Asymptotic Stability of Nonlinear Schrödinger Equations with Potential*

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November 14, 2004

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Abstract

We prove asymptotic stability of trapped solitons in the generalized nonlinear Schrödinger equation with a potential in dimension 1 and for even potential and even initial conditions.

1 Introduction

In this paper we study the generalized nonlinear Schrödinger equation with a potential

$$i\frac{\partial\psi}{\partial t} = -\psi_{xx} + V_h\psi - f(|\psi|^2)\psi \tag{1}$$

in dimension 1. Here $V_h : \mathbb{R} \to \mathbb{R}$ is a family of external potentials, $\psi_{xx} = \partial_x^2\psi$, and $f(s)$ is a nonlinearity to be specified later. Such equations arise in the theory of Bose-Einstein condensation [1], nonlinear optics, theory of water waves [2]and in other areas. To fix ideas we assume the potentials to be of the form $V_h(x) := V(hx)$ with V smooth and decaying at ∞. Thus for $h = 0$, Equation (1) becomes the standard generalized nonlinear Schrödinger equation (gNLS)

$$i\frac{\partial\psi}{\partial t} = -\psi_{xx} + \mu\psi - f(|\psi|^2)\psi, \tag{2}$$

*This paper is a part of the first author's Ph.D thesis.

†Supported by NSERC under Grant NA7901.

‡Supported by NSF under Grant DMS-0400526.

[1]In this case Equation (1) is called the Gross-Pitaevskii equation.

[2]In these two areas one usually takes $V_h = 0$, but taking into account impurities and/or variations in geometry of the medium one arrives at (1) with, in general, a time-dependent V_h.

where $\mu = V(0)$. For a certain class of nonlinearities, $f(|\psi|^2)$ (see Section 2), there is an interval $\mathcal{I}_0 \subset \mathbb{R}$ such that for any $\lambda \in \mathcal{I}_0$ Equation (2) has solutions of the form $e^{i(\lambda-\mu)t}\phi_0^\lambda(x)$ where $\phi_0^\lambda \in \mathcal{H}_2(\mathbb{R})$ and $\phi_0^\lambda > 0$. Such solutions (in general without the restriction $\phi_0^\lambda > 0$) are called the *solitary waves* or *solitons* or, to emphasize the property $\phi_0^\lambda > 0$, the *ground states*. For brevity we will use the term *soliton* applying it also to the function ϕ_0^λ without the phase factor $e^{i(\lambda-\mu)t}$.

Equation (2) is translationally and gauge invariant. Hence if $e^{i(\lambda-\mu)t}\phi_0^\lambda(x)$ is a solution for Equation (2), then so is $e^{i(\lambda-\mu)t}e^{i\alpha}\phi_0^\lambda(x+a)$, for any $\lambda \in \mathcal{I}_0,\ a \in \mathbb{R},\ \alpha \in [0, 2\pi)$. This situation changes dramatically when the potential V_h is turned on. In general, as was shown in [FW, Oh1, ABC] out of the three-parameter family $e^{i(\lambda-\mu)t}e^{i\alpha}\phi_0^\lambda(x+a)$ only a discrete set of two parameter families of solutions to Equation (1) bifurcate: $e^{i\lambda t}e^{i\alpha}\phi_h^\lambda(x)$, $\alpha \in [0, 2\pi)$ and $\lambda \in \mathcal{I}$ for some $\mathcal{I} \subseteq \mathcal{I}_0$, with $\phi_h^\lambda \in \mathcal{H}_2(\mathbb{R})$ and $\phi_h^\lambda > 0$. Each such family corresponds to a different critical point of the potential $V_h(x)$. It was shown in [Oh2] that the solutions corresponding to minima of $V_h(x)$ are orbitally (Lyapunov) stable and to maxima, orbitally unstable. We call the solitary wave solutions described above which correspond to the minima of $V_h(x)$ *trapped solitons* or just *solitons* of Equation (1) omitting the last qualifier if it is clear which equation we are dealing with.

The main result of this paper is a proof that the trapped solitons of Equation (1) are asymptotically stable. The latter property means that if an initial condition of (1) is sufficiently close to a trapped soliton then the solution converges in some weighted $\mathcal{L}^2$ space to, in general, another trapped soliton of the same two-parameter family. In this paper we prove this result under the additional assumption that the potential and the initial condition are even. This limits the number of technical difficulties we have to deal with. In the subsequent paper we remove this restriction and allow the soliton to 'move'.

In fact, in this paper we prove a result more general than asymptotic stability of trapped solitons. Namely, we show that if the initial conditions are of the form

$$\psi_0 = e^{i\gamma_0}(\phi_h^{\lambda_0} + \chi_0),$$

with χ_0 being small in the space $(1+x^2)\mathcal{H}^1$, $\gamma_0 \in \mathbb{R}$ and $\lambda_0 \in \mathcal{I}$ ($\mathcal{I}$ will be defined later). Then the solution, $\psi(t)$, of Equation (1) can be written as

$$\psi(t) = e^{i\gamma(t)}(\phi_h^{\lambda(t)} + \chi(t)), \tag{3}$$

where $\gamma(t) \in \mathbb{R}$, $\chi(t) \to 0$ in some local norm, and $\lambda(t) \to \lambda_\infty$ for some λ_∞ as $t \to \infty$.

We observe that (1) is a Hamiltonian system with conserved energy (see Section 2) and, though orbital (Lyapunov) stability is expected, the asymptotic stability is a subtle matter. To have asymptotic stability the system should be able to dispose of excess of its energy, in our case, by radiating it to infinity. The infinite dimensionality of a Hamiltonian system in question plays a crucial role here.

First attack on the asymptotic stability in infinite dimensional Hamiltonian systems was made in the pioneering work of Soffer and Weistein [SW1] where the asymptotic stability of nonlinear bound states was proved for the nonlinear Schrödinger equation with a potential and a weak nonlinearity in the dimensions higher than or equal to 3. Asymptotic stability of moving solitons in the (generalized) nonlinear Schrödinger equation without potential and dimension 1 was first proven by Buslaev and Perelman [BP1]. The above results were significantly extended by Soffer and Weinstein, Buslaev and Perelman, Tsai and Yau, Buslaev and Sulem, Cuccagna(see [SW2, SW3, SW4, BP2, TY1, TY2, TY3, BS, Cu1, Cu2, Cu3]). Related results in multi-soliton dynamics were obtained by Perelman, and Rodnianski, Schlag and Soffer (see [Pere, RSS1, RSS2]). Deift and Zhou (see [DeZh])used a different approach, inverse scattering method, to asymptotic behavior of solitons of the 1-dimensional nonlinear Schrödinger equations.

Among earlier work we should mention the works of Shatah and Strauss, Weinstein, Grillakis, Shatah and Strauss on orbital stability (see [SS, We1, We2, GSS1, GSS2]) whose results were extended by Comech and Pelinosky, Comech, Cuccagna, Pelinovsky and Vougalter, Cuccagna and Pelinovsky, and Schlag (see [CP, CO, Cu1, CPV, CuPe, Sch]).

There is an extensive physics literature on the subject; some of the references can be found in Grimshaw and Pelinovsky [GP].

Long-term dynamics of solitons in external potentials is determined by Bronski and Jerrard, Fröhlich, Tsai and Yau, Keraani, and Fröhlich, Gustafson, Jonsson and Sigal (see [BJ, FTY, Ker, FGJS]).

Our approach is built on the beautiful theory of one-dimensional Schrödinger operators developed by Buslaev and Perelman, and Buslaev and Sulem(see [BP1, BP2, BS]). One of the key points in this approach is obtaining suitable (and somewhat surprising) estimates on the propagator for the linearization of Equation (1) around the soliton family $e^{i\gamma}\phi_h^\lambda$. One of the difficulities here lies in the fact that the corresponding generator, $L(\lambda)$, is not self-adjoint. To obtain the desired estimates one develops the spectral representation for the propagator (in terms of the boundary values of the resolvent; this can be also extended to other functions of the generator (see Subsection 5.2)) and then estimates the integral kernel of the resolvent using estimates on various solutions of the corresponding spectral problem $(L(\lambda) - \sigma)\xi = 0$ (Appendix A). These estimates are close to the correponding estimates of [BP1, BP2, BS]. Since these estimate are somewhat involved we take pain to provide a detailed and readable account. Note that, independently, W. Schlag [Sch] has developped spectral representation similiar to ours (see Subsection 5.2) and Goldberg and Schlag [GS] obtained (by a different technique) estimates on the propagators of the one-dimensional, self-adjoint (scalar) Schrödinger operators similiar to some of our estimates (see Subsection 5.1) but under more general assumptions on the potential than in our (non-self-adjoint, vector) case.

The paper is organized as follows: in Section 2 we describe the Hamiltonian structure of Equation (1), cite a well-posedness result, formulate our conditions on the nonlinearity and the potential and our main result. In Section 3 we describe the spectral structure of the linearized equation around the trapped soliton. In Section 4 we decompose the solution into a part moving in the 'soliton manifold' and a simplectically orthogonal fluctuation and find the equations for the soliton parameters and for the fluctuation. In the same section we estimate the soliton parameters and the fluctuation assuming certain estimates on the linearized propagators (i.e. the solutions of the linearized equation). The latter estimates are proven in Section 5, modulo estimates of the generalized eigenfunctions which are obtained in Appendix A. In Appendix B we analyze the implicit conditions on the nonlinearity and the potential made in Section 3.

As customary we often denote derivatives by subindices as in $\phi^\lambda_\lambda = \frac{d}{d\lambda}\phi^\lambda$ and $\phi^\lambda_x = (\frac{d}{dx}\phi^\lambda)$ for $\phi^\lambda = \phi^\lambda(x)$. The Sobolev and L^2 spaces are denoted by $\mathcal{H}^1$ and $\mathcal{L}^2$ respectively.

Acknowledgment

We are grateful to S. Cuccagna, V. Vougalter and, especially, V.S. Buslaev for fruitful discussions. In particular, V.S.Buslaev taught us beautiful ODE techniques many of which are due to him and his collaborators.

This paper is a part of the first author's Ph.D requirement. While the paper has been prepared for publication the authors received the preprints by Comech and Pelinovsky [CP], Schlag [Sch] and Goldberg and Schlag [GS] whose results overlap with some of the results of the present paper, namely, with the description of the linearized spectrum (Section 3) and with construction of functions of nonself-adjoint operators (Subsection 5.2) and with an estimate of one-dimensional propagators (Subsection 5.1) respectively. The authors are grateful to Comech, Pelinovsky and Schlag for communicating their results prior to publications.

2 Properties of (1), Assumptions and Results

In this section we discuss some general properties of Equation (1) and formulate our results.

2.1 Hamiltonian Structure

Equation (1) is a Hamiltonian system on Sobolev space $\mathcal{H}^1(\mathbb{R},\mathbb{C})$ viewed as a real space $\mathcal{H}^1(\mathbb{R},\mathbb{R})\oplus\mathcal{H}^1(\mathbb{R},\mathbb{R})$ with the inner product $(\psi,\phi) = Re\int_{\mathbb{R}}\bar{\psi}\phi$ and with the simpletic form $\omega(\psi,\phi) = Im\int_{\mathbb{R}}\bar{\psi}\phi$. The Hamiltonian functional is:

$$H(\psi) := \int [\frac{1}{2}(|\psi_x|^2 + V_h|\psi|^2) - F(|\psi|^2)],$$

where $F(u) := \frac{1}{2}\int_0^u f(\xi)d\xi$.

Equation (1) has the time-translational and gauge symmetries which imply the the following conservation laws: for any $t \geq 0$, we have

(CE) conservation of energy:

$$H(\psi(t)) = H(\psi(0));$$

(CP) conservation of the number of particles:

$$N(\psi(t)) = N(\psi(0)),$$

where $N(\psi) := \int |\psi|^2$.

We need the following condition on the nonlinearity f for the global well-posedness of (1).

(fA) The nonlinearity f is locally Lipschitz and $f(\xi) \leq c(1 + |\xi|^q)$ for some $c > 0$ and $q < 2$.

The following theorem is proved in [Oh3, Caz].

Theorem *Assume that the nonlinearity f satisfies the condition (fA), and that the potential V is bounded. Then Equation (1) is globally well posed in $\mathcal{H}^1$, i.e. the Cauchy problem for Equation (1) with initial datum $\psi(0) \in \mathcal{H}^1$ has a unique solution $\psi(t)$ in the space $\mathcal{H}^1$ and this solution depends continuously on $\psi(0)$.*

Moreover $\psi(t)$ satisfies the conservation laws (CE) and (CP).

If $\psi(0)$ has a finite norm $\|(1 + |x|)\psi(0)\|_2$, then we have the following estimates:

$$\|(1 + |x|)\psi(t)\|_2 \leq e(\|\psi(0)\|_{\mathcal{H}^1})[\|(1 + |x|)\psi(0)\|_2 + t\|\psi(0)\|_{\mathcal{H}^1}], \tag{4}$$

where $e : \mathcal{R}_+ \to \mathcal{R}_+$ is a smooth function.

2.2 Existence and Stability of Solitons

In this subsection we discuss the problem of existence and stability of solitons.

It is proved in [BP1, BL] that if the nonlinearity f in Equation (1) is smooth, real and satisfies the following condition

(fB) There is an interval $\mathcal{I}_0 \in \mathbb{R}^+$ s.t. for any $\lambda \in \mathcal{I}_0$

$$U(\phi, \lambda) := -\lambda\phi^2 + \int_0^{\phi^2} f(\xi) d\xi$$

has a positive root and the smallest positive root $\phi_0(\lambda)$ satisfies $U_\phi(\phi_0(\lambda), \lambda) > 0$,

then for any $\lambda \in \mathcal{I}_0$ there exists a unique solution of Equation (2) of the form $e^{i(\lambda-\mu)t}\phi_0^\lambda$ with $\phi_0^\lambda \in \mathcal{H}^2$ and $\phi_0^\lambda > 0$. Such solutions are called the solitary waves or solitons or to emphasize that $\phi_0^\lambda > 0$, the ground states. For brevity we use the term soliton and we apply it to the function ϕ_0^λ. Note the function ϕ_0^λ satisfies the equation:

$$-(\phi_0^\lambda)_{xx} + \lambda\phi_0^\lambda - f((\phi_0^\lambda)^2)\phi_0^\lambda = 0. \tag{5}$$

Remark 1. *If $f(\xi) = c\xi^p + o(\xi^p)$ with $c,\ p > 0$, then Condition (fB) is satisfied for $\lambda \in (0, \delta)$ with δ sufficiently small.*

When the potential V is present, then some of the solitons above bifurcate into solitons for Equation (1). Namely, similarly as in [FW, Oh1] one can show that if f satisfies the following condition,

(fC) f is smooth, $f(0) = 0$ and there exists $p \geq 1$, such that $|f^{'}(\xi)| \leq c(1+|\xi|^p)$,

and if V satisfies the condition

(VA) V is smooth and 0 is a non-degenerate local minimum of V,

and if the soliton, ϕ_0^λ, exists for Equation (5), then for any $\lambda \in \mathcal{I}_{0V} := \{\lambda|\lambda > \inf_{x\in\mathbb{R}}\{V(x)\}\} \cap \{\lambda|\lambda + V(0) \in \mathcal{I}_0\}$ there exists a soliton ϕ_h^λ satisfying the equation

$$-\frac{d^2}{dx^2}\phi_h^\lambda + (\lambda + V_h)\phi_h^\lambda - f((\phi_h^\lambda)^2)\phi_h^\lambda = 0$$

and which is of the form $\phi_h^\lambda = \phi_0^{\lambda+V(0)} + O(h^{3/2})$ where ϕ_0^λ is the soliton of Equation (5). (The subindex should not be confused with the derivative in h.)

Under more restrictive conditions on the nonlinearity f one can show as in [GSS1, FGJS, We2] that the soliton ϕ_h^λ is a minimizer of the energy functional $H(\psi)$ for a fixed number of particles $N(\psi) = constant$ if and only if

$$\frac{d}{d\lambda}\|\phi_h^\lambda\|_2^2 > 0. \tag{6}$$

The latter condition is also equivalent to the orbital stability of ϕ_h^λ. In what follows we set

$$\mathcal{I} = \{\lambda \in \mathcal{I}_{0V} : \frac{\partial}{\partial\lambda}\|\phi_h^\lambda\|_2 > 0\}. \tag{7}$$

Observe that there exist some constants $c,\ \delta > 0$ such that

$$|\phi_h^\lambda(x)| \leq ce^{-\delta|x|} \text{ and } |\frac{d}{d\lambda}\phi_h^\lambda| \leq ce^{-\delta|x|}, \tag{8}$$

and similarly for the derivatives of ϕ_h^λ and $\frac{d}{d\lambda}\phi_h^\lambda$. The first estimate can be found in [GSS1] and the second estimate follows from the fact that the function $\frac{d}{d\lambda}\phi_h^\lambda$ satisfies the equation

$$[-\frac{d^2}{dx^2} + V_h + \lambda - f((\phi_h^\lambda)^2) - 2f^{'}((\phi_h^\lambda)^2)(\phi_h^\lambda)^2]\frac{d}{d\lambda}\phi_\lambda^h = -\phi_h^\lambda$$

and standard arguments.

For our main result we will also require the following condition on the potential V :

(VB) $|V(x)| \leq ce^{-\alpha|x|}$ for some $c, \alpha > 0$.

2.3 Linearized Operator and Spectral Conditions

In our analysis we use some implicit spectral conditions on the Fréchet derivative $\partial G(\phi_h^\lambda)$ of the map

$$G(\psi) = -i(-\frac{d^2}{dx^2} + \lambda + V_h)\psi + if(|\psi|^2)\psi \tag{9}$$

appearing on the right hand side of Equation (1). We compute

$$\partial G(\phi_h^\lambda)\chi = -i(-\frac{d^2}{dx^2} + \lambda + V_h)\chi + if((\phi_h^\lambda)^2)\chi + 2if^{'}((\phi_h^\lambda)^2)(\phi_h^\lambda)^2 Re\chi. \tag{10}$$

This is a real linear but not complex linear operator. To convert it to a linear operator we pass from complex functions to real vector-functions:

$$\chi \longleftrightarrow \vec{\chi} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},$$

where $\chi_1 = Re\chi$ and $\chi_2 = Im\chi$. Then

$$\partial G(\phi_h^\lambda)\chi \longleftrightarrow L(\lambda)\vec{\chi}$$

where

$$L(\lambda) := \begin{pmatrix} 0 & L_-(\lambda) \\ -L_+(\lambda) & 0 \end{pmatrix}, \tag{11}$$

with

$$L_-(\lambda) := -\frac{d^2}{dx^2} + V_h + \lambda - f((\phi_h^\lambda)^2), \tag{12}$$

and

$$L_+(\lambda) := -\frac{d^2}{dx^2} + V_h + \lambda - f((\phi_h^\lambda)^2) - 2f^{'}((\phi_h^\lambda)^2)(\phi_h^\lambda)^2. \tag{13}$$

Then we extend the operator $L(\lambda)$ to the complex space $\mathcal{H}^2(\mathbb{R},\mathbb{C}) \oplus \mathcal{H}^2(\mathbb{R},\mathbb{C})$. By a general result (see e.g. [RSIV]), $\sigma_{ess}(L(\lambda)) = (-i\infty, -i\lambda] \cap [i\lambda, i\infty)$ if the potential V_h in Equation (1) decays at ∞.

We show in the next section that the operator $L(\lambda)$ has at least four usual or associated eigenvectors: the zero eigenvector $\begin{pmatrix} 0 \\ \phi_h^\lambda \end{pmatrix}$ and associated zero eigenvector $\begin{pmatrix} \frac{d}{d\lambda}\phi_h^\lambda \\ 0 \end{pmatrix}$ related to the gauge symmetry $\psi(x,t) \rightarrow e^{i\alpha}\psi(x,t)$ of the original equation, and two eigenvectors with $O(h^2)$ eigenvalues originating from the zero eigenvector $\begin{pmatrix} \partial_x\phi_0^\lambda \\ 0 \end{pmatrix}$ of the $V = 0$ equation due to the translational symmetry of that equation and associated zero eigenvector $\begin{pmatrix} 0 \\ x\phi_0^\lambda \end{pmatrix}$ related to the boost transformation $\psi(x,t) \rightarrow e^{ibx}\psi(x,t)$ coming from the Galilean symmetry of the $V = 0$ equation.

Besides of eigenvalues, the operator $L(\lambda)$ may have resonances at the tips, $\pm i\lambda$, of its essential spectrum (those tips are called thresholds). The definition of the resonance is as follows:

Definition 1. *A function $h \neq 0$ is called a resonance of $L(\lambda)$ at $i\lambda$ if and only if h is C^2, is bounded and satisfies the equation*

$$(L(\lambda) - i\lambda)h = 0.$$

Similarly we define a resonance at $-i\lambda$.

In what follows we make the following spectral assumptions:

(SA) Dimension of the generalized eigenvector space for isolated eigenvectors is 4,

(SB) $L(\lambda)$ has no embedded eigenvalues,

(SC) $L(\lambda)$ has no resonances at $\pm i\lambda$.

Condition (SA) is satisfied for a large class of nonlinearities, but it is not generic. For some open set of nonlinearities the operator $L(\lambda)$ might have other purely imaginary, isolated eigenvalues besides those mentioned above. Our technique can be extended to this case. For the consideration of space this will be done elsewhere.

Conjecture 1. *Conditions (SB) and (SC) are satisfied for generic nonlinearities f and potentials V provided that V decays exponentially fast at ∞.*

There are standard techniques for proving the (SB) part of this conjecture. This will be addressed elsewhere.

The following results support the (SC) part of the conjecture. Introduce the family of operators

$$L_{general}(U) := L_0 + U,$$

where

$$L_0 := \begin{pmatrix} 0 & -\frac{d^2}{dx^2} + \beta \\ \frac{d^2}{dx^2} - \beta & 0 \end{pmatrix} \text{ and } U := \begin{pmatrix} 0 & V_1 \\ -V_2 & 0 \end{pmatrix}$$

parameterized by $\beta > 0$, $s \in \mathbb{C}$ and the functions $V_1(x)$ and $V_2(x)$ satisfying

$$|V_1(x)|, |V_2(x)| \leq ce^{-\alpha|x|} \tag{14}$$

for some constants $c, \alpha > 0$. Then we have

Proposition 2.3.1. *(A) If (SB) and (SC) are satisfied for a given U^0, then (SB) and (SC) are satisfied for any U such that $\|e^{\alpha|x|}(U - U^0)\|_{\mathcal{L}^\infty}$ is sufficiently small, where α is the same as in Equation (14).*

(B) If for some U^0 the operator $L_{general}(U^0)$ has a resonance at $i\beta$ (or at $-i\beta$) and if

$$\int_{-\infty}^{\infty} V_1^0(x) + V_2^0(x)dx \neq 0,$$

then there exists a small neighborhood $\mathcal{A} \subset \mathbb{C}$ of 1 such that $L_{general}(sU^0)$ has no resonance at $i\beta$ for $s \in \mathcal{A} \backslash \{1\}$.

