Mass Renormalization in Non-relativistic Quantum Electrodynamics with Spin 1/2

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Abstract

The effective mass $m_{\rm eff}$ of the the Pauli-Fierz Hamiltonain with ultraviolet cutoff Λ and the bare mass m in nonrelativistic QED with spin 1/2 is investigated. Analytic properties of $m_{\rm eff}$ in coupling constant e are shown and explicit forms of constants $a_1(\Lambda/m)$ and $a_2(\Lambda/m)$ depending on Λ/m such that

$$m_{\rm eff}/m = 1 + a_1(\Lambda/m)e^2 + a_2(\Lambda/m)e^4 + \mathcal{O}(e^6)$$

are given. It is shown that the spin interaction enhances the effective mass and that there exist strictly positive constants b_1, b_2, c_1 and c_2 such that

$$b_1 \le \lim_{\Lambda\to\infty} \frac{a_1(\Lambda/m)}{\log(\Lambda/m)} \le b_2, \quad -c_1 \le \lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m)}{(\Lambda/m)^2} \le -c_2.$$

In particular $a_2(\Lambda/m)$ does not diverges as $\pm[\log(\Lambda/m)]^2$ but $-(\Lambda/m)^2$.

1 Introduction

One electron with the bare mass m interacting with a quantized radiation field carries a virtual cloud of photons. This virtual cloud enhances the bare mass m to an effective

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mass $m_{\rm eff} \geq m$. In this paper the effective mass is defined as the inverse of the Hessian of the ground state energy of the Hamiltonian at total momentum zero. The electron includes spin and the quantized radiation field has an infrared cutoff κ and an ultraviolet cutoff Λ. We are interested in the asymptotic behavior of $m_{\rm eff}$ as ultraviolet cutoff Λ goes to infinity. It is admittedly a subtle problem. Our contribution in this paper is to investigate the asymptotics of the fourth order of a coupling constant e of the effective mass $m_{\rm eff}$. Although the purpose of this paper is to prove the same things as the spinless case established in [8], the result is completely different from it. In the spinless case its divergence is $\sqrt{\Lambda}$ and in the present case Λ^2 as $\Lambda \to \infty$.

It is proven in [1] that the authentic Pauli-Fierz model in non-relativistic QED can be derived from QED by a scaling limit. QED is renormalizable, and then the Pauli-Fierz model may be expected to be also renormalizable. In facts, there exists no fermion loop but only photon absorption-emission diagrams in the non-relativistic QED. Although QED is believed to be trivial after the renormalization, there is a strong belief that the Pauli-Fierz model with spin is renormalizable and the renormalized theory is non-trivial. Nevertheless we show that there exist divergences in higher order perturbation expansions, which cannot be removed by the conventional renormalization scheme. So it is inconceivable that the effective mass $m_{\rm eff}$ of the Pauli-Fierz Hamiltonian with spin can be renormalized in our procedure.

In the spinless case, the e^2 order of the ground state energy $E = \sum_{n=0}^\infty \frac{e^n}{n!} E_{(n)}$ of the Pauli Fierz Hamiltonian with total momentum zero is identically zero, the inclusion of spin, however, presents that $\frac{1}{2} E_{(2)} \neq 0$ and it diverges like Λ^2 as Λ goes to infinity. This divergence can be hidden as a constant to define a Wick product. It begins to appear, however, in the connected diagrams of higher orders. Since the Pauli-Fierz Hamiltonian before the scaling limit must have substantially many cutoffs which may be shielded by the scaling limit, the appearance of such divergences in higher order diagrams is not a contradiction. In fact without these cutoffs, the scaling limit would not exist. As a consequence $E_{(2)}$ causes e^4 order of $m_{\rm eff}$ to include Λ^2 divergence.

The effective mass and the ground state energy of the Pauli-Fierz Hamiltonian are studied in e.g., [2, 3, 4, 6, 8, 7, 11, 12, 15]. We note that a precise grasp of the definition of the effective mass m_{eff} is also a problem. Actually there are several ways to define the effective mass. Alternative definition of the effective mass is given in e.g., Lieb-Loss [12] and Lieb [11]. It is worth checking that both of definitions coincide with each others under some conditions, e.g., $\Lambda \ll 1$. Catto-Hainzl [3] expand the ground state energy of the Pauli-Fierz Hamiltonian with spin and without infrared cutoff up to e^4 order. Hainzl and Seiringer [6] give an exact form of the coefficient of e^2 order of the effective mass. Moreover other related work is Spohn [15] where upper and lower bounds of the effective mass of the Nelson model are studied. See also [16] for recent movement and progress in this field.

Our results are summarized as follows:

Theorem A (Analytic properties) The effective mass m_{eff} divided by the bare mass m, m_{eff}/m, is a function of e^2 and Λ/m, and analytic in e^2 for $e^2 < e_{00}$ with some $e_{00} > 0$ depending on Λ.

Theorem B ($(\Lambda/m)^2$-divergence) Let

$$\frac{m_{\text{eff}}}{m} = 1 + a_1(\Lambda/m)e^2 + a_2(\Lambda/m)e^4 + \mathcal{O}(e^6).$$

Then there exist strictly positive constants b_i and c_i, $i = 1, 2$, such that

$$b_1 \leq \lim_{\Lambda\to\infty} \frac{a_1(\Lambda/m)}{\log(\Lambda/m)} \leq b_2, \quad -c_1 \leq \lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m)}{(\Lambda/m)^2} \leq -c_2.$$

Theorem C (Enhancement by spin) Let $m_{\text{eff}}^{\text{spinless}}$ be the effective mass for the spinless Pauli-Fierz Hamiltonian. Then $m_{\text{eff}} > m_{\text{eff}}^{\text{spinless}}$ follows for e with a sufficiently small $|e|$ but $\neq 0$.

Theorem A, B and C are converted to Corollary 2.2, Theorem 2.9 and Corollary 3.6, respectively. We organize the paper as follows: Section 2 is devoted to expanding m_{eff} up to e^4. In section 3 the divergence of the forth order of m_{eff} is estimated. Finally in section 4, some additional arguments on the scaling limit and open problems on this model are given.

1.1 Definitions of the non-relativistic QED

Let $\mathcal{F}$ be the boson Fock space over $L^2(\mathbb{R}^3\times\{1,2\})$ given by

$$\mathcal{F} \equiv \bigoplus_{n=0}^{\infty}\left[\otimes_s^n L^2(\mathbb{R}^3\times\{1,2\})\right],$$

where $\otimes_s^n$ denotes the n-fold symmetric tensor product and $\otimes_s^0 L^2(\mathbb{R}^3\times\{1,2\}) \equiv \mathbb{C}$. The Fock vacuum $\Omega \in \mathcal{F}$ is defined by $\Omega \equiv \{1,0,0,..\}$. Let $a(f)$, $f \in L^2(\mathbb{R}^3\times\{1,2\})$, be the creation operator and $a^*(f)$, $f \in L^2(\mathbb{R}^3\times\{1,2\})$, the annihilation operator on $\mathcal{F}$ defined by

$$\begin{aligned}
&(a^*(f)\Psi)^{(n+1)} \equiv \sqrt{n+1}S_{n+1}(f\otimes\Psi^{(n)}),\\
&D(a^*(f)) = \left\{\Psi\in\mathcal{F}|\sum_{n=0}^{\infty}(n+1)\|S_{n+1}(f\otimes\Psi^{(n)})\|^2_{\otimes^n L^2(\mathbb{R}^3\times\{1,2\})}<\infty\right\},
\end{aligned}$$

and $a(f) = [a^*(\bar{f})]^*$, where S_n denotes the symmetrizer, $D(X)$ the domain of operator X, and $\|\cdot\|_{\mathcal{K}}$ the norm on $\mathcal{K}$. The scalar product on $\mathcal{K}$ is denoted by $(f,g)_{\mathcal{K}}$ which is linear in g and anti-linear in f. They satisfy canonical commutation relations:

$$[a(f),a^*(g)] = (\bar{f},g)_{L^2(\mathbb{R}^3\times\{1,2\})},\quad [a(f),a(g)]=0,\quad [a^*(f),a^*(g)]=0,$$

where $[X,Y]=XY-YX$. Formal kernels of $a(f)$ and $a^*(f)$ are denoted by $a(k,j)$ and $a^*(k,j)$, $(k,j)\in\mathbb{R}^3\times\{1,2\}$, respectively. Then we write as $\sum_{j=1,2}\int a^\sharp(k,j)f(k,j)dk$ for $a^\sharp(f)$. The linear hull of $\{a^*(f_1)\cdots a^*(f_n)\Omega|f_j\in L^2(\mathbb{R}^3\times\{1,2\}), j\geq 0\}$ is dense in $\mathcal{F}$. Let $T: L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ be a self-adjoint operator. We define $\Gamma(e^{itT})$ by

$$\Gamma(e^{itT})a^*(f_1)\cdots a^*(f_n)\Omega \equiv a^*(e^{itT}f_1)\cdots a^*(e^{itT}f_n)\Omega.$$

Thus $\Gamma(e^{itT})$ turns out to be a strongly continuous one-parameter unitary group in t, which implies that there exists a self-adjoint operator $d\Gamma(T)$ on $\mathcal{F}$ such that

$$\Gamma(e^{itT}) = e^{itd\Gamma(T)},\quad t\in\mathbb{R}.$$

The self-adjoint operator $d\Gamma(T)$ is referred to the second quantization of T. We define a Hilbert space $\mathcal{H}$ by

$$\mathcal{H} \equiv \mathbb{C}^2\otimes\mathcal{F}.$$

We identify $\mathcal{H}$ as

$$\mathcal{H} \cong \bigoplus_{n=0}^{\infty} \left[\mathbb{C}^2 \otimes (\otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\}))\right] \equiv \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}.$$

The Pauli-Fierz Hamiltonian with total momentum $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ is given by a symmetric operator on $\mathcal{H}$:

$$H_m(e,p) \equiv \frac{1}{2m}\left\{\sum_{\mu=1}^{3} \sigma_\mu \otimes (p_\mu - P_{\mathrm{f}\mu} - eA_{\hat{\varphi}\mu})\right\}^2 + 1 \otimes H_\mathrm{f},$$

where $m > 0$ and $e \in \mathbb{R}$ denote the mass and the charge of an electron, respectively, $\sigma \equiv (\sigma_1, \sigma_2, \sigma_3)$ the 2×2 Pauli-matrices given by

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the free Hamiltonian H_f, the momentum operator P_f and the quantum radiation field $A_{\hat{\varphi}\mu}$ are given by

$$H_\mathrm{f} \equiv d\Gamma(\omega), \quad P_{\mathrm{f}\mu} \equiv d\Gamma(k_\mu),$$
$$A_{\hat{\varphi}\mu} \equiv \frac{1}{\sqrt{2}}\sum_{j=1,2}\int \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} e_\mu(k,j)(a^*(k,j) + a(k,j))dk, \quad \mu = 1,2,3.$$

Here $e(k,j)$, $j = 1,2$, denotes polarization vectors such that three vectors $e(k,1)$, $e(k,2)$, $k/|k|$ form a right-handed system in $\mathbb{R}^3$, i.e.,

$$|e(k,j)| = 1, \quad j = 1,2, \quad e(k,1) \cdot e(k,2) = 0, \quad e(k,1) \times e(k,2) = k/|k|.$$

We fix polarization vectors, e.g., for $k = (|k|\cos\theta\cos\phi, |k|\sin\theta\cos\phi, |k|\cos\theta)$,

$$e(k,1) = (-\sin\theta, \cos\theta, 0), \quad e(k,2) = (-\cos\theta\sin\phi, -\sin\theta\sin\phi, \cos\phi),$$

where $0 \le \theta \le 2\pi$, $-\pi/2 \le \phi \le \pi/2$. Note that $e(-k,1) = e(k,1)$, $e(-k,2) = -e(k,2)$. Since

$$[P_{\mathrm{f}\mu}, A_{\hat{\varphi}\nu}] = \frac{1}{\sqrt{2}}\sum_{j=1,2}\int \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} k_\mu e_\nu(k,j)(a^*(k,j) - a(k,j))dk,$$

it follows that

$$H_m(e,p) = \frac{1}{2m}\left\{1\otimes\sum_{\mu=1}^{3}(p_\mu - P_{\mathrm{f}\mu} - eA_{\hat\varphi\mu})^2\right\} + 1\otimes H_\mathrm{f} - \frac{e}{2m}\sum_{\mu=1}^{3}\sigma_\mu\otimes B_\mu,$$

where B_μ denotes the quantum magnetic field given by

$$B_{\hat\varphi\mu} \equiv \frac{i}{\sqrt{2}}\sum_{j=1,2}\int\frac{\hat\varphi(k)}{\sqrt{\omega(k)}}(k\times e(k,j))_\mu(a^*(k,j)-a(k,j))dk.$$

We simply write $H_m(e,p)$ as

$$H_m(e,p) = \frac{1}{2m}(p - P_\mathrm{f} - eA_{\hat\varphi})^2 + H_\mathrm{f} - \frac{e}{2m}\sigma B_{\hat\varphi}, \quad p\in\mathbb{R}^3.$$

The spinless Pauli-Fierz Hamiltonian is given by

$$H_m^{\mathrm{spinless}}(e,p) = \frac{1}{2m}(p - P_\mathrm{f} - eA_{\hat\varphi})^2 + H_\mathrm{f}$$

which acts on $\mathcal{F}$. We assume that $\hat\varphi(k) = \overline{\hat\varphi(k)} = \hat\varphi(-k)$ and

$$\int_{\mathbb{R}^3}(\omega(k)^{-2}+\omega(k))|\hat\varphi(k)|^2dk < \infty.$$

Then $H_m(e,p)$ is self-adjoint on $D(\frac{1}{2m}{P_\mathrm{f}}^2 + H_\mathrm{f})$ for $(e,p)\in\mathbb{R}\times\mathbb{R}^3$ such that $|e|<\mathbf{e}$ and $|p|<\mathbf{p}$ with some constants $\mathbf{e}$ and $\mathbf{p}$. This is proven by the Kato-Rellich theorem and using the inequalities

$$\begin{aligned}&\|a^*(f)\Psi\|_\mathcal{F} \le \|f/\sqrt{\omega}\|_{L^2(\mathbb{R}^3\times\{1,2\})}\|H_\mathrm{f}^{1/2}\Psi\|_\mathcal{F} + \|f\|_{L^2(\mathbb{R}^3\times\{1,2\})}\|\Psi\|_\mathcal{F},\\ &\|a(f)\Psi\|_\mathcal{F} \le \|f/\sqrt{\omega}\|_{L^2(\mathbb{R}^3\times\{1,2\})}\|H_\mathrm{f}^{1/2}\Psi\|_\mathcal{F}.\end{aligned}$$

1.2 Mass renormalization

In what follows we set

$$\hat\varphi(k) \equiv \begin{cases} 0, & |k|<\kappa,\\ 1/\sqrt{(2\pi)^3}, & \kappa\le|k|\le\Lambda,\\ 0, & |k|>\Lambda.\end{cases} \tag{1.1}$$

The effective mass $m_\mathrm{eff} = m_\mathrm{eff}(e^2,\Lambda,\kappa,m)$ is defined by

$$\frac{1}{m_\mathrm{eff}} \equiv \frac{1}{3}\Delta_p\inf\sigma(H_m(e,p))\lceil_{p=0}.$$

Removal of the ultraviolet cutoff Λ through mass renormalization means to find sequences $\{m\}$ and $\{\Lambda\}$ such that $\Lambda \to \infty$ and $m \to 0$, and $m_{\rm eff}$ converges to some positive constant. To achieve this, we want to find constants $\beta < 0$ and $b > 0$ such that

$$\lim_{\Lambda\to\infty} m_{\rm eff}(e^2, \Lambda, \kappa\Lambda^\beta, (b\Lambda)^\beta) = m_{\rm ph}, \tag{1.2}$$

where $m_{\rm ph} > 0$ is a given constant. We will see later that $m_{\rm eff}/m$ is a function of e^2, Λ/m and κ/m. Set

$$f(e^2, \Lambda/m, \kappa/m) \equiv \frac{m_{\rm eff}}{m}. \tag{1.3}$$

The analysis of (1.2) can be reduced to find constants $0 \le \gamma < 1$ and $0 < b_0 < \infty$ such that

$$\lim_{\Lambda\to\infty} \frac{f(e^2, \Lambda/m, \kappa/m)}{(\Lambda/m)^\gamma} = b_0. \tag{1.4}$$

If we succeed in finding constants γ and b_0 such as in (1.4), then, taking

$$\beta \equiv \frac{-\gamma}{1-\gamma} < 0, \quad b \equiv 1/b_1^{1/\gamma},$$

we see that

$$\lim_{\Lambda\to\infty} m_{\rm eff}(e^2, \Lambda, \kappa\Lambda^\beta, (b\Lambda)^\beta) = \lim_{\Lambda\to\infty} b_0 \left(\frac{\Lambda}{b_1^{1/\gamma}}\right)^\beta \left(\frac{\Lambda}{(\Lambda/b_1^{1/\gamma})^\beta}\right)^\gamma = b_0 b_1,$$

where b_1 is a parameter, which is adjusted such as $b_0 b_1 = m_{\rm ph}$. Hence we will be able to establish (1.2). The mass renormalization is, however, a subtle problem, and unfortunately we can not yet find constants γ and b_0 such as in (1.4). It is proven that f is analytic in e with $|e|$ sufficiently small under some conditions. Let us $\kappa > 0$ be fixed and set

$$f(e^2, \Lambda/m, \kappa/m) = \sum_{n=0}^\infty a_n(\Lambda/m) e^{2n}.$$

In the previous paper [8] it is proven that

$$a_2(\Lambda/m) \approx \sqrt{\Lambda/m}, \quad \Lambda \to \infty, \tag{1.5}$$

for a spinless Pauli-Fierz Hamiltonian, where $X \approx Y$ means that there exist positive constants x_1 and x_2 such that

$$x_1 \le \lim_{\Lambda\to\infty} (X/Y) \le x_2.$$

(1.5) suggests that $\gamma \geq 1/2$ for $e \neq 0$, and that γ is not analytic in e since $\gamma = 0$ for $e = 0$. We are interested in knowing the dependence of spin for the asymptotics of $a_2(\Lambda/m)$ as $\Lambda \to \infty$. Although it is desirable that $\gamma < 1$, actually in this paper we shall show that

$$a_2(\Lambda/m) \approx -(\Lambda/m)^2, \quad \Lambda \to \infty.$$

This suggests that $\gamma \geq 2$ and, then mass renormalization such as (1.2) may not work.

