3D crystal: how flat its flat facets are?

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Abstract

We investigate the hypothesis that the (random) crystal of the $(-)$-phase inside the $(+)$-phase of the 3D canonical Ising model has flat facets. We argue that it might need to be weakened, due to the possibility of formation of extra monolayer on a facet. We then prove this weaker hypothesis for the Solid-On-Solid model.

1 Introduction

Consider the classical Ising model $\{\sigma_s = \pm 1\}$, given by the Hamiltonian

$$H^{\text{Ising}}(\sigma) = -\sum_{\substack{s,t\in\mathbb{Z}^d:\\ |s-t|=1}} \sigma_s \sigma_t,$$

in the cubic box $\mathbb{T}_N \subset \mathbb{Z}^d$ of side N, with periodic boundary conditions and at the temperature β^{-1}, which is low enough. Let us impose the canonical constraint:

$$\sum_{s\in\mathbb{T}_N} \sigma_s = b\,|\mathbb{T}_N|,$$

where $|\mathbb{T}_N|$ is the volume of the box $\mathbb{T}_N$, the constant b satisfies $\left(1-\left(\frac{1}{d}\right)^{d/(d-1)}\right) < b < m(\beta)$, and where $m(\beta)$ is the spontaneous magnetization. Then the typical configuration σ under the Gibbs canonical distribution will have a crystal: namely, it will have one large contour $\Gamma(\sigma)$ (which is a surface of codimension one, of linear size $\sim N$ and of volume $\sim N^d$), randomly located, such that inside Γ we will see the minus-phase, while outside Γ the plus-phase.[1]

We are interested in the geometry of the (random) crystal $\Gamma(\sigma)$. It is known that under a certain scaling the shape of Γ tends to a non-random limit. Namely, if one shifts Γ so that its center of mass will be at the origin, and then scales Γ by a factor of $\frac{1}{N}$ in every direction, then in the limit $N\to\infty$ the random surface $\frac{1}{N}\Gamma$ will approach the non-random surface $W_d(\beta)$, the well-known Wulff shape. (In fact, $W_d(\beta)$ depends also on b, but since this dependence is just a linear scaling, we will omit it.) The meaning of the word "approach" depends on the dimension d. In dimension 2 the surface $W_2(\beta)$ is just an analytic curve, and a question of convergence of Γ to $W_2(\beta)$ is studied in [DKS, IS] in great detail. Namely, with probability going to 1 as $N\to\infty$, one can shift the curve $NW_2(\beta)$ in such a way that the contour Γ will lie inside the $N^{3/4}$-neighborhood of $NW_2(\beta)$. In dimension 3 the known results hold in a weaker L^1 sense: one should pass from the configuration σ to its integrated magnetization profile, which is a function $M_\sigma(x)\in[-1,+1]$

[1] Of course we will have the separation of phases for all values of b. Our restriction excludes the case when the minus-phase arranges itself into a strip, wrapping around the torus.

on the unit torus $\mathcal{T}^3$. Then the L^1 distance between $\frac{1}{m(\beta)} M_\sigma(x)$, properly shifted, and the signed characteristic function $2\mathbf{I}_{W_3(\beta)} - 1$ of the inside of the surface $W_3(\beta)$ goes to zero as $N \to \infty$, see [CP, B, BIV]. Notice that on a suitable coarse grained scale, refined results on the stability of the Wulff crystal w.r.t. the Hausdorff distance were obtained in [BI].

Unlike the curve $W_2(\beta)$, the surface $W_3(\beta)$ is not analytic; moreover, it contains flat pieces – called facets – provided that the temperature β^{-1} is below certain critical temperature T_r – called roughening temperature. It is known rigorously that $T_r \geq T_c(2)$, see [BFL, BFM], where we denote by $T_c(d)$ the critical temperature of the d-dimensional Ising model. It is an open question whether T_r is equal to $T_c(3)$ or is strictly less, as the common belief is. The shape of the facets of $W_3(\beta)$ is also given by the Wulff construction, see [M] or [S1], sect. 2.5.

On the microscopic level, it was proven by Dobrushin [D] that at sufficiently low temperatures, rigid interfaces occur for some Gibbs measures with specific choices of boundary conditions. Therefore it is a natural question to ask, in which sense the flat facets observed in the macroscopic crystals and the microscopic rigid interfaces are related. In this paper we want to discuss the question of whether or not the random crystals $\Gamma(\sigma)$ themselves have flat facets, for N large. Clearly, the results concerning the L^1-convergence of $\Gamma(\sigma)$ to $W_3(\beta)$ are perfectly consistent with either behavior. Some time ago one of us made the following conjecture, see [S1], sect. 3.4:

Conjecture 1 ***(Probably wrong)*** *Let the temperature β^{-1} be low enough. Then the following event has probability approaching* 1 *as $N \to \infty$:*
There exist six distinct 2D planes $L_i = L_i(\sigma) \subset \mathbb{T}_N$, $i = 1, ..., 6$, two for each coordinate direction, such that the intersections $L_i \cap \Gamma(\sigma)$ are flat facets of $\Gamma(\sigma)$. Namely, for every i
i) $\operatorname{diam}(L_i \cap \Gamma(\sigma)) \geq C_1(\beta) \operatorname{diam}(\Gamma(\sigma))$, *with $C_1(\beta) \to \sqrt{2/3}$ as $\beta \to \infty$;*
ii) $\frac{\operatorname{Area}(L_i \cap \Gamma(\sigma))}{[\operatorname{diam}(L_i \cap \Gamma(\sigma))]^2} \geq C_2(\beta)$, *with $C_2(\beta) \to 1/2$ as $\beta \to \infty$, where by* $\operatorname{Area}(L_i \cap \Gamma(\sigma))$ *we mean the number of plaquettes of $\Gamma(\sigma)$, belonging to the plane L_i;*
iii) The asymptotic shape of the facets $L_i \cap \Gamma(\sigma)$ is given by the corresponding Wulff construction, see [M] or [S1], sect. 2.5.

We believe now that the above statement is a little bit too strong to be true. More precisely, it is almost true, except that one of the above 6 facets has an extra monolayer of the height one! So our refined conjecture looks as follows:

Conjecture 2 *(**Hopefully correct**) Let the temperature β^{-1} be low enough. Then the following event has probability approaching 1 as $N \to \infty$:*
There exist six distinct 2D planes $L_i = L_i(\sigma) \subset \mathbb{T}_N$, $i = 1, ..., 6$, two for each coordinate direction, such that the intersections $L_i \cap \Gamma(\sigma)$ are flat facets of $\Gamma(\sigma)$ in the following sense:

- *for every i $\operatorname{diam}(L_i \cap \Gamma(\sigma)) \geq C_1(\beta) \operatorname{diam}(\Gamma(\sigma))$, with $C_1(\beta) \to \sqrt{2/3}$ as $\beta \to \infty$;*
- *for every i except $i = i_0 = i_0(\sigma)$ $\frac{\operatorname{Area}(L_i \cap \Gamma(\sigma))}{[\operatorname{diam}(L_i \cap \Gamma(\sigma))]^2} \geq C_2(\beta)$, with $C_2(\beta) \to 1/2$ as $\beta \to \infty$;*
- *$\frac{\operatorname{Area}\left(L_{i_0} \cap \Gamma(\sigma)\right) + \operatorname{Area}\left(\left(L_{i_0} + \mathbf{n}_{i_0}\right) \cap \Gamma(\sigma)\right)}{\left[\operatorname{diam}\left(L_{i_0} \cap \Gamma(\sigma)\right)\right]^2} \geq C_2(\beta)$, where $\mathbf{n}_{i_0}$ is the unit vector orthogonal to L_{i_0} and pointing "away from" $\Gamma(\sigma)$.*

The meaning of the last statement is that on the facet $L_{i_0} \cap \Gamma(\sigma)$ there is another "monoatomic" layer of our crystal, having the shape $(L_{i_0} + \mathbf{n}_{i_0}) \cap \Gamma(\sigma)$. The limiting values $\sqrt{2/3}$ and $1/2$ are coming from the fact that in the limit $\beta \to \infty$ we expect $\Gamma(\sigma)$ to approach the shape of the cube.