Remark 2. *1. For $U = 0$, the operator $L_{general}(U) = L_0$ has resonances at $\pm i\beta$. Hence Statement (B) shows that the operators $L_{general}(sU)$ with $s \neq 0$ and sufficiently small have no resonance at $\pm i\beta$.*

2. *It is proved in [Kau] that if $f(u) = u$ and $V(x) = 0$ in Equation (1), then Conditions (SA) and (SB) hold, but Condition (SC) fails. Proposition 2.3.1 (B) implies that Equation (1) with $f(u) = u$ and $V(x) = sV^0(x)$, for a large class of $V^0(x)$ and for $s \neq 0$ sufficiently small, satisfies (SB) and (SC). It can be proved that for a large subclass of potentials $V^0(x)$ Condition (SA) remains to be satisfied. For consideration of space it will be done elsewhere.*

3. *Equation (1) with $f(u) = u^2$ and $V(x) = 0$ is integrable. It can be shown that the operator $L(\lambda)$ in this case satisfies the conditions (SB) and (SC). However, this equation fails Condition (6) (it is a critical NLS) and (SA) (its generalized zero eigenvector space is of dimension 6.) It is easy to stabilize this equation by changing the nonlinearity slightly, say, taking $f(u) = u^{2-\epsilon}$ or $f(u) = u^2 - \epsilon u^4$. The resulting equations satisfy (6), (SB) and (SC) but not (SA). Specifically, if the nonlinearity $f(u) = u^{2-\epsilon}$ and the potential $V = 0$, then Equation (1) has a standing wave solution $\psi(x,t) = e^{it}\phi(x)$ with*

$$\phi(x) = (12 - 4\epsilon)^{-\frac{1}{4-2\epsilon}}[e^{(2-\epsilon)x} + e^{-(2-\epsilon)x}]^{-\frac{1}{2-\epsilon}}.$$

Then by Proposition 2.3.1 Statement (A) and an explicit form of the soliton ϕ the corresponding linearized operator,

$$\begin{pmatrix} 0 & -\frac{d^2}{dx^2} + 1 - \phi^{4-2\epsilon} \\ \frac{d^2}{dx^2} - 1 + (5-2\epsilon)\phi^{4-2\epsilon} & 0 \end{pmatrix},$$

has no resonances at $\pm i$ provided that $\epsilon > 0$ is sufficiently small.

2.4 Main theorem

We state the main theorem of this paper.

Theorem 2.4.1. *Assume Conditions (VA), (VB), (fA)-(fC) and (SA)-(SC) and assume that the nonlinearity f is a polynomial of degree $p \geq 4$. Assume the external potential V is even, and $\lambda \in \mathcal{I}$ with $\mathcal{I}$ defined in Equation (7). There exists a constant $\delta > 0$ such that if $\psi(0)$ is even and satisfies*

$$\inf_{\gamma \in \mathcal{R}} \{\|x^2(e^{i\gamma}\psi(0) - \phi_h^\lambda)\|_2 + \|e^{i\gamma}\psi(0) - \phi_h^\lambda\|_{\mathcal{H}^1}\} \leq \delta,$$

then there exists a constant $\lambda_\infty \in \mathcal{I}$, such that

$$\inf_{\gamma \in \mathcal{R}} \|(1+|x|)^{-\nu}(\psi(t) - e^{i\gamma}\phi^{\lambda_\infty})\|_2 \to 0$$

as $t \to \infty$ where $\nu > 3.5$, in other words, the trapped soliton is asymptotically stable.

3 Properties of Operator $L(\lambda)$

In this section we find eigenvectors and define the essential spectrum subspace of $L(\lambda)$. Here we do not assume that the potential V is even. Our main theorem is:

Theorem 3.0.2. *If V satisfies Conditions (VA) and (VB) and if $\lambda \in \mathcal{I}$, then $L(\lambda)$ has 3 independent eigenvectors and one associated eigenvector with small eigenvalues: one eigenvector $\begin{pmatrix} 0 \\ \phi_h^\lambda \end{pmatrix}$ and one associated eigenvector $\begin{pmatrix} \frac{d}{d\lambda}\phi_h^\lambda \\ 0 \end{pmatrix}$ with eigenvalue 0, both of which are even if V is even; 2 independent eigenvectors with small non-zero imaginary eigenvalues, which are odd if V is even.*

Proof. The proof is based on the following facts: the operator $L(\lambda)$ has the eigenvector $\begin{pmatrix} 0 \\ \phi_h^\lambda \end{pmatrix}$:

$$L(\lambda) \begin{pmatrix} 0 \\ \phi_h^\lambda \end{pmatrix} = 0,$$

related to the gauge symmetry of the map $G(\psi)$ (see Equation (9)), and associated zero eigenvector $\begin{pmatrix} \frac{d}{d\lambda}\phi_h^\lambda \\ 0 \end{pmatrix}$:

$$L(\lambda) \begin{pmatrix} \frac{d}{d\lambda}\phi_h^\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_h^\lambda \end{pmatrix}.$$

Moreover, for $h = 0$, the operator has the zero eigenvector $\begin{pmatrix} \partial_x \phi_0^{\lambda+V(0)} \\ 0 \end{pmatrix}$:

$$L^{h=0}(\lambda) \begin{pmatrix} \partial_x \phi_0^{\lambda+V(0)} \\ 0 \end{pmatrix} = 0,$$

coming from the translational symmetry of the map $G(\psi)$ and the associated zero eigenvector $\begin{pmatrix} 0 \\ x\phi_0^{\lambda+V(0)} \end{pmatrix}$:

$$L^{h=0}(\lambda) \begin{pmatrix} 0 \\ x\phi_0^{\lambda+V(0)} \end{pmatrix} = \begin{pmatrix} 2\partial_x \phi_0^{\lambda+V(0)} \\ 0 \end{pmatrix},$$

coming from the boost transformation.

The first two properties above yield the first part of the theorem. The last two properties and elementary perturbation theory will yield the second part of this theorem.

To prove the second part of the theorem we first observe that since the operator $L(\lambda)$ is of the form $L(\lambda) = JH(\lambda)$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the anti-self-adjoint matrix and $H(\lambda) = \begin{pmatrix} L_+(\lambda) & 0 \\ 0 & L_-(\lambda) \end{pmatrix}$ is a real self-adjoint operator, the spectrum of $L(\lambda)$ is symmetric with respect to the real and imaginary axis. Hence the eigenvectors $\begin{pmatrix} \partial_x \phi_0^{\lambda+V(0)} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x\phi_0^{\lambda+V(0)} \end{pmatrix}$ for $h=0$ give rise to either two pure imaginary or two real eigenvalues. We claim for $V^{''}(0)>0$ the former case takes place; and for $V^{''}(0)<0$, the latter one. To prove this we use the Feshbach projection method (see [GuSi]) with the projections $\bar{P}:=I-P$ and

$$P := \text{Projection on Span}\{\begin{pmatrix} 0 \\ \phi_h^\lambda \end{pmatrix}, \begin{pmatrix} \partial_\lambda \phi_h^\lambda \\ 0 \end{pmatrix}, \begin{pmatrix} \partial_x \phi_h^\lambda \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x\phi_h^\lambda \end{pmatrix}\}.$$

Then the eigenvalue equation $L(\lambda)\psi=\mu\psi$ is equivalent to the nonlinear eigenvalue problem

$$(PL(\lambda)P - W)\phi = \mu\phi$$

where $\phi \in \text{Span}P$ and $W := PL(\lambda)\bar{P}(\bar{P}L(\lambda)\bar{P}-\mu)^{-1}\bar{P}L(\lambda)P$. It is easy to see that there exists some constant $\delta_1,\ \delta_2>0$ such that if h is sufficiently small and if $|\mu|\le\delta_1$ then for $n=0,1,2$

$$\|\partial_\mu^n(\bar{P}L(\lambda)\bar{P}-\mu)^{-1}\|_{\mathcal{L}^2\cap \text{Range}\bar{P}\to\mathcal{L}^2\cap\text{Range}\bar{P}} \le \delta_2. \tag{15}$$

We claim that

$$\|W\| = O(h^3).$$

Indeed, similarly as in [Oh1] we can get that $L(\lambda)=L^{h=0}(\lambda)+O(h^{3/2})$ and $P=P^{h=0}+O(h^{3/2})$. Therefore

$$PL(\lambda)\bar{P} = P^{h=0}L^{h=0}(\lambda)\bar{P}^{h=0} + O(h^{3/2}) = O(h^{3/2})$$

and similarly $\bar{P}L(\lambda)P=O(h^{3/2})$. Since we look for small eigenvalues μ we could use Estimate (15) to prove $\partial_\mu^n W = O(h^3)$, $n=0,1,2$. We have the following observations for the term $PL(\lambda)P$:

$$PL(\lambda)P\begin{pmatrix} 0 \\ \phi_h^\lambda \end{pmatrix} = 0,\ PL(\lambda)P\begin{pmatrix} \partial_x\phi_h^\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_h^\lambda \end{pmatrix}$$

$$PL(\lambda)P\begin{pmatrix} 0 \\ x\phi_h^\lambda \end{pmatrix} = \begin{pmatrix} 2\partial_x\phi_h^\lambda \\ 0 \end{pmatrix},$$

$$PL(\lambda)P\begin{pmatrix} \partial_x\phi_h^\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h^2V''(0)x\phi_h^\lambda \end{pmatrix} + O(h^3)$$

The operator $PL(\lambda)P+W$ restricted to the 4-dimensional space $RanP$ has the 4×4 matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h^2V''(0) \\ 0 & 0 & 2 & 0 \end{pmatrix} + O(h^3).$$

By a standard contraction argument we could prove that there are four eigenvalues, i.e. four values of μ: $0+O(h^{3/2}), 0+O(h^{3/2}), \pm\sqrt{-2h^2V''(0)}+O(h^{3/2})$. Since we already know that $L(\lambda)$ has an eigenvalue 0 with multiplicity 2, the other two eigenvalues are $\pm\sqrt{-2h^2V''(0)}+O(h^{3/2})$. □

Corollary 3.0.3. *There exist a real function* ξ_1 *and an imaginary function* η_1 *such that* $\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}$ *is the eigenvector of* $L(\lambda)$ *with small, nonzero and imaginary eigenvalue* $i\epsilon_1$*. Therefore*

$$L(\lambda)\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = i\epsilon_1 \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}, \; L(\lambda)\begin{pmatrix} \xi_1 \\ -\eta_1 \end{pmatrix} = -i\epsilon_1 \begin{pmatrix} \xi_1 \\ -\eta_1 \end{pmatrix} \tag{16}$$

For the operator $L(\lambda)$ is not self-adjoint we define the projection onto the pure point spectrum subspace of $L(\lambda)$ as:

$$P_d^{L(\lambda)} := \frac{1}{2i\pi}\int_\Gamma (L(\lambda) - z)^{-1} dz,$$

where curve Γ is a small circle around 0 :

$$\Gamma := \{z||z| = \min(\lambda, 2\epsilon_1)\},$$

where, recall ϵ_1 from Equation (16).

Proposition 3.0.4. *In the Dirac notation*

$$\begin{aligned} P_d^{L(\lambda)} \quad = \quad & \frac{1}{\langle \phi_h^\lambda, \frac{d}{d\lambda}\phi_h^\lambda\rangle}(\left| \begin{matrix} 0 \\ \phi_h^\lambda \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{d}{d\lambda}\phi_h^\lambda \\ 0 \end{matrix} \right| + \left| \begin{matrix} \frac{d}{d\lambda}\phi_h^\lambda \\ 0 \end{matrix} \right\rangle \left\langle \begin{matrix} 0 \\ \phi_h^\lambda \end{matrix} \right|) \\ & + \frac{1}{2\langle \xi_1, \eta_1\rangle}(\left| \begin{matrix} \xi_1 \\ \eta_1 \end{matrix} \right\rangle \left\langle \begin{matrix} -\eta_1 \\ \xi_1 \end{matrix} \right| - \left| \begin{matrix} \xi_1 \\ -\eta_1 \end{matrix} \right\rangle \left\langle \begin{matrix} \eta_1 \\ \xi_1 \end{matrix} \right|). \end{aligned}$$

The proof of this proposition is straightforward but tedious, and given in Appendix C where it is proved in a more general setting.

Definition 2. *We define the essential spectrum subspace of* $L(\lambda)$ *as Range*$(1 - P_d^{L(\lambda)})$*, where* P_d^λ *is defined before Proposition 3.0.4. And we define the operator*

$$P_{ess}^{L(\lambda)} := 1 - P_d^{L(\lambda)}. \tag{17}$$

4 Re-parametrization of $\psi(t)$

In this section we introduce a convenient decomposition of the solution $\psi(t)$ to Equation (1) into a solitonic component and a simplectically fluctuation.

4.1 Decomposition of $\psi(t)$

In this subsection we decompose $\psi(t)$, and derive equations of each component. From now on we fix one sufficiently small h, and we will drop the subindex h and denote ϕ_h^λ by ϕ^λ, and $\frac{d}{d\lambda}\phi_h^\lambda$ by ϕ_λ^λ.

Theorem 4.1.1. *Assume V and $\psi(0)$ are even. There exists a constant $\delta > 0$, so that if the initial datum $\psi(0)$ satisfies $\inf_{\gamma\in\mathcal{R}} \|\psi(0) - e^{i\gamma}\phi^\lambda\|_{\mathcal{H}^1} < \delta$, then there exist differentiable functions $\lambda,\ \gamma : \mathbb{R}^+ \to \mathbb{R}$, such that*

$$\psi(t) = e^{i\int_0^t \lambda(t)dt + i\gamma(t)}(\phi^{\lambda(t)} + R), \tag{18}$$

where R is in the essential spectrum subspace, i.e.

$$Im\langle R, i\phi^\lambda\rangle = Im\langle R, \phi_\lambda^\lambda\rangle = 0. \tag{19}$$

Proof. By the Lyapunov stability (see [Oh2, GSS1]), $\forall\ \epsilon > 0$, there exists a constant δ, such that if $\inf_{\gamma\in R} \|\psi(0) - e^{i\gamma}\phi^\lambda\|_{\mathcal{H}^1} < \delta$, then $\forall\ t > 0$, $\inf_{\gamma} \|\psi(t) - e^{i\gamma}\phi^\lambda\|_{\mathcal{H}^1} < \epsilon$. The decompositions (18) (19) follow from Splitting Theorem in [FGJS] and the fact that $\psi(t)$ are even while all the eigenvectors, besides even eigenvectors $\begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix}$ and $\begin{pmatrix} \phi_\lambda^\lambda \\ 0 \end{pmatrix}$, are odd. □

Plug Equation (18) into Equation (1) to obtain:

$$\begin{aligned} &-\dot\gamma(\phi^\lambda + R) + i(\dot\lambda\phi_\lambda^\lambda + R_t) \\ &= -R_{xx} + \lambda R + V_h R - f(|\phi^\lambda|^2)R - f'(|\phi^\lambda|^2)(\phi^\lambda)^2(R + \bar R) + N(R), \end{aligned} \tag{20}$$

where

$$N(R) = -f(|\psi|^2)(\phi^\lambda + R) + f(|\phi^\lambda|^2)(\phi^\lambda + R) + f'(|\phi^\lambda|^2)(\phi^\lambda)^2(R + \bar R).$$

Passing from complex functions $R = R_1 + iR_2$ to real vector-functions $\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$ we obtain

$$\frac{d}{dt}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 0 & L_-(\lambda) \\ -L_+(\lambda) & 0 \end{pmatrix}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} ImN(R) \\ -ReN(R) \end{pmatrix} + \begin{pmatrix} -\dot\lambda\phi_\lambda^\lambda \\ -\dot\gamma\phi^\lambda \end{pmatrix}. \tag{21}$$

Differentiating $Im\langle R, i\phi^\lambda\rangle = 0$ (see Decomposition 19) with respect to t, we get

$$Im\langle R_t, \phi^\lambda\rangle + \dot\lambda Im\langle R, i\phi_\lambda^\lambda\rangle = 0. \tag{22}$$

Multiply Equation (20) by $i\phi^\lambda$ and use Equation (22) to obtain:

$$\dot\lambda\langle\phi_\lambda^\lambda, \phi^\lambda\rangle - \dot\lambda Re\langle R, \phi_\lambda^\lambda\rangle - \dot\gamma Im\langle R, \phi^\lambda\rangle = Im\langle N(R), \phi^\lambda\rangle.$$

By similar reasoning the relation $Im\langle R, \phi_\lambda^\lambda\rangle = 0$ implies that

$$-\dot\gamma\langle\phi^\lambda, \phi_\lambda^\lambda\rangle - \dot\gamma Re\langle R, \phi_\lambda^\lambda\rangle + \dot\lambda Im\langle R, \phi_{\lambda\lambda}^\lambda\rangle = Re\langle N(R), \phi_\lambda^\lambda\rangle.$$

Combine the last two equations into a matrix form:

Lemma 4.1.2. *The parameters λ and γ fixed by Equations (22) (19) satisfy the equations:*

$$\left[\begin{array}{cc} \langle\phi^\lambda_\lambda,\phi^\lambda\rangle - Re\langle R,\phi^\lambda_\lambda\rangle & -Im\langle R,\phi^\lambda\rangle \\ -Im\langle R,\phi^\lambda_{\lambda\lambda}\rangle & \langle\phi^\lambda_\lambda,\phi^\lambda\rangle + Re\langle R,\phi^\lambda_\lambda\rangle \end{array}\right]\left[\begin{array}{c} \dot\lambda \\ \dot\gamma \end{array}\right] = \left[\begin{array}{c} Im\langle N(R),\phi^\lambda\rangle \\ -Re\langle N(R),\phi^\lambda_\lambda\rangle \end{array}\right]. \tag{23}$$

By our requirement and orbital stability of the solitons, $\langle\phi^\lambda_\lambda,\phi^\lambda\rangle > \epsilon_0 > 0$ for some constant ϵ_0, and $\langle R,\phi^\lambda_\lambda\rangle$ and $\langle R,\phi^\lambda_{\lambda\lambda}\rangle$ are small. Thus the matrix on the left hand side is invertible and

$$\| \left[\begin{array}{cc} \langle\phi^\lambda_\lambda,\phi^\lambda\rangle - Re\langle R,\phi^\lambda_\lambda\rangle & -Im\langle R,\phi^\lambda\rangle \\ -Im\langle R,\phi^\lambda_{\lambda\lambda}\rangle & \langle\phi^\lambda_\lambda,\phi^\lambda\rangle + Re\langle R,\phi^\lambda_\lambda\rangle \end{array}\right]^{-1} \| \leq c \tag{24}$$

for some $c > 0$ independent of time t.

4.2 Change of Variables

In this subsection we study key Equation (21). The study is complicated by the fact that the linearized operator $L(\lambda(t))$ depends on time t. To circumvent this difficulty we rearrange Equation (21) as follows. We fix time $T > 0$, and define function g^T by

$$e^{i\int_0^t \lambda(s)ds + i\gamma(t)} R =: e^{i\lambda_1 t + i\gamma_1} g^T \tag{25}$$

where $\gamma_1 = \gamma(T)$ and $\lambda_1 = \lambda(T)$. Denote

$$\Delta_1 := -\int_0^t \lambda(t)dt - \gamma(t) + \lambda_1 t + \gamma_1. \tag{26}$$

From Equations (25) and (26), we derive the equation for g^T. Let $g^T = g_1^T + ig_2^T$, then Equation (21) implies

$$\frac{d}{dt}\left(\begin{array}{c} g_1^T \\ g_2^T \end{array}\right) = L(\lambda_1)\left(\begin{array}{c} g_1^T \\ g_2^T \end{array}\right) + \left(\begin{array}{c} ImD \\ -ReD \end{array}\right), \tag{27}$$

where

$$D = D_1 + D_2 + D_3,$$
$$D_1 = \dot\gamma\phi^\lambda e^{-i\Delta_1} - i\dot\lambda\phi^\lambda_\lambda e^{-i\Delta_1},$$
$$\begin{aligned} D_2 \;&=\; [f(|\phi^{\lambda_1}|^2) + f'(|\phi^{\lambda_1}|^2)(\phi^{\lambda_1})^2 - f(|\phi^\lambda|^2) - f'(|\phi^\lambda|^2)(\phi^\lambda)^2]g^T \\ &+\; [f'(|\phi^{\lambda_1}|^2)(\phi^{\lambda_1})^2 - f'(|\phi^\lambda|^2)(\phi^\lambda)^2]\bar{g^T} + f'(|\phi^\lambda|^2)(\phi^\lambda)^2[1 - e^{-2i\Delta_1}]\bar{g^T}, \end{aligned}$$
$$D_3 = e^{-i\Delta_1}N(R).$$

We need to decompose g^T along the point spectrum and essential spectrum subspaces of the operator $L(\lambda_1)$. Since g^T is even and $\langle\phi^{\lambda_1},\phi^{\lambda_1}_{\lambda_1}\rangle > 0$, there are differentiable real functions $k_1^T,\ k_2^T:\ [0,T]\to\mathbb{R}$ such that

$$g^T = ik_1^T\phi^{\lambda_1} + k_2^T\phi^{\lambda_1}_{\lambda_1} + h^T, \tag{28}$$

and h^T is in the essential spectrum subspace of $L(\lambda_1)$, where, recall P_{ess} from Equations (17).

Lemma 4.2.1. *The functions k_1^T, k_2^T and $h^T = h_1^T + ih_2^T$ satisfy the following equations:*

$$\begin{bmatrix} -\sin(\Delta_1)\langle\phi^{\lambda_1},\phi^{\lambda}\rangle, & \cos(\Delta_1)\langle\phi_{\lambda_1}^{\lambda_1},\phi^{\lambda}\rangle \\ \cos(\Delta_1)\langle\phi_{\lambda}^{\lambda},\phi^{\lambda_1}\rangle, & \sin(\Delta_1)\langle\phi_{\lambda_1}^{\lambda_1},\phi_{\lambda}^{\lambda}\rangle \end{bmatrix} \begin{bmatrix} k_1^T \\ k_2^T \end{bmatrix} = -\begin{bmatrix} Re\langle e^{i\Delta_1}h^T,\phi^{\lambda}\rangle \\ Im\langle e^{i\Delta_1}h^T,\phi_{\lambda}^{\lambda}\rangle \end{bmatrix}, \tag{29}$$

$$\frac{d}{dt}\begin{pmatrix} h_1^T \\ h_2^T \end{pmatrix} = L(\lambda_1)\begin{pmatrix} h_1^T \\ h_2^T \end{pmatrix} + P_{ess}\begin{pmatrix} ImD \\ -ReD \end{pmatrix}. \tag{30}$$

Proof. By Equation (19), we have the following two equations:

$$\begin{aligned} 0 &= Im\langle R, i\phi^{\lambda}\rangle = Re\langle e^{i\Delta_1}g^T,\phi^{\lambda}\rangle \\ &= k_2^T\cos(\Delta_1)\langle\phi_{\lambda_1}^{\lambda_1},\phi^{\lambda}\rangle - k_1^T\sin(\Delta_1)\langle\phi^{\lambda_1},\phi^{\lambda}\rangle + Re\langle e^{i\Delta_1}h^T,\phi^{\lambda}\rangle; \end{aligned}$$

$$\begin{aligned} 0 &= Im\langle R,\phi_{\lambda}^{\lambda}\rangle \\ &= k_2^T\sin(\Delta_1)\langle\phi_{\lambda_1}^{\lambda_1},\phi_{\lambda}^{\lambda}\rangle + k_1^T\cos(\Delta_1)\langle\phi^{\lambda_1},\phi_{\lambda}^{\lambda}\rangle + Im\langle e^{i\Delta_1}h^T,\phi_{\lambda}^{\lambda}\rangle. \end{aligned}$$

Since $\begin{pmatrix} 0 \\ \phi^{\lambda_1} \end{pmatrix}$ and $\begin{pmatrix} \phi_{\lambda_1}^{\lambda_1} \\ 0 \end{pmatrix}$ are eigenvectors of $L(\lambda_1)$, Equation (27) implies Equation (30). ∎

When $|\lambda-\lambda_1|$ is small, $\langle\phi_{\lambda_1}^{\lambda_1},\phi^{\lambda}\rangle,\ \langle\phi_{\lambda}^{\lambda},\phi^{\lambda_1}\rangle > \epsilon_0 > 0$ for some constant ϵ_0. Thus in this case the matrix

$$\begin{bmatrix} -\sin(\Delta_1)\langle\phi^{\lambda_1},\phi^{\lambda}\rangle, & \cos(\Delta_1)\langle\phi_{\lambda_1}^{\lambda_1},\phi^{\lambda}\rangle \\ \cos(\Delta_1)\langle\phi_{\lambda}^{\lambda},\phi^{\lambda_1}\rangle, & \sin(\Delta_1)\langle\phi_{\lambda_1}^{\lambda_1},\phi_{\lambda}^{\lambda}\rangle \end{bmatrix}$$

has an inverse uniformly bounded in t and T.

