltonian H. Repeating the non-decay measurement with the period t/n, we can describe the time evolution over the interval $[0, t]$ of a state originally in the subspace $P\mathfrak{H}$ by the interlaced product $(Pe^{-itH/n}P)^n$; the question is how this operator will behave as $n \to \infty$. By QZE in the sense [Fr71] we mean that such a limit will preserve the norm of the original state, while quantum Zeno dynamics refers to the limiting operator itself and its dependence upon the time t.

It is clear that there are Hamiltonians for which the question does not make sense, a trivial example being constructed using the momentum operator which generated a shift group in $L^2(\mathbb{R})$, hence it is reasonable to assume that H is semibounded (for a physicist it means, of course, semibounded from below). However, this assumption is not sufficient to justify the "natural" guess about the limiting operator, namely that it might be equal to e^{-itH_P} with $H_P = PHP$; notice that in general the last operator may not even be closed. Examples can be found – see [Ex, Rem. 2.4.9] or [MS03] – that the limit of $(Pe^{-itH/n}P)^n$ may not exist if the operator PHP is "too small", or more exactly, if $\mathrm{dom}(H^{1/2}) \cap \mathfrak{h}$ is not dense in $\mathfrak{h}$. It appears thus to be more reasonable to replace it by $H_P = (\sqrt{H}P)^*\sqrt{H}P$ assuming that the closed operator $\sqrt{H}P$ is densely defined, then the question is in which sense the limit of $(Pe^{-itH/n}P)^n$ exists as $n \to \infty$ and whether it equals e^{-itH_P} with the last named generator.

A partial answer has been given in [EI03] where the limit existence is demonstrated for almost all t in the strong operator topology, along a subsequence $\{n'\}$ of natural numbers[1], or more generally, without taking a subsequence but with respect to a slightly weaker topology which involves averaging in the parameter t, see Remark 3.5. The reason behind this restriction is that the exponential function involved in the interlaced product gives rise to oscillations which are not easy to deal with. One of the main ingredients in the present paper is a simple observation that one can avoid the mentioned problem when $f(s) = e^{-is}$ is replaced by other functions which will allow us to use a positivity-type argument. We will illustrate this idea in the next section on the case when the unitary group is replaced by the resolvent, i.e. $f(s) = (1+is)^{-1}$. While such a formula is not directly related to our physical motivation described above, it represents a mathematical interest of its own[2].

In Sec. 3 we return to the exponential function and show that the strong convergence can be demonstrated if we modify the original problem setting by multiplying the unitary group with a spectral projection of the Hamiltonian corresponding to a suitable interval, with the sequence of this intervals expanding as $n \to \infty$ – cf. Thm 3.3. The question is whether such a product formula has something in common with the Zeno problem which assumes repeated non-decay measurements. The meaning of the factors which enter the modified product is obvious: we start with ascertaining that the state is in the unstable-system subspace $\mathfrak{h} = P\mathfrak{H}$ and let it evolve for the time t/n. Then, before checking again that it has not decayed we perform an

[1]This fact was omitted in the first version of the paper from which the claim was reproduced in the review [Sch04]. A complete answer is at the present time known only in the case when P is finite-dimensional – cf. [EI03].

[2]In particular, our argument provides an easy way to show that for resolvents in the recent result of [Ca04] one need not resort to subsequences or to a weaker topology.

additional yes-no experiment [Ja] which consists of "energy filtering", that is, we let the system pass if the measured energy value falls into the interval $[0, \pi n/t)$. After the non-decay measurement characterized by the projection P we let the system evolve again for the time t/n and keep repeating the whole procedure over the whole time interval $[0, t]$.

Let us mention that from the mathematical point of view there is a relation between the Zeno product formula with an energy filtering and the unitary Lie-Trotter product formula. In fact, the ideas to employ a spectral cut-off together with the exponential function, and to replace the unitary group by a the resolvent are not new: they were used to derive a modification of the unitary Lie-Trotter formula in [Ich80] and [La81, La85], respectively, both for the form sum of two non-negative self-adjoint operators.

Notice now that the combination of the energy filtering and non-decay measurement following immediately one after another can be regarded as *a single generalized measurement.* In fact, a product of two, in general non-commuting[3] projections represents the simplest non-trivial example of generalized observables[4] realized as positive maps of the respective space of density matrices [Da1, Sec. 2.1]. Thus our mathematical result corresponds to a modified Zeno situation with such generalized measurements, which depend on n and tend to the standard non-decay yes-no experiment as $n \to \infty$.

In the last two sections we will show how the convergence properties of such product formulæ can be improved, in particular, demonstrating more restrictive assumptions on the operators involved which allow us to pass to the operator-norm topology. We will also comment briefly on possible extensions of the present results.

2 Resolvent case

Let $H = H^* \geq 0$ be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ and let P be an orthogonal projection onto a subspace $\mathfrak{h} \subseteq \mathfrak{H}$. The positivity assumption is made only for convenience; the validity of the results will

[3]We are primarily interested, of course, in the nontrivial case when the P does not commute with H, and thus also with the spectral projections $E_H([0, \pi n/t))$.

[4]Since the spectral projections involved commute with the evolution operator, one can also replace the product $PE_H([0, \pi t/n))$ in our formulæ by $E_H([0, \pi t/n))PE_H([0, \pi t/n))$. Such generalized observables represented by symmetrized projection products have been recently studied as *almost sharp quantum effects* – cf. [AG04].