2 Effective Mass and Analyticity

2.1 Analytic properties

Let

$$T_m(e,p) \equiv \frac{1}{2}(p - P_{\mathrm{f}} - eA_{\hat{\varphi}_m})^2 + H_{\mathrm{f}} - \frac{e}{2}\sigma B_{\hat{\varphi}_m},$$

where

$$\hat{\varphi}_m(k) \equiv \hat{\varphi}(mk) = \begin{cases} 0, & |k| < \kappa/m, \\ 1/\sqrt{(2\pi)^3}, & \kappa/m \leq |k| \leq \Lambda/m, \\ 0, & |k| > \Lambda/m. \end{cases} \tag{2.1}$$

Since $A_{\hat{\varphi}} \cong mA_{\hat{\varphi}_m}$, $B_{\hat{\varphi}} \cong mB_{\hat{\varphi}_m}$, $H_{\mathrm{f}} \cong mH_{\mathrm{f}}$ and $P_{\mathrm{f}} \cong mP_{\mathrm{f}}$, where $X \cong Y$ denotes the unitary equivalence between X and Y, we see that

$$H_m(e,p) \cong mT_m(e, p/m). \tag{2.2}$$

The set of operators $J = (J_1, J_2, J_3)$ on $\mathcal{H}$ is defined by

$$J_\mu \equiv d\Gamma((k \times (-i\nabla_k))_\mu) + \frac{1}{2}\sigma_\mu, \quad \mu = 1,2,3.$$

In [9] it is written informally as

$$\begin{aligned} d\Gamma((k \times (-i\nabla_k))_\mu) = &-i \sum_{j=1,2} \int a^*(k,j)(k \times \nabla_k)_\mu a(k,j) dk \\ &+ i \int \frac{k_\mu}{|k|}\{a^*(k,2)a(k,1) - a^*(k,1)a(k,2)\} dk. \end{aligned}$$

Let $n \in \mathbb{R}^3$ with $|n| = 1$, $\theta \in \mathbb{R}$, and $\mathrm{R} = (R_{\mu\nu})_{1 \leq \mu,\nu \leq 3}$ be the 3×3 matrix describing the rotation around n through angle θ. Since $e(\mathrm{R}k, j) = \mathrm{R}e(k,j)$ for $k \neq |k|n$, $j = 1,2$, we have

$$e^{i\theta n\cdot J}A_\mu e^{-i\theta n\cdot J}=\sum_{\nu=1}^3 R_{\mu\nu}A_\nu,\quad e^{i\theta n\cdot J}P_{\mathrm{f}\mu}e^{-i\theta n\cdot J}=\sum_{\nu=1}^3 R_{\mu\nu}P_{\mathrm{f}\nu},\quad e^{i\theta n\cdot J}H_{\mathrm{f}}e^{-i\theta n\cdot J}=H_{\mathrm{f}},$$
$$e^{i\theta n\cdot J}B_\mu e^{-i\theta n\cdot J}=\sum_{\nu=1}^3 R_{\mu\nu}B_\nu,\quad e^{i\theta n\cdot J}\sigma_\mu e^{-i\theta n\cdot J}=\sum_{\nu=1}^3 R_{\mu\nu}\sigma_\nu.$$

Hence

$$e^{i\theta n\cdot J}T_m(e,p)e^{-i\theta n\cdot J}=T_m(e,\mathrm{R}^{-1}p).$$

In particular for $p\neq(0,0,p_3)$,

$$e^{i\theta_p n_p\cdot J}T_m(e,p)e^{-i\theta_p n_p\cdot J}=T_m(e,|p|n_z), \tag{2.3}$$

where $\theta_p=\arccos\{(n_z,p)/|p|\}$, $n_z=(0,0,1)$, and $n_p=p\times n_z/|p\times n_z|$. It is also seen that for $p\in\mathbb{R}^3$,

$$T_m(e,|p|n_z)\cong T_m(e,-|p|n_z).$$

Then

$$H_m(e,p)\cong mT_m(e,(|p|/m)n_z),\quad p\in\mathbb{R}^3.$$

Let $:X:$ be the Wick product of X. We define

$$H(e,\epsilon)\equiv :T_m(e,\epsilon n_z):,\quad \epsilon\in\mathbb{R}.$$

Set

$$E(e,\epsilon)\equiv\inf\sigma(H(e,\epsilon)).$$

Since

$$T_m(e,\epsilon n_z)=H(e,\epsilon)+e^2\frac{2}{3}\|\hat\varphi/\sqrt{2\omega}\|^2_{L^2(\mathbb{R}^3)}$$

and

$$\inf\sigma(H_m(e,p))=\inf\sigma(mT_m(e,(|p|/m)n_z))=mE(e,|p|/m)-e^2\frac{2}{3}m\|\hat\varphi/\sqrt{2\omega}\|^2_{L^2(\mathbb{R}^3)},$$

we have

$$\frac{m}{m_{\mathrm{eff}}}=\partial_\epsilon^2E(e,\epsilon)\lceil_{\epsilon=0}.$$

Then it is also seen that m/m_{eff} depends on Λ/m and κ/m. We study $:H(e,\epsilon):$ for $H_m(e,p)$ for simplicity. We set $A\equiv A_{\hat\varphi_m}$ and $B\equiv B_{\hat\varphi_m}$.

We review analytic properties of both of $E(e,\epsilon)$ and a ground state of $H(e,\epsilon)$. In [8] the two fold degeneracy of a ground state has been proven for the case where $\omega(k) = \sqrt{|k|^2+\nu^2}$, $\nu>0$, and $\kappa=0$. With a small modification of [8] we can also prove the two fold degeneracy of a ground state for our case, i.e., $\omega(k)=|k|$ and $\kappa>0$, and show its analytic properties. As is seen below in our case $E(e,\epsilon)$ i s not an isolated eigenvalue and is degenerate. Then it is *not* clear that a ground state of $H(e,\epsilon)$ and $E(e,\epsilon)$ are analytic in e and ϵ. Let

$$\varphi_+ \equiv \begin{pmatrix}1\\0\end{pmatrix}\otimes\Omega, \quad \varphi_- \equiv \begin{pmatrix}0\\1\end{pmatrix}\otimes\Omega.$$

We shall prove the following lemma.

Lemma 2.1 *There exist constants $e_0>0$ and $\epsilon_0>0$ such that for $(e,\epsilon)\in\mathcal{D}_a\equiv\{(e,\epsilon)\in\mathbb{R}^2||e|<e_0,|\epsilon|<\epsilon_0\}$, (1) the dimension of $\mathrm{Ker}(H(e,\epsilon)-E(e,\epsilon))$ is two, (2) $E(e,\epsilon)$ is an analytic function of e^2 and ϵ^2, (3) there exists a strongly analytic ground state of $H(e,\epsilon)$.*

From this lemma the following corollary immediately follows.

Corollary 2.2 *Effective mass $m_{\rm eff}$ is analytic of e^2 on $\{e\in\mathbb{R}^3|e^2<e_{00}\}$ with some $e_{00}>0$.*

To prove Lemma 2.1 we need several lemmas. Let

$$\mathcal{F}_0\equiv\mathcal{F}(L^2(\mathbb{R}^3_\kappa\times\{1,2\})), \quad \mathcal{F}_\kappa\equiv\mathcal{F}(L^2(\mathbb{R}^{3\perp}_\kappa\times\{1,2\})),$$

where $\mathbb{R}^3_\kappa=\{k\in\mathbb{R}^3||k|\le\kappa\}$. Then it follows that $\mathcal{F}\cong\mathcal{F}_\kappa\otimes\mathcal{F}_0$ and

$$\mathcal{H}\cong\mathcal{H}_\kappa\otimes\mathcal{F}_0, \tag{2.4}$$

where $\mathcal{H}_\kappa\equiv\mathbb{C}^2\otimes\mathcal{F}_\kappa$. It is seen that $H(e,\epsilon)$ is reduced by $\mathcal{H}_\kappa$. Let

$$K(e,\epsilon)\equiv H(e,\epsilon)\lceil_{\mathcal{H}_\kappa}.$$

Under the identification (2.4), we have

$$H(e,\epsilon)\cong K(e,\epsilon)\otimes 1_{\mathcal{F}_0}+1_{\mathcal{H}_\kappa}\otimes(H_{\rm f}\lceil_{\mathcal{F}_0}). \tag{2.5}$$

Since $\sigma(H_{\rm f}\lceil_{\mathcal{F}_0})=[0,\infty)$, we have by (2.5),

$$\inf\sigma(K(e,\epsilon))=E(e,\epsilon).$$

Proposition 2.3 *(1) It follows that*

$$\inf \sigma_{\rm ess}(K(e,|p|)) - \inf \sigma(K(e,|p|)) \geq \inf_{|k|>\kappa} \{E(e,|p-k|) + \omega(k) - E(e,|p|)\}.$$

In particular, assume that $\inf_{|k|>\kappa}\{E(e,|p-k|)+\omega(k)-E(e,|p|)\}>0$. *Then* $K(e,\pm|p|)$ *has a ground state in* $\mathcal{H}_\kappa$. *(2) The dimension of* Ker $(K(e,\pm|p|) - \inf\sigma(K(e,\pm|p|))$ *is two.*

Proof: (1) and (2) follow in the similar manner to [5, Theorem 2.3] and [9], respectively. □

Since $E(e,\epsilon)$ is continuous in (e,ϵ), and $E(e,|p-k|)+\omega(k)-E(e,|p|) = |k|$ for $(e,p) = (0,0) \in \mathbb{R}\times\mathbb{R}^3$, it is seen that

$$\left\{ (e,p) \in \mathbb{R}\times\mathbb{R}^3 \,\middle|\, \inf_{|k|>\kappa} \{E(e,|p-k|)+\omega(k)-E(e,|p|)\} > 0 \right\} \neq \emptyset.$$

Thus by Proposition 2.3 we have the corollary.

Corollary 2.4 *There exist constants* $\epsilon^* > 0$ *and* $e^* > 0$ *such that* $K(e,\epsilon)$ *has a two-fold ground state for* $(e,\epsilon) \in \mathcal{D}_g \equiv \{(e,\epsilon)\in\mathbb{R}^2 | |e| < e^*, |\epsilon| < \epsilon^*\}$.

Let $\varphi_{{\rm g},\kappa}(e,\epsilon) \in \mathcal{H}_\kappa$ be a ground state of $K(e,\epsilon)$, i.e., $K(e,\epsilon)\varphi_{{\rm g},\kappa}(e,\epsilon) = E(e,\epsilon)\varphi_{{\rm g},\kappa}(e,\epsilon)$. Since

$$e^{i\theta n_z\cdot J} H(e,\epsilon) e^{-i\theta n_z\cdot J} = H(e,\epsilon), \quad \theta\in\mathbb{R}^3,$$

and $e^{i\theta n_z\cdot J}$ is reduced by $\mathcal{H}_\kappa$, it follows that

$$e^{i\theta(p/|p|)\cdot J} K(e,\epsilon) e^{-i\theta(p/|p|)\cdot J} = K(e,\epsilon) \tag{2.6}$$

on $\mathcal{H}_\kappa$. Note that $\sigma(n_z\cdot J) = \mathbb{Z}_{1/2} \equiv \{\pm1/2, \pm3/2, ...\}$ and $\sigma(n_z \cdot J\lceil_{\mathcal{H}_\kappa}) \subset \mathbb{Z}_{1/2}$. Then $K(e,\epsilon)$ and $\mathcal{H}_\kappa$ are decomposed as

$$\mathcal{H}_\kappa = \bigoplus_{z\in\sigma(n\cdot J\lceil_{\mathcal{H}_\kappa})} \mathcal{H}_\kappa(z), \qquad K(e,\epsilon) = \bigoplus_{z\in\sigma(n\cdot J\lceil_{\mathcal{H}_\kappa})} K_z(e,\epsilon).$$

Note that $\mathcal{H}_\kappa(z)$ is independent of ϵ. Let

$$K_-(e,\epsilon) \equiv \bigoplus_{z\leq -1/2} K_z(e,\epsilon), \qquad K_+(e,\epsilon) \equiv \bigoplus_{z\geq 1/2} K_z(e,\epsilon).$$

Then

$$K(e,\epsilon) = K_-(e,\epsilon) \oplus K_+(e,\epsilon).$$

Lemma 2.5 *Assume that* $(e,\epsilon)\in\mathcal{D}_g$. *Then* $K_\pm(e,\epsilon)$ *has a unique ground state* $\varphi_{\mathrm{g},\kappa_\pm}(e,\epsilon)$ *such that* $\varphi_{\mathrm{g},\kappa_\pm}(e,\epsilon)\in\mathcal{H}_\kappa(\pm 1/2)$, *and* $\inf\sigma(K_\pm(e,\epsilon))=E(e,\epsilon)$ *follows, i.e.,*

$$\begin{aligned}&\mathrm{Ker}(K(e,\epsilon)-E(e,\epsilon))\\&=[\mathcal{H}_\kappa(-1/2)\cap\mathrm{Ker}(K(e,\epsilon)-E(e,\epsilon))]\oplus[\mathcal{H}_\kappa(1/2)\cap\mathrm{Ker}(K(e,\epsilon)-E(e,\epsilon))].\end{aligned}$$

Proof: It is proven in the similar manner as [9, Theorem 1.2]. □

Lemma 2.6 *For* $(e,\epsilon)\in\mathcal{D}_g$, $\inf\sigma(K_\pm(e,\epsilon))$ *is an isolated eigenvalue.*

Proof: By Proposition 2.3, $\delta\equiv\inf\sigma_{\mathrm{ess}}(K(e,\epsilon))-\inf\sigma(K(e,\epsilon))>0$, which implies that $\inf\sigma_{\mathrm{ess}}(K_\pm(e,\epsilon))-\inf\sigma(K_\pm(e,\epsilon))\geq\delta>0$. Then the lemma follows. □

Now we are in the position to show analytic properties of $\varphi_{\mathrm{g},\kappa}$ on e.

Lemma 2.7 *There exist constants* $e_0>0$ *and* $\epsilon_0>0$ *such that* $\varphi_{\mathrm{g},\kappa_\pm}(\cdot)$ *is strongly analytic, and* $E(\cdot)$ *analytic on* $\mathcal{D}_a\equiv\{(e,\epsilon)\in\mathbb{R}^2||e|<e_0,|\epsilon|<\epsilon_0\}\subset\mathcal{D}_g$.

