Renormalization and Diophantine invariant KAM tori of Hamiltonian flows

Saša Kocić
Department of Physics, University of Texas at Austin, Austin, TX 78712, USA
E-mail: kocic@physics.utexas.edu
(Dated: October 19, 2004)

We apply a renormalization scheme to construct invariant tori with Diophantine frequency vectors for two degree-of-freedom, near-integrable analytic Hamiltonians. We prove that the tori are analytic functions.

1. INTRODUCTION

Half a century ago, Kolmogorov [1] stated a theorem about the persistence of quasi-periodic motion (invariant tori) with sufficiently incommensurate frequency vectors in the phase space of a (non-degenerate) integrable Hamiltonian system under small analytic perturbations. Several years later, a detailed proof of the theorem was given and the requirement of analyticity was weakened by Arnol'd [2] and Moser [3]. It was the beginning of a series of rigorous results and methods that are collectively referred to as the KAM theory.

Almost at the same time, a different simple and novel idea was born in theoretical physics - renormalization. Originally introduced in quantum field theory and statistical mechanics, it later became a powerful tool for studying dynamical systems on progressively smaller spatial scales and longer time scales.

We prove the existence of KAM tori with Diophantine frequency vectors, invariant under the flow of near-integrable two degree-of-freedom analytic Hamiltonian functions satisfying a certain non-degeneracy condition, using the rigorous renormalization scheme formulated in [4]. We further prove that the invariant tori are analytic.

The renormalization scheme used here applies to two degree-of-freedom analytic Hamiltonians and is associated to Diophantine frequency vectors that form a set of full Lebesgue measure. The renormalization scheme can be considered a generalization of the renormalization scheme for analytic Hamiltonians formulated by Koch [5], associated to frequency vectors of a specific type (called Koch type in [6]). In the case of two degree-of-freedom ($d = 2$) Hamiltonians, those are the vectors whose slopes are quadratic irrational numbers. We have proved in [4] that the renormalization has a trivial integrable limit set. In the case of $d = 2$ and the golden mean winding number, Koch proved the existence of a non-trivial fixed point of the renormalization [7].

The method of construction of invariant tori used here is based on a renormalization scheme designed as a differentiable *dynamical* system on a space of analytic Hamiltonians. A similar method has been used in [8] to prove the existence of non-differentiable golden invariant tori associated to the critical fixed point of the renormalization group transformations. A different method for construction of invariant tori, based on the analogy with quantum field theory and the summation of the (generically divergent) Lindstedt series, was used by Gallavotti *et al* [9, 10]. Related results for near-integrable area-preserving maps, proofs of existence of golden invariant curves by means of renormalization methods, have been obtained by MacKay [11] and Khanin and Sinai [12].

Our renormalization scheme [4] consists of successive application of the operators $\mathcal{R}_n$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. This sequence of operators has been defined between Banach spaces of analytic Hamiltonian functions on complex neighborhoods of $\mathbb{T}^2 \times \{0\}$, where $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ and 0 is the origin in $\mathbb{R}^2$. These neighborhoods are referred to as the phase space. The domain of the n^{th}-step renormalization operator $\mathcal{R}_n$ has been restricted to Hamiltonians H_n that are close to an integrable Hamiltonian H_n^0. The latter has an invariant torus of frequency ω_n located at $p = 0$ and frequencies of nearby tori *twisted* in the direction of Ω_n. The operator $\mathcal{R}_n$ maps a Hamiltonian H_n, close to H_n^0, into $H_{n+1} = \mathcal{R}_n(H_n)$, close to $H_{n+1}^0 = \mathcal{R}_n(H_n^0)$.

An important feature of this operator is a canonical transformation of the (lifted) phase space, that brings a Hamiltonian closer to integrable. This transformation eliminates the irrelevant, *non-resonant*, modes of the Hamiltonian. The *resonant* modes, which cause the problem of small denominators in conventional KAM theory, are kept. The fact that the elimination of the resonant modes is not necessary, contrary to the usual KAM theory approach, is due to the fact that the renormalization operator also involves a scaling of the (lifted) phase space. On the Fourier modes of the Hamiltonians the scaling acts by mapping some of the *resonant* modes into *non-resonant*. The processes of elimination and rescaling, that are the main constituents of the renormalization operator, resemble the block spin transformations in the theory of critical phenomena.

The paper is organized as follows. In section 2, we review the formulation of the renormalization scheme introduced in [4]. In section 3, we prove (Theorem 3.8) the existence of invariant tori of a class of Diophantine frequency vectors, for analytic Hamiltonians sufficiently close to integrable. We also prove (Theorem 3.12) that the invariant tori are analytic functions.

2. THE RENORMALIZATION SCHEME

We review briefly the construction of the renormalization scheme (see [4] for more details). The purpose of this review is to introduce the notation and state the main fact (Theorem 2.8) about the convergence of the orbits of near-integrable Hamiltonians.

2.1. The integrable Hamiltonians

We consider a sequence of integrable Hamiltonians $H_n^0 = \omega_n \cdot p + \frac{1}{2}(\Omega_n \cdot p)^2$, $n \in \mathbb{N}_0$, which is associated to a sequence of pairs of vectors (ω_n, Ω_n) in $\mathbb{R}^2$. The pairs (ω_n, Ω_n) are generated from a pair of vectors $(\omega, \Omega) = (\omega_0, \Omega_0)$, consisting of a Diophantine vector $\omega \in \mathbb{R}^2$ and a *suitable* vector $\Omega \in \mathbb{R}^2$.

Definition 2.1 *A vector $\omega \in \mathbb{R}^2$ is called Diophantine of order $\beta \geq 0$ if there exists $C > 0$, such that $|\omega \cdot \nu| > C\|\nu\|^{-(1+\beta)}$, for all $\nu \in \mathbb{Z}^2 \backslash \{0\}$. The set of all Diophantine vectors of order β will be denoted by $DC(\beta)$.*

Here and in the following, $\|\cdot\|$ denotes the ℓ^1-norm of a vector and "*" stands for transpose.

Definition 2.2 *A vector $\Omega \in \mathbb{R}^2$ is called suitable if it is of the form $\Omega = (x, y)^*$, with $|x| \geq |y| \geq 0$, $xy \leq 0$ and $\|\Omega\| = 1$.*

We assume that $\omega \in \mathbb{R}^2$ is of the form $\omega = \ell(1, \alpha)^*$, where $\ell \in \mathbb{R}^+$ and α is an irrational number greater than 1. Further, we construct the vectors $\omega_n \in \mathbb{R}^2$ from the continued fraction expansion of α. The unique continued fraction expansion [13] of such an irrational $\alpha \in \mathbb{R}$ is given by

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}}\,, \tag{2.1}$$

where $a_n \in \mathbb{Z}^+$, $n \geq 0$. We also write $\alpha = [a_0, a_1, \ldots]$. We define $\omega_n = \ell(1, \alpha_n)^*$, where $\alpha_n = [a_n, a_{n+1}, \ldots]$ and $\ell \in \mathbb{R}^+$. These vectors could also be defined recursively, via $\omega_{n+1} = \alpha_{n+1} T_n^{-1} \omega_n$, using the matrix

$$T_n = \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix} \in GL(2, \mathbb{Z})\,. \tag{2.2}$$

Similarly, the vectors Ω_n are defined recursively by $\Omega_n = T_{n-1}^{-1}\Omega_{n-1}/\|T_{n-1}^{-1}\Omega_{n-1}\|$, for $n \in \mathbb{N}$.

Notice that if ω_0 is Diophantine of order β, so is every ω_n, $n \in \mathbb{N}$. If Ω_0 is suitable, so is every Ω_n, $n \in \mathbb{N}$.

Let us define $\tilde{A}_n = \prod_{i=0}^n \alpha_i$. One can show [4] that $\tilde{A}_n$ grows at least exponentially with n, i.e. $\tilde{A}_n \geq \gamma^n$, where $\gamma = (1 + \sqrt{5})/2$ is the golden mean.

The crucial fact is that the growth of $\tilde{A}_n$ with n can be controlled in the case of a Diophantine vector ω_0. The Diophantine property of $\omega \in DC(\beta)$, can be expressed as the existence of a constant $\tilde{K} > 0$, such that

$$\tilde{A}_{n+1} < \tilde{K}\tilde{A}_n^{1+\beta}\,, \tag{2.3}$$

for all $n \in \mathbb{N}_0$.