4.3 Estimates of the Parameters λ, γ and the Function R

In this subsection we will estimate the parameters $\lambda(t)$, $\gamma(t)$ and the function $R(t)$.

Proposition 4.3.1. *Let $\nu > 7/2$ and $\rho_\nu := (1+|x|)^{-\nu}$. We have for time $t \ge 0$,*

$$|\dot{\lambda}(t)| + |\dot{\gamma}(t)| \le c(1+t)^{-3},$$

$$\|\rho_\nu R\|_2 \le c(1+t)^{-3/2}$$

where the constant c is independent of t.

The proof of this proposition is based on estimates of the evolution operator

$$U(t) = e^{tL(\lambda_1)}$$

which we formulate now. Note that $U(t)$ is defined in a standard way (see Lemma 5.1.1 for detailed definition). Recall that the operator P_{ess}, defined in Equation (17), is the projection onto the essential spectrum subspace $\mathcal{H}_{pp}(L^*(\lambda))^{\perp}$.

We prove in Section 5 that in the 1-dimensional case $U(t)$ satisfies the following estimates:

$$\|\rho_\nu U(t)P_{ess}h\|_2 \leq c(1+t)^{-\frac{3}{2}}\|\rho_{-2}h\|_2; \tag{31}$$

$$\|\rho_\nu U(t)P_{ess}h\|_2 \leq c(1+t)^{-3/2}(\|\rho_{-2}h\|_1 + \|h\|_2); \tag{32}$$

$$\|U(t)P_{ess}h\|_{\mathcal{L}^\infty} \leq ct^{-1/2}(\|\rho_{-2}h\|_1 + \|h\|_2); \tag{33}$$

$$\|U(t)P_{ess}h\|_{\mathcal{L}^\infty} \leq c(1+t)^{-\frac{1}{2}}\|\rho_{-2}h\|_{\mathcal{H}^1}; \tag{34}$$

where $\nu > 7/2$. Recall $\rho_\nu(x) = (1+|x|)^{-\nu}$.

Proof of Proposition 4.3.1 We will estimate the following quantities:

$$m_1^T(t) = \|\rho_\nu h^T\|_2, \quad m_2^T(t) = \|g^T\|_{\mathcal{L}^\infty},$$

$$M_1(T) = \sup_{\tau\leq T}(1+\tau)^{3/2}m_1^T(\tau), \quad M_2(T) = \sup_{\tau\leq T}(1+\tau)^{1/2}m_2^T(\tau),$$

where ν is a constant greater than 3.5.

Note various constants c used below do not depend on t or T.

The matrix on the left hand side of Equation (29) has a uniformly bounded inverse. Hence using the definition of M_1 we obtain

$$|k_1^T| + |k_2^T| \leq cM_1(1+t)^{-\frac{3}{2}}. \tag{35}$$

From Equations (23), (24), (25) and (28), we have

$$|\dot\lambda| + |\dot\gamma| \leq c\|\rho_\nu R\|_2^2 \leq c(|k_1^T| + |k_2^T| + \|\rho_\nu h^T\|_2)^2.$$

Hence by the definition of M_1,

$$|\dot\lambda| + |\dot\gamma| \leq cM_1^2(1+t)^{-3}. \tag{36}$$

By the definition of g^T in (25), $\|\rho_\nu R\|_2 = \|\rho_\nu g^T\|_2$. By the decomposition (28), we have

$$\|\rho_\nu R\|_2 \leq c(|k_1^T| + |k_2^T|) + \|\rho_\nu h^T\|_2 \leq c(1+t)^{-3/2}M_1. \tag{37}$$

We need to estimate Δ_1 given in Equation (26). By the observations that

$$\lambda(t) - \lambda_1 = -\int_t^T \dot\lambda(\tau)d\tau \text{ and } \gamma(t) - \gamma_1 = -\int_t^T \dot\gamma(\tau)d\tau$$

and Estimate (36), we have

$$\begin{aligned}
|e^{i\Delta_1} - 1| &= |e^{-\int_0^t i(\lambda(t)-\lambda_1)dt - \int_0^t i\dot\gamma(t)dt} - 1| \\
&= |e^{-i\int_0^t\int_t^T \dot\lambda(\tau)d\tau dt - i\int_t^T \dot\gamma(t)dt} - 1| \\
&\leq cM_1^2.
\end{aligned}$$

Now we estimate h^T. Using the Duhamel principle, we rewrite Equation (30) as

$$\begin{pmatrix} h_1^T \\ h_2^T \end{pmatrix} = U(t)\begin{pmatrix} h_1^T(0) \\ h_2^T(0) \end{pmatrix} + \int_0^t U(t-\tau)P_{ess}\begin{pmatrix} ImD \\ -ReD \end{pmatrix}d\tau,$$

where, recall, $U(t)=e^{tL(\lambda_1)}$. Using Estimate (33), we obtain

$$\begin{array}{rcl} \|h^T\|_{\mathcal{L}^\infty} & \leq & \|U(t)h^T(0)\|_{\mathcal{L}^\infty}+\int_0^t\sum_{n=1}^3\|U(t-\tau)P_{ess}\left(\begin{array}{c} ImD_n \\ -ReD_n \end{array}\right)\|_{\mathcal{L}^\infty}d\tau \\ & \leq & \|U(t)h^T(0)\|_{\mathcal{L}^\infty} \\ & & +c\int_0^t(1+|t-\tau|)^{-1/2}\sum_{n=1}^3(\|\rho_{-2}D_n\|_1+\|D_n\|_2)d\tau, \end{array} \tag{38}$$

and using Estimate (32) we derive

$$\begin{array}{rcl} \|\rho_\nu h^T\|_2 & \leq & \|\rho_\nu U(t)h^T(0)\|_2+\int_0^t\sum_{n=1}^3\|\rho_\nu U(t-\tau)P_{ess}\left(\begin{array}{c} ImD_n \\ -ReD_n \end{array}\right)\|_2 d\tau \\ & \leq & \|\rho_\nu U(t)h^T(0)\|_2 \\ & & +c\int_0^t(1+|t-\tau|)^{-3/2}\sum_{n=1}^3(\|\rho_{-2}D_n\|_1+\|D_n\|_2)d\tau. \end{array} \tag{39}$$

Next we estimate $\|\rho_{-2}D_n\|_1+\|D_n\|_2$, $n=1,2,3$.

By Estimate (36) we can estimate D_1 :

$$\|\rho_{-2}D_1\|_1+\|D_1\|_2\leq c(|\dot{\gamma}|+|\dot{\lambda}|)\leq cM_1^2(1+t)^{-3}.$$

For D_2,

$$\|\rho_{-2}D_2\|_1+\|D_2\|_2\leq cM_1^2(1+t)^{-\frac{3}{2}}.$$

For D_3, recall f is a polynomial, so we can take out the terms containing at least one power of ϕ_λ, denote it by D_I, and $D_{II}:=D_3-D_I$. Since the leading term of D_I is $c|g^T|^2\phi^\lambda$, we have

$$\|D_I\|_2+\|\rho_{-2}D_I\|_1\leq cM_2M_1(1+t)^{-2}. \tag{40}$$

Here we have ignored the higher order terms which are estimated by $P(M_2)M_1(1+t)^{-3}$, where the function P is a polynomial such that $P(0)=0$.

Since $c|g^T|^{2p+1}$ is the leading-order term of D_{II},

$$\|D_{II}\|_2+\|(1+|x|)^2D_{II}\|_1\leq c(\|g^T\|_{\mathcal{L}^\infty}^{2p}\|g^T\|_2+\|g^T\|_\infty^{2p-1}\|(1+|x|)g^T\|_2^2).$$

Since $\|g^T\|_2\leq c$, $\|(1+|x|)g^T\|_2\leq c(1+t)$ by Equation (4), and $\|g^T\|_{\mathcal{L}^\infty}^{2p-1}\leq M_2^{2p-1}(1+t)^{-p+1/2}$, and using that $p\geq 4$ we have

$$\begin{array}{rl} & \|D_{II}\|_2+\|(1+|x|)^2D_{II}\|_1 \\ \leq & c(M_2^{2p}(1+t)^{-p}+M_2^{2p-1}(1+t)^{-p+1/2}(1+t)^2) \\ \leq & c(M_2^{2p}(1+t)^{-4}+M_2^{2p-1}(1+t)^{-\frac{3}{2}}). \end{array} \tag{41}$$

By Estimates (40) and (41) and the fact that $D_3=D_I+D_{II}$, we obtain

$$\|D_3\|_2+\|\rho_{-2}D_3\|_1\leq c(1+t)^{-3/2}(M_2M_1+M_2^{2p}+M_2^{2p-1}).$$

This finishes the estimates of the D_i's. Now we return to the estimation of the h^T.

By Estimate (39), we have

$$\begin{array}{rl} & \|\rho_\nu h^T\|_2 \\ \leq & c(1+t)^{-3/2}(\|g^T(0)\|_2 + \|\rho_{-2}g^T(0)\|_1) + \\ & c\int_0^t (M_2M_1 + M_1^2 + M_2^{2p} + M_2^{2p-1})\frac{1}{(1+t-\tau)^{3/2}(1+\tau)^{3/2}}d\tau, \end{array}$$

so

$$\begin{array}{rcl} (1+t)^{3/2}\|\rho_\nu h^T\|_2 & \leq & c(\|g^T(0)\|_2 + \|\rho_{-2}g^T(0)\|_1) \\ & & +c(M_2M_1 + M_1^2 + M_2^{2p} + M_2^{2p-1}). \end{array}$$

Remember the definition of M_1 and

$$S = \|\rho_{-2}g^T(0)\|_1 + \|g^T(0)\|_{\mathcal{H}^1}.$$

Then we have

$$M_1 \leq cS + c(M_2M_1 + M_1^2 + M_2^{2p} + M_2^{2p-1}). \tag{42}$$

By Estimate (38), we have

$$\begin{array}{rcl} \|h^T\|_{\mathcal{L}^\infty} & \leq & c(1+t)^{-\frac{1}{2}}(\|\rho_{-2}g^T(0)\|_1 + \|g^T(0)\|_{\mathcal{H}^1}) \\ & & +\int_0^t \frac{d\tau}{(t-\tau)^{1/2}}(\|D\|_2 + \|\rho_{-2}D\|_1). \end{array}$$

By the definition of M_2 we have

$$M_2 \leq cS + c(M_2M_1 + M_1^2 + M_2^{2p} + M_2^{2p-1}). \tag{43}$$

By Equation (25) we have

$$S = \|\rho_{-2}g^T(0)\|_1 + \|g^T(0)\|_{\mathcal{H}^1} = \|\rho_{-2}R(0)\|_1 + \|R(0)\|_{\mathcal{H}^1}.$$

Thus S only depends on $R(0)$.

If $M_n(0)$, $(n = 1, 2)$ are sufficiently close to zero, by Estimates (42) and (43) we have shown that $M_1(T), M_2(T) \leq \mu(S)S$ for any time T, where $\mu(S)$ is a function that is bounded for sufficiently small S. Thus we have shown that

$$M_1(T) + M_2(T) \leq c(\|\rho_{-2}R(0)\|_1 + \|R(0)\|_{\mathcal{H}^1}).$$

The last estimate together with Estimates (36) and (37) implies Proposition 4.3.1.

□

4.4 Proof of Theorem 2.4.1

In this subsection we prove our main Theorem 2.4.1. To this end we use Proposition 4.3.1. Since

$$|\dot{\lambda}| + |\dot{\gamma}| \leq c(1+t)^{-3}$$

for some $c > 0$, there exist $\lambda_\infty,\ \gamma_\infty$ such that

$$|\lambda(t) - \lambda_\infty| + |\gamma(t) - \gamma_\infty| \leq c(1+t)^{-2}. \tag{44}$$

Recall that the solutions $\psi(t)$ can be written as in Equation (18)

$$\psi(t) = e^{-i\int_0^t \lambda(t)dt + i\gamma(t)}(\phi^{\lambda(t)} + R).$$

By Proposition 4.3.1 and Equation (44) we have for $\nu > 3.5$

$$\begin{array}{ll} & \|\rho_\nu(\phi^{\lambda(t)} + R - \phi^{\lambda_\infty})\|_2 \\ \leq & \|\rho_\nu(\phi^{\lambda(t)} - \phi^{\lambda_\infty})\|_2 + \|\rho_\nu R\|_2 \\ \leq & c[(1+t)^{-2} + (1+t)^{-3/2}] \\ \leq & c(1+t)^{-3/2}, \end{array}$$

which implies Theorem 2.4.1.

5 Estimates On Propagators for Matrix Schrödinger Operators

5.1 Formulation of the Main Result

In this subsection we prove Estimates (31)-(34) on the propagator $U(t) = e^{tL(\lambda_1)}$, where $L(\lambda)$ is defined in Definition 11. Actually we study more general propagators generated by the operators

$$L_{general} = \begin{pmatrix} 0 & L_2 \\ -L_1 & 0 \end{pmatrix},$$

where

$$L_1 := -\frac{d^2}{dx^2} + V_1 + \beta, \text{ and } L_2 := -\frac{d^2}{dx^2} + V_2 + \beta.$$

Here the constant $\beta > 0$, and the functions V_1 and V_2 are even, real and satisfy the estimates:

$$|V_1(x)|,\ |V_2(x)| \leq ce^{-\alpha|x|}$$

for some constants $c,\ \alpha > 0$.

By standard arguments (see, e.g. [RSIV]) we have that

$$\sigma_{ess}(L_{general}) = i(-\infty, -\beta] \cup i[\beta, \infty).$$

The points $-i\beta$ and $i\beta$ are called thresholds. They affect the long time behavior of the semigroup $e^{tL_{general}}$ in a crucial way.

The following notion will play an important role:

Definition 3. *A function $h \neq 0$ is called the resonance of $L_{general}$ at $i\beta$ (or $-i\beta$), if and only if h is bounded and satisfies*

$$(L_{general} - i\beta I)h = 0 \quad (or\ (L_{general} + i\beta I)h = 0).$$

Lemma 5.1.1. *The operator $L_{general}$ generates a semigroup, $e^{tL_{general}}$, $t \geq 0$.*

Proof. We write the operator $L_{general}$ as

$$L_{general} = L_0 + U,$$

where

$$L_0 := \left(\begin{array}{ll} 0 & -\frac{d^2}{dx^2} + \beta \\ \frac{d^2}{dx^2} - \beta & 0 \end{array} \right), \; U := \left(\begin{array}{ll} 0 & V_1 \\ -V_2 & 0 \end{array} \right).$$

It is easy to verify that the operator L_0 is a generator of a (C_0) contraction semigroup (see, e.g. [Gold, RSII]). Also the operator $U : \mathcal{L}^2 \to \mathcal{L}^2$ is bounded. By [[Gold], Theorem 6.4 and [RSII]] $L_{general} = L_0 + U$ generates a (C_0) semigroup. $\square$

Let P_{ess} be the projection onto the essential spectrum subspace of $L_{general}$, where, recall the definition of P_{ess} in Equation (17).

Theorem 5.1.2. *Assume that $L_{general}$ has no resonances at $\pm i\beta$, no eigenvalues embedded in the essential spectrum, and has no eigenvalues with non-zero real parts. Then $\forall \mu > 3.5$ there exists a constant $c = c(\mu) > 0$ such that*

$$\|\rho_\mu e^{tL_{general}} P_{ess} h\|_2 \leq c(1+t)^{-3/2} \|\rho_{-2} h\|_2; \tag{45}$$

$$\|\rho_\mu e^{tL_{general}} P_{ess} h\|_2 \leq c(1+t)^{-3/2} (\|\rho_{-2} h\|_1 + \|h\|_2) \tag{46}$$

$$\|e^{tL_{general}} P_{ess} h\|_{\mathcal{L}^\infty} \leq c(1+t)^{-1/2} \|\rho_{-2} h\|_{\mathcal{H}^1}, \tag{47}$$

$$\|e^{tL_{general}} P_{ess} h\|_{\mathcal{L}^\infty} \leq ct^{-1/2} (\|h\|_2 + \|\rho_{-2} h\|_1), \tag{48}$$

where, recall $\rho_\nu(x) = (1+|x|)^{-\nu}$.

The proof of an equivalent theorem of Theorem 5.1.2 is given in Subsection 5.3.

It is more convenient to transform first the operator $L_{general}$ as

$$H := -iT^* L_{general} T, \tag{49}$$

where the 2×2 matrix

$$T := \frac{1}{\sqrt{2}} \left(\begin{array}{ll} 1 & i \\ i & 1 \end{array} \right).$$

Compute the matrix operator H:

$$H = H_0 + W, \tag{50}$$

where

$$H_0 := \left(\begin{array}{ll} -\frac{d^2}{dx^2} + \beta & 0 \\ 0 & \frac{d^2}{dx^2} - \beta \end{array} \right), \; W := 1/2 \left(\begin{array}{ll} V_3 & -iV_4 \\ -iV_4 & -V_3 \end{array} \right), \tag{51}$$

with the functions $V_4 := V_1 - V_2$, $V_3 := V_2 + V_1$. From the properties of functions V_1, V_2 we have

$$|V_4(x)|,\ |V_3(x)| \leq ce^{-\alpha|x|} \tag{52}$$

for some constants $c, \alpha > 0$. Hence $\sigma_{ess}(H) = (-\infty, -\beta] \cup [\beta, \infty)$. The assumptions on $L_{general}$ are transported to H as following: H has no resonances at $\pm\beta$, and has only finitely many eigenvalues which are in the interval $(-\beta, \beta)$.

Clearly the operator H also generates a semigroup e^{-itH}. To prove the theorem above we relate the propagator e^{-itH} to the resolvent $(H - \lambda \pm i0)^{-1}$ of the generator H on the essential spectrum.

We introduce some notations that will be used below: Let $F = [f_{ij}]$ be an $n \times m$ matrix with entry $f_{ij} \in B$, where B is a normed space, then

$$\|F\|_B := \sum_{i,j} \|f_{ij}\|_B.$$

The Hölder inequality for such vector-valued functions reads: if the constants $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, the vectors $F_1 \in \mathcal{L}^p$ and $F_2 \in \mathcal{L}^q$, then

$$\|F_1 F_2\|_{\mathcal{L}^1} \leq \|F_1\|_{\mathcal{L}^p} \|F_2\|_{\mathcal{L}^q}.$$

5.2 The Spectral Representation and the Integral Kernel of the Propagator $e^{-itH}P_{ess}$

In this subsection we compute the spectral representation and the integral kernel of $e^{-itH}P_{ess}$, where P_{ess} is the projection onto the essential spectrum subspace of operator H which is unbounded and non-self-adjoint. The main theorem is:

Theorem 5.2.1. *Let ϵ_0 be a small positive number. Then*

$$\begin{aligned} &e^{-itH}P_{ess} = \lim_{K\to\infty} \lim_{\epsilon\to 0^+} \\ &\tfrac{1}{2i\pi}\{\textstyle\int_{\beta-\epsilon_0}^{K} + \int_{-K}^{-\beta+\epsilon_0}\} e^{-it\lambda}[(H - \lambda - i\epsilon)^{-1} - (H - \lambda + i\epsilon)^{-1}]d\lambda, \end{aligned} \tag{53}$$

where the limits on the right hand side are strong limits. The limits are independent of ϵ_0 because

$$(H \pm z + i0)^{-1} = (H \pm z - i0)^{-1}$$

for any z in the interval $[\beta - \epsilon_0, \beta)$.

Remark 3. *Clearly Equation (53) can be extended to functions $f(\lambda)$ such that $\int |\hat{f}(t)|dt \leq \infty$ by*

$$f(H)P_{ess} = \frac{1}{2i\pi}\{\int_{\beta}^{\infty} + \int_{-\infty}^{-\beta}\} f(\lambda)[(H - \lambda - i0)^{-1} - (H - \lambda + i0)^{-1}]d\lambda,$$

where the function $\hat{f}$ is the Fourier transform of f. It can also be extended to other classes of functions using that if $\lambda \in (-\infty, -\beta] \cup [\beta, \infty)$ and $g \in \mathcal{C}_0^\infty$ then

$$\langle g, \frac{1}{2\pi i}\sigma_3[(H - \lambda - i0)^{-1} - (H - \lambda + i0)^{-1}]g\rangle \geq 0, \tag{54}$$

where $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Since the operator H plays an important role in the nonlinear theory we discuss this extension elsewhere. It is not used in this paper.

We divide the proof of Theorem 5.2.1 into two parts: in Lemmas 5.2.2 and 5.2.3 we prove that the limits on the right hand side exist; then we prove that the left and right hand sides are equal.

For $\theta \in \mathbb{R}$, we define the space $\mathcal{L}^{2,\theta} := (1+|x|)^\theta \mathcal{L}^2$:

$$\|g\|_{\mathcal{L}^{2,\theta}} = \|(1+|x|)^\theta g\|_2.$$

Lemma 5.2.2. *Let $\beta^{'} < \beta$ with $\beta - \beta^{'}$ sufficiently small. If K is a sufficiently large constant, then for any function $g \in \mathcal{L}^{2,2}$*

$$\lim_{\epsilon \to 0^+} \int_{\beta'}^{K} e^{-it(\lambda - i\epsilon)}(H - \lambda + i\epsilon)^{-1} g d\lambda \tag{55}$$

exists in the space $\mathcal{L}^{2,-2}$.

Proof. Let $f, g \in \mathcal{L}^{2,2}$. The function

$$u(z) := \langle f, e^{-itz}(H-z)^{-1} g\rangle$$

is analytic on the set

$$\Gamma := \{z | Imz > 0,\ \ Rez \geq \beta^{'}\}.$$

Therefore

$$\begin{array}{rl} & \int_{\beta'}^{K} u(z - i0)dz \\ = & \int_{\beta'-i\epsilon}^{K-i\epsilon} u(z)dz + \int_{K-i\epsilon}^{K} u(z)dz + \int_{\beta'}^{\beta^{'}-i\epsilon} u(z)dz. \end{array} \tag{56}$$

Moreover, since $\sigma_{ess}(H) = (-\infty, -\beta] \cup [\beta, \infty)$, we have for a $\epsilon_0 \in (0, \beta)$ that

$$|u(z)| \leq c\|f\|_2 \|g\|_2$$

in the interval $[\beta^{'} - i\epsilon, \beta^{'}]$. Hence

$$|\int_{\beta'}^{\beta^{'}-i\epsilon} u(z)dz| \leq c\epsilon \|f\|_2 \|g\|_2. \tag{57}$$

Consider $u(z)$ in the interval $[K - i\epsilon, K]$. We claim that in this interval

$$|u(z)| \leq c\|f\|_{\mathcal{L}^{2,2}} \|g\|_{\mathcal{L}^{2,2}}.$$

Indeed, the integral kernel of $(H_0 - z)^{-1}$ is

$$G_0(x,y,z) = \frac{1}{2i\pi} \begin{pmatrix} \frac{e^{-\sqrt{z-\beta}|x-y|}}{\sqrt{z-\beta}} & 0 \\ 0 & -\frac{e^{-\sqrt{z+\beta}|x-y|}}{\sqrt{z+\beta}} \end{pmatrix}, \tag{58}$$

where $\sqrt{z-\beta}$ and $\sqrt{\beta-z}$ are defined in such a way that their real parts are nonnegative. Hence for $z \in K - i[0,\epsilon]$ the operators $(H_0-z)^{-1}: \mathcal{L}^{2,2} \to \mathcal{L}^{2,-2}$ are uniformly bounded in $|Imz|$, and converge to zero as $K \to \infty$. Thus the operator $1+(H_0-z)^{-1}W : \mathcal{L}^{2,-2} \to \mathcal{L}^{2,-2}$ has a bounded inverse for sufficiently large Rez. Hence the equation

$$(H-z)^{-1} = (1+(H_0-z)^{-1}W)^{-1}(H_0-z)^{-1}$$

implies that the operators $(H-z)^{-1} : \mathcal{L}^{2,2} \to \mathcal{L}^{2,-2}$ are uniformly bounded for $|Rez|$ large and $Imz \neq 0$. Since $f, g \in \mathcal{L}^{2,2}$, our claim follows.