At present we have no proof of this conjecture, and our paper is a result of an attempt to prove it. Namely, we prove here a weaker statement, and for a simpler – SOS – model. More precisely, we show that in the "canonical" SOS-model indeed a flat facet is formed, which may have an extra monolayer of particles. We formulate our result in the next section. In section 3 we further discuss it and we make various comments concerning the validity of the conjectures above.

The heuristic explanation of our result is simple. Imagine that on one facet of the crystal $\Gamma(\sigma)$ we have two monolayers – the top one, $F_1(\sigma)$, located over the second one, $F_2(\sigma)$, with the size of the second one significantly smaller than the size of $\Gamma(\sigma)$ itself. Then we can enlarge $F_2(\sigma)$ to the full size of the facet of $\Gamma(\sigma)$, diminishing at the same time the monolayer $F_1(\sigma)$. It might even be that by that procedure the monolayer $F_1(\sigma)$ will disappear completely. But in any case this procedure decreases the surface energy of the crystal. The reason for that is the same as for the fact that merging together two droplets into a larger one decreases the overall surface energy. In fact, we need here a slightly more general statement, that if the possible growth of the larger droplet is constrained by the container, then still to grow it to the maximal possible size, while diminishing the smaller one correspondingly is energetically favorable. In the present paper it will be enough for us to have a zero-temperature analog of this statement, which is the content of the Lemma 8 below. This statement for the general case can be proven by the methods of the paper [SS1].

The second result of our paper deals with the question about the range of fluctuations of the random crystal around its limit shape. As we have said above, in the 3D case the known results about the closeness of the random crystal to its asymptotic shape are obtained only for the L^1 distance between them, while in the 2D case they are known to hold for the Hausdorff distance. Probably one cannot hope to extend this result to the 3D case at all subcritical temperatures. However it is reasonable to expect that such result does hold at very low temperatures. That was suggested already in the book [DKS]. Namely, though the solution of the Wulff variational problem is not stable in the Hausdorff distance, due to the possibility that thin long hairs can appear on the crystal, at low temperatures these hairs are highly improbable due to their energetic cost. Here we give an extra reason to believe it by proving "No Hairs" theorem, that for the low temperature SOS model in the $N \times N$ box the random surface fluctuates away from the flat facet by less than $C \ln N$, for some $C < \infty$, and so the low temperature SOS crystal is always "clean-shaven".

We finish this introduction by pointing out the technical innovations of the present paper. Usually, to prove a result of such kind, one has to obtain the lower estimate on the probability of "nice" behavior of the random surface we are interested in, together with the upper estimate on the probability of its "ugly" behavior. Here the latter is easy, while the former is very hard, since this is the question about the typical behavior of the collection of contours which are strongly interacting, see [FPS]. We manage to establish our result by having only the upper estimate. This is both the strong and the weak point of our approach; we prove our theorem, but we do not have the technique to obtain the complete control over our model.

2 Statement of the Main Result

Let $\varphi = \{\varphi_s \in \mathbb{Z}\}$ be an integer valued random field, defined for $s \in \mathbb{Z}^2$. Its distribution is defined by the "Solid-on-Solid" Hamiltonian

$$H(\varphi) = \sum_{\substack{s,t \in \mathbb{Z}^2: \\ |s-t|=1}} |\varphi_s - \varphi_t| .$$

Namely, let $\Lambda \subset \mathbb{Z}^2$ be a finite box, $|\Lambda| < \infty$, the configuration (boundary condition) $\psi_.$ be given outside Λ, and the parameter $\beta > 0$ (inverse temperature) is fixed. Then we define the distribution $Q_{\beta,\Lambda,\psi}$ on the configurations $\Omega_\Lambda = \{\varphi_s : s \in \Lambda\}$ by

$$Q_{\beta,\Lambda,\psi}(\varphi) = \exp\left\{-\beta H_\Lambda\left(\varphi \,\middle|\, \psi\right)\right\} \ / \ Z(\beta, \Lambda, \psi) . \tag{1}$$

Here

$$H_\Lambda\left(\varphi \,\middle|\, \psi\right) = \sum_{\substack{s\in\Lambda, t \in \mathbb{Z}^2: \\ |s-t|=1}} |(\varphi \vee \psi)_s - (\varphi \vee \psi)_t| ,$$

$(\varphi \vee \psi)_t$ equals to φ_t for $t \in \Lambda$ and to ψ_t for $t \notin \Lambda$, and the partition function $Z(\beta, \Lambda, \psi)$ is a normalizing factor, making (1) a probability distribution.

Our model of the crystal will be the distribution obtained from $Q_{\beta,\Lambda,\psi}$ by a suitable conditioning. Namely, we will consider the case when

- $\Lambda = \Lambda_N = \{s : 1 \leq s_i \leq N, i = 1, 2\}$ is a square $N \times N$,
- $\psi_. \equiv 0$,
- the volume constraint

$$\mathbb{V}_N(\varphi) = \sum_{s \in \Lambda} \varphi_s \geq \lambda N^3 \tag{2}$$

  is imposed, with $\lambda > 0$ fixed.

We denote the conditional distribution $Q_{\beta,\Lambda_N,\psi=0}\left(\varphi \,\middle|\, \mathbb{V}_N(\varphi) \geq \lambda N^3\right)$ by $P_{\beta,N}(\varphi)$. We do not keep λ in this notation, since it will be fixed throughout the paper. We will use the notation $Q_{\beta,N}$ for the unconditional distribution

$Q_{\beta,\Lambda_N,\psi=0}(\cdot)$. We define the crystal $\mathsf{C}(\varphi)$ to be the body below the graph of φ:

$$\mathsf{C}(\varphi) = \left\{(s,h) \in \mathbb{Z}^3 : s \in \Lambda, 0 \leq h \leq \varphi(s)\right\}.$$

To formulate our results about the facets we have to introduce the level sets. So for every φ and every integer $l > 0$ we denote by $D(\varphi, l)$ the subset of all sites s in Λ, where $\varphi(s) \geq l$. We identify $D(\varphi, l)$ with the union of the closed unit squares centered at the corresponding points s. The connected components of the topological boundary of $D(\varphi, l)$ will be called contours. The set of all contours will be denoted by $\Delta(\varphi, l)$. The sets $D(\varphi, l)$ can be disconnected; we denote by $D_i(\varphi, l)$ the collections of connected components of $D(\varphi, l)$, $i = 1, 2, ...$ which are mutually external. They will be called **sections**. By $\partial D_i(\varphi, l)$ we denote the outer component of the boundary of the section $D_i(\varphi, l)$. The set of contours $\partial D_i(\varphi, l)$ coincides with the set of external contours of the family $\Delta(\varphi, l)$. A section $D_i(\varphi, l)$ will be called **large,** if

$$|\partial D_i(\varphi, l)| \geq K \ln N, \tag{3}$$

where K is some big constant, to be chosen later. Otherwise it is called **small**.

Consider now the level $L = L(\varphi)$, which is defined to be the **maximal value** of l-s, satisfying the following condition:

- $|D(\varphi, l)| \geq a(\beta) N^2$, where $a(\beta)$ is some small quantity, $a(\beta) \to 0$ as $\beta \to \infty$, to be defined later.

Denote by $\mathbf{F}_1(\varphi)$ the level set $D(\varphi, L(\varphi))$, and introduce also the notation $\mathbf{F}_i(\varphi)$ for the level sets $D(\varphi, L(\varphi) - i + 1)$. Our initial hypothesis was that the level set $\mathbf{F}_1(\varphi)$ – the "First Facet" – is the facet sought, in the sense that $|\mathbf{F}_1(\varphi)| \geq (1 - a(\beta)) N^2$. However at present we cannot prove nor disprove this statement, and we think that it is not valid. In particular we cannot rule out the case of $|\mathbf{F}_1(\varphi)| \sim N^2/2$, say. Still, we can show that a sharply localized jump of the function $|D(\varphi, l)|$ happens for typical φ-s:

Theorem 3 *Suppose the temperature is low enough. Then for the typical crystal* ***the "Second Facet" is large****:*

$$P_{\beta,N}\left\{\varphi : |\mathbf{F}_2(\varphi)| \geq (1 - a(\beta)) N^2\right\} \to 1 \text{ as } N \to \infty,$$

with some $a(\beta) \to 0$ as $\beta \to \infty$.