The growth of the product $\prod_{i=0}^{n} \|T_i^{-1}\Omega_i\|$ can also be controlled in that case. We have the following bound (Proposition 2.4 in [4]),

$$\frac{1+\alpha_0}{4\alpha_0}(\tilde{A}_n + \tilde{A}_{n-1}) < \prod_{i=0}^{n} \|T_i^{-1}\Omega_i\| < (\tilde{A}_n + \tilde{A}_{n-1}) \,. \tag{2.4}$$

The above bounds play an essential role in the construction of the renormalization scheme.

The integrable Hamiltonians H_n^0 do not satisfy Kolmogorov's non-degeneracy condition, but they do have a *twist* in the Ω_n direction, i.e. they satisfy

$$\left|\frac{\partial^2 H_n^0}{\partial p_i \partial p_j}\right| = 0, \qquad \left|\frac{\partial^2 H_n^0}{\partial(\Omega_n \cdot p)^2}\right| = 1 \neq 0. \tag{2.5}$$

2.2. Construction of the renormalization scheme

We begin by defining domains and function spaces for our Hamiltonians. Further, we construct a one-step renormalization transformation and present the main result from [4], about the convergence of the renormalization orbits of the Hamiltonians towards the orbit of the integrable Hamiltonian H_0^0.

Consider the normalized vector $\hat{\omega}_n = \omega_n/\|\omega_n\|$, $n \in \mathbb{N}_0$, and let the vectors ω_n' and Ω_n', be defined by the relations $\omega_n' \cdot \Omega_n = 0$, $\omega_n' \cdot \hat{\omega}_n = 1$, $\Omega_n' \cdot \hat{\omega}_n = 0$ and $\Omega_n' \cdot \Omega_n = 1$.

Definition 2.3 *Given a pair of positive numbers $\rho = (\rho_1, \rho_2) \in \mathbb{R}^{+2}$, define*

$$\begin{aligned} \mathcal{D}_{n,1}(\rho_1) &= \{q \in \mathbb{C}^2 : |\mathrm{Im}\, \omega_n' \cdot q| < \rho_1,\ |\mathrm{Im}\, \Omega_n' \cdot q| < \rho_1\} \\ \mathcal{D}_{n,2}(\rho_2) &= \{p \in \mathbb{C}^2 : |\hat{\omega}_n \cdot p| < \rho_2,\ |\Omega_n \cdot p| < \rho_2\}, \end{aligned} \tag{2.6}$$

and the set $\mathcal{D}_n(\rho) = \mathcal{D}_{n,1}(\rho_1) \times \mathcal{D}_{n,2}(\rho_2)$.

The Hamiltonians that we consider are lifted to analytic functions $H_n : \mathcal{D}_n(\rho) \to \mathbb{C}$, which are 2π-periodic in both q variables. We will refer to these functions as Hamiltonians as well. They can be expanded in Fourier-Taylor series

$$H_n(q,p) = \sum_{(\nu,k)\in I} (H_n)_{\nu,k} (\hat{\omega}_n \cdot p)^{k_1} (\Omega_n \cdot p)^{k_2} e^{iq\cdot\nu}\,,$$

where $k = (k_1, k_2) \in \mathbb{N}_0^2$, $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ and $I = \mathbb{Z}^2 \times \mathbb{N}_0^2$. We refer to each term in this sum as a mode of the Hamiltonian H_n, and to the associated k and ν as the mode indices.

Definition 2.4 *Given $\rho > 0$, componentwise, define $\mathcal{A}_n(\rho)$ to be the Banach space of functions H_n that are analytic on $\mathcal{D}_n(\rho)$, extend continuously to the boundary of $\mathcal{D}_n(\rho)$, and have finite norm*

$$\|H_n\|_{n,\rho} = \sum_{(\nu,k)\in I} |(H_n)_{\nu,k}| \rho_2^{\|k\|} e^{\rho_1(|\hat{\omega}_n\cdot\nu| + |\Omega_n\cdot\nu|)}\,.$$

Define the projection operators $\mathbb{P}_n^k$ on $\mathcal{A}_n(\rho)$ by $\mathbb{P}_n^k H_n = (H_n)_{0,k}(\hat{\omega}_n \cdot p)^{k_1}(\Omega_n \cdot p)^{k_2}$, and let $\mathbb{E} = \sum_{k\in\mathbb{N}_0^2} \mathbb{P}_n^k$. Define also the functionals $\mathfrak{p}_n^k : \mathcal{A}_n(\rho) \to \mathbb{C}$, by setting $\mathfrak{p}_n^k H_n = (H_n)_{0,k}$.

We focus on Hamiltonians $H_n = H_n^0 + h_n$ in $\mathcal{A}_n(\rho)$, which are close to the integrable Hamiltonian H_n^0.

The main part of the n^{th}-step renormalization operator $\mathcal{R}_n$ is the map $H_n \mapsto H_n \circ \mathcal{T}_n$, where $\mathcal{T}_n$ is the linear map on $\mathbb{C}^2 \times \mathbb{C}^2$, defined by $\mathcal{T}_n(q,p) = (T_n q, \mu_n T_n^{*-1} p)$, with $\mu_n = \alpha_{n+1}^{-1} \|T_n^{-1}\Omega_n\|^{-2}$. This "scaling map" represents a singular operator, as the matrix T_n has an expanding eigendirection. The map is, however, analyticity improving when the domain of the operator is restricted to Hamiltonians which contain only modes for which either $\|k\|$ is large or ν is almost perpendicular to ω. These modes and the Hamiltonians are called *resonant*.

Definition 2.5 *Given $\sigma, \kappa > 0$ and vectors $\omega_n, \Omega_n \in \mathbb{R}^2$, we define the non-resonant index set,*

$$I_n^- = \{(\nu, k) \in I : |\omega_n \cdot \nu| > \sigma |\Omega_n \cdot \nu|,\ |\omega_n \cdot \nu| > \kappa \|k\|\}.$$

The resonant index set is defined as its complement, $I_n^+ = I \backslash I_n^-$. The corresponding projection operators on $\mathcal{A}_n(\rho)$, $\mathbb{I}_n^-$ and $\mathbb{I}_n^+ = \mathbb{I} - \mathbb{I}_n^-$, are defined by setting

$$(\mathbb{I}_n^- H_n)(q, p) = \sum_{(\nu,k)\in I_n^-} (H_n)_{\nu,k} (\hat{\omega}_n \cdot p)^{k_1} (\Omega_n \cdot p)^{k_2} e^{iq\cdot\nu}.$$

As a part of the n^{th}-step renormalization operator, we further include a translation in the variable $(\Omega_{n+1} \cdot p)$ to prevent the n^{th}-step renormalization operator from having an expanding eigendirection. The existence of the non-zero quadratic part of the integrable Hamiltonian H^0_{n+1} is essential for the construction of such a transformation. At each renormalization step, we perform a canonical transformation of the phase space that completely eliminates all the *non-resonant* modes of a Hamiltonian. This map can be constructed, by a Nash-Moser type iteration. Finally, we include a time-rescaling and an additional, nonlinear, scaling of the action variables, in order to preserve the normalization $H^0_{n+1} = \mathcal{R}_n(H^0_n)$.