Proof: We write $K(e,\epsilon)$ as $K(e,\epsilon)=K_0+K_{\mathrm{I}}$, where

$$\begin{aligned}K_0&\equiv\frac{1}{2}(\epsilon n_z-P_{\mathrm{f}})^2+H_{\mathrm{f}},\\K_{\mathrm{I}}&\equiv -e(\epsilon n_z-P_{\mathrm{f}})\cdot A+\frac{e^2}{2}:AA:-\frac{e}{2}\sigma B.\end{aligned}$$

For (e,ϵ) with both of $|e|$ and $|\epsilon|$ sufficiently small, we see that

$$\|K_{\mathrm{I}}\Psi\|_{\mathcal{H}_\kappa}\leq a\|K_0\Psi\|_{\mathcal{H}_\kappa}+b\|\Psi\|_{\mathcal{H}_\kappa}\tag{2.7}$$

for $\Psi\in D(K_0)$ with $a<1$ and $b\geq 0$. It can be seen by (2.7), [14, p.16 Lemma] and [14, Theorem XII.9] that $K_\pm(e,\epsilon)$ is an analytic family in the sense of Kato. Since $E(e,\epsilon)$ is an isolated non-degenerate eigenvalue of $K_\pm(e,\epsilon)$ by Lemmas 2.5 and 2.6, $\varphi_{\mathrm{g},\kappa_\pm}(e,\epsilon)$ is strongly analytic in e, and $E(e,\epsilon)$ analytic in e by [14, Theorem XII.8]. Analyticity for ϵ is also proven in the similar manner to e. Then the proof is complete. □

Under the identification $\mathcal{H}\cong\mathcal{H}_\kappa\otimes\mathcal{F}_0$, we define

$$\varphi_{\mathrm{g}_\pm}(e,\epsilon)\equiv\varphi_{\mathrm{g},\kappa_\pm}(e,\epsilon)\otimes\Omega_\kappa,\tag{2.8}$$

where Ω_κ denotes the Fock vacuum in $\mathcal{F}_0$.

Proof of Lemma 2.1

Since $H(e,\epsilon)-E(e,\epsilon)=(K(e,\epsilon)-E(e,\epsilon))\otimes 1_{\mathcal{F}_0}+1_{\mathcal{H}_\kappa}\otimes(H_{\rm f}\lceil_{\mathcal{F}_0})$ and the dimension of $\mathrm{Ker}(K(e,\epsilon)-E(e,\epsilon))$ is two and that of $\mathrm{Ker}(H_{\rm f}\lceil_{\mathcal{F}_0})$ is one, it follows that the dimension of $\mathrm{Ker}(H(e,\epsilon)-E(e,\epsilon))$ is two. The analytic property of $\varphi_{{\rm g}_\pm}(\cdot)$ follows from Lemma 2.7 and (2.8). Thus the lemma is proven. □

Let $\epsilon=0$ and set

$$\varphi_{\rm g}(e,0)=\sum_{n=0}^\infty \frac{e^n}{n!}\varphi_{(n)}.$$

Lemma 2.8 *There exists a strongly analytic ground state $\varphi_{\rm g}(e,0)$ such that*

$$\varphi_{(0)}=\varphi_+, \tag{2.9}$$

$$(\varphi_\pm,\varphi_{(n)})_{\mathcal{H}}=0,\quad n=1,2,3. \tag{2.10}$$

Proof: Let $\varphi_{\rm g}(e,0)=\varphi_{{\rm g},\kappa_+}(e,0)\otimes\Omega_\kappa$. Since $\varphi_-\in\mathcal{H}(-1/2)$ and $\varphi_{\rm g}(e,0)\in\mathcal{H}(1/2)$, (2.9) and (2.10) for φ_- follow. We shall prove (2.10) for φ_+. Let $f_{\rm g}(e)=\sum_{n=0}^\infty \frac{a_n}{n!}e^n$ be an analytic function and set $f_{\rm g}(e)\varphi_{\rm g}(e,0)=\sum_{n=0}^\infty \frac{e^n}{n!}\tilde{\varphi}_{(n)}$. We can adjust a_n to satisfy $(\varphi_\pm,\tilde{\varphi}_{(n)})_{\mathcal{H}}=0$ for $n=1,2,3$. Thus redefining $\varphi_{\rm g}(e,0)$ by $f_{\rm g}(e)\varphi_{\rm g}(e,0)$, we have (2.10). □

2.2 Main theorem

By Lemma 2.1, $m/m_{\rm eff}$ is analytic in e near $e=0$. Let

$$m_{\rm eff}/m=\sum_{n=0}^\infty a_n(\Lambda/m)e^{2n}.$$

Theorem 2.9 *There exist constants $b_1>0, b_2>0, c_1>0$ and $c_2>0$ such that*

$$b_1\le\lim_{\Lambda\to\infty}\frac{a_1(\Lambda/m)}{\log(\Lambda/m)}\le b_2, \tag{2.11}$$

$$-c_1\le\lim_{\Lambda\to\infty}\frac{a_2(\Lambda/m)}{(\Lambda/m)^2}\le -c_2. \tag{2.12}$$

Remark 2.10 *This theorem implies that $a_2(\Lambda/m)<0$ for large Λ. Then the reader may think that this contradicts to the fact that $m_{\rm eff}>m$ for $e^2>0$. However, the expansion is asymptotic and the leading term $a_1(\Lambda/m)$ is positive, then this negative sign is not a contradiction.*

To prove this theorem we derive exact forms of $a_2(\Lambda/m)$ in the next section.

2.3 Expansions

In what follows we assume that ground state $\varphi_{\rm g}=\sum_{n=0}^\infty \frac{e^n}{n!}\varphi_{(n)}$ satisfies (2.9) and (2.10). Let us define H, E and $\varphi_{\rm g}$ by

$$\begin{aligned}
&H\equiv H(e,0)=H_0+eH_{\rm I}+\frac{e^2}{2}H_{\rm II},\\
&E\equiv E(e,0)=\inf\sigma(H),\\
&\varphi_{\rm g}\equiv\varphi_{\rm g}(e,0)=\sum_{n=0}^\infty\frac{e^n}{n!}\varphi_{(n)},
\end{aligned}$$

where

$$\begin{aligned}
&H_0\equiv H_{\rm f}+\frac{1}{2}{P_{\rm f}}^2,\\
&H_{\rm I}\equiv H_{\rm I}^{(1)}+H_{\rm I}^{(2)},\quad H_{\rm I}^{(1)}=AP_{\rm f},\quad H_{\rm I}^{(2)}\equiv-\frac{1}{2}\sigma B,\\
&H_{\rm II}\equiv :AA:=A^+A^++2A^+A^-+A^-A^-.
\end{aligned}$$

Here we put

$$A^+\equiv\frac{1}{\sqrt{2}}\sum_{j=1,2}\int\frac{\hat\varphi_m(k)}{\sqrt{\omega(k)}}e(k,j)a^*(k,j)dk,\quad A^-\equiv\frac{1}{\sqrt{2}}\sum_{j=1,2}\int\frac{\hat\varphi_m(k)}{\sqrt{\omega(k)}}e(k,j)a(k,j)dk.$$

Lemma 2.11 *It follows that $E_{(0)}=E_{(2m+1)}=0$, $m\geq 0$, and*

$$\frac{1}{2}E_{(2)}=(\varphi_{(0)},H_{\rm I}^{(2)}\varphi_{(1)})_{\mathcal H}=-(\varphi_{(0)},\left(-\frac{\sigma B}{2}\right)H_0^{-1}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)})_{\mathcal H}\neq 0. \tag{2.13}$$

Moreover

$$\varphi_{(0)}=\varphi_+, \tag{2.14}$$

$$\varphi_{(1)}=-H_0^{-1}H_{\rm I}^{(2)}\varphi_{(0)}, \tag{2.15}$$

$$\varphi_{(2)}=H_0^{-1}(-H_{\rm II}\varphi_{(0)}+2H_{\rm I}H_0^{-1}H_{\rm I}^{(2)}\varphi_{(0)}+E_{(2)}\varphi_{(0)}), \tag{2.16}$$

$$\varphi_{(3)}=3H_0^{-1}(-H_{\rm II}\varphi_{(1)}-H_{\rm I}\varphi_{(2)}+E_{(2)}\varphi_{(1)}). \tag{2.17}$$

In particular $\varphi_{(1)}\in\mathcal F^{(1)}$, $\varphi_{(2)}\in\mathcal F^{(2)}$ and $\varphi_{(3)}\in\mathcal F^{(1)}\oplus\mathcal F^{(3)}$.

Remark 2.12 *Although H_0^{-1} is an unbounded operator, (2.15)-(2.17) are well defined by the fact that*

$$D(H_0^{-1}) \supset \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm fin} \middle| \Psi^{(0)} = 0, {\rm supp}_{k\in\mathbb{R}^{3n}} \Psi^{(n)}(k,j) \not\ni \{0\}, j \in \{1,2\}^n, n \geq 1 \right\}. \tag{2.18}$$

Proof: It is obvious that $E_{(0)} = 0$, and by the unitary equivalence between H and H with e replaced by $-e$, $E_{(2m+1)} = 0$ follows. Let $\varphi_{\rm g}'$ and E' be ${\rm s}-d\varphi_{\rm g}/de$ and dE/de, respectively, and $\Phi \in D(H)$. Then

$$(H\Phi, \varphi_{\rm g})_{\mathcal{H}} = E(\Phi, \varphi_{\rm g})_{\mathcal{H}}. \tag{2.19}$$

Take derivatives in e on the both sides of (2.19). Then we have

$$(H'\Phi, \varphi_{\rm g})_{\mathcal{H}} + (H\Phi, \varphi_{\rm g}')_{\mathcal{H}} = E'(\Phi, \varphi_{\rm g}')_{\mathcal{H}} + E(\Phi, \varphi_{\rm g}')_{\mathcal{H}}, \tag{2.20}$$

$$\begin{aligned}&(H''\Phi, \varphi_{\rm g})_{\mathcal{H}} + 2(H'\Phi, \varphi_{\rm g}')_{\mathcal{H}} + (H\Phi, \varphi_{\rm g}'')_{\mathcal{H}} \\ &\qquad = E''(\Phi, \varphi_{\rm g})_{\mathcal{H}} + 2E'(\Phi, \varphi_{\rm g}')_{\mathcal{H}} + E(\Phi, \varphi_{\rm g}'')_{\mathcal{H}},\end{aligned} \tag{2.21}$$

$$\begin{aligned}&(H'''\Phi, \varphi_{\rm g})_{\mathcal{H}} + 3(H''\Phi, \varphi_{\rm g}')_{\mathcal{H}} + 3(H'\Phi, \varphi_{\rm g}'')_{\mathcal{H}} + (H\Phi, \varphi_{\rm g}''')_{\mathcal{H}} \\ &\qquad = E'''(\Phi, \varphi_{\rm g})_{\mathcal{H}} + 3E''(\Phi, \varphi_{\rm g}')_{\mathcal{H}} + 3E'(\Phi, \varphi_{\rm g}'')_{\mathcal{H}} + E(\Phi, \varphi_{\rm g}''')_{\mathcal{H}},\end{aligned} \tag{2.22}$$

where $H' = H_{\rm I} + eH_{\rm II}$ and $H'' = H_{\rm II}$. By (2.20) we see that $\varphi_{\rm g}' \in D(H)$ with

$$H'\varphi_{\rm g} + H\varphi_{\rm g}' = E'\varphi_{\rm g} + E\varphi_{\rm g}'. \tag{2.23}$$

By (2.21) and (2.22), we also see that $\varphi_{\rm g}'' \in D(H)$ and $\varphi_{\rm g}''' \in D(H)$ with

$$H''\varphi_{\rm g} + 2H'\varphi_{\rm g}' + H\varphi_{\rm g}'' = E''\varphi_{\rm g} + 2E'\varphi_{\rm g}' + E\varphi_{\rm g}'', \tag{2.24}$$

$$H'''\varphi_{\rm g} + 3H''\varphi_{\rm g}' + 3H'\varphi_{\rm g}'' + H\varphi_{\rm g}''' = E'''\varphi_{\rm g} + 3E''\varphi_{\rm g}' + 3E'\varphi_{\rm g}'' + E\varphi_{\rm g}''' \tag{2.25}$$

From (2.24) it follows that

$$(\varphi_{\rm g}, H''\varphi_{\rm g})_{\mathcal{H}} + (\varphi_{\rm g}, 2H'\varphi_{\rm g}')_{\mathcal{H}} + (\varphi_{\rm g}, H\varphi_{\rm g}'')_{\mathcal{H}} = E''(\varphi_{\rm g}, \varphi_{\rm g})_{\mathcal{H}} + (\varphi_{\rm g}, 2E'\varphi_{\rm g}')_{\mathcal{H}} + (\varphi_{\rm g}, E\varphi_{\rm g}'')_{\mathcal{H}}. \tag{2.26}$$

Put $e = 0$ in (2.26). Then

$$(\varphi_{(0)}, H_{\rm II}\varphi_{(0)})_{\mathcal{H}} + (\varphi_{(0)}, 2H_{\rm I}\varphi_{(1)})_{\mathcal{H}} + (\varphi_{(0)}, H_0\varphi_{(2)})_{\mathcal{H}} = E_{(2)}(\varphi_{(0)}, \varphi_{(0)})_{\mathcal{H}}. \tag{2.27}$$

Since the left-hand side of (2.27) equals to $2(\varphi_{(0)}, H_{\mathrm{I}}^{(2)}\varphi_{(1)})_{\mathcal{H}}$, it follows that $E_{(2)} = 2(\varphi_{(0)}, H_{\mathrm{I}}^{(2)}\varphi_{(1)})_{\mathcal{H}}$. By (2.23), (2.24) and (2.25) with $e=0$, we have

$$\begin{aligned}
&H_{\mathrm{I}}^{(2)}\varphi_{(0)} + H_0\varphi_{(1)} = 0,\\
&H_{\mathrm{II}}\varphi_{(0)} + 2H_{\mathrm{I}}\varphi_{(1)} + H_0\varphi_{(2)} = E_{(2)}\varphi_{(0)},\\
&3H_{\mathrm{I}}\varphi_{(2)} + 3H_{\mathrm{II}}\varphi_{(1)} + H_0\varphi_{(3)} = 3E_{(2)}\varphi_{(0)}.
\end{aligned}$$

Then

$$\begin{aligned}
\varphi_{(1)} &= -H_0^{-1}H_{\mathrm{I}}^{(2)}\varphi_{(0)} + a\varphi_+ + b\varphi_-,\\
\varphi_{(2)} &= -H_0^{-1}(H_{\mathrm{II}}\varphi_{(0)} + 2H_{\mathrm{I}}\varphi_{(1)} - E_{(2)}\varphi_{(0)}) + a'\varphi_+ + b'\varphi_-,\\
\varphi_{(3)} &= H_0^{-1}(-3H_{\mathrm{I}}\varphi_{(2)} - 3H_{\mathrm{II}}\varphi_{(1)} + 3E_{(2)}\varphi_{(0)}) + a''\varphi_+ + b''\varphi_-
\end{aligned}$$

with some constants a, b, a', b', a'' and b''. Since we see that $-H_0^{-1}H_{\mathrm{I}}^{(2)}\varphi_{(0)} \perp \mathcal{F}^{(0)}$, $-H_0^{-1}(H_{\mathrm{II}}\varphi_{(0)}+2H_{\mathrm{I}}\varphi_{(1)}-E_{(2)}\varphi_{(0)}) \perp \mathcal{F}^{(0)}$ and $H_0^{-1}(-3H_{\mathrm{I}}\varphi_{(2)}-3H_{\mathrm{II}}\varphi_{(1)}+3E_{(2)}\varphi_{(0)}) \perp \mathcal{F}^{(0)}$, it follows that $a=b=a'=b'=a''=b''=0$ by (2.10). Then (2.15), (2.16) and (2.17) follow. Then the lemma is proven. □

Lemma 2.13 *Let* $\varphi_{\mathrm{g}}' = \mathrm{s}\text{–}\partial\varphi_{\mathrm{g}}(e,\epsilon)/\partial\epsilon\lceil_{\epsilon=0}$. *Then* $(P_{\mathrm{f}} + eA)_{\mu=3}\varphi_{\mathrm{g}} \in D((H-E)^{-1})$ *with*

$$\varphi_{\mathrm{g}}' = (H-E)^{-1}(P_{\mathrm{f}} + eA)_{\mu=3}\varphi_{\mathrm{g}}, \tag{2.28}$$

and

$$\frac{m}{m_{\mathrm{eff}}} = 1 - 2\frac{(\varphi_{\mathrm{g}}, (P_{\mathrm{f}} + eA)_3{\varphi_{\mathrm{g}}}')_{\mathcal{H}}}{(\varphi_{\mathrm{g}}, \varphi_{\mathrm{g}})_{\mathcal{H}}}. \tag{2.29}$$

