On Hamiltonian bifurcations of invariant tori with a Floquet multiplier -1

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Abstract

Nearly integrable Hamiltonian systems are considered, for which the unperturbed system has a lower–dimensional torus not satisfying the second Mel'nikov condition ; on a 2:1 covering space a suitable choice of the toral angles yields a vanishing Floquet exponent. A nilpotent Floquet matrix $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ leads to the quasi–periodic analogue of the period–doubling bifurcation, so particular emphasis is given to the case $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ of vanishing normal linear behaviour. The actions conjugate to the toral angles unfold the various ways in which the degenerate torus becomes normally elliptic, hyperbolic or parabolic. With a KAM–theoretic approach it is then shown that this bifurcation scenario survives a non–integrable perturbation, parametrised by pertinent large Cantor sets.

The bifurcation scenario is governed by the "first" unimodal planar singularity $\frac{1}{24}p^4 \pm \frac{1}{24}q^4 + \frac{\mu}{4}p^2q^2$, which has co–dimension 8 with respect to all planar singularities. In the present context this high number is reduced to co–dimension 3 since the π–rotation on the 2:1 covering space has to be respected, and in case the Hamiltonian system is reversible there is a further reduction by 1 and the co–dimension becomes 2 . In such low co–dimensions it becomes more transparent why the modulus μ — although playing a prominent rôle during the KAM iteration — is of limited influence on the dynamical implications.

1 Setting of the problem

It is the period–doubling bifurcation that distinguishes the bifurcation theory of periodic orbits from the bifurcation theory of equilibria. A periodic orbit changes its stability, thereby generating a periodic orbit of twice the period. A convenient way to study this phenomenon is to use a Poincaré section ; a fixed point of the (Poincaré) map changes its stability and generates an orbit of period two. In this way a complete analysis could be given in [Meyer;70,75], concerning area–preserving maps and hence Hamiltonian systems in two degrees of freedom. For the generalization to d degrees of freedom see [Bridges,Furter;93].

In the dissipative setting one can (generically) use a centre manifold to make -1 the sole Floquet multiplier of the bifurcating periodic orbit. See *e.g.* [Newhouse,Palis,Takens;83] for a complete analysis of the period–doubling bifurcation in this context. It has been in the dissipative setting that a generalization to quasi–periodic orbits has first been obtained, in [Braaksma,Broer,Huitema;90]. Instead of period doubling it seems more appropriate to speak of frequency halving in this case, where an invariant torus changes its stability and generates an invariant torus with one frequency divided by 2 . The Hamiltonian analogue has been obtained in [Broer,Hanßmann,You;03], see Propositions 2.1 and 2.2 below.

During the period–doubling bifurcation the persistence of the periodic orbit, having Floquet multiplier -1 , is not in question. For invariant tori we always need Diophantine conditions on the frequencies $\omega_1, \ldots, \omega_n$ to have effective bounds away from internal resonance — in the same way as the period of a periodic orbit should not tend to infinity. When proving persistence of invariant tori with purely imaginary Floquet exponents, or normal frequencies $\Omega_1, \ldots, \Omega_m$, one usually assumes Diophantine conditions of the form

$$|k_1\omega_1 + \ldots + k_n\omega_n \; + \; \ell_1\Omega_1 + \ldots + \ell_m\Omega_m| \;\; \geq \;\; \frac{\gamma}{|k|^\tau} \qquad \forall_{k\in\mathbb{Z}^n\setminus\{0\}} \;\; \forall_{\ell\in\mathbb{Z}^m,\,|\ell|\leq 2} \tag{1}$$

with $\gamma > 0$ and $\tau > n-1$, cf. [Broer,Huitema,Sevryuk;96] and references therein. Now a Floquet multiplier -1 corresponds to a lattice

$$(k_1\omega_1 + \ldots + k_n\omega_n)\pi\mathrm{i} \; + \; 2\pi\mathrm{i}\,\omega_1\,\mathbb{Z} \; + \; \ldots \; + \; 2\pi\mathrm{i}\,\omega_n\,\mathbb{Z} \quad , \qquad k \in \{0,1\}^n\setminus\{0\}$$

of Floquet exponents from which one may derive the corresponding normal frequency Ω_1 . Therefore (1) with $\ell_1 = 2$ is not satisfied. The "traditional" approach used in *e.g.* [Broer, Huitema,Sevryuk;96], [Rüssmann;01] to prove persistence of invariant tori does so by preserving the normal linear behaviour as well. In this way violations of the Diophantine conditions (1) indicate bifurcations. However, as shown in [Bourgain;97], [Xu,You;01] the mere persistence of invariant tori (without any control on the normal behaviour) can already be secured by imposing (1) only with $|\ell| \leq 1$. Since we are interested in persistence of the whole bifurcation scenario we will in fact even have to control the nonlinear normal behaviour.

Given a non–degenerate integrable Hamiltonian system, we know from KAM theory that most maximal invariant tori survive a small (Hamiltonian) perturbation, cf. [Broer, Huitema,Sevryuk;96] or references therein. These maximal tori are the regular fibres of the *ramified torus bundle* defined by the integrable system. The singular fibres of this bundle are given by the lower dimensional invariant tori together with their stable and unstable manifolds. In this way the ramified torus bundle stratifies the phase space of a d–degrees–of–freedom Hamiltonian system into d–parameter families of (maximal) d–tori, $(d-1)$–parameter families of (lower dimensional) $(d-1)$–tori, ..., 1–parameter families of periodic orbits and isolated equilibria. Thus, while we expect the bifurcations of periodic orbits of a generic Hamiltonian system to be of co–dimension one, there will be bifurcations of higher co–dimensions in the various families of invariant tori. Of particular interest are the $(d-1)$–tori since these determine how the maximal tori are distributed in phase space. This makes it worthwhile to understand the persistence of families of lower dimensional invariant tori under small perturbations.

Bifurcations of invariant tori occur where the Diophantine conditions (1) are not satisfied. For $\ell = 0$ this leads to internal resonance and we expect such tori to break up. This leads especially to dynamical instabilities when there are normal–internal resonances as well, see [Litvak–Hinenzon,Rom–Kedar;02a,02b,04] for a first study of the ensuing possibilities. Let us have a closer look at the four types of normal–internal resonances.

In case the right hand side of (1) vanishes with $|\ell| = 1$ one can achieve $\Omega_1 = 0$ after a better suited choice of the toral angles, cf. [Xu,You;01], [Broer,Hanßmann,Jorba,Villanueva,Wagener;03]. When this zero eigenvalue is coming from a nilpotent linearization $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ we speak of a *normally parabolic* torus and the ensuing bifurcations have been dealt with in [Hanßmann;98], [Broer,Hanßmann,You;03]. These bifurcations stem from planar singularities $\mathcal{H}(q,p) = \frac{1}{2}p^2 + \mathcal{F}(q)$ where $\mathcal{F}$ has a vanishing 2–jet. For a similar treatment of the quasi–periodic bifurcations triggered by the *simple* planar singularities in the case of a vanishing normal linear part $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ see [Broer,Hanßmann,You;04], [Hanßmann;03].

For invariant tori of dimension $n \leq d-2$ there are (at least) two normal degrees of freedom and the right hand side of (1) can vanish with two different non–zero coefficients of ℓ. In well–chosen co–ordinates this reads $\Omega_1 = \Omega_2$ or $\Omega_1 = -\Omega_2$, see [Xu,You;01]. While the latter case is related to a quasi–periodic Hamiltonian Hopf bifurcation, cf. [Broer,Hoo,Naudot;04], [Broer,Hanßmann,Hoo;04], it is shown in [de Jong;99] that the torus does not bifurcate in the former case. Much more degenerate would be $\Omega_1 = 0 = \Omega_2$ or in fact any combination of violations of (1) with different $\ell \in \mathbb{Z}^m$. But already in five degrees of freedom one may have 1–parameter families of invariant 3–tori with this high level of degeneracy.

The last normal–internal resonance (and in fact the only alternative to $\Omega = 0$ for families of $(d-1)$–tori) is

$$2\Omega_1 \quad = \quad k_1\omega_1 + \ldots + k_n\omega_n$$

with at least one of the integer coefficients k_i odd. In reminiscence of the periodic case we speak of a Floquet multiplier -1. The aim of this paper is to study resulting bifurcations. By restricting to n–tori in $d = n+1$ degrees of freedom we exclude the possibility of additional normal–internal resonances. Such interaction of different normal–internal resonances would lead to dramatically more complicated dynamics. This restriction also prevents further hyperbolic and/or elliptic directions normal to the invariant torus. Concerning normal hyperbolicity this is not a real limitation since one would immediately restrict to a centre manifold, cf. [Mielke;91]. It is more subtle to include additional normal frequencies $\Omega_2, \ldots, \Omega_m$ (that *do* satisfy (1)) as these would have to be taken care of throughout the KAM iteration scheme given in Section 5. See also [Broer,Chow,Kim,Vegter;93], [Gelfreich,Lerman;02]. Restricting to bifurcations of n–tori in $n+1$ degrees of freedom keeps such technical details at a minimum. For the same reason we remain in the real analytic category, though generalizations to the context of infinite and finite differentiability as in [Broer,Huitema,Takens;90] exist.

The dynamics around tori with Floquet multiplier -1 is best studied on a 2:1 covering space, see [Broer,Hanßmann,Jorba,Villanueva,Wagener;03], [Broer,Hanßmann,You;03] or Section 2 below. A typical Hamiltonian function to which our results apply is then given by

$$H(x,y,q,p) \;=\; h(y) \;+\; \frac{a}{24}p^4 \;+\; \frac{b}{24}q^4 \;+\; \frac{\mu}{4}p^2q^2 \;+\; \frac{y_1}{2}p^2 \;+\; \frac{y_2}{2}q^2 \;+\; y_3pq \tag{2}$$

where the coefficients $a, b, \mu \neq 0$ satisfy $9\mu^2 \neq ab$. The phase space is $\mathcal{P} = \mathbb{T}^n \times \mathsf{Y} \times \mathsf{S}$, where $\mathbb{T}^n$ denotes an n–torus, $\mathsf{Y} \subseteq \mathbb{R}^n$ contains the conjugate actions and $\mathsf{S} \subseteq \mathbb{R}^2$ is the "normal symplectic space" where the bifurcations take place — respecting invariance under the π–rotation $(q,p) \mapsto (-q,-p)$. This latter property allows to return from the 2:1 cover $\mathcal{P}$ back to its base space. For a slight generalization of (2) see Theorem 4.1.

The normal linear behaviour of the invariant n–torus $y = 0$ of the Hamiltonian vector field X_H defined by (2) vanishes. This is more degenerate than the normally parabolic case treated in [Broer,Hanßmann,You;03] and also makes the three unfolding terms p^2, q^2 and pq in (2) necessary, cf. [Broer,Chow,Kim,Vegter;93]. In the reversible context only the two unfolding terms p^2 and q^2 are needed, for this reason we consider the reversible problem also in its own right.

The "normal symplectic terms" $\mathcal{H}(q,p) = \frac{a}{24}p^4 + \frac{b}{24}q^4 + \frac{\mu}{4}p^2q^2$ in (2) are motivated by singularity theory. Indeed, a non–vanishing 2–jet would lead to a normally elliptic, hyperbolic or parabolic invariant torus $y = 0$ and once the 2–jet vanishes the 3–jet has to vanish as well since the third order terms are not invariant under the π–rotation $(q,p) \mapsto (-q,-p)$. The above expression is then the least degenerate possibility. In fact it merely corresponds to a homogeneous 4th order polynomial in general position : a suitable rotation of the (q,p)–plane always allows one to transform away the p^3q and pq^3 terms. Similarly, one could scale p and q to achieve $a = \pm 1$ and $b = \pm 1$. We refrain from this further simplification since such an independent scaling does not respect the symplectic structure. Furthermore, in applications it is more convenient to simply have to check the signs of a and b (and of certain expressions involving μ as well) than to redo the tedious computations to read the "type" of the bifurcation off from the "minimal set of coefficients". For this reason we also refrain from using a scaling $(q,p) \mapsto (\alpha q, \frac{1}{\alpha}p)$ followed, if necessary, by a multiplication of the Hamiltonian by -1 (reversing time) to achieve $a = +1$. Note that non–symplectic co–ordinate transformations may in fact be used without causing problems in the present one–degree–of–freedom context, cf. [Broer,Chow,Kim,Vegter;93].

However, after a and b have been made ± 1 the coefficient of p^2q^2 cannot be further simplified. This means that $\mathcal{H}$ does not define a simple singularity as the coefficient μ constitutes a *modulus*. Indeed, this modulus derives directly from the grandfather of all moduli, the cross ratio. Considering $\mathcal{H}$ as a complex function defined on $\mathbb{C}^2$ this is immediate ; in the real category things are a bit more subtle as pointed out in [Wassermann;77] (see also [Duistermaat;74]).

The planar singularity $\mathcal{H}$ is 4–*determined*, see [Wassermann;77], [Arnol'd,Gusein–Zade,Varchenko;82] : if the 4–jet of a singularity $\mathcal{F}$ is a homogeneous polynomial of order four in general position then $\mathcal{F}$ is *right equivalent* to $\mathcal{H}$. This means that there exists a co–ordinate transformation ψ in the (q, p)–plane that not only gets rid of the p^3q and pq^3 terms, but transforms away all higher order terms of $\mathcal{F}$ as well. While one should not expect ψ to preserve the symplectic structure, ψ nevertheless maps level sets of $\mathcal{F}$ to level sets of $\mathcal{H}$ and thereby provides an *equivalence* of the flows defined by the Hamiltonian vector fields $X_{\mathcal{F}}$ and $X_{\mathcal{H}}$. See again [Broer,Chow,Kim,Vegter;93] for further details. The equivalence ψ is a *conjugacy* (also preserving the time parametrisation of the phase curves) if and only if the symplectic structure is preserved.

If we consider the *universal unfolding* $\mathcal{H}^\lambda$ of $\mathcal{H}$ the above considerations remain true for all small perturbations $\mathcal{F}$ of $\mathcal{H}$. Given $\mathcal{F}$ there is a paramter value λ_0 such that $\mathcal{F}$ is right equivalent to $\mathcal{H}^{\lambda_0}$ and hence leads to an equivalent flow. This observation motivates the choice (2) of a Hamiltonian in the neighbourhood of (a 2:1 lift of) a lower dimensional torus with Floquet multiplier -1 .

When unfolding a unimodal singularity like $\mathcal{H}$ the modal parameter has to be unfolded as well and one might expect an additional term $\frac{y_4}{4}p^2q^2$ to be necessary in (2). However, as shown in Section 3 the unfolding terms p^2, q^2 (and pq in case the system is not reversible) are sufficient to obtain all the phase portraits "contained" in $X_{\mathcal{H}}$. In this way we end up with the five different cases of *figures 3.1–3.5.*

This paper is organized as follows. In the next section we review the Hamiltonian period–doubling bifurcation and its quasi–periodic counterpart. The dynamics of the integrable system defined by (2) is analysed in Section 3. For the reversible case complete bifurcation diagrams are given. The main result is stated in Section 4, where non–integrable perturbations of these dynamics are considered. The problem is led back to a normal form theorem, the proof of which constitutes Section 5.

2 The Hamiltonian frequency–halving bifurcation

Adjacent to the most degenerate invariant n–tori $y_1 = y_2 = y_3 = 0$ of the dynamics defined by (2) are normally parabolic tori, *e.g.* given by $y_2 = y_3 = 0$, $y_1 \neq 0$, and these undergo frequency–halving bifurcations. Reviewing the results of [Broer,Hanßmann,You;03] allows us to fix notation and *e.g.* clarify the relation between the "initial" phase space $\mathcal{B}$ and its 2:1 cover $\mathcal{P}$.

2.1 The period–doubling bifurcation

In the simplest example of a Hamiltonian period–doubling bifurcation a periodic orbit loses its stability, turning from elliptic to hyperbolic, and a second family of elliptic periodic orbits with twice the period detaches from the bifurcating periodic orbits. See *e.g.* [Abraham,Marsden;78], [Arnol'd,Kozlov,Neishtadt;85], [Meyer,Hall;92] for more details. An essential ingredient is that a periodic orbit is topologically non–trivial, and correspondingly there is no simple model equation on a topologically trivial phase space.