Definition 2.6 *On an open ball in $\mathbb{I}_n^+\mathcal{A}_n(\rho)$, $\rho > 0$, componentwise, the n^{th}-step renormalization operator $\mathcal{R}_n$ is defined by the following action on a Hamiltonian H_n,*

$$\mathcal{R}_n(H_n) = \frac{\theta'_n}{\mu'_n}(H_n \circ \Lambda_n - \mathbb{P}^{(0,0)}_{n+1} H_n \circ \Lambda_n),$$

where $\Lambda_n = \mathcal{T}_n \circ V_{H'_{n+1}} \circ U_{H''_{n+1}} \circ S_{\tilde{H}_{n+1}}$. Here $H'_{n+1} = (\theta_n/\mu_n) H_n \circ \mathcal{T}_n$, with $\theta_n = \alpha_{n+1}$, $H''_{n+1} = H'_{n+1} \circ V_{H'_{n+1}}$ and $\tilde{H}_{n+1} = H''_{n+1} \circ U_{H''_{n+1}}$. The transformation $V_{H'_{n+1}} : \mathcal{D}_{n+1}(\varrho) \to \mathcal{D}_{n+1}(\tilde{\rho})$, where $3/2\rho > \tilde{\rho} > \varrho > \varrho' > \rho$, componentwise, represents the translation $V_{H'_{n+1}}(q, p) = (q, p - v_{H'_{n+1}} \Omega'_{n+1})$, with $v_{H'_{n+1}} \in \mathbb{C}$ determined by the equation $\mathfrak{p}^{(0,1)}_{n+1} H'_{n+1} \circ V_{H'_{n+1}} = 0$. The map $U_{H''_{n+1}} : \mathcal{D}_{n+1}(\varrho') \to \mathcal{D}_{n+1}(\varrho)$ is a canonical transformation that satisfies $\mathbb{I}^-_{n+1}(H''_{n+1} \circ U_{H''_{n+1}}) = 0$. The transformation $S_{\tilde{H}_{n+1}} : \mathcal{D}_{n+1}(\rho) \to \mathcal{D}_{n+1}(\varrho')$ represents a scaling $S_{\tilde{H}_{n+1}}(q, p) = (q, z_{\tilde{H}_{n+1}} p)$ of the action variables. The scaling parameters θ'_n and μ'_n take the values $\theta'_n = \theta_n \cdot \tau_{\tilde{H}_{n+1}}$ and $\mu'_n = \mu_n \cdot z_{\tilde{H}_{n+1}}$. The parameters $z_{\tilde{H}_{n+1}}, \tau_{\tilde{H}_{n+1}} \in \mathbb{C}$ are determined such that $\mathfrak{p}^{(0,2)}_{n+1} H_{n+1} = 1/2$ and $\mathfrak{p}^{(1,0)}_{n+1} H_{n+1} = 1$, where $H_{n+1} = \mathcal{R}_n(H_n)$.

Under the assumption that ρ, σ, κ are appropriately chosen constants, it has been shown in [4] that the n^{th}-step renormalization operator is well-defined, analytic map on an open ball about the integrable Hamiltonian H^0_n, of radius $\zeta_n = C'/(\alpha_n \alpha_{n+1})^2$, with sufficiently small $C' > 0$.

Remark 2.7 *The fact that the n^{th}-step renormalization operator is defined on a set of resonant Hamiltonians is not a restriction for $n \geq 1$, as the renormalized Hamiltonians are always resonant. In order to extend the definition of the 0^{th}-step renormalization operator to an open ball in $\mathcal{A}_n(\rho)$, one can include a pre-renormalization step consisting of a canonical transformation that eliminates the non-resonant modes of the Hamiltonians. The stated results would qualitatively not depend on such a transformation.*

We have further proved in [4] that, if ρ, σ, κ are appropriately chosen constants (independent of the renormalization step) and H_0 is sufficiently close to H^0_0, the successive application of the one-step renormalization operators is well-defined and the renormalization orbits of all Hamiltonians in a neighborhood of the integrable Hamiltonian H^0_0 approach the sequence of integrable Hamiltonians H^0_n, $n \in \mathbb{N}_0$. This sequence is the trivial limit set of the renormalization. We recall Lemma 5.3 in [4].

Theorem 2.8 *Let $\omega_0 \in DC(\beta)$, $0 \leq \beta < \sqrt{2} - 1$, and let $\rho_1 > 0$. For sufficiently large $\tau > 2$, there exist $\rho_2, \sigma, \kappa > 0$, such that if $C > 0$ is sufficiently small, $H_0 = H^0_0 + h_0 \in \mathbb{I}^+_0 \mathcal{A}_0(\rho)$ and $\|h_0\|_{0,\rho} \leq C^2 < 1$, then $\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho} \leq C\tilde{A}_n^{-\tau} < \zeta_n/2$, $n \geq 0$, where $\zeta_n = C'/(\alpha_n \alpha_{n+1})^2$, and $C' > 0$ (as chosen earlier). Furthermore, if $\beta < (\sqrt{161} - 11)/10$, then $\|h_n\|_{n,\rho} \leq C\tilde{A}^{-2}_{n-1} < \zeta_n/2$.*

In the following, we assume that the considered Hamiltonians H_0 belong to the open ball $B^+_{0,\rho}(C^2)$ of Hamiltonians in $\mathbb{I}^+_0\mathcal{A}_0(\rho)$, of radius $C^2>0$, centered at H^0_0, and satisfy the normalization conditions $\mathfrak{p}^{(1,0)}_0 H_0 = 1$ and $\mathfrak{p}^{(0,2)}_0 H_0 = 1/2$. We further assume that $C'>0$, $0 \le \beta < (\sqrt{161}-11)/10$, $\tau > 2+\beta$, $\rho > 0$, componentwise, and $\sigma, \kappa > 0$ are fixed constants chosen such that the orbit $H_n = H^0_n + h_n$, $n \in \mathbb{N}_0$, of a near-integrable Hamiltonian $H_0 \in \mathbb{I}^+_0\mathcal{A}_0(\rho)$, satisfies

$$\|h_n\|_{n,\rho} \le C\tilde{A}^{-2}_{n-1} < \zeta_n/2, \qquad \|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho} \le C\tilde{A}^{-\tau}_n < \zeta_n/2. \tag{2.7}$$

3. INVARIANT TORI

In this section, we construct the invariant tori of a class of Diophantine frequency vectors for near-integrable Hamiltonians approaching the trivial limit set under the renormalization. More precisely, we prove that every Hamiltonian H_n, of the renormalization orbit of any $H_0 \in B^+_{0,\rho}(C^2)$, with sufficiently small $C>0$, has an invariant torus of Diophantine frequency vector ω_n. We further prove that the invariant tori of the Hamiltonians H_0 are analytic.

3.1. Definitions of invariant tori and formal identities

We start by providing some definitions and some formal identities that will be used in the rest of the paper.

Definition 3.1 *Given $\delta_1 > 0$, define*

$$\mathcal{D}_{n,0}(\delta_1) = \{q \in \mathbb{C}^2 : |\mathrm{Im}\,\omega'_n \cdot q| < \delta_1,\ |\mathrm{Im}\,\Omega'_n \cdot q| = 0\} \times \{0\}\,. \tag{3.1}$$

Definition 3.2 *Given $r>0$ and $\delta_1>0$, let $\mathcal{A}_{n,0}(\delta_1)$ be the space of functions f, analytic on $\mathcal{D}_{n,0}(\delta_1)$, 2π-periodic in both q-variables, with the finite norm*

$$\|f\|_{n,\delta_1} = \sum_{\nu\in\mathbb{Z}^2} |f_\nu|(1+|\Omega_n\cdot\nu|)^r e^{\delta_1|\hat{\omega}_n\cdot\nu|}\,, \tag{3.2}$$

where f_ν are the Fourier coefficients of f.

Definition 3.3 *We say that H_n has an invariant torus with frequency vector ω_n if there exists a continuous map $\Gamma_n : \mathcal{D}_{n,0}(\delta_1) \to \mathcal{D}_n(\rho)$, and a continuous function $t:\mathbb{R}\to\mathbb{R}$, such that for all $s\in\mathbb{R}$,*

$$\Phi^{t(s)}_n \circ \Gamma_n = \Gamma_n \circ \Psi^s_n. \tag{3.3}$$

Here Φ_n is the flow for the Hamiltonian H_n and Ψ_n is the flow for the Hamiltonian $K_n = \omega_n\cdot p$, i.e. $\Psi^s_n(q,0) = (q+\omega_n s, 0)$.