This result means that the crystal $\mathsf{C}(\varphi)$ indeed has a horizontal facet, in the following sense: the level of height $L(\varphi)-1$ is almost filled with sites, whereas at the levels above $L(\varphi)$ only few sites belong to the crystal. We do not know what happens at the level $L(\varphi)$, i.e. how big the First Facet really is.

It is known that the SOS-model undergoes the roughening transition in temperature, see [FrSp]. At low temperatures the (unconstrained) SOS-model (without condition (2)) with zero boundary conditions is localized, while at high temperatures it diverges logarithmically with N. It is reasonable to conjecture that the roughening temperature T_r^{SOS} is critical for our problem as well. In particular, it will mean that for every temperature $\beta^{-1} > T_r^{SOS}$, every $\varepsilon > 0$ and for every pair m, n of integers

$$P_{\beta,N}\left\{\varphi : \frac{|D(\varphi, L(\varphi)+m)|}{|D(\varphi, L(\varphi)-n)|} < 1-\varepsilon\right\} \to 0 \text{ as } N \to \infty, \tag{4}$$

for any value of the parameter $a > 0$, used in the definition of the level height $L(\varphi)$. As our theorem shows, the behavior opposite to (4) takes place at low temperatures, and we conjecture that it is the case for all temperatures below T_r^{SOS}.

To formulate the No Hairs theorem we introduce the boundary $\partial\mathbf{F}_i(\varphi)$ of the i-th facet to be just the boundary $\partial D(\varphi, L(\varphi)-i+1)$. The theorem states that inside $\partial\mathbf{F}_2(\varphi)$ the surface φ is almost flat, up to logarithmic excitations.

Theorem 4 *There exists β_0 such that for any $\beta > \beta_0$, one can find $C > 0$ for which the following holds*

$$P_{\beta,N}\{\varphi : \exists s \in \mathrm{Int}\,(\partial\mathbf{F}_2(\varphi)), \quad |\varphi_s - L(\varphi)| > C\ln N\} \to 0 \quad \text{as } N \to \infty.$$

3 Zero temperature Ising crystal

In this section we discuss the relations between the Conjectures 1 and 2 above. The Conjecture 2 is clearly a weaker statement, so it is not surprising that we can prove its SOS-counterpart, while we can not prove nor disprove the SOS-version of Conjecture 1. The real reason why the Conjecture 2 is simpler is the fact that it is valid at zero temperature, while the Conjecture 1 is definitely not.

The question about the shape of the crystal in the canonical Ising model becomes in the case of zero temperature the question about the isoperimetric problem in $\mathbb{Z}^3$. Namely, we are looking into the following problem: let K be an integer, and we consider the family $\tilde{\mathcal{V}}_K$ of all subsets $V \subset \mathbb{Z}^3$ containing precisely K sites. For every V we define the value $|\partial V|$ to be the number of plaquettes (of the dual lattice) in the boundary of V; in other words, $|\partial V|$ is the area of the surface ∂V. We define $\mathcal{V}_K \subset \tilde{\mathcal{V}}_K$ to be the subset consisting of *minimal* V-s:

$$V \in \mathcal{V}_K \Leftrightarrow |\partial V| = \min_{W \in \tilde{\mathcal{V}}_K} |\partial W| .$$

In the following we will not distinguish the elements of $\mathcal{V}_K$ which differ by translation only, thus $\mathcal{V}_K$ becomes a finite set, so we can endow it with a uniform probability distribution. We want to take K to infinity and to look on the typical properties of the crystal shapes. However what we will see depends on the values of K. In case $K = M^3$ with M integer, the set $\mathcal{V}_K$ contains just one element, so the situation is trivial. To get some interesting behavior one has to choose the subsequence $K_n \to \infty$ in a special way. There are many different options here, and we describe just one of them.

In the formulation of the theorem, which follows, the expressions "square with rounded corners" and "cube with rounded corners" are used. They mean here the following. Let k be an integer, and $Y_1, ..., Y_4$ be four Young diagrams with the total number of cells less than k. Then the square $k \times k$ with four diagrams $Y_1, ..., Y_4$ removed from its four corners is our "square with rounded corners". In the same way, a cube $k \times k \times k$ with rounded corners is obtained from k-cube by removing eight 3D Young diagrams (called also "skyscrapers") $S_1, ..., S_8$, with the total volume below k, from its eight corners. We will call these diagrams as *defects*.

Let $0 < \mu < 1$ be a fixed number. We take

$$K_n = n^3 + k(k-1) + 1,$$

where $k = [\mu n]$ denote the integer part of μn.

Theorem 5 *As $n \to \infty$, a typical random shape V, drawn from the uniform distribution on $\mathcal{V}_{K_n}$, can be described as follows:*

V is a "cube with rounded corners" of size n, to one (random) face of which a monolayer is attached, which is a "square with rounded corners" of

size k. These roundings have asymptotic shapes as $n \to \infty$: namely, let $x > 0$ satisfies

$$(k-4x)^2 = \frac{2^{11} 3^3 (\zeta(3))^2}{\pi^6} x^3 \tag{5}$$

(so x is of the order of $k^{2/3}$). Then each rounded corner of the square, scaled down by a factor $x^{1/2}$, has asymptotic shape given by the Vershik curve,

$$\exp\left\{-\tfrac{\pi}{\sqrt{6}}u\right\} + \exp\left\{-\tfrac{\pi}{\sqrt{6}}v\right\} = 1, \tag{6}$$

while each rounded corner of the cube, scaled down by a factor $(k-4x)^{1/3}$, has asymptotic shape given by the Cerf-Kenyon surface, see [CK], Theorems 1.2 and 1.3.

The curve (6) was obtained in [VKer]. The proof of the above result will appear later, see [S2]. The equation (5) is related with the asymptotic numbers of partitions and plane partitions of a large integer, see [S1], Sect. 4.1 and 4.2.

4 Proof of the Second Facet theorem

We first prove a weaker statement, which, in fact, contains the main part of the proof of our result. Define $\mathbf{E}_j(\varphi)$ as the interior volume of all the external contours $\partial_i D(\varphi, L(\varphi)-j+1)$ of the level set $D(\varphi, L(\varphi)-j+1)$.

Theorem 6 *For any $a > 0$, there is β_0 such that*

$$\forall \beta \geq \beta_0, \quad P_{\beta,N}\left\{\varphi : \mathbf{E}_2(\varphi) \geq (1-a)N^2\right\} \to 1 \text{ as } N \to \infty.$$

Proof. The proof relies on energy estimates of the contours lying in the first and second facets. For a given height configuration φ, we denote by $\{\gamma_i\}_{i \leq K_1}$ the set of all external contours of the family $\Delta(\varphi, L(\varphi))$. These are just the external boundaries $\partial D_i(\varphi, L(\varphi))$ of the connected components of $\mathbf{F}_1(\varphi)$. Similarly, $\{\Gamma_i\}_{i \leq K_2}$ will refer to the external contours in $\mathbf{F}_2(\varphi)$. By construction the contours satisfy a compatibility condition, namely for any γ_i there exists Γ_j such that γ_i lies inside Γ_j.