To get a better understanding of the dynamics one passes to a 2:1 covering of the phase space, cf. [Braaksma,Broer,Huitema;90], [Broer,Vegter;92], [Broer,Hanßmann,Jorba,Villanueva,Wagener;03]. In this way the bifurcating family of periodic orbits is lifted to a family of periodic orbits with twice the period, while the periodic orbits that branch off and already have twice the period get lifted to 2 such periodic orbits. Thus, on the covering space all periods are (approximately) the same and one can work with the phase space $\mathcal{P} = \mathbb{T} \times \mathbb{R} \times \mathbb{R}^2$ with symplectic form $\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}q \wedge \mathrm{d}p$. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a circle (or 1–torus), and the projection from this 2:1 covering to the base space is obtained by identifying points that are mapped onto each other under the deck transformation

$$S: \quad \begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{P} \\ (x, y, q, p) & \mapsto & (x + \frac{1}{2}, y, -q, -p) \end{array} \quad . \tag{3}$$

The involution S generates the deck group $\mathbb{Z}_2 = \{\mathrm{id}, S\}$ and the base space therefore reads $\mathcal{B} = \mathcal{P}/\mathbb{Z}_2$. In particular, the bifurcating family of periodic orbits is given by $(q, p) = (0, 0)$ — on both the base space $\mathcal{B}$ and its covering $\mathcal{P}$. A simple model Hamiltonian for the period–doubling bifurcation is now given by

$$H(x, y, q, p) \;=\; \omega y \;+\; \frac{a}{2}p^2 \;+\; \frac{b}{24}q^4 \;+\; \frac{y}{2}q^2 \tag{4}$$

with non–zero coefficients a, b and ω . Note that H is invariant under (3) and hence indeed defines a Hamiltonian system on the base space $\mathcal{B}$. The dynamics defined by (4) on $\mathcal{P}$ display a periodic Hamiltonian pitchfork bifurcation, cf. [Rimmer;78,83] ; as y passes through 0 the central periodic orbit loses its stability, giving rise to two new stable periodic orbits. Note that (4) is in fact *reversible*, *i.e.* invariant under *e.g.* the reversing symmetry

$$R: \quad \begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{P} \\ (x, y, q, p) & \mapsto & (-x, y, q, -p) \end{array}$$

as is often the case for simple model equations, cf. [Broer,Hoveijn,van Noort,Vegter;99]. Already adding a term like $p^3 q$ to (4) would break this symmetry, while keeping the invariance under (3) and still inducing a period–doubling bifurcation on the base space $\mathcal{B}$. As is well–known, cf. [Meyer;70], [Abraham,Marsden;78], this bifurcation survives any (sufficiently small) perturbation — in particular the system may cease to be integrable.

2.2 The frequency–halving bifurcation

For the quasi–periodic analogue of the Hamiltonian period–doubling bifurcation we already take the 2:1 covering space as a starting point. Denoting by $\mathbb{T}^n = \mathbb{R}^n/_{\mathbb{Z}^n}$ the standard n–torus this phase space is given by

$$\mathcal{P} \;=\; \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2 \tag{5}$$

with symplectic form $\sum \mathrm{d}x_i \wedge \mathrm{d}y_i \;+\; \mathrm{d}q \wedge \mathrm{d}p$. The base space on which the actual frequency–halving bifurcation takes place is again of the form

$$\mathcal{B} \;=\; \mathcal{P}/_{\mathbb{Z}_2} \quad ,$$

but the involution (3) gets replaced by

$$\begin{array}{rccc} \sigma : & \mathcal{P} & \longrightarrow & \mathcal{P} \\ & (x_1, x_2, \ldots, x_n, y_1, \ldots, y_n, q, p) & \mapsto & (x_1 + \frac{1}{2}, x_2, \ldots, x_n, y_1, \ldots, y_n, -q, -p) \end{array} \tag{6}$$

to generate the deck group $\mathbb{Z}_2 = \{\mathrm{id}, \sigma\}$. The Hamiltonian (4) is subject only to the obvious adaption and becomes

$$H(x, y, q, p) \;=\; (\omega \mid y) \;+\; \frac{a}{2}p^2 \;+\; \frac{b}{24}q^4 \;+\; \frac{y_1}{2}q^2 \tag{7}$$

where $(.. \mid ..)$ denotes the standard inner product. Thus, it is only the first frequency $2\omega_1$ that gets halved when the y–family of invariant tori $(q, p) = (0, 0)$ is lifted from $\mathcal{B}$ to $\mathcal{P}$. In applications this situation is easily achieved by a suitable choice of the toral angles $x_1, \ldots, x_n$, cf. [Broer,Hanßmann,Jorba,Villanueva,Wagener;03]. Similar to the periodic case $n = 1$ the model Hamiltonian H is reversible with respect to

$$\begin{array}{rccc} \rho : & \mathcal{P} & \longrightarrow & \mathcal{P} \\ & (x_1, \ldots, x_n, y_1, \ldots, y_n, q, p) & \mapsto & (-x_1, \ldots, -x_n, y_1, \ldots, y_n, q, -p) \end{array} \quad , \tag{8}$$

i.e. the function H is invariant under the larger group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by ρ and σ .

The dynamics defined by (7) on $\mathcal{P}$ is readily analysed. Since x does not explicitly enter the Hamiltonian function H , the dynamical system is equvariant with respect to the free n–torus action

$$\begin{array}{rccc} \tau : & \mathbb{T}^n \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ & (\xi, (x, y, q, p)) & \mapsto & (x + \xi, y, q, p) \end{array} \tag{9}$$

and reducing this symmetry amounts to fixing the value of the momentum mapping

$$\begin{array}{rccc} J : & \mathcal{P} & \longrightarrow & \mathbb{R}^n \\ & (x, y, q, p) & \mapsto & y \end{array}$$

and ignoring the cyclic variables $x \in \mathbb{T}^n$. This leads to the n–parameter family of one–degree–of–freedom systems with Hamiltonian

$$\mathcal{H}^y(q,p) \;=\; \frac{a}{2}p^2 \;+\; \frac{b}{24}q^4 \;+\; \frac{y_1}{2}q^2 \tag{10}$$

where the constant term $(\omega \mid y)$ has been dropped. As variation of $y_2, \ldots, y_n$ has no consequences, this family in fact behaves like the 1–parameter family with Hamiltonian $\mathcal{H}^{y_1}$. When varying y_1 a Hamiltonian pitchfork bifurcation takes place. To fix thoughts let $a, b > 0$. Then the origin $(q,p) = (0,0)$ is elliptic for $y_1 > 0$ and there are no further equilibria. At $y_1 = 0$ the origin becomes parabolic, with linearization $\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$, and for $y_1 < 0$ the origin has turned into a hyperbolic equilibrium. Moreover, the pair $(q,p) = (\pm\sqrt{-\frac{6y_1}{b}}, 0)$ of elliptic equilibria bifurcates off from the origin. Since the system is equivariant with respect to the π–rotation

$$(q,p) \;\mapsto\; (-q,-p) \tag{11}$$

these two additional equilibria have the same energy. This becomes particularly important when a and b have different signs since in this case the two newly–born equilibria are hyperbolic and hence connected by heteroclinic orbits.

$\{\downarrow,\rightarrow\}$	u	v	w
u	0	w	$2u$
v	$-w$	0	$-2v$
w	$-2u$	$2v$	0

Table 2.1: The Poisson structure on $\mathcal{C}$.

If one reduces the discrete symmetry (11) on $\mathbb{R}^2$ one is led to a non–smooth phase space $\mathcal{C}$ with a conical singularity. Indeed, while the group $\mathbb{Z}_2$ generated by (11) acts freely on $\mathbb{R}^2 \setminus \{(0,0)\}$, the origin has non–trivial isotropy and is therefore reduced to the tip of the cone

$$\mathcal{C} \;=\; \left\{ (u,v,w) \in \mathbb{R}^3 \;\middle|\; 2uv = \frac{w^2}{2} \,, \quad u,v \geq 0 \right\} \tag{12}$$

under the reduction mapping

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathcal{C} \\ (q,p) & \mapsto & (\frac{q^2}{2}, \frac{p^2}{2}, pq) \end{array} \quad . \tag{13}$$

In fact it is this same reduced phase space $\mathcal{C}$ one would end up with when reducing the non–free $\mathbb{T}^n$–action on the base space $\mathcal{B}$ (without first lifting to the 2:1 cover $\mathcal{P}$) as it is precisely the y–family of n–tori given by $(q,p) = (0,0)$ that has the non–trivial isotropy group $\mathbb{Z}_2$. Under (13) the Hamiltonian (10) turns into

$$\mathcal{H}^{y_1}(u,v,w) \;=\; av \;+\; \frac{b}{6}u^2 \;+\; y_1 u \tag{14}$$

and since the Poisson structure on $\mathcal{C}$ is given by *table 2.1* the reduced equations of motion are

$$\begin{array}{rclcl} \dot{u} & = & \{u, \mathcal{H}^{y_1}\} & = & aw \\ \dot{v} & = & \{v, \mathcal{H}^{y_1}\} & = & -\frac{b}{3}uw \; - \; y_1 w \\ \dot{w} & = & \{w, \mathcal{H}^{y_1}\} & = & 2av \; - \; \frac{2b}{3}u^2 \; - \; 2y_1 u \quad . \end{array}$$

This can be used to confirm that, again for $a, b > 0$, the singular equilibrium loses its stability as y_1 passes from positive to negative values and a (regular) elliptic equilibrium $(u, v, w) = (-\frac{3y_1}{b}, 0, 0)$, $y_1 < 0$, branches off from the singular point. The pre–image of this elliptic equilibrium under the reduction mapping (13) consists precisely of the two elliptic equilibria $(q, p) = (\pm\sqrt{-\frac{6y_1}{b}}, 0)$ we had found on the 2:1 cover $\mathbb{R}^2$ of $\mathcal{C}$. In particular, when a and b have different signs these equilibria are hyperbolic and the two heteroclinic orbits on the 2:1 cover get reduced to one homoclinic orbit on the singular reduced phase space $\mathcal{C}$.

Thus, it might be merely a matter of taste whether one prefers to analyse the one–degree–of–freedom problem directly on $\mathcal{C}$ or on its 2:1 cover, although it seems that for Hamiltonians on $\mathcal{C}$ that are more complicated than (14) it is simpler (or at least more routine) to consider the $\mathbb{Z}_2$–invariant counterpart on the smooth reduced phase space $\mathbb{R}^2$. However, when it comes to comparing the base space $\mathcal{B}$ and its 2:1 cover $\mathcal{P}$ as to which is best fit for the perturbation analysis in $n+1$ degrees of freedom, the topologically trivial product space $\mathcal{P}$ is clearly the preferred choice. In this way the following result is proven in [Broer,Hanßmann,You;03] (though not explicitly stated).

Proposition 2.1 *Let $\mathsf{Y} \subseteq \mathbb{R}^n$ a neighbourhood of the origin and $\mathsf{S} \subseteq \mathbb{R}^2$ be a neighbourhood of the origin that is invariant under the π–rotation* (11)*. Supply $\mathcal{P} = \mathbb{T}^n \times \mathsf{Y} \times \mathsf{S}$ with the symplectic structure $\sum \mathrm{d}x_i \wedge \mathrm{d}y_i \, + \, \mathrm{d}q \wedge \mathrm{d}p$ and consider a (real analytic) integrable Hamiltonian function $\tilde{H}$ on the base space $\mathcal{B} = \mathcal{P}/_{\mathbb{Z}_2}$ defined by the deck transformation* (6)*. Let H denote the lift of $\tilde{H}$ to $\mathcal{P}$ and assume that as a Taylor series in q and p it starts as*

$$H(x, y, q, p) \;\; = \;\; h(y) \; + \; \frac{a(y)}{2}p^2 \; + \; \frac{b(y)}{24}q^4 \; + \; \frac{c(y)}{2}q^2 \; + \; \ldots$$

with $c(0) = 0$, but $a(0) \neq 0$ and $b(0) \neq 0$. Furthermore assume that the mapping

$$\begin{array}{rcl} \mathsf{Y} & \longrightarrow & \mathbb{RP}^n \\ y & \mapsto & [c(y) : \omega_1(y) : \ldots : \omega_n(y)] \end{array}$$

is a submersion, where $\omega_i(y) = \frac{\partial h}{\partial y_i}(y)$. Then a sufficiently small perturbation of $\tilde{H}$ undergoes a Hamiltonian frequency–halving bifurcation.

Let us sketch parts of the proof that will be helpful later on, thereby also explaining the meaning of the conclusion.

The bifurcation occurs at the hyperplane $c^{-1}(0) \subseteq \mathsf{Y}$ as $c(y)$ changes from positive to negative values. Note that we refrain from using $Dc(0) \neq 0$ to achieve $c(y) \equiv y_1$ by some co–ordinate change. Indeed, to make such a transformation preserve the symplectic structure, the conjugate angles $x \in \mathbb{T}^n$ would change as well and we would lose the simple form (6) of the deck transformation. Still, with one direction transversal to $c^{-1}(0)$ serving as bifurcation parameter there are only $n-1$ directions left in Y to control the n frequencies $\omega_1, \ldots, \omega_n$ of the unperturbed lower dimensional tori

$$\mathbb{T}^n \times \{y\} \times \{0\} \quad \subseteq \quad \mathcal{P} \quad .$$

Already in the periodic case $n = 1$ one is interested in equivalences rather than conjugacies between the unperturbed and the perturbed flow, allowing the use of a Poincaré section to *e.g.* prove that period–doubling bifurcations are structurally stable phenomena in Hamiltonian systems. It is not the frequency vector ω itself we want to control, but only the mutual ratio $[\omega_1(y) : \ldots : \omega_n(y)]$.

We therefore follow [Broer,Huitema,Takens;90], [Hanßmann;98] and consider $\hat{H} := \eta H$ with an additional parameter η varying in a small interval around 1 . Then we choose new co–ordinates (λ, ν) on the (y, η)–space defining $\lambda := \hat{c}(y, \eta) = \eta c(y)$ and $\Sigma := \ker D\hat{c}(0, 1)$ with $\nu_1, \ldots, \nu_n$ serving as co–ordinates on Σ . Letting (λ, ν) play the rôle of external parameters we now have to study small perturbations of the integrable $(n+1)$–parameter family

$$N(x, y, q, p; \lambda, \nu) \;\; = \;\; (\nu \mid y) \; + \; \frac{a(\nu)}{2} p^2 \; + \; \frac{b(\nu)}{24} q^4 \; + \; \frac{\lambda}{2} q^2 \tag{15}$$

of Hamiltonians defined on $\mathbb{T}^n \times \mathsf{Y} \times \mathsf{S} \times \Lambda \times \Sigma$, where $\Lambda \subseteq \mathbb{R}$ is a small interval around the origin. While we cannot expect invariant n–tori $\mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \times \{\nu\}$ with resonant frequency vector ν to survive small perturbations, we know from [Broer,Hanßmann,You;03] that these normally parabolic n–tori do persist if ν satisfies Diophantine conditions of the form

$$|\, (k \mid \nu) \,| \;\geq\; \frac{\gamma}{|k|^\tau} \qquad \forall_{k \in \mathbb{Z} \setminus \{0\}} \tag{16}$$

with $\gamma > 0$ and $\tau > n - 1$. Indeed, there is a Whitney–C^∞–smooth co–ordinate transformation Φ that is symplectic for fixed ν such that in the new co–ordinates the lower order terms of the perturbed Hamiltonian are again of the form (15) and in particular integrable — provided that ν satisfies (16).

In these new co–ordinates it easily follows that the central torus $\mathbb{T}^n \times \{0\} \times \{0\} \times \{\lambda\} \times \{\nu\}$ changes its normal behaviour as λ passes through zero and an additional pair of invariant n–tori branches off. In this way one has persistence of normally hyperbolic tori for internal frequencies ν that satisfy the same Diophantine conditions (16) as the persistent normally parabolic tori. For normally elliptic tori one has to exclude strong resonances involving the normal frequency Ω as well by imposing on (ν, Ω) the Diophantine conditions of the form

$$|\, (k \mid \nu) + \ell\Omega \,| \;\geq\; \frac{\gamma}{|k|^\tau} \qquad \forall_{k \in \mathbb{Z} \setminus \{0\}} \quad \forall_{\ell \in \{0, \pm 1, \pm 2\}} \tag{17}$$

with $\gamma > 0$ and $\tau > n - 1$.

On the base space $\mathcal{B}$ this provides the bifurcation scenario of a family of invariant tori turning from elliptic to hyperbolic (or *vice versa*, depending on the sign of the coefficient $a(0)$), and at the intermediate parabolic tori a family of elliptic tori branches off (or a family of hyperbolic tori, depending on the sign of the product $a(0) \cdot b(0)$ of the coefficients) with one of its (internal) frequencies halved. The normally parabolic, hyperbolic and elliptic invariant n–tori have frequencies satisfying the Diophantine conditions (16) and (17), respectively. In particular they form Cantor families.

2.3 Bifurcations of higher co–dimension

With periodic orbits one encounters (in a generic Hamiltonian system) only bifurcations of co–dimension one, since periodic orbits form 1–parameter families. Invariant n–tori form n–parameter families, parametrised by the actions conjugate to the toral angles. While degenerate n–tori of co–dimension n that one encounters in such families are isolated, and hence may disappear in resonance gaps, the normally degenerate invariant n–tori of co–dimension $n-1$ already form 1–parameter families and should persist on a pertinent Cantor subset defined by Diophantine conditions of the form (16) on the internal frequencies. For normally parabolic tori the following generalization of Proposition 2.1 is proven in [Broer,Hanßmann,You;03].