Proof: Since $H(e,\epsilon)\varphi_{\mathrm{g}}(e,\epsilon) = E(e,\epsilon)\varphi_{\mathrm{g}}(e,\epsilon)$, in the similar manner as (2.23)-(2.25), we have

$$H'(e,\epsilon)\varphi_{\mathrm{g}}(e,\epsilon) + H(e,\epsilon)\varphi_{\mathrm{g}}'(e,\epsilon) = E'(e,\epsilon)\varphi_{\mathrm{g}}(e,\epsilon) + E(e,\epsilon)\varphi_{\mathrm{g}}'(e,\epsilon), \tag{2.30}$$

$$\begin{aligned}
&H''(e,\epsilon)\varphi_{\mathrm{g}}(e,\epsilon) + 2H'(e,\epsilon)\varphi_{\mathrm{g}}'(e,\epsilon) + H(e,\epsilon)\varphi_{\mathrm{g}}''(e,\epsilon)\\
&\qquad = E''(e,\epsilon)\varphi_{\mathrm{g}}(e,\epsilon) + 2E'(e,\epsilon)\varphi_{\mathrm{g}}'(e,\epsilon) + E(e,\epsilon)\varphi_{\mathrm{g}}''(e,\epsilon)
\end{aligned} \tag{2.31}$$

where $\varphi_{\mathrm{g}}'(e,\epsilon) = \mathrm{s}\text{–}\partial\varphi_{\mathrm{g}}(e,\epsilon)/\partial\epsilon$, $H'(e,\epsilon) = -(P_{\mathrm{f}} + eA)_3$ and $H''(e,\epsilon) = 0$. Putting $\epsilon = 0$ in (2.30) and (2.31), we see that

$$(H-E)\varphi_{\mathrm{g}}' = (P_{\mathrm{f}} + eA)_{\mu=3}\varphi_{\mathrm{g}}, \tag{2.32}$$

$$E''\varphi_{\mathrm{g}} = (H-E)\varphi_{\mathrm{g}}'' + \varphi_{\mathrm{g}} - 2(P_{\mathrm{f}} + eA)_3\varphi_{\mathrm{g}}'. \tag{2.33}$$

Since $m/m_{\rm eff} = E''$ by (2.2), we have

$$\frac{m}{m_{\rm eff}} = \frac{(\varphi_{\rm g}, E''\varphi_{\rm g})_{\mathcal H}}{(\varphi_{\rm g}, \varphi_{\rm g})_{\mathcal H}}.$$

Thus the lemma follows from (2.32) and (2.33). $\square$

Remark 2.14 *Combining (2.29) with (2.28) we have*

$$\frac{m}{m_{\rm eff}} = 1 - 2\frac{((P_{\rm f}+eA)_3\varphi_{\rm g}, (H-E)^{-1}(P_{\rm f}+eA)_3\varphi_{\rm g})_{\mathcal H}}{(\varphi_{\rm g}, \varphi_{\rm g})_{\mathcal H}}. \tag{2.34}$$

Or, by a symmetry we can also express $m/m_{\rm eff}$ such as

$$\frac{m}{m_{\rm eff}} = 1 - \frac{2}{3}\sum_{\mu=1}^{3}\frac{((P_{\rm f}+eA)_\mu\varphi_{\rm g}, (H-E)^{-1}(P_{\rm f}+eA)_\mu\varphi_{\rm g})_{\mathcal H}}{(\varphi_{\rm g}, \varphi_{\rm g})_{\mathcal H}}. \tag{2.35}$$

Corollary 2.15 *The effective mass satisfies that $m_{\rm eff} > m$ for $e \neq 0$ but $|e|$ is sufficiently small.*

Proof: Since $((P_{\rm f}+eA)_\mu\varphi_{\rm g}, (H-E)^{-1}(P_{\rm f}+eA)_\mu\varphi_{\rm g})_{\mathcal H} \geq 0$, it follows that $m_{\rm eff} \geq m$ from (2.35). For $e = 0$, $m_{\rm eff} = m$. Since $m_{\rm eff}$ is analytic in e, the corollary follows. $\square$

Theorem 2.16 *The effective mass is expanded as*

$$\frac{m}{m_{\rm eff}} = 1 - \frac{2}{3}c_1(\Lambda/m)e^2 - \frac{2}{3}c_2(\Lambda/m)e^4 + \mathcal{O}(e^6), \tag{2.36}$$

or

$$\frac{m_{\rm eff}}{m} = 1 + \frac{2}{3}c_1(\Lambda/m)e^2 + \left(\frac{2}{3}c_2(\Lambda/m) + \left(\frac{2}{3}\right)^2 c_1(\Lambda/m)^2\right)e^4 + \mathcal{O}(e^6), \tag{2.37}$$

where

$$\begin{aligned} c_1(\Lambda/m) &\equiv \sum_{\mu=1}^{3}(\Psi^\mu_{(1)}, \tilde H_{(0)}\Psi^\mu_{(1)})_{\mathcal H}, \\ c_2(\Lambda/m) &\equiv \sum_{\mu=1}^{3}\Big\{(\Psi^\mu_{(1)}, \tilde H_{(2)}\Psi^\mu_{(1)})_{\mathcal H} - (\Psi^\mu_{(1)}, \tilde H_{(0)}\Psi^\mu_{(1)})_{\mathcal H}(\varphi_{(1)}, \varphi_{(1)})_{\mathcal H} + 2\Re(\Psi^\mu_{(2)}, \tilde H_{(1)}\Psi^\mu_{(1)})_{\mathcal H} \\ &\qquad + (\Psi^\mu_{(2)}, \tilde H_{(0)}\Psi^\mu_{(2)})_{\mathcal H} + 2\Re(\Psi^\mu_{(3)}, \tilde H_{(0)}\Psi^\mu_{(1)})_{\mathcal H}\Big\}. \end{aligned} \tag{2.38}$$

Here

$$\Psi^\mu_{(n)} \equiv \frac{1}{(n-1)!}A_\mu\varphi_{(n-1)} + \frac{1}{n!}P_{{\rm f}\mu}\varphi_{(n)}, \quad n = 1,2,3, \quad \mu = 1,2,3,$$

$$\tilde H_{(0)} = H_0^{-1},\ \tilde H_{(1)} = -H_0^{-1}H_{\rm I}H_0^{-1},\ \tilde H_{(2)} = -\frac{1}{2}H_0^{-1}H_{\rm II}H_0^{-1} + H_0^{-1}(H_{\rm I}H_0^{-1}H_{\rm I} - \left(-\frac{E_{(2)}}{2}\right))H_0^{-1}.$$

Proof: We have

$$\frac{1}{(\varphi_{\rm g},\varphi_{\rm g})_{\mathcal{H}}} = 1 - e^2(\varphi_{(1)},\varphi_{(1)})_{\mathcal{H}} - e^4(\frac{1}{4}(\varphi_{(2)},\varphi_{(2)})_{\mathcal{H}} + \frac{1}{6}(\varphi_{(1)},\varphi_{(3)})_{\mathcal{H}}) + \mathcal{O}(e^6) \tag{2.39}$$

and

$$(P_{\rm f}+eA)_\mu\varphi_{\rm g} = e\Psi^\mu_{(1)} + e^2\Psi^\mu_{(2)} + e^3\Psi^\mu_{(3)} + \mathcal{O}(e^4). \tag{2.40}$$

Note also that $\Psi^\mu_{(1)}\in\mathcal{F}^{(1)}$, $\Psi^\mu_{(2)}\in\mathcal{F}^{(2)}$ and $\Psi^\mu_{(3)}\in\mathcal{F}^{(1)}\oplus\mathcal{F}^{(3)}$. Let us set

$$\varphi'_{\rm g} \equiv \sum_{n=0}^\infty \frac{e^n}{n!}\phi_{(n)}, \quad \mu=1,2,3. \tag{2.41}$$

Substituting (2.39), (2.40) and (2.41) into (2.29), we have

$$\begin{aligned}\frac{m}{m_{\rm eff}} = {}& 1 - 2(\Psi^3_{(1)},\phi_{(1)})_{\mathcal{H}}e^2 \\ & -2\left\{\frac{1}{6}(\Psi^3_{(1)},\phi_{(3)})_{\mathcal{H}} + \frac{1}{2}(\Psi^3_{(2)},\phi_{(2)})_{\mathcal{H}} + \frac{1}{6}(\Psi^3_{(3)},\phi_{(1)})_{\mathcal{H}}\right\}e^4 + \mathcal{O}(e^6).\end{aligned} \tag{2.42}$$

Then we shall see the explicit form of $\phi_{(n)}$ below. The identity

$$((H-E)\Phi,\varphi'_{\rm g})_{\mathcal{H}} = ((P_{\rm f}+eA)_3\Phi,\varphi_{\rm g})_{\mathcal{H}}, \quad \Phi\in D(H), \tag{2.43}$$

is derived from (2.32). Putting $e=0$ in (2.43), we have $H_0\phi_{(0)}=0$, which implies that $\phi_{(0)} = a_0\varphi_- + b_0\varphi_+$ with some constants a_0 and b_0. We differentiate both sides of (2.43) at $e=0$. Then we have

$$\begin{aligned} &H_{\rm I}\phi_{(0)} + H_0\phi_{(1)} = A_3\varphi_{(0)} + P_{\rm f3}\varphi_{(1)}, \\ &(H_{\rm II}-E_{(2)})\phi_{(0)} + 2H_{\rm I}\phi_{(1)} + H_0\phi_{(2)} = 2A_3\varphi_{(1)} + P_{\rm f3}\varphi_{(2)}, \\ &3(H_{\rm II}-E_{(2)})\phi_{(1)} + 3H_{\rm I}\phi_{(2)} + H_0\phi_{(3)} = 3A_3\varphi_{(2)} + P_{\rm f3}\varphi_{(3)},\end{aligned}$$

from which it follows that

$$\begin{aligned} \phi_{(1)} &= H_0^{-1}(A_3\varphi_{(0)} + P_{\rm f3}\varphi_{(1)} - H_{\rm I}\phi_{(0)}) + a_1\varphi_- + b_1\varphi_+, \\ \phi_{(2)} &= H_0^{-1}(2A_3\varphi_{(1)} + P_{\rm f3}\varphi_{(2)} - 2H_{\rm I}\phi_{(1)}) + a_2\varphi_- + b_2\varphi_+, \\ \phi_{(3)} &= H_0^{-1}(3A_3\varphi_{(2)} + P_{\rm f3}\varphi_{(3)} - 3H_{\rm I}\phi_{(2)} - 3(H_{\rm II}-E_{(2)})\phi_{(1)}) + a_3\varphi_- + b_3\varphi_+,\end{aligned}$$

where a_j, b_j, $j = 1, 2, 3$, are some constants. Then it is obtained that

$$\phi_{(1)} = H_0^{-1}\Psi^3_{(1)} + R_1,$$
$$\phi_{(2)} = H_0^{-1}(2\Psi^3_{(2)} - 2H_{\mathrm{I}}H_0^{-1}\Psi^3_{(1)}) + R_2,$$
$$\phi_{(3)} = H_0^{-1}(6\Psi^3_{(3)} - 6H_{\mathrm{I}}H_0^{-1}\Psi^3_{(2)} + 6H_{\mathrm{I}}H_0^{-1}H_{\mathrm{I}}H_0^{-1}\Psi^3_{(1)} - 3(H_{\mathrm{II}} - E_{(2)})\Psi^3_{(1)}) + R_3.$$

Here

$$R_1 \equiv -H_0^{-1}H_{\mathrm{I}}(a_0\varphi_- + b_0\varphi_+) + a_1\varphi_- + b_1\varphi_+,$$
$$R_2 \equiv 2H_0^{-1}H_{\mathrm{I}}H_0^{-1}H_{\mathrm{I}}(a_0\varphi_- + b_0\varphi_+) - 2H_0^{-1}H_{\mathrm{I}}(a_1\varphi_- + b_1\varphi_+) + a_2\varphi_- + b_2\varphi_+,$$
$$\begin{aligned} R_3 \equiv\ & -6H_0^{-1}H_{\mathrm{I}}H_0^{-1}H_{\mathrm{I}}H_0^{-1}H_{\mathrm{I}}(a_0\varphi_- + b_0\varphi_+) + 3(H_{\mathrm{II}} - E_{(2)})H_0^{-1}H_{\mathrm{I}}(a_0\varphi_- + b_0\varphi_+) \\ & +6H_0^{-1}H_{\mathrm{I}}H_0^{-1}H_{\mathrm{I}}(a_1\varphi_- + b_1\varphi_+) - 3H_0^{-1}(H_{\mathrm{II}} - E_{(2)})(a_1\varphi_- + b_1\varphi_+) \\ & -3H_0^{-1}H_{\mathrm{I}}(a_2\varphi_- + b_2\varphi_+) + a_3\varphi_- + b_3\varphi_+. \end{aligned}$$

It is directly seen that the contributions of constants $a_0, b_0, ..., a_3, b_3$ to (2.42) are

$$(\Psi^3_{(1)}, R_1)_{\mathcal{H}} = (\Psi^3_{(1)}, R_3)_{\mathcal{H}} = (\Psi^3_{(2)}, R_2)_{\mathcal{H}} = (\Psi^3_{(3)}, R_1)_{\mathcal{H}} = 0.$$

Then (2.42) coincides with (2.36) by a symmetry. Hence the lemma follows. □

We shall compute $c_2(\Lambda/m)$. We set

$$\tilde{H}_{(0)} \equiv H_1, \quad \tilde{H}_{(1)} \equiv H_2 + H_3, \quad \tilde{H}_{(2)} \equiv H_4 + H_5 + H_6 + H_7 + H_8,$$

where we put

$$H_1 = H_0^{-1}, \quad H_2 = -H_0^{-1}P_{\mathrm{f}}AH_0^{-1}, \quad H_3 = -H_0^{-1}\left(-\frac{\sigma B}{2}\right)H_0^{-1},$$
$$H_4 = \frac{1}{2}H_0^{-1}(-H_{\mathrm{II}})H_0^{-1}, \quad H_5 = H_0^{-1}P_{\mathrm{f}}AH_0^{-1}P_{\mathrm{f}}AH_0^{-1},$$
$$H_6 = H_0^{-1}\left(-\frac{\sigma B}{2}\right)H_0^{-1}P_{\mathrm{f}}AH_0^{-1}, \quad H_7 = H_6{}^* = H_0^{-1}P_{\mathrm{f}}AH_0^{-1}\left(-\frac{\sigma B}{2}\right)H_0^{-1},$$
$$H_8 = H_0^{-1}\left\{\left(-\frac{\sigma B}{2}\right)H_0^{-1}\left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right)\right\}H_0^{-1}.$$

We list groups of vectors by which φ_{g} is expanded:

$$\varphi_{(1)} = -H_0^{-1}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)},$$

$$\begin{aligned}
\varphi_{(2)} &= H_0^{-1}(-H_{\rm II})\varphi_{(0)} + 2H_0^{-1}P_{\rm f}AH_0^{-1}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)} \\
&\quad +2H_0^{-1}\left\{\left(-\frac{\sigma B}{2}\right)H_0^{-1}\left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right)\right\}\varphi_{(0)}, \\
\varphi_{(3)} &= -3H_0^{-1}(-H_{\rm II})H_0^{-1}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)} + 3H_0^{-1}P_{\rm f}AH_0^{-1}(-H_{\rm II})\varphi_{(0)} \\
&\quad +6H_0^{-1}P_{\rm f}AH_0^{-1}P_{\rm f}AH_0^{-1}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)} \\
&\quad +6H_0^{-1}P_{\rm f}AH_0^{-1}\left\{\left(-\frac{\sigma B}{2}\right)H_0^{-1}\left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right)\right\}\varphi_{(0)} \\
&\quad +3H_0^{-1}\left(-\frac{\sigma B}{2}\right)H_0^{-1}(-H_{\rm II})\varphi_{(0)} + 6H_0^{-1}\left(-\frac{\sigma B}{2}\right)H_0^{-1}P_{\rm f}AH_0^{-1}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)} \\
&\quad +6H_0^{-1}\left(-\frac{\sigma B}{2}\right)H_0^{-1}\left\{\left(-\frac{\sigma B}{2}\right)H_0^{-1}\left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right)\right\}\varphi_{(0)}.
\end{aligned}$$