Since $f, g \in \mathcal{L}^{2,2}$, $u(z)$ is bounded in the interval from $K - i\epsilon$ to K. Hence

$$|\int_{K-i\epsilon}^{K} u(z)dz| \leq c\epsilon \|f\|_{\mathcal{L}^{2,2}} \|g\|_{\mathcal{L}^{2,2}}. \tag{59}$$

Equations (56), (57), (59) imply Equation (55). □

Lemma 5.2.3. *For any $g \in \mathcal{L}^{2,2}$,*

$$\lim_{K_1\to\infty} \int_{\beta}^{K_1} e^{-it\lambda}[(H-\lambda+i0)^{-1} - (H-\lambda-i0)^{-1}]g d\lambda \tag{60}$$

exists in the norm $\mathcal{L}^{2,-2}$.

Proof. It is sufficient to prove the following statement: for a fixed $g \in \mathcal{L}^{2,2}$ and large constant K_2, the integral

$$\int_{K_2}^{K_1} |\langle f, e^{-it\lambda}[(H-\lambda+i0)^{-1} - (H-\lambda-i0)^{-1}]g\rangle| d\lambda \tag{61}$$

converges uniformly as $K_1 \to \infty$ in any $f \in \mathcal{L}^{2,2}$ such that $\|f\|_{\mathcal{L}^{2,2}} = 1$.

Since we are only concerned with the convergence of (61), we always assume the constant K_2 is sufficiently large so that if $\lambda \geq K_2$ then

$$1 + (H_0 - \lambda \pm i0)^{-1}W : \ \mathcal{L}^{2,-2} \to \mathcal{L}^{2,-2}$$

is invertible.

First using the second resolvent equation and formula

$$(H-\lambda \pm i0)^{-1} = [1 + W(H_0 - \lambda \pm i0)^{-1}]^{-1}(H_0 - \lambda \pm i0)^{-1}$$

we obtain

$$\begin{aligned} & (H-\lambda+i0)^{-1} - (H-\lambda-i0)^{-1} \\ = \ & (1+(H_0-\lambda+i0)^{-1}W)^{-1} \\ & [(H_0-\lambda+i0)^{-1} - (H_0-\lambda-i0)^{-1}] \\ & (1+W(H_0-\lambda-i0)^{-1})^{-1}. \end{aligned}$$

Next, by a standard argument we derive

$$\begin{array}{rl} & \frac{1}{2\pi i}[(H_0-\lambda+i0)^{-1}-(H_0-\lambda-i0)^{-1}] \\ = & \left(\begin{array}{ll} \frac{\cos k(x-y)}{k} & 0 \\ 0 & 0 \end{array}\right) \\ = & \frac{1}{2k}[\left(\begin{array}{l} e^{ikx} \\ 0 \end{array}\right)\left(\begin{array}{ll} e^{-iky}, & 0 \end{array}\right)+\left(\begin{array}{l} e^{-ikx} \\ 0 \end{array}\right)\left(\begin{array}{ll} e^{iky}, & 0 \end{array}\right)]. \end{array}$$

Since $f,\ g\in\mathcal{L}^{2,2}$, the following functions are well defined

$$f_\lambda^*:=[1+W^*(H_0-\lambda-i0)^{-1}]^{-1}f,$$

$$g_\lambda:=[1+W(H_0-\lambda-i0)^{-1}]^{-1}g,$$

and $f_\lambda^*, g_\lambda\in\mathcal{L}^{2,2}$. Furthermore by Equation (58) we obtain that for large λ

$$\begin{array}{lll} \|f-f_\lambda^*\|_{\mathcal{L}^{2,2}} & \leq & \frac{c}{|k|}\|f\|_{\mathcal{L}^{2,2}}, \\ \|g-g_\lambda\|_{\mathcal{L}^{2,2}} & \leq & \frac{c}{|k|}\|g\|_{\mathcal{L}^{2,2}}, \end{array} \tag{62}$$

where, recall $k=\sqrt{\lambda-\beta}$. Therefore

$$\begin{array}{rl} & \int_{K_1}^{K_2}|\langle f,(H-\lambda+i0)^{-1}-(H-\lambda-i0)^{-1}]g\rangle|dk^2 \\ \leq & \int_{K_1}^{K_2}|\langle f_\lambda^*,\left(\begin{array}{l} e^{ikx} \\ 0 \end{array}\right)\rangle\langle\left(\begin{array}{l} e^{ikx} \\ 0 \end{array}\right),g_\lambda\rangle| \\ & +|\langle f_\lambda^*,\left(\begin{array}{l} e^{-ikx} \\ 0 \end{array}\right)\rangle\langle\left(\begin{array}{l} e^{-ikx} \\ 0 \end{array}\right),g_\lambda\rangle|dk. \end{array}$$

We only consider the first term of the right hand side, we claim that

$$\begin{array}{rl} & \int_{K_1}^{K_2}|\langle f_\lambda^*,\left(\begin{array}{l} e^{ikx} \\ 0 \end{array}\right)\rangle\langle\left(\begin{array}{l} e^{ikx} \\ 0 \end{array}\right),g_\lambda\rangle|dk \\ \leq & c\int_{K_1}^{K_2}a_{K_1}^{-2}\frac{\|g\|^2_{\mathcal{L}^{2,2}}}{k^2}+(a_{K_1}^2+b_{K_1}^2)\frac{\|f\|^2_{\mathcal{L}^{2,2}}}{k^2} \\ & +b_{K_1}^{-2}|\langle g,\left(\begin{array}{l} e^{ikx} \\ 0 \end{array}\right)\rangle|^2+(a_{K_1}^2+b_{K_1}^2)|\langle\left(\begin{array}{l} e^{ikx} \\ 0 \end{array}\right),f\rangle|^2dk, \end{array} \tag{63}$$

where

$$a_K:=(\int_K^\infty\frac{\|g\|^2_{\mathcal{L}^{2,2}}}{k^2}dk)^{1/10},$$

$$b_K:=(\int_K^\infty|\langle g,\left(\begin{array}{l} e^{ikx} \\ 0 \end{array}\right)\rangle|^2dk)^{1/10}.$$

Thus it is easy to see that as $K_1,K_2\to\infty$, $a_{K_1},b_{K_1}\to 0$. For the four terms on the right side of Estimate (63),

$$a_{K_1}^{-2}\int_{K_1}^{K_2}\frac{\|g\|^2_{\mathcal{L}^{2,2}}}{k^2}dk\leq a_{K_1}^8;$$

$$(a_{K_1}^2+b_{K_1}^2)\int_{K_1}^{K_2}\frac{\|f\|^2_{\mathcal{L}^{2,2}}}{k^2}dk\leq(a_{K_1}^2+b_{K_1}^2)\|f\|^2_{\mathcal{L}^{2,2}}\int_{K_1}^{K_2}\frac{1}{k^2}dk;$$

$$b_{K_1}^{-2}\int_{K_1}^{K_2}|\langle g,\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle|^2 dk \leq b_{K_1}^8;$$

$$(a_{K_1}^2+b_{K_1}^2)\int_{K_1}^{K_2}|\langle\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix},f\rangle|^2 dk \leq (a_{K_1}^2+b_{K_1}^2)\|f\|_2^2.$$

Therefore

$$\int_{K_1}^{K_2}|\langle f_\lambda^*,\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle\langle\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix},g_\lambda\rangle|dk \to 0$$

for fixed $g \in \mathcal{L}^{2,2}$, and the decay is independent of f.

What left is to prove Estimate (63): Indeed,

$$\begin{aligned}
& |\langle f_\lambda^*,\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle\langle\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix},g_\lambda\rangle| \\
= \; & |\langle f_\lambda^*-f+f,\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle\langle\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix},g_\lambda-g+g\rangle| \\
\leq \; & a_{K_1}|\langle f_\lambda^*-f,\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle a_{K_1}^{-1}\langle\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix},g_\lambda-g\rangle| \\
& +b_{K_1}|\langle f,\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle b_{K_1}^{-1}\langle\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix},g\rangle| \\
& +a_{K_1}|\langle f_\lambda^*-f,\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle a_{K_1}^{-1}\langle\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix},g\rangle| \\
& +b_{K_1}|\langle f,\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle b_{K_1}^{-1}\langle\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix},g_\lambda-g\rangle|.
\end{aligned}$$

By using Hölder Inequality we obtain Estimate (63). □

Lemmas 5.2.2 and 5.2.3 show the existence of the limits on the right hand side of Equation (53). Now we prove Equation (53):

Instead of the unbounded, non-self adjoint operator H, we consider the bounded, non-self adjoint operators

$$K_{\epsilon,\kappa} := (1-i\epsilon(H+i\kappa))^{-1}$$

for any $\epsilon,\ \kappa \in \mathbb{R}$, where $K_{\epsilon,\kappa}$ is well defined because H has no complex spectrum.

For the operator $K_{\epsilon,\kappa}$, it is not invertible at the set (its spectrum)

$$\{(1-i\epsilon(a_n+i\kappa))^{-1}|\ a_n \text{ is eigenvalue of } H\} \ \cup\ \gamma_2\ \cup\ \gamma_3,$$

where γ_2 and γ_3 are the curves:

$$\gamma_2 := \{(1-i\epsilon(\mu+i\kappa))^{-1}|\ \mu \geq \beta\},\quad \gamma_3 := \{(1-i\epsilon(\mu+i\kappa))^{-1}|\ \lambda \leq -\beta.\}.$$

Also it is easy to see that $P_d^{K_{\epsilon,\kappa}} = P_d^H$, where, recall, $P_d^{K_{\epsilon,\kappa}}$ from Definition 2. By Definition 2, $K_{\epsilon,\kappa}$ has the same essential spectrum subspace to H. Denote the projection onto essential spectrum subspace of $K_{\epsilon,\kappa}$ by $P_{ess}^{K_{\epsilon,\kappa}}$, then

$$P_{ess}^{K_{\epsilon,\kappa}} = P_{ess}.$$

Let f be an entire function on $\mathbb{C}$. Then we can define the operator $f(K_{\epsilon,\kappa})$ by the Taylor series (see e.g. [JMS]), moreover one has

$$f(K_{\epsilon,\kappa})P_{ess}^{K_{\epsilon,\kappa}} = \frac{1}{2i\pi}\oint_{\gamma_1} f(\lambda)(K_{\epsilon,\kappa}-\lambda)^{-1}d\lambda, \tag{64}$$

where γ_1 is a contour around the curves γ_2, γ_3, but leaving $\{(1-i\epsilon(a_n + i\kappa))^{-1}|\ a_n$ is eigenvalue of $H\}$ outside.

In order to get a similar formula as in Equation (53) we want to transform the right hand side of Equation (64): First we notice that

$$(K_{\epsilon,\kappa}-\lambda)^{-1} = -\frac{1}{\lambda}(H+i\kappa-\frac{1}{i\epsilon}+\frac{1}{i\lambda\epsilon})^{-1}(H+i\kappa-\frac{1}{i\epsilon}).$$

There exists an $\epsilon_0>0$ such that if $|\lambda|\le\epsilon_0$ and $\lambda\in\gamma_2\cup\gamma_3$, then

$$(H+i\kappa-\frac{1}{i\epsilon}+\frac{1}{i(\lambda\pm 0)\epsilon})^{-1} : \mathcal{L}^{2,2}\to\mathcal{L}^{2,-2} \tag{65}$$

are well defined and

$$\|\frac{1}{\lambda}(H+i\kappa-\frac{1}{i\epsilon}+\frac{1}{i(\lambda\pm 0)\epsilon})^{-1}\|_{\mathcal{L}^{2,2}\to\mathcal{L}^{2,-2}} \le \frac{c}{\sqrt{|\lambda|}}$$

for some constant c. Therefore the integral

$$\int_{\substack{\lambda\in\gamma_2\cup\gamma_3,\\ |\lambda|\le\epsilon_0}} \frac{f(\lambda)}{\lambda}(H+i\kappa-\frac{1}{i\epsilon}+\frac{1}{i(\lambda\pm 0)\epsilon})^{-1}d\lambda$$

exists. Based on the arguments above we can deform the contour γ_1 as

$$\begin{aligned} f(K_{\epsilon,\kappa})P_{ess}^{K_{\epsilon,\kappa}} &= \tfrac{1}{2i\pi}\textstyle\int_{\gamma_4+\gamma_5} f(\lambda)(K_{\epsilon,\kappa}-\lambda)^{-1}d\lambda \\ &+\tfrac{1}{2i\pi}\textstyle\int_{\substack{\lambda\in\gamma_2\cup\gamma_3,\\ |\lambda|\le\epsilon_0}} f(\lambda)[(K_{\epsilon,\kappa}-\lambda+i0)^{-1}-(K_{\epsilon,\kappa}-\lambda-i0)^{-1}]d\lambda \end{aligned} \tag{66}$$

where γ_4,γ_5 are the contours around the spectral points $\gamma_2\cap\{\lambda||\lambda|>\epsilon_0\}$, $\gamma_3\cap\{\lambda||\lambda|>\epsilon_0\}$ respectively, and all other spectral points of $K_{\epsilon,\kappa}$ are kept outside. Since we proved that when $|\lambda|\le\epsilon_0$ the operators $(K_{\epsilon,\kappa}-\lambda\pm i0)^{-1} : (1-\frac{d^2}{dx^2})^{-1}\mathcal{L}^{2,2}\to\mathcal{L}^{2,-2}$ which justifies the following calculation

$$\begin{aligned} & (K_{\epsilon,\kappa}-\lambda-i0)^{-1}-(K_{\epsilon,\kappa}-\lambda+i0)^{-1} \\ = & -\tfrac{1}{i\epsilon\lambda^2}[(H+i\kappa-\tfrac{1}{i\epsilon}+\tfrac{1}{i\epsilon(\lambda-i0)})^{-1}-(H+i\kappa-\tfrac{1}{i\epsilon}+\tfrac{1}{i\epsilon(\lambda+i0)})^{-1}]. \end{aligned}$$

By the change of variable $z=\frac{1}{i\epsilon\lambda}+i\kappa+\frac{i}{\epsilon}$, we have

$$\begin{aligned} & \tfrac{1}{2i\pi}\textstyle\int_{\substack{\lambda\in\gamma_2\cup\gamma_3\\ |\lambda|\le\epsilon_0}} f(\lambda)[(K_{\epsilon,\kappa}-\lambda+i0)^{-1}-(K_{\epsilon,\kappa}-\lambda-i0)^{-1}]d\lambda \\ = & \tfrac{1}{2i\pi}(\textstyle\int_{\kappa_1}^{\infty}+\int_{-\infty}^{\kappa_2})f((1-i\epsilon(z+i\kappa))^{-1})((H-z-i0)^{-1}-(H-z+i0)^{-1})dz. \end{aligned} \tag{67}$$

where κ_1,κ_2 are the points such that $|\frac{1}{i\epsilon\kappa_n}+i\kappa+\frac{i}{\epsilon}|=\epsilon_0$ $(n=1,2)$.

On the other hand

$$\begin{array}{rl} & \frac{1}{2i\pi}\int_{\gamma_4+\gamma_5} f(\lambda)(K_{\epsilon,\kappa}-\lambda)^{-1}d\lambda \\ = & -\frac{1}{2i\pi}\int_{\gamma_4+\gamma_5}\frac{f(\lambda)}{\lambda}[H+i\kappa-\frac{1}{i\epsilon}+\frac{1}{i\lambda\epsilon}]^{-1}(H+i\kappa-\frac{1}{i\epsilon})d\lambda \\ = & \frac{1}{2i\pi}\int_{\gamma_4+\gamma_5}\frac{f(\lambda)}{i\lambda^2\epsilon}[H+i\kappa-\frac{1}{i\epsilon}+\frac{1}{i\lambda\epsilon}]^{-1}d\lambda. \end{array} \tag{68}$$

Let $z=i\kappa-\frac{1}{i\epsilon}+\frac{1}{i\lambda\epsilon}$, the equation equals to

$$\begin{array}{rl} & \frac{1}{2i\pi}\int_{\gamma_6+\gamma_7} f(\frac{1}{i\epsilon(z-i\kappa+\frac{1}{i\epsilon})})(H-z)^{-1}dz \\ = & \frac{1}{2i\pi}(\int_{\beta}^{\kappa_1}+\int_{\kappa_2}^{-\beta})f((1-i\epsilon(z+i\kappa))^{-1})[(H-z-i0)^{-1}-(H-z+i0)^{-1}]dz. \end{array} \tag{69}$$

where γ_6 and γ_7 are corresponding to the curves γ_4 and γ_5, the constants $\kappa_1,\ \kappa_2$ are the same as that in Equation (67).

By Equations (66) (67) (68) and (69) we get that for any entire function f,

$$f(K_{\epsilon,\kappa})P_{ess}=\frac{1}{2i\pi}(\int_{\beta}^{\infty}+\int_{-\infty}^{-\beta})f((1-i\epsilon(z+i\kappa))^{-1})[(H-z-i0)^{-1}-(H-z+i0)^{-1}]dz. \tag{70}$$

By the results in [Gold, RSII] we have that for some $\kappa_0\in\mathbb{R}$

$$\begin{array}{rcl} e^{-itH} & = & s-\lim\limits_{\epsilon\to 0^+} e^{-itH(1-i\epsilon(i\kappa_0+H))^{-1}} \\ & = & s-\lim\limits_{\epsilon\to 0^+} e^{\frac{t}{\epsilon}+(t\kappa_0+\frac{t}{\epsilon})(1-i\epsilon(H+i\kappa_0))^{-1}}. \end{array} \tag{71}$$

Since the function $f_\epsilon(z):=e^{\frac{t}{\epsilon}+(-it\kappa_0+\frac{t}{\epsilon})z}$ is analytic, we could use Equation (67):

$$\begin{array}{rl} & e^{-itH(1-i\epsilon(i\kappa_0+H))^{-1}}P_{ess} \\ = & \frac{1}{2i\pi}(\int_{\beta}^{+\infty}+\int_{-\infty}^{-\beta})e^{-it\frac{z}{1-i\epsilon(z+i\kappa_0)}}((H-z-i0)^{-1}-(H-z+i0)^{-1})dz. \end{array}$$

Now let $\epsilon\to 0^+$, the left hand side is e^{-itH} by Equation (71), and the right hand side is

$$\frac{1}{2i\pi}(\int_{\beta}^{+\infty}+\int_{-\infty}^{-\beta})e^{-itz}((H-z-i0)^{-1}-(H-z+i0)^{-1})dz$$

by Equation (70) and some arguments similar to the proof of Lemmas 5.2.2 and 5.2.3. The theorem follows.

Let $L^{\gamma}:=e^{\frac{\gamma}{4}|x|}\mathcal{L}^2$ with the norm

$$\|g\|_{L^{\gamma}}:=\|e^{\frac{\gamma}{4}|x|}g\|_{\mathcal{L}^2}. \tag{72}$$

It will be proved in the discussion after Lemma A.3.10 Appendix A, that for any $\lambda>\beta$, the operator $1+(H_0-\lambda+i0)^{-1}W:\ L^{-\alpha}\to L^{-\alpha}$ has a bounded inverse, where, recall the constant α from Equation (88). Define

$$e(\cdot,k):=[1+(H_0-\lambda+i0)^{-1}W]^{-1}\left(\begin{array}{c} e^{ikx} \\ 0 \end{array}\right), \tag{73}$$

and

$$e^{\%}(x,k) := \sigma_3 e(x,k)$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

There are two terms on the right hand side of Equation (53), we denote the first term by $e^{-itH}P_{ess}^{+}$.

Lemma 5.2.4.

$$e^{-iHt}P_{ess}^{+}f = \frac{1}{2\pi}\int_{k\geq 0} e^{-ik^2 t}[\langle e^{\%}(x,k), f\rangle e(\cdot,k) + \langle e^{\%}(-x,k)f\rangle e(-\cdot,k)]dk. \tag{74}$$

Proof. Let $f,\ g \in L^{\alpha}$, and define

$$f_\lambda^* := (1 + W^*(H_0 - \lambda - i0)^{-1})^{-1} f,$$
$$g_\lambda := (1 + W(H_0 - \lambda - i0)^{-1})^{-1} g.$$

Thus $f_\lambda^*,\ g_\lambda \in L^{\alpha}$. Therefore all the following computations make sense:

$$\begin{aligned}
& \langle f, -i((H-\lambda-i0)^{-1} - (H-\lambda+i0)^{-1})g\rangle \\
= \ & \langle f_1, -i[(H_0-\lambda-i0)^{-1} - (H_0-\lambda+i0)^{-1}]g_1\rangle \\
= \ & \tfrac{1}{2k}\langle f_1, \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle\langle \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}, g_1\rangle + \tfrac{1}{2k}\langle f_1, \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}\rangle\langle \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}, g_1\rangle \\
= \ & \tfrac{1}{2k}\langle f, (1 + (H_0-\lambda+i0)^{-1}W)^{-1}\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\rangle \\
& \langle (1 + (H_0-\lambda+i0)^{-1}W^*)^{-1}\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}, g\rangle \\
& + \tfrac{1}{2k}\langle f, (1 + (H_0-\lambda+i0)^{-1}W)^{-1}\begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}\rangle \\
& \langle (1 + (H_0-\lambda+i0)^{-1}W^*)^{-1}\begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}, g\rangle \\
= \ & \tfrac{1}{2k}(\langle f, e(\cdot,k)\rangle\langle e^{\%}(\cdot,k), g\rangle + \langle f, e(-\cdot,k)\rangle\langle e^{\%}(-\cdot,k), g\rangle).
\end{aligned}$$

Then Equation (74) follows. ∎

Taking $t = 0$ in Equation (53) we obtain

$$P_{ess} = P_{ess}^{+} + P_{ess}^{-},$$

where the operator P_{ess}^{+} is given by

$$\begin{aligned}
P_{ess}^{+}f &= \lim_{K\to\infty}\lim_{\epsilon\to 0^+}\frac{1}{2i\pi}\int_{\beta-\epsilon_0}^{K}[(H-\lambda-i\epsilon)^{-1} - (H-\lambda+i\epsilon)^{-1}]d\lambda, \\
&= \tfrac{1}{2\pi}\int_{k\geq 0}\langle e^{\%}(x,k), f\rangle e(\cdot,k) + \langle e^{\%}(-x,k)f\rangle e(-\cdot,k)dk
\end{aligned} \tag{75}$$

for any sufficiently small $\epsilon_0 > 0$, and similarly for P_{ess}^{-}. In fact, the operators $P_{ess}^{\pm}$ are spectral projections corresponding to the branches $[\beta,\infty)$ and $(-\infty,-\beta]$ of the essential spectrum of the operator H.

5.3 Proof of Theorem 5.1.2

In this subsection we prove key Theorem 5.1.2. Recall the definition of functions $e(x, k)$ in Equation (73). To this end we use the following technical results proven in Appendix A.1.