Proposition 2.2 *Let $\mathsf{Y} \subseteq \mathbb{R}^n$ a neighbourhood of the origin and $\mathsf{S} \subseteq \mathbb{R}^2$ be a neighbourhood of the origin that is invariant under the π–rotation* (11). *Supply $\mathcal{P} = \mathbb{T}^n \times \mathsf{Y} \times \mathsf{S}$ with the symplectic structure $\sum \mathrm{d}x_i \wedge \mathrm{d}y_i + \mathrm{d}q \wedge \mathrm{d}p$ and consider a (real analytic) integrable Hamiltonian function $\tilde{H}$ on the base space $\mathcal{B} = \mathcal{P}/\mathbb{Z}_2$ defined by the deck transformation* (6). *Let H denote the lift of $\tilde{H}$ to $\mathcal{P}$ and assume that as a Taylor series in q and p it starts as*

$$H(x, y, q, p) \;=\; h(y) + \frac{a(y)}{2}p^2 + \frac{b(y)}{d!}q^d + \sum_{j=1}^{\frac{d}{2}-1} \frac{c_j(y)}{(2j)!}q^{2j} + \ldots$$

where $d \leq 2n$ is even and the coefficient functions satisfy $a(0), b(0) \neq 0$ and $c(0) = 0$. Furthermore the mapping

$$\begin{array}{rccc} c: & \mathsf{Y} & \longrightarrow & \mathbb{R}^{\frac{d}{2}-1} \\ & y & \mapsto & c(y) \end{array}$$

is a submersion and the $\binom{n+2}{2}$ vectors

$$\frac{\partial^{|\ell|}}{\partial y^\ell} \begin{pmatrix} c \\ \omega \end{pmatrix} \quad , \quad |\ell| \leq 2$$

span $\mathbb{R}^{\frac{d}{2}-1} \times \mathbb{R}^n$, where $\omega(y) = Dh(y)$. Then a sufficiently small perturbation of $\tilde{H}$ undergoes a degenerate Hamiltonian frequency–halving bifurcation.

The strategy of the proof is again to first analyse how the integrable $(n+\frac{d}{2}-1)$–parameter family

$$N(x,y,q,p;\lambda,\omega) \quad = \quad (\omega \mid y) \; + \; \frac{a(\omega)}{2}p^2 \; + \; \frac{b(\omega)}{d!}q^d \; + \; \sum_{j=1}^{\frac{d}{2}-1} \frac{\lambda_j}{(2j)!}q^{2j} \tag{18}$$

of Hamiltonians defined on $\mathbb{T}^n \times \mathsf{Y} \times \mathsf{S} \times \Lambda \times \Sigma$ behaves under small perturbations, *i.e.* to consider $\lambda \in \Lambda \subseteq \mathbb{R}^{\frac{d}{2}-1}$ and the frequencies $\omega \in \Sigma \subseteq \mathbb{R}^n$ as external parameters. The parameters λ unfold the bifurcation diagram of the most degenerate normally parabolic tori $\mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \times \{\omega\}$. Indeed, the $\mathbb{Z}_2$–invariant universal unfolding of the planar singularity $\frac{a}{2}p^2 + \frac{b}{d!}q^d$ is given by

$$\frac{a}{2}p^2 \; + \; \frac{b}{d!}q^d \; + \; \sum_{j=1}^{\frac{d}{2}-1} \frac{\lambda_j}{(2j)!}q^{2j} \quad ,$$

see [Beer;76], [Wassermann;77]. Here $\mathbb{Z}_2$ is generated by the π–rotation (11). This yields a whole hierarchy of normally parabolic invariant tori, while the open domains of the bifurcation diagram parametrise normally elliptic and hyperbolic n–tori. Since $c : \mathsf{Y} \longrightarrow \mathbb{R}^{\frac{d}{2}-1}$ is a submersion we can use $\frac{d}{2}-1$ of the actions y conjugate to the toral angles x to account for the unfolding parameters λ . This leaves $n - \frac{d}{2} + 1$ "free directions" in Y which are not enough to control the n frequencies. For $d = 4$ we were able to control the frequency ratios instead but this fails for $d \geq 6$ as well. Therefore, we relax the control on the perturbed tori and only aim to show that a perturbed system does have Cantor families of invariant n–tori — without trying to further connect the perturbed tori to the unperturbed ones. The Cantor families of lower dimensional tori surviving a perturbation of (18) are again defined by the Diophantine conditions (16) and (17). What we want to prevent is that $\omega : \mathsf{Y} \longrightarrow \mathbb{R}^n$ maps large portions into the resonance gaps, and this is achieved by the conditions on the second derivative. Indeed, these imply that the image of Y is "sufficiently curved" and does not lie in any hyperplane passing through the origin. Thus, where a resonance gap is entered the curvature forces the image of Y to leave the resonance gap so fast that the majority of frequency vectors $\omega(y)$ are Diophantine.

For this line of argumentation to work one has to choose the constant $\tau > 2n-1$ in (16), while still $\gamma > 0$. In particular the relative measure of Diophantine frequencies still tends to 1 as $\gamma \searrow 0$.

3 Bifurcations in the unperturbed system

Before considering perturbations, we should first get a good understanding of the dynamics defined by the integrable Hamiltonian (2) itself. To this end we reduce to one degree of freedom, analyse that planar "backbone" system and then reconstruct the full (integrable) flow. Particular emphasis will be given to the reversible case.

3.1 Reduction to one degree of freedom

Motivated by the proofs of Propositions 2.1 and 2.2 we decouple the question how the actions y conjugate to the toral angles x can account for the necessary parameters and consider the model family

$$N(x, y, q, p; \lambda, \omega) = (\omega \mid y) + \frac{a}{24}p^4 + \frac{b}{24}q^4 + \frac{\mu}{4}p^2q^2 + \frac{\lambda_1}{2}p^2 + \frac{\lambda_2}{2}q^2 + \lambda_3 pq \quad (19)$$

of Hamiltonians defined on $\mathbb{T}^n \times \mathsf{Y} \times \mathsf{S} \times \Lambda \times \Sigma$. Here $\mathsf{Y} \subseteq \mathbb{R}^n$ is a neighbourhood of the origin and $\mathsf{S} \subseteq \mathbb{R}^2$ a neighbourhood of the origin that is invariant under the π–rotation (11) ; as always $\mathcal{P} = \mathbb{T}^n \times \mathsf{Y} \times \mathsf{S}$ is supplied with the standard symplectic structure. Recall that N is invariant under the deck transformation (6) and hence defines a Hamiltonian system on $\mathcal{B} = \mathcal{P}/_{\mathbb{Z}_2}$ as well — this is in fact how N was constructed. On $\mathcal{P}$ we can reduce the free torus action (9) and obtain

$$\mathcal{N}^\lambda(q, p) = \frac{a}{24}p^4 + \frac{b}{24}q^4 + \frac{\mu}{4}p^2q^2 + \frac{\lambda_1}{2}p^2 + \frac{\lambda_2}{2}q^2 + \lambda_3 pq \quad . \quad (20)$$

A further reduction of the discrete symmetry (11) on $\mathbb{R}^2$ would lead to the Hamiltonian

$$\mathcal{N}^\lambda(u, v, w) = \frac{a}{6}v^2 + \frac{b}{6}u^2 + \frac{\mu}{4}w^2 + \lambda_1 v + \lambda_2 u + \lambda_3 w \quad (21)$$

on the non–smooth phase space $\mathcal{C}$ defined in (12). This would also be the result of a direct reduction of the (non–free) $\mathbb{T}^n$–action on $\mathcal{B}$.

The three coefficients a, b and μ in (20) are of different nature. Indeed, while for a and b only the (relative) sign is of importance, μ is a modulus of the $\mathbb{Z}_2$–invariant singularity $\mathcal{N}^0$, see [Beer;76]. Next to $a \neq 0$ and $b \neq 0$ we furthermore impose $9\mu^2 \neq ab$ — a restriction only if $a \cdot b > 0$.

In applications Hamiltonian systems are often in addition reversible. To impose reversibility with respect to the involution ρ defined in (8) on the Hamiltonian N one has to restrict to $\lambda_3 = 0$ in (19). After reduction to one degree of freedom this leads to the replacement

$$\mathcal{H}^{\lambda_1,\lambda_2} = \mathcal{N}^{\lambda_1,\lambda_2,0}$$

in (20), cf. [Wassermann;77]. The reversibility with respect to

$$(q, p) \mapsto (q, -p) \quad (22)$$

together with the (non–reversing) symmetry (11) yields a symmetry group $\mathbf{Z}_2 \times \mathbf{Z}_2$ of $\mathcal{H}^\lambda$. We remark that on the cone $\mathcal{C}$ this reduces to the symmetry group $\mathbb{Z}_2$ of $\mathcal{H}^\lambda$ generated by

$$(u, v, w) \quad \mapsto \quad (u, v, -w) \quad .$$

Having fewer (unfolding) parameters usually simplifies the analysis. Therefore, we first study the reversible family $\mathcal{H}^\lambda$ before considering (20) in Section 3.3.

3.2 Analysis in the reversible case

The aim of this section is to show that the bifurcation diagram of the one–degree–of–freedom Hamiltonian system defined by

$$\mathcal{H}^\lambda(q,p) \;=\; \frac{a}{24}p^4 + \frac{b}{24}q^4 + \frac{\mu}{4}p^2q^2 + \frac{\lambda_1}{2}p^2 + \frac{\lambda_2}{2}q^2 \tag{23}$$

is given by *figures 3.1–3.5*, under the condition that $a, b, \mu \neq 0$ and $9\mu^9 \neq ab$. In particular μ merely plays a rôle similar to a and b and does not act as a modulus — there is no continuum of bifurcation diagrams. The Hamiltonian vector field $X_{\mathcal{H}^\lambda}$ has the equations of motion

$$\begin{aligned} \dot{q} &= \frac{a}{6}p^3 + \frac{\mu}{2}pq^2 + \lambda_1 p \\ \dot{p} &= -\frac{b}{6}q^3 - \frac{\mu}{2}p^2q - \lambda_2 q \end{aligned}$$

whence the linearization

$$DX_{\mathcal{H}^\lambda}(q,p) \;=\; \begin{pmatrix} \mu pq & \frac{a}{2}p^2 + \frac{\mu}{2}q^2 + \lambda_1 \\ -\frac{\mu}{2}p^2 - \frac{b}{2}q^2 - \lambda_2 & -\mu pq \end{pmatrix} \tag{24}$$

has at the origin the eigenvalues $\pm\sqrt{-\lambda_1\lambda_2}$: the equilibrium at the origin is elliptic for $\lambda_1\lambda_2 > 0$ and hyperbolic for $\lambda_1\lambda_2 < 0$ while bifurcations take place at the λ_1– and λ_2–axes.

3.2.1 Cases with $a > 0$ and $b > 0$

Let us first study cases where a and b have the same sign and, to fix thoughts, concentrate on $a, b > 0$ (if both are negative simply reverse time by multiplying (23) with -1 , thereby changing the sign of μ and also reversing the orientations of the λ_1– and λ_2–axes). Then additional pairs of equilibria on the q– and p–axes are given by

$$(q,p) \;=\; (\pm\sqrt{-\frac{6\lambda_2}{b}},0) \qquad \text{for} \quad \lambda_2 < 0 \tag{25}$$

$$(q,p) \;=\; (0,\pm\sqrt{-\frac{6\lambda_1}{a}}) \qquad \text{for} \quad \lambda_1 < 0 \quad . \tag{26}$$

Lemma 3.1 *The equilibria* (25),(26) *originate in Hamiltonian pitchfork bifurcations at the origin.*

Proof. The four connected components of

$$\left\{ \lambda \in \mathbb{R}^2 \;\middle|\; \lambda_1 \cdot \lambda_2 = 0\,,\, \lambda_1 + \lambda_2 \neq 0 \right\} \tag{27}$$

leave us with four cases.

(*i*) $\lambda_1 > 0$: the lowest order terms are of the form (10), with positive coefficients of p^2 and q^4 , and λ_2 playing the rôle of the bifurcation parameter y_1 . As λ_2 turns from positive to negative values, the elliptic equilibrium at the origin becomes parabolic at $\lambda_2 = 0$ and gives rise to the pair (25) of elliptic equilibria.

(*ii*) $\lambda_1 < 0$: now the coefficients of p^2 and q^4 have different signs, whence the pair (25) consists of hyperbolic equilibria.

(*iii*) $\lambda_2 > 0$: the lowest order terms are of the form (10), but with the rôles of p and q interchanged. Both coefficients are positive and the bifurcation takes place as λ_1 passes through 0 .

(*iv*) $\lambda_2 < 0$: similar to (*ii*) the pair (26) consists after the bifurcation of hyperbolic equilibria ; **q.e.d.**

For equilibria outside the co–ordinate axes one has to solve the two equations

$$\begin{aligned} \frac{a}{6}p^2 + \frac{\mu}{2}q^2 + \lambda_1 &= 0 \\ \frac{\mu}{2}p^2 + \frac{b}{6}q^2 + \lambda_2 &= 0 \end{aligned}$$

and obtains the quartet of equilibria

$$(q,p) \;=\; \left(\pm\sqrt{\frac{6a\lambda_2 - 18\mu\lambda_1}{9\mu^2 - ab}}\,,\, \pm\sqrt{\frac{6b\lambda_1 - 18\mu\lambda_2}{9\mu^2 - ab}} \right) \tag{28}$$

provided that both expressions under the square root are positive. The equilibria (28) are again born in Hamiltonian pitchfork bifurcations, we will prove this for each case separately. Both square roots vanish simultaneously if and only if $(\lambda_1, \lambda_2) = (0,0)$.

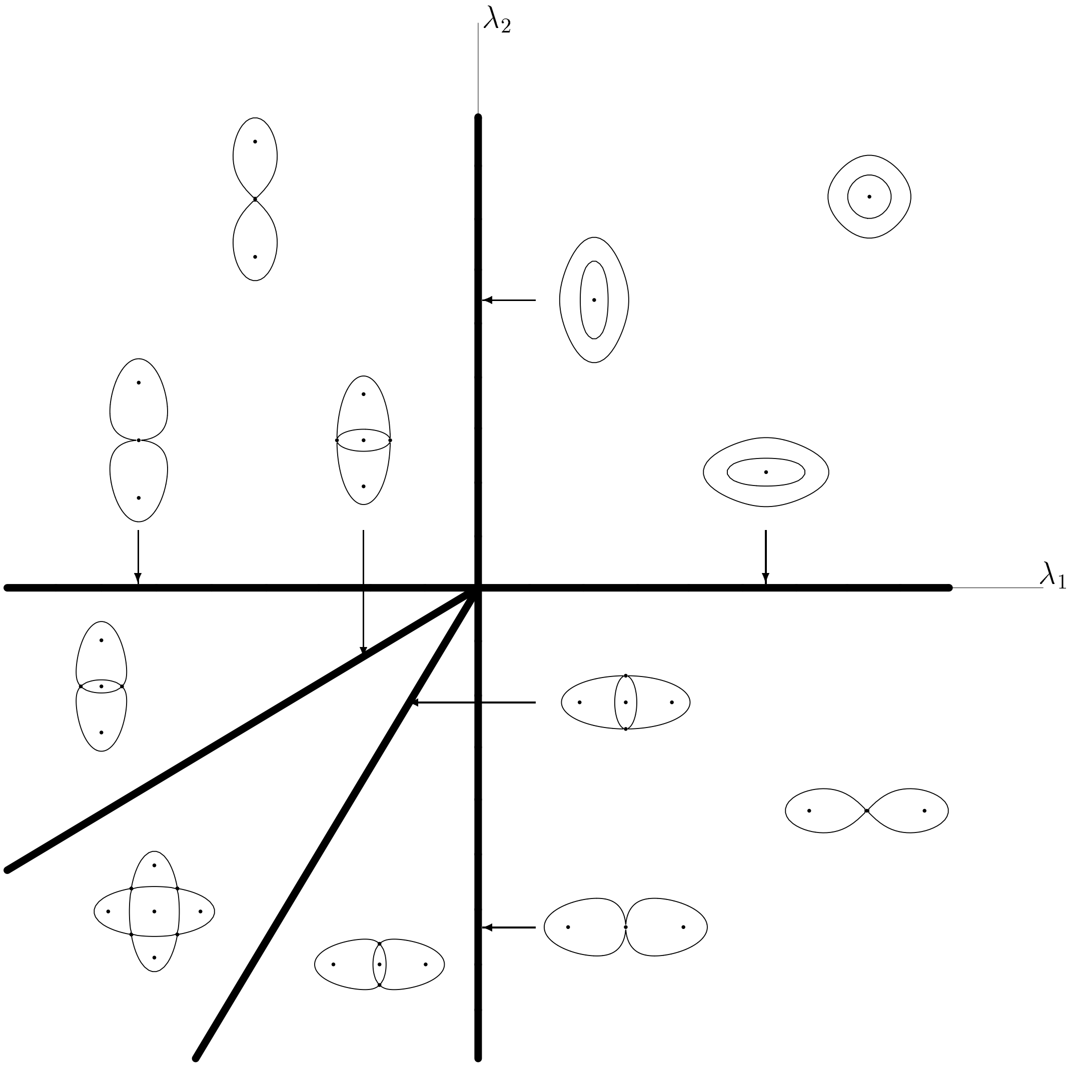

Figure 3.1: Bifurcation diagram of (23) for $a, b > 0$, $\mu > \frac{1}{3}\sqrt{ab}$. For $\lambda = 0$ the phase portrait looks like the one in the first quadrant.