We introduce two events; the first one, $\mathcal{S}$, consists of configurations such that the volume contribution to $\mathbf{E}_1(\varphi)$ of external small contours of the first facet is larger than $\frac{a}{2}N^2$, while the second, $\mathcal{L}$, corresponds to the configurations for which the volume of the external large contours in the first facet is above $\frac{a}{2}N^2$, and also the volume of the external large contours in the second facet is smaller than $(1-a)\,N^2$:

$$\mathcal{S} = \left\{ \varphi;\quad \sum_{\gamma_i \text{ small}} |\mathrm{Int}\,(\gamma_i)| \geq \frac{a}{2}N^2 \right\}, \tag{7}$$

$$\mathcal{L} = \left\{ \varphi;\quad \sum_{\gamma_i \text{ large}} |\mathrm{Int}(\gamma_i)| \geq \frac{a}{2}N^2, \quad \sum_{\Gamma_j \text{ large}} |\mathrm{Int}(\Gamma_j)| < (1-a)N^2 \right\}. \tag{8}$$

By construction, $|\mathbf{E}_1(\varphi)| \geq |\mathbf{F}_1(\varphi)| \geq aN^2$, so we can write

$$P_{\beta,N}\left\{\varphi : |\mathbf{E}_2(\varphi)| < (1-a)\,N^2\right\} \leq P_{\beta,N}(\mathcal{S}) + P_{\beta,N}(\mathcal{L})\,.$$

Thus, to complete the proof, it is enough to show that for β large enough, there exists $c = c(a,\beta) > 0$ such that

$$P_{\beta,N}(\mathcal{L}) \leq \exp(-cN)\,, \tag{9}$$

$$P_{\beta,N}(\mathcal{S}) \leq \exp\left(-c\left(\frac{N}{\ln N}\right)^2\right). \tag{10}$$

The estimate (9) on the large contours will be obtained in Subsection 4.3 and the estimate (10) on the small contours in Subsection 4.4. ■

4.1 A priory estimates on the height of the facet

We start with very elementary estimates. Every SOS-surface $\varphi \in \Omega_{\Lambda_N}$ is made from N^2 horizontal plaquettes and a number of vertical ones; we denote this last one by $S(\varphi)$. Evidently, the distribution $Q_{\beta,N}(\varphi)$ equals to $\exp\{-\beta S(\varphi)\}$, up to normalization factor. Standard Peierls and counting arguments lead to the following simple estimate:

$$Q_{\beta,N}(\varphi : S(\varphi) = S) \leq 3^{N^2+S}\exp\{-\beta S\}\,. \tag{11}$$

Under $P_{\beta,N}$, the facet should be located with a high probability at a height of order N. But for us a weaker statement will be sufficient. For $K > 1$ and $k > 0$ we define

$$\mathcal{H} = \{\varphi;\ k \leq L(\varphi) \leq KN\} .$$

Proposition 7 *For any β large enough, any k fixed and $K \geq K(\beta)$ large enough*

$$P_{\beta,N}(\mathcal{H}) \geq 1 - \exp(-\beta' N^2), \tag{12}$$

where β' diverges with β.

Proof. Our claim follows easily from (11) . Indeed, the property $L(\varphi) > KN$ implies that

$$S(\varphi) \geq 4K\sqrt{a(\beta)}N^2,$$

since for every level l below $L(\varphi)$ we have $|D(\varphi, l)| \geq a(\beta) N^2$. On the other hand, if the surface φ is such that all its sections $D_i(\varphi, l)$ have the area below $a(\beta) N^2$ for $l \geq k$, then

$$S(\varphi) \geq \frac{\lambda N^3 - kN^2}{a(\beta) N^2} 4\sqrt{a(\beta)}N = C(\beta) N^2, \text{ with } C(\beta) \to \infty \text{ as } \beta \to \infty.$$

Therefore

$$P_{\beta,N}\{\varphi : L(\varphi) \notin [k, KN]\} \leq \frac{Q_{\beta,N}(S(\varphi) \geq \tilde{C}(\beta) N^2)}{Q_{\beta,N}(\ \mathbb{V}_N \geq \lambda N^3)} \leq \exp\left(-\beta' N^2\right) ,$$

since the denominator is always larger than $\exp(-\beta\lambda N^2)$. ■

4.2 Isoperimetric inequality - zero temperature case

Here we prove the statement mentioned in the introduction, that merging two droplets together decreases the surface energy. More generally, just increasing the bigger one at the expense of the smaller one still makes the energy smaller. We prove here the corresponding statement for the 2D zero temperature Ising model only, while some generalizations are available by using the technique of [SS1].

For an integer V we define $L = L(V)$ to be the largest integer such that $L^2(V) \leq V$, and we introduce $r = r(V) = V - L^2(V)$. We denote by $p = p(V)$ the length of the shortest closed path on the lattice $\mathbb{Z}^2$, surrounding V unit plaquettes. Clearly,

$$p(V) = \begin{cases} 4L(V) & \text{if } V = L^2(V), \\ 4L(V) + 2 & \text{if } 0 < r(V) \leq L(V), \\ 4L(V) + 4 & \text{if } L(V) < r(V) \leq 2L(V). \end{cases}$$

We will call $p(V)$ the surface energy of the droplet V. In what follows we will identify the integers V with the collections of plaquettes from $\mathbb{Z}^2$ with perimeter $p(V)$, which will be called also droplets. Now, let $V_1 \leq V_2$ be two integers, and we suppose that for some N and some (small) $\rho > 0$ we have

$$V_1 \geq \rho N^2,$$

$$V_2 \leq N^2.$$

The second condition means that the larger droplet V_2 fits inside the volume $N \times N$, and the first one – that the smaller droplet is not too small.

Let now D be any integer, satisfying the conditions

$$V_1 \geq D \geq \rho N^2.$$

Lemma 8 *The transfer of the amount D from the droplet V_1 to V_2 decreases the total surface energy: there exists a constant $\varkappa = \varkappa(\rho) > 0$, such that*

$$(1 - \varkappa(\rho))(p(V_1) + p(V_2)) \geq p(V_1 - D) + p(V_2 + D).$$

Proof. We will show that the difference

$$p(V_1 - D) + p(V_2 + D) - p(V_1) - p(V_2)$$

is of the order of $p(V_1) + p(V_2)$ and negative. Since the function $p(x)$ equals approximately to $4\sqrt{x}$ – more precisely,

$$4\sqrt{x} \leq p(x) < 4\sqrt{x} + 4, \tag{13}$$

– it is enough to show that the difference

$$\sqrt{V_1 - D} + \sqrt{V_2 + D} - \sqrt{V_1} - \sqrt{V_2}$$

is of the order of $\sqrt{V_1}+\sqrt{V_2}$ and negative. Let us rewrite the difference as

$$\sqrt{V_1}\left(\sqrt{1-\frac{D}{V_1}}-1\right)+\sqrt{V_2}\left(\sqrt{1+\frac{D}{V_2}}-1\right)$$

and use the Taylor expansion of the function $\sqrt{1+x}$. We get

$$\begin{aligned}&\sqrt{V_1}\left(\sqrt{1-\frac{D}{V_1}}-1\right)+\sqrt{V_2}\left(\sqrt{1+\frac{D}{V_2}}-1\right)\\ &=\frac{1}{2}\left(-\frac{D}{\sqrt{V_1}}+\frac{D}{\sqrt{V_2}}\right)-\frac{1}{8}\left(\frac{D^2}{V_1\sqrt{V_1}}+\frac{D^2}{V_2\sqrt{V_2}}\right)+\ldots \qquad (14)\end{aligned}$$

Now, since $V_1<V_2$, the contents of all the odd brackets are negative, while the even coefficients are also negative, so the difference is negative as well. Finally, since all the values V_1,V_2 and D are of the same order, the second term $\frac{D^2}{V_1\sqrt{V_1}}+\frac{D^2}{V_2\sqrt{V_2}}$ is of the order of $\sqrt{V_1}+\sqrt{V_2}$, and the proof follows. ■

4.3 Estimates on the large contours

In this section we will prove the estimate (9).