From the above expressions of $\varphi_{(1)}, \varphi_{(2)}, \varphi_{(3)}$, it follows that for $\mu = 1, 2, 3$,

$$\Psi^\mu_{(1)} \equiv \Phi^\mu_{(1)} + \Phi^\mu_{(2)}, \quad \Psi^\mu_{(2)} \equiv \sum_{i=2}^{6}\Phi^\mu_{(i)}, \quad \Psi^\mu_{(3)} \equiv \sum_{i=7}^{16}\Phi^\mu_{(i)},$$

where

$$\begin{aligned}
\Phi^\mu_{(1)} &= A^+_\mu\varphi_{(0)} \\
\Phi^\mu_{(2)} &= \frac{1}{2}P_{{\rm f}\mu}H_0^{-1}\sigma B^+\varphi_{(0)} \\
\Phi^\mu_{(3)} &= \frac{1}{2}A_\mu H_0^{-1}\sigma B^+\varphi_{(0)} \\
\Phi^\mu_{(4)} &= -\frac{1}{2}P_{{\rm f}\mu}H_0^{-1}A^+A^+\varphi_{(0)} \\
\Phi^\mu_{(5)} &= -\frac{1}{2}P_{{\rm f}\mu}H_0^{-1}P_{\rm f}AH_0^{-1}\sigma B^+\varphi_{(0)} \\
\Phi^\mu_{(6)} &= P_{{\rm f}\mu}H_0^{-1}\left\{\left(-\frac{\sigma B}{2}\right)H_0^{-1}\left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right)\right\}\varphi_{(0)} \\
&= \frac{1}{4}P_{{\rm f}\mu}H_0^{-1}\sigma B^+H_0^{-1}\sigma B^+\varphi_{(0)} \\
\Phi^\mu_{(7)} &= -\frac{1}{2}A_\mu H_0^{-1}A^+A^+\varphi_{(0)} \\
\Phi^\mu_{(8)} &= -\frac{1}{2}A_\mu H_0^{-1}P_{\rm f}AH_0^{-1}\sigma B^+\varphi_{(0)} \\
\Phi^\mu_{(9)} &= \frac{1}{2}A_\mu 2H_0^{-1}\left\{\left(-\frac{\sigma B}{2}\right)H_0^{-1}\left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right)\right\}\varphi_{(0)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4} A_\mu H_0^{-1} \sigma B^+ H_0^{-1} \sigma B^+ \varphi_{(0)} \\
\Phi^\mu_{(10)} &= -\frac{1}{4} P_{\mathrm{f}\mu} H_0^{-1} :AA: H_0^{-1} \sigma B^+ \varphi_{(0)} \\
\Phi^\mu_{(11)} &= -\frac{1}{2} P_{\mathrm{f}\mu} H_0^{-1} P_\mathrm{f} A H_0^{-1} A^+ A^+ \varphi_{(0)} \\
\Phi^\mu_{(12)} &= -\frac{1}{2} P_{\mathrm{f}\mu} H_0^{-1} P_\mathrm{f} A H_0^{-1} P_\mathrm{f} A H_0^{-1} \sigma B^+ \varphi_{(0)} \\
\Phi^\mu_{(13)} &= \frac{1}{6} P_{\mathrm{f}\mu} 3 H_0^{-1} P_\mathrm{f} A 2 H_0^{-1} \left\{ \left(-\frac{\sigma B}{2}\right) H_0^{-1} \left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right) \right\} \varphi_{(0)} \\
&= \frac{1}{4} P_{\mathrm{f}\mu} H_0^{-1} P_\mathrm{f} A H_0^{-1} \sigma B^+ H_0^{-1} \sigma B^+ \varphi_{(0)} \\
\Phi^\mu_{(14)} &= \frac{1}{4} P_{\mathrm{f}\mu} H_0^{-1} \sigma B H_0^{-1} A^+ A^+ \varphi_{(0)} \\
\Phi^\mu_{(15)} &= \frac{1}{4} P_{\mathrm{f}\mu} H_0^{-1} \sigma B H_0^{-1} P_\mathrm{f} A H_0^{-1} \sigma B^+ \varphi_{(0)} \\
\Phi^\mu_{(16)} &= \frac{1}{6} P_{\mathrm{f}\mu} 3 H_0^{-1} \left(-\frac{\sigma B}{2}\right) 2 H_0^{-1} \left\{ \left(-\frac{\sigma B}{2}\right) H_0^{-1} \left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right) \right\} \varphi_{(0)} \\
&= -\frac{1}{8} P_{\mathrm{f}\mu} H_0^{-1} \sigma B H_0^{-1} \sigma B^+ H_0^{-1} \sigma B^+ \varphi_{(0)}
\end{aligned}
$$

Remark 2.17 *We give a remark on terms $\Phi^\mu_{(6)}, \Phi^\mu_{(9)}, \Phi^\mu_{(13)}, \Phi^\mu_{(16)}$. Since by (2.13)*

$$
(\varphi_{(0)}, \left\{ \left(-\frac{\sigma B}{2}\right) H_0^{-1} \left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right) \right\} \varphi_{(0)})_\mathcal{H} = 0,
$$

we see that $\left\{ \left(-\frac{\sigma B}{2}\right) H_0^{-1} \left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right) \right\} \varphi_{(0)}$ *is perpendicular to $\mathcal{F}^{(0)}$. Thus we have*

$$
\left\{ \left(-\frac{\sigma B}{2}\right) H_0^{-1} \left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right) \right\} \varphi_{(0)} = \frac{1}{4} \sigma B^+ H_0^{-1} \sigma B^+ \varphi_{(0)} \in \mathcal{F}^{(2)}.
$$

Substituting $H_1, ..., H_8$ and $\Phi^\mu_{(1)}, ..., \Phi^\mu_{(16)}$ into (2.38), we have the lemma below:

Lemma 2.18 *The coefficient $c_2(\Lambda/m)$ is given by*

$$
\begin{aligned}
c_2(\Lambda/m) = \sum_{\mu=1}^{3} \Bigg\{ & (\sum_{i=1}^{2} \Phi^\mu_{(i)}, \sum_{\ell=4}^{8} H_\ell \sum_{i=1}^{2} \Phi^\mu_{(i)})_\mathcal{H} + (\sum_{i=1}^{2} \Phi^\mu_{(i)}, H_1 \sum_{i=1}^{2} \Phi^\mu_{(i)})_\mathcal{H} \\
& (\sum_{i=3}^{6} \Phi^\mu_{(i)}, (H_2 + H_3) \sum_{i=1}^{2} \Phi^\mu_{(i)})_\mathcal{H} + (\sum_{i=3}^{6} \Phi^\mu_{(i)}, H_1 \sum_{i=3}^{6} \Phi^\mu_{(i)})_\mathcal{H} + (\sum_{i=7}^{16} \Phi^\mu_{(i)}, H_1 \sum_{i=1}^{2} \Phi^\mu_{(i)})_\mathcal{H} \Bigg\} . \quad (2.44)
\end{aligned}
$$

From (2.44) we see that $c_2(\Lambda/m)$ is decomposed into 76 terms.

2.4 Classification of the Divergent Terms

Though we have 76 terms of the form $(\Phi^\mu_{(\ell)}, H_m\Phi^\mu_{(n)})_{\mathcal{H}}$ in $c_2(\Lambda/m)$, not all of them are important. Namely $(\Phi^\mu_{(\ell)}, H_m\Phi^\mu_{(n)})_{\mathcal{H}}$ and its complex conjugate $(\Phi^\mu_{(n)}, H^*_m\Phi^\mu_{(\ell)})_{\mathcal{H}}$ appear and we take the real part of them.

Lemma 2.19 *It follows that*
$$\sum_{\substack{(\Phi^\mu_{(\ell)},\mathrm{H_m}\Phi^\mu_{(\mathrm{n})})_{\mathcal{H}} \text{ contains} \\ \text{odd number of } \sigma\mathrm{B}}} (\Phi^\mu_{(\ell)}, H_m\Phi^\mu_{(n)})_{\mathcal{H}} = 0.$$

Proof: Let $\gamma \equiv C\sigma_3$, where C stands for a complex-conjugacy operator on $\mathcal{F}$: $C^2 = 1$, $CJ = -JC$, $J = \sqrt{-1}$. We choose $\varphi_{(0)}$ so that $\gamma\varphi_{(0)} = \varphi_{(0)}$. Then γ is an anti-unitary operator such that $\gamma\sigma_\mu\gamma^* = \sigma_\mu$, $\gamma A_\mu\gamma^* = A_\mu$, $\gamma B_\mu\gamma^* = -B_\mu$ and $(\psi,\phi)_{\mathcal{F}} = (\gamma\phi,\gamma\psi)_{\mathcal{F}}$. Thus terms containing odd number of σB are purely imaginary. Then the sum of contributions of terms containing odd number of σB is zero. The proof is complete. □

By this lemma it is enough to consider terms containing even number of σB's. Our results are summarized on the lists for $c_2(\Lambda/m)$ which consist of five groups. The columns of σB (resp. P_{f}, H_0^{-1}) mean the number of σB (resp. P_{f}, H_0^{-1}) in term $(\Phi^\mu_{(\ell)}, H_m\Phi^\mu_{(n)})$. In Tables 1-5, we exhibit only the terms containing even number of σB. On these lists, order means an upper or a lower bound for the leading divergent term of the expectation values $(\Phi_{(\ell)}, H_{(m)}\Phi_{(n)})$ listed on the row. Note that they are one of $[\log(\Lambda/m)]^2$, $\sqrt{\Lambda/m}$ and $(\Lambda/m)^2$.

3 Estimates of 38 terms

We classify these 38 terms into 3 types by the numbers of σB's. Type I consists of terms which contain no σB and thus has been already calculated in [8]. Type II consists of terms which contain two σB's and two A_μ's. Finally Type III consists of terms which contain four σB's only, and diverges most strongly.

3.1 Terms with four σB's or $\frac{1}{2}E_{(2)}$

Though the conventional power-counting theorem says that these 38 quantities diverge at most like $[\log(\Lambda/m)]^2$ as $\Lambda \to \infty$, this does not work in some cases. The first violation

No.	Term	σB	$P_{\rm f}$	H_0^{-1}	Order
(1)	$-(\Phi^{\mu}_{(1)}, H_1\Phi^{\mu}_{(1)})_{\mathcal{H}}(\varphi_{(1)}, \varphi_{(1)})_{\mathcal{H}}$	2	0	3	$[\log(\Lambda/m)]^2$
(2)	$-(\Phi^{\mu}_{(2)}, H_1\Phi^{\mu}_{(2)})_{\mathcal{H}}(\varphi_{(1)}, \varphi_{(1)})_{\mathcal{H}}$	4	2	5	$[\log(\Lambda/m)]^2$

Table 1: $(\sum_{i=1}^{2} \Phi^{\mu}_{(i)}, H_1 \sum_{i=1}^{2} \Phi^{\mu}_{(i)})_{\mathcal{H}}$

No.	Term	σB	$P_{\rm f}$	H_0^{-1}	Order
(1)	$(\Phi^{\mu}_{(3)}, H_1\Phi^{\mu}_{(3)})_{\mathcal{H}}$	2	0	3	$[\log(\Lambda/m)]^2$
(2)	$(\Phi^{\mu}_{(5)}, H_1\Phi^{\mu}_{(3)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(3)	$(\Phi^{\mu}_{(3)}, H_1\Phi^{\mu}_{(5)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(4)	$(\Phi^{\mu}_{(4)}, H_1\Phi^{\mu}_{(4)})_{\mathcal{H}}$	0	2	3	$+\sqrt{\Lambda/m}$
(5)	$(\Phi^{\mu}_{(6)}, H_1\Phi^{\mu}_{(4)})_{\mathcal{H}}$	2	2	4	$-\sqrt{\Lambda/m}$
(6)	$(\Phi^{\mu}_{(4)}, H_1\Phi^{\mu}_{(6)})_{\mathcal{H}}$	2	2	4	$-\sqrt{\Lambda/m}$
(7)	$(\Phi^{\mu}_{(5)}, H_1\Phi^{\mu}_{(5)})_{\mathcal{H}}$	2	4	5	$+\sqrt{\Lambda/m}$
(8)	$(\Phi^{\mu}_{(6)}, H_1\Phi^{\mu}_{(6)})_{\mathcal{H}}$	4	2	5	$[\log(\Lambda/m)]^2$

Table 2: $(\sum_{i=3}^{6} \Phi^{\mu}_{(i)}, H_1 \sum_{i=3}^{6} \Phi^{\mu}_{(i)})_{\mathcal{H}}$

No.	Term	σB	$P_{\rm f}$	H_0^{-1}	Order
(1)	$(\Phi^{\mu}_{(1)}, H_4\Phi^{\mu}_{(1)})_{\mathcal{H}}$	0	0	2	$[\log(\Lambda/m)]^2$
(2)	$(\Phi^{\mu}_{(2)}, H_4\Phi^{\mu}_{(2)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(3)	$(\Phi^{\mu}_{(1)}, H_5\Phi^{\mu}_{(1)})_{\mathcal{H}}$	0	2	3	$\log(\Lambda/m)$
(4)	$(\Phi^{\mu}_{(2)}, H_5\Phi^{\mu}_{(2)})_{\mathcal{H}}$	2	4	5	$[\log(\Lambda/m)]^2$
(5)	$(\Phi^{\mu}_{(2)}, H_6\Phi^{\mu}_{(1)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(6)	$(\Phi^{\mu}_{(1)}, H_6\Phi^{\mu}_{(2)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(7)	$(\Phi^{\mu}_{(2)}, H_7\Phi^{\mu}_{(1)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(8)	$(\Phi^{\mu}_{(1)}, H_7\Phi^{\mu}_{(2)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(9)	$(\Phi^{\mu}_{(1)}, H_8\Phi^{\mu}_{(1)})_{\mathcal{H}}$	2	0	3	$[\log(\Lambda/m)]^2$
(10)	$(\Phi^{\mu}_{(2)}, H_8\Phi^{\mu}_{(2)})_{\mathcal{H}}$	4	2	5	$[\log(\Lambda/m)]^2$

Table 3: $(\sum_{i=1}^{2} \Phi^{\mu}_{(i)}, \sum_{\ell=4}^{8} H_\ell \sum_{i=1}^{2} \Phi^{\mu}_{(i)})_{\mathcal{H}}$

No.	Term	σB	P_{f}	H_0^{-1}	Order
(1)	$(\Phi^\mu_{(4)}, H_2\Phi^\mu_{(1)})_{\mathcal{H}}$	0	2	3	$\log(\Lambda/m)$
(2)	$(\Phi^\mu_{(6)}, H_2\Phi^\mu_{(1)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(3)	$(\Phi^\mu_{(3)}, H_2\Phi^\mu_{(2)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(4)	$(\Phi^\mu_{(5)}, H_2\Phi^\mu_{(2)})_{\mathcal{H}}$	2	4	5	$\sqrt{\Lambda/m}$
(5)	$(\Phi^\mu_{(3)}, H_3\Phi^\mu_{(1)})_{\mathcal{H}}$	2	0	3	$[\log(\Lambda/m)]^2$
(6)	$(\Phi^\mu_{(5)}, H_3\Phi^\mu_{(1)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(7)	$(\Phi^\mu_{(4)}, H_3\Phi^\mu_{(2)})_{\mathcal{H}}$	2	2	4	$\sqrt{\Lambda/m}$
(8)	$(\Phi^\mu_{(6)}, H_3\Phi^\mu_{(2)})_{\mathcal{H}}$	4	2	5	$-\sqrt{\Lambda/m}$

Table 4: $(\sum_{i=3}^{6}\Phi^\mu_{(i)}, (H_2+H_3)(\sum_{i=1}^{2}\Phi^\mu_{(i)}))_{\mathcal{H}}$

No.	Term	σB	P_{f}	H_0^{-1}	Order
(1)	$(\Phi^\mu_{(7)}, H_1\Phi^\mu_{(1)})_{\mathcal{H}}$	0	0	2	$[\log(\Lambda/m)]^2$
(2)	$(\Phi^\mu_{(9)}, H_1\Phi^\mu_{(1)})_{\mathcal{H}}$	2	0	3	$[\log(\Lambda/m)]^2$
(3)	$(\Phi^\mu_{(11)}, H_1\Phi^\mu_{(1)})_{\mathcal{H}}$	0	2	3	$= 0$
(4)	$(\Phi^\mu_{(13)}, H_1\Phi^\mu_{(1)})_{\mathcal{H}}$	2	2	4	$= 0$
(5)	$(\Phi^\mu_{(15)}, H_1\Phi^\mu_{(1)})_{\mathcal{H}}$	2	2	4	$= 0$
(6)	$(\Phi^\mu_{(8)}, H_1\Phi^\mu_{(2)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(7)	$(\Phi^\mu_{(10)}, H_1\Phi^\mu_{(2)})_{\mathcal{H}}$	2	2	4	$= 0$
(8)	$(\Phi^\mu_{(12)}, H_1\Phi^\mu_{(2)})_{\mathcal{H}}$	2	4	5	$[\log(\Lambda/m)]^2$
(9)	$(\Phi^\mu_{(14)}, H_1\Phi^\mu_{(2)})_{\mathcal{H}}$	2	2	4	$[\log(\Lambda/m)]^2$
(10)	$(\Phi^\mu_{(16)}, H_1\Phi^\mu_{(2)})_{\mathcal{H}}$	4	2	5	$-(\Lambda/m)^2$

Table 5: $(\sum_{i=7}^{16}\Phi^\mu_{(i)}, H_1\sum_{i=1}^{2}\Phi^\mu_{(i)})_{\mathcal{H}}$

takes place in the region where two momenta $k_1 \in \mathbb{R}^3$ and $k_2 \in \mathbb{R}^3$ of two photons have opposite directions and $|k_1+k_2|$ becomes small. Such a violation yields $\sqrt{\Lambda/m}$ divergence, which is investigated in [9]. The second violation takes place when the integrand is far from symmetric; each term $(\Phi^\mu_{(i)}, H_j\Phi^\mu_{(k)})_{\mathcal{H}}$, roughly speaking, has the form

$$\mathcal{E} = \int_{\kappa/m\le|k_1|,|k_2|\le\Lambda/m} dk_1dk_2|k_1|^{-a}|k_2|^{-b},$$

where $1 \le a$, $1 \le b$, $a+b=6$ (see Appendix). Then

$$c_1[\log(\Lambda/m)]^2 \le \mathcal{E} \le c_2(\Lambda/m)^2$$

depending on a and b, with some constants c_1 and c_2. Thus we restrict ourselves to the most singular and complicated contributions of order e^4, which come from the terms of type III and $E_{(2)}/2$. See Table 6. Note that $E_{(2)}/2 \approx -(\Lambda/m)^2$ as $\Lambda\to\infty$ which may be, however, regarded as a subtraction needed to *Wick order* $\left(-\frac{\sigma B}{2}\right) H_0^{-1} \left(-\frac{\sigma B}{2}\right)$. Actually all the term $\Phi^\mu_{(i)}$ has no $E_{(2)}/2$, but only H_8 contains it.