extend easily to any semibounded H. We set

$$F(\tau) := P(I + i\tau H)^{-1}P : \mathfrak{h} \longrightarrow \mathfrak{h}\,, \quad \tau > 0\,, \tag{2.1}$$

and

$$S(\tau) := \frac{I_{\mathfrak{h}} - F(\tau)}{\tau} : \mathfrak{h} \longrightarrow \mathfrak{h}\,, \quad \tau > 0\,, \tag{2.2}$$

where $I_{\mathfrak{h}}$ is the identity operator in the subspace $\mathfrak{h}$. In the following we use the notation PXP for the restriction of an operator X acting in $\mathfrak{H}$ to the subspace $\mathfrak{h}$, i.e., we set $PXP := PX \upharpoonright \mathfrak{h}$. Let us assume that

$$\mathrm{dom}(T) := \mathrm{dom}(\sqrt{H}) \cap \mathfrak{h} \tag{2.3}$$

is dense in $\mathfrak{h}$. We set $T : \mathfrak{h} \longrightarrow \mathfrak{H}$,

$$Tf := \sqrt{H}f, \quad f \in \mathrm{dom}(T). \tag{2.4}$$

Since the operator $\sqrt{H}$ is closed the operator T is also closed. Let

$$K := T^*T : \mathfrak{h} \longrightarrow \mathfrak{h} \tag{2.5}$$

which defines a non-negative self-adjoint operator in $\mathfrak{h}$. Notice that K corresponds to the sesquilinear form $\mathfrak{k}$,

$$\mathfrak{k}(f, g) = (\sqrt{H}f, \sqrt{H}g)\,, \quad f, g \in \mathrm{dom}(T)\,. \tag{2.6}$$

Remark 2.1 This definition allows us to write the operator $H_P = (\sqrt{H}P)^*\sqrt{H}P$ mentioned in the introduction as $K \oplus 0$ and its exponential as $e^{-itK} \oplus I_{\mathfrak{h}^\perp}$. We will concentrate on the nontrivial part of the problem only and formulate the claims below for convergence in $\mathfrak{h}$.

¿From (2.2) on gets in a straightforward way the representation

$$S(\tau) = \tau P \frac{H^2}{I + \tau^2 H^2} P + iP \frac{H}{I + \tau^2 H^2} P\,, \tag{2.7}$$

hence

$$iI_{\mathfrak{h}} + S(\tau) = iI_{\mathfrak{h}} + \tau P \frac{H^2}{I + \tau^2 H^2} P + iP \frac{H}{I + \tau^2 H^2} P\,. \tag{2.8}$$

Then obviously we have the representation

$$
\begin{aligned}
iI_{\mathfrak{h}} + S(\tau) = &\left(I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P\right)^{1/2} \times \\
&\times \{iI_{\mathfrak{h}} + M(\tau)\} \left(I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P\right)^{1/2}, \quad \tau > 0\,,
\end{aligned}
\tag{2.9}
$$

where we put

$$
\begin{aligned}
M(\tau) := \tau &\left(I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P\right)^{-1/2} \times \\
&\times P\frac{H^2}{I+\tau^2H^2}P \left(I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P\right)^{-1/2}, \quad \tau > 0\,.
\end{aligned}
$$

The relation (2.9) yields

$$
\begin{aligned}
(iI_{\mathfrak{h}} + S(\tau))^{-1} = &\left(I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P\right)^{-1/2} \times \\
&\times \{iI_{\mathfrak{h}} + M(\tau)\}^{-1} \left(I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P\right)^{-1/2}, \quad \tau > 0\,.
\end{aligned}
\tag{2.10}
$$

Lemma 2.2 *Let H be a non-negative self-adjoint operator in $\mathfrak{H}$ and let $\mathfrak{h}$ be a subspace of $\mathfrak{H}$. If $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ is dense in $\mathfrak{h}$, then*

$$
\operatorname*{s-lim}_{\tau\to 0} \left(I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P\right)^{-1} = (I_{\mathfrak{h}} + K)^{-1}. \tag{2.11}
$$

Proof. Let us introduce a family of sesquilinear forms:

$$
\mathfrak{k}_\tau(f,g) = \left(\left(I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P\right) f, g\right), \quad f,g \in \mathfrak{h}\,, \quad \tau > 0\,.
$$

A straightforward computation shows that sequence is monotone,

$$
\mathfrak{k}_{\tau_1}(f,f) \le \mathfrak{k}_{\tau_2}(f,f)\,, \quad 0 < \tau_2 \le \tau_1\,, \quad f \in \mathfrak{h}\,.
$$

Since $\mathfrak{k}_\tau \le \mathfrak{k}$ for $\tau > 0$, and $\lim_{\tau\to 0} \mathfrak{k}_\tau(f,f) = \mathfrak{k}(f,f)$, $f \in \mathrm{dom}(\mathfrak{k})$, the assertion expressed by (2.11) follows from Theorem VIII.3.13 of [Ka]. □

Further we need the following general but simple statement.

Lemma 2.3 *Let* $\{X(\tau)\}_{\tau>0}$, $\{Y(\tau)\}_{\tau>0}$, *and* $\{A(\tau)\}_{\tau>0}$ *be families of bounded non-negative self-adjoint operators in* $\mathfrak{h}$ *such that the condition*

$$0 \le X(\tau) \le A(\tau) \le Y(\tau)\,, \quad \tau > 0\,,$$

is satisfied. If $s-\lim_{\tau\to 0} X(\tau) = s-\lim_{\tau\to 0} Y(\tau) = A$, *where* A *is a bounded self-adjoint operator in* $\mathfrak{h}$, *then* $s-\lim_{\tau\to 0} A(\tau) = A$.

Proof. Since for each $f \in \mathfrak{h}$ we have

$$(X(\tau)f, f) \le (A(\tau)f, f) \le (Y(\tau)f, f)\,, \quad \tau > 0\,,$$

we get $\lim_{\tau\to 0}(A(\tau)f, f) = (Af, f)$, $f \in \mathfrak{h}$, or $w-\lim_{\tau\to 0} A(\tau) = A$. By the Löwner-Heinz inequality [BS] we find

$$0 \le X(\tau)^{1/2} \le A(\tau)^{1/2} \le Y(\tau)^{1/2}, \quad \tau > 0\,.$$

Since $s-\lim_{\tau\to 0} X(\tau)^{1/2} = s-\lim_{\tau\to 0} Y(\tau)^{1/2} = A^{1/2}$, we obtain the weak convergence $w-\lim_{\tau\to 0} A(\tau)^{1/2} = A^{1/2}$. By

$$\left\|A(\tau)^{1/2}f - A^{1/2}f\right\|^2 = (A(\tau)f, f) - 2\Re\mathrm{e}\,(A(\tau)^{1/2}f, A^{1/2}f) + (Af, f)$$

we get $\text{s-lim}_{\tau\to 0} A(\tau)^{1/2} = A^{1/2}$, which yields $\text{s-lim}_{\tau\to 0} A(\tau) = A$. $\square$

Let us now consider the operator family

$$L(\tau) := P\frac{H}{I + \tau^2 H^2}P + \tau P\frac{H^2}{I + \tau^2 H^2}P : \mathfrak{h} \longrightarrow \mathfrak{h}\,, \quad \tau > 0\,, \tag{2.12}$$

for which we can prove a claim analogous to Lemma 2.2.