For a given integer $\ell\geq 2$ and a compatible collection of large contours $(\gamma,\Gamma)=(\{\gamma_i\},\{\Gamma_j\})$ we denote by $\varphi\sim(\gamma,\Gamma,\ell)$ the height configurations φ which satisfy:

- the volume constraint $\mathbb{V}_N(\varphi)=\sum_{s\in\Lambda}\varphi_s\geq\lambda N^3$,
- $L(\varphi)=\ell$,
- the only exterior large contours on the level sets $\mathbf{F}_1$ and $\mathbf{F}_2$ are given by (γ,Γ).

Then we have

$$P_{\beta,N}(\mathcal{L})\leq P_{\beta,N}(\mathcal{H}^c)+\frac{1}{Q_{\beta,N}(\,\mathbb{V}_N\geq\lambda N^3)}\sum_{\ell=2}^{KN}\ \sum_{(\gamma,\Gamma)\in\mathcal{L}}\ \sum_{\varphi\sim(\gamma,\Gamma,\ell)}Q_{\beta,N}(\varphi)\,.$$

For a given triplet (γ,Γ,ℓ), we define the erasing map

$$\varphi \rightsquigarrow \hat{\varphi} = \Big(\hat{\varphi}_s = \varphi_s - \sum_i 1_{\{s\in \mathrm{Int}(\gamma_i)\}} - \sum_j 1_{\{s\in \mathrm{Int}(\Gamma_j)\}} \Big)_{s\in\Lambda} . \tag{15}$$

It maps injectively the set $\{\varphi \sim (\gamma,\Gamma,\ell)\}$ into the set

$$\Big\{ \mathbb{V}_N(\varphi) \geq \lambda N^3 - \sum_i |\mathrm{Int}(\gamma_i)| - \sum_j |\mathrm{Int}(\Gamma_j)| \Big\} .$$

Evidently,

$$Q_{\beta,N}(\varphi) = \exp\big(-\beta \sum_i |\gamma_i| - \beta \sum_j |\Gamma_j| \big) \; Q_{\beta,N}(\hat{\varphi}) .$$

Therefore we have the "Peierls estimate"

$$\begin{aligned} P_{\beta,N}(\mathcal{L}) \leq P_{\beta,N}(\mathcal{H}^c) + \sum_{\ell=2}^{KN} \sum_{(\gamma,\Gamma)\in\mathcal{L}} & \exp\big(-\beta \sum_i |\gamma_i| - \beta \sum_j |\Gamma_j| \big) \\ & \times \frac{Q_{\beta,N}\big(\mathbb{V}_N(\varphi) \geq \lambda N^3 - \sum_i |\mathrm{Int}(\gamma_i)| - \sum_j |\mathrm{Int}(\Gamma_j)| \big)}{Q_{\beta,N}\big(\mathbb{V}_N(\varphi) \geq \lambda N^3 \big)} . \end{aligned} \tag{16}$$

The important quantity is the total volume of the interiors of the contours, thus we are going to average on all the possible contour shapes in order to retain only the information on the volume. Fix $V \in [\frac{a}{2}N^2, (1-a)N^2]$ and consider the collection (γ_i) of the large contours such that

$$\sum_i |\mathrm{Int}(\gamma_i)| = V. \tag{17}$$

The optimal shape for a contour of volume V is a square of side length $L = L(V)$ with (possibly) an additional layer of $r(V)$ sites such that

$$V = L(V)^2 + r(V) , \qquad r(V) \in \{0,\ldots,2L(V)\} .$$

In this case the following isoperimetric inequality holds uniformly over the collection (γ_i) which satisfy the volume constraint (17) :

$$\sum_i |\gamma_i| \geq 4L(V) \ .$$

Let $\beta' = \beta - 10$. Summing over all the collections of contours such that $\sum_i |\mathrm{Int}(\gamma_i)| = V$, we get

$$\sum_{(\gamma_i)} \exp\big(-\beta \sum_i |\gamma_i|\big) \leq \exp(-\beta' 4L(V)) \left(\sum_{(\gamma_i)} \prod_i \exp\big(-10|\gamma_i|\big) \right)$$
$$\leq \exp(-4\beta' L(V)),$$

where the final inequality is obtained by taking into account the entropy of a single large contour

$$\sum_{(\gamma_i)} \prod_i \exp\big(-10|\gamma_i|\big) \leq \left(1 + \sum_{\ell \geq K \ln N} 3^\ell \exp(-10\ell)\right)^{N^2} - 1 < 1.$$

Plugging this inequality in (16), we get

$$\begin{aligned} P_{\beta,N}(\mathcal{L}) &\leq P_{\beta,N}(\mathcal{H}^c) \\ &+ \sum_{\ell=2}^{N^3} \sum_{\sqrt{\frac{a}{2}}N \leq L_1 \leq L_2 \leq \sqrt{1-a}N} (2L_1+1)(2L_2+1)\exp\big(-4\beta'(L_1+L_2)\big) \\ &\times \frac{Q_{\beta,N}\big(\mathbb{V}_N(\varphi) \geq \lambda N^3 - (L_1+1)^2 - (L_2+1)^2\big)}{Q_{\beta,N}\big(\mathbb{V}_N(\varphi) \geq \lambda N^3\big)}. \end{aligned} \tag{18}$$

Here we have indexed the volume V_1 of the large contours in $\mathbf{F}_1(\varphi)$ by the parameter L_1 such that $V_1 = L_1^2 + r_1$ (with $0 \leq r_1 \leq 2L_1$). Thus for a given L_1, there is at most $(2L_1+1)$ corresponding quasi-cubes. Similarly, the contours in $\mathbf{F}_2$ are indexed by L_2.

The final step is to show that for $\sqrt{\frac{a}{2}}N \leq L_1 \leq L_2 \leq \sqrt{1-a}N$ we have

$$\frac{Q_{\beta,N}\big(\mathbb{V}_N(\varphi) \geq \lambda N^3 - (L_1+1)^2 - (L_2+1)^2\big)}{Q_{\beta,N}\big(\mathbb{V}_N(\varphi) \geq \lambda N^3\big)} \leq \exp\big(4(\beta' - 10)(L_1+L_2)\big). \tag{19}$$

This inequality combined with (18) will imply (9).

For this we will use our Lemma 8, with $D = \min\{(L_1+1)^2, N^2 - (L_2+1)^2\}$. From it we know that for β and N large enough

$$4(1 - \frac{20}{\beta})\big(L_1 + L_2\big) \geq p\big((L_1+1)^2 - D\big) + p\big((L_2+1)^2 + D\big). \tag{20}$$

As a consequence the function $a(\beta)$ of the Theorem 3 must be chosen in such a way that

$$\varkappa(a(\beta)) > 20/\beta. \tag{21}$$

Any choice of the function $a(\beta) \to 0$ as $\beta \to \infty$, consistent with (21), is allowed in Theorem 3. So $a(\beta)$ vanishes as β diverges, but nevertheless it cannot be too small. The heuristic reason is that for any finite β the macroscopic crystal has rounded edges; thus there exists a constant $a'(\beta) > 0$ such that it is no longer favorable to erase level sets which have the volume larger than $(1 - a'(\beta)) N^2$.

Now to any height configuration φ in $\{\mathbb{V}_N(\varphi) \geq \lambda N^3 - (L_1+1)^2 - (L_2 + 1)^2\}$, we associate the configuration $\check{\varphi}$, defined by

$$\check{\varphi}_s = \varphi_s + 1_{\{s\in[(L_2+1)^2+D]\}} + 1_{\{s\in[(L_1+1)^2-D]\}}, \quad \forall s = (s_1, s_2) \in \Lambda, \tag{22}$$

where $[n]$ denotes the square droplet $n \times n$. Here one needs of course to fix the position of the smaller droplet $[(L_1+1)^2 - D]$ to be inside the larger one, $[(L_2+1)^2 + D]$. This however holds automatically, since either the droplet $[(L_1+1)^2 - D]$ is empty, or $[(L_2+1)^2 + D] = N \times N$.