Term	σB	$P_{\rm f}$	H_0^{-1}
$\mathcal{E}_0 = (\Phi^\mu_{(1)}, H_8\Phi^\mu_{(1)})_{\mathcal{H}}$	2	0	3
$\mathcal{E}_1 = (\Phi^\mu_{(6)}, H_3\Phi^\mu_{(2)})_{\mathcal{H}}$	4	2	5
$\mathcal{E}_2 = (\Phi^\mu_{(6)}, H_1\Phi^\mu_{(6)})_{\mathcal{H}}$	4	2	5
$\mathcal{E}_3 = (\Phi^\mu_{(2)}, H_8\Phi^\mu_{(2)})_{\mathcal{H}}$	4	2	5
$\mathcal{E}_4 = (\Phi^\mu_{(16)}, H_1\Phi^\mu_{(2)})_{\mathcal{H}}$	4	2	5

Table 6: Terms with four σB and $E_{(2)}/2$

3.2 Lower and upper bounds

Lemma 3.1 *The 35 terms except for $\mathcal{E}_0$, $\mathcal{E}_3$ and $\mathcal{E}_4$ satisfy that*

$$\lim_{\Lambda\to\infty} \frac{|(\Phi^\mu_{(i)}, H_j\Phi^\mu_{(k)})_{\mathcal{H}}|}{\sqrt{\Lambda/m}} < \infty, \quad \mu=1,2,3,$$

where $(i,j,k) \ne (1,8,1),(2,8,2),(16,1,2)$.

Though this lemma can be proven in principle in the similar manner to [9], we sketch the outline of our calculations in the Appendix for the sake of the reader who may not be familiar with the perturbative calculations of the field theory. We recommend the reader to read the Appendix in advance.

Remark 3.2 *Each upper bound of* $(\Phi^\mu_{(i)}, H_j \Phi^\mu_{(k)})_{\mathcal{H}}$ *is listed in Tables 1-5. We note that* $(\Phi^\mu_{(i)}, H_1 \Phi^\mu_{(1)})$, $i = 11, 13, 15$ *in Table 5 are automatically zero since* $P_{\mathrm{f}} A \varphi_{(0)} = A P_{\mathrm{f}} \varphi_{(0)} = 0$. *Moreover* $(\Phi^\mu_{(10)}, H_1 \Phi^\mu_{(2)})_{\mathcal{H}} = 0$ *since the integrand is odd in momentum* k_i.

We write down explicit forms of $\mathcal{E}_0, \mathcal{E}_3$ and $\mathcal{E}_4$ which contain most singular divergence:

$$\begin{aligned}
\mathcal{E}_0 &= \frac{1}{4}\sum_{\mu=1}^{3}(\varphi_{(0)}, A^-_\mu H_0^{-1}\left\{\sigma B H_0^{-1}\sigma B - 4\left(-\frac{E_{(2)}}{2}\right)\right\} H_0^{-1} A^+_\mu \varphi_{(0)})_{\mathcal{H}} \\
&= \frac{1}{4}\sum_{\mu=1}^{3}(\varphi_{(0)}, A^-_\mu H_0^{-1}\sigma B^- H_0^{-1}\sigma B^+ H_0^{-1} A^+_\mu \varphi_{(0)})_{\mathcal{H}} - \left(-\frac{E_{(2)}}{2}\right) E_a, \\
\mathcal{E}_3 &= \frac{1}{16}\sum_{\mu=1}^{3}(\varphi_{(0)}, \sigma B^- H_0^{-1} P_{\mathrm{f}\mu} H_0^{-1}\left\{\sigma B H_0^{-1}\sigma B - 4\left(-\frac{E_{(2)}}{2}\right)\right\} H_0^{-1} P_{\mathrm{f}\mu} H_0^{-1}\sigma B^+ \varphi_{(0)})_{\mathcal{H}} \\
&= \frac{1}{16}\sum_{\mu=1}^{3}(\varphi_{(0)}, \sigma B^- P_{\mathrm{f}\mu}(H_0^{-1})^2 \sigma B^- H_0^{-1}\sigma B^+ (H_0^{-1})^2 P_{\mathrm{f}\mu}\sigma B^+ \varphi_{(0)})_{\mathcal{H}} - \frac{1}{4}\left(-\frac{E_{(2)}}{2}\right) E_b, \\
\mathcal{E}_4 &= -\frac{1}{16}(\varphi_{(0)}, \sigma B^- (H_0^{-1})^3 P_{\mathrm{f}}{}^2 \sigma B^- H_0^{-1}\sigma B^+ H_0^{-1}\sigma B^+ \varphi_{(0)})_{\mathcal{H}}.
\end{aligned}$$

Here we have used that the contribution from $\sigma B^+ H_0^{-1} \sigma B^-$ in $\{\cdots\}$ in $\mathcal{E}_1$ and $\mathcal{E}_3$ vanishes, since $\sigma B^- H_0^{-1} A^+_\mu \varphi_{(0)} = 0$ and $\sigma B^- (H_0^{-1})^2 P_{\mathrm{f}\mu} \sigma B^+ \varphi_{(0)} = 0$, and we put

$$\begin{aligned}
-\frac{E_{(2)}}{2} &= \frac{1}{4}\sum_{\mu=1}^{3}(\varphi_{(0)}, \sigma B^-_\mu H_0^{-1}\sigma B^+_\mu \varphi_{(0)})_{\mathcal{H}}, \\
E_a &\equiv \sum_{\mu=1}^{3}(\varphi_{(0)}, A^-_\mu (H_0^{-1})^2 A^+_\mu \varphi_{(0)})_{\mathcal{H}}, \\
E_b &\equiv \sum_{\mu=1}^{3}(\varphi_{(0)}, \sigma B^-_\mu (H_0^{-1})^4 P_{\mathrm{f}}{}^2 \sigma B^+_\mu \varphi_{(0)})_{\mathcal{H}}.
\end{aligned}$$

Lemma 3.3 *Asymptotic behaviors of* $E_{(2)}$, E_a *and* E_b *are (1)* $\lim_{\Lambda\to\infty}\left|\frac{E_{(2)}}{(\Lambda/m)^2}\right| < \infty$, *(2)* $\lim_{\Lambda\to\infty} E_a < \infty$, *(3)* $\lim_{\Lambda\to\infty} E_b < \infty$.

Proof: Let us express the expectation value of 2×2 matrix O with respect to $\binom{1}{0}$ by

$$\langle O \rangle_E \equiv (\binom{1}{0}, O\binom{1}{0}).$$

Using the pull-through formula etc. (see Appendix), we see that

$$\begin{aligned} E_{(2)} &= -\frac{1}{2}\sum_{\ell=1,2}\int_{\kappa/m\le|k|\le\Lambda/m}\frac{1}{(2\pi)^3 2\omega(k)}\frac{\langle(\sigma\cdot[k\times e(k,\ell)])^2\rangle_E}{|k|^2/2+|k|}dk \\ &= -\frac{1}{8\pi^2}\int_{\kappa/m}^{\Lambda/m}\frac{r^2}{r+2}dr. \end{aligned}$$

Then (1) follows. (2) and (3) follow from

$$\lim_{\Lambda\to\infty}|E_a|\le \text{const.}\times\int_{\kappa/m}^{\infty}\frac{1}{r(r+2)^2}dr<\infty$$

and

$$\lim_{\Lambda\to\infty}|E_b|\le \text{const.}\times\int_{\kappa/m}^{\infty}\frac{1}{r(r+2)^2}dr<\infty.$$

Thus the lemma is proven. □

Using formulae

$$(\sigma\cdot p)(\sigma\cdot q)=-(\sigma\cdot q)(\sigma\cdot p)+2(p,q)=(p,q)+i\sigma\cdot(p\times q),\quad (\sigma\cdot p)^2=|p|^2, \tag{3.1}$$

$$(k_1\times e(k_1,\ell_1),k_2\times e(k_2,\ell_2))=(k_1,k_2)(e(k_1,\ell_1),e(k_2,\ell_2))-(k_1,e(k_2,\ell_2))(k_2,e(k_1,\ell_1)), \tag{3.2}$$

$$\sum_{\ell_2=1,2}(k_1,e(k_2,\ell_2))^2=|k_1|^2-(k_1,\hat{k}_2)^2,\quad \hat{k}=k/|k| \tag{3.3}$$

and

$$\sum_{\ell_1,\ell_2=1,2}((k_1\times e(k_1,\ell_1),(k_2\times e(k_2,\ell_2))^2=|k_1|^2|k_2|^2(1+(\hat{k}_1,\hat{k}_2)^2)$$

we can directly see that

$$\mathcal{E}_0=\frac{1}{4}\int d\mathbf{k}\frac{1}{|k_1||k_2|}\left\{\frac{4|k_2|^2}{\mathcal{E}(k_1)^2\mathcal{E}(k_1,k_2)}-\frac{4|k_2|^2}{\mathcal{E}(k_1)^2\mathcal{E}(k_2)}+\frac{(k_1,k_2)(1+(\hat{k}_1,\hat{k}_2))}{\mathcal{E}(k_1)\mathcal{E}(k_2)\mathcal{E}(k_1,k_2)}\right\} \tag{3.4}$$

$$\begin{aligned} \mathcal{E}_3 &= \frac{1}{16}\int d\mathbf{k}\frac{1}{|k_1||k_2|}\left\{\frac{4|k_1|^4|k_2|^2}{\mathcal{E}(k_1)^4\mathcal{E}(k_1,k_2)}-\frac{4|k_1|^4|k_2|^2}{\mathcal{E}(k_1)^4\mathcal{E}(k_2)}\right. \\ &\qquad \left.+\frac{\left(-2|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)+\langle\sigma(k_1,k_2)\rangle_E\right)(k_1,k_2)}{\mathcal{E}(k_1)\mathcal{E}(k_2)\mathcal{E}(k_1,k_2)^3}\right\}, \\ &= \frac{1}{16}\int d\mathbf{k}\frac{1}{|k_1||k_2|}\left\{\frac{4|k_1|^4|k_2|^2}{\mathcal{E}(k_1)^4\mathcal{E}(k_1,k_2)}-\frac{4|k_1|^4|k_2|^2}{\mathcal{E}(k_1)^4\mathcal{E}(k_2)}\right. \\ &\qquad \left.+\frac{-2|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)(k_1,k_2)}{\mathcal{E}(k_1)\mathcal{E}(k_2)\mathcal{E}(k_1,k_2)^3}\right\}, \end{aligned} \tag{3.5}$$

$$\mathcal{E}_4 = -\frac{1}{16}\int d\mathbf{k}\frac{1}{|k_1||k_2|}\left\{\frac{4|k_1|^4|k_2|^2}{\mathcal{E}(k_1)^4\mathcal{E}(k_1,k_2)} + \frac{\left(-2|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)+\langle\sigma(k_1,k_2)\rangle_E\right)|k_1|^2}{\mathcal{E}(k_1)^3\mathcal{E}(k_2)\mathcal{E}(k_1,k_2)}\right\}, \tag{3.6}$$

where $\int d\mathbf{k} f(k_1,k_2) = \int_{\kappa/m\le|k_1|,|k_2|\le\Lambda/m}\frac{1}{4(2\pi)^6}f(k_1,k_2)dk_1dk_2$ and

$$\begin{aligned}\mathcal{E}(k) &= \frac{1}{2}|k|^2+\omega(k),\\ \mathcal{E}(k_1,k_2) &= \frac{1}{2}|k_1+k_2|^2+\omega(k_1)+\omega(k_2).\end{aligned}$$

Moreover

$$\sigma(k_1,k_2) \equiv \sum_{\ell_1,\ell_2=1,2}\sigma\cdot[(k_1\times e(k_1,\ell_1))\times(k_2\times e(k_2,\ell_2))]((k_1\times e(k_1,\ell_1))\cdot(k_2\times e(k_2,\ell_2)).$$

It is seen that $\sigma(k_1,k_2) = -\sigma(k_2,k_1)$, $\sigma(k_1,k_2)=\sigma(-k_1,k_2)$ and $\sigma(k_1,k_2)=\sigma(k_1,-k_2)$, from which it follows that

$$\int d\mathbf{k}\frac{1}{|k_1||k_2|}\frac{(k_1,k_2)\langle\sigma(k_1,k_2)\rangle_E}{\mathcal{E}(k_1)\mathcal{E}(k_2)\mathcal{E}(k_1,k_2)^3} = 0.$$

We used this fact in the computations of $\mathcal{E}_3$.

3.3 Origin of the Λ^2 Divergence of m_{eff}

The term $\mathcal{E}_3$ has the term $-\frac{1}{2}E_{(2)}E_b$ which diverges as $(\Lambda/m)^2$ as $\Lambda\to\infty$. We intuitively see that this divergence is canceled in the way mentioned below. The first term of the integrand of $\mathcal{E}_3$ behaves as

$$\frac{1}{|k_1||k_2|}\frac{4|k_1|^4|k_2|^2}{\mathcal{E}(k_1)^4\mathcal{E}(k_1,k_2)} \approx \frac{1}{|k_1||k_2|}\frac{32}{\mathcal{E}(k_1)^2} \tag{3.7}$$

for $|k_1|/|k_2| \ll 1$, and then

$$\int_{\substack{\kappa/m\,\le\,|k_1|,\,|k_2|\,\le\,\Lambda/m\\ |k_1|/|k_2|\,\ll\,1}}\frac{1}{|k_1||k_2|}\frac{32}{\mathcal{E}(k_1)^2}dk_1dk_2 \approx (\Lambda/m)^2$$

as $\Lambda\to\infty$. On the other hand, this divergence is canceled by $-\frac{1}{2}E_{(2)}E_b$, since it has the integral kernel

$$\frac{1}{|k_1||k_2|}\frac{4|k_1|^4}{\mathcal{E}(k_1)^4}\frac{|k_2|^2}{\mathcal{E}(k_2)} \approx \frac{1}{|k_1||k_2|}\frac{32}{\mathcal{E}(k_1)^2}$$

for $|k_1|/|k_2| \ll 1$. Hence we shall see that $\mathcal{E}_3$ diverges as $\sqrt{\Lambda/m}$ but not $(\Lambda/m)^2$. The divergence $(\Lambda/m)^2$ of $\mathcal{E}_0$ is also canceled by $-\frac{1}{2}E_{(2)}E_a$. The integrand of $\mathcal{E}_4$ has the same kernel as (3.7). The divergence $-(\Lambda/m)^2$ of $\mathcal{E}_4$ coming from (3.7), however, remains without being subtracted. Then we obtain that $\mathcal{E}_4 \approx -(\Lambda/m)^2$ as $\Lambda \to \infty$. The actual calculations of $\mathcal{E}_i$ are done by this idea and straightforward.