Lemma 2.4 *Under the assumptions of Lemma 2.2 we have*

$$\operatorname*{s-lim}_{\tau\to 0}(I_{\mathfrak{h}} + L(\tau))^{-1} = (I_{\mathfrak{h}} + K)^{-1}. \tag{2.13}$$

Proof. First we note that

$$(I_{\mathfrak{h}} + L(\tau))^{-1} \le \left(I_{\mathfrak{h}} + P\frac{H}{I + \tau^2 H^2}P\right)^{-1}, \quad \tau > 0\,. \tag{2.14}$$

Since

$$I_{\mathfrak{h}} + P\frac{H}{I + \tau^2 H^2}P \le I_{\mathfrak{h}} + K\,, \quad \tau > 0\,,$$

we find

$$I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P + \tau P\frac{H^2}{I+\tau^2H^2}P \le I_{\mathfrak{h}} + K + \tau P\frac{H^2}{I+\tau^2H^2}P\,, \quad \tau > 0\,,$$

which in turn yields

$$\left(I_{\mathfrak{h}} + K + \tau P\frac{H^2}{I+\tau^2H^2}P\right)^{-1} \le (I_{\mathfrak{h}} + L(\tau))^{-1}, \quad \tau > 0\,.$$

Hence we get a two-sided estimate,

$$\begin{aligned}\left(I_{\mathfrak{h}} + K + \tau P\frac{H^2}{I+\tau^2H^2}P\right)^{-1} &\le \\ \le (I_{\mathfrak{h}} + L(\tau))^{-1} \le \left(I_{\mathfrak{h}} + P\frac{H}{I+\tau^2H^2}P\right)^{-1}, &\quad \tau > 0\,,\end{aligned} \tag{2.15}$$

where the left-hand side can be represented as follows,

$$\begin{aligned}\left(I_{\mathfrak{h}} + K + \tau P\frac{H^2}{I+\tau^2H^2}P\right)^{-1} &= (I_{\mathfrak{h}} + K)^{-1/2} \times \\ \times \left(I_{\mathfrak{h}} + \tau (I_{\mathfrak{h}} + K)^{-1/2}P\frac{H^2}{I+\tau^2H^2}P(I_{\mathfrak{h}} + K)^{-1/2}\right)^{-1} &\times \\ \times (I_{\mathfrak{h}} + K)^{-1/2}, \quad \tau > 0\,.\end{aligned} \tag{2.16}$$

By virtue of (2.6) we infer that

$$\begin{aligned}&(\sqrt{H}P(I_{\mathfrak{h}} + K)^{-1/2}f, \sqrt{H}P(I_{\mathfrak{h}} + K)^{-1/2}f) = \\ &\quad = \mathfrak{k}((I_{\mathfrak{h}} + K)^{-1/2}f, (I_{\mathfrak{h}} + K)^{-1/2}f) = \|\sqrt{K}(I_{\mathfrak{h}} + K)^{-1/2}f\|^2 \le \|f\|^2,\end{aligned}$$

for any $f \in \mathfrak{h}$, which yields

$$\|Wf\| \le \|f\|\,, \quad f \in \mathfrak{h}\,,$$

where $W := \sqrt{H}P(I_{\mathfrak{h}} + K)^{-1/2}$. From the identity

$$\begin{aligned}&\left(\frac{\tau H}{I+\tau^2H^2}Wf, Wg\right) = \\ &\quad \left(\frac{\tau H}{I+\tau^2H^2}\sqrt{H}P(I_{\mathfrak{h}} + K)^{-1/2}f, \sqrt{H}P(I_{\mathfrak{h}} + K)^{-1/2}g\right)\end{aligned}$$

we obtain

$$\tau(I_{\mathfrak{h}} + K)^{-1/2} P \frac{H^2}{I + \tau^2 H^2} P (I_{\mathfrak{h}} + K)^{-1/2} = W^* \frac{\tau H}{I + \tau^2 H^2} W \, .$$

Since

$$\operatorname*{s-lim}_{\tau \to 0} \frac{\tau H}{1 + \tau^2 H^2} = 0,$$

and W is bounded we conclude that

$$\operatorname*{s-lim}_{\tau \to 0} (I_{\mathfrak{h}} + K)^{-1/2} P \frac{\tau H^2}{I + \tau^2 H^2} P (I_{\mathfrak{h}} + K)^{-1/2} = 0 \, .$$

Using the representation (2.16) we arrive at

$$\operatorname*{s-lim}_{\tau \to 0} \left(I_{\mathfrak{h}} + K + \tau P \frac{H^2}{I + \tau^2 H^2} P \right)^{-1} = (I_{\mathfrak{h}} + K)^{-1} \, ;$$

taking now into account the inequalities (2.15), Lemma 2.2 and applying Lemma 2.3 we complete the proof. ☐

Now we are in position to prove the main result of this section.

Theorem 2.5 *Let H be a non-negative self-adjoint operator in $\mathfrak{H}$ and let $\mathfrak{h}$ be a closed subspace of $\mathfrak{H}$. If $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ is dense in $\mathfrak{h}$, then we have for any $t_0 > 0$ the limit*

$$\operatorname*{s-lim}_{n \to \infty} \left(P(I + itH/n)^{-1} P \right)^n = e^{-itK} \, ,$$

uniformly in $t \in [0, t_0]$, where the generator K is defined by (2.6).