Theorem 5.3.1. *Assume that there are no embedded eigenvalues in the essential spectrum and there are no resonances at the tips, $\pm\beta$, of the essential spectrum. Then Equation (73) defines smooth functions $e(x, k)$ which are generalized eigenfunctions of the operator H : $He(\cdot, k) = \lambda e(\cdot, k)$ with $k = \sqrt{\lambda - \beta}$. If $k \geq 0$, we have the following estimates:*

$$\sup_{k\geq 0} |\frac{d^n}{dk^n} e(x, k)| \leq c\rho_{-n}; \tag{76}$$

$$\sup_{k\geq 0} |\frac{d^n}{dk^n}(e(x, k)/k)| \leq c\rho_{-n-1}; \tag{77}$$

$$[\int_{k\geq 0} |\frac{d^n}{dk^n}(e(x, k)/k)|^2 dk]^{1/2} \leq c\rho_{-n-1}; \tag{78}$$

$$e(\cdot, 0) = 0; \tag{79}$$

$$\|\langle e^{\%}(\cdot, k), h\rangle\|_{\mathcal{H}^2} \leq c\|\rho_{-2}h\|_2, \tag{80}$$

where $n = 0, 1, 2$, all c's are constants independent of x, and recall $\rho_\nu(x) = (1 + x^2)^{-\nu}$.

Before starting proving Theorem 5.1.2, we state the following standard estimate:

$$\|e^{it\frac{d^2}{dx^2}} f\|_{\mathcal{L}^\infty} \leq ct^{-1/2}\|f\|_{\mathcal{L}^1}$$

valid for some constant c and for any $f \in \mathcal{L}^1$. This estimate implies that if a function g satisfies $\int e^{ikx} g(k)dk \in \mathcal{L}^1$ and is even, then

$$|\int e^{-ik^2t} g(k)dk| \leq ct^{-1/2}\| \int \cos(kx)g(k)dk\|_{\mathcal{L}^1}. \tag{81}$$

Proof of Estimate (45). We will only consider the first term on the right hand side of $e^{-iHt}P_{ess}$ in Equation (53) and the first term of the right hand side of Equation (74). To simplify the notation we denote

$$h^{\#}(k) := \langle e^{\%}(\cdot, k), h\rangle.$$

Also for a function $g(x, k)$ of two variables such that if for each fixed x (or k), $g(x, \cdot) \in B$ (or $g(\cdot, k) \in B$), then we define $\|g\|_{B^k} = \|g(x, \cdot)\|_B$ (or $\|g\|_{B^x} = \|g(\cdot, k)\|_B$).

Since $e(x, 0) = 0$ by Estimate (79), we can integrate by part and use Estimate (81) to obtain:

$$\begin{aligned} & |\tfrac{1}{2\pi} \int 2ik \tfrac{e(x,k)}{2ik} e^{-ik^2t} h^{\#}(k)dk| \\ = & |\tfrac{1}{2\pi} t^{-1} \int \tfrac{d}{dk}(\tfrac{e(x,k)h^{\#}(k)}{2ik}) e^{-ik^2t}dk| \\ \leq & ct^{-3/2}\|\tfrac{d}{dk}(\tfrac{e(x,k)h^{\#}(k)}{2ik})\|_{\mathcal{L}^1(dk)}, \end{aligned} \tag{82}$$

where $\hat{g}(x) := \int_0^\infty \cos(kx)g(k)dk$. Since $\|\hat{g}_2\|_1 \leq \|g_2\|_{\mathcal{H}^1}$, we have

$$\begin{array}{rl} & \|\frac{d}{dk}\frac{\widehat{(e(x,k)h^{\#}(k))}}{2ik}\|_{(\mathcal{L}^1)^k} \\ \leq & c\sum_{n=0}^{2}\|\frac{d^n}{dk^n}\frac{e(x,k)}{2ik}h^{\#}(k)\|_{\mathcal{L}^2(dk)} \\ \leq & c\|h^{\#}\|_{\mathcal{H}^2}\sum_{n=0}^{2}\|\frac{d^n}{dk^n}(e/k)\|_{\mathcal{L}^\infty(k)}. \end{array}$$

By Estimate (77),

$$\sum_{n=0}^{2}\|\frac{d^n}{dk^n}(e/k)\|_{\mathcal{L}^\infty(dk)} \leq c(1+|x|)^3,$$

and by Estimate (80)

$$\|h^{\#}\|_{\mathcal{H}^2} \leq c\|\rho_{-2}h\|_2.$$

The last four estimates imply that

$$\int \langle e^{\%}(\cdot,k),h\rangle e(x,k)e^{-ik^2t}dk| \leq ct^{-3/2}(1+|x|)^3\|\rho_{-2}h\|_2.$$

Similarly we estimate the other terms in Equations (74), (53) to get

$$|e^{-itH}P_{ess}h| \leq ct^{-3/2}(1+|x|)^3\|\rho_{-2}h\|_2.$$

Therefore if $\mu > 3.5$, we have

$$\|\rho_\mu e^{-iHt}P_{ess}h\|_2 \leq ct^{-3/2}\|\rho_{-2}h\|_2.$$

By Lemma 60 we have $\|\rho_\mu e^{-iHt}P_{ess}h\|_2 \leq c\|\rho_{-2}h\|_2$. The last two equations give

$$\|\rho_\mu e^{-iHt}P_{ess}h\|_2 \leq c(1+t)^{-3/2}\|\rho_{-2}h\|_2$$

for some c. This estimate is equivalent to Estimate (45).
Proof of Estimate (46): We start from the last line of Equation (82),

$$\begin{array}{rl} & \|\frac{d}{dk}\frac{\widehat{(e(x,k)h^{\#}(k))}}{2ik}\|_{\mathcal{L}^1(dk)} \\ \leq & c\sum_{n=0}^{2}\|\frac{d^n}{dk^n}\frac{e(x,k)}{2ik}h^{\#}(k)\|_{\mathcal{L}^2(dk)} \\ \leq & c\sum_{n=0}^{2}\|\frac{d^n}{dk^n}h^{\#}\|_{\mathcal{L}^\infty}\sum_{n=0}^{2}\|\frac{d^n}{dk^n}(e/k)\|_2. \end{array}$$

By Estimate (80),

$$\sum_{n=0}^{2}\|\frac{d^n}{dk^n}h^{\#}\|_{\mathcal{L}^\infty} \leq c\|\rho_{-2}h\|_1,$$

and by Estimate (78),

$$\sum_{n=0}^{2}\|\frac{d^n}{dk^n}(e/k)\|_2 \leq c(1+|x|)^3.$$

Therefore we have as before

$$\|\rho_\nu e^{-iHt}P_{ess}h\|_2 \leq ct^{-3/2}\|\rho_{-2}h\|_1.$$

Since $\|\rho_\nu e^{-iHt}P_{ess}h\|_2 \leq c\|\rho_{-2}h\|_2$, Estimate (46) is proved.
Proof of Estimate (47). By the Duhamel principle,

$$\begin{array}{rl} & \|e^{-itH}P_{ess}\rho_2 h\|_{\mathcal{L}^\infty} \\ \leq & \|e^{-itH_0}P_{ess}\rho_2 h\|_{\mathcal{L}^\infty} + \|\int_0^t e^{-i(t-s)H_0}P_{ess}We^{-isH}P_{ess}\rho_2 h ds\|_{\mathcal{L}^\infty}. \end{array} \tag{83}$$

Based on the estimates

$$\|h\|_{\mathcal{L}^\infty} \leq c\|h\|_{\mathcal{H}^1} \text{ and} \|e^{it\frac{d^2}{dx^2}}\|_{\mathcal{L}^1\to\mathcal{L}^\infty} \leq ct^{-1/2},$$

we have

$$\|e^{-itH_0}P_{ess}\rho_2 h\|_{\mathcal{L}^\infty} \leq c(1+t)^{-1/2}\|h\|_{\mathcal{H}^1} \tag{84}$$

and

$$\begin{array}{rl} & \|\int_0^t e^{-i(t-s)H_0}P_{ess}We^{-isH}P_{ess}\rho_2 h ds\|_{\mathcal{L}^\infty} \\ \leq & c\int_0^t |t-s|^{-1/2}\|We^{-isH}P_{ess}\rho_2 h\|_{\mathcal{L}^1}ds. \end{array}$$

Furthermore, by Estimate (88) we have $|W| \leq c\rho_6$. Hence

$$\|We^{-isH}P_{ess}\rho_2 h\|_{\mathcal{L}^1} \leq c\|\rho_4 e^{-isH}P_{ess}\rho_2 h\|_2.$$

By Estimate (45) we have

$$\|\rho_4 e^{-isH}P_{ess}\rho_2 h\|_2 \leq c(1+s)^{3/2}\|h\|_2.$$

Thus

$$\begin{array}{rl} & \int_0^t \|e^{-i(t-s)H_0}P_{ess}\|_{\mathcal{L}^1\to\mathcal{L}^\infty}\|We^{-isH}P_{ess}\rho_2 h\|_{\mathcal{L}^1}ds \\ \leq & c\int_0^t \frac{1}{|t-s|^{1/2}}\frac{1}{(1+|s|)^{-3/2}}ds\|h\|_2 \\ \leq & c(1+t)^{-1/2}\|h\|_2. \end{array} \tag{85}$$

Estimates (83), (84) and (85) imply the inequality

$$\|e^{-itH}P_{ess}\rho_2 h\|_{\mathcal{L}^\infty} \leq c(1+t)^{-1/2}\|h\|_{\mathcal{H}^1}$$

which is equivalent to Estimate (47).
Proof of Estimate (48). By the Duhamel Principle

$$\|e^{-itH}P_{ess}h\|_{\mathcal{L}^\infty} \leq \|e^{-itH_0}P_{ess}h\|_{\mathcal{L}^\infty} + \|\int_0^t e^{-i(t-s)H_0}P_{ess}We^{-isH}P_{ess}hds\|_{\mathcal{L}^\infty}.$$

For the first term we have

$$\|e^{-itH_0}P_{ess}h\|_{\mathcal{L}^\infty} \leq ct^{-1/2}\|h\|_1;$$

and for the second term we have

$$\begin{array}{rl} & \|\int_0^t e^{-i(t-s)H_0}P_{ess}We^{-isH}P_{ess}hds\|_{\mathcal{L}^\infty} \\ \leq & c\int_0^t \frac{1}{|t-s|^{1/2}}\|\rho_{-2}We^{-isH}P_{ess}h\|_2. \end{array}$$

By Estimate(46),

$$\|\rho_{-2}We^{-isH}P_{ess}h\|_2 \le c(1+s)^{-3/2}(\|\rho_{-2}h\|_1 + \|h\|_2).$$

Therefore

$$\begin{array}{ll} & \|e^{-itH}P_{ess}h\|_{\mathcal{L}^\infty} \\ \le & c(t^{-1/2} + \int_0^t \frac{1}{|t-s|^{1/2}}(1+s)^{-3/2}ds)(\|h\|_2 + \|\rho_{-2}h\|_1) \\ \le & ct^{-1/2}(\|h\|_2 + \|\rho_{-2}h\|_1), \end{array}$$

which gives the last Estimate (48).

□

A Proof of Theorem 5.3.1

In this appendix we study the functions

$$e(x,k) := [1 + (H_0 - \lambda + i0)^{-1}W]^{-1}\left(\begin{array}{c} e^{ikx} \\ 0 \end{array}\right)$$

with $k = \sqrt{\lambda - \beta}$ introduced in Subsection 5.3. A simple manipulation shows that $He(\cdot,k) = \lambda e(\cdot,k)$, i.e. $e(\cdot,k)$ are generalized eigenfunctions of the operator H corresponding to spectral points $\lambda = k^2 + \beta$. We begin with some auxiliary results on solutions to the equation $H\xi = \lambda\xi$.

A.1 Generalized Eigenfunctions of H

In this subsection we study solutions of the equation $H\xi = \lambda\xi$, considered as a differential equation, with λ in an appropriate domain of the complex plane $\mathbb{C}$.

From now on we will only consider the positive branch of the essential spectrum subspace. So we always assume $Re\lambda \ge \beta > 0$ and $Im\lambda$ is sufficiently small. The negative branch is treated exactly the same. If $Re\lambda \ge \beta$ we define two functions $\sqrt{\lambda - \beta}$ and $\sqrt{\lambda + \beta}$ such that they are analytic and if $\lambda - \beta > 0$ (or $\lambda + \beta > 0$) then $\sqrt{\lambda - \beta} > 0$ (or $\sqrt{\lambda + \beta} > 0$).

Define the domain

$$\Omega := \{\lambda | Re\lambda \ge \beta, |Im\sqrt{\lambda - \beta}| + |Im\sqrt{\lambda + \beta}| \le \frac{\alpha}{4}\}, \tag{86}$$

where, recall α from (88). We always denote

$$k := \sqrt{\lambda - \beta} \text{ and } \mu := \sqrt{\lambda + \beta}. \tag{87}$$

Hence the function $\mu = \sqrt{2\beta - k^2}$ is analytic in $k = \sqrt{\lambda - \beta},\ \lambda \in \Omega$.

Below we will use the space $\mathcal{L}^{\infty,\beta} := e^{-\beta|x|}\mathcal{L}^2$ with the norm

$$\|W\|_{\mathcal{L}^{\infty,\beta}} = \|e^{-\beta|x|}W\|_{\mathcal{L}^\infty}.$$

We formulate the main result of this appendix. Recall the definition of the operator $H := H_0 + W$, where

$$H_0 := \left(\begin{array}{ll} -\frac{d^2}{dx^2} + \beta & 0 \\ 0 & \frac{d^2}{dx^2} - \beta \end{array} \right), \quad W := 1/2 \left(\begin{array}{ll} V_3 & -iV_4 \\ -iV_4 & -V_3 \end{array} \right),$$

with the constant $\beta > 0$, the functions V_3, V_4 even, smooth, real and decaying exponentially fast at ∞ :

$$|V_4(x)|,\ |V_3(x)| \leq ce^{-\alpha|x|} \tag{88}$$

for some constants $c,\ \alpha > 0$. Without loss of generality we assume $\alpha < \beta$. The following is the main theorem:

Theorem A.1.1. *If $\lambda \in \Omega$, then the equation*

$$(H - \lambda)\phi = 0 \tag{89}$$

has C^3 solutions $\phi_1(\cdot, \mu, W)$, $\psi_1(\cdot, k, W)$, $\psi_2(\cdot, k, W)$ and $\xi_1(\cdot, \mu, W)$ which are analytic in k and satisfy the following estimates: there exist constants $R_1, c, \epsilon_0 > 0$ such that for $\forall\ x > R_1,\ \ \lambda \geq \beta$

$$|\frac{d^n}{dk^n}(\phi_1(x, \mu, W)e^{\mu x} - \left(\begin{array}{c} 0 \\ 1 \end{array} \right))| \leq ce^{-\epsilon_0 x}, \tag{90}$$

$$|\frac{d^n}{dk^n}(\psi_1(x, k, W)e^{-ikx} - \left(\begin{array}{c} 1 \\ 0 \end{array} \right))| \leq ce^{-\epsilon_0 x}, \tag{91}$$

$$|\frac{d^n}{dk^n}(\psi_2(x, k, W)e^{ikx} - \left(\begin{array}{c} 1 \\ 0 \end{array} \right))| \leq ce^{-\epsilon_0 x} \tag{92}$$

where $n = 0, 1, 2$.

Also

$$\lim_{x \to +\infty} \xi_1(x, \mu, W)e^{-\mu x} = \left(\begin{array}{c} 0 \\ 1 \end{array} \right). \tag{93}$$

For any constant R_2, there exists a constant $c_2 > 0$, such that if $\beta \leq \lambda \leq \beta + 1$ and $x \geq R_2$, then

$$|\frac{d^n}{dk^n}\phi_1(x, \mu, W)| \leq c_2, \tag{94}$$

and

$$|\frac{d^n}{dk^n}\psi_1(x, k, W)| \leq c_2(1 + |x|)^n, \tag{95}$$

$$|\frac{d^n}{dk^n}\psi_2(x, k, W)| \leq c_2(1 + |x|)^n, \tag{96}$$

where $n = 0, 1, 2$.

Moreover the following maps are continuous

$$\mathcal{L}^{\infty,\alpha} \ni W \to \tilde{\phi}_1(W) \in \mathcal{C}^2 \ \textit{and}\ \mathcal{L}^{\infty,\alpha} \ni W \to \tilde{\psi}_1(W) \in \mathcal{C}^2 \tag{97}$$

are continuous: where, recall the constant α from Equation (88),

$$\tilde{\phi}_1(W) := (\frac{d}{dx}\phi_1(x, \sqrt{2\beta}, W)|_{x=0}, \phi_1(x, \sqrt{2\beta}, W)|_{x=0}),$$

$$\tilde{\psi}_1(W) := (\frac{d}{dx}\psi_1(x, 0, W)|_{x=0}, \psi_1(x, 0, W)|_{x=0}).$$

A proof of this theorem follows from Propositions A.1.3, A.1.5 and A.1.6.

When the potential W is fixed, for brevity we use for the solutions above the notations $\phi_1(x,\mu), \psi_1(x,k), \psi_2(x,k), \xi_1(x,\mu)$ respectively if there is no confusion.

Since we need to study the analyticity of all these functions, and derive them by convergent sequences, we use frequently the following lemma:

Lemma A.1.2. *If $\{f_n\}_{n=0}^{\infty}$ is a sequence of analytic functions, and*

$$\sum_n \|f_n\|_{\mathcal{L}^\infty} < \infty,$$

then $\sum_n f_n$ is an analytic function.

First let's look at ϕ_1:

Proposition A.1.3. *Recall $\mu = \sqrt{\lambda+\beta}$. There is a solution $\phi_1(\cdot,\mu)$ to Equation (89) which satisfies the following integral equation:*

$$\phi_1(x,\mu) = \begin{pmatrix} 0 \\ e^{-\mu x} \end{pmatrix} - \int_x^{+\infty} \begin{pmatrix} \frac{\sin k(x-y)}{k} & 0 \\ 0 & -\frac{e^{-\mu|x-y|}-e^{\mu|x-y|}}{2\mu} \end{pmatrix} W(y)\phi_1(y,\mu)dy, \tag{98}$$

Moreover $\phi_1(x,\cdot)$ is analytic in k, and satisfies Estimates (90) (94), and (97).

Proof. First we need to prove the existence of solutions $\phi_1(x,\mu)$ to Equation (98), which can be rewritten as

$$\psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + A_\lambda \psi, \tag{99}$$

where $\psi(x,\mu) := e^{\mu x}\phi_1(x,\mu)$, and the operator $A_\lambda : \mathcal{L}^\infty([T,\infty)) \to \mathcal{L}^\infty([T,\infty))$ is defined as

$$(A_\lambda f)(x) := e^{\mu x}\int_x^\infty \begin{pmatrix} \frac{\sin k(x-y)}{k} & 0 \\ 0 & -\frac{e^{-\mu(|x-y|)}-e^{\mu|x-y|}}{2\mu} \end{pmatrix} e^{-\mu y} W(y) f(y) dy$$

with $T \in (-\infty,\infty)$ an arbitrary constant. We show for a sufficiently large n, $\|A_\lambda^n\| < 1$, therefore Equation (99) has a unique solution in $\mathcal{L}^\infty([T,\infty))$.

Observe that $Re\mu > 0$ and $|\frac{\sin k(x-y)}{k}| \le c_1(1+|x|+|y|)e^{\mu|x-y|}$. Thus if $x \in [T,\infty)$, then

$$|A_\lambda \psi(x)| \le c(T)(1+|x|)\int_x^\infty (1+|y|)e^{-\alpha|y|}dy \|\psi\|_{\mathcal{L}^\infty([T,\infty))} \tag{100}$$

for some $c(T)$ independent of λ, where, recall α from Inequality (88).

Therefore

$$\begin{array}{rcl} |A_\lambda^n \psi(x)| & \leq & c^n(T)(1+|x|)\int_x^\infty (1+|x_1|)e^{-\alpha|x_1|}dx_1 \int_{x_1}^\infty (1+|x_2|)e^{-\alpha|x_2|}dx_2 \\ & & \cdots \int_{x_n}^\infty (1+|y|)e^{-\alpha|y|}dy\|\psi\|_{\mathcal{L}^\infty([T,\infty))} \\ & = & \frac{(1+|x|)c^n(T)}{n!}(\int_x^\infty (1+|x|)e^{-\alpha|x|}dx)^n\|\psi\|_{\mathcal{L}^\infty([T,\infty))}. \end{array}$$

Hence if $n \in \mathbb{N}$ is sufficiently large, then

$$\|A_\lambda^n\|_{\mathcal{L}^\infty([T,\infty))\to\mathcal{L}^\infty([T,\infty))} < 1.$$

By the Neumann series, there exists a unique $\psi \in \mathcal{L}^\infty([T,\infty))$. Thus there exists a function $\phi_1(x,\mu)$ defined in the interval $x \in [T,\infty)$ which is solution to Equation (98). Since T is an arbitrary constant, $\phi_1(x,\mu)$ is well defined for $x \in (-\infty,\infty)$.

For the analyticity of $\phi_1(x,\cdot)$: Observe that $e^{\mu x}\phi_1(x,\mu) = \sum_{n=0}^{+\infty} A_\lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the sequence converges absolutely in the $\mathcal{L}^\infty$ norms. Moreover each function $A_\lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is analytic in k, so by Lemma A.1.2 $\phi_1(x,\cdot)$ is analytic.

For Estimate (90): if the constant T is sufficiently large, then by Estimate (100) we have that $\forall\ x > T$,

$$\|A_\lambda\|_{\mathcal{L}^\infty((x,\infty))\to\mathcal{L}^\infty((x,\infty))} \leq ce^{-\epsilon_0 x}$$

for some constants $c,\ \epsilon_0$ independent of x and λ.

By a direct calculation we can prove that $(H-\lambda)\phi_1(\cdot,\mu) = 0$.

To prove (97), we only need to consider the case $\lambda = \beta$. □

Using $\phi_1(\cdot,\mu)$, we define another solution $\phi_2(\cdot,\mu)$ to Equation (89) by

$$\phi_2(x,\mu) := \phi_1(-x,\mu).$$

To prove the existence of solutions $\psi_1,\ \psi_2,\ \xi_1$ in Theorem A.1.1, we prove first their existence on the domain $[R,+\infty)$, where R is a large constant. Then we continue the solutions to the interval $(-\infty,\infty)$ by ODE theories. To this end the following lemma will be used:

Lemma A.1.4. *Let $a \geq b$ be constants and $\Omega_2 \subset \mathbb{C}$ be a bounded closed set on the complex plane. Define an 4×4 matrix $T(x,k) := [T_{ij}(x,k)]$ such that each entry T_{ij} is C^2 continuous in the variable $x \in [a,b]$, and analytic in the variable $k \in \Omega_2$.*

Let $X(x,k) : [a,b]\times\Omega_2 \to \mathbb{C}^4$ be a solution to the ODE system

$$\frac{dX(\cdot,k)}{dx} = T(\cdot,k)X(\cdot,k), \tag{101}$$

with an initial datum $X(a,k)$.

If $X(a,\cdot)$ is an analytic function of $k \in \Omega_2$, then X is C^2 in $x \in [a,b]$, and analytic in $k \in \Omega_2$.

Proof. We can rewrite Equation (101) as

$$X(\cdot,k)=X(a,k)+A_kX(\cdot,k),$$

where $A_k:\ \mathcal{L}^\infty([a,b])\to\ \mathcal{L}^\infty([a,b])$ is an operator defined by

$$A_k(X)(x)=\int_a^x T(y,k)X(y)dy.$$

We can get easily that

$$|A_k(X)(x)|\le c\int_a^x dx\|X\|_{\mathcal{L}^\infty},$$

where c is independent of x and k.

Thus there exists an integer $m\in N$, such that $\|A_k^m\|_{\mathcal{L}^\infty([a,b])\to\mathcal{L}^\infty([a,b])}<1$. By the same strategy as in Proposition A.1.3, we can get the existence, smoothness and analyticity of $X(x,k)$. □

Proposition A.1.5. *There exist solutions ψ_1 and ψ_2 to Equation (89) which are analytic in k and satisfy Estimates (91), (92), (95), (96) and (97).*

If $\lambda=\beta$ then there is a solution η such that

$$\eta(x)=[\begin{pmatrix} x\\ 0\end{pmatrix}+O(e^{-\gamma x})], \tag{102}$$

as $x\to-\infty$, where $\gamma>0$ is a constant.