The correspondence $\varphi \rightsquigarrow \check{\varphi}$ maps injectively the set $\{\mathbb{V}_N(\varphi) \geq \lambda N^3 - (L_1+1)^2 - (L_2+1)^2\}$ into $\{\mathbb{V}_N(\varphi) \geq \lambda N^3\}$. Furthermore, the energy difference between the height configurations φ and $\check{\varphi}$ is bounded by $p((L_1+1)^2 - D) + p((L_2+1)^2 + D)$, so

$$\begin{aligned} &\frac{Q_{\beta,N}\left(\varphi : \mathbb{V}_N(\varphi) \geq \lambda N^3 - (L_1+1)^2 - (L_2+1)^2\right)}{Q_{\beta,N}\left(\varphi : \mathbb{V}_N(\varphi) \geq \lambda N^3\right)} \\ &\leq \exp\left\{\beta\left[p\left((L_1+1)^2 - D\right) + p\left((L_2+1)^2 + D\right)\right]\right\}. \end{aligned}$$

Combining this inequality with (20), we conclude that (19) holds.

4.4 Estimates on the small contours

This subsection contains the proof of the estimate (10).

We follow the scheme of the proof used to control the phase of small contours in the Ising model (see [SS2]). Define the subset $\Lambda^{(0)} = (K_N \times \mathbb{Z})^2 \cap \Lambda$, where $K_N = 2K \ln N$. For any site s in Λ such that $\|s\|_\infty < K_N$, the shift of $\Lambda^{(0)}$ by s is denoted by $\Lambda^{(s)}$. To any collection of contours $\gamma = (\gamma_i)$, the number of sites in $\Lambda^{(s)}$ belonging to the interior of a contour in γ is denoted by $\mathcal{N}_s$.

If φ belongs to S and γ is the collection of all the small contours in $\mathbf{F}_1(\varphi)$, then $\sum_{\|s\|_\infty \leq K_N} \mathcal{N}_s \geq \frac{a}{2} N^2$. Thus there exists at least one site s such that

$$\mathcal{N}_s \geq \frac{a}{2} \left(\frac{N}{K_N} \right)^2 . \tag{23}$$

Denote by Δ^s the set of collections of exterior small contours $\gamma^{(s)} = (\gamma_i^{(s)})$, such that the condition (23) is fulfilled. Now for $\gamma^{(s)} \in \Delta^s$ and $\ell \geq 2$ we introduce the sets $\mathcal{S}\left(\gamma^{(s)}, \ell\right)$ of the height configurations φ, which satisfy the properties:

- $L(\varphi) = \ell, \ \mathbb{V}_N(\varphi) \geq \lambda N^3$,
- the contours from $\gamma^{(s)}$ are among the exterior contours of $\mathbf{F}_1(\varphi)$.

We get

$$P_{\beta,N}(\mathcal{S}) \leq P_{\beta,N}(\mathcal{H}^c) + \sum_{\ell=2}^{N^3} \sum_{\|s\|_\infty \leq K_N} \sum_{\gamma^{(s)} \in \Delta^s} \sum_{\varphi \sim \mathcal{S}\left(\gamma^{(s)}, \ell\right)} P_{\beta,N}(\varphi) \,.$$

We proceed as in the previous subsection and erase all the small contours. belonging to the set $\gamma^{(s)}$. The total volume contribution of these contours is always smaller than N^2, so we get

$$P_{\beta,N}(\mathcal{S}) \leq P_{\beta,N}(\mathcal{H}^c) + \sum_{\ell=2}^{N^3} \sum_{\|s\|_\infty \leq K_N} \sum_{\gamma^{(s)} \in \Delta^s} \exp\big(-\beta \sum_i |\gamma_i^{(s)}| \big) \tag{24}$$
$$\times \frac{Q_{\beta,N}\big(\varphi : \ \mathbb{V}_N(\varphi) \geq \lambda N^3 - N^2\big)}{Q_{\beta,N}\big(\varphi : \ \mathbb{V}_N(\varphi) \geq \lambda N^3\big)}.$$

Using a shift of the height configurations by 1, we easily see that

$$\frac{Q_{\beta,N}\big(\mathbb{V}_N(\varphi) \geq \lambda N^3 - N^2\big)}{Q_{\beta,N}\big(\mathbb{V}_N(\varphi) \geq \lambda N^3\big)} \leq \exp(4\beta N) \,.$$

In order to complete the derivation of (10), it is enough to prove that there is $c > 0$ such that uniformly in s,

$$\sum_{\gamma^{(s)} \in \Delta^s} \exp\left(-\beta \sum_i |\gamma_i^{(s)}| \right) \leq \exp\left(-cN^2/K_N^2 \right) . \tag{25}$$

The occurrence of a small contour surrounding a site i_0 is bounded by

$$\sum_{\Gamma \ni i_0} \exp(-\beta|\Gamma|) \leq \sum_{\ell \geq 4} \ell 3^\ell \exp(-\beta\ell) = q_\beta ,$$

where q_β vanishes as β goes to infinity.

Denote the number of sites N^2/K_N^2 in $\Lambda^{(s)}$ by M_N. The small contours surrounding different sites of $\Lambda^{(s)}$ are independent, so we will obtain the bound (25) from an upper bound for a large deviation of a system of M_N independent random variables. We have

$$\begin{aligned} \sum_{\gamma^{(s)} \in \Delta^s} \exp\left(-\beta \sum_i |\gamma_i^{(s)}| \right) &\leq \sum_{k \geq \frac{a}{2} M_N} \binom{M_N}{k} q_\beta^k \\ &\leq \left(\frac{1}{1 - q_\beta} \right)^{M_N} \sum_{k \geq \frac{a}{2} M_N} \binom{M_N}{k} q_\beta^k \left(1 - q_\beta\right)^{M_N - k} , \end{aligned}$$

where the last sum is the probability of the following event:

Let $\xi_1, ..., \xi_{M_N}$ be i.i.d. random variables, taking values 1 with probability q_β and 0 with probability $1 - q_\beta$. Then $\sum_{k \geq \frac{a}{2} M_N} \binom{M_N}{k} q_\beta^k \left(1 - q_\beta\right)^{M_N - k} = \Pr\left\{ \xi_1 + ... + \xi_{M_N} \geq \frac{a}{2} M_N \right\}$. It is well known that such probability can be estimated from above by $\exp(-c_\beta M_N)$, where c_β is a positive constant for β large enough. For a reference one can consult, for example, Lemma 10 and Corollary 11 of [MRSV]. On the other hand $q_\beta \to 0$ as $\beta \to \infty$, so the estimate (25) follows.

4.5 End of the proof

Thus far we have proven that the quantity $\mathbf{E}_2(\varphi)$ – the area of the external contours $\{\Gamma_i\}_{i \leq K_2}$ of the second facet $\mathbf{F}_2(\varphi)$ – is typically above the level $(1 - a(\beta)) N^2$. We are going to explain now that this in fact easily implies that the area of the facet $\mathbf{F}_2(\varphi)$ itself has to be above the level $(1-2a(\beta))\,N^2$. Indeed, suppose that two events happen:

$$\mathbf{E}_2(\varphi) \geq (1-a(\beta))\,N^2, \text{ and } |\mathbf{F}_2(\varphi)| \leq (1-2a(\beta))\,N^2. \tag{26}$$

We will show that its probability vanishes as $N \to \infty$.

To see this, let us introduce the *second order* external contours $\left\{\tilde{\Gamma}_j\right\}_{j \leq \tilde{K}_2}$, by defining them to be all the external contours of the collection $\Delta(\varphi, L(\varphi)-1) \setminus \{\Gamma_i\}_{i \leq K_2}$. (So the contours $\{\Gamma_i\}_{i \leq K_2}$ should be called the *first order* external contours.)