Proof. Since

$$\begin{aligned} &(I_{\mathfrak{h}} + L(\tau))^{-1} = \\ &\quad \left(I_{\mathfrak{h}} + P \frac{H}{I + \tau^2 H^2} P \right)^{-1/2} (I + M(\tau))^{-1} \left(I_{\mathfrak{h}} + P \frac{H}{I + \tau^2 H^2} P \right)^{-1/2} \end{aligned}$$

holds for any $\tau > 0$, by Lemmata 2.2 and 2.4 we infer that

$$\operatorname*{w-lim}_{\tau \to 0} (I_{\mathfrak{h}} + M(\tau))^{-1} = I_{\mathfrak{h}} \, ,$$

which yields

$$\operatorname*{s-lim}_{\tau\to 0}\left(I_{\mathfrak{h}} - (I_{\mathfrak{h}} + M(\tau))^{-1}\right)^{1/2} = 0\,.$$

Hence the above limit is strong at the same time,

$$\operatorname*{s-lim}_{\tau\to 0}(I_{\mathfrak{h}} + M(\tau))^{-1} = I_{\mathfrak{h}}\,.$$

By the resolvent identity this gives

$$\operatorname*{s-lim}_{\tau\to 0}(iI_{\mathfrak{h}} + M(\tau))^{-1} = -iI_{\mathfrak{h}}\,;$$

then using the representation (2.10) and applying Lemma 2.2 we find

$$\operatorname*{s-lim}_{\tau\to 0}(iI_{\mathfrak{h}} + S(\tau))^{-1} = -i(I_{\mathfrak{h}} + K)^{-1}$$

or, by the resolvent identity again,

$$\operatorname*{s-lim}_{\tau\to 0}(I_{\mathfrak{h}} + S(\tau))^{-1} = (I_{\mathfrak{h}} + iK)^{-1}\,.$$

The sought result then follows from Chernoff's theorem [Ch74], see also Lemma 3.29 of [Da2]. $\square$

3 Exponential case

Let us now pass to the second problem described in the introduction. We shall consider the operator family

$$F(\tau) := PE_H(\Delta_\tau)e^{-i\tau H}P : \mathfrak{h} \longrightarrow \mathfrak{h}\,, \quad \tau > 0\,,$$

where $E_H(\cdot)$ is the spectral measure corresponding to H and

$$\Delta_\tau := [0, \pi/\tau)\,.$$

Furthermore, in analogy with (2.2) we set

$$S(\tau) := \frac{I_{\mathfrak{h}} - F(\tau)}{\tau} : \mathfrak{h} \longrightarrow \mathfrak{h}\,, \quad \tau > 0\,.$$

This operator family has the representation

$$S(\tau) = \frac{1}{\tau}P(I - E_H(\Delta_\tau)\cos(\tau H))P + i\frac{1}{\tau}PE_H(\Delta_\tau)\sin(\tau H)P\,.$$

By the factorization

$$\begin{aligned} iI_{\mathfrak{h}} + S(\tau) = &\left(I_{\mathfrak{h}} + \frac{1}{\tau} P E_H(\Delta_\tau)\sin(\tau H)P\right)^{1/2} \times \\ &\times \{iI_{\mathfrak{h}} + M(\tau)\}\left(I_{\mathfrak{h}} + \frac{1}{\tau} P E_H(\Delta_\tau)\sin(\tau H)P\right)^{1/2}, \end{aligned}$$

where

$$\begin{aligned} M(\tau) := &\left(I_{\mathfrak{h}} + \frac{1}{\tau} P E_H(\Delta_\tau)\sin(\tau H)P\right)^{-1/2} \times \\ &\times \frac{1}{\tau} P(I - E_H(\Delta_\tau)\cos(\tau H))P\left(I_{\mathfrak{h}} + \frac{1}{\tau} P E_H(\Delta_\tau)\sin(\tau H)P\right)^{-1/2}, \end{aligned}$$

one gets

$$\begin{aligned} (iI_{\mathfrak{h}} + S(\tau))^{-1} = &\left(I_{\mathfrak{h}} + \frac{1}{\tau} P E_H(\Delta_\tau)\sin(\tau H)P\right)^{-1/2} \times \\ &\times \{iI_{\mathfrak{h}} + M(\tau)\}^{-1}\left(I_{\mathfrak{h}} + \frac{1}{\tau} P E_H(\Delta_\tau)\sin(\tau H)P\right)^{-1/2}. \end{aligned} \tag{3.1}$$

Recall that a piecewise smooth[5] function $f : \mathbb{R}_+ \to [0,1]$ is called a *Kato function* if $f(0) = 1$ and $f'(0+) = -1$.

Lemma 3.1 *Let H be a non-negative self-adjoint operator in $\mathfrak{H}$ and let $\mathfrak{h}$ a subspace of $\mathfrak{H}$. If $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ is dense in $\mathfrak{h}$ and a Kato function g satisfies the condition*

$$\sup_{\lambda>0} \frac{1-g(\lambda)}{\lambda} \le 1\,,$$

then we have

$$\operatorname*{s-lim}_{\tau\to 0}\left(I_{\mathfrak{h}} + \frac{1}{\tau} P(I - g(\tau H))P\right)^{-1} = (I_{\mathfrak{h}} + K)^{-1}. \tag{3.2}$$

Proof. By assumption of the function g we have

$$\left(\frac{1}{\tau} P(I - g(\tau H))Pf, f\right) \le (\sqrt{H}Pf, \sqrt{H}Pf)\,, \quad f \in \mathrm{dom}(T)\,,$$

[5] This notion appears in the literature with different regularity requirements; the present form is sufficient for purposes of the present paper – see [Za, Chap. 4].

or equivalently

$$\frac{1}{\tau}P(I-g(\tau H))P \le K\,, \quad \tau>0\,,$$

which implies

$$(I_{\mathfrak{h}}+K)^{-1} \le \left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))P\right)^{-1}\,, \quad \tau>0\,. \tag{3.3}$$

Furthermore, we have

$$\frac{1}{\tau}P(I-g(\tau H))E_H([0,a))P \le \frac{1}{\tau}P(I-g(\tau H))P\,, \quad a>0\,,$$

which shows that

$$\begin{aligned}\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))P\right)^{-1} &\le \\ \times\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))E_H([0,a))P\right)^{-1}&, \quad a>0\,.\end{aligned} \tag{3.4}$$