The flow Φ_n for the Hamiltonian H_n and the flow Φ_{n+1} for the renormalized Hamiltonian $H_{n+1} = \mathcal{R}_n(H_n)$ can be related by

$$\Lambda_n \circ \Phi^t_{n+1} = \Phi^{\theta'_n t}_n \circ \Lambda_n. \tag{3.4}$$

This identity is valid for $t\in\mathbb{C}$ on any domain where the compositions are well-defined. The identity formally follows from

$$\frac{d}{dt} f\circ\Phi^{\theta'_n t}_n \circ \Lambda_n \,|_{t=0} = \theta'_n\{f,H_n\}\circ\Lambda_n = \frac{\theta'_n}{\mu'_n}\{f\circ\Lambda_n, H_n\circ\Lambda_n\} = \{f\circ\Lambda_n, \mathcal{R}_n(H_n)\},$$

where we have used the identity ${\mu'_n}^{-1}\{f\circ\Lambda_n, g\circ\Lambda_n\} = \{f,g\}\circ\Lambda_n$, for complex valued functions f and g defined on a neighborhood of $\mathbb{T}^2\times\mathbb{R}^2$. Here $\{f,g\} = \nabla_q f\cdot\nabla_p g - \nabla_q g\cdot\nabla_p f$ is the Poisson bracket of the functions f and g.

Now, we have the identities

$$\Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1} \circ \Psi_n^s = \Lambda_n \circ \Gamma_{n+1} \circ \Psi_{n+1}^{\alpha_{n+1}^{-1}s} \circ \mathcal{T}_n^{-1}$$
$$= \Lambda_n \circ \Phi_{n+1}^{t_{n+1}(\alpha_{n+1}^{-1}s)} \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1} = \Phi_n^{\theta'_n t_{n+1}(\alpha_{n+1}^{-1}s)} \circ \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1}.$$

Thus, formally, if Γ_{n+1} is an invariant torus with frequency vector ω_{n+1} of H_{n+1}, then

$$\Gamma_n = \mathcal{M}_n(\Gamma_{n+1}) = \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1} \tag{3.5}$$

is the invariant torus with frequency vector ω_n of H_n. The flow parameters t_n and t_{n+1} are related by $t_n(s) = \theta'_n t_{n+1}(\alpha_{n+1}^{-1}s)$. One solution of this functional equation is $t_n(s) = \xi s / \prod_{i=1}^n \tau_i$, where ξ is a constant.

3.2. Properties of the maps $\mathcal{M}_n$

Introduce the coordinates $x_1 = \omega'_n \cdot q$, $x_2 = \Omega'_n \cdot q$, $x_3 = \hat{\omega}_n \cdot p$ and $x_4 = \Omega_n \cdot p$, $n \in \mathbb{N}_0$. Denote the partial derivative with respect to x_i as ∂_i, $i = 1,2,3,4$. We can define the components of a vector valued function $f = (f^q, f^p)$ on $\mathcal{D}_{n,0}(\delta_1)$, $\delta_1 > 0$, as $f^1 = \omega'_n \cdot f^q$, $f^2 = \Omega'_n \cdot f^q$, $f^3 = \hat{\omega}_n \cdot f^p$, $f^4 = \Omega_n \cdot f^p$.

Definition 3.4 *Given $r > 0$ and $\delta_1 > 0$, let $\mathcal{B}_{n,0}(\delta_1)$ be the Banach space of vector-valued functions $f = (f^q, f^p)$ on $\mathcal{D}_{n,0}(\delta_1)$, whose components f^i, $i = 1,2,3,4$, belong to $\mathcal{A}_{n,0}(\delta_1)$, with the norm*

$$\|f\|_{n,\delta_1} = \max\{c_n\|f^1\|_{n,\delta_1}, \|f^2\|_{n,\delta_1}, \|f^3\|_{n,\delta_1}, \|f^4\|_{n,\delta_1}\}, \tag{3.6}$$

where $c_n = \tilde{A}_n \prod_{i=0}^{n-1} \|T_i^{-1}\Omega_i\|$.

Here, we study the contracting properties of the maps $\mathcal{M}_n$ defined formally by the action on a vector valued function $F = I + f$, where I is the identity map and $f \in \mathcal{B}_{n+1,0}(\delta_1)$,

$$\mathcal{M}_n(F) = \Lambda_n \circ F \circ \mathcal{T}_n^{-1}. \tag{3.7}$$

The weight factor c_n has been introduced in the norm (3.6) in order to make the map $\mathcal{M}_n$ a contraction; if the norm were defined without that factor, the scaling map in Λ_n would actually expand the first component of F.

Similarly, for a vector valued function $g = (g^q, g^p)$ on $\mathcal{D}_n(\rho)$ with components $g^1 = \omega'_n \cdot g^q$, $g^2 = \Omega'_n \cdot g^q$, $g^3 = \hat{\omega}_n \cdot g^p$ and $g^4 = \Omega_n \cdot g^p$ in $\mathcal{A}_n(\rho)$, define the norms

$$\|g\|_{n,\rho} = \max\{c_n\|g^1\|_{n,\rho}, \|g^2\|_{n,\rho}, \|g^3\|_{n,\rho}, \|g^4\|_{n,\rho}\}, \tag{3.8}$$

and

$$\begin{aligned}\|g\|'_{n,\rho} = \ &\max\{\|\partial_1 g^1\|_{n,\rho} + c_n \textstyle\sum_{i=2}^4 \|\partial_i g^1\|_{n,\rho}, c_n^{-1}\|\partial_1 g^2\|_{n,\rho} + \sum_{i=2}^4 \|\partial_i g^2\|_{n,\rho},\\ &c_n^{-1}\|\partial_1 g^3\|_{n,\rho} + \textstyle\sum_{i=2}^4 \|\partial_i g^3\|_{n,\rho}, c_n^{-1}\|\partial_1 g^4\|_{n,\rho} + \sum_{i=2}^4 \|\partial_i g^4\|_{n,\rho}\}.\end{aligned} \tag{3.9}$$

Before proving that the maps $\mathcal{M}_n$ are contractions, let us state the following Proposition. The proof of this Proposition is straightforward and will be omitted.

Proposition 3.5 *Let $\delta_1 > 0$, $\delta_1^< = \delta_1\|\omega_{n+1}\|/(\alpha_{n+1}\|\omega_n\|) < 2\delta_1/3$ and $\rho > 0$, componentwise, with $\rho_1 > \delta_1$. Let also $f_n, g_n \in \mathcal{A}_{n,0}(\delta_1)$, $h_n \in \mathcal{A}_n(\rho)$ and $X_n, Y_n \in \mathcal{A}^2_{n,0}(\delta_1)$, such that $\delta_1 + \|\omega'_n \cdot X_n\|_{n,\delta_1} < \rho_1$, $\|\Omega'_n \cdot X_n\|_{n,\delta_1} < \rho_1$, $\|\hat{\omega}_n \cdot Y_n\|_{n,\delta_1} < \rho_2$ and $\|\Omega_n \cdot Y_n\|_{n,\delta_1} < \rho_2$. If $U(q,0) = (q + X_n(q,0), Y_n(q,0))$ is a given change of variables, then*

(i) $|f_n(q,0)| \le \|f_n\|_{n,\delta_1}$, $\forall (q,0) \in \mathcal{D}_{n,0}(\delta_1)$,

(ii) $f_n g_n \in \mathcal{A}_{n,0}(\delta_1)$ *and* $\|f_n g_n\|_{n,\delta_1} \le \|f_n\|_{n,\delta_1}\|g_n\|_{n,\delta_1}$,

(iii) $\|f_{n+1} \circ \mathcal{T}_n^{-1}\|_{n,\delta_1} \le \|T_n^{-1}\Omega_n\|^r \|f_{n+1}\|_{n+1,\delta_1^<}$,

(iv) $\|h_n \circ U\|_{n,\delta_1} \le c_{\rho_1}(\|\Omega'_n \cdot X_n\|_{n,\delta_1})\|h_n\|_{n,\rho}$,

where $c_{\rho_1}(s) = \sup_{t\ge 0}(1+t)^r e^{-(\rho_1 - s)t}$, *for* $|s| < \rho_1$.

Denote by $B_n(b)$ an open ball of radius $b > 0$ in the affine space $I + \mathcal{B}_{n,0}(\delta_1)$, centered at I.