Under (26) we have that

$$\sum_{j \leq \tilde{K}_2} |\mathrm{Int}(\tilde{\Gamma}_j)| \geq aN^2.$$

Note also that the erasing map is now given by

$$\varphi \rightsquigarrow \check{\varphi} = \left(\check{\varphi}_s = \varphi_s + \sum_{j \leq \tilde{K}_2} 1_{\{s \in \mathrm{Int}(\tilde{\Gamma}_j)\}}\right)_{s \in \Lambda},$$

(compare with (15)), so

$$\mathbb{V}_N(\check{\varphi}) \geq \mathbb{V}_N(\varphi) \geq \lambda N^3. \tag{27}$$

As above, we split the second order external contours $\left\{\tilde{\Gamma}_j\right\}_{j \leq \tilde{K}_2}$ into small and large ones. We introduce two events; the first one, $\tilde{\mathcal{S}}$, consists of configurations such that their small second order external contours of the second facet have a total volume larger than $\frac{a}{2}N^2$, while the second, $\tilde{\mathcal{L}}$, corresponds to the configurations with the volume of the large second order external contours of the second facet is above $\frac{a}{2}N^2$:

$$\tilde{\mathcal{S}} = \left\{\varphi;\quad \sum_{\tilde{\Gamma}_j \text{ small}} |\mathrm{Int}(\tilde{\Gamma}_j)| \geq \frac{a}{2}N^2\right\},$$

$$\tilde{\mathcal{L}} = \left\{\varphi;\quad \sum_{\tilde{\Gamma}_j \text{ large}} |\mathrm{Int}(\tilde{\Gamma}_j)| \geq \frac{a}{2}N^2\right\}.$$

Then we can estimate the probabilities $P_{\beta,N}(\tilde{\mathcal{L}}),\ P_{\beta,N}(\tilde{\mathcal{S}})$ by repeating the estimates for $P_{\beta,N}(\mathcal{L}),\ P_{\beta,N}(\mathcal{S})$, obtained above. In fact, the corresponding estimates are even simpler, because the analogs of the estimates (18), (24), do not contain the factors $\frac{Q_{\beta,N}\left(\varphi:\ \mathbb{V}_N(\varphi)\geq\lambda N^3-\cdot\right)}{Q_{\beta,N}\left(\varphi:\ \mathbb{V}_N(\varphi)\geq\lambda N^3\right)}$, due to (27).

5 Proof of the No Hairs theorem

The excitations of the microscopic crystal around the second facet will be called hairs. There are two kinds of hairs: the up-hairs and the down-hairs. The up-hair of the SOS-surface φ is a sequence $\gamma_1,\dots,\gamma_H$ of contours, such that:

- $$\gamma_i \in \partial D\left(\varphi, L\left(\varphi\right)+i\right), \tag{28}$$
- contours γ_i are ordered by inclusion, which means that Int $(\gamma_{i+1}) \subseteq$ Int (γ_i) for all $i \geq 1$,
- Int $(\gamma_1) \subseteq$ Int $(\partial \mathbf{F}_2(\varphi))$,
- the sequence $\gamma_1,\dots,\gamma_H$ is maximal, in the sense that there is no longer sequence of external contours, satisfying all of the above, of which our sequence is a subsequence.

We denote such an up-hair by $\Gamma = \{\gamma_1,\dots,\gamma_H\}$.

The down-hair is defined as a similar sequence $\hat{\Gamma} = \{\hat{\gamma}_1,\dots,\hat{\gamma}_H\}$ of contours, except that in (28) the sign is opposite: $\hat{\gamma}_i \in \partial D\left(\varphi, L\left(\varphi\right)-i\right)$. The value H will be called the length of the hair.

Clearly, our statement is equivalent to proving that the probability of occurrence of a hair with length $H > C \ln N$ vanishes in the limit as $N \to \infty$. We denote such an event by A_C.

In what follows we will treat only the up-hairs, since the case of the down-hairs is simpler, as there is no volume constraint (see the argument of the subsection 4.5); therefore, in the rest of this section we will use the term "hair" instead of "up-hair".

By our definitions we have that $|\mathrm{Int}(\gamma_1)| \le aN^2$. We introduce now the sequence

$$v_r(N) = a\frac{N^2}{2^r},\ r = 1, 2, ...,$$

and we will characterize each hair by the amounts $H_r(\Gamma)$ of its contours γ_i such that

$$v_{r+1}(N) < |\mathrm{Int}\ (\gamma_i)\,| \le v_r(N)\,. \tag{29}$$

Naturally, we need only these $v_r(N)$-s which are ≥ 1, so we define R_N to be the largest value of r such that the scale $v_r(N) \ge 1$. We fix also the intermediate scale $R'_N < R_N$:

$$R'_N = \frac{1}{\ln 2}\ln\left(\frac{aN^2}{C_1(\ln N)^2}\right).$$

Then for every $r \le R'_N$ we have

$$v_r(N) \ge C_1(\ln N)^2\,,$$

and the choice of the constant C_1 is made in such a way that the contours of volume larger than $C_1(\ln N)^2$ are large contours, i.e. their perimeter is larger than $K \ln N$.

Finally we introduce the sequence $h_r(N)$, defined as follows:

$$h_r(N) = \begin{cases} 4, & \text{if } \ r < R'_N\,, \\ C_2\, 2^{r/2} N^{-1} \ln N, & \text{if } \ r \ge R'_N\,, \end{cases}$$

where C_2 is chosen such that

$$h_{R'_N}(N) = \frac{\sqrt{a}\, C_2 \ln N}{\sqrt{v_{R'_N}(N)}} = \frac{\sqrt{a}}{\sqrt{C_1}} C_2 \ge 10\,. \tag{30}$$

By definition,

$$\sum_{r=0}^{R_N} h_r(N) \le 4R'_N + \sum_{r=R'_N}^{R_N} C_2 2^{r/2} N^{-1} \ln N \le C_3 \ln N\,,$$

for some $C_3 > 0$.

Let us fix the constant C of our theorem to be much larger than C_3. If the length of Γ exceeds $C \ln N$, then we define the value r_0 as the first index r for which the bound $H_r(\Gamma) < h_r(N)$ is violated. Then define ℓ as the first index such that

$$|\mathrm{Int}\,(\gamma_\ell)| \leq v_{r_0}(N)\,.$$

To summarize, to any Γ with length larger than $C \ln N$ we associate a pair (r_0, ℓ) and the subsequence of contours $\{\gamma_\ell, \gamma_{\ell+1}, \ldots, \gamma_{\ell+h_{r_0}-1}\} \subset \Gamma$, for which (29) holds. We denote by $\big(r_0, \ell, \{\gamma_\ell, \ldots, \gamma_{\ell+h_{r_0}-1}\}\big)$ the class of all such hairs Γ.

The strategy of the proof will be to apply a Peierls type estimate (under the volume constraint) to the section of the hair made of the exterior contours $\{\gamma_\ell, \ldots, \gamma_{\ell+h_{r_0}-1}\}$.

$$\begin{aligned} P_{\beta,N}\left\{\mathcal{A}_C\right\} \leq & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (31) \\ \leq \sum_{r_0=1}^{R_N} \sum_{\ell=1}^{C_3 \ln N} \sum_{\{\gamma_i\}} & P_{\beta,N}\left\{\varphi \text{ has a hair } \Gamma \text{ in the class } \big(r_0, \ell, \{\gamma_\ell, \ldots, \gamma_{\ell+h_{r_0}-1}\}\big)\right\}\,, \end{aligned}$$

where the sum is over the collections of contours $\{\gamma_\ell, \ldots, \gamma_{\ell+h_{r_0}-1}\}$. In order to estimate (31), two cases have to be distinguished. Either r_0 is smaller than R'_N and all the contours $\gamma_\ell, \ldots, \gamma_{\ell+h_{r_0}-1}$ are large, or r_0 is larger than R'_N and one has to rely on more delicate estimates, taking into account the fact that these contours might be small.

Case 1 : $r_0 \leq R'_N$.