For positive μ the square roots in (28) vanish in the third (and in the first) quadrant of the λ–plane. This leads to additional bifurcation lines in the third quadrant, given by

$$3\mu\lambda_1 \;=\; a\lambda_2 \;<\; 0 \quad \text{and} \quad 3\mu\lambda_2 \;=\; b\lambda_1 \;<\; 0 \quad . \tag{29}$$

Note that these two half lines coincide exactly when $\mu = \frac{1}{3}\sqrt{ab}$.

Lemma 3.2 *For $\mu > \frac{1}{3}\sqrt{ab}$ the two hyperbolic equilibria on the q–axis (on the p–axis) simultaneously undergo Hamiltonian pitchfork bifurcations at $3\mu\lambda_2 = b\lambda_1 < 0$ (at $3\mu\lambda_1 = a\lambda_2 < 0$, respectively). The only other bifurcations are those detailed in Lemma 3.1.*

Proof. The co–ordinate transformation

$$\Phi \; : \; \left\{ \begin{array}{rcl} Q & = & q - (1 + \frac{\mu}{4\lambda_2} p^2) \sqrt{-\frac{6\lambda_2}{b}} \\ P & = & p \end{array} \right. \tag{30}$$

turns (23) into

$$\mathcal{H}^\lambda(Q,P) \;=\; -\lambda_2 Q^2 \;+\; \frac{1}{24\,b}(ab - 9\mu^2)P^4 \;+\; \frac{1}{2b}(b\lambda_1 - 3\mu\lambda_2)P^2 \;+\; h.o.t. \tag{31}$$

where the constant term $-\frac{3\lambda^2}{2b}$ is omitted and the *higher order terms* amount to

$$\frac{1}{6}\sqrt{-6b\lambda_2}Q^3 \;-\; \frac{\mu}{2}P^2Q^2 \;-\; \frac{\mu^2}{16}\sqrt{\frac{-6}{b\lambda_2}}P^4Q \;+\; \frac{b}{24}(Q + \frac{\mu}{4}\sqrt{\frac{-6}{b\lambda_2}}P^2)^4 \quad .$$

In this way we recover the normal form (10) with a positive coefficient $-\lambda_2 > 0$ of the quadratic term Q^2 and a negative coefficient $\frac{1}{24\,b}(ab - 9\mu^2) < 0$ of the quartic term P^4 . Therefore, during the Hamiltonian pitchfork bifurcation at $b\lambda_1 - 3\mu\lambda_2 = 0$ a pair of hyperbolic equilibria detaches from the "origin" $(Q,P) = (0,0)$, *i.e.* from the equilibrium at $(q,p) = (+\sqrt{-\frac{6\lambda_2}{b}}, 0)$. Using the π–rotation (11) we obtain the same result at $(q,p) = (-\sqrt{-\frac{6\lambda_2}{b}}, 0)$. A similar computation shows that the equilibria (26) become parabolic at $a\lambda_2 - 3\mu\lambda_1 = 0$ where the equilibria (28) "return to the co–ordinate axes".

Outside the six half lines defined by (27) and (29) all equilibria are structurally stable. To rule out additional global bifurcations we have a closer look at the hyperbolic equilibria.

The equilibria (28) are hyperbolic throughout their existence. Since these equilibria are mapped to each other under the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the π–rotation (11) and the reflection (22), they always have the same energy and thus form a heteroclinic cycle. "Between" the two half lines (29) the other five equilibria are elliptic and no global bifurcation is possible.

Similarly, in the two wedges "between" the two half lines (29) and the negative co–ordinate axes the only hyperbolic equilibria are (25) or (26) and form a $\mathbb{Z}_2$–invariant heteroclinic cycle.

For $\lambda_1 \cdot \lambda_2 < 0$ the origin is the sole hyperbolic equilibrium, and there are no hyperbolic equilibria for (λ_1, λ_2) in the first quadrant ; **q.e.d.**

Lemma 3.3 *For $0 < \mu < \frac{1}{3}\sqrt{ab}$ there is a connection bifurcation at the half line $\sqrt{a}\,\lambda_2 = \sqrt{b}\,\lambda_1 < 0$. The only other bifurcations are the Hamiltonian pitchfork bifurcations of Lemma 3.1 and at the two half lines* (29).

Proof. For (λ_1, λ_2) near the half line $3\mu\lambda_2 = b\lambda_1 < 0$ the co–ordinate transformation (30) again yields the normal form (31) of the Hamiltonian near the equilibrium $(q,p) = (+\sqrt{-\frac{6\lambda_2}{b}}, 0)$, but now with a positive coefficient $\frac{1}{24\,b}(ab - 9\mu^2) > 0$ of the quartic term P^4 . Since the coefficient $-\lambda_2$ of the quadratic term Q^2 is positive as well, the additional equilibria (28) are elliptic, while the equilibria (25) turn from elliptic to hyperbolic. A similar Hamiltonian pitchfork bifurcation takes place at the half line $3\mu\lambda_1 = a\lambda_2 < 0$.

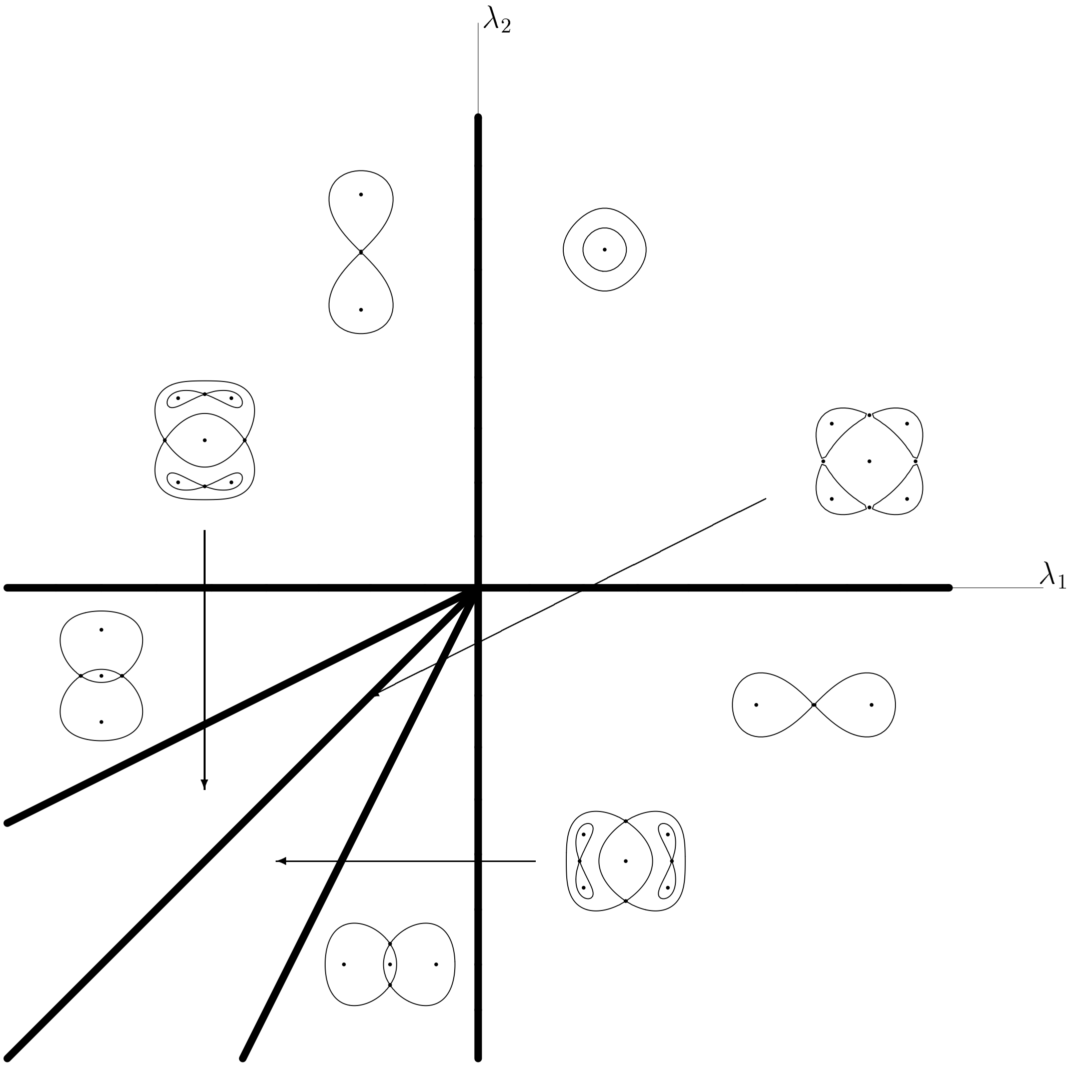

Figure 3.2: Bifurcation diagram of (23) for $a, b > 0$, $0 < \mu < \frac{1}{3}\sqrt{ab}$. For $\lambda = 0$ the phase portrait looks like the one in the first quadrant.

"Between" the two half lines (29) the equilibria (25) and (26) form two pairs of hyperbolic equilibria with energies

$$h_1^\lambda \;=\; \mathcal{H}^\lambda(0, \pm\sqrt{-\frac{6\lambda_1}{a}}) \;=\; -\frac{3\lambda_1^2}{2a} \quad \text{and} \quad h_2^\lambda \;=\; -\frac{3\lambda_2^2}{2b} \quad .$$

Where $h_1^\lambda = h_2^\lambda$ this leads to a heteroclinic cycle connecting all four equilibria. The transversality condition

$$(0,0) \;\neq\; \nabla_\lambda(h_1^\lambda - h_2^\lambda) \;=\; (-\frac{3\lambda_1}{a}, \frac{3\lambda_2}{b})$$

ensures that this happens in a non–degenerate way ; **q.e.d.**

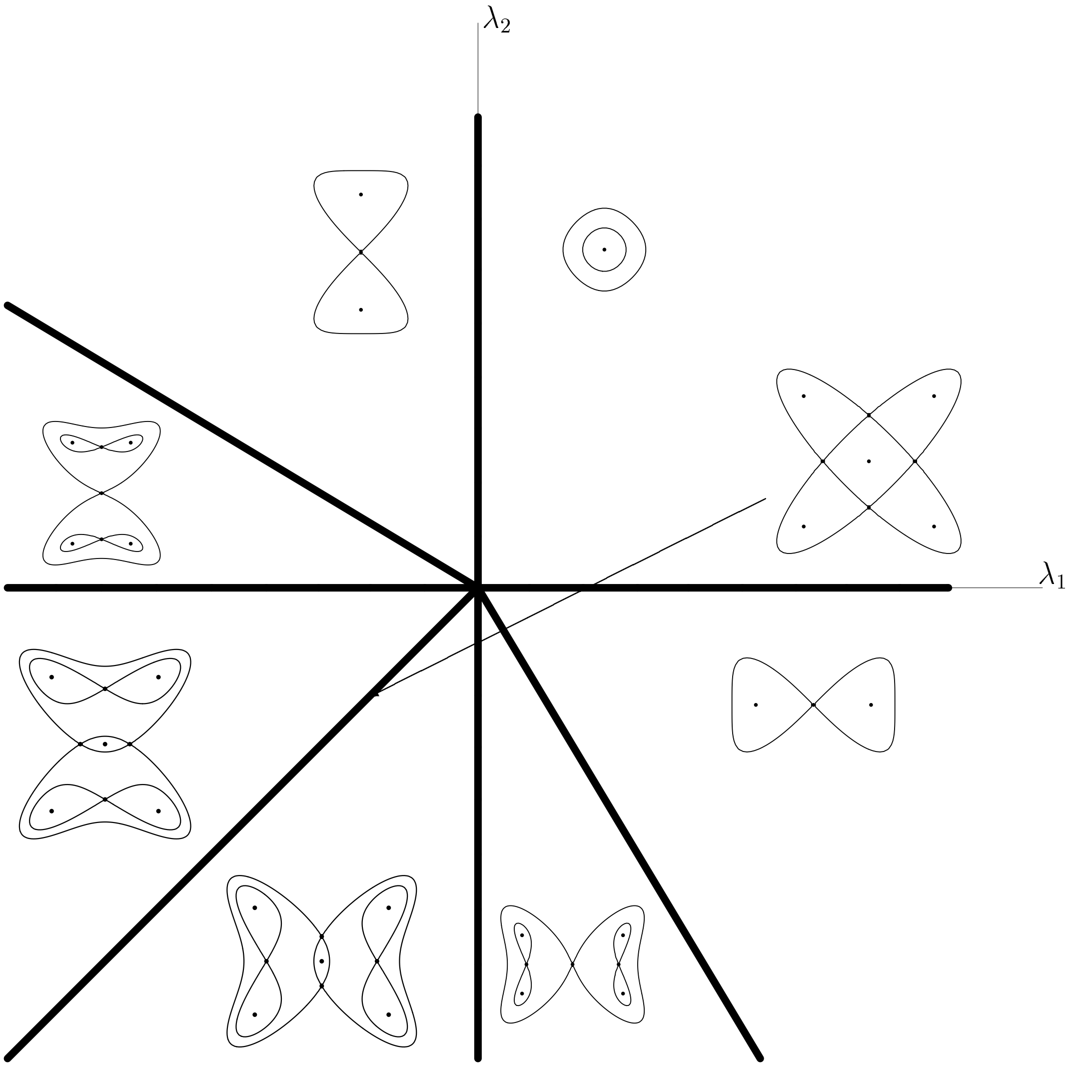

Figure 3.3: Bifurcation diagram of (23) for $a, b > 0\,,\ -\frac{1}{3}\sqrt{ab} < \mu < 0$. For $\lambda = 0$ the phase portrait looks like the one in the first quadrant.

For negative μ the square roots in (28) vanish in the second and fourth quadrant of the λ–plane, leading to the bifurcation lines

$$\left\{\ \lambda \in \mathbb{R}^2 \ \middle|\ 3\mu\lambda_2 = b\lambda_1\,,\ \lambda_1 > 0\,,\ \lambda_2 < 0\ \right\} \tag{32}$$

$$\left\{\ \lambda \in \mathbb{R}^2 \ \middle|\ 3\mu\lambda_1 = a\lambda_2\,,\ \lambda_1 < 0\,,\ \lambda_2 > 0\ \right\} \quad . \tag{33}$$

Lemma 3.4 *For $a, b > 0\,,\ -\frac{1}{3}\sqrt{ab} < \mu < 0$ there is a connection bifurcation at the half line $\sqrt{a}\,\lambda_2 = \sqrt{b}\,\lambda_1 < 0$. The only other bifurcations are the Hamiltonian pitchfork bifurcations of Lemma 3.1 and at the two half lines* (32) *and* (33).

Proof. After the co–ordinate transformation (30), for (λ_1, λ_2) near (32), we have positive coefficients $-\lambda_2 > 0$ and $\frac{1}{24\,b}(ab - 9\mu^2) > 0$ of both the quadratic term Q^2 and the quartic term P^4 in the normal form (31). Similarly for the bifurcating equilibria (26) as (λ_1, λ_2) crosses (33). Therefore the equilibria (28) are always elliptic, while in the third quadrant of the λ–plane (25) and (26) form two pairs of hyperbolic equilibria. From here on the second paragraph of the proof of Lemma 3.3 applies verbatim ; **q.e.d.**

We see that letting μ pass through zero merely results in a change of order in which the local bifurcations occur on a circular path around $(\lambda_1, \lambda_2) = (0, 0)$. Recall that the passage through $\mu = +\frac{1}{3}\sqrt{ab}$ had resulted in the additional global bifurcation at

$$\left\{ \lambda \in \mathbb{R}^2 \;\middle|\; \sqrt{a}\,\lambda_2 = \sqrt{b}\,\lambda_1 \,,\, \lambda_1 < 0 \,,\, \lambda_2 < 0 \right\} \quad . \tag{34}$$

The passage through $\mu = -\frac{1}{3}\sqrt{ab}$ turns out to be much more spectacular as one might argue that the equilibria (28) “disappear through infinity”. In particular there are two more half lines

$$\left\{ \lambda \in \mathbb{R}^2 \;\middle|\; b\lambda_1 = (3\mu - \sqrt{9\mu^2 - ab})\lambda_2 \,,\, \lambda_1 > 0 \,,\, \lambda_2 < 0 \right\} \tag{35}$$

$$\left\{ \lambda \in \mathbb{R}^2 \;\middle|\; a\lambda_2 = (3\mu - \sqrt{9\mu^2 - ab})\lambda_1 \,,\, \lambda_1 < 0 \,,\, \lambda_2 > 0 \right\} \tag{36}$$

where global bifurcations occur.