Lemma 3.6 *Let* $r < 1$ *and let* $0 < \delta_1 < \rho_1$. *For sufficiently small* $b, C > 0$ *and all Hamiltonians* $H_n = \mathcal{R}_{n-1} \circ \cdots \circ \mathcal{R}_0(H_0)$, $n \in \mathbb{N}_0$, *with* H_0 *in* $B^+_{0,\rho}(C^2) \subset \overset{+}{\mathbb{I}}_0 \mathcal{A}_0(\rho)$, *the map* $\mathcal{M}_n$ *is a contraction with the contraction rate* $a < 1$ *(independent of* H_0 *and* n*) from* $B_{n+1}(b)$ *into* $B_n(b)$.

Proof: Let $F_{n+1} \in B_{n+1}(b)$ and let $f_{n+1} = F_{n+1} - I$. Define the map $\mathcal{N}_n$ by

$$\mathcal{N}_n(f_{n+1}) = \mathcal{M}_n(F_{n+1}) - I = \Lambda_n \circ (I + f_{n+1}) \circ \mathcal{T}_n^{-1} - I\,. \tag{3.10}$$

The fact that the maps $V_{H'_{n+1}}$, $U_{H''_{n+1}}$ and $S_{\tilde{H}_{n+1}}$, included in Λ_n, are close to the identity motivates us to write $\Lambda_n = \mathcal{T}_n \circ (I + g_{n+1})$. Therefore,

$$\mathcal{N}_n(f_{n+1}) = \mathcal{T}_n \circ (f_{n+1} + g_{n+1} \circ (I + f_{n+1})) \circ \mathcal{T}_n^{-1}. \tag{3.11}$$

As the maps $V_{H'_{n+1}}$, $U_{H''_{n+1}}$ and $S_{\tilde{H}_{n+1}}$ depend only on the q-dependent part of the Hamiltonian, $\|g^i_{n+1}\|_{n+1,\rho} \le C_1\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho}$, for $i = 1,2,3,4$, i.e. $\|g^i_{n+1}\|_{n+1,\rho} \le CC_1\tilde{A}_n^{-\tau}$, where $C_1 > 0$ is an n-independent constant. Here, we assume that $C > 0$ is sufficiently small such that the estimates (2.7) are valid. Using the Diophantine bound (2.3) and the inequalities (2.4), we find that, if $\tau \ge 2 + \beta$, then

$$\|g_{n+1}\|_{n+1,\rho} \le \tilde{A}_{n+1} \prod_{i=0}^{n} \|T_i^{-1}\Omega_i\| \frac{CC_1}{\tilde{A}_n^{\tau}} \le \frac{CC_2}{\tilde{A}_n^{\tau-2-\beta}} \le CC_2, \tag{3.12}$$

where $C_2 > 0$ is a constant. These inequalities show that the growth of the weight factor with n is slower that the decrease of the norm of the q-dependent part of Hamiltonian.

The derivative of the operator $\mathcal{N}_n$ at f_{n+1} is given by its action on an arbitrary $\tilde{f} \in B_{n+1}(b)$,

$$D\mathcal{N}_n(f_{n+1})\tilde{f} = \mathcal{T}_n \circ (\tilde{f} + Dg_{n+1} \circ (I + f_{n+1})\tilde{f}) \circ \mathcal{T}_n^{-1}. \tag{3.13}$$

Using the Proposition 3.5, we find that for $(\delta_1, 0) < \rho' < \rho$, componentwise,

$$\begin{aligned}\|D\mathcal{N}_n(f_{n+1})\tilde{f}\|_{n,\delta_1} \le{}& \|T_n^{-1}\Omega_n\|^{r-1}\|\tilde{f}\|_{n+1,\delta_1^<} \\ &+ \|T_n^{-1}\Omega_n\|^{r-1} c_{\rho'_1}(\|f^2_{n+1}\|_{n+1,\delta_1^<})\|g_{n+1}\|'_{n+1,\rho'}\|\tilde{f}\|_{n+1,\delta_1^<},\end{aligned} \tag{3.14}$$

providing that $\delta_1^< + \|f^1_{n+1}\|_{n+1,\delta_1^<} < \rho'_1$, $\|f^2_{n+1}\|_{n+1,\delta_1^<} < \rho'_1$, $\|f^3_{n+1}\|_{n+1,\delta_1^<} < \rho'_2$ and $\|f^4_{n+1}\|_{n+1,\delta_1^<} < \rho'_2$. These conditions are satisfied for sufficiently small $b > 0$. Using Cauchy estimates on the norms of the derivatives $\|\partial_i g^j_{n+1}\|_{n+1,\rho'}$, for $i,j = 1,2,3,4$, we find that $\|g_{n+1}\|'_{n+1,\rho'} \le C_3\|g_{n+1}\|_{n+1,\rho}$, with $C_3 > 0$. Thus, as $r < 1$ and $\|T_n^{-1}\Omega_n\| \ge 3/2$, for sufficiently small $C > 0$, the operator norm $\|D\mathcal{N}_n(f_{n+1})\|$ can be bounded by a constant $a < 1$.

As $\|\mathcal{N}_n(0)\|_{n,\delta_1} \le \|T_n^{-1}\Omega_n\|^{r-1}c_{\rho_1}(0)\|g_{n+1}\|_{n+1,\rho}$, for sufficiently small $C > 0$, $\|\mathcal{N}_n(0)\|_{n,\delta_1} < (1-a)b$. This shows that $\mathcal{M}_n$ maps $B_{n+1}(b)$ into $B_n(b)$ and contracts distances at least by a factor $a < 1$. □

3.3. Construction of invariant tori

Let $H_n \in \overset{+}{\mathbb{I}}_n \mathcal{A}_n(\rho)$, $n \in \mathbb{N}_0$, be the renormalization orbit of a given Hamiltonian $H_0 \in B^+_{0,\rho}(C^2)$.

Proposition 3.7 *Let* $0 < \delta_1 < \rho_1$ *and* $\rho_2 > 0$. *For sufficiently small* $b > 0$, *the following holds. If* $H_0 \in B^+_{0,\rho}(C^2)$, $|t_m - s_m| \le C_4\|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho}$ *and* $|t_m| < C_5$, *for some* $C_4, C_5 > 0$, $t_m, s_m \in \mathbb{R}$ *and for all* $m \in \mathbb{N}_0$, *and if* $C > 0$ *is sufficiently small, then*

$$\|(\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m} - I)\|_{m,\delta_1} \le b, \tag{3.15}$$

where $\Phi_{m,0}$ *is the flow for the Hamiltonian* H_m^0 *restricted to* $\mathcal{D}_{m,0}(\delta_1)$.

Proof: Let $s \in \mathbb{R}$ be given and let $(q,0) \in \mathcal{D}_{m,0}(\delta_1)$. The flow Φ_m for the Hamiltonian H_m satisfies the equation,

$$\frac{d}{dt}(\Phi_m^t \circ \Phi_{m,0}^{-s} - I)(q,0) = (\mathbb{J} \cdot \nabla H_m \circ \Phi_m^t \circ \Phi_{m,0}^{-s})(q,0), \tag{3.16}$$

where $\mathbb{J}(q,p) = (p,-q)$ and $\nabla = (\nabla_q, \nabla_p)$. Introducing the function

$$\Upsilon_s(q,t) = (\Phi_m^t \circ \Phi_{m,0}^{-s} - I)(q,0), \tag{3.17}$$

2π-periodic in both q-variables, we can integrate the equation (3.16) to obtain the integral equation

$$\Upsilon_s(q,t) = -(s\omega_m, 0) + \int_0^t dt' \{\mathbb{J} \cdot \nabla H_m \circ [(q,0) + \Upsilon_s(q,t')]\}. \tag{3.18}$$

This equation can be viewed as the fixed point equation of the functional Ξ_s, defined by the action

$$\Xi_s(\Upsilon_s(q,t), q) = -(s\omega_m, 0) + \int_0^t dt' \{\mathbb{J} \cdot \nabla H_m \circ [(q,0) + \Upsilon_s(q,t')]\}, \tag{3.19}$$

on a space of functions $\Upsilon_s : \mathcal{D}_{m,0}(\delta_1) \times J \to \mathbb{C}^2 \times \mathbb{C}^2$, $J = [0,t] \subset \mathbb{R}$, with the norm

$$|||\Upsilon_s||| \;=\; \sup_{t' \in J} \max\{c_m \|\Upsilon_s^1(\cdot,t')\|_{m,\delta_1}, \|\Upsilon_s^2(\cdot,t')\|_{m,\delta_1}, c_m \|\Upsilon_s^3(\cdot,t')\|_{m,\delta_1}, c_m \|\Upsilon_s^4(\cdot,t')\|_{m,\delta_1}\}, \tag{3.20}$$

where $\Upsilon_s^1 = \omega_m' \cdot \Upsilon_s^q, \Upsilon_s^2 = \Omega_m' \cdot \Upsilon_s^q$, $\Upsilon_s^3 = \hat{\omega}_m \cdot \Upsilon_s^p, \Upsilon_s^4 = \Omega_m \cdot \Upsilon_s^p$, and $\Upsilon_s = (\Upsilon_s^q, \Upsilon_s^p)$.