It is obvious that

$$\operatorname*{s-lim}_{\tau\to 0}\frac{1}{\tau}P(I-g(\tau H))E_H([0,a))P = PHE_H([0,a))P\,, \quad a>0\,,$$

and moreover,

$$\operatorname*{s-lim}_{\tau\to 0}\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))E_H([0,a))P\right)^{-1} = (I_{\mathfrak{h}}+PHE_H([0,a))P)^{-1} \tag{3.5}$$

holds for all $a>0$ by Chernoff's theorem [Ch74], see also Lemma 3.29 of [Da2]. Since $\{PHE_H([0,a))P\}_{a>0}$, is a monotone increasing family of operators, which obeys $PHE_H([0,a))P \le K$ and

$$\lim_{a\to\infty}(E_H([0,a))\sqrt{H}f,\sqrt{H}f) = (\sqrt{H}f,\sqrt{H}f)\,, \quad f\in\mathrm{dom}(T)\,,$$

by Theorem VIII.3.13 of [Ka] we get that

$$\operatorname*{s-lim}_{a\to\infty}(I_{\mathfrak{h}}+PHE_H([0,a))P)^{-1} = (I_{\mathfrak{h}}+K)^{-1}.$$

¿From (3.4) and (3.5) we obtain

$$\begin{aligned}\limsup_{\tau\to 0}\left(\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))P\right)^{-1}f,f\right) &\leq \\ &\leq \left((I_{\mathfrak{h}}+PHE_H([0,a))P)^{-1}f,f\right)\,,\quad a>0,\quad f\in\mathfrak{h}\,,\end{aligned}$$

which yields

$$\limsup_{\tau\to 0}\left(\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))P\right)^{-1}f,f\right)\leq\left((I_{\mathfrak{h}}+K)^{-1}f,f\right)\,,\quad f\in\mathfrak{h}\,.$$

Using (3.3) we obtain

$$\left((I_{\mathfrak{h}}+K)^{-1}f,f\right)\leq\liminf_{\tau\to 0}\left(\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))P\right)^{-1}f,f\right)\,,\quad f\in\mathfrak{h}\,.$$

and therefore

$$\lim_{\tau\to 0}\left(\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))P\right)^{-1}f,f\right)=\left((I_{\mathfrak{h}}+K)^{-1}f,f\right)\,,\quad f\in\mathfrak{h}\,.$$

This yields

$$\operatorname*{w-lim}_{\tau\to 0}\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))P\right)^{-1}=(I_{\mathfrak{h}}+K)^{-1}\,. \tag{3.6}$$

Using once more the Löwner-Heinz inequality and taking into account (3.3) and (3.4) we infer that

$$\begin{aligned}(I_{\mathfrak{h}}+K)^{-1/2}\leq\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))P\right)^{-1/2} &\leq \\ \leq\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))E_H([0,a))P\right)^{-1/2}&,\quad a>0\,,\quad \tau>0\,.\end{aligned}$$

Mimicking the reasoning of the previous section we get

$$\operatorname*{w-lim}_{\tau\to 0}\left(I_{\mathfrak{h}}+\frac{1}{\tau}P(I-g(\tau H))P\right)^{-1/2}=(I_{\mathfrak{h}}+K)^{-1/2}\,, \tag{3.7}$$

and (3.6) in combination with (3.7) finally yield the relation (3.2) $\square$

Now we set

$$g(\lambda) := \begin{cases} 1 - \sin(\lambda) & \dots \quad \lambda \in [0, \pi) \\ 1 & \dots \quad \lambda \in [\pi, \infty) \end{cases}$$

One easily verifies that $g(\cdot)$ is a Kato function, and by functional calculus

$$I - g(\tau H) = \sin(\tau H) E_H(\Delta_\tau)\,.$$

Furthermore, we have

$$\sup_{\lambda > 0} \frac{1 - g(\lambda)}{\lambda} = \sup_{\lambda > 0} \frac{\sin(\lambda)}{\lambda} \chi_{[0,\pi)}(\lambda) \le 1\,.$$

This allows us to apply Lemma 3.1 which yields

$$\text{s-lim} \left(I_{\mathfrak{h}} + \frac{1}{\tau} P E_H(\Delta_\tau) \sin(\tau H) P \right)^{-1} = (I_{\mathfrak{h}} + K)^{-1}. \tag{3.8}$$

Next we set

$$L(\tau) := \frac{1}{\tau} P E_H(\Delta_\tau) \sin(\tau H) P + \frac{1}{\tau} P (I - E_H(\Delta_\tau) \cos(\tau H)) P\,.$$

Lemma 3.2 *Under the assumptions of Lemma 2.2 we have*

$$\operatorname*{s-lim}_{\tau \to 0} (I_{\mathfrak{h}} + L(\tau))^{-1} = (I_{\mathfrak{h}} + K)^{-1}. \tag{3.9}$$

Proof. We note that

$$(I_{\mathfrak{h}} + L(\tau))^{-1} \le \left(I_{\mathfrak{h}} + \frac{1}{\tau} P E_H(\Delta_\tau) \sin(\tau H) P \right)^{-1}. \tag{3.10}$$

Since

$$\frac{1}{\tau} P E_H(\Delta_\tau) \sin(\tau H) P \le K\,,$$

we have also a lower to the left-hand side of (3.10),

$$(I_{\mathfrak{h}} + K + \frac{1}{\tau} P (I - E_H(\Delta_\tau) \cos(\tau H)) P)^{-1} \le (I_{\mathfrak{h}} + L(\tau))^{-1}. \tag{3.11}$$

Now we use the representation

$$(I_{\mathfrak{h}} + K + \frac{1}{\tau}P(I - E_H(\Delta_\tau)\cos(\tau H))P)^{-1} = \\ = (I_{\mathfrak{h}} + K)^{-1/2}(I_{\mathfrak{h}} + N(\tau))^{-1}(I_{\mathfrak{h}} + K)^{-1/2}\,, \tag{3.12}$$

where

$$N(\tau) := \frac{1}{\tau}(I_{\mathfrak{h}} + K)^{-1/2}P(I - E_H(\Delta_\tau)\cos(\tau H))P(I_{\mathfrak{h}} + K)^{-1/2}.$$