Lemma 3.5 *For $\mu < -\frac{1}{3}\sqrt{ab}$ the bifurcation diagram consists of connection bifurcations at* (34),(35),(36) *and of Hamiltonian pitchfork bifurcations at* (27),(32),(33).

Proof. The coefficients of the normal form (31) obtained after the co–ordinate transformation (30) reveal that the equilibria (28) are hyperbolic throughout their existence, which is this time the area “between” (32) and (33) that includes the first quadrant. While this has no influence on the formation of the heteroclinic cycle formed by the hyperbolic equilibria (25) and (26) in the third quadrant, this leads to additional bifurcations in the second and fourth quadrant when the energy

$$h^\lambda \;=\; \frac{3(b\lambda_1^2 + a\lambda_2^2 - 6\mu\lambda_1\lambda_2)}{2(9\mu^2 - ab)} \tag{37}$$

of the equilibria (28) coincides with the energy 0 of the hyperbolic equilibrium at the origin. This happens at (35) and (36), satisfying the transversality condition

$$(0, 0) \;\neq\; \nabla_\lambda h^\lambda \;=\; \frac{3}{9\mu^2 - ab}(b\lambda_1 - 3\mu\lambda_2, a\lambda_2 - 3\mu\lambda_1)$$

at every point of these two sets ; **q.e.d.**

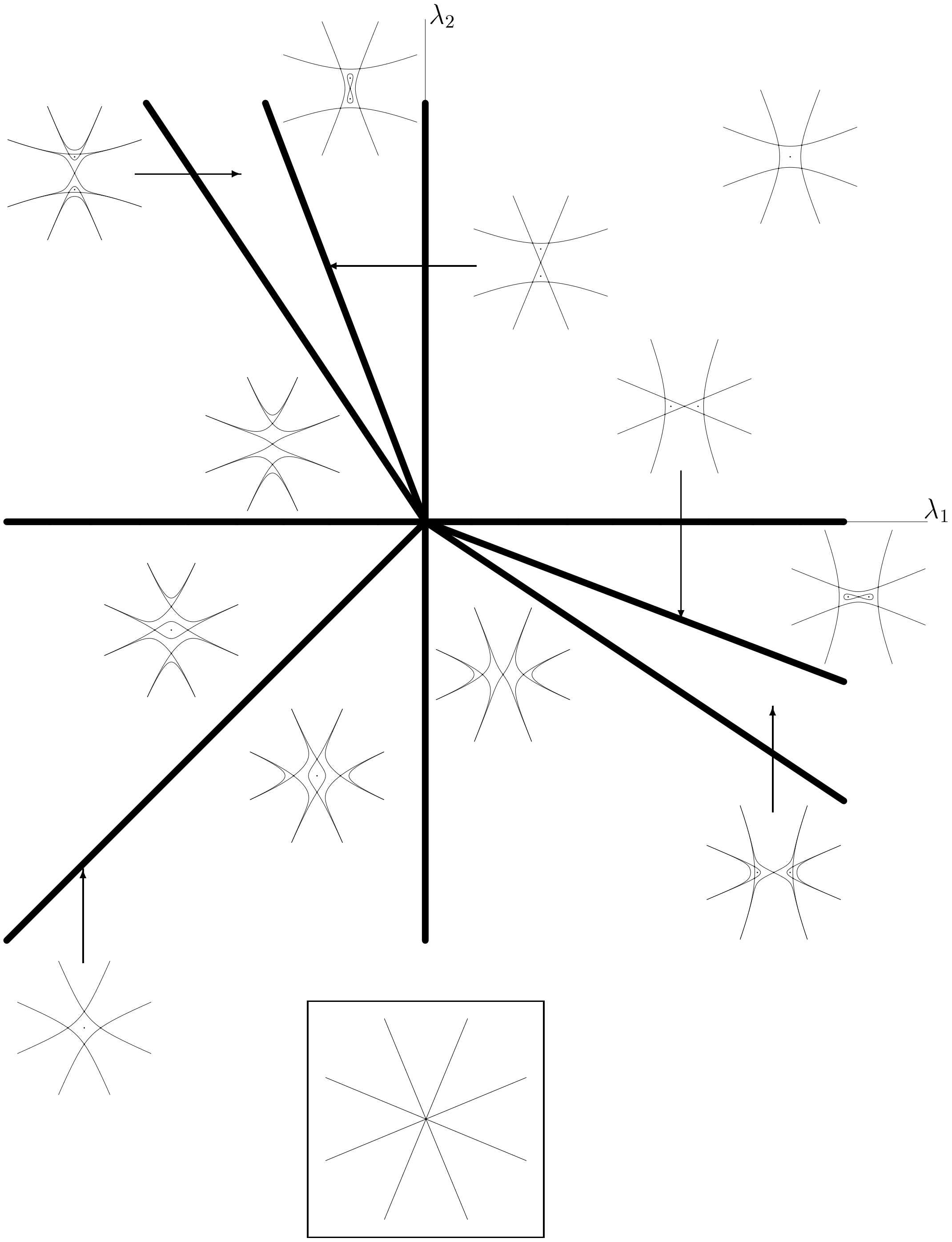

Figure 3.4: Bifurcation diagram of (23) for $a, b > 0$, $\mu < -\frac{1}{3}\sqrt{ab}$. The phase portrait for $\lambda = 0$ is given below in the box.

3.2.2 Cases with $a \cdot b < 0$

When a and b have different signs the condition $9\mu^2 \neq ab$ is empty. We can always arrange $a > 0$, $b < 0$ and $\mu > 0$ by, if necessary, multiplying (23) with -1 and interchanging the rôles of q and p (thereby rearranging the λ_1– and λ_2–axes). This case had already been studied in [Broer,Lunter,Vegter;98]. The pairs of equilibria on the q– and p–axes have energies $\mathcal{H}^\lambda(0, \pm\sqrt{-\frac{6\lambda_1}{a}}) = -\frac{3\lambda_1^2}{2a} < 0$ and $\mathcal{H}^\lambda(\pm\sqrt{-\frac{6\lambda_2}{b}}, 0) = -\frac{3\lambda_2^2}{2b} > 0$. Therefore they do not lead to global bifurcations (in fact where (25) or (26) are hyperbolic all other equilibria are elliptic). Where they exist the equilibria (28) are hyperbolic and have energy h^λ given by (37). Along these lines one proves the following result.

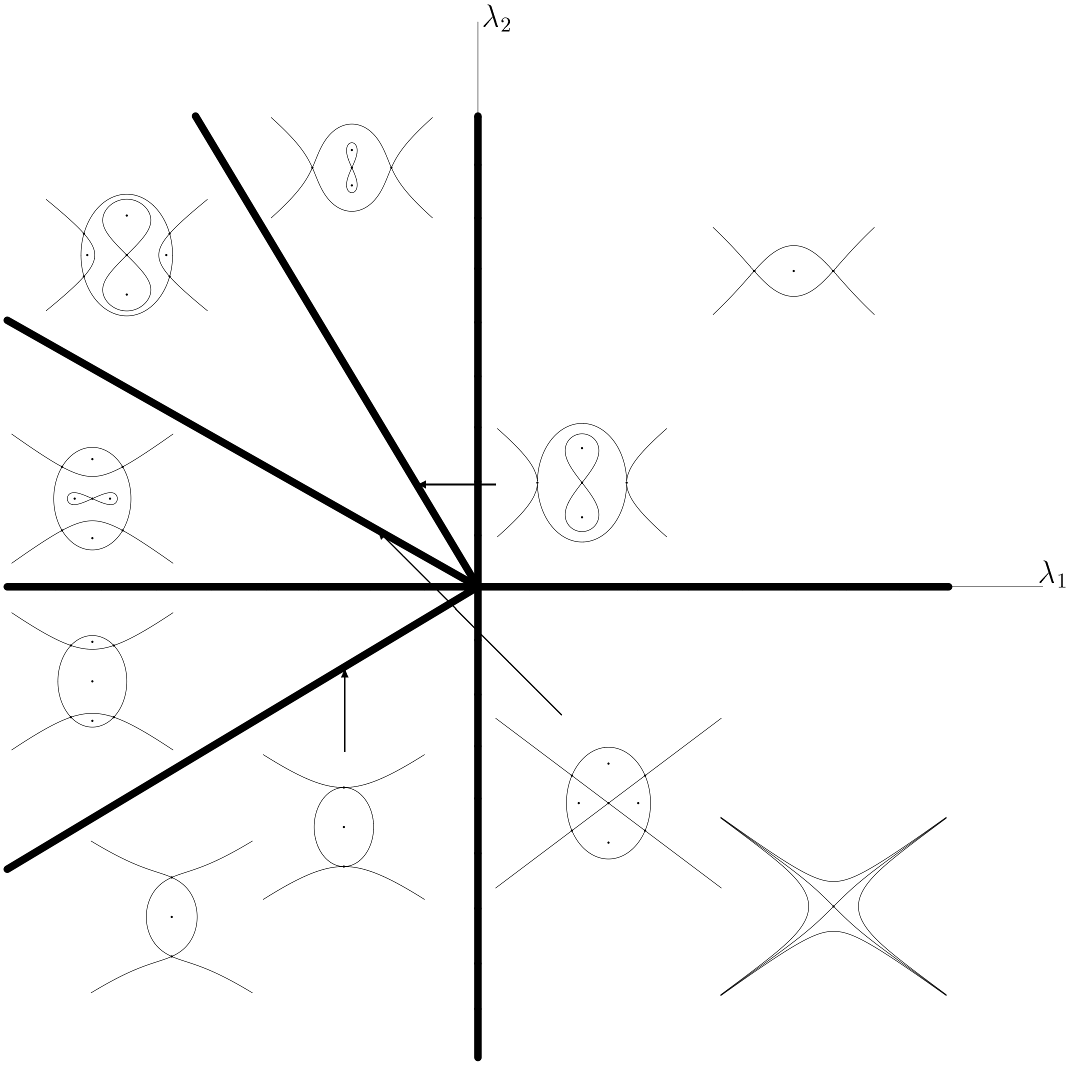

Figure 3.5: Bifurcation diagram of (23) for $a > 0$, $b < 0$, $\mu > 0$. For $\lambda = 0$ the phase portrait looks like the one in the fourth quadrant.

Lemma 3.6 *For* $a > 0$, $b < 0$, $\mu > 0$ *the bifurcation diagram consists of a connection bifurcation at*

$$\left\{ \lambda \in \mathbb{R}^2 \;\middle|\; a\lambda_2 = (3\mu - \sqrt{9\mu^2 - ab})\lambda_1 \, , \, \lambda_1 < 0 \, , \, \lambda_2 > 0 \right\}$$

and of Hamiltonian pitchfork bifurcations at (27) *and at*

$$\left\{ \lambda \in \mathbb{R}^2 \;\middle|\; 3\mu\lambda_2 = b\lambda_1 \, , \, \lambda_1 < 0 \, , \, \lambda_2 > 0 \right\}$$
$$\left\{ \lambda \in \mathbb{R}^2 \;\middle|\; 3\mu\lambda_1 = a\lambda_2 \, , \, \lambda_1 < 0 \, , \, \lambda_2 < 0 \right\} \quad .$$

3.3 The universal 3–parameter unfolding

The extra unfolding term $\lambda_3 pq$ in (20) breaks the reversibility of (23). To get an overview on the ensuing dynamical changes it is helpful to explicitly reduce the $\mathbb{Z}_2$–symmetry generated by the π–rotation (11) and consider the flows of (21) with $\lambda_3 = 0$ and $\lambda_3 \neq 0$ on the reduced phase space $\mathcal{C}$ defined by (12). For instance, the "primary" Hamiltonian pitchfork bifurcations of Lemma 3.1 remain unchanged for $\lambda_3 \neq 0$, while the "secondary" Hamiltonian pitchfork bifurcations (which generate the four equilibria (28)) become cuspoidal bifurcations of parabolic equilibria and split into centre–saddle bifurcations. Similarly, the open regions in the (λ_1, λ_2)–plane where the equilibria (28) form heteroclinic cycles become hypersurfaces (*i.e.* planes) of connection bifurcations in the $(\lambda_1, \lambda_2, \lambda_3)$–space. On the other hand there are no degenerate Hamiltonian pitchfork bifurcations since that would require degenerate fourth order terms. Instead of a detailed computation necessary to construct the 3–dimensional bifurcation diagrams we content ourselves with a more abstract result.

Proposition 3.7 *For fixed* $a, b, \mu \neq 0$ *with* $9\mu^2 \neq ab$ *the* 3*–parameter family of* $\mathbb{Z}_2$*–equivariant Hamiltonian systems defined by* (20) *is structurally stable.*

Proof. Let $\mathcal{H}^{\lambda_1,\lambda_2,\lambda_3}$ denote a small $\mathbb{Z}_2$–invariant perturbation of (20). From singularity theory, cf. [Beer;76], [Poènaru;76], we obtain a right equivalence between $\mathcal{H}^\lambda$ and a Hamiltonian function $\widehat{\mathcal{N}}^\lambda$ in normal form (20). This right equivalence of planar singularities is a C^∞–equivalence of the two local flows defined by these Hamiltonians, see [Broer,Chow,Kim,Vegter;93] (only if preserving the symplectic structure it is a C^∞–conjugacy, with no time re–parametrisation necessary). Compared to $\mathcal{N}^\lambda$, in $\widehat{\mathcal{N}}^\lambda$ the coefficient μ of p^2q^2 has to be changed to a new value $\hat{\mu}$ (while $\hat{a} = a$ and $\hat{b} = b$). Therefore another re–parametrisation in λ is necessary to let the bifurcation diagrams of $\widehat{\mathcal{N}}^\lambda$ and $\mathcal{N}^\lambda$ match. Since all occurring singularities outside $\lambda = 0$ are simple there are local equivalences around the equilibria which can be joined to global C^∞–equivalences of the two flows. In case $\hat{\mu} \neq \mu$ the two flows at $\lambda = 0$ are not C^∞–equivalent. This can easily be seen for $a, b > 0$ and $\mu, \hat{\mu} < -\frac{1}{3}\sqrt{ab}$ where the stable and unstable manifolds form 4 straight lines with different cross ratios. In the other cases two or all four lines become imaginary, but still prevent a C^k–equivalence for any $k \geq 1$. However, it is straighforward to construct a C^0–equivalence between the two flows ; **q.e.d.**

The above proof shows that diffeomorphisms are too rigid to deal with the planar singularities (and their unfoldings) underlying the local bifurcations in one degree of freedom, whence one has to work with homeomorphisms instead. In fact, the greater flexibility of the latter not only takes care of the modulus μ in $\mathcal{N}^0$, but of all moduli that exist with respect to right equivalences of planar singularities given by diffeomorphisms, cf. [Damon;88], [du Plessis,Wall;95].

3.4 Reconstruction of the full flow

For fixed $(\lambda, \omega) \in \Lambda \times \Sigma \subseteq \mathbb{R}^3 \times \mathbb{R}^n$ it is straightforward to reconstruct the flow of the Hamiltonian system defined by (19) on $\mathcal{P} = \mathbb{T}^n \times \mathsf{Y} \times \mathsf{S}$. Indeed, one merely has to superpose the quasi–periodic flow

$$x(t) \; = \; x_0 + t\omega \quad , \quad y(t) \; = \; y_0$$

to the phase portraits of the one–degree–of–freedom problem (20).

To pass to the base space $\mathcal{B}$ we have to identify points on $\mathcal{P}$ that are mapped onto each other under the deck transformation (6). In this way pairs of invariant tori project down to a single torus in the base. The invariant tori of the family $\mathbb{T}^n \times \mathsf{Y} \times \{0\}$ are each projected to one invariant torus with the frequency ω_1 multiplied by 2 . In this way *e.g.* the bifurcation diagrams in *figures 3.1–3.5* give rise to similar bifurcation diagrams of (Y–families of) invariant tori.

4 Quasi–periodic persistence

Given a perturbation of (2) that is invariant not only under the involution (6), but also with respect to the $\mathbb{T}^n$–action (9) we can use the latter action to reduce to one degree of freedom and conclude that such an integrable perturbation leads to similar phase portraits of invariant tori. To what extent can one prove such a result for non–integrable perturbations ?

We cannot expect resonant tori to survive general perturbations. To obtain a sufficiently strong form of non–resonance we impose Diophantine conditions of the form (16) (and (17) where normally elliptic tori are concerned) with $\gamma > 0$ and $\tau > 2n - 1$. In this way the smooth families of invariant tori of (2) are restricted to Cantor families. To prevent that the most degenerate tori are isolated (and hence may fall into a resonance gap) the number $n+1$ of degrees of freedom has to be at least five. This allows to prove a result along the lines of Proposition 2.2.