Proof. We will only prove the existence of solutions $\psi_1(x,k)$, that of $\psi_2(x,k)$ is almost the same.

First we will prove the existence of a function $\psi_1(\cdot,k)$ on the domain $x\in[R,+\infty)$, $\lambda\in\Omega$ satisfying the following equation

$$\begin{aligned}\psi_1(x,k)\ &=\ e^{ikx}\begin{pmatrix}1\\0\end{pmatrix}-\int_x^{+\infty}\begin{pmatrix}-\frac{\sin k(x-y)}{k} & 0\\ 0 & -\frac{1}{2\mu}e^{\mu(x-y)}\end{pmatrix}W(y)\psi_1(y,k)dy\\ &-\ \int_R^x\begin{pmatrix}0&0\\0&-\frac{1}{2\mu}e^{-\mu(x-y)}\end{pmatrix}W(y)\psi_1(y,k)dy,\end{aligned}$$

where R is a sufficiently large constant.

Define $\psi_k(x):=e^{-ikx}\psi_1(x,k)$. We could rewrite the equation as

$$\psi_k(x)=\begin{pmatrix}1\\0\end{pmatrix}-A_{k1}\psi_k-A_{k2}\psi_k,$$

where the operators A_{k1} and $A_{k2}:\mathcal{L}^\infty([R,\infty))\to\mathcal{L}^\infty([R,\infty))$ are defined as:

$$(A_{k1}\psi)(x)=\int_x^{+\infty}\begin{pmatrix}-\frac{\sin k(x-y)}{k} & 0\\ 0 & -\frac{1}{2\mu}e^{\mu(x-y)}\end{pmatrix}W(y)e^{-ik(x-y)}\psi(y)dy$$

and

$$(A_{k2}\psi)(x) = \int_R^x \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2\mu}e^{-\mu(x-y)} \end{pmatrix} W(y)e^{-ik(x-y)}\psi(y)dy.$$

We claim that if the constant R is sufficiently large, then

$$\|A_{k1}\|_{\mathcal{L}^\infty([R,\infty))\to\mathcal{L}^\infty([R,\infty))} + \|A_{k2}\|_{\mathcal{L}^\infty([R,\infty))\to\mathcal{L}^\infty([R,\infty))} < 1$$

for any k. Indeed, by the properties of the domain Ω from (86) and that $|W(x)| \leq e^{-\alpha|x|}$ we have

$$|A_{k1}\psi(x)| \leq c_1 \int_x^\infty e^{-\frac{\alpha}{2}|x|}dx\|\psi\|_{\mathcal{L}^\infty} \leq c_1 e^{-\epsilon_0|x|}\|\psi\|_{\mathcal{L}^\infty}, \tag{103}$$

$$|A_{k2}\psi(x)| \leq c_1 \int_R^x e^{-c_2|x-y|}e^{-\frac{\alpha}{2}|y|}dy\|\psi\|_{\mathcal{L}^\infty} \leq c_1 e^{-\epsilon_0|x|}\|\psi\|_{\mathcal{L}^\infty}. \tag{104}$$

for some constants $c_1,\ c_2,\ \epsilon_0 > 0$. Hence if R is sufficient large, $\|A_{k1}\psi\|_{\mathcal{L}^\infty([R,\infty))} + \|A_{k2}\psi\|_{\mathcal{L}^\infty([R,\infty))} \leq 1$. By the contraction lemma we could get that ψ_k exists and

$$\psi_k(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{n=1}^{+\infty}(A_{k1} + A_{k2})^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The estimate (91) is from Estimates (103) and (104).

When x is not necessarily large we estimate $\psi_1(x,k)$ by Lemma A.1.4: $\forall R_4 > 0$, if $x \in [R_3, -R_4]$ and if $\lambda \in [\beta, \beta+1]$, then we have

$$|\frac{d^n}{dk^n}\psi_1(x,k)| \leq c$$

for some constant c, and $n = 0, 1, 2$. Thus Estimate (95) is proven.

To prove Claim (97), we only need to consider the case $\lambda = \beta$, thus it is easier to prove it.

By the similar strategy we can find a solution η to $(H-\beta)\eta = 0$ such that it satisfies the estimate (102) and the following equation:

$$\begin{aligned} \eta(x) &= \begin{pmatrix} x \\ 0 \end{pmatrix} - \int_x^{+\infty} \begin{pmatrix} -\frac{\sin k(x-y)}{k} & 0 \\ 0 & -\frac{1}{2\mu}e^{\mu(x-y)} \end{pmatrix} W(y)\eta(y)dy \\ &- \int_R^x \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2\mu}e^{-\mu(x-y)} \end{pmatrix} W(y)\eta dy. \end{aligned}$$

By a direct calculation we could prove that $(H-\lambda)\psi_1(\cdot,k) = 0$ and $(H-\beta)\eta = 0$. □

Proposition A.1.6. *Recall $\mu = \sqrt{\lambda+\beta}$. There exists a solution $\xi_1(x,\mu)$ to Equation (89) which is analytic in k and as $x \to +\infty$*

$$\xi_1(x,\mu) = e^{\mu x}(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(e^{-\epsilon(\lambda)x})) \tag{105}$$

for some $\epsilon(\lambda) > 0$.

Proof. We follow an idea from [BP1]. We will prove that if the constant R is sufficiently large, then there exists a function ξ_1 such that

$$\begin{aligned}\xi_1(x,\mu) &= e^{\mu x}\begin{pmatrix}0\\1\end{pmatrix}+\int_x^\infty\begin{pmatrix}0 & 0\\ 0 & -\frac{1}{2\mu}e^{\mu(x-y)}\end{pmatrix}W(y)\xi_1(y,\mu)dy\\ &+\int_R^x\begin{pmatrix}\frac{\sin k(x-y)}{k} & 0\\ 0 & -\frac{1}{2\mu}e^{-\mu(x-y)}\end{pmatrix}W(y)\xi_1(y,\mu)dy.\end{aligned}$$

If we could prove the existence, it is easy to prove that

$$(H-\lambda)\xi_1(\cdot,\mu)=0.$$

Let $\psi=e^{-\mu x}\xi_1(x,\mu)$, then this equation can be rewritten as

$$\psi=\begin{pmatrix}0\\1\end{pmatrix}+A_{\lambda 1}\psi+A_{\lambda 2}\psi,$$

$A_{\lambda 1}$ and $A_{\lambda 2}:\mathcal{L}^\infty([R,\infty))\to\mathcal{L}^\infty([R,\infty))$ are operators defined as:

$$(A_{\lambda 1}\psi)(x)=\int_x^\infty\begin{pmatrix}0 & 0\\ 0 & -\frac{1}{2\mu}\end{pmatrix}W(y)\psi(y)dy,$$

and

$$(A_{\lambda 2}\psi)(x)=\int_R^x\begin{pmatrix}\frac{\sin k(x-y)}{k} & 0\\ 0 & -\frac{1}{2\mu}e^{-\mu(x-y)}\end{pmatrix}e^{-\mu(x-y)}W(y)\psi(y)dy.$$

As usual we want to find a large number R, s.t. $\|A_{\lambda 1}+A_{\lambda 2}\|_{\mathcal{L}^\infty([R,\infty))\to\mathcal{L}^\infty([R,\infty))}<1$. Then we can implement the contraction argument.

For $A_{\lambda 1}$,

$$|A_{\lambda 1}\psi(x)|\le c\int_x^\infty e^{-\alpha y}dy\|\psi\|_{\mathcal{L}^\infty([R,\infty))}, \tag{106}$$

thus if R is sufficiently large, then

$$\|A_{\lambda 1}\|_{\mathcal{L}^\infty([R,\infty))\to\mathcal{L}^\infty([R,\infty))}<1.$$

For $A_{\lambda 2}$, since there exists some constant $\epsilon>0$ such that $Re(\mu-\pm ik)>\epsilon>0$, we obtain

$$|(A_{\lambda 2}\psi)(x)|\le c(1+|x|)\int_R^x(1+|y|)e^{-\epsilon(\lambda)(x-y)}e^{-\alpha|y|}dy\|\psi\|_{\mathcal{L}^\infty([R,\infty))} \tag{107}$$

for some constant c independent of R. Thus if $R\to+\infty$, then

$$\max_{x\in[R,\infty)}\{\int_R^x e^{-\epsilon(\lambda)(x-y)}e^{-\alpha|x|}dy\}\to 0,$$

i.e.

$$\|A_{\lambda 2}\|_{\mathcal{L}^\infty([R,\infty))\to\mathcal{L}^\infty([R,\infty))}\to 0$$

as $R \to +\infty$.

We choose a large constant R such that

$$\|A_{\lambda 1} + A_{\lambda 2}\|_{\mathcal{L}^\infty([R,\infty)) \to \mathcal{L}^\infty([R,\infty))} < 1.$$

The existence of the solution follows by a standard contraction argument.

Since each function $(A_{\lambda 1} + A_{\lambda 2})^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is analytic in k, then by Lemma A.1.2 $\xi_1(x, \cdot) = \sum_{n=0}^{\infty} (A_{\lambda 1} + A_{\lambda 2})^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is analytic.

Estimate (105) can be proved by Estimates (106) and (107). □

We define another solution to (89) by

$$\xi_2(x, \mu) := \xi_1(-x, \mu).$$

A.2 Generalized Wronskian

A generalized Wronskian function is defined in the next lemma, whose proof is straightforward and is omitted here:

Lemma A.2.1. *If X_1 and X_2 satisfy $(H - \lambda)X_i = 0$, $(i = 1, 2)$ then*

$$W(X_1, X_2) := \partial_x X_1^T X_2 - \partial_x X_2^T X_1 = Const.$$

Define two 2×2 matrices

$$\begin{array}{lcl} F_1(x,k) & := & \left[\ \psi_1(x,k), \phi_1(x,\mu) \ \right], \\ F_2(x,k) & := & \left[\ \psi_1(-x,\mu), \phi_2(x,\mu) \ \right]. \end{array} \tag{108}$$

The 2×2 matrix $D(k) := (\partial_x F_1^T) F_2 - F_1^T (\partial_x F_2)$ is independent of x because each entry is a Wronskian function. Observe that $D(k)$ is a symmetric matrix. Let

$$D(k) =: \begin{pmatrix} D_{11}(k) & D_{12}(k) \\ D_{12}(k) & D_{22}(k) \end{pmatrix}. \tag{109}$$

The entry $D_{22}(k) = W(\phi_1(\cdot, \mu), \phi_2(\cdot, \mu))$ will play an important role later.

Under an assumption that there are no eigenvalues embedded in the essential spectrum one can prove, by strategies similar to that in [Rau, RSS1], that $det D(k) \neq 0$ and the operator $1 + (H_0 - \lambda + i0)^{-1} W$ is invertible in some sense for any $\lambda > \beta$. In this paper we approach these problems in a different way and we do not use the assumption on the embeded eigenvalues.

We have the following result:

Theorem A.2.2. *H has a resonance at the point β if and only if $det D(0) = 0$.*

Proof. First we prove the sufficient condition, i.e. assume H has no resonance at the point β. Since the vectors

$$\phi_2(\cdot, \sqrt{2\beta}), \ \psi_2(-\cdot, 0), \ \eta, \ \xi_2(\cdot, \sqrt{2\beta})$$

form an basis in the solution space $(H-\beta)\varphi = 0$, there exist 2×2 matrices A_2 and B_2, such that

$$\left[\psi_1(\cdot,0),\phi_1(\cdot,\sqrt{2\beta})\right] = \left[\psi_2(-\cdot,0),\phi_2(\cdot,\sqrt{2\beta})\right]A_2 + \left[\eta,\xi_2(\cdot,\sqrt{2\beta})\right]B_2.$$

We claim $detB_2 \neq 0$.

Indeed if $detB_2 = 0$, we could choose an invertible matrix B_1 so that

$$B_2B_1 = \begin{bmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned}[\gamma_1,\gamma_2] &:= [\psi_1(\cdot,0),\phi_1(\cdot,\sqrt{2\beta})]B_1 \\ &= [\psi_2(-\cdot,0),\phi_2(\cdot,\sqrt{2\beta})]A_2B_1 + [\eta_3,0]\end{aligned}$$

for some function η_3, which implies the function γ_2 is bounded at $-\infty$. Since we already know that $\psi_1(\cdot,0)$ and $\phi_1(\cdot,\sqrt{2\beta})$ are bounded at $+\infty$, γ_2 is a resonance at β. This contradicts to the fact that there are no resonances at β. Therefore B_2 is invertible.

Re-compute $D(0)$ to prove that it is invertible:

$$\begin{aligned}D(0) &= [\tfrac{d\psi_1(x,0)}{dx},\tfrac{d\phi_1(x,\sqrt{2\beta})}{dx}]^T[\psi_2(-x,0),\phi_2(x,\sqrt{2\beta})] \\ &\quad -[\psi_1(x,0),\phi_1(x,\sqrt{2\beta})]^T[\tfrac{d\psi_2(x,0)}{dx},\tfrac{d\phi_2(-x,\sqrt{2\beta})}{dx}] \\ &= B_2^T\{[\tfrac{d\eta}{dx},\tfrac{d\xi_2(x,\sqrt{2\beta})}{dx}]^T[\psi_2(-x,0),\phi_2(x,\sqrt{2\beta})] \\ &\quad -[\eta,\xi_2(x,\sqrt{2\beta})]^T[\tfrac{d\psi_2(-x,0)}{dx},\tfrac{d\phi_2(x,\sqrt{2\beta})}{dx}]\} \\ &= B_2^T\begin{bmatrix} 1 & * \\ 0 & 2\sqrt{2\beta}\end{bmatrix},\end{aligned}$$

where $*$ is an unimportant constant. Therefore $detD(0)\neq 0$.

Now, we prove the necessary condition: Suppose $detD(0)\neq 0$. Since the vectors

$$\phi_1(\cdot,\sqrt{2\beta}),\ \psi_1(\cdot,0),\ \eta(-\cdot),\ \xi_2(-\cdot,\sqrt{2\beta})$$

form a basis to the solution space for the equation

$$(H-\beta)\varphi = 0,$$

we only need to consider the linear combination of these vectors when we look for a resonance. First we exclude the vectors having $\eta(-\cdot)$, $\xi_2(-\cdot,\sqrt{2\beta})$ components because at ∞ the first one blows up exponentially fast, and the second blows up at the rate of x, then any linear combination with them is unbounded, i.e. is not an resonance.

$\phi_1(\cdot,\sqrt{2\beta})$ could not be a resonance otherwise $D_{22}(0) = D_{12}(0) = 0$ which implies $detD(0) = 0$. We claim that for any scalar z, $\psi_1(\cdot,0)+z\phi_1(\cdot,\sqrt{2\beta})$ could not be a resonance: indeed, if it is a resonance, then let

$$G_1 = F_1\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix},\ G_2 = F_2\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix},$$

where, recall F_1, F_2 from Equation (108). Let

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} := W(G_1, G_2) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} D(0) \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

where

$$a_3 = W(\psi_1(\cdot, 0) + z\phi_1(\cdot, \sqrt{2\beta}), \phi_2(\cdot, \sqrt{2\beta})) = 0$$

and

$$a_1 = W(\psi_1(\cdot, 0) + z\phi_1(\cdot, \sqrt{2\beta}), \psi_1(-\cdot, 0) + z\phi_2(\cdot, \sqrt{2\beta})) = 0$$

which implies that $det D(0) = 0$. □

Recall that the space L^γ defined in Equation (72). The following lemma explains the choice of the space $L^{-\alpha/2}$ in the next section.

Lemma A.2.3. *If a function $\phi \in L^{-\alpha/2}$ satisfies $(H - \lambda)\phi = 0$ and if $\lambda > \beta$, then*

$$W(\phi, \phi_1(\pm\cdot, \mu)) = 0, \tag{110}$$

and

$$\begin{aligned} \phi &= b_{+1}\phi_1(\cdot, \mu) + b_{+2}\psi_1(\cdot, k) + b_{+3}\psi_2(\cdot, k) \\ &= b_{-1}\phi_1(-\cdot, \mu) + b_{-2}\psi_1(-\cdot, k) + b_{-3}\psi_2(-\cdot, k) \end{aligned} \tag{111}$$

for some constants $b_{\pm 1},\ b_{\pm 2},\ b_{\pm 3}$.

Proof. The vectors $\{\phi_1(\cdot, \mu), \psi_1(\cdot, k), \psi_2(\cdot, k), \xi_1(\cdot, \mu)\}$ form a basis to the solution space of $(H - \lambda)\phi = 0$. Moreover $\phi_1(\cdot, \mu), \psi_1(\cdot, k), \psi_2(\cdot, k) \in L^{-\alpha/2}([0, +\infty))$ while $\xi_1(\cdot, \mu) \notin L^{-\alpha/2}([0, +\infty))$ by the fact that $Re\mu > \alpha$, which follows from the assumption that $\alpha < \beta$ made after Equation (88). This implies the + part of Equation (111).

The + part of Equation (110) follows from Equation (111) and the following results:

$$W(\phi_1(\cdot, \mu), \phi_1(\cdot, \mu)) = W(\psi_1(\cdot, k), \phi_1(\cdot, \mu)) = W(\psi_2(\cdot, k), \phi_1(\cdot, \mu)) = 0$$

while

$$W(\xi_1(\cdot, \mu), \phi_1(\cdot, \mu)) \neq 0.$$

The $-$ part of the lemma is proven similarly. □

A.3 Generalized Eigenfunction $e(x, k)$

In this subsection we prove that the function $e(x, k)$ in Equation (73) is well defined. Recall the definition of the domain Ω from Equation (86). For any $\lambda \in \Omega$ we define the operator $\mathcal{R}^+(\lambda) : L_\alpha \to L_{-\alpha}$ by its integral kernel

$$G_k^+(x, y) = \begin{pmatrix} \frac{e^{ik|x-y|}}{2k} & 0 \\ 0 & -\frac{e^{-\mu|x-y|}}{2i\mu} \end{pmatrix}, \tag{112}$$

where, recall that $k = \sqrt{\lambda - \beta}$. The operator $\mathcal{R}^+(\lambda)$ is continuation to the resolvent $(H_0 - \lambda)^{-1}$, $\lambda \in \Omega \cap \mathbb{C}^+$ in the following sense. Observe that $\sigma(H_0) = \sigma_{ess}(H_0) = (-\infty, -\beta] \cup [\beta, \infty)$. For any functions $f,\ g \in L^\alpha$ and $\lambda \in \mathbb{C}^+ \cap \Omega$, the quadratic form $\langle f, \mathcal{R}^+(\lambda) g\rangle$, $\lambda \in \Omega$, is an analytic continuation of the quadratic form $\langle f, (H_0 - \lambda)^{-1} g\rangle$ from $\Omega \cap \mathbb{C}^+$ to Ω.

Similarly we define $\mathcal{R}^-(\lambda)$ using the integral

$$G_k^-(x,y) = \begin{pmatrix} -\frac{e^{-ik|x-y|}}{2k} & 0 \\ 0 & -\frac{e^{-\mu|x-y|}}{2i\mu} \end{pmatrix}.$$

It is the analytic continuation of the resolvent $(H_0 - \lambda)^{-1}$ from $\lambda \in \Omega \bigcap \mathbb{C}^-$ to Ω.

Equation (112) and Inequality (88) imply that if $\lambda \in \Omega$, then $\mathcal{R}^\pm(\lambda) W : L^{-\alpha/2} \to L^{-\alpha/2}$ are compact operators (in fact, trace class operators, see [RSI]).

The following theorem is the main result of this subsection:

Theorem A.3.1. *If $\lambda > \beta$ is not an eigenvalues of H embedded in the essential spectrum, then the operators*

$$1 + \mathcal{R}^+(\lambda) W :\ L^{-\alpha/2} \to L^{-\alpha/2} \tag{113}$$

are invertible, and the functions $e(\cdot, k)$ in Equation (73) are well defined and can be written as

$$e(x,k) = -i \frac{2D_{22}(k)k}{detD(k)} \eta(x,k), \tag{114}$$

where $\eta(x,k) = \psi_1(x,k) - \frac{D_{12}(k)}{D_{22}(k)} \phi_1(x,\mu)$.

The proof of Theorem A.3.1 will be after Lemma A.3.10, and will use the results from Lemma A.3.6, Propositions A.3.8, A.3.9 and Lemma A.3.10.

The following simple lemma whose proof is obvious is important in this subsection:

Lemma A.3.2. *Let C be the operator of complex conjugating, and*

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $\lambda > \beta$, then

$$\sigma_3 \mathcal{R}^\pm(\lambda) \sigma_3 = \mathcal{R}^\pm(\lambda),\ \sigma_3 W \sigma_3 = W^*,$$

and therefore

$$C\sigma_3(1 + \mathcal{R}^\pm(\lambda) W) C\sigma_3 = 1 + \mathcal{R}^\mp(\lambda) W,$$

$$\sigma_3(1 + \mathcal{R}^\pm(\lambda) W^*) \sigma_3 = 1 + \mathcal{R}^\pm(\lambda) W.$$

Corollary A.3.3. *If $\lambda > \beta$, then $\phi_1(x,\mu) = -\sigma_3 \overline{\phi}_1(x,\mu)$.*

We start with studying the analytic function $D(k)$ introduced in Equation (109).

Lemma A.3.4. *If there exists some $\lambda_1 > \beta$ such that $1+\mathcal{R}^+(\lambda_1)W$ is not invertible, then either λ_1 is an eigenvalue of H, or $D_{22}(\sqrt{\lambda_1-\beta}) \neq 0$.*

Proof. Assume λ_1 is not an eigenvalue of H and assume by contradiction that $D_{22}(\sqrt{\lambda_1-\beta})=0$.

By the definition of $D_{22}(k)$ and Lemma A.2.3 we can get that $\phi_1(x) := \phi_1(x,\sqrt{\beta+\lambda_1})$ is a bounded function, i.e.

$$\phi_1(x) = c_1\phi_1(-x) + c_2\psi_2(-x,\sqrt{\lambda_1-\beta}) + c_3\psi_1(-x,\sqrt{\lambda_1-\beta})$$

for some constants $c_i,\ i=1,2,3$. Furthermore, since $1+\mathcal{R}(\lambda_1)W$ is not invertible, there exists a function $g \in L^{-\alpha/2}\backslash\mathcal{L}^2$ such that $(1+\mathcal{R}(\lambda_1)W)g=0$. By elementary calculations we get that

$$g(x) = c_4\phi_1(x) + c_5\psi_1(x,\sqrt{\lambda_1-\beta}) = c_6\phi_1(-x) + c_7\phi_1(-x,\sqrt{\lambda_1-\beta})$$

for some constants $c_n,\ n=4,5,6,7$.

Since g and ϕ_1 satisfy the equation $(H-\lambda_1)\psi=0$, $W(\phi_1,g)$ is independent of x. Therefore

$$0 = W(\phi_1,g) = 2c_7c_3\sqrt{\lambda_1-\beta}\,i,$$

then either $c_7=0$ or $c_3=0$. Similarly by calculating $W(\phi_1,g(-\cdot))$, we can get that either $c_5=0$ or $c_2=0$. Hence by Corollary A.3.3 either $g\in\mathcal{L}^2$ and is therefore an eigenfunction of H or $\phi_1\in\mathcal{L}^2$ and is therefore an eigenfunction of H. By the assumption of the lemma, $D_{22}(\sqrt{\lambda_1-\beta})\neq 0$. ☐

Since $\mathcal{R}^+(\lambda)W$ is analytic in a neighborhood of the semi-axis $[\beta,\infty)$ and since $\|\mathcal{R}^+(\lambda)W\|_{L^{-\alpha/2}\to L^{\alpha/2}} \to 0 \to 0$ as $\lambda\to\infty$ the operator $1+\mathcal{R}^+(\lambda)$ are not invertible for at most finite number of points $\lambda\in[\beta,\infty)$.