In this case $h_{r_0} = 4$ and we have

$$\sum_{i=0}^{3} |\mathrm{Int}\,(\gamma_{\ell+i})| \leq 4 v_{r_0}(N)\,,$$

$$\sum_{i=0}^{3} |\gamma_{\ell+i}| \geq 4(4\sqrt{v_{r_0+1}(N)})\,.$$

Applying the Peierls estimate as in the proof of Theorem 6 (see relation (15)), we get for every $r_0 = [0, R'_N]$ and every height $\ell \le C_3 \ln N$

$$\sum_{\{\gamma_i\}} P_{\beta,N}\left\{\Gamma \in (r_0, \ell, \{\gamma_\ell, \ldots, \gamma_{\ell+3}\})\right\} \tag{32}$$
$$\le \sum_{\{\gamma_i\}} \exp\left(-\beta \sum_{i=0}^{3} |\gamma_{\ell+i}|\right) \frac{Q_{\beta,N}\left(\mathbb{V}_N(\varphi) \ge \lambda N^3 - 4 v_{r_0}(N)\right)}{Q_{\beta,N}\left(\mathbb{V}_N(\varphi) \ge \lambda N^3\right)} .$$

The loss of volume can be compensated by adding a single square contour of side length $4[\sqrt{4 v_{r_0}(N)}] + 1$, in the same manner as in relation (22) above. Furthermore the entropy of the four large contours can be easily bounded, so that we obtain

$$\sum_{r_0=0}^{R'_N} \sum_{\ell=0}^{C_3 \ln N} \sum_{\{\gamma_i\}} P_{\beta,N}\left\{\Gamma \sim (r_0, \ell, \{\gamma_\ell, \ldots, \gamma_{\ell+3}\})\right\} \tag{33}$$
$$\le \sum_{r_0=0}^{R'_N} \sum_{\ell=0}^{C_3 \ln N} N^8 \exp\left(-16(\beta - \ln 3)\sqrt{v_{r_0+1}(N)} + 4\beta\sqrt{4 v_{r_0}(N)}\right)$$
$$\le \exp\left(-C_\beta \ln N\right) ,$$

where C_β is a positive constant for β large enough.

Case 2 : $r_0 > R'_N$.

First notice that for any collection of contours in the r.h.s. of (31), we have for the area

$$\sum_{i=\ell}^{\ell+h_{r_0}-1} |\mathrm{Int}(\gamma_i)| \le v_{r_0}(N)\, h_{r_0}(N) ,$$

while for the boundary we get

$$\sum_{i=\ell}^{\ell+h_{r_0}-1} |\gamma_i| \ge 4\sqrt{v_{r_0+1}(N)}\, h_{r_0}(N) .$$

Since

$$v_r(N) h_r(N) = a\frac{N^2}{2^r} C_2\, 2^{r/2} N^{-1} \ln N = \frac{a C_2}{2^{r/2}} N \ln N,$$

for $r > R'_N = \frac{1}{\ln 2} \ln\left(\frac{aN^2}{C_1(\ln N)^2}\right)$ we have

$$v_r(N)h_r(N) < \frac{aC_2}{\sqrt{\frac{aN^2}{C_1(\ln N)^2}}} N \ln N = \sqrt{aC_1}C_2(\ln N)^2.$$

On the other hand,

$$\sqrt{v_{r+1}(N)}h_r(N) = \sqrt{a\frac{N^2}{2^{r+1}}}C_2\, 2^{r/2}N^{-1}\ln N = \sqrt{\frac{a}{2}}C_2\ \ln N. \tag{34}$$

Thus the energy $\sum_{i=\ell}^{\ell+h_{r_0}-1} |\gamma_i|$ of the collection of contours exceeds

$$4\sqrt{\frac{a}{2}}C_2\ \ln N,$$

and therefore, due to (30), is much larger than the one of a single square contour with the same area, since its energy equals

$$4\sqrt{v_{r_0}(N)h_{r_0}(N)} < 4\sqrt{\sqrt{aC_1}C_2}\ln N.$$

Nevertheless these contours can be small, and one has to estimate their entropy carefully.

The number $\mathcal{N}(L_\ell,\ldots,L_{\ell+h_{r_0}-1})$ of compatible contours $\{\gamma_\ell,\ldots,\gamma_{\ell+h_{r_0}-1}\}$ with respective length $\{L_\ell,\ldots,L_{\ell+h_{r_0}-1}\}$ can be estimated by

$$\mathcal{N}(L_\ell,\ldots,L_{\ell+h_{r_0}-1}) \leq N^2\, 3^{L_\ell} \prod_{i=\ell+1}^{\ell+h_{r_0}-1} \left(v_{r_0}(N)\, 3^{L_i}\right) .$$

To see this, one chooses first the contour γ_ℓ of length L_ℓ at a random position in the box, then the other contours pile up above it, so their entropy is given by the second term, where $v_{r_0}(N)$ is the maximal area of γ_ℓ.

We proceed as in (32) and get for a given r_0 and height ℓ

$$\sum_{\{\gamma_i\}} P_{\beta,N}\left\{\Gamma \in \left(r_0,\ell,\{\gamma_\ell,\ldots,\gamma_{\ell+h_{r_0}-1}\}\right)\right\}$$

$$\leq \sum_{\{\gamma_i\}} \exp\left(-\beta\sum_{i=0}^{h_{r_0}-1}|\gamma_{\ell+i}|\right)\ \frac{Q_{\beta,N}\left(\mathbb{V}_N(\varphi)\geq\lambda N^3 - v_{r_0}(N)h_{r_0}(N)\right)}{Q_{\beta,N}\left(\mathbb{V}_N(\varphi)\geq\lambda N^3\right)}$$

$$\leq \sum_{\{L_i\}} N^2 \exp\left(h_{r_0}(N)\ln\left(v_{r_0}(N)\right) + (\ln 3-\beta)\sum_{i=\ell}^{\ell+h_{r_0}-1} L_i + 4\beta\sqrt{v_{r_0}(N)h_{r_0}(N)}\right) ,$$

where each L_i ranges in $[4\sqrt{v_{r_0+1}(N)}, 4\sqrt{v_{r_0}(N)}]$, and so

$$\sum_{i=\ell}^{\ell+h_{r_0}-1} L_i \geq 4\sqrt{v_{r_0+1}(N)}\, h_{r_0}(N).$$

Summing over the sequence $\{L_i\}$, we see that

$$\sum_{\{\gamma_i\}} P_{\beta,N} \left\{\Gamma \sim \left(r_0, \ell, \{\gamma_\ell, \ldots, \gamma_{\ell+h_{r_0}-1}\}\right)\right\} \tag{35}$$
$$\leq N^2 \exp\left(h_{r_0}(N)\sqrt{v_{r_0+1}(N)}\left(\frac{\ln\left(v_{r_0}(N)\right)}{\sqrt{v_{r_0+1}(N)}} - 4(\beta-10) + 4\beta\sqrt{\frac{2}{h_{r_0}(N)}}\right)\right).$$

Recall that $h_{r_0}(N) > h_{R'_N}(N) \geq 10$ (see (30)), $h_{r_0}(N)\sqrt{v_{r_0+1}(N)} = \sqrt{\frac{a}{2}}C_2 \ln N$ (see (34)), while $\frac{\ln\left(v_{r_0}(N)\right)}{\sqrt{v_{r_0+1}(N)}} \leq 1$. Thus summing (35) over the indices r_0 and ℓ, we obtain

$$\begin{aligned}
&\sum_{r_0=R'_N+1}^{R_N} \sum_{\ell=0}^{C_3 \ln N} \sum_{\{\gamma_i\}} P_{\beta,N} \left\{\Gamma \sim \left(r_0, \ell, \{\gamma_\ell, \ldots, \gamma_{\ell+h_{r_0}-1}\}\right)\right\} \\
&\leq \sum_{r_0=R'_N+1}^{R_N} \sum_{\ell=0}^{C_3 \ln N} N^2 \exp\left(\sqrt{\frac{a}{2}}C_2 \ln N\left(1 - 4(\beta-10) + \frac{4\beta}{\sqrt{5}}\right)\right) \\
&\leq \exp\left(-C_\beta \ln N\right),
\end{aligned}$$

where C_β is positive constant for β large enough.

Combining this with (33), we conclude the proof.

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