We consider an open ball of functions Υ_s in that space that satisfy $|||\Upsilon_s||| < b$, with $0 < b < \min\{\rho_1 - \delta_1, \rho_2\}$. This justifies the formal use of equations (3.18) and (3.19), as for any given $t' \in J$, $\|\Upsilon_s(\cdot,t')\|_{m,\delta_1} \le |||\Upsilon_s|||$.

Now, if $\Upsilon_s^{(1)}$ and $\Upsilon_s^{(2)}$ are two functions from that ball, we have

$$\Xi_s(\Upsilon_s^{(2)}(q,t), q) - \Xi_s(\Upsilon_s^{(1)}(q,t), q) = \int_0^t dt' \int_{\Upsilon_s^{(1)}(q,t')}^{\Upsilon_s^{(2)}(q,t')} d\Upsilon \cdot \nabla(\mathbb{J} \cdot \nabla H_m) \circ [(q,0) + \Upsilon].$$

Thus,

$$|||\Xi_s(\Upsilon_s^{(2)}, \cdot) - \Xi_s(\Upsilon_s^{(1)}, \cdot)||| \le |t| \cdot \|\Upsilon_s^{(2)} - \Upsilon_s^{(1)}\| \cdot \|\nabla(\mathbb{J} \cdot \nabla H_m) \circ [(q,0) + \Upsilon]\|, \tag{3.21}$$

where

$$\begin{aligned} \|\nabla(\mathbb{J} \cdot \nabla H_m) \circ [\Upsilon + (q,0)]\| \le \max\{ & c_m \|\partial_2 \partial_3 H_m\|_{m,\rho'} + \sum_{i=1, i\neq 2}^4 \|\partial_i \partial_3 H_m\|_{m,\rho'}, \\ \|\partial_2 \partial_4 H_m\|_{m,\rho'} + c_m^{-1} \sum_{i=1, i\neq 2}^4 \|\partial_i \partial_4 H_m\|_{m,\rho'}, & \; c_m \|\partial_2 \partial_1 H_m\|_{m,\rho'} + \sum_{i=1, i\neq 2}^4 \|\partial_i \partial_1 H_m\|_{m,\rho'}, \\ & c_m \|\partial_2 \partial_2 H_m\|_{m,\rho'} + \sum_{i=1, i\neq 2}^4 \|\partial_i \partial_2 H_m\|_{m,\rho'}\}, \end{aligned}$$

with $b + \delta_1 < \rho_1' < \rho_1$ and $b < \rho_2' < \rho_2$.

Using Cauchy estimates on the norms of the derivatives and the fact that $\|\partial_2 \partial_i H_m\|_{m,\rho'} \le C_6 \|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho}$, $i = 1,2,3$, where $C_6 > 0$, this norm can be bounded by an m-independent constant. Here, we again use the fact that $\|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho}$ drops faster with m, than c_m increases. The inequality (3.21) then implies that, if $|t| = |t_m| < C_5$, $H_0 \in B_{0,\rho}^+(C^2)$ and $C > 0$ is sufficiently small, Ξ_s is a contraction with a contraction rate $\bar{a} < 1$.

Notice that

$$\Xi_s(0,q) = ((t-s)\omega_m, 0) + \mathbb{J} \cdot \nabla h_m(q,0) t, \tag{3.22}$$

and that $\nabla \mathbb{E} h_m(q,0) = 0$. Therefore, since $|t_m - s_m| \le C_4 \|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho}$, where $C_4 > 0$ and $|t_m| < C_5$, if $t = t_m$ and $s = s_m$, then, for sufficiently small $C > 0$, we have $|||\Xi_{s_m}(0,q)||| < (1-\bar{a})b$.

This shows that $\Xi_{s_m}(\cdot, q)$ is a contraction on a ball of radius $b > 0$, and has a unique fixed point Υ_{s_m}, of norm $|||\Upsilon_{s_m}||| < b$, that satisfies the equation (3.18). The claim follows from the fact that for $|t_m| < C_5$, $\|\Upsilon_{s_m}(\cdot, t_m)\|_{m,\delta_1} \leq |||\Upsilon_{s_m}|||$. □

Let, in the following, $b > 0$ and $C > 0$ be chosen sufficiently small such that the assumptions of Lemma 3.6 and Proposition 3.7 are satisfied. Let F_m, $m \in \mathbb{N}_0$, be arbitrary maps in $B_m(b)$. Define

$$\Gamma_{n,m} = (\mathcal{M}_n \circ \cdots \circ \mathcal{M}_{m-1})(F_m), \tag{3.23}$$

for $0 \leq n < m$, or, taking into account the definition (3.7) of the operators $\mathcal{M}_n$,

$$\Gamma_{n,m} = \Lambda_n \circ \cdots \circ \Lambda_{m-1} \circ F_m \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_n^{-1}. \tag{3.24}$$

Here, the maps Λ_n and the operators $\mathcal{M}_n$ are associated to the sequence of Hamiltonians H_n, $n \in \mathbb{N}_0$.

Theorem 3.8 *Let $0 < \delta_1 < \rho_1$. If $H_0 \in B^+_{0,\rho}(C^2)$ and $b, C > 0$ are sufficiently small, the limits $\Gamma_n = \lim_{m\to\infty} \Gamma_{n,m}$, $n \in \mathbb{N}_0$, exist in $B_n(b)$, and satisfy*

$$\|\Gamma_n - I\|_{n,\delta_1} \leq b. \tag{3.25}$$

Γ_n is an invariant torus of H_n with frequency vector ω_n. The invariant tori satisfy

$$\Gamma_n = \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1}. \tag{3.26}$$

Proof: By Lemma 3.6, if $n < m < i$, then $\|\Gamma_{n,i} - \Gamma_{n,m}\|_{n,\delta_1} \leq 2ba^{m-n}$. Thus, the sequence $m \mapsto \Gamma_{n,m}$ is Cauchy and converges in this norm to a limit Γ_n. Moreover, this limit is independent of F_m. The relationship (3.26) follows from the continuity of $\mathcal{M}_n$.

Now, we want to show that, in the above norm, $\Gamma_n \to I$, when $n \to \infty$. Consider a one-parameter family of Hamiltonians $H_n(s) = H_n^0 + s(H_n - H_n^0)$, $s \in \mathbb{C}$, and let $\Lambda_n(s)$ and $\Gamma_n(s)$ be the corresponding one parameter families of maps. The map $s \mapsto \Lambda_n(s)$ is analytic in the domain $|s| < \zeta_n \tilde{A}_{n-1}^2/C$ containing the unit disc. By uniform convergence, $s \mapsto \Gamma_n(s)$ is analytic from the same domain into $B_n(b)$. As $\Gamma_n(0) = I$, by Schwartz's lemma, $\|\Gamma_n(1) - I\|_{n,\delta_1} \leq 2bC/(\zeta_n \tilde{A}_{n-1}^2)$. The inequality (3.25) then follows from Theorem 2.8.