Assume now that some point $\lambda_1>\beta$ is not an eigenvalue of H, and $1+\mathcal{R}^+(\lambda_1)W$ is not invertible, then $D_{22}(\sqrt{\lambda_1-\beta})\neq 0$. Since D_{22} is an analytic function of k, there exists a small neighborhood Ω_1 of λ_1 such that $\forall\lambda\in\Omega_1$, $D_{22}(k)\neq 0$, and $\forall\lambda\in\Omega_1\backslash\{\lambda_1\}$, $1+\mathcal{R}^+(\lambda)W$ is invertible as an operator from $L^{-\alpha/2}\to L^{-\alpha/2}$. Hence we have

Lemma A.3.5. *The function $e(x,k) := (1+\mathcal{R}^+(\lambda)W)^{-1}\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}$ is well defined for all $\Omega_1\backslash\{\lambda_1\}$ and belongs to $L^{-\alpha/2}$. Moreover it satisfies the equation $(H-\lambda)e(\cdot,k)=0$.*

Next using the lemma above we derive additional properties of the function $e(x,k)$ for $\lambda\in\Omega\backslash\{\lambda_1\}$.

Lemma A.3.6. *If at some point $\lambda\in\Omega$ the operator $1+\mathcal{R}^+(\lambda)W$ defined in Equation (113) is invertible, then there are functions $s,\ a$ of k, such that*

$$e(x,k) = s(k)\psi_1(x,k) + a(k)\phi_1(x,\mu).$$

Especially, $s(k)\neq 0$ if λ is sufficiently large.

Proof. Since for $\lambda \in \Omega_1 \backslash \{\lambda_1\}$, $(H-\lambda)e(\cdot,k) = 0$ and $e(\cdot,k) \in L^{-\alpha/2}$ we have by Lemma A.2.3 that

$$e(\cdot,k) = s(k)\psi_1(\cdot,k) + a(k)\phi_2(\cdot,\mu) + b(k)\psi_2(\cdot,k)$$

for some functions $s,\ a,\ b$. Thus we only need to prove that $b = 0$.

From the properties of ψ_1, ψ_2, ϕ_1 we can get that

$$e(\cdot,k) = s(k)\left(\begin{array}{c} e^{ikx} \\ 0 \end{array}\right) + b(k)\left(\begin{array}{c} e^{-ikx} \\ 0 \end{array}\right) + O(e^{-\frac{\alpha}{2}x}) \tag{115}$$

as $x \to +\infty$. From the definition of the domains Ω and Ω_1, we see that $|Imk| < \alpha/2$.

On the other hand

$$\begin{aligned} \left(\begin{array}{c} e^{ikx} \\ 0 \end{array}\right) &= e(x,k) + \mathcal{R}^+(\lambda)We(\cdot,k) \\ &= e(x,k) + e^{ikx}\left(\begin{array}{c} a_1(k) \\ 0 \end{array}\right) + O(e^{-\frac{\alpha}{2}x}) \end{aligned} \tag{116}$$

for some constant $a_1(k)$ as $x \to +\infty$. Comparing these two equations (115), (116), we find that $b = 0$.

By the fact that

$$\lim_{\lambda\to\infty} \|\mathcal{R}^+(\lambda)We(\cdot,k)\|_{\mathcal{L}^\infty} = 0$$

and Equation (116), we can get that $s(k)$ could not be zero if λ is large. $\square$

Lemma A.3.7. *The analytic function $D_{22}(k)$ can be zero at only a discrete subset of Ω.*

Proof. We prove by contradiction. Suppose not, then $D_{22}(k) = 0$ globally. By Lemma A.2.3

$$W(e(\cdot,k), \phi_1(-\cdot,\mu)) = 0$$

for any λ provided that $e(x,k)$ is well defined. We proved in Lemma A.3.6 that for large λ, $e(x,k)$ is well defined and $e(x,k) = s(k)\psi_1(x,k) + a(k)\phi_1(x,u)$ for some constants $s(k) \neq 0$ and $a(k)$. Since

$$D_{22}(k) := W(\phi_1(x,\mu), \phi_1(-x,\mu)) = 0,$$

we have

$$D_{12}(k) = D_{21}(k) := W(\psi_1(x,k), \phi_1(-x,k)) = 0$$

for any large k, where, recall the definition of $D(k)$ in Equation (109). Thus $DetD(k) = 0$ globally which contradicts to the fact that $detD(0) \neq 0$. Thus $D_{22}(k) = 0$ only at a discrte subset. $\square$

Proposition A.3.8. *Equation (114) holds for any $\lambda \in \Omega$ if the opeator $1 + \mathcal{R}^+(\lambda)W$ defined in Equation (113) is invertible.*

Proof. Define Ω_2 be the subset of Ω such that if $\lambda \in \Omega_3$ then the operator $1 + \mathcal{R}^+(\lambda)W$ is invertible and $D_{22}(\sqrt{\lambda - \beta}) \neq 0$. By Lemma A.3.7 we can see that $\Omega \backslash \Omega_2$ is a discrete subset of Ω, and $\Omega_1 \subset \Omega_2$. Therefore $e(x, k)$ is well defined if $k^2 + \beta = \lambda \in \Omega_2$. By the fact of $e(\cdot, k) \in L^{-\alpha/2}$ and Lemmas A.2.3, A.3.6 there exist functions s, a, k_1, k_2 of the variable k such that

$$\begin{aligned} e(x,k) &= s(k)\psi_1(x,k) + a(k)\phi_1(x,\mu) \\ &= \psi_2(-x,k) + k_1(k)\psi_1(-x,k) + k_2(k)\phi_2(x,k). \end{aligned} \tag{117}$$

If $\lambda \in \Omega_2$, then the fact $D_{22}(k) \neq 0$ implies that $s(k) \neq 0$. Define $s_1(k) := \frac{1}{s(k)}$ and $k_3(k) := \frac{a(k)}{s(k)}$. Then

$$e(x,k) = \frac{1}{s_1(k)}[\psi_1(x,k) + k_3(k)\phi_1(x,\mu)].$$

We claim $k_3(k) = -\frac{D_{12}(k)}{D_{22}(k)}$. Indeed, consider the matrix

$$\begin{aligned} (s_1(k)e(x,k), \phi_1(x,\mu)) &= (\psi_1(x,k), \phi_1(x,\mu)) \begin{pmatrix} 1 & 0 \\ k_3(k) & 1 \end{pmatrix} \\ &= F_1(x,k) \begin{pmatrix} 1 & 0 \\ k_3(k) & 1 \end{pmatrix}. \end{aligned}$$

Recall F_1 and F_2 from Equation (108). In the following computation we use the fact $W(e(\cdot,k), \phi_2(\cdot,\mu)) = 0$:

$$\begin{aligned} &\begin{pmatrix} D_{11}(k) & D_{12}(k) \\ D_{12}(k) & D_{22}(k) \end{pmatrix} = \partial_x F_1^T F_2 - F_1^T \partial_x F_2 \\ = &\begin{pmatrix} 1 & -k_3(k) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_{11}(k) - \frac{D_{12}^2(k)}{D_{22}(k)} & 0 \\ 0 & D_{22}(k) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k_3(k) & 1 \end{pmatrix}. \end{aligned}$$

This equality implies that $D_{12}(k) + k_3(k)D_{22}(k) = 0$ or equivalently $k_3(k) = -\frac{D_{12}(k)}{D_{22}(k)}$. By Equation (117),

$$\begin{aligned} &\psi_1(x,k) - \tfrac{D_{12}(k)}{D_{22}(k)}\phi_1(x,\mu) \\ = \;& s_1(k)\psi_2(-x,k) + s_1(k)k_1(k)\psi_1(-x,k) + s_1(k)k_2(k)\phi_2(x,\mu). \end{aligned}$$

We use a Wronskian function to derive an expression for $s_1(k)$:

$$2iks_1(k) = W(\psi_1(\cdot,k) - \frac{D_{12}(k)}{D_{22}(k)}\phi_1(\cdot,\mu), \psi_1(-\cdot,k)) = D_{11}(k) - \frac{D_{12}^2(k)}{D_{22}(k)}.$$

Since the right hand side is analytic, $s_1(k)$ is meromorphic.

Therefore we proved Equation 114 if $\lambda \in \Omega_2$. Since $D_{22}(k) = 0$ only at a discrete subset of Ω and $\eta(x,k)$ and $s_1(k)$ are meromorphic functions, Equation 114 holds if the function $e(x,k)$ is well defined. □

Since $1+\mathcal{R}^+(\lambda)W$ is not invertible at λ_1, we expect that $[1+\mathcal{R}^+(\lambda)W]^{-1}\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}$ blows up in some sense at $\lambda = \lambda_1$. We want to determine the nature of this blow up.

Proposition A.3.9. *If $1 + \mathcal{R}^+(\lambda)W$ is not invertible at some point $\lambda_1 > \beta$, then:*

$$s_1(\sqrt{\lambda_1 - \beta}) = 0;$$
$$[1 + \mathcal{R}^+(\lambda_1)W]\eta(\cdot, \sqrt{\lambda_1 - \beta}) = 0; \tag{118}$$

and

$$\eta(\cdot, \sqrt{\lambda_1 - \beta}) \in L^{-\alpha/2},$$

where α is given in Equation (88) and $L^\gamma := e^{\gamma/4|x|}\mathcal{L}^2$

Proof. (1) The proof of $\eta(\cdot, \sqrt{\lambda_1 - \beta}) \in L^{-\alpha/2}$ is easy: the analytic function

$$W(\eta(\cdot, k), \phi_2(\cdot, \mu)) = 0$$

for $\lambda \in \Omega_1 \backslash \{\lambda_1\}$, therefore for any $\lambda \in \Omega$. By Lemma A.2.3 this implies that

$$\eta(\cdot, k) = k_1\psi_1(-\cdot, k) + k_2\psi_2(-\cdot, k) + k_3\phi_2(\cdot, \mu)$$

for some $k_n,\ n = 1, 2, 3$. Thus if $\lambda > \beta$, then η is bounded at $-\infty$. By the definition of $\eta(\cdot, k)$ it is bounded at $+\infty$. Therefore $\eta(\cdot, k) \in \mathcal{L}^\infty \subset L^{-\alpha/2}$ if $\lambda > \beta$.

(2) To prove $(1 + \mathcal{R}^+(\lambda_1)W)\eta(\cdot, \sqrt{\lambda_1 - \beta}) = 0$, we use that $1 + \mathcal{R}^+(\lambda_1)W$ is not invertible, and therefore there exists a function $g \in L^{-\alpha/2}$ such that $(1 + \mathcal{R}^+(\lambda_1)W)g = 0$ which implies that

$$g = z_1\psi_1(\cdot, \sqrt{\lambda_1 - \beta}) + z_2\phi_1$$

for some constants z_1 and z_2 by a similar argument as in Equation (116). Since $D_{22}(\sqrt{\lambda_1 - \beta}) \neq 0,\ z_1 \neq 0$. Thus without loss of generality we assume $z_1 = 1$. We claim

$$z_2 = -\frac{D_{12}(\sqrt{\lambda_1 - \beta})}{D_{22}(\sqrt{\lambda_1 - \beta})} \tag{119}$$

or equivalently $g = \eta(\cdot, \sqrt{\lambda_1 - \beta})$ (see Equation (**??**)). Indeed, using

$$\eta(\cdot, \sqrt{\lambda_1 - \beta}) - g = [z_2 + \frac{D_{12}(\sqrt{\lambda_1 - \beta})}{D_{22}(\sqrt{\lambda_1 - \beta})}]\phi_1$$

and using the Wronskian function to calculate $z_2 + \frac{D_{12}(k)}{D_{22}(k)}$, we obtain

$$[z_2+\frac{D_{12}(\sqrt{\lambda_1 - \beta})}{D_{22}(\sqrt{\lambda_1 - \beta})}]D_{22}(\sqrt{\lambda_1 - \beta}) = W(\eta(\cdot, \sqrt{\lambda_1 - \beta}) - g, \phi_2(\cdot, \sqrt{\beta + \lambda_1})) = 0.$$

Since $D_{22}(\sqrt{\lambda_1 - \beta}) \neq 0$ as proven in Lemma A.3.4, we have Equation (119). Therefore we have $g = \eta(\cdot, \sqrt{\lambda_1 - \beta})$.

(3) The equation $s_1(\sqrt{\lambda_1-\beta})=0$ follows from the following three facts

$$(1+\mathcal{R}^+(\lambda)W)\eta(\cdot,k)=s_1(k)\left(\begin{array}{c} e^{ikx} \\ 0 \end{array}\right)$$

which follows from Equations (73) and (**??**),

$$\lim_{k\to\sqrt{\lambda_1-\beta}}\eta(\cdot,k)=\eta(\cdot,\sqrt{\lambda_1-\beta}) \text{ in } L^{-\alpha/2}$$

which can be proved by Lemma A.1.4 and the Dominated Convergence Theorem, and Equation (118).

□

In the following lemma we prove that $s_1(k)$ could not be zero:

Lemma A.3.10. *There exist functions $s,\ a,\ b,\ \gamma$ such that*

$$\begin{aligned} e(x,k) &= s(k)\psi_1(x,k)+a(k)\phi_1(x,\mu) \\ &= b(k)\phi_2(x,\mu)+\psi_2(-x,k)+r(k)\psi_1(-x,k), \end{aligned}$$

where $|s(k)|^2+|r(k)|^2=1$, and $\bar{s}r+\bar{r}s=0$.

Proof. By Lemmas A.2.3, A.3.6 there exist functions $s,\ a,\ b,\ c$ and r such that

$$\begin{aligned} e(x,k) &= s(k)\psi_1(x,k)+a(k)\phi_1(x,\mu) \\ &= b(k)\phi_2(x,\mu)+c(k)\psi_2(-x,k)+r(k)\psi_1(-x,k). \end{aligned}$$

One can show that $c=1$ by a similar expansion as in Equation (116) at $-\infty$.

We divide the proof into two cases: $s(k)=0$ and $s(k)\neq 0$.

(1) If $s(k)=0:$ then $a(k)\neq 0$, thus

$$\frac{e(x,k)}{a(k)}=\phi_1(x,k).$$

By Corollary A.3.3,

$$-\sigma_3\overline{\frac{e(x,k)}{a(k)}}=\frac{e(x,k)}{a(k)}.$$

Thus

$$-\frac{1}{\bar{a}(k)}=\frac{r(k)}{a(k)},$$

which implies $|r(k)|=1$. This proves the lemma when $s(k)=0$.

(2) If $s(k)\neq 0:$ It is easy to get that

$$\begin{aligned} e(-x,k) &= \psi_2(x,k)+r(k)\psi_1(x,k)+b(k)\phi_1(x,\mu) \\ &= s(k)\psi_1(-x,k)+a(k)\phi_2(x,\mu) \end{aligned}$$

is a solution to $H-\lambda$. There exist b_1, b_2 such that

$$\begin{aligned}\sigma_3\bar{e}(-x,k) &= \psi_1(x,k)+\bar{r}(k)\psi_2(x,k)+b_1(k)\phi_1(x,\mu)\\ &= \bar{s}(k)\psi_2(-x,k)+b_2(k)\phi_2(x,\mu)\end{aligned}$$

satisfies $(H_0-\lambda+W)\sigma_3\bar{e}(\cdot,k)=0$.

Therefore

$$e(x,k)=\frac{1}{\bar{s}(k)}\sigma_3\bar{e}(-x,k)+\frac{r(k)}{s(k)}e(-x,k)+d(k)\phi_1(x,\mu) \tag{120}$$

for some $d(k)$. We claim that if $d(k)\neq 0$ then $D_{22}(k)=0$. Indeed, we already know that $e(\cdot,k)\in L^{-\alpha/2}$. So $\phi_1(\pm\cdot,\mu)\in L^{-\alpha/2}$. Thus by Lemma A.2.3, $D_{22}(k)=W(\phi_1(\cdot,\mu),\phi_1(-\cdot,\mu))=0$.

Therefore if $D_{22}(k)\neq 0$, then $d(k)=0$. Then Equation (120) implies

$$\frac{1}{\bar{s}(k)}+\frac{r^2(k)}{s(k)}=s(k),$$

and

$$\frac{r(k)}{s(k)}=-\frac{\bar{r}(k)}{\bar{s}(k)}.$$

Therefore

$$|s(k)|^2+|r(k)|^2=1, \text{ and } s(k)\bar{r}(k)+\bar{s}(k)r(k)=0.$$

Since s and r are meromorphic functions of k, and since $D_{22}(k)=0$ only at discrete points, the formula works for all k.

□

Proof of Theorem A.3.1: By Lemma A.3.10 and the proof of Proposition A.3.8, one can obtain:

$$s(k)=\frac{1}{s_1(k)}=i\frac{2D_{22}(k)k}{detD(k)}, \tag{121}$$

and

$$|s(k)|^2=|\frac{1}{s_1(k)}|^2\leq 1. \tag{122}$$

And by Equation (122) we have that $s_1(\sqrt{\lambda_1-\beta})\neq 0$. Hence the operator $1+\mathcal{R}^+(\lambda)W$ must be invertible at the point $\lambda_1>\beta$, otherwise there is a contradiction by Proposition A.3.9.

□

Also by Lemma A.3.10 and the proof of Proposition A.3.8 we have that

$$a(k)=-s(k)\frac{D_{12}(k)}{D_{22}(k)}=-\frac{2ikD_{12}(k)}{detD(k)}. \tag{123}$$

Moreover $s,\ a,\ b,\ r$ are meromorphic functions of k. Since $detD(0)\neq 0$ they are analytic functions of k in a neighborhood of 0.

Proposition A.3.11. *If H has no resonance at β, then*

$$e(\cdot, 0) = 0.$$

Proof. By Theorem A.2.2 $detD(0) \neq 0$, and by Lemma A.3.10 and Equations (121) (123)

$$e(x,k) = \frac{2ikD_{22}(k)}{detD(k)}\psi_1(x,k) - \frac{2ikD_{12}(k)}{detD(k)}\phi_1(x,k).$$

Hence $e(x,0) = 0$. □

A.4 Estimates on $e(x,k)$

In this subsection we estimate the eigenfunctions

$$e(x,k) = [1 + \mathcal{R}^+(\lambda)W]^{-1}\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}$$

for all $\lambda > \beta$, which are well defined as proved in the last subsection.

Theorem A.4.1. *If $\lambda \geq \beta$, then*

$$\|\frac{d^n}{dk^n}(e(x,k)/k)\|_{\mathcal{L}^2(dk)} \leq c(1+|x|)^{n+1};$$

$$\|\frac{d^n}{dk^n}(e(x,k)/k)\|_{\mathcal{L}^\infty(dk)} \leq c(1+|x|)^{n+1};$$

$$\|f^\#\|_{\mathcal{H}^2} \leq c\|\rho_{-2}f\|_2;$$

$$|\frac{d^n}{dk^n}f^\#| \leq c\|\rho_{-n}f\|_1,$$

where $n = 0,1,2$, the constant c is independent of x, and, recall, $\rho_\nu = (1+|x|)^{-\nu}$.

The estimates in Theorem A.4.1 will be proved in Propositions A.4.2 and A.4.4 and Corollary A.4.5.

Proposition A.4.2.

$$\|\frac{d^n}{dk^n}e(x,\cdot)\|_{\mathcal{L}^\infty(dk)} \leq c(1+|x|)^n,$$

$$\|\frac{d^n}{dk^n}(e(x,\cdot)/k)\|_{\mathcal{L}^\infty(dk)} \leq c(1+|x|)^{n+1},$$

where c is a constant independent of x, n and λ. $n = 0,1,2,3$ and $\lambda > \beta > 0$.

Proof. Since we proved $e(x,0) = 0$, $\|\frac{d^n}{dk^n}(e(x,k)/k)\|_{\mathcal{L}^\infty(dk)}$ can be estimated by $\|\frac{d^{n+1}}{dk^{n+1}}e(x,k)\|_{\mathcal{L}^\infty(dk)}$.

We divide the proof into two cases: $\lambda > \beta + \epsilon_0$ and $\beta \leq \lambda \leq \beta + \epsilon_0$, where ϵ_0 is a small positive number to be specified later.

(1) If $\lambda > \beta + \epsilon_0$, then

$$\begin{aligned} e(x,k) &= \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} - \mathcal{R}^+(\lambda) W e(x,k) \\ &= \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} - \int_{-\infty}^{\infty} \begin{pmatrix} \frac{e^{ik|x-y|}}{2k} & 0 \\ 0 & -\frac{e^{-\mu|x-y|}}{2i\mu} \end{pmatrix} W e(y,k) dy. \end{aligned}$$

We estimate $\|e(x,k)\|_{\mathcal{L}^\infty(dx)}$ by $\|e(\cdot,k)\|_{L^{-\alpha/2}}$:

$$\|e(\cdot,k)\|_{\mathcal{L}^\infty(dx)} \le 1 + \frac{c}{|k|}\left(\int_{-\infty}^{\infty} |W(y) e^{\frac{\alpha}{2}|y|}|^2 dy\right)^{1/2} \|e(\cdot,k)\|_{L^{-\alpha/2}(dx)},$$

where the constant c is independent of λ.

$\forall\ \epsilon_0 > 0$, there exists a constant $c(\epsilon_0) > 0$, such that if $\lambda > \beta + \epsilon_0$, then

$$\|(1 + \mathcal{R}^+(\lambda) W)^{-1}\|_{L^{-\alpha/2} \to L^{-\alpha/2}} \le c(\epsilon_0).$$

Thus if $\lambda > \beta + \epsilon_0$ then we have $\|e^{-\alpha/2|\cdot|} e(\cdot,k)\|_{\mathcal{L}^2(dx)} \le c(\epsilon_0)$, hence $\|e(\cdot,k)\|_{\mathcal{L}^\infty(dx)} \le c(\epsilon_0)$.

For $\frac{d}{dk} e(x,k)$, we need Fubini's Theorem to justify the following computation. Since it is tedious and not hard, we do not want to do it.

$$\begin{aligned} \tfrac{d}{dk} e(x,k) &= ix \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} - \int_{-\infty}^{\infty} A(x,y,k) W e(y,k) dy \\ &- \int_{-\infty}^{\infty} \begin{pmatrix} \frac{e^{ik|x-y|}}{2k} & 0 \\ 0 & \frac{e^{-\mu|x-y|}}{2\mu} \end{pmatrix} W \tfrac{d}{dk} e(x,k) dy, \end{aligned}$$

where

$$A(x,y,k) := \begin{pmatrix} \frac{ik|x-y|e^{ik|x-y|} - k e^{ik|x-y|}}{2k} & 0 \\ 0 & -\frac{ik|x-y|e^{-\mu|x-y|} - ik e^{-\mu|x-y|}}{2i\mu^3} \end{pmatrix}.$$

Similar reasoning proves that if $\lambda > \beta + \epsilon_0$, then

$$\|\frac{d^n}{dk^n} e(x,\cdot)\|_{\mathcal{L}^\infty(dx)} \le c(\epsilon_0)(1+|x|)^n,$$

where the constant c is independent of x, $n = 0, 1, 2, 3$.

(2) After finishing the estimates of $e(x,k)$ when $\lambda \ge \beta + \epsilon_0$, we consider the cases $\beta \le \lambda \le \beta + \epsilon_0$. We choose ϵ_0 so small such that if $\lambda - \beta \le \epsilon_0$, then $detD(k) \ne 0$. When we estimate the functions $\phi_1,\ \phi_2,\ \psi_1,\ \psi_2$, we always divide the domain $(-\infty,\infty)$ into two parts: $(-\infty,0],\ [0,\infty)$. We will use the same strategy to estimate $e(x,k)$ when k is small.