3.4 Proof of Theorem 2.9

Lemma 3.4 *There exist positive constants c_1 and c_2 such that*

$$-c_1 \le \lim_{\Lambda\to\infty} \frac{\mathcal{E}_4}{(\Lambda/m)^2} \le -c_2. \tag{3.8}$$

Lemma 3.5 *It follows that*

$$\lim_{\Lambda\to\infty} \frac{\mathcal{E}_0}{[\log(\Lambda/m)]^2} < \infty, \tag{3.9}$$

and

$$\lim_{\Lambda\to\infty} \frac{\mathcal{E}_3}{[\log(\Lambda/m)]^2} < \infty. \tag{3.10}$$

Proof of Theorem 2.9

It is obtained that

$$\begin{aligned} a_1(\Lambda/m) &= (2/3)c_1(\Lambda/m), \\ a_2(\Lambda/m) &= (2/3)c_2(\Lambda/m) + (2/3)^2 c_1(\Lambda/m)^2 \end{aligned} \tag{3.11}$$

in Theorem 2.16. A direct calculation yields that

$$c_1(\Lambda/m) = \sum_{\mu=1}^{3}((A_\mu + P_{\mathrm{f}} H_0^{-1}\left(-\frac{\sigma B}{2}\right))\varphi_{(0)}, H_0^{-1}(A_\mu + P_{\mathrm{f}} H_0^{-1}\left(-\frac{\sigma B}{2}\right))\varphi_{(0)})_{\mathcal{H}} \tag{3.12}$$

$$= \sum_{\mu=1}^{3}(A_\mu\varphi_{(0)}, H_0^{-1}A_\mu\varphi_{(0)})_{\mathcal{H}} + \sum_{\mu=1}^{3}(P_{\mathrm{f}\mu} H_0^{-1}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)}, H_0^{-1}P_{\mathrm{f}\mu} H_0^{-1}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)})_{\mathcal{H}}$$

$$= 2\int \frac{\hat{\varphi}_m(k)}{2(2\pi)^3\omega(k)} \frac{1}{\frac{1}{2}|k|^2+\omega(k)} dk + \frac{1}{4}\int \frac{\hat{\varphi}_m(k)}{2(2\pi)^3\omega(k)} \frac{2|k|^2}{(\frac{1}{2}|k|^2+\omega(k))^3} dk$$

$$= \frac{4\pi}{(2\pi)^3}\int_{\kappa/m}^{\Lambda/m} \frac{r}{\frac{1}{2}r^2+r} dr + \frac{1}{4}\frac{4\pi}{(2\pi)^3}\int_{\kappa/m}^{\Lambda/m} \frac{r^5}{(\frac{1}{2}r^2+r)^3} dr.$$

Thus we have

$$a_1(\Lambda/m) = \frac{2}{3}\frac{4\pi}{(2\pi)^3}\left(\int_{\kappa/m}^{\Lambda/m} \frac{r}{\frac{1}{2}r^2+r} dr + \frac{1}{4}\int_{\kappa/m}^{\Lambda/m} \frac{r^5}{(\frac{1}{2}r^2+r)^3} dr\right) \tag{3.13}$$

and there exist positive constants b_1 and b_2 such that

$$b_1 \le \lim_{\Lambda\to\infty} \frac{a_1(\Lambda/m)}{\log(\Lambda/m)} \le b_2. \tag{3.14}$$

Our analysis in Lemmas 3.1, 3.4 and 3.5 implies that there exist constants c_1 and c_2 such that

$$-c_1 \le \lim_{\Lambda\to\infty} \frac{2\Re(\Phi^\mu_{(16)}, H_1\Phi^\mu_{(2)})_{\mathcal{H}}}{(\Lambda/m)^2} \le -c_2$$

and

$$\lim_{\Lambda\to\infty} \frac{(\Phi^\mu_{(i)}, H_j\Phi^\mu_{(k)})_{\mathcal{H}}}{(\Lambda/m)^2} = 0$$

for $(i,j,k) \neq (16,1,2)$. Thus by (2.44) we have

$$-c_1 \le \lim_{\Lambda\to\infty} \frac{c_2(\Lambda/m)}{(\Lambda/m)^2} \le -c_2. \tag{3.15}$$

Hence

$$-c_1 \le \lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m)}{(\Lambda/m)^2} \le -c_2$$

follows from (3.11), (3.14) and (3.15). $\square$

From (3.12) we see that e^2 order of the effective mass, $m_{\text{eff}}^{\text{spinless}}$, for the spinless Pauli-Fierz Hamiltonian is $\frac{2}{3}\frac{4\pi}{(2\pi)^3}\int_{\kappa/m}^{\Lambda/m} \frac{r}{\frac{1}{2}r^2+r}dr$. By (3.13) we have a corollary.

Corollary 3.6 *It follows that $m_{\text{eff}} > m_{\text{eff}}^{\text{spinless}}$ for e with a sufficiently small $|e|$ but $\neq 0$.*

Proof: We have

$$\frac{d}{de^2}(m_{\text{eff}} - m_{\text{eff}}^{\text{spinless}})\Big\lceil_{e^2=0} = \frac{1}{6}\frac{4\pi}{(2\pi)^3}\int_{\kappa/m}^{\Lambda/m} \frac{r^5}{(\frac{1}{2}r^2+r)^3}dr > 0.$$

Since $m_{\text{eff}} = m_{\text{eff}}^{\text{spinless}}$ for $e=0$ and both of m_{eff} and $m_{\text{eff}}^{\text{spinless}}$ are analytic in e^2, the corollary follows. $\square$

3.5 Proof of Lemmas 3.4 and 3.5

To prove Lemmas 3.4 and 3.5, we prepare some notation. To estimate $\mathcal{E}_i$, $i=0,3,4$, we introduce the polar coordinate $(r_1,\theta,\phi,r_2,\theta_2,\phi_2)$, where $r_i=|k_i|$, $i=1,2$, $0\le\theta_2\le\pi$,

$0 \le \phi_2 \le 2\pi$, $\cos\theta = (\hat{k}_1, \hat{k}_2)$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$. We introduce

$$\mathcal{R} = \mathcal{R}(r_1, r_2) \equiv \frac{1}{2}(r_1^2 + r_2^2) + r_1 + r_2.$$
$$\mathcal{E}_\pm = \mathcal{E}_\pm(r_1, r_2) \equiv \mathcal{R} \pm r_1 r_2 = \frac{1}{2}(r_1 \pm r_2)^2 + r_1 + r_2$$

Then $\mathcal{E}(k_1, k_2) = \mathcal{R}(r_1, r_2) + r_1 r_2 \cos\theta$. We set

$$\mathcal{K} = \mathcal{K}(r_1, r_2) \equiv \int_0^\pi \frac{1}{\mathcal{R} + r_1 r_2 \cos\theta} r_1 r_2 \sin\theta d\theta,$$
$$\hat{\mathcal{K}} = \hat{\mathcal{K}}(r_1, r_2) \equiv \int_0^\pi \frac{(1 - \cos^2\theta)\cos\theta}{(\mathcal{R} + r_1 r_2 \cos\theta)^3} r_1 r_2 \sin\theta d\theta.$$

Then we have

$$\begin{aligned}
\mathcal{K} &= \log\frac{\mathcal{E}_+}{\mathcal{E}_-} = \log\frac{\mathcal{R} + r_1 r_2}{\mathcal{R} - r_1 r_2}, \\
\hat{\mathcal{K}} &= -\frac{1}{r_1^3 r_2^3}\left[6 r_1 r_2 + \frac{2 r_1^3 r_2^3}{\mathcal{E}_+ \mathcal{E}_-} - 3\mathcal{R}\log\frac{\mathcal{E}_+}{\mathcal{E}_-}\right]
\end{aligned}$$

from which it follows that for $\zeta \equiv r_1 r_2 / \mathcal{R} < 1$,

$$\mathcal{K}(r_1, r_2) = 2\sum_{n=0}^{\infty} \frac{1}{2n+1}\zeta^{2n+1}, \quad \hat{\mathcal{K}}(r_1, r_2) = -\frac{1}{R^2}\left[\frac{4}{5}\zeta^2 + \mathcal{O}\left(\zeta^4\right)\right] \tag{3.16}$$

By the Taylor expansion of $\mathcal{K}(r_1, r_2) = \mathcal{K}(r_1, r_1/x)$ and $2r_1 r_2/\mathcal{E}(r_2) = 4x(1+(2x/r_1))^{-1}$ by $x = r_1/r_2$ (note that $\zeta = 2x - 4(1 + r_1^{-1})x^2 + \mathcal{O}(x^3)$), we have for $r_1/r_2 < 1$ and $r_1 \gg 1$,

$$\mathcal{K}(r_1, r_2) = 4x - \frac{8}{r_1}x^2 + (\frac{4}{3} + \mathcal{O}(\frac{1}{r_1}))x^3 + \mathcal{O}(x^4), \tag{3.17}$$
$$\frac{2r_1 r_2}{\mathcal{E}(r_2)} = 4x - \frac{8}{r_1}x^2 + \frac{16}{r_1^2}x^3 + \mathcal{O}(\frac{x^4}{r_1^3}). \tag{3.18}$$

From (3.17) and (3.18) it follows that for $r_1/r_2 < 1$ and $r_1 \gg 1$,

$$\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)} = (\frac{4}{3} + \mathcal{O}(\frac{1}{r_1}))\frac{r_1^3}{r_2^3} + \mathcal{O}(\frac{r_1^4}{r_2^4}). \tag{3.19}$$

3.5.1 Proof of Lemma 3.5

Proof: For notational simplicity we set $m = \kappa = 1$. Replacing (k_1, k_2) by $|k_1||k_2|$ in (3.4), we have

$$\begin{aligned}
|\mathcal{E}_0| &\le \frac{1}{4}\int d\mathbf{k}\frac{1}{|k_1||k_2|}\left(\frac{4|k_2|^2}{\mathcal{E}(k_1)^2\mathcal{E}(k_1,k_2)} - \frac{4|k_2|^2}{\mathcal{E}(k_1)^2\mathcal{E}(k_2)} + \frac{2|k_1||k_2|}{\mathcal{E}(k_1)\mathcal{E}(k_2)\mathcal{E}(k_1,k_2)}\right) \\
&= \frac{1}{2(2\pi)^4}\int_1^\Lambda\int_1^\Lambda dr_1 dr_2 \left\{\frac{4r_2^2}{\mathcal{E}(r_1)^2}\left(\mathcal{K}(r_1,r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)}\right) + \frac{2r_1 r_2}{\mathcal{E}(r_1)\mathcal{E}(r_2)}\mathcal{K}(r_1,r_2)\right\}.
\end{aligned} \tag{3.20}$$

In [9, Lemma 4.2 (1)] it is shown that for the second integrand of (3.20),

$$\int_1^\Lambda \int_1^\Lambda dr_1 dr_2 \frac{2r_1 r_2}{\mathcal{E}(r_1)\mathcal{E}(r_2)} \mathcal{K}(r_1, r_2) \le C \log \Lambda, \tag{3.21}$$

where and hereafter, C denotes a non-zero constant which may be different from line to line. We decompose the integral region $\mathcal{I} \equiv \{(r_1, r_2) | 1 \le r_1, r_2 \le \Lambda\}$ into three regions:

$$\begin{aligned} \mathrm{I} &= \{(r_1, r_2) \in \mathcal{I} | \lambda^{-1} < r_1/r_2 < \lambda\}, \\ \mathrm{II}_1 &= \{(r_1, r_2) \in \mathcal{I} | r_1/r_2 \le \lambda^{-1}\}, \\ \mathrm{II}_2 &= \{(r_1, r_2) \in \mathcal{I} | \lambda \le r_1/r_2\}, \end{aligned}$$

where $\lambda \ge 3$ is a fixed positive constant. By $1/\mathcal{E}(r) \le 2/r^2$ and (3.19),we have

$$\left| \frac{4r_2^2}{\mathcal{E}(r_1)^2} \left(\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)} \right) \right| \le C \frac{1}{r_1 r_2}, \quad (r_1, r_2) \in \mathrm{II}_1,$$

and by $\mathcal{K}(r_1, r_2) \le C \log(r_1 + r_2 + 1)$,

$$\left| \frac{4r_2^2}{\mathcal{E}(r_1)^2} \left(\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)} \right) \right| \le C \left(\frac{r_2^2}{r_1^4} \log \left[r_1 (1 + \frac{1}{r_1} + \frac{r_2}{r_1}) \right] + \frac{r_2}{r_1^3} \right), \quad (r_1, r_2) \in \mathrm{II}_2 \cup \mathrm{I}.$$

Thus

$$\int_{\mathrm{II}_1} dr_1 dr_2 \left| \frac{4r_2^2}{\mathcal{E}(r_1)^2} \left(\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)} \right) \right| \le C \int_1^\Lambda dr_2 \int_1^{r_2/\lambda} dr_1 \frac{1}{r_1 r_2} \le C [\log \Lambda]^2, \tag{3.22}$$

$$\int_{\mathrm{I}} dr_1 dr_2 \left| \frac{4r_2^2}{\mathcal{E}(r_1)^2} \left(\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)} \right) \right| \le C \int_1^\Lambda dr_1 \int_{r_1/\lambda}^{\lambda r_1} dr_2 (\frac{\log r_1}{r_1^2} + \frac{1}{r_1^2}) \le C [\log \Lambda]^2 \tag{3.23}$$

and

$$\int_{\mathrm{II}_2} dr_1 dr_2 \left| \frac{4r_2^2}{\mathcal{E}(r_1)^2} \left(\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)} \right) \right| \le C \int_1^\Lambda dr_1 \int_1^{r_1/\lambda} dr_2 (\frac{\log r_1}{r_1^2} + \frac{1}{r_1^2}) \le C [\log \Lambda]^2 \tag{3.24}$$

Hence (3.9) follows from (3.21),(3.22), (3.23) and (3.24). In the similar way as that of $\mathcal{E}_0$, we can prove (3.10). It is immediate to see that $0 < \mathcal{E}_3$ is bounded from above by

$$\begin{aligned} &\frac{1}{16} \int d\mathbf{k} \frac{1}{|k_1||k_2|} \left\{ \frac{4|k_1|^4 |k_2|^2}{\mathcal{E}(k_1)^4 \mathcal{E}(k_1, k_2)} - \frac{4|k_1|^4 |k_2|^2}{\mathcal{E}(k_1)^4 \mathcal{E}(k_2)} - \frac{2|k_1|^3 |k_2|^3 (1 - (\hat{k}_1, \hat{k}_2)^2)(\hat{k}_1, \hat{k}_2)}{\mathcal{E}(k_1)\mathcal{E}(k_2)\mathcal{E}(k_1, k_2)^3} \right\} \\ &\le \frac{1}{8(2\pi)^4} \int_1^\Lambda \int_1^\Lambda dr_1 dr_2 \left\{ \frac{4r_1^4 r_2^2}{\mathcal{E}(r_1)^4} \left(\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)} \right) - 8r_1 r_2 \hat{\mathcal{K}}(r_1, r_2) \right\} \end{aligned} \tag{3.25}$$

where $-\hat{\mathcal{K}}(r_1, r_2) > 0$ should be noted. For the second integrand of (3.25) it can be seen in the similar manner as [9, Lemma 4.2 (4)] that

$$\int_1^\Lambda \int_1^\Lambda dr_1 dr_2 r_1 r_2 \left|\hat{\mathcal{K}}(r_1, r_2)\right| \le C \int_1^\Lambda \int_1^\Lambda dr_1 dr_2 \frac{1}{\mathcal{E}_-(r_1, r_2)} \le C\sqrt{\Lambda}. \tag{3.26}$$

Note that

$$\left|\frac{4r_1^4 r_2^2}{\mathcal{E}(r_1)^4}\left(\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)}\right)\right| \le 4\left|\frac{4r_2^2}{\mathcal{E}(r_1)^2}\left(\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)}\right)\right|.$$

Then

$$\int_1^\Lambda \int_1^\Lambda \left|\frac{4r_1^4 r_2^2}{\mathcal{E}(r_1)^4}\left(\mathcal{K}(r_1, r_2) - \frac{2r_1 r_2}{\mathcal{E}(r_2)}\right)\right| \le C[\log \Lambda]^2 \tag{3.27}$$

follows from (3.22), (3.23) and (3.24). Hence (3.10) follows from (3.26) and (3.27). □

3.5.2 Proof of Lemma 3.4

Proof: Let us decompose $\mathcal{E}_4$ as $\mathcal{E}_4 = \mathcal{E}_{41} + \mathcal{E}_{42}$, where

$$\mathcal{E}_{41} = -\frac{1}{16}\int d\mathbf{k} \frac{1}{|k_1||k_2|} \frac{4|k_1|^4|k_2|^2}{\mathcal{E}(k_1)^4 \mathcal{E}(k_1, k_2)},$$

$$\mathcal{E}_{42} = -\frac{1}{16}\int d\mathbf{k} \frac{1}{|k_1||k_2|} \frac{\left(-2|k_1|^2|k_2|^2(1 - (\hat{k}_1, \hat{k}_2)^2) + \langle\sigma(k_1, k_2)\rangle_E\right)|k_1|^2}{\mathcal{E}(k_1)^3 \mathcal{E}(k_2)\mathcal{E}(k_1, k_2)}.$$