It remains to be proved that Γ_n is an invariant torus of H_n with frequency vector ω_n. First, we formally have the identities

$$\begin{aligned}
\Phi_n^{t_n} \circ \Gamma_{n,m} \circ \Psi_n^{-s_n} &= \Phi_n^{t_n} \circ \Lambda_n \circ \cdots \circ \Lambda_{m-1} \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_n^{-1} \circ \Psi_n^{-s_n} \\
&= \Lambda_n \circ \Phi_{n+1}^{\theta_n'^{-1} t_n} \circ \cdots \circ \Lambda_{m-1} \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \Psi_{n+1}^{-\alpha_{n+1}^{-1} s_n} \circ \mathcal{T}_n^{-1} \\
&= \Lambda_n \circ \cdots \circ \Lambda_{m-1} \circ \Phi_m^{t_m} \circ \Psi_m^{-s_m} \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_n^{-1},
\end{aligned} \tag{3.27}$$

on any domain where the compositions are well-defined, where $t_m = \theta_{m-1}'^{-1} \cdots \theta_0'^{-1} t_0$ and $s_m = \alpha_m^{-1} \cdots \alpha_1^{-1} s_0$, $m \in \mathbb{N}$. As the map Λ_n maps $\mathcal{D}_n(\rho)$ into $\mathcal{D}_n(\rho')$, with $\rho' < \rho$, componentwise, for t_0 in an open interval containing zero and H_0 sufficiently close to H_0^0, the equality $\Phi_n^{t_n} \circ \Lambda_n = \Lambda_n \circ \Phi_{n+1}^{\theta_n'^{-1} t_n}$, $n \in \mathbb{N}_0$, is an identity between maps on $\mathcal{D}_n(\rho)$, with a range contained in $\mathcal{D}_n(\rho)$.

As both sides of (3.27) are the maps on $\mathcal{B}_{n,0}(\delta_1)$, we have the identity

$$\Phi_n^{t_n} \circ \Gamma_{n,m} \circ \Psi_n^{-s_n} = (\mathcal{M}_n \circ \cdots \circ \mathcal{M}_{m-1})(\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m}).$$

Here, we have also used the fact that I is the invariant torus of frequency ω_m of H_m^0. It suffices to show that the map $\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m}$ belongs to the domain of $\mathcal{M}_{m-1}$, for sufficiently large m.

Let $t_0 = \xi s_0$, where $\xi = \prod_{j=1}^{\infty} \tau_j = \lim_{k\to\infty} \prod_{j=1}^{k} \tau_j$ and $\tau_j = \tau_{\tilde{H}_j}$. The existence of this limit follows from the convergence of $\sum_{j=0}^{\infty} |\tau_{j+1} - 1|$ and the mean value theorem. The convergence of the sum itself follows from the fact that $|\tau_{j+1} - 1| \leq C_7 \|(\mathbb{I} - \mathbb{E})h_j\|_{j,\rho}$, where $C_7 > 0$, and Theorem 2.8. We have,

$$\frac{\tilde{A}_m}{\alpha_0}(t_m - s_m) = t_0 \prod_{i=1}^{m} \tau_i^{-1} - s_0 = \left(\prod_{i=m+1}^{\infty} \tau_i - 1 \right) s_0 \leq s_0 C_8 \|(\mathbb{I} - \mathbb{E}) H_m\|_{m,\rho},$$

with $C_8 > 0$.

Thus, for sufficiently small $b, C > 0$, the assumptions of Proposition 3.7 are satisfied. The same Proposition then implies that $\|\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m} - I\|_{m,\delta_1} < b$ and that the map $\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m}$ belongs to the domain of $\mathcal{M}_{m-1}$. The right hand side of (3.27) converges in $\mathcal{B}_{n,0}(\delta_1)$ to Γ_n when $m \to \infty$. As the convergence implies pointwise convergence and since the maps $\Phi_n^{t_n}$ and $\Psi_n^{s_n}$ are both continuous and invertible, we conclude that $\Phi_n^{t_n} \circ \Gamma_n \circ \Psi_n^{-s_n} = \Gamma_n$. Thus, Γ_n is an invariant torus of frequency ω_n, associated to the Hamiltonian H_n. $\square$

3.4. Analyticity of invariant tori

Define the translation $J_v : (q,p) \mapsto (q+v,p)$, for $v \in \mathbb{R}^2$, a one parameter family of Hamiltonians $H_n(v) = H_n \circ J_v$ and the map $\mathcal{J}_v : H_n \mapsto H_n \circ J_v^{-1}$. Let $\Lambda_n(v)$ be the one parameter family of maps introduced in Definition 2.6, associated to $H_n(v)$. Let $0 < \rho < \varrho' < \varrho < \tilde{\rho} < 3\rho/2$, componentwise, be chosen as in Definition 2.6.

Proposition 3.9 *Let $v \in \mathbb{R}^2$. For sufficiently small $c > 0$ and for all Hamiltonians H_n in the ball $B_{n,\varrho}(c) \subset \mathcal{A}_n(\varrho)$, of radius c, centered at H_n^0, the identity $U_H \circ J_v = J_v \circ U_{H\circ J_v}$, is valid on $\mathcal{D}_n(\varrho')$, $n \in \mathbb{N}$.*

Proof: The proof of this proposition follows from the construction of the transformation U_{H_n} that satisfies the equation $\mathbb{I}^- H_n \circ U_{H_n} = 0$. In [4], this transformation is constructed as the composition of a sequence of canonical transformations U_ϕ generated by functions ϕ. As the translation $\mathcal{J}_v$ commutes with $\mathbb{I}^\pm$ and differentiation, we have $\phi_{H_n\circ J_v} = \phi_{H_n} \circ J_v$, where ϕ_{H_n} is the generating function associated to H_n. Then, $U_{\phi_{H_n}} \circ J_v = J_v \circ U_{\phi_{H_n\circ J_v}}$. The claim follows, by taking the limit of the product of the above mentioned sequence. $\square$

Proposition 3.10 *Let $v \in \mathbb{R}^2$. For sufficiently small $C > 0$ and for all Hamiltonians $H_0 \in B_{0,\rho}^+(C^2) \subset \mathbb{I}_0^+\mathcal{A}_n(\rho)$, if the sequence H_n, $n \in \mathbb{N}_0$, is the renormalization orbit of H_0, then the identity $\Lambda_n \circ J_v = J_{T_n v} \circ \Lambda_n(T_n v)$, is valid on $\mathcal{D}_n(\rho)$, $n \in \mathbb{N}$.*

Proof: Notice first that $\mathcal{T}_n \circ J_v = J_{T_n v} \circ \mathcal{T}_n$ and that $V_{H'_{n+1}}$ and $S_{\tilde{H}_{n+1}}$ commute with J_v. Further, notice that $V_{H'_{n+1}\circ J_v} = V_{H'_{n+1}}$ and $S_{\tilde{H}_{n+1}\circ J_v} = S_{\tilde{H}_{n+1}}$. Recall that $\Lambda_n = \mathcal{T}_n \circ V_{H'_{n+1}} \circ U_{H''_{n+1}} \circ S_{\tilde{H}_{n+1}}$, $H'_{n+1} = (\theta_n/\mu_n) H_n \circ \mathcal{T}_n$, $H''_{n+1} = H'_{n+1} \circ V_{H'_{n+1}}$ and $\tilde{H}_{n+1} = H''_{n+1} \circ U_{H''_{n+1}}$. Let $H'_{n+1}(v) = (\theta_n/\mu_n) H_n(v) \circ \mathcal{T}_n$, $H''_{n+1}(v) = H'_{n+1}(v) \circ V_{H'_{n+1}(v)}$ and $\tilde{H}_{n+1}(v) = H''_{n+1}(v) \circ U_{H''_{n+1}(v)}$. Using the previous relations and the Proposition 3.9, we find that $\Lambda_n \circ J_v = J_{T_n v} \circ \mathcal{T}_n \circ V_{H'_{n+1}(T_n v)} \circ U_{H''_{n+1}(T_n v)} \circ S_{\tilde{H}_{n+1}(T_n v)}$, and the claim immediately follows. $\square$

Proposition 3.11 *Let $v \in \mathbb{R}^2$. For sufficiently small $C > 0$ and for all Hamiltonians $H_0 \in B_{0,\rho}^+(C^2) \subset \mathbb{I}_0^+\mathcal{A}_n(\rho)$, we have $J_v \circ \Gamma_0(H_0 \circ J_v) = \Gamma_0(H_0) \circ J_v$. Here, by writing $\Gamma_0(H_0)$, we have emphasized that the invariant torus Γ_0 is associated to a Hamiltonian H_0.*