This operator can be rewritten in the form

$$N(\tau) = \frac{1}{\tau}(I_{\mathfrak{h}} + K)^{-1/2}PE_H([\pi/\tau,\infty))P(I_{\mathfrak{h}} + K)^{-1/2} + \\ + \frac{1}{\tau}(I_{\mathfrak{h}} + K)^{-1/2}PE_H(\Delta_\tau)(I - \cos(\tau H))P(I_{\mathfrak{h}} + K)^{-1/2},$$

and using the contraction operator $W := \sqrt{H}P(I_{\mathfrak{h}} + K)^{-1/2}$ we get

$$\frac{1}{\tau}(I_{\mathfrak{h}} + K)^{-1/2}PE_H([\pi/\tau,\infty))P(I_{\mathfrak{h}} + K)^{-1/2} = W^*(\tau H)^{-1}E_H([\pi/\tau,\infty))W\,.$$

Now since $\|(\tau H)^{-1}E_H([\pi/\tau,\infty))\| \le \frac{1}{\pi} \le 1$, we obtain the estimate

$$\frac{1}{\tau}\|(I_{\mathfrak{h}} + K)^{-1/2}PE_H([\pi/\tau,\infty))P(I_{\mathfrak{h}} + K)^{-1/2}f\| \le \|E_H([\pi/\tau,\infty))Wf\|\,,$$

which yields

$$\lim_{\tau\to 0}\frac{1}{\tau}\|(I_{\mathfrak{h}} + K)^{-1/2}PE_H([\pi/\tau,\infty))P(I_{\mathfrak{h}} + K)^{-1/2}f\| \\ \le \lim_{\tau\to 0}\|E_H([\pi/\tau,\infty))Wf\| = 0\,.$$

Moreover, by virtue of the limit

$$\operatorname*{s-lim}_{\tau\to 0}\frac{1}{\tau}(I - \cos(\tau H))f = 0\,, \quad f \in \mathrm{dom}(H)\,,$$

we infer that

$$\operatorname*{s-lim}_{\tau\to 0}\frac{1}{\tau}(I_{\mathfrak{h}} + K)^{-1/2}PE_H(\Delta_\tau)(I - \cos(\tau H))P(I_{\mathfrak{h}} + K)^{-1/2} = 0\,,$$

hence

$$\operatorname*{s-lim}_{\tau\to 0} N(\tau) = 0\,.$$

Taking now into account the representation (3.12) we find

$$\operatorname*{s-lim}_{\tau\to 0}(I_{\mathfrak{h}} + K + \frac{1}{\tau}P(I - E_H(\Delta_\tau)\cos(\tau H))P)^{-1} = (I_{\mathfrak{h}} + K)^{-1}\,;$$

from (3.10) and (3.11) in combination with (3.8) and Lemma 2.3 we conclude finally that (3.9) is valid. □

Theorem 3.3 *Let H be a non-negative self-adjoint operator on $\mathfrak{H}$ and let $\mathfrak{h}$ be a closed subspace of $\mathfrak{H}$. If the set* $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ *is dense in $\mathfrak{h}$, then*

$$\operatorname*{s-lim}_{n\to\infty} \left(PE_H([0, \pi n/t))e^{-itH/n}P\right)^n = e^{-itK}$$

uniformly in $t \in [0, t_0]$ for any $t_0 > 0$, where K is defined by (2.6).

Proof. Since

$$\begin{aligned}(I_{\mathfrak{h}} + L(\tau))^{-1} = &\left(I_{\mathfrak{h}} + \frac{1}{\tau}PE_H(\Delta_\tau)\sin(\tau H)P\right)^{-1/2} \times \\ &\times \{I_{\mathfrak{h}} + M(\tau)\}^{-1}\left(I_{\mathfrak{h}} + \frac{1}{\tau}PE_H(\Delta_\tau)\sin(\tau H)P\right)^{-1/2},\end{aligned}$$

one can use (3.8) and (3.9) to conclude that

$$\operatorname*{w-lim}_{\tau\to 0}\left(I_{\mathfrak{h}} + M(\tau)\right)^{-1} = I_{\mathfrak{h}}\,.$$

Repeating the argument from proof of Theorem 2.5 we get

$$\operatorname*{s-lim}_{\tau\to 0}\left(I_{\mathfrak{h}} + M(\tau)\right)^{-1} = I_{\mathfrak{h}}\,.$$

Then (3.1) and (3.8) yield

$$\operatorname*{s-lim}_{\tau\to 0}\left(iI_{\mathfrak{h}} + S(\tau)\right)^{-1} = -i(I_{\mathfrak{h}} + K)^{-1},$$

or equivalently

$$\operatorname*{s-lim}_{\tau\to 0}\left(I_{\mathfrak{h}} - iS(\tau)\right)^{-1} = (I_{\mathfrak{h}} + K)^{-1}.$$

The last limit together with Chernoff's theorem [Ch74], see also Lemma 3.29 of [Da2], complete the proof in the exponential case. □

Remark 3.4 The conclusion of this section can be extended to other families of "filter windows". Inspecting the proof, one sees that $\Delta_\tau := [0, \pi/\tau)$ can be replaced by any family $\{\tilde{\Delta}_\tau\}$ such that $\tilde{\Delta}_\tau \subset \Delta_\tau$ and $\tilde{\Delta}_\tau \to \mathbb{R}_+$ as $\tau \to 0+$.

Remark 3.5 The result given in Theorem 3.3 is in a sense complementary to the conclusions of [EI03] where the formula without the spectral projection $E_H([0, \pi n/t))$ was proved, however, the convergence was demonstrated only in a weaker topology, namely that of $L^2_{\text{loc}}(\mathbb{R}; \mathcal{H})$. Recall that it implies existence of a set $M \subset \mathbb{R}$ of Lebesgue measure zero and a strictly increasing sequence $\{n'\}$ of positive integers along which $(Pe^{-itH/n'}P)^{n'} \to e^{-itK}$ holds strongly for all $t \in \mathbb{R} \setminus M$. A complete result without resorting to a subsequence and exclusion of a zero-measure set still waits for its proof.