In Lemma A.3.10, we prove that if $x \in [0,+\infty)$,

$$e(x,k) = \frac{-2ikD_{22}(k)}{detD(k)} \psi_1(x,k) + \frac{2ikD_{12}(k)}{detD(k)} \phi_1(x,\mu).$$

If $x \in [0, +\infty)$, by Theorem A.1.1 we have

$$\|\frac{d^n}{dk^n}e(x,k)\|_{\mathcal{L}^\infty(dk)} \leq c(1+|x|)^n$$

for $n = 0, 1, 2, 3$. Similarly if $x \in (-\infty, 0]$, then

$$\|\frac{d^n}{dk^n}e(x,k)\|_{\mathcal{L}^\infty(dk)} \leq c(1+|x|)^n$$

for $n = 0, 1, 2$.

Conclusion: there exists a constant $c > 0$, such that if $\lambda \geq \beta$, then

$$\|\frac{d^n}{dk^n}e(x,k)\|_{\mathcal{L}^\infty(dk)} \leq c(1+|x|)^n$$

where $n = 0, 1, 2, 3$. □

In the following lemma we decompose $e(x,k)$ into several parts which are easier to understand.

Lemma A.4.3. *If $x \geq 0$ and $\lambda > \beta$, then there exists function s_2 such that*

$$|\frac{d^n}{dk^n}[e(x,k) - s_2(k)\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}]| \leq c\frac{1}{1+|k|}e^{-\epsilon_0|x|}. \tag{124}$$

If $x \leq 0$ and $\lambda > \beta$, then there exists a function $\gamma_2(k)$ such that

$$|\frac{d^n}{dk^n}[e(x,k) - \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} - \gamma_2(k)\begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}]| \leq c\frac{1}{1+|k|}e^{-\epsilon_0|x|}.$$

Also $|\frac{d^n}{dk^n}s_2(k)|, \quad |\frac{d^n}{dk^n}\gamma_2(k)| \leq c$.

All $c, \ \epsilon_0$ used do not depend on x and k; and $n = 0, 1, 2$.

Proof. We only prove Estimate (124), the proof of the second estimate is similar. As in the proof of Proposition A.4.2, we divide the proof into two parts, $\lambda > \beta + \epsilon_0$ and $\beta \leq \lambda \leq \beta + \epsilon_0$.

(1) When $\lambda > \beta + \epsilon_0$, we start with the definition of $e(x,k)$:

$$\begin{aligned}
& e(x,k) - \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \\
= \ & -\tfrac{1}{2k}\int_{-\infty}^{\infty}\begin{pmatrix} e^{ik|x-y|} & 0 \\ 0 & -\frac{ke^{-\mu|x-y|}}{\mu} \end{pmatrix} W(y)e(y,k)dy \\
= \ & -\tfrac{1}{2k}\int_{-\infty}^{\infty}\begin{pmatrix} 0 & 0 \\ 0 & -\frac{ke^{-\mu|x-y|}}{\mu} \end{pmatrix} W(y)e(y,k)dy \\
& -\tfrac{1}{2k}e^{ikx}\int_{-\infty}^{\infty}\begin{pmatrix} e^{-iky} & 0 \\ 0 & 0 \end{pmatrix} W(y)e(y,k)dy \\
& -\tfrac{1}{2k}\int_{x}^{\infty}\begin{pmatrix} 2\sin k(x-y) & 0 \\ 0 & 0 \end{pmatrix} W(y)e(y,k)dy
\end{aligned}$$

All the three terms on the right hand side are nice functions, so we could use Fubini's Theorem to make the following calculations:
For the first term, if $x \in [0, +\infty)$ we have that

$$|\frac{d^n}{dk^n}\int_{-\infty}^{\infty}\begin{pmatrix} 0 & 0 \\ 0 & -\frac{ke^{-\mu|x-y|}}{\mu}\end{pmatrix} W(y)e(y,k)dy| \le ce^{-\frac{\alpha}{4}|x|}$$

by Proposition A.4.2; for the second term:

$$-\frac{e^{ikx}}{2k}\int_{-\infty}^{\infty}\begin{pmatrix} e^{-iky} & 0 \\ 0 & 0\end{pmatrix} W(y)e(y,k)dy = \begin{pmatrix} s_2(k) \\ 0 \end{pmatrix} e^{ikx},$$

where s_2 have the estimate

$$|\frac{d^n}{dk^n}s_2(k)| \le c(\epsilon_0)$$

for all $\lambda > \beta + \epsilon_0$;
and the third term:

$$|\frac{d^n}{dk^n}\int_{x}^{\infty}\begin{pmatrix} 2\sin k(x-y) & 0 \\ 0 & 0\end{pmatrix} W(y)e(y,k)dy| \le ce^{-\frac{\alpha}{4}|x|}.$$

All the constants c used above are independent of k, x and n, where $n = 0, 1, 2$.

(2) We consider the case $\beta \le \lambda \le \beta + \epsilon_0$.

Since $detD(0) \neq 0$, there exist $\epsilon_0,\ \delta > 0$ such that if $|\lambda - \beta| \le \epsilon_0$, then $|detD(k)| \ge \delta$.

If $\beta \le \lambda \le \beta + \epsilon_0$, then Estimate (124) can be proven by Lemma A.3.10 in which the fact that $s,\ a_1,\ a_2,\ \gamma$ are analytic functions of k is proved and the following fact: if $x > 0$, by Lemma A.1.4 and Estimate (91) we have

$$|\frac{d^n}{dk^n}(\psi_1(x,k) - \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix})| \le ce^{-\epsilon_0 x},$$

for some constant c independent of x and λ, $n = 0, 1, 2$. Similar estimates can be gotten for $\psi_2(\pm\cdot, k)$.

□

We can prove the third estimate of Theorem A.4.1 by using the estimates made in Lemma A.4.3.

Proposition A.4.4.

$$\|f^{\#}\|_{\mathcal{H}^2} \le c\|(1 + |\cdot|)^2 f\|_2,$$

where, recall that

$$f^{\#}(k) = \int_{-\infty}^{\infty}\bar{e}^{\%}(x,k)\cdot f(x)dx.$$

Proof. Decompose $f^{\#}$ into two parts,

$$
\begin{aligned}
& |\tfrac{d^2}{dk^2}\int_{-\infty}^{\infty}\bar{e}^{\%}(x,k)\cdot f(x)dx| \\
\leq\ & |\tfrac{d^2}{dk^2}\int_{0}^{\infty}\bar{e}^{\%}(x,k)\cdot f(x)dx| + |\tfrac{d^2}{dk^2}\int_{-\infty}^{0}\bar{e}^{\%}(x,k)\cdot f(x)dx|.
\end{aligned}
$$

By Lemma A.4.3,

$$
\begin{aligned}
& |\tfrac{d^2}{dk^2}\int_{0}^{\infty}\bar{e}^{\%}(x,k)\cdot f(x)dx| \\
\leq\ & |\tfrac{d^2}{dk^2}\int_{0}^{\infty}(\bar{e}^{\%}(x,k)-\bar{s}_2(k)\begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix})\cdot f(x)dx| \\
& +|\tfrac{d^2}{dk^2}\int_{0}^{\infty}\bar{s}_2(k)\begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}\cdot f(x)dx| \\
\leq\ & c\int_{0}^{\infty}\tfrac{e^{-\epsilon_0|x|}}{1+|k|}|f(x)|dx + c|\int_{0}^{+\infty}\begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}\cdot(1+|x|)^2 f(x)dx| \\
\leq\ & c\tfrac{1}{1+|k|}\|f\|_2 + c|\int_{0}^{+\infty}\begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}\cdot(1+|x|)^2 f(x)dx|.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& |\tfrac{d^2}{dk^2}\int_{-\infty}^{0}\bar{e}^{\%}(x,k)\cdot f(x)dx| \\
\leq\ & c\|f\|_2\tfrac{1}{1+|k|} + c|\int_{-\infty}^{0}\begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}\cdot(1+|x|)^2 f(x)dx| \\
& +c|\int_{-\infty}^{0}\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\cdot(1+|x|)^2 f(x)dx|.
\end{aligned}
$$

Combine the two parts together:

$$
\begin{aligned}
& |\tfrac{d^2}{dk^2}\int_{-\infty}^{\infty}\bar{e}^{\%}(x,k)\cdot f(x)dx| \\
\leq\ & c\|f\|_2\tfrac{1}{1+|k|} + c|\int_{-\infty}^{\infty}\begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}\cdot(1+|x|)^2 f(x)dx| \\
& +c|\int_{-\infty}^{0}\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}\cdot(1+|x|)^2 f(x)dx|.
\end{aligned}
$$

We conclude that $\|\frac{d^2}{dk^2}f^{\#}\|_2 \leq c\|(1+|\cdot|)^2 f\|_2$.

It is easier to prove $\|f^{\#}\|_2 \leq c\|f\|_2$, thus

$$\|f^{\#}\|_{\mathcal{H}^2} \leq c\|(1+|x|)^2 f\|_2.$$

□

Corollary A.4.5.

$$\|\frac{d^n}{dk^n}(e(x,k)/k)\|_{\mathcal{L}^2(dk)} \leq c(1+|x|)^{n+1},$$

$$|\frac{d^n}{dk^n}f^{\#}| \leq c\|(1+|x|)^n f\|_1,$$

where $n=0,1,2$.

Proof. When $|k| \leq 1$, by Proposition A.4.2 we have

$$|\frac{d^n}{dk^n}(e(x,k)/k)| \leq c(1+|x|)^{n+1}.$$

When $k > 1$, by Proposition A.4.2 we have

$$|\frac{d^n}{dk^n}(e(x,k)/k)| \leq c\frac{1}{1+|k|}(1+|x|)^n.$$

Therefore

$$\|\frac{d^n}{dk^n}(e(x,k)/k)\|_{\mathcal{L}^2(dk)} \leq c(1+|x|)^{n+1}.$$

Recall $f^{\#}(k) = \langle e^{\%}(\cdot,k), f\rangle$. By Proposition A.4.2,

$$|\frac{d^n}{dk^n}f^{\#}(k)| \leq c\int |\frac{d^n}{dk^n}e^{\%}(x,k)||f(x)|dx \leq c\int (1+|x|)^n|f(x)|dx.$$

□

This completes the proof of Theorem A.4.1.

B Proof of Statements (A) and (B) of Proposition 2.3.1

In this appendix we prove Proposition 2.3.1 translated in the context of the operator H, i.e. for the family of operators $H(W) := H_0 + W$ where the operators H_0, W are defined in Equation (51).
Proof of (A). Suppose for some W_0 Statement (SB) and (SC) are satisfied. We use the notations and estimates from Subsection A.1. The Wronskian depends on the potential W and we display this dependence explicitly by writing $D(k,W)$ for $D(k)$. By Definitions (108) (109) we can see that $detD(0,W)$ is a continuous functional of the functions $\frac{d^n}{dx^n}\phi_1(x,\sqrt{2\beta},W)|_{x=0}$ and $\frac{d^n}{dx^n}\psi_1(x,0,W)|_{x=0}$ where $n=0,1$. By Estimates (97), the last two functions are continuous in variable W. Thus if $detD(0,W_0)\neq 0$ for some W_0, then there exists a constant $\epsilon > 0$ such that if the function W satisfies $\|e^{\alpha|x|}(W-W_0)\|_{\mathcal{L}^\infty} \leq \epsilon$, then $detD(0,W)\neq 0$. By Theorem A.2.2 $H(W)$ has no resonance at the point β. This completes the proof of (A).
Proof of (B). We fix the function W. It is not hard to prove that the functions $\frac{d^n}{dx^n}\phi_1(x,\sqrt{2\beta},sW)$, $\frac{d^n}{dx^n}\psi_1(x,0,sW)$ $(n=0,1)$ are analytic in the variable $s\in\mathbb{C}$. Therefore the function $detD(0,sW)$ is analytic in s as well. Thus $detD(0,sW)$ is either identically zero or vanishes at most at a discrete set of s. It is left to prove the first case does not occur if $\int_{-\infty}^{\infty} V_3 \neq 0$, where, recall $V_3 = V_1 + V_2$ from Transformation (49).

The proof is based on the following facts valid for sufficiently small s:

(I) The Wronskian function $W(\phi_1(\cdot,\sqrt{2\beta},sW),\phi_1(-\cdot,\sqrt{2\beta},sW))\neq 0$ which will be proved in Lemma **B.2** below;

(II) As shown in Lemma **B.1** below there exists a solution $\varphi_1(x, sW)$ with the following properties:

(DI) for each s there exist constants $c_1(s), c_2(s)$ such that $c_1(s) \neq 0$ if $\int_{-\infty}^{\infty} V_3 \neq 0$ and

$$\varphi_1(x, sW) - c_1(s) \begin{pmatrix} x \\ 0 \end{pmatrix} - c_2(s) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

decays exponentially fast at $-\infty$;

(DII) for any $x \in \mathbb{R}$ we have

$$|\varphi_1(x, sW)| \leq c(1 + |x|).$$

The function

$$\varphi_1(x, sW) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

decays exponentially fast at $+\infty$.

Given these facts we see that

$$W(\varphi_1(\cdot, sW), \varphi_1(-\cdot, sW)) \neq 0, W(\phi_1(\cdot, \sqrt{2\beta}, sW), \phi_1(-\cdot, \sqrt{2\beta}, sW)) \neq 0,$$

$$W(\varphi_1(\cdot, sW), \phi_1(\pm\cdot, \sqrt{2\beta}, sW)) = 0.$$

Since $\varphi_1(x, s) = \psi_1(x, 0, sW) + c_3(s)\phi_1(x, \sqrt{2\beta}, sW)$ for some constant $c_3(s)$ and recalling the definition of $D(0)$ hence $D(0, sW)$ from Equation (109) we can get that $detD(0, sW) \neq 0$. This implies that the operator $H(sW)$ has resonance only at discrete values of s. The statement (B) is proved (assuming Lemmas **B.1** and **B.2** below).

Lemma B.1. *There exists a solution $\varphi_1(\cdot, sW)$ of the equation*

$$[H(sW) - \beta]\varphi_1(\cdot, sW) = 0$$

with the properties stated in (DI) and (DII).

Proof. Recall the definition

$$H(sW) = H_0 + sW$$

with

$$H_0 = \begin{pmatrix} -\frac{d^2}{dx^2} + \beta & 0 \\ 0 & \frac{d^2}{dx^2} - \beta \end{pmatrix}, \ W = 1/2 \begin{pmatrix} V_3 & -iV_4 \\ -iV_4 & -V_3 \end{pmatrix},$$

the functions V_3, V_4 are smooth, even, and decay exponentially fast at ∞.

We could rewrite the equation for

$$\varphi_1(\cdot, sW) =: \begin{pmatrix} \varphi_{11}(\cdot, sW) \\ \varphi_{12}(\cdot, sW) \end{pmatrix}$$

as

$$\begin{pmatrix} \varphi_{11}(\cdot, sW) \\ \varphi_{12}(\cdot, sW) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s/2 \begin{pmatrix} \int_x^\infty \int_y^\infty V_3(t)\varphi_{11}(t, sW) - iV_4(t)\varphi_{12}(t, sW)dt \\ (-\frac{d^2}{dx^2} + 2\beta)^{-1}(iV_4\varphi_{11}(\cdot, sW) + V_3\varphi_{12}(\cdot, sW)) \end{pmatrix};$$

The proof of the existence of $\varphi_1(\cdot, sW)$ and the fact that $|\varphi_1(x, sW)| \le c(1+|x|)$ is easy because when s is small we could use the contraction lemma. We will not go into the details because we solve similar problems many times.

Since the Wronskian function $W(\varphi_1(x, sW), \varphi_1(-x, sW))$ is independent of x and analytic in s, it can be expanded in the variable s. We only need to compute the first two terms of $\varphi_1(x, sW)$ in terms of s to prove (DII):

$$\varphi_1(x, sW) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s/2 \begin{pmatrix} \int_x^\infty \int_y^\infty V_3(t)dtdy \\ i(-\frac{d^2}{dx^2} + 2\beta)^{-1}V_4 \end{pmatrix} + O(s^2).$$

Thus

$$\begin{aligned} & W(\varphi_1(\cdot, sW), \varphi_1(-\cdot, sW)) \\ = \; & \tfrac{d}{dx}\varphi_1^T(x, sW)\varphi_1(-x, sW) - \varphi_1^T(x, sW)\tfrac{d}{dx}\varphi_1(-x, sW) \\ = \; & -s\int_x^\infty V_3(t)dt - s\int_{-x}^\infty V_3(t)dt + O(s^2) \\ = \; & -s\int_x^\infty V_3(t)dt - s\int_{-\infty}^x V_3(-t)dt + O(s^2) \\ = \; & -s\int_{-\infty}^{+\infty} V_3 + o(s^2) + O(s^2). \end{aligned}$$

□

Lemma B.2. *If s is sufficiently small, then*

$$W(\phi_1(\cdot, \sqrt{2\beta}, sW), \phi_1(-\cdot, \sqrt{2\beta}, sW)) \neq 0.$$

Proof. For $n = 0, 1$, as $s \to 0$,

$$\frac{d^n}{dx^n}\phi_1(\cdot, \sqrt{2\beta}, sW) \to \frac{d^n}{dx^n}\phi_1(\cdot, \sqrt{2\beta}, 0)$$

in the $\mathcal{L}^\infty([0,\infty))$ norm. By Proposition A.1.3 we could get easily that

$$\phi_1(x, \sqrt{2\beta}, 0) = \begin{pmatrix} 0 \\ e^{-\sqrt{2\beta}x} \end{pmatrix}.$$

Next we use the Wronskian function again: as $s \to 0$,

$$\begin{aligned} & W(\phi_1(\cdot, \sqrt{2\beta}, s), \phi_1(-\cdot, \sqrt{2\beta}, s)) \\ = \; & \tfrac{d}{dx}\phi_1^T(x, \sqrt{2\beta}, s)\phi_1(-x, \sqrt{2\beta}, s) - \phi_1^T(\cdot, \sqrt{2\beta}, s)\tfrac{d}{dx}\phi_1(-\cdot, \sqrt{2\beta}, s)|_{x=0} \\ \to \; & -2\sqrt{2\beta}. \end{aligned}$$

Thus we proved the lemma. □

C Proof of Proposition 3.0.4

In this appendix we will prove Proposition 3.0.4 in a more general setting. We base our arguments on a general form of the operator $L_{general}$ given in Subsection 5.1.

Lemma C.1. *For any constant $\lambda_0 \notin (-i\infty, -i\beta] \cup [i\beta, i\infty)$, the operator-valued function $(L_{general} - \lambda_0 + z)^{-1}$ is an analytic function of z in a small neighborhood of 0. Furthermore*

$$(L(\lambda) - \lambda_0 + z)^{-1} = \sum_{n=m_0}^{+\infty} z^n K_n,$$

where $m_0 > -\infty$ is an integer and K_n's are operators.

Proof. Recall $L_{general} = L_0 + U$, where

$$L_0 := \begin{pmatrix} 0 & -\frac{d^2}{dx^2} + \beta \\ \frac{d^2}{dx^2} - \beta & 0 \end{pmatrix}, \ U := \begin{pmatrix} 0 & V_1 \\ -V_2 & 0 \end{pmatrix},$$

λ is a positive constant, V_1 and V_2 are smooth functions decaying exponentially fast at ∞. Since it is hard to get the Laurent series of $(L_{general} - \lambda_0 - z)^{-1}$ directly we make a transformation:

$$(L(\lambda) - \lambda_0 + z)^{-1} = (1 + (L_0 - \lambda_0 + z)^{-1} U)^{-1} (L_0 - \lambda_0 + z)^{-1}.$$

We make expansion on each term: The operators $(L_0 - \lambda_0 + z)^{-1}$ have no singularity when z is sufficiently small, so there exist operators $K_{1,n}$ $(n = 0, 1 \cdots)$ such that

$$(L_0 - \lambda_0 + z)^{-1} = \sum_{n=0}^{\infty} z^n K_{1,n}.$$

$(L_0 - \lambda_0 + z)^{-1} U$ are trace class operators, thus by [RSI], Theorem VI.14

$$(1 + (L_0 - \lambda_0 + z)^{-1} U)^{-1} = \sum_{n=m_0}^{\infty} z^n K_{2,n}$$

where $K_{2,n}$ $(n = m_0, m_0 + 1 \cdots)$ are operators and $m_0 > -\infty$ is an integer. The lemma is proved. $\square$

The following is the main theorem of this section.

Proposition C.2. *Suppose A is an operator having a complex number θ as an isolated eigenvalue and*

$$(A - \theta - z)^{-1} = \sum_{n=m}^{+\infty} A_n z^n,$$

where $m > -\infty$ is an integer and A_n are operators. Then we have the following three results:

(EnA) A_{-1} *is a projection operator:*

$$P_\theta^A := \frac{1}{2i\pi}\oint_{|x-\theta|=\epsilon}(A-x)^{-1}dx = A_{-1},$$

where $\epsilon > 0$ is a sufficiently small constant. $RangeP_\theta^A$ is the space of eigenvectors and associated eigenvectors of A with the eigenvalue θ, i.e.

$$RangeA_{-1} = \{x|(A-\theta)^k x = 0 \text{ for some positive integer } k\}.$$

Assume, if the operator P_θ^A is finite dimensional:

$$RangeP_\theta^A = \{\xi_1, \cdots, \xi_n\},$$

then the operator A^* has n independent eigenvectors and associated eigenvectors $\eta_1, \cdots, \eta_n$ with the eigenvalue $\bar{\theta}$;

(EnB) *The $n \times n$ matrix $T = [T_{ij}]$ where $T_{ij} := \langle \eta_i, \xi_j \rangle$ is invertible,*

(EnC) *The operator P_θ^A is of the form:*

$$P_\theta^A f = (\xi_1, \cdots, \xi_n) T^{-1} \begin{pmatrix} \langle \eta_1, f \rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle \eta_n, f \rangle \end{pmatrix}. \tag{125}$$

Proof. The proof of (EnA) is well known (see, e.g. [RSIV, Kato]). To prove (EnB), assume $detT = 0$. Then there exist constants $b_1, \cdots, b_n$ such that

$$b_1\eta_1 + \cdots + b_n\eta_n \perp \xi_1, \cdots, \xi_n$$

and at least one of the constants $b_1, \cdots, b_n$ is not zero. Since $P_\theta^A f \in \mathrm{Span}\{\xi_1, \cdots, \xi_n\}$ for any vector f, we have

$$\begin{aligned} 0 &= \langle b_1\eta_1 + b_2\eta_2 + \cdots + b_n\eta_n, P_\theta^A f\rangle \\ &= \langle P_{\bar{\theta}}^{A^*}(b_1\eta_1 + b_2\eta_2 + \cdots + b_n\eta_n), f\rangle \\ &= \langle b_1\eta_1 + b_2\eta_2 + \cdots + b_n\eta_n, f\rangle. \end{aligned}$$

Thus

$$b_1\eta_1 + \cdots + b_n\eta_n = 0.$$

This contradicts to the fact that the vectors $\eta_1, \cdots, \eta_n$ are linearly independent. Therefore we proved that the matrix T is invertible. For (EnC). For any vector f, $P_\theta^A f \in \mathrm{Span}\{\xi_1, \cdots, \xi_n\}$. Therefore they is a $n \times 1$ scalar matrix $\begin{pmatrix} a_1, \cdots, a_n \end{pmatrix}^T$ such that

$$P_\theta^A f = (\xi_1, \cdots, \xi_n) \begin{pmatrix} a_1, \cdots, a_n \end{pmatrix}^T.$$

What is left is to compute a concrete form of a_i's. We have the following formula

$$\langle \eta_i, f\rangle = \langle P_{\tilde{\theta}}^{A^*}\eta_i, f\rangle = \langle \eta_i, P_\theta^A f\rangle,$$

while

$$\langle \eta_i, P_\theta^A f\rangle = (\langle \eta_i, \xi_1\rangle, \cdots, \langle \eta_i, \xi_n\rangle)\begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix}$$

which implies Formula (125). □

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