We have the inequality

$$|\mathcal{E}_{42}| \le \frac{1}{16}\int d\mathbf{k} \frac{1}{|k_1||k_2|}\left(\frac{6|k_1|^4|k_2|^2}{\mathcal{E}(k_1)^3 \mathcal{E}(k_2)\mathcal{E}(k_1, k_2)}\right) = \frac{1}{16}\frac{(4\pi)(2\pi)}{(2\pi)^6}\int_1^\Lambda \int_1^\Lambda dr_1 dr_2 \frac{\mathcal{K}(r_1, r_2)}{\mathcal{E}(r_1)}.$$

Here we used the inequality $|\langle\sigma(k_1, k_2)\rangle_E| \le 4|k_1|^2|k_2|^2$. It is seen that

$$\int_1^\Lambda \int_1^\Lambda dr_1 dr_2 \frac{\mathcal{K}(r_1, r_2)}{\mathcal{E}(r_1)} \le C[\log \Lambda]^2. \tag{3.28}$$

We shall estimate $\mathcal{E}_{41}$. We have

$$\mathcal{E}_{41} = \frac{1}{16}\frac{(4\pi)(2\pi)}{(2\pi)^6}\int_1^\Lambda \int_1^\Lambda dr_1 dr_2 \frac{4r_1^2 r_2^2}{\mathcal{E}(r_1)^4}\mathcal{K}(r_1, r_2).$$

Since

$$\frac{4r_1^2 r_2^2}{\mathcal{E}(r_1)^4}\mathcal{K}(r_1, r_2) \le C\frac{r_2^2}{r_1^6}\log(r_1 + r_2 + 1),$$

we have

$$\int_{\mathrm{I}\cup\mathrm{II}_2} dr_1 dr_2 \frac{4r_1^2 r_2^2}{\mathcal{E}(r_1)^4}\mathcal{K}(r_1,r_2) < \infty. \tag{3.29}$$

Moreover

$$\mathcal{K}(r_1,r_2) \le C\frac{r_1}{r_2}, \quad (r_1,r_2)\in \mathrm{II}_1,$$

we have

$$\int_{\mathrm{II}_1} dr_1 dr_2 \frac{4r_1^2 r_2^2}{\mathcal{E}(r_1)^4}\mathcal{K}(r_1,r_2) < C\int_1^\Lambda dr_2 \int_1^{r_2/\lambda} dr_1 \frac{r_2}{r_1^5} \le C\Lambda^2. \tag{3.30}$$

Hence by (3.28), (3.29) and (3.30),

$$\mathcal{E}_4 \le C\Lambda^2 \tag{3.31}$$

follows. In the region II_1 it follows that

$$4\frac{r_1}{r_2} - C\left(\frac{r_1}{r_2}\right)^2 \le \mathcal{K}(r_1,r_2). \tag{3.32}$$

Since $2/3 \le r^2/\mathcal{E}(r) \le 2$, we see together with (3.32) that

$$\frac{4r_1^2 r_2^2}{\mathcal{E}(r_1)^4}\mathcal{K}(r_1,r_2) \ge C\left\{\frac{r_1}{r_2} - \left(\frac{r_1}{r_2}\right)^2\right\}\frac{r_2^2}{r_1^6} = C\frac{r_2}{r_1^5} - C\frac{1}{r_1^4}, \quad (r_1,r_2)\in\mathrm{II}_1.$$

Then we have

$$\int_{\mathrm{II}_1} dr_1 dr_2 \frac{4r_1^2 r_2^2}{\mathcal{E}(r_1)^4}\mathcal{K}(r_1,r_2) \ge C\int_1^\Lambda dr_1 \int_{\lambda r_1}^\Lambda dr_2 \left(\frac{r_2}{r_1^5} - \frac{1}{r_1^4}\right) \ge C(\Lambda^2 - \Lambda),$$

which implies that

$$C\Lambda^2 \le \mathcal{E}_4. \tag{3.33}$$

Hence the lemma follows from (3.31) and (3.33). □

4 Conclusion and Discussion

We have shown that the second order $a_2(\Lambda/m)$ of $m_{\text{eff}}/m = \sum_{n=0}^\infty a_n(\Lambda/m)e^{2n}$ diverges like $-(\Lambda/m)^2$, which seems not to be renormalized by the conventional renormalization scheme of the field theory. The traditional power counting theorem means that this quantity diverges like $[\log(\Lambda/m)]^2$ at most. However, in the present case, the integrands are not symmetric in momenta k_i, contrary to the case of the covariant perturbation theory. This seems to be a reason why the conventional power counting theorem does not hold.

As was discussed in [1], the Pauli-Fierz model is obtained from the Maxwell-Dirac Hamiltonian

$$H_{\mathrm{DM}} = \sum_{\mu=1}^{3} \alpha_\mu (p - eA)_\mu + \beta m + V + H_{\mathrm{f}}, \tag{4.1}$$

where V is a suitable scalar potential, and $(\alpha_1, \alpha_2, \alpha_3, \beta)$ are 4×4 symmetric matrices satisfying anti-commutation relations. This Hamiltonian is self-adjoint if suitable cutoffs are introduced. To get the scaling limit, we let $\kappa > 0$ be the speed of light, and define

$$H_{\mathrm{DM}}^{(\kappa)} = \kappa \sum_{\mu=1}^{3} \alpha_\mu (p - eA^{(\kappa)})_\mu + \kappa^2 \beta m - \kappa^2 m + V_{0,\kappa} + H_{\mathrm{f}}, \tag{4.2}$$

where $-\kappa^2 m$ is introduced to ensure the ground state energy is zero except for negative energies of Dirac's sea of positrons. Moreover suitable cutoffs depending on κ are introduced into A_μ and V_0 so that $A_\mu^{(\kappa)}$ turns out to be bounded as $\|A_\mu^{(\kappa)}\| < K(\kappa)$, which is achieved by truncating the spectrum of $|A_\mu|$ and $|V_0|$ more than $K(\kappa)$. Then it is proven that the scaling limit of $H_{\mathrm{DM}}^{(\kappa)}$ converges to the Pauli-Fierz Hamiltonian with spin 1/2 in the strong resolvent sense. So it is not clear to what extent the original properties of the model are kept in this limit. Though it is not yet proven and this Hamiltonian may not have a ground state as the quantum field model of ϕ_4^3, we expect that H_{DM} is renormalizable since the Feynman diagrams appearing in this model are a subset of QED in which no fermion-loop diagrams appear. These arguments will lead us to ask a number of questions:

(1) Is the so-called covariant renormalization method in QED directly related to the renormalization of the Hamiltonian?

(2) Renormalizability of QED seems to be ensured by an indefinite metric. Does the introduction of the indefinite metric [10] cures the renormalizability of the Pauli-Fierz model ?

(3) Is the model (4.1) renormalizable ?

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A Sketch of the Calculations of Lemma 3.1

To estimate $\mathcal{E} = (\Phi_{(\ell)}, H_{(m)}\Phi_{(n)})$, we represent A_μ and σB_μ in terms of $a^\#(k,\ell)e_\mu(k,\ell)$ and $a^\#(k,\ell)[\sigma\cdot(ik\times e(k,\ell))]$, and use the pull-through formulae

$$\begin{aligned} a(k,\ell)(\frac{1}{2}{P_{\mathrm{f}}}^2 + H_{\mathrm{f}})^{-1} &= (\frac{1}{2}(P_{\mathrm{f}}+k)^2 + \omega(k) + H_{\mathrm{f}})^{-1}a(k,\ell), \\ a(k,\ell)P_{\mathrm{f}\mu} &= (P_{\mathrm{f}\mu} + k_\mu)a(k,\ell) \end{aligned}$$

and canonical commutation relations:

$$[a(k_1,\ell_1), a^*(k_2,\ell_2)] = \delta(k_1-k_2)\delta_{\ell_1,\ell_2}.$$

The terms containing no σB have been estimated in [9]. The terms containing two σB consist of two A_μ's, two σB's and 0 or two or four P_{f}'s. Let $k_i, e(k_i,\ell_i)$, $i=1,2$, be two independent momenta and polarizations vectors of photons. Then using (3.1)–(3.3), we have the following factors $P(k_i)$ coming from the polarization vectors $e_i = e(k_i,\ell_i)$, the Pauli-spin matrices $\sigma\cdot(k_i\times e_i)$, and k_i coming from P_{f}:

$$\begin{aligned} &\sum_{\ell_1,\ell_2=1,2}[\sigma\cdot(k_1\times e_1)]^2(e_2,e_2) = 4|k_1|^2 \\ &\sum_{\ell_1,\ell_2=1,2}[\sigma\cdot(k_1\times e_1)][\sigma(k_2\times e_2)](e_1,e_2) = 2(k_1,k_2) + i\sigma\cdot(\cdots) \\ &\sum_{\ell_1,\ell_2=1,2}[\sigma\cdot(k_1\times e_1)][\sigma(k_2\times e_2)](k_2,e_1)(k_1,e_2) = -(k_1,k_2)(1-(\hat{k}_1,\hat{k}_2)^2) + i\sigma\cdot(\cdots) \end{aligned}$$

The sum of the contributions from $i\sigma(\cdots)$ is zero. We have $\mathcal{E}^p(k_1,k_2)\mathcal{E}^q(k_1)\mathcal{E}^r(k_2)$ in the denominator coming from $H_{(1)} = H_0^{-1}$ where $p+q+r\ge 3$ is equal to the number of H_0^{-1}. Let q' and r' be the numbers of σB and P_{f}. Then $p+q+r-(q'+r')/2 = 3$ and $q' = 0,2,4$, and we see that $\mathcal{E}$ is represented as sum of the following integrals:

$$\frac{1}{4(2\pi)^6}\int\frac{1}{\omega(k_1)\omega(k_2)}\frac{P(k_i)}{\mathcal{E}^p(k_1,k_2)\mathcal{E}^q(k_1)\mathcal{E}^r(k_2)}dk_1dk_2 \tag{1.1}$$

where $P(k_i)$ is a polynomial of k_i of degree $q'+r'$. We estimate (1.1). Changing variables (k_1,k_2) in (1.1) to the polar coordinate stated in the proofs of Lemmas 3.4 and 3.5, and integrating with respect to $0\le\phi,\phi_2\le 2\pi$ and $0\le\theta_2\le\pi$, we have $dk_1=4\pi r_1^2dr_1$ and $dk_2=2\pi r_2^2\sin\theta dr_2d\theta$ so that $(k_1,k_2)=r_1r_2\cos\theta$, $r_i=|k_i|$. Then we see that the integral over $0<\theta<\pi$ takes one of the following forms:

$$
\begin{aligned}
\mathcal{K}_{1,p} &= \int_0^\pi \frac{1}{(\mathcal{R}+r_1r_2\cos\theta)^p} r_1r_2\sin\theta d\theta,\\
\hat{\mathcal{K}}_{1,p} &= \int_0^\pi \frac{\cos}{(\mathcal{R}+r_1r_2\cos\theta)^p} r_1r_2\sin\theta d\theta,\\
\mathcal{K}_{2,p} &= \int_0^\pi \frac{1-\cos^2\theta}{(\mathcal{R}+r_1r_2\cos\theta)^p} r_1r_2\sin\theta d\theta,\\
\hat{\mathcal{K}}_{2,p} &= \int_0^\pi \frac{(1-\cos^2\theta)\cos\theta}{(\mathcal{R}+r_1r_2\cos\theta)^p} r_1r_2\sin\theta d\theta
\end{aligned}
$$

where we recall $\mathcal{R}=\frac{1}{2}(r_1^2+r_2^2)+r_1+r_2$. Then we have

$$
\mathcal{K}_{1,p}(r_1,r_2) = \begin{cases} \log\dfrac{\mathcal{E}_+}{\mathcal{E}_-} & p=1\\ \dfrac{1}{p-1}\left(\dfrac{1}{\mathcal{E}_-^{p-1}}-\dfrac{1}{\mathcal{E}_+^{p-1}}\right) & p>1 \end{cases}
$$

$$
\mathcal{K}_{2,p}(r_1,r_2) = \frac{1}{(r_1r_2)^2}\times\begin{cases} \left[-\mathcal{E}_+\mathcal{E}_-\log\dfrac{\mathcal{E}_+}{\mathcal{E}_-}+2\mathcal{R}r_1r_2\right] & p=1\\ \left[2\mathcal{R}\log\dfrac{\mathcal{E}_+}{\mathcal{E}_-}-4r_1r_2\right] & p=2\\ \left[\dfrac{2\mathcal{R}r_1r_2}{\mathcal{E}_+\mathcal{E}_-}-\log\dfrac{\mathcal{E}_+}{\mathcal{E}_-}\right] & p=3 \end{cases}
$$

$$
\begin{aligned}
\hat{\mathcal{K}}_{1,p}(r_1,r_2) &= \frac{1}{r_1r_2}\times\begin{cases} \left[2r_1r_2-\mathcal{R}\log\dfrac{\mathcal{E}_+}{\mathcal{E}_-}\right] & p=1\\ \left[\log\dfrac{\mathcal{E}_+}{\mathcal{E}_-}-\mathcal{R}\left(\dfrac{1}{\mathcal{E}_-}-\dfrac{1}{\mathcal{E}_+}\right)\right] & p=2 \end{cases}\\
\hat{\mathcal{K}}_{2,3}(r_1,r_2) &= \frac{1}{(r_1r_2)^3}\left[3\mathcal{R}\log\frac{\mathcal{E}_+}{\mathcal{E}_-}-6r_1r_2-\frac{2r_1^3r_2^3}{\mathcal{E}_+\mathcal{E}_-}\right]
\end{aligned}
$$

where $\mathcal{E}_\pm = \mathcal{R} \pm r_1 r_2$. They have the following Taylor expansions by $\zeta = r_1 r_2/\mathcal{R}$:

$$\mathcal{K}_{1,p}(r_1,r_2) = \begin{cases} \sum_{n=0}^{\infty} \frac{2}{2n+1}\zeta^{2n+1} & p=1 \\ \frac{-2}{p-1}\frac{1}{\mathcal{R}^{p-1}}\sum_{n=0}^{\infty}\begin{pmatrix} -p+1 \\ 2n+1 \end{pmatrix}\zeta^{2n+1} & p>1 \end{cases}$$

$$\mathcal{K}_{2,p}(r_1,r_2) = \begin{cases} \frac{4}{3}\zeta + \frac{2}{3}\zeta^3 + \sum_{n=2}^{\infty}(1-\zeta^{-2})\frac{2}{2n+1}\zeta^{2n+1} & p=1 \\ \frac{4}{\mathcal{R}}\sum_{n=1}^{\infty}\frac{1}{2n+1}\zeta^{2n-1} & p=2 \\ \frac{2}{\mathcal{R}^2}\sum_{n=1}^{\infty}\left(1-\frac{1}{2n+1}\right)\zeta^{2n-1} & p=3 \end{cases}$$

$$\hat{\mathcal{K}}_{1,p}(r_1,r_2) = \begin{cases} -\sum_{n=1}^{\infty}\frac{2}{2n+1}\zeta^{2n} & p=1 \\ -\frac{2}{\mathcal{R}}\sum_{n=1}^{\infty}(1-\frac{1}{2n+1})\zeta^{2n} & p=2 \end{cases}$$

$$\hat{\mathcal{K}}_{2,3}(r_1,r_2) = -\frac{2}{\mathcal{R}^2}\sum_{n=2}^{\infty}\left(1-\frac{3}{2n+1}\right)\zeta^{2(n-1)}$$

Note that

$$\zeta = \frac{2x}{1+(2+\frac{1}{r_1})x+(1+\frac{1}{r_1})x^2} = 2x-(4+\frac{2}{r_1})x^2+O(x^3)$$

$x = r_1/r_2$. Since $\mathcal{K}>0$ (resp. $\hat{\mathcal{K}}<0$) are odd (resp. even) in ζ, the first terms are enough. We decompose $[1,\Lambda/m]\times[1,\Lambda/m]$ into the symmetric region I and the asymmetric regions II_1 and II_2 and we use the Taylor expansion in the regions II_1 and II_2. Then the following integrals are fundamental and left to the reader as an exercise:

$$\int_{[1,\Lambda/m]\times[1,\Lambda/m]}\frac{1}{\mathcal{R}-r_1r_2}dr_1dr_2 \approx \sqrt{\Lambda/m},$$
$$\int_{[1,\Lambda/m]\times[1,\Lambda/m]}\frac{1}{r_1r_2}\mathcal{K}_{1,1}(r_1,r_2)dr_1dr_2 \approx \log(\Lambda/m)$$

as $\Lambda\to\infty$. Thus the remaining integrals with two σB are done following the methods described in Section 3.5. The terms with four σB's, i.e., $(\Phi^\mu_{(6)}, H_3\Phi^\mu_{(2)})_{\mathcal{H}}$ and $(\Phi^\mu_{(6)}, H_1\Phi^\mu_{(6)})_{\mathcal{H}}$ are similarly estimated.

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