4 Operator norm convergence

In the following we are going to ask how the convergence properties of the above derived product formula can be improved under more restrictive assumptions assumptions. To this end we set

$$F_0(\tau) := Pe^{-i\tau H}P : \mathfrak{h} \longrightarrow \mathfrak{h}\,, \quad \tau > 0\,,$$

and

$$S_0(\tau) := \frac{I_{\mathfrak{h}} - F_0(\tau)}{\tau} : \mathfrak{h} \longrightarrow \mathfrak{h}\,, \quad \tau > 0\,.$$

We will show first that if $\sqrt{H}$ is defined everywhere on $\mathfrak{h}$ one can get rid of the spectral projections used in the previous section.

Theorem 4.1 *Let H be a non-negative self-adjoint operator on $\mathfrak{H}$ and let $\mathfrak{h}$ be a closed subspace of $\mathfrak{H}$. If $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H})$, then*

$$\text{s-}\lim_{n\to\infty} \left(Pe^{-itH/n}P\right)^n = e^{-itK} \tag{4.1}$$

uniformly in $t \in [0, t_0]$ for any $t_0 > 0$ where K is defined by (2.6).

Proof. To begin with, note that $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H})$ implies that $T = \sqrt{H}P$ is a bounded operator, and consequently, $K = T^*T$ is also bounded. Then we may employ the representation

$$(I_{\mathfrak{h}} - F_0(\tau))f = i\int_0^\tau ds\, T^* e^{-isH} Tf\,, \quad \tau > 0\,, \quad f \in \mathfrak{h}\,, \tag{4.2}$$

or

$$S_0(\tau)f - iKf = \frac{i}{\tau}\int_0^\tau ds\, T^*\left(e^{-isH} - I_{\mathfrak{h}}\right)Tf\,, \quad \tau > 0\,, \quad f \in \mathfrak{h}\,,$$

which provides us with the estimate

$$\|S_0(\tau)f - iKf\|_{\mathfrak{h}} \le \sup_{s\in[0,\tau]} \|T^*\left(e^{-isH} - I_{\mathfrak{h}}\right)Tf\|_{\mathfrak{h}} \quad \tau > 0\,, \quad f \in \mathfrak{h}\,.$$

Since

$$\lim_{\tau\to 0}\sup_{s\in[0,\tau]} \|T^*\left(e^{-isH} - I_{\mathfrak{h}}\right)Tf\|_{\mathfrak{h}} = 0\,, \quad f \in \mathfrak{h}\,,$$

we immediately conclude that

$$\operatorname*{s-lim}_{\tau\to 0} S_0(\tau) = iT^*T = iK\,.$$

In this way we obtain the relation

$$\operatorname*{s-lim}_{\tau\to 0}(I_{\mathfrak{h}} + S_0(\tau))^{-1} = (I_{\mathfrak{h}} + iK)^{-1},$$

and using Chernoff's theorem [Ch74] one more time we prove (4.1). □

Corollary 4.2 *Let H be a non-negative self-adjoint operator in $\mathfrak{H}$ and let $\mathfrak{h}$ be a closed subspace of $\mathfrak{H}$ such that $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H})$. If in addition the operator T is compact or $\mathfrak{h} \subseteq \mathrm{dom}(H^\alpha)$ is satisfied for some $\alpha \in (\frac{1}{2}, 1]$, then*

$$\lim_{n\to\infty} \left\|\left(Pe^{-itH/n}P\right)^n - e^{-itK}\right\|_{\mathcal{B}(\mathfrak{h})} = 0 \tag{4.3}$$

holds uniformly in $t \in [0, t_0]$ for any $t_0 > 0$.

Proof. Since

$$S_0(\tau)f - iKf = \frac{i}{\tau}\int_0^\tau ds\, T^*\left(e^{-isH} - I_{\mathfrak{h}}\right)Tf\,, \quad \tau > 0\,, \quad f \in \mathfrak{h}\,,$$

we can write down the estimate

$$\|S_0(\tau) - iK\|_{\mathcal{B}(\mathfrak{H})} \le \sup_{s\in[0,\tau]} \left\|T^*\left(e^{-isH} - I_{\mathfrak{h}}\right)T\right\|_{\mathcal{B}(\mathfrak{H})}\,, \quad \tau > 0\,. \tag{4.4}$$

If T is compact, then

$$\lim_{\tau\to 0}\sup_{s\in[0,\tau]}\left\|T^*\left(e^{-isH}-I_{\mathfrak{h}}\right)T\right\|_{\mathcal{B}(\mathfrak{H})}=0\,;$$

the argument is analogous to that in the proof of Proposition 2.69 in [Za]. In combination with (4.4) this yields

$$\lim_{\tau\to 0}\|S_0(\tau)-iK\|_{\mathcal{B}(\mathfrak{H})}=0. \tag{4.5}$$

If $\mathfrak{h}\subseteq \mathrm{dom}(H^\alpha)$ for some $\alpha\in(\frac{1}{2},1]$, then we set $T_\alpha:=H^\alpha P$ and $K_\alpha=T_\alpha^*T_\alpha$. Notice that T_α is a bounded operator. From (4.4) we obtain the estimate

$$\|S_0(\tau)-iK\|_{\mathcal{B}(\mathfrak{H})}\leq \sup_{s\in[0,\tau]} s^{2\alpha-1}\left\|T_\alpha^*\,\frac{e^{-isH}-I_{\mathfrak{h}}}{(sH)^{2\alpha-1}}\,T_\alpha\right\|_{\mathcal{B}(\mathfrak{H})}\,,\quad \tau>0\,.$$

Using the inequality

$$\sup_{\lambda\geq 0}\left|\frac{e^{-i\lambda}-1}{\lambda^{2\alpha-1}}\right|\leq 2^{2(1-\alpha)}$$

we find

$$\|S_0(\tau)-iK\|_{\mathcal{B}(\mathfrak{H})}\leq 2^{2(1-\alpha)}\tau^{2\alpha-1}\|K_\alpha\|_{\mathcal{B}(\mathfrak{h})}\,,\quad \tau>0\,,$$

which yields again the relation (4.5).