Theorem 4.1 *For $n \geq 4$ let $\mathsf{Y} \subseteq \mathbb{R}^n$ be a neighbourhood of the origin and $\mathsf{S} \subseteq \mathbb{R}^2$ a neighbourhood of the origin that is invariant under the π–rotation (11). Supply $\mathcal{P} = \mathbb{T}^n \times \mathsf{Y} \times \mathsf{S}$ with the symplectic structure $\sum \mathrm{d}x_i \wedge \mathrm{d}y_i + \mathrm{d}q \wedge \mathrm{d}p$ and consider a (real analytic) integrable Hamiltonian function $\tilde{H}$ on the base space $\mathcal{B} = \mathcal{P}/_{\mathbb{Z}_2}$ defined by the deck transformation (6). Let H denote the lift of $\tilde{H}$ to $\mathcal{P}$ and assume that as a Taylor series in q and p it starts as*

$$H(x, y, q, p) \;=\; h(y) + \frac{a(y)}{24}p^4 + \frac{b(y)}{24}q^4 + \frac{\mu(y)}{4}p^2q^2 + \frac{c_1(y)}{2}p^2 + \frac{c_2(y)}{2}q^2 + c_3(y)pq + \ldots$$

with $a(0), b(0), \mu(0) \neq 0$ and $9(\mu(0))^2 \neq a(0) \cdot b(0)$. The three coefficients $c(y)$ satisfy $c(0) = 0$ and the mapping

$$\begin{array}{rccc} c: & \mathsf{Y} & \longrightarrow & \mathbb{R}^3 \\ & y & \mapsto & c(y) \end{array}$$

is a submersion. Furthermore the $\binom{n+2}{2}$ vectors

$$\frac{\partial^{|\ell|}}{\partial y^\ell} \begin{pmatrix} c \\ \omega \end{pmatrix} \quad , \quad |\ell| \leq 2 \tag{38}$$

span $\mathbb{R}^3 \times \mathbb{R}^n$, where $\omega(y) = Dh(y)$. Then a sufficiently small perturbation of $\tilde{H}$ has a flow with Cantor families of invariant n–tori of the following Hausdorff dimensions.

The most degenerate invariant n–tori are parametrised by Cantor sets of dimension $n-3$ and there are Cantor sets of dimension $n-1$ parametrising n–tori that undergo (non–degenerate) frequency–halving bifurcations. Invariant n–tori that undergo a (quasi–periodic) centre–saddle bifurcation are parametrised by Cantor sets of dimension $n-1$ and the Cantor sets of dimension $n-2$ where such families meet parametrise cuspoidal bifurcations of normally parabolic n–tori.

In the reversible case we can be a bit more specific since the corresponding bifurcation diagrams were explicitly given in the previous section. The key observation is that the reversing symmetry (8) can be respected during every step of the proof of Theorem 4.1. As usual the extra symmetry (8) lowers the co–dimension of occurring bifurcations, so invariant lower dimensional tori with vanishing normal linear behaviour may already occur in reversible Hamiltonian systems with four degrees of freedom.

Corollary 4.2 *For $n \geq 3$ let $\mathcal{B}$ and $\mathcal{P}$ be as in Theorem 4.1. Consider a (real analytic) integrable Hamiltonian function $\tilde{H}$ on $\mathcal{B}$ with lift H on $\mathcal{P}$. Assume that H starts as a Taylor series in q and p as*

$$H(x,y,q,p) \;=\; h(y) \;+\; \frac{a(y)}{24}p^4 \;+\; \frac{b(y)}{24}q^4 \;+\; \frac{\mu(y)}{4}p^2q^2 \;+\; \frac{c_1(y)}{2}p^2 \;+\; \frac{c_2(y)}{2}q^2 \;+\; \ldots$$

with $a(0), b(0), \mu(0) \neq 0$ and $9(\mu(0))^2 \neq a(0)\cdot b(0)$. The two coefficients $c(y)$ satisfy $c(0) = 0$ and the mapping

$$c : \; \mathsf{Y} \;\longrightarrow\; \mathbb{R}^2$$

is a submersion. Furthermore the $\binom{n+2}{2}$ vectors

$$\frac{\partial^{|\ell|}}{\partial y^\ell}\begin{pmatrix} c \\ \omega \end{pmatrix} \quad , \quad |\ell| \leq 2$$

span $\mathbb{R}^2 \times \mathbb{R}^n$, where $\omega(y) = Dh(y)$. Then the flow of a sufficiently small reversible perturbation of $\tilde{H}$ has invariant lower dimensional tori organized according to the bifurcation diagrams in figures 3.1–3.5. The most degenerate n–tori are parametrised by Cantor sets of dimension $n-2$ and there are Cantor sets of dimension $n-1$ parametrising n–tori that undergo frequency–halving or quasi–periodic Hamiltonian pitchfork bifurcations.

Note that we did not address the subordinate global bifurcations. Indeed, the circumstances of the formation of heteroclinic orbits change drastically under non–integrable perturbations. In the generic case the stable and unstable manifolds that coincide for the unperturbed system have transversal intersections, which are expected to be exponentially small. As a result the region in parameter space where connection bifurcations occur get replaced by open regions in the form of an (exponentially small) "horn", cf. [Broer,Roussarie;01]. The boundary lines of the horn stand for "first" heteroclinic tangencies (there are no heteroclinic orbits for parameter values outside the horn) and within the horn "secondary" heteroclinic tangencies are *abundant*. But while such scenarios require certain genericity conditions on the perturbation, the conclusions of Theorem 4.1 and Corollary 4.2 are valid for all (sufficiently small) perturbations.

4.1 Simplification to a parameter–dependent form

The difficult part in proving Theorem 4.1 is to keep track of the most degenerate "object" in the perturbed system. Our first step is therefore to look for the invariant n–tori with normal linear behaviour $\binom{00}{00}$ on the 2:1 cover.

To simplify the situation we decouple the persistence problem from the way the persisting tori are parametrised by the actions y conjugate to the toral angles x . Next to treating the frequencies $\omega_1, \ldots, \omega_n$ as external parameters we introduce unfolding parameters λ_1, λ_2 and λ_3 . This allows us to bring the lower order terms of the unperturbed Hamiltonian in Theorem 4.1 into the form (19) defined on $\mathcal{P} \times \Lambda \times \Sigma$. Here $\Lambda \subseteq \mathbb{R}^3$ is a neighbourhood of the origin and $\Sigma \subseteq \mathbb{R}^n$ is a (possibly unbounded) domain of frequency vectors. By Σ_γ we denote the set of those frequencies that satisfy the Diophantine conditions (16) and further restrict to $\Sigma'_\gamma := \left\{ \omega \in \Sigma_\gamma \;\middle|\; d(\omega, \partial\Sigma) \geq \gamma \right\}$. Furthermore $|..|_{_A}$ stands for the supremum norm on the set A .

Theorem 4.3 *Let the coefficient functions $a, b, \mu : \Sigma \longrightarrow \mathbb{R}$ in the normal form*

$$N(x,y,q,p;\lambda,\omega) \;=\; (\omega \mid y) + \frac{a(\omega)}{24}p^4 + \frac{b(\omega)}{24}q^4 + \frac{\mu(\omega)}{4}p^2q^2 + \frac{\lambda_1}{2}p^2 + \frac{\lambda_2}{2}q^2 + \lambda_3 pq \quad (39)$$

satisfy the bounds

$$|a|_{_\Sigma} \,,\; |b|_{_\Sigma} \,,\; |\mu|_{_\Sigma} \,,\; |Da|_{_\Sigma} \,,\; |Db|_{_\Sigma} \,,\; |D\mu|_{_\Sigma} \,,\; \left|\frac{1}{a}\right|_{_\Sigma} ,\; \left|\frac{1}{b}\right|_{_\Sigma} ,\; \left|\frac{1}{\mu}\right|_{_\Sigma} ,\; \left|\frac{1}{9\mu^2 - ab}\right|_{_\Sigma} \;<\; C$$

for some constant $C > 0$. Then there exists a small positive constant $\delta > 0$, independent of Σ , with the following property. For any real analytic perturbation $H = N + P$ of (39) with

$$|P|_{_{\mathbb{T}^n \times \mathsf{Y} \times \mathsf{S} \times \Lambda \times \Sigma}} \;<\; \delta$$

that is invariant under the deck transformation σ defined in (6) there exists a σ–equivariant Gevrey–regular diffeomorphism Φ on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^3 \times \Sigma$ such that

1). *Φ is real analytic for fixed ω .*

2). *Φ is symplectic for fixed (λ, ω) .*

3). *Φ is C^∞–close to the identity.*

4). *On $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^3 \times \Sigma'_\gamma \,\cap\, \Phi^{-1}(\mathbb{T}^n \times \mathsf{Y} \times \mathsf{S} \times \Lambda \times \Sigma)$ one can split $H \circ \Phi = N_\infty + P_\infty$ into an integrable part N_∞ and higher order terms P_∞ . Here N_∞ has the same form (39) as N . The x–dependence is pushed into the higher order terms, i.e. $\dfrac{\partial^{|j|+k+l+|m|} P_\infty}{\partial y^j \partial p^k \partial q^l \partial \lambda^m}(x,0,0,0;0,\omega) \;=\; 0$ for all $(x,\omega) \in \mathbb{T}^n \times \Sigma'_\gamma$ and all j,k,l,m satisfying $4|j| + k + l + 2|m| \leq 4$.*

We prove Theorem 4.3 in the next section, using a KAM iteration scheme. Let us first use this result to derive Theorem 4.1.

The line of argumentation follows the sketch of proof we gave for Proposition 2.2. In the new co–ordinates provided by Φ the parameters λ unfold the bifurcation diagram of the most degenerate invariant tori $\mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \times \{\omega\}$, $\omega \in \Sigma'_\gamma$. This yields the whole hierarchy of normally parabolic invariant tori undergoing bifurcations that we encountered in Section 3. The open domains of the bifurcation diagrams parametrise normally hyperbolic and elliptic n–tori. For persistence the latter have to satisfy the additional Diophantine conditions (17) involving the normal frequency.

Since $c : \mathsf{Y} \longrightarrow \mathbb{R}^3$ is a submersion, 3 of the $n \geq 4$ actions y conjugate to the toral angles x account for the unfolding parameters λ . Choosing the constant $\tau > 2n - 1$ in (16) and (17), the curvature condition on the $\binom{n+2}{2}$ vectors (38) ensures that the pre–image of Diophantine tori under the frequency mapping $\omega : \mathsf{Y} \longrightarrow \mathbb{R}^n$ still has large relative measure (tending to 1 as $\gamma \to 0$).

5 Proof of Theorem 4.3

Our aim is to find a co–ordinate transformation Φ that transforms away — to some extent — the small perturbation P of the integrable normal form N and thus allows to recover the bifurcation scenario. The idea is to construct Φ as a limit of transformations $(\Phi_i)_{i \in \mathbb{N}}$ that approximately solve the equation $H \circ \Phi = N$. For this KAM iteration scheme we follow the quite universal set–up of [Moser;67], [Pöschel;89], [Broer,Huitema,Takens;90] in the form given in [Broer,Hanßmann,You;03].

At each iteration step we perform two operations. First the "lower order terms" are made x–independent, and then these "lower order terms" are put into normal form (39). The latter is achieved by an explicit combination (47) of two transvections, but for the former we have to deal with small denominators to solve a homological equation. This imposes restrictions on the domains where Φ_i is defined.

Indeed at each iteration step we can use an "ultraviolet cut–off" η_i for the order of the resonances so that we only have to deal with finitely many of them whence Φ_i is defined on an open neighbourhood $U_{r_i}(\Sigma'_\gamma)$ of our set Σ'_γ of Diophantine frequencies. In the limit we need $\eta_i \stackrel{i \to \infty}{\longrightarrow} \infty$ whence these neighbourhoods shrink down to the Cantor set Σ'_γ . In fact the domain D_i of definition of Φ_i has to shrink in each of its components, and we use a carefully balanced interplay between geometrically and exponentially fast converging sequences (43) to obtain (via the Inverse Approximation Lemma of [Wagener;03]) the desired smoothness of the limit transformation.

We can restrict ourselves to fixing the Diophantine constant $\gamma = 1$ when proving Theorem 4.3. This allows for a more transparent argumentation where the sizes r_i of the shrinking neighbourhoods of Σ'_1 are effectively decoupled from the Diophantine constant $\gamma > 0$. To extend the result thus obtained to the Σ'_γ of Theorem 4.3 one can resort to a simple scaling argument, cf. [Broer,Hanßmann,You;03] or [Broer,Huitema,Takens;90], [Braaksma,Broer,Huitema;90].

At the ith iteration step we construct a diffeomorphism $\Psi_i : D_{i+1} \longrightarrow \Psi_i(D_{i+1}) \subseteq D_i$ with $H_i \circ \Psi_i = H_{i+1}$. By composition we then define

$$\Phi_{i+1} \; := \; \Psi_0 \circ \Psi_1 \circ \ldots \circ \Psi_i \quad : \quad D_{i+1} \; \longrightarrow \; D_0$$

and give the domains D_i the product structure

$$D_i \;=\; \mathcal{T}_i \times \mathcal{D}_i \times \mathcal{U}_i \;\subseteq\; \mathbb{K}^n \times \mathbb{C}^{n+5} \times \mathbb{C}^n$$

where $\mathbb{K} = \mathbb{T} + \mathrm{i}\mathbb{R}$ is the complex cylinder to the real circle $\mathbb{T} = \mathbb{R}/_{\mathbb{Z}}$. The reason to consider our Hamiltonians $H_i = N_i + P_i$ on (shrinking) complex domains is that this allows to control derivatives by the supremum norm using Cauchy's inequality. We use an exponentially fast converging sequence $\varepsilon_i \to 0$ to define

$$\mathcal{D}_i \;=\; \left\{ \, (y, q, p; \lambda) \in \mathbb{C}^{n+5} \;\middle|\; |y| \le \varepsilon_i^4 \,,\, |q|, |p| \le \varepsilon_i \,,\, |\lambda| \le \varepsilon_i^2 \, \right\}$$

and also put $\delta_i := \varepsilon_i^{4+\xi}$ with $0 < \xi < 1$ for a bound

$$|P_i|_{D_i} \;\le\; \delta_i$$

on the perturbation part P_i of H_i . The aim of the ith iteration step is to achieve this same bound with i replaced by $i+1$. In the angular direction we use a geometrically fast converging sequence $s_i \to 0$ for the open neighbourhoods

$$\mathcal{T}_i \;=\; U_{\kappa + s_i}(\mathbb{T}^n)$$

of radius $\kappa + s_i$ in view of the Paley–Wiener estimate on exponential decay of the Fourier coefficients of P_i . Since

$$\bigcap D_i \;=\; U_\kappa(\mathbb{T}^n) \times \{(0, 0, 0; 0)\} \times \Sigma'_1$$

the limit of the Φ_i will still be analytic in x . We use polynomial truncations in y, q, p, λ to define the limit Φ_∞ , with coefficient functions defined on $\mathcal{T}_i \times \mathcal{U}_i$. To obtain (Whitney)–Gevrey–regularity of the limit of these coefficient functions on $U_\kappa(\mathbb{T}^n) \times \Sigma'_1$ we let $r_i \to 0$ geometrically fast and define

$$\mathcal{U}_i \;=\; U_{r_i}(\Sigma'_1)$$

as an open neighbourhood of radius r_i . For $\mathcal{U}_0$ we can even use $U_{r_0}(\Sigma)$ as the Diophantine conditions have not yet entered.

An additional complication is that a mere polynomial truncation of the Φ_i would cease to preserve the symplectic structure. For this reason we introduce generating functions S_i of the Φ_i , the polynomial truncations $\widetilde{S}_i$ of which generate symplecto–morphisms as well. The limit $\widetilde{S}_\infty$ of these then generates the desired Φ_∞ . As $\widetilde{S}_\infty$ is polynomial in y, q, p, λ , analytic in x and (Whitney)–Gevrey–regular in ω we conclude that Φ_∞ is (Whitney)–Gevrey–regular and even analytic in x, y, q, p, λ — as stated in Theorem 4.3.