Proof: Let $v \in \mathbb{R}^2$ be fixed. Since $\mathcal{J}_v^{-1}$ is an isometry on $\mathcal{A}_n(\rho)$, for sufficiently small $C > 0$ and for all Hamiltonians $H_0 \in B_{0,\rho}^+(C^2)$, $\mathcal{J}_v^{-1}(H_0)$ belongs to $B_{0,\rho}^+(C^2)$ and the orbit $\mathcal{R}_{n-1} \circ \cdots \circ \mathcal{R}_0(\mathcal{J}_v^{-1}(H_0))$, $n \in \mathbb{N}$, satisfies the bounds analogous to the bounds (2.7). Thus, we can construct the maps

$$\Gamma_{0,m}(H_0 \circ J_v) = \Lambda_0(v) \circ \cdots \circ \Lambda_{m-1}(T_{m-2}^{-1} \ldots T_0^{-1} v) \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_0^{-1}, \tag{3.28}$$

for $m \in \mathbb{N}$. By Theorem 3.8, the limit $\Gamma_0(H_0 \circ J_v) = \lim_{m\to\infty} \Gamma_{0,m}(H_0 \circ J_v)$ is an invariant torus of $H_0 \circ J_v$ with frequency vector ω_0. Using Proposition 3.10, we find

$$\begin{aligned}
\Gamma_{0,m} \circ J_v &= \Lambda_0 \circ \cdots \circ \Lambda_{m-1} \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_0^{-1} \circ J_v \\
&= J_v \circ \Lambda_0(v) \circ \cdots \circ \Lambda_{m-1}(T_{m-2}^{-1} \ldots T_0^{-1} v) \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_0^{-1} \\
&= J_v \circ \Gamma_{0,m}(H_0 \circ J_v).
\end{aligned} \tag{3.29}$$

Further, by taking the limit $m \to \infty$ of this identity, we find that $J_v \circ \Gamma_0(H_0 \circ J_v) = \Gamma_0(H_0) \circ J_v$. This holds for all Hamiltonians H_0 in a ball $B_{0,\rho}^+(C^2)$ independent of $v \in \mathbb{R}^2$. $\square$

The relationship between $\Gamma_0(H_0 \circ J_v)$ and $\Gamma_0(H_0)$, valid *a priori* for $v \in \mathbb{R}^2$, can be used to analytically continue $\Gamma_0(H_0)$ away from $\mathbb{R}^2 \times \{0\}$.

In the following, given any two sets X and Y, if $x \in X$, then $\mathcal{E}_x$ denotes the evaluation functional defined by $\mathcal{E}_x f = f(x)$, for all functions $f \in Y^X$.

Theorem 3.12 *Let $\tilde{\rho}_1 - \rho_1 > \delta_1 > \delta_1' > 0$ and $\tilde{\rho}_2 - \rho_2 > 0$. There exists an open neighborhood B of H_0^0 in $\mathcal{A}_0(\tilde{\rho})$, such that for every Hamiltonian $H_0 \in B$, the map*

$$\mathcal{G}_{H_0}(q,p) = \mathcal{E}_0 J_q \circ \Gamma_0(H_0 \circ J_q), \qquad q \in \mathcal{D}_{0,1}(\delta_1'), \quad p \in \mathbb{C}^2, \tag{3.30}$$

defines an analytic function $\mathcal{G}_{H_0}$ on $\mathcal{D}_{0,1}(\delta_1') \times \mathbb{C}^2$ whose restriction to $\mathbb{R}^2 \times \{0\}$ coincides with $\Gamma_0(H_0)$.

Proof: Let $\delta = (\delta_1, 0)$. The map $(q, H) \mapsto H \circ J_q$ is analytic from $\mathcal{D}_{0,1}(\delta_1) \times \mathcal{A}_0(\tilde{\rho})$ into $\mathcal{A}_0(\rho)$. This follows from the fact that the map $\mathcal{J}_q^{-1}$ is bounded with norm 1 from $\mathcal{A}_0(\rho + \delta)$ to $\mathcal{A}_0(\rho)$, for every $q \in \mathcal{D}_{0,1}(\delta_1)$, and the differentiation is bounded from $\mathcal{A}_0(\tilde{\rho})$ to $\mathcal{A}_0(\rho + \delta)$.

Define the map $G : H_0 \mapsto G(H_0)$, by setting $G(H_0)(q,p) = \Gamma_0(H_0)(q,0) - S_0(q,p)$, for H_0 in the domain of Γ_0. Here $S_0(q,p) = (q,0)$. As $\mathcal{J}_q^{-1}$ is bounded from $\mathcal{A}_0(\tilde{\rho})$ to $\mathcal{A}_0(\rho)$, there exists an open ball B in $\mathcal{A}_0(\tilde{\rho})$ containing H_0^0, such that for every $q \in \mathcal{D}_{0,1}(\delta_1)$, both H_0 and $\mathcal{J}_q^{-1} H_0$ belong to the domain of Γ_0. Since Γ_0 depends analytically on H_0, so does G, and as, $\mathcal{E}_0$ is bounded, the map $(q, H_0) \mapsto \mathcal{E}_0 G(\mathcal{J}_q^{-1} H_0)$ is analytic on $\mathcal{D}_{0,1}(\delta_1) \times B$. The analyticity of $\mathcal{G}_{H_0}$ for $H_0 \in B$ follows from the identity $(\mathcal{G}_{H_0} - S_0)(q,p) = \mathcal{E}_0 G(\mathcal{J}_q^{-1} H_0)$.

Using Proposition 3.11, we find

$$\mathcal{E}_{(q,0)} \Gamma_0(H_0) = \mathcal{E}_0(\Gamma_0(H_0) \circ J_q) = \mathcal{E}_0 J_q \circ \Gamma_0(H_0 \circ J_q) = \mathcal{G}_{H_0}(q,0), \tag{3.31}$$

for all $q \in \mathbb{R}^2$ and $H_0 \in B$. This shows that $\Gamma_0(H_0)$ and $\mathcal{G}_{H_0}$ agree on $\mathbb{R}^2 \times \{0\}$ as claimed. $\square$

Acknowledgements

I would like to thank Professor Hans Koch for fruitful discussions.

---

[1] A. N. Kolmogorov, On conservation of quasiperiodic motions under small perturbations of the Hamiltonian, *Dokl. Akad. Nauk SSSR* **98** (1954), 527-530.
[2] V. I. Arnold, Proof of A. N. Kolmogorov's Theorem on the preservation of quasiperiodic motions under small perturbations of the Hamiltonian, *Usp. Mat. Nauk. SSSR* **18**(5) (1963), 13-40.
[3] J. Moser, On invariant curves of area-preserving mappings of an annulus, *Nachr. Akad. Wiss. Gött,II, Math. Phys.* **K1** (1962), 1-20.
[4] S. Kocić, Renormalization of Hamiltonians for Diophantine frequency vectors, *mp-arc 04-237* (2004).
[5] H. Koch, A renormalization group for Hamiltonians, with applications to KAM tori, *Ergod. Th.* & *Dynam. Sys.* **19** (1999), 475-521.
[6] J. Lopes-Dias, Renormalization of flows on the multidimensional torus close to a KT frequency vector, *Nonlinearity* **15** (2002), 647-664.
[7] H. Koch, A Renormalization Group Fixed Point Associated with the Breakup of Golden Invariant Tori, *mp-arc 02-175* (2002).
[8] H. Koch, Existence of critical invariant tori, *mp-arc 04-210* (2004).
[9] G. Gallavotti, G. Gentile, V. Mastropietro, Field theory and KAM tori, *Math. Phys. Electron. J.* **1** (1995), paper 5.
[10] G. Gallavotti and G. Gentile, Hyperbolic low-dimensional invariant tori and summations of divergent series, *Commun. Math. Phys.* **227** (2002), 421-460.
[11] R. S. MacKay, A renormalization approach to invariant circles in area-preserving maps, *Physica* **7D** (1983), 283-300.
[12] K. M. Khanin and Ya. G. Sinai, The renormalization group method and Kolmogorov-Arnol'd-Moser theory, in *Nonlinear Phenomena in Plasma Physics and Hydrodynamics*, edited by R. Z. Sagdeev (Mir, Moscow, 1986), 93-118.
[13] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers (Oxford Science Publications, 1990).