Next we use the expansions of the operator families in question,

$$e^{-tS_0(t/n)}-e^{-itK}=\sum_{j=0}^{\infty}\frac{(-t)^k}{j!}\left(S_0(t/n)^j-(iK)^j\right)\,,$$

which together with (4.5) allows us to conclude that

$$\lim_{n\to\infty}\left\|e^{-tS_0(t/n)}-e^{-itK}\right\|_{\mathcal{B}(\mathfrak{h})}=0 \tag{4.6}$$

holds for any $t>0$, uniformly in $t\in[0,t_0]$. We shall combine it with the telescopic estimate

$$\begin{aligned}&\left\|F_0(t/n)^n-e^{-itK}\right\|_{\mathcal{B}(\mathfrak{h})}\leq \\ &\leq \left\|F_0(t/n)^n-e^{-tS_0(t/n)}\right\|_{\mathcal{B}(\mathfrak{h})}+\left\|e^{-tS_0(t/n)}-e^{-itK}\right\|_{\mathcal{B}(\mathfrak{h})}\,,\end{aligned} \tag{4.7}$$

where the first term can be treated as in Lemma 3.27 of [Da2],

$$\left\|\left(F_0(t/n)^n-e^{-tS_0(t/n)}\right)f\right\|_{\mathfrak{h}}\leq\sqrt{n}\,\|(F_0(t/n)-I_{\mathfrak{h}})\,f\|_{\mathfrak{h}}\,,\quad f\in\mathfrak{h}\,. \tag{4.8}$$

Using the representation (4.2) with $\tau = t/n$, we can estimate the right-hand side of (4.8) by

$$\|(F_0(t/n) - I_{\mathfrak{h}})\, f\|_{\mathfrak{h}} \le \int_0^{t/n} ds\, \|T^* e^{-isH} T f\|_{\mathfrak{h}}\,, \quad f \in \mathfrak{h}\,,$$

which gives

$$\|(F_0(t/n) - I_{\mathfrak{h}})\, f\|_{\mathfrak{h}} \le \frac{t}{n} \|K\|_{\mathcal{B}(\mathfrak{h})} \|f\|\,, \quad f \in \mathfrak{h}\,.$$

Inserting this estimate into (4.8) we obtain

$$\left\|F_0(t/n)^n - e^{-tS_0(t/n)}\right\|_{\mathcal{B}(\mathfrak{h})} \le \frac{t}{\sqrt{n}} \|K\|_{\mathcal{B}(\mathfrak{h})}$$

which yields

$$\lim_{n\to\infty} \left\|F_0(t/n)^n - e^{-tS_0(t/n)}\right\|_{\mathcal{B}(\mathfrak{h})} = 0 \tag{4.9}$$

for any $t > 0$, uniformly in $t \in [0, t_0]$. Taking into account (4.6), (4.7) and (4.9) we arrive at the sought relation (4.3). $\square$

Remark 4.3 Obviously, the conclusion (4.3) is valid if $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H})$ and $\mathfrak{h}$ is a finite dimensional subspace. Indeed, in this case the operator T is finite dimensional, and therefore compact. This gives an alternative proof of the result derived in Sec. 5 of [EI03].

Remark 4.4 In connection with the last remark let us mention that in the finite-dimensional case there is one more way to prove the claim suggested by G.M. Graf and A. Guekos [GG04]. The argument is based on the observation that

$$\lim_{t\to 0} t^{-1} \left\|Pe^{-itH}P - Pe^{-itK}P\right\|_{\mathcal{B}(\mathfrak{h})} = 0 \tag{4.10}$$

implies $\left\|(Pe^{-itH/n}P)^n - Pe^{-itK}\right\|_{\mathcal{B}(\mathfrak{h})} = n\, o(t/n)$ as $n \to \infty$ by means of a natural telescopic estimate. To establish (4.10) one first proves that

$$t^{-1}\left[(f, Pe^{-itH}Pg)_{\mathcal{B}(\mathfrak{h})} - (f, g)_{\mathcal{B}(\mathfrak{h})} - it(\sqrt{H}Pf, \sqrt{H}Pg)_{\mathcal{B}(\mathfrak{h})}\right] \longrightarrow 0$$

as $t \to 0$ for all f, g from $\mathrm{dom}(\sqrt{H}P)$ which coincides in this case with $\mathfrak{h}$ by the closed-graph theorem. The last expression is equal to

$$\left(\sqrt{H}Pf, \left[\frac{e^{-itH} - I}{tH} - i\right] \sqrt{H}Pg\right)_{\mathcal{B}(\mathfrak{h})}$$

and the square bracket tends to zero strongly by the functional calculus, which yields the sought conclusion. In the same way we find that

$$t^{-1}\left[(f,Pe^{-itK}Pg)_{\mathcal{B}(\mathfrak{h})}-(f,g)_{\mathcal{B}(\mathfrak{h})}-it(\sqrt{K}f,\sqrt{K}g)_{\mathcal{B}(\mathfrak{h})}\right]\longrightarrow 0$$

holds as $t\to 0$ for any vectors $f,g\in\mathfrak{h}$. Next we note that $(\sqrt{K}f,\sqrt{K}g)_{\mathcal{B}(\mathfrak{h})} = (\sqrt{H}Pf,\sqrt{H}Pg)_{\mathcal{B}(\mathfrak{h})}$, and consequently, the expression contained in (4.10) tends to zero weakly as $t\to 0$, however, in a finite dimensional $\mathfrak{h}$ the weak and operator-norm topologies are equivalent.

5 Concluding remarks

In this paper we have been able to make a derive a pair of new product formulæ related to the Zeno problem. While the results of Section 4 contribute to the problem in its standard form, important questions about the sense in which Zeno dynamics exists remain open.

At the same time we demonstrated here that the problem can be interpreted in a broader sense, with suitable generalized observables replacing the usual non-decay measurement. We have analyzed here the simplest nontrivial situation when we deal with a product of two non-commuting projections. One can ask, of course, what will happen if one considers instead more complicated measurement characterized by positive-operator-valued measures (POVM), as analyzed and classified in the book [Da1]. This question goes beyond the scope of the present paper; we hope to return to it is a later publication.

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