5.1 The iteration step

During a single step of the KAM iteration we construct a co–ordinate transformation Ψ_i that turns the given Hamiltonian $H_i = N_i + P_i$ into a "new" Hamiltonian $H_{i+1} = N_{i+1} + P_{i+1}$ in such a way that $|P_{i+1}|_{D_{i+1}}$ is much smaller than $|P_i|_{D_i}$. In this process we concentrate on the "lower order terms" of P_i while the "higher order terms" are directly relegated to P_{i+1} . To make these notions precise we write P_i as a Taylor series

$$P_i(x, y, q, p; \lambda, \omega) \quad = \sum_{|j|+k+l+|m|=0}^{\infty} P_{i,jklm}(x, \omega)\, y^j p^k q^l \lambda^m$$

and expand the coefficient functions as Fourier series

$$P_{i,jklm}(x, \omega) \quad = \quad \sum_{h \in \mathbb{Z}^n} P^h_{i,jklm}(\omega)\, \mathrm{e}^{2\pi\sqrt{-1}(h|x)} \quad .$$

We introduce the truncation

$$R_i \quad = \quad \sum_{|h| \leq \eta_i} R^h_i \mathrm{e}^{2\pi\sqrt{-1}(h|x)} \tag{40}$$

with

$$\begin{aligned} R^h_i \quad &= \quad \sum_{|j|=1} P^h_{i,j000}(\omega)\, y^j \; + \; \sum_{k+l=1}^{4} P^h_{i,0kl0}(\omega)\, p^k q^l \\ &\quad + \sum_{|m|=1} \sum_{k+l=1}^{2} P^h_{i,0klm}(\omega)\, p^k q^l \lambda^m \; + \; \sum_{|m|=0}^{2} P^h_{i,000m}(\omega)\, \lambda^m \end{aligned}$$

and consider $P_i - R_i$ as the "higher order terms". Here the "ultraviolet cut–off" η_i is the maximal order $|h| = |h_1| + \ldots + |h_n|$ of the resonances we have to cope with at the ith stage.

Let F_i be a function defined in a domain $D \subseteq D_i$ with Hamiltonian vector field X_{F_i} and flow φ^i_t . For the time–1–flow φ^i_1 we then have

$$\begin{aligned} H_i \circ \varphi^i_1 \quad &= \quad (N_i + R_i) \circ \varphi^i_1 \; + \; (P_i - R_i) \circ \varphi^i_1 \\ &= \quad N_i \; + \; R_i \; + \; \{N_i, F_i\} \; + \; \{R_i, F_i\} \\ &\quad + \int_0^1 (1-t)\{\{N_i + R_i, F_i\}, F_i\} \circ \varphi^i_t \, \mathrm{d}t \; + \; (P_i - R_i) \circ \varphi^i_1 \\ &= \quad N_i \; + \; R_i \; + \; \{N_i, F_i\} \; + \; \bar{P}_i \end{aligned} \tag{41}$$

where $\bar{P}_i$ will be relegated to P_{i+1} . The usual aim in the KAM iteration step is to define F_i as a solution of the so–called homological equation

$$N_i + R_i + \{N_i, F_i\} \quad = \quad N_{i+1} \quad .$$

In the present bifurcational context the normal form N_i is nonlinear in p and q whence the Poisson bracket $\{N_i, F_i\}$ contains again "higher order terms" that can be relegated to P_{i+1} . Furthermore the "lower order terms" are not yet in normal form (39). We therefore split the iteration step into two parts.

1. We truncate $\{N_i, F_i\}$ at the same order as N_i and R_i and solve the "intermediate homological equation"

$$N_i \;+\; R_i \;+\; \{N_i, F_i\} \;=\; \bar{N}_i \tag{42}$$

up to that order. The "intermediate" $\bar{N}_i$, already independent of x , but not yet normalized in p and q , is defined as the sum $N_i + R_i^0$ of the average of the truncation (40) added to the old normal form. The solution of (42) leads to small denominators that can be estimated by means of the Diophantine conditions (16).

2. Then we transform $\bar{N}_i$ into normal form (39). This passage from $\bar{N}_i$ to N_{i+1} does not involve small denominators, but requires methods from singularity theory instead. Specifically, since only "lower order terms" are involved, this is achieved explicitly by a combination (47) of two transvections (while more general transformations from singularity theory usually do not preserve the symplectic structure, cf. [Broer,Chow,Kim,Vegter;93]). The translations used in [Broer,Hanßmann,You;03] would not respect the deck transformation (6) and it is indeed this symmetry that prevents the occurrence of many "lower order terms" in $\bar{N}_i$. In the reversible case we only have to use a re–parametrisation of λ_1, λ_2 to take care of terms in $\bar{N}_i$ that are not in normal form.

We now define the geometrically fast and exponentially fast converging sequences that entered the domains D_i , the bounds of $|P_i|$ and the "ultraviolet cut–off". Let $s_0 < \min\{\kappa, \frac{1}{2}\}$ and $\delta_0 < \frac{1}{2}$ be given, then we put

$$\begin{aligned} s_i &= \frac{s_0}{4^i} \\ r_i &= s_i^{2\tau+2} \\ \eta_i &= [s_i^{-2}] \\ \delta_i &= \delta_{i-1}^{\frac{\chi}{4+\xi}}\,\delta_{i-1} \;=\; \delta_0^{(1+\frac{\chi}{4+\xi})^i} \\ \varepsilon_i &= \delta_i^{\frac{1}{4+\xi}} \end{aligned} \tag{43}$$

with $0 < \chi < \xi < 1$. The constants in the estimates occurring during the iteration step will be absorbed in s_0 and δ_0 , leading to inequalities of the form

$$s_0 \;\leq\; c \quad , \qquad \delta_0 \;\leq\; c \quad , \qquad \delta_0^{\zeta} \;\leq\; c\, s_0$$

with constants $c > 0$ and exponents $\zeta > 0$. The only exception is an inequality

$$s_0 \;<\; \frac{1}{c - \zeta \ln(\delta_0)}$$

that serves to estimate the higher order Fourier coefficients of P_i . Since $\delta_0^{\zeta}\,\ln(\delta_0) \xrightarrow{\delta_0 \to 0} 0$ for all $\zeta > 0$ it is possible to find small s_0, δ_0 satisfying all these inequalities — δ_0 is the small positive constant δ sought for in Theorem 4.3. Since $(s_i)_i$ decreases geometrically and $(\delta_i)_i$ decreases exponentially fast we also include factors $\frac{1}{s_i}$ in the "generic constants", taking care that at each iteration step the total number of such factors remains finite and independent of i .

We are now able to formulate the iteration step in our proof of Theorem 4.3. Here and in the following lemmata and propositions we refrain from tediously repeating the assumptions we made so far.

Proposition 5.1 *There are $s_0 > 0$ and $\delta_0 > 0$ such that the following holds true. Given $H_i = N_i + P_i$ as above, there is a co–ordinate transformation*

$$\begin{array}{cccc} \Psi_i : & D_{i+1} & \longrightarrow & D_i \\ & (x, y, q, p; \lambda, \omega) & \mapsto & (\psi_{\lambda,\omega}(x, y, q, p); \lambda_+, \omega_+) \end{array}$$

with symplectic $\psi_{\lambda,\omega}$ such that $H_{i+1} := H_i \circ \Psi_i$, defined on D_{i+1} , has the form

$$H_{i+1} \; = \; N_{i+1} \; + \; P_{i+1}$$

with N_{i+1} in normal form (39) and perturbation P_{i+1} satisfying

$$\begin{aligned} |P_{i+1}|_{D_{i+1}} &\leq \delta_{i+1} \\ |a_{i+1} - a_i|_{\mathcal{U}_{i+1}} &\leq \varepsilon_i \\ |b_{i+1} - b_i|_{\mathcal{U}_{i+1}} &\leq \varepsilon_i \\ |\mu_{i+1} - \mu_i|_{\mathcal{U}_{i+1}} &\leq \varepsilon_i \quad . \end{aligned}$$

Moreover,

$$\left| \frac{\partial^{|j|+k+l+|m|} P_{i+1}}{\partial y^j \partial p^k \partial q^l \partial \lambda^m} \right|_{D_{i+1}} \leq \varepsilon_{i+1}^{4+\xi-\|jklm\|}$$

where $\|jklm\| := 4|j| + k + l + 2|m| \leq 4$.

The coefficient functions a_i, b_i and μ_i are of order one, *i.e.* they satisfy bounds as formulated in Theorem 4.3. The estimates by ε_i on the differences $|a_{i+1} - a_i|$, $|b_{i+1} - b_i|$ and $|\mu_{i+1} - \mu_i|$ imply that the same is true for a_{i+1}, b_{i+1} and μ_{i+1} as well, and also for the (existing) limit functions a_∞, b_∞ and μ_∞ .

The truncation (40) is inspired by (39) being a quasi–homogeneous polynomial with weight $(4, 1, 1; 2)$ in the variables $(y, q, p; \lambda)$, cf. [Arnol'd,Gusein–Zade,Varchenko;82], [Broer,Hanßmann,You;03], which in turn induces the weighted order $\|jklm\|$ on indices $j \in \mathbb{N}_0^n$, $k, l \in \mathbb{N}_0$, $m \in \mathbb{N}_0^3$ of monomials $y^j p^k q^l \lambda^m$ whence N_i has (weighted) order 4 . Note that the weights also entered the definition of the shrinking domains $\mathcal{D}_i$ and that the truncation (40) reads

$$R_i \;=\; \sum_{\|jklm\|=0}^{4} \sum_{|h| \leq \eta_i} P_{i,jklm}^h(\omega)\, \mathrm{e}^{2\pi\sqrt{-1}(h|x)} y^j p^k q^l \lambda^m \quad . \tag{44}$$

We also introduce the shorthand notation

$$P^h_{i,\nu} \;=\; \sum_{\|jklm\|=\nu} P^h_{i,jklm}(\omega)\, y^j p^k q^l \lambda^m$$

and define $\mathcal{F}_\nu = \{$Quasi–homogeneous polynomials of weighted order $> \nu\}$.

5.1.1 The "intermediate" homological equation

For the detailed single iteration step proving Proposition 5.1 we drop the index i and use the so–called "+"–notation, replacing occurrences of the index $i+1$ by an index $+$. This allows us to use the common abbreviation $\mathrm{i} = \sqrt{-1}$ in the Fourier exponents and also emphasizes that the constants in our estimates have to be independent of the iteration step i . The generic letter "c" is used where we do not need to remember the value of such a constant, and we also use the shorthand $A \leq\!\cdot\; B$ for $A \leq c \cdot B$.

Most of our estimates rely on Cauchy's inequality and require that we shrink the domain a bit. Therefore we define nested domains

$$D^\iota \;=\; U_{\kappa+s-\frac{\iota s}{8}}(\mathbb{T}^n) \times \mathcal{D}^\iota \times U_{r-\frac{\iota r}{8}}(\mathbb{T}^n) \quad , \qquad \iota = 1, \ldots, 6$$

with

$$\mathcal{D}^\iota \;=\; \Big\{ (y,q,p;\lambda) \;\Big|\; |y| \leq (2^{7-\iota}\varepsilon_+)^4 \,,\, |q|, |p| \leq 2^{7-\iota}\varepsilon_+ \,,\, |\lambda| \leq (2^{7-\iota}\varepsilon_+)^2 \Big\}$$

that satisfy $D_+ \subset D^6 \subset \ldots \subset D^1 \subset D$.

Lemma 5.2 *For the truncation* (44) *we have*

$$\begin{aligned} |R|_{D^1} &\leq\!\cdot\;\; \delta \\ |P-R|_{D^1} &\leq\!\cdot\;\; \alpha^{1-\xi}\delta_+ \end{aligned}$$

where $\alpha = \varepsilon^{\frac{\chi}{4+\xi}}$.

Proof. Since $|R|_{D^1} \leq |R-P|_{D^1} + |P|_{D^1}$ we only have to show the second inequality. Note that

$$P - R \;=\; \sum_{\substack{\nu\leq 4\\ |h|>\eta}} P^h_\nu \mathrm{e}^{2\pi\mathrm{i}(h|x)} \;+\; \sum_{\substack{\nu>4\\ h\in\mathbb{Z}^n}} P^h_\nu \mathrm{e}^{2\pi\mathrm{i}(h|x)} \quad .$$

For the first term we use the Paley–Wiener estimate $|P^h_{jklm}(\omega) y^j p^k q^l \lambda^m| \leq |P|_D\, \mathrm{e}^{-2\pi|h|(\kappa+s)}$ and an upper bound of the number of vectors $h \in \mathbb{Z}^n$ with $|h| = \nu$ to derive

$$\begin{aligned} \Big|\sum_{\substack{\|jklm\|\leq 4\\ |h|>\eta}} P^h_{\|jklm\|} \mathrm{e}^{2\pi\mathrm{i}(h|x)}\Big| \;&\leq\; \sum_{|h|>\eta} |P|_D\, \mathrm{e}^{-2\pi|h|(\kappa+s)} \mathrm{e}^{2\pi|\mathrm{Im}\,x|\,|h|} \\ &\leq\; \delta \sum_{\nu=\eta+1}^{\infty} \nu^n \mathrm{e}^{-\frac{\pi}{6}\nu s} \;\leq\; \delta \int_\eta^\infty \zeta^n \mathrm{e}^{-\frac{\pi}{6}\zeta s}\, \mathrm{d}\zeta \\ &\leq\!\cdot\;\; \eta^n\, \mathrm{e}^{-\frac{\pi\eta s}{6}}\, \delta \;\leq\!\cdot\;\; s^{-2n}\, \mathrm{e}^{-\frac{\pi}{6s}}\, \delta \;\leq\!\cdot\;\; \varepsilon\delta \end{aligned}$$

where the last inequality is given by Lemma 5.10 of [Braaksma,Broer;87]. For the second term we use $\varepsilon_+ = \alpha\varepsilon$ and obtain

$$\begin{aligned} |\sum_{\substack{\|jklm\|>4 \\ h\in\mathbf{Z}^n}} P^h_{\|jklm\|} \mathrm{e}^{2\pi\mathrm{i}(h|x)}| &\leq \left| \int \frac{\partial^{|j|+k+l+|m|}}{\partial y^j \partial p^k \partial q^l \partial\lambda^m} \sum_{\substack{\|jklm\|>4 \\ h\in\mathbf{Z}^n}} P^h_{\|jklm\|} \mathrm{e}^{2\pi\mathrm{i}(h|x)} \, \mathrm{d}y^j \mathrm{d}p^k \mathrm{d}q^l \mathrm{d}\lambda^m \right| \\ &\leq\!\cdot \quad \frac{(\alpha\varepsilon)^5}{\varepsilon^5} |\sum_{\nu=0}^{\infty} \sum_{|h|\in\mathbb{Z}^n} P^h_\nu \mathrm{e}^{2\pi\mathrm{i}(h|x)}| \quad \leq\!\cdot \quad \alpha^5\delta \end{aligned}$$

where $\int$ stands for

$$\underbrace{\int_0^{y_1}\cdots\int_0^{y_1}}_{j_1}\cdots\underbrace{\int_0^{y_n}\cdots\int_0^{y_n}}_{j_n}\underbrace{\int_0^{p}\cdots\int_0^{p}}_{k}\underbrace{\int_0^{q}\cdots\int_0^{q}}_{l}\underbrace{\int_0^{\lambda_1}\cdots\int_0^{\lambda_1}}_{m_1}\cdots\underbrace{\int_0^{\lambda_3}\cdots\int_0^{\lambda_3}}_{m_3}$$

with $\|jklm\| = 5$ **;** **q.e.d.**

To solve (42) up to "higher order terms" we let

$$F \;=\; \sum_{\nu=0}^{4} \sum_{0<|h|\leq\eta} F^h_\nu \mathrm{e}^{2\pi\mathrm{i}(h|x)}$$

with

$$F^h_\nu \;=\; \sum_{\|jklm\|=\nu} F^h_{jklm}(\omega)\, y^j p^k q^l \lambda^m$$

be the solution of

$$N \;+\; R \;+\; \{N,F\} \;=\; \bar N \qquad (\mathrm{mod}\,\mathcal{F}_4) \quad ,$$

i.e. up to weighted order 4 . Since

$$\begin{aligned} \{N,F\} &= -\frac{\partial N}{\partial y}\frac{\partial F}{\partial x} + \frac{\partial N}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial N}{\partial p}\frac{\partial F}{\partial q} \\ &= -2\pi\mathrm{i}(h\mid\omega)F \;+\; \{\mathcal{N},F\} \end{aligned}$$

with $\mathcal{N} = \frac{a(\omega)}{24}p^4 + \frac{b(\omega)}{24}q^4 + \frac{\mu(\omega)}{4}p^2q^2 + \frac{\lambda_1}{2}p^2 + \frac{\lambda_2}{2}q^2 + \lambda_3 pq$, the F^h_ν can be recursively defined by

$$2\pi\mathrm{i}(h\mid\omega)F^h_\nu \;=\; P^h_\nu \;+\; \{\mathcal{N},F^h_{\nu-2}\}$$

which leads to

$$F^h_\nu \;=\; \sum_{\iota=0}^{2}\left(\frac{1}{2\pi\mathrm{i}(h\mid\omega)}\right)^{\iota+1} \underbrace{\{\mathcal{N},\ldots,\{\mathcal{N}}_{\iota},P^h_{\nu-2\iota}\underbrace{\}\cdots\}}_{\iota}$$

where we encounter the small denominators. Here we have defined $P^h_\nu = 0$ if $\nu < 0$. Since we only solved (42) up to the weighted order 4 , the higher order terms

$$\{\mathcal{N}, \sum_{\substack{0<|h|\leq\eta \\ 2<\nu\leq 4}} F^h_\nu \mathrm{e}^{2\pi \mathrm{i}(h|x)}\}$$

have to be included in the new perturbation P_+ . For Poisson brackets with $\mathcal{N}$ we have the inequality

$$|\{\mathcal{N}, G\}|_{D^\iota} \quad \leq\cdot \quad \varepsilon^2 \, |G|_{D^{\iota-1}} \quad . \tag{45}$$

Lemma 5.3 *For the function F we have*

$$|F|_{D^3} \quad \leq\cdot \quad \delta$$

and

$$\left|\frac{\partial^{|j|+k+l+|m|} F}{\partial y^j \partial p^k \partial q^l \partial \lambda^m}\right|_{D^4} \quad \leq\cdot \quad \varepsilon^{-\nu}\delta$$

if $\nu = \|jklm\| \leq 4$.

Proof. Using (45) and the Paley–Wiener estimate we obtain

$$\begin{aligned} |F|_{D^3} \quad &\leq \quad \sum_{\nu=0}^{4} \sum_{0<|h|\leq\eta} \sum_{\iota=0}^{2} \left|\frac{1}{2\pi \mathrm{i}(h \mid \omega)}\right|^{\iota+1} \left|\{\mathcal{N}, \ldots, \{\mathcal{N}, P^h_{\nu-2\iota}\} \cdots\} \mathrm{e}^{2\pi \mathrm{i}(h|x)}\right|_{D^3} \\ &\leq\cdot \quad \sum_{\nu=0}^{4} \sum_{0<|h|\leq\eta} \sum_{\iota=0}^{2} |2\pi(h \mid \omega)|^{-\iota-1} \varepsilon^{2\iota} \, |P|_{D^1} \, \mathrm{e}^{-\frac{\pi}{4}|h|s} \quad . \end{aligned}$$

To estimate the small denominators we take $w \in \Sigma'_1$ with $|\omega - w| \leq \frac{r}{2}$ and use $\eta = [r^{\frac{-1}{\tau+1}}]$ to derive

$$|(h \mid \omega)| \;\geq\; |(h \mid w)| \;-\; |\omega - w| \cdot |h| \;\geq\; |h|^{-\tau} - \frac{r}{2}\eta \;\geq\; \frac{1}{2}|h|^{-\tau}$$

whence the inequality

$$\sum_{h\in\mathbf{Z}} |h|^{3\tau} \, \mathrm{e}^{-\frac{\pi}{4}|h|s} \quad \leq\cdot \quad s^{-4\tau}$$

yields the first inequality. The second inequality then follows from Cauchy's estimate;
q.e.d.

Denote by

$$\begin{aligned} \|X_F\|_D \quad &:= \quad \max\left\{ |\frac{\partial F}{\partial y}|_D \, , \, \varepsilon^{-4}|\frac{\partial F}{\partial x}|_D \, , \, \varepsilon^{-2}|\frac{\partial F}{\partial p}|_D \, , \, \varepsilon^{-2}|\frac{\partial F}{\partial q}|_D \right\} \\ \Uparrow D_\nu X_F \Uparrow_D \quad &:= \quad \max_{|j|+k+l\leq\nu} \left\{ |\frac{\partial^{|j|+k+l} G}{\partial y^j \partial p^k \partial q^l}|_D \right\} \qquad \text{for} \quad \nu \geq 1 \end{aligned}$$

where G stands for either of $\frac{\partial F}{\partial y}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial p}, \frac{\partial F}{\partial q}$.

Lemma 5.4 *The Hamiltonian vector field X_F satisfies*

$$\|X_F\|_{D^4} \leq. \varepsilon^\xi \quad , \qquad \Uparrow D_\nu X_F \Uparrow_{D^4} \ \leq. \varepsilon^\xi \quad \forall_{\nu \geq 1} \quad .$$

Proof. This is an immediate consequence of Lemma 5.3 and $\delta = \varepsilon^{4+\xi}$. Indeed, the expressions $\frac{\partial^{|j|+k+l}}{\partial y^j \partial p^k \partial q^l} G$ with $G \in \{\frac{\partial F}{\partial y}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial p}, \frac{\partial F}{\partial q}\}$ are either partial derivatives of F of weighted order ≤ 4 , or vanish identically ; **q.e.d.**

Hence, the flow φ_t of X_F satisfies $\|\varphi_t - \mathrm{id}\|_{D^4} \leq c\,|t|\,\varepsilon^\xi$ as well whence

$$\psi_t : \ (x, y, q, p; \lambda, \omega) \ \mapsto \ (\varphi_t(x, y, q, p); \lambda, \omega)$$

maps D^5 into D^4 for $-1 \leq t \leq 1$. For the time–1–map ψ_1 we define the norm

$$\|\psi_1\|_{C^{jkl}(D)} \ := \ \max_{0 \leq t \leq 1} \left\| \frac{\partial^{|j|+k+l} \psi_t}{\partial y^j \partial p^k \partial q^l} \right\|_D \quad .$$

Lemma 5.5 *The co–ordinate transformation ψ_1 satisfies*

$$\|\psi_1 - \mathrm{id}\|_{C^{jkl}(D^5)} \quad \leq. \quad \varepsilon^\xi \quad .$$

The proof follows that of Lemma 6.5 in [Broer,Hanßmann,You;03] almost verbatim and is omitted.

5.1.2 Transformation of $\bar{N}$ into normal form

So far we have solved the small divisor problem (42) to construct a change of variables ψ_1 that transforms away the x–dependence of the lower order terms entering in $\bar{N}$. We stress that F inherits from P the invariance under the deck transformation (6), which reads

$$P^h_{jklm} \ = \ (-1)^{k+l} \mathrm{e}^{\pi \mathrm{i} h_1} P^h_{jklm} \qquad \forall_{h,j,k,l,m} \quad .$$

In particular the time–1–flow φ_1 is σ–equivariant, being the lift to $\mathcal{P}$ of the Hamiltonian flow defined by the projection of F from $\mathcal{P}$ to the base space $\mathcal{B}$. The symmetry σ also puts severe restrictions on the contribution R^0 to $\bar{N}$ as all coefficients P^0_{jklm} with odd $k+l$ vanish. In this way we are left with

$$R^0 \ = \ \sum_{|j|=1} P^0_{j000} y^j \ + \ \sum_{k+l=4} P^0_{0kl0} p^k q^l \ + \ \sum_{k+l=2} P_{kl}(\lambda) p^k q^l$$

where

$$P_{kl}(\lambda) \ = \ \sum_{|m|=0}^{1} P^0_{0klm} \lambda^m \quad . \tag{46}$$

Here and below we completely suppress the ω–dependence. We also ignore the constant terms $P^0_{000m}\lambda^m$. Indeed, since it is *a priori* clear that these do not add up to ∞ we did not burden the presentation with the introduction of a constant term $e(\lambda,\omega)$ in (39) to compensate for these. The symplectic change of co–ordinates we use to transform away the two terms p^3q and pq^3 is a combination of two transvections

$$\phi \; : \; \left\{ \begin{array}{rcl} q_1 & = & q \, + \, up \\ p_1 & = & (1+uv)p \, + \, vq \end{array} \right. \tag{47}$$

where u and v are implicitly defined by $f(u,v) = 0$ with

$$\begin{array}{rcl} f_1(u,v) & = & \left(\dfrac{\mu}{2} \, + \, 2P^0_{0220}\right)(1+2uv)u \, + \, \left(\dfrac{a}{6} \, + \, 4P^0_{0400}\right)v \, - \, P^0_{0310}(1+4uv) \\ & & + \, P^0_{0130}(3u^2+4u^3v) \, - \, \left(\dfrac{b}{6} \, + \, 4P^0_{0040}\right)u^3 \\ f_2(u,v) & = & \left(\dfrac{b}{6} \, + \, 4P^0_{0040}\right)(1+uv)^3u \, + \, \left(\dfrac{\mu}{2} \, + \, 2P^0_{0220}\right)(1+3uv+2u^2v^2)v \\ & & - \, P^0_{0130}(1+6uv+9u^2v^2+4u^3v^3) \, + \, P^0_{0310}(3v^2+4uv^3) \\ & & - \, \left(\dfrac{a}{6} \, + \, 4P^0_{0400}\right)v^3 \quad . \end{array}$$

This is possible since

$$\det Df(0,0) \;\; = \;\; \frac{1}{36}(\mu^2 - 9ab) \; + \; \mathcal{O}(\delta)$$

is bounded away from zero. We thus arrive at the new normal form $N_+ = \bar{N} + \phi$ if we put

$$\begin{array}{rcl} a_+ & = & a \, + \, 24\,P^0_{0400} \, - \, 24\,P^0_{0310}u \, + \, 6(\mu+4P^0_{0220})u^2 \\ & & - \, 24\,P^0_{0130}u^3 \, + \, (b+24P^0_{0040})u^4 \\ b_+ & = & (b+24P^0_{0040})(1+uv)^4 \, - \, 24\,P^0_{0130}(1+uv)^3v \, + \, 6(\mu+4P^0_{0220})(1+uv)^2v^2 \\ & & - \, 24\,P^0_{0310}(1+uv)v^3 \, + \, (a+24P^0_{0400})v^4 \\ \mu_+ & = & (\mu+4P^0_{0220})(1+6uv+6u^2v^2) \, - \, 12\,P^0_{0130}(u+3u^2v+2u^3v^2) \\ & & - \, 12\,P^0_{0310}(v+2uv^2) \, + \, (b+24P^0_{0040})(u+u^2v)^2 \, + \, (a+24P^0_{0400})v^2 \end{array}$$

and furthermore use (46) to re–parametrise

$$\begin{array}{rcl} \omega_i^+ & = & \omega_i \, + \, P^0_{j000} \qquad \text{with} \quad j = j(i) = (0,\ldots,0,\overset{i}{1},0,\ldots,0) \\ \lambda_1^+ & = & \lambda_1 \, + \, 2\,P_{20}(\lambda) \, - \, 2(\lambda_3+P_{11}(\lambda))u \, + \, (\lambda_2+2P_{02}(\lambda))u^2 \\ \lambda_2^+ & = & (\lambda_2+2P_{02}(\lambda))(1+uv)^2 \, - \, 2(\lambda_3+P_{11}(\lambda))(v+uv^2) \, + \, (\lambda_1+2P_{20}(\lambda))v^2 \\ \lambda_3^+ & = & (\lambda_3+P_{11}(\lambda))(1+2uv) \, - \, (\lambda_2+2P_{02}(\lambda))(u+u^2v) \, - \, (\lambda_1+2P_{20}(\lambda))v \end{array}$$

with a non–singular Jacobian.

In the reversible case the monomials p^3q, pq^3 and pq are excluded since they are not invariant under (8). This simplifies all formulas as ϕ and λ_3 become obsolete and we can put $u = v = 0 = \lambda_3$ above.

5.1.3 Estimates of the iteration step

We now compose our map $\Psi : D^6 \longrightarrow D^1 \subseteq D$ (which we later restrict to D_+) from $\psi_1 : D^5 \longrightarrow D^1$ and the mapping

$$(x, y, q, p; \lambda^+, \omega^+) \mapsto (x, y, \phi(q, p); \lambda, \omega)$$

from D^6 to D^5 . To this end we need the inequalities

$$\begin{aligned} |P^0_{j000}| &\leq\cdot \frac{\delta}{\varepsilon^4} < \frac{r}{8} \\ |\lambda - \lambda^+| &\leq\cdot \frac{\delta}{\varepsilon^2} < (1-2\alpha)^2\varepsilon^2 \end{aligned}$$

and similar to Lemma 5.5 we can estimate Ψ :

Lemma 5.6 *The co–ordinate transformation Ψ satisfies*

$$\|\Psi - \mathrm{id}\|_{C^{jkl}(D^6)} \leq\cdot \varepsilon^\xi \quad .$$

Proof. Since ϕ is a linear map, the estimate follows from Lemma 5.5 and the chain rule ; **q.e.d.**

The new perturbation is

$$P_+ = \left(\bar{P} + \{ \mathcal{N}, \sum_{\substack{0<|h|\leq\eta \\ 2<\nu\leq 4}} F^h_\nu e^{2\pi i(h|x)} \} \right) \circ \phi$$

with $\bar{P}$ defined in (41) and, as in [Broer,Hanßmann,You;03], can be estimated by

$$|P_+|_{D_+} \leq |P_+|_{D^6} \leq\cdot \alpha^{(1-\xi)(\xi-\chi)} \varepsilon^\chi \delta < \delta_+ \quad .$$

Using Cauchy's inequality on $\mathcal{D}_+ \subset D^6$ we furthermore obtain

$$\left|\frac{\partial^{|j|+k+l+|m|} P_+}{\partial y^j \partial p^k \partial q^l \partial \lambda^m}\right|_{D_+} \leq \varepsilon_+^{-\nu} \delta_+$$

for $\nu = \|jklm\|$. This concludes the proof of Proposition 5.1.

5.2 Iteration and convergence

Composing $\Phi_{i+1} := \Psi_0 \circ \Psi_1 \circ \ldots \circ \Psi_i$ the co–ordinate transformations given by Proposition 5.1 we obtain a change of variables that transforms H_0 into $H_{i+1} = N_{i+1} + P_{i+1}$. Our aim is to find a "limit" Φ_∞ and this necessitates a bound on the C^ν–norm

$$\|\Phi_i\|_{C^\nu(D_i)} \quad = \quad \max_{|j|+k+l\leq\nu} \left\| \frac{\partial^{|j|+k+l}\Phi_i}{\partial y^j\,\partial p^k\,\partial q^l} \right\|_{D_i} \quad .$$

Lemma 5.7 *The co–ordinate transformation Φ_i satisfies*

$$\|\Phi_i\|_{C^\nu(D_i^6)} \;\leq\; c \qquad \forall_{i\in\mathbb{N}}$$

with a suitable constant $c = c(\nu)$.

Proof. The estimate follows from Lemma 5.6 and the chain rule ; **q.e.d.**

To construct a limit of the Φ_i we have to pass to a polynomial truncation of fixed degree to obtain Φ_∞ by means of coefficient functions defined on

$$\bigcap_i \mathcal{T}_i \times \mathcal{U}_i \quad = \quad \overline{U_\kappa(\mathbb{T}^n)} \times \Sigma_1' \quad .$$

Since we want to preserve the symplectic structure we follow [Hanßmann;98], [Broer,Hanßmann,You;03] and pass to a generating function S_i of Φ_i , which we truncate to $\widetilde{S}_i$ and the limit $\widetilde{S}_\infty$ then generates the desired limit transformation Φ_∞ . Indeed, writing $\Phi_i : (x, y, p, q; \lambda, \omega) \mapsto (X, Y, P, Q)$ the 1–form

$$\sum_{\nu=1}^{n} (y_\nu - Y_\nu)\,\mathrm{d}x_\nu \;+\; (X_\nu - x_\nu)\,\mathrm{d}Y_\nu \quad+\quad (Q-q)\,\mathrm{d}P \;+\; (p-P)\,\mathrm{d}q \tag{48}$$

is exact for fixed (λ, ω) and thus defines $S_i = S_i(x, Y, q, P)$. As the estimates of Proposition 5.1 should remain valid we truncate at order 5 in $(Y, q, P; \lambda)$ and drop quadratic and higher order terms in $\lambda_1, \lambda_2, \lambda_3$. The coefficient functions $S_i^{jklm}(x, \omega)$ are holomorphic.

Lemma 5.8 *The finitely many sequences $(\widetilde{S}_i^{jklm})_{i\in\mathbb{N}}$ of coefficient functions are uniformly convergent on $\overline{U_\kappa(\mathbb{T}^n)} \times \Sigma_1'$.*

The proof follows that of Lemma 6.9 in [Broer,Hanßmann,You;03] almost verbatim and is omitted.

The Inverse Approximation Lemma of [Wagener;03] yields (Whitney)–Gevrey–regular limit functions $\widetilde{S}_\infty^{jklm}$, which constitute the coefficients of a generating function

$$\widetilde{S}_\infty \;:\; U_\kappa(\mathbb{T}^n) \times \mathbb{C}^n \times \mathbb{C}^2 \times \mathbb{C}^3 \times \Sigma_1' \;\longrightarrow\; \mathbb{C}$$

of Φ_∞ which is polynomial in Y, q, P, λ and analytic in x and can be extended to the whole frequency space $\mathbb{R}^n$ using Whitney's Extension Theorem, cf. [Bonet,Braun,Meise,Taylor;91]. Similarly we obtain (Whitney)–Gevrey–regular limit transformations for ω and λ and coefficient functions a_∞, b_∞ and μ_∞ of the (limit) normal form N_∞ . The definition $P_\infty := H_0 \circ \Phi_\infty - N_\infty$ concludes the proof of Theorem 4.3.

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