

Analytic Smoothing Effect for the Schrödinger Equation with Long-Range Perturbation

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Abstract

We study microlocal analytic singularity of solutions to Schrödinger equation with analytic coefficients. Using microlocal weight estimate developed for estimating the phase space tunneling, we prove microlocal smoothing estimates that generalize results by L. Robbiano and C. Zuily. We suppose the Schrödinger operator is a long-range type perturbation of the Laplacian, and we employ positive commutator type estimates to prove the smoothing property.

1 Introduction

It is well-known that solutions to the Schrödinger equation have infinite propagation speed, and hence we cannot expect propagation of singularity theorems similar to that for the wave equation. Instead, local smoothing effect has been used to study the local smoothness of solutions to Schrödinger equations. The smoothing effects for the Schrödinger equation has been a very rich source of investigations during the last past years: see, e.g., [Ze, Sjl, Yam, Yaj1, HaKa1, Yaj2, GiVe, KPV, KaSa, CKS, KaTa, KaYa, KRY, HaKa2, Wu, RoZu1, RoZu2, KaWa, MRZ, Do1, RoZu3, HaWu, Na2, Na3, Do2]. In particular, Craig, Kappeler and Strauss [CKS] showed that this effect may be considered as a microlocal phenomenon, and this observation inspired series of investigations, both in the C^∞ -case (in particular [Wu, HaWu]) and in the analytic case ([RoZu1, RoZu2, RoZu3]). While the papers [Wu, HaWu, RoZu3] address the case of the Laplacian associated to a certain generalization of asymptotically flat metrics (the so-called *scattering metrics*, defined on a compact manifold with boundary, where the boundary

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plays the role of the infinity), the papers [RoZu1, RoZu2] concentrate more specifically on the case of asymptotically flat metrics on \mathbb{R}^n . In all of these five papers, the perturbations are assumed to be of short-range type (in the sense that $\sigma > 1$ in Assumption A). Their methods rely on a special notion of wave front set (the *quadratic scattering wave front set*) for which results of propagation are proved by using constructions of microlocal parametrices. As it is often the case with such a method, construction and the computations are relatively complicated. Nakamura [Na2] gave a simpler proof in the case of asymptotically flat metrics with long-range C^∞ -class perturbations, using a different notion of wave front set (the *homogeneous wave front set*) for which the result of propagation is obtained through positive commutator type argument (which is similar to the original proof of the propagation of singularity theorem by Hörmander [Ho]). More precise characterization of the C^∞ -wave front set of solutions to Schrödinger equations is studied in [Na3] using an Egorov theorem type argument.

The purpose of this paper is to prove a theorem similar to [Na2] for the analytic singularity. The proof is relatively simple, and we recover the results of [RoZu1, RoZu2] without using a construction of parametrix, but rather using microlocal energy estimates in the same spirit as in [Ma2]. We note this method has been employed to give simpler argument to study the analytic wave front set, or, more generally, of the microsupport of solutions of analytic partial differential equations (see, e.g., [Ma2] Chapter 4). In this paper, we apply a generalization of the method to study the analytic homogeneous wave front set and analytic smoothing effects.

It is not completely clear to us whether our notion of wave front set coincides exactly or not with that of [RoZu1, RoZu2], but we will see that the results of these papers can be recovered from our main theorem. Moreover, our argument allows us to generalize these results to the case of long-range perturbations in a sense similar to that of [Na2] (that is, the perturbation is not only assumed to decay more slowly, but the coefficients of the lower order terms are indeed allowed to have some polynomial growth at infinity). In addition, we can also relax the assumptions of analyticity on the coefficients of the operator, in the sense that we assume that they can be extended holomorphically to a strip in \mathbb{C}^n around \mathbb{R}^n (rather than a complex sector as in [RoZu1, RoZu2]).

In the next section we specify our assumptions, introduce the notion of analytic homogeneous wave front set, and state our main results. Section 3 is devoted to a discussion of microlocal exponential weight estimate, in a spirit similar to that of [Ma2], but with a Bargmann transform that involves an additional parameter. In Section 4, we prove our main results, and we give the details on the argument to recover the result of [RoZu2] in Appendix A. The other 3 appendices discuss proof of technical lemmas.

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2 Notations and Main Result

We consider the analytic wave front set of solutions to Schrödinger equation with variable coefficients. Namely, we set

$$P = \frac{1}{2} \sum_{j,k=1}^n a_{jk}(x) D_j D_k + \sum_{j=1}^n a_j(x) D_j + a_0(x)$$

on $\mathcal{H} = L^2(\mathbb{R}^n)$, where $D_j = -i\partial_{x_j}$. We suppose the coefficients $\{a_\alpha(x)\}$ satisfy the following assumptions. For $\nu > 0$ we denote

$$\Gamma_\nu = \{z \in \mathbb{C}^n \mid |\operatorname{Im} z| < \nu\}.$$

Assumption A. For each α , $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$, the $a_{jk}(x)$'s are real-valued and the matrix $(a_{jk}(x))_{1 \leq j,k \leq n}$ is symmetric and positive definite. We set $b_j(x) = \operatorname{Re} a_j(x)$, $c_j(x) = \operatorname{Im} a_j(x)$ for $j = 0, 1, \dots, n$ and we assume $a_{jk}(x)$, $b_j(x)$, $c_j(x)$ are extended to holomorphic functions on Γ_ν with some $\nu > 0$. Moreover, there is $\sigma > 0$ such that for all $\beta \in \mathbb{Z}_+^n$,

$$\begin{aligned} |\partial_z^\beta (a_{jk}(z) - \delta_{jk})| &\leq C_\beta \langle z \rangle^{-\sigma-|\beta|}, & j, k = 1, \dots, n, \\ |\partial_z^\beta b_j(z)| &\leq C_\beta \langle z \rangle^{1-\sigma-|\beta|}, & j = 1, \dots, n, \\ |\partial_z^\beta c_j(z)| &\leq C_\beta \langle z \rangle^{-\sigma-|\beta|}, & j = 1, \dots, n, \\ |\partial_z^\beta b_0(z)| &\leq C_\beta \langle z \rangle^{2-\sigma-|\beta|}, \\ |\partial_z^\beta c_0(z)| &\leq C_\beta \langle z \rangle^{1-\sigma-|\beta|} \end{aligned}$$

for $z \in \Gamma_\nu$ with some $C_\beta > 0$.

We consider the solution $u(t) \in C([0, T]; L^2(\mathbb{R}^n))$ to the time-dependent Schrödinger equation:

$$\begin{cases} i \frac{\partial u}{\partial t} = Pu & (t \in (0, T)); \\ u|_{t=0} = u_0, \end{cases} \quad (2.1)$$

where $u_0 \in L^2(\mathbb{R}^n)$. We denote the L^2 -norm by $\|\cdot\|$ without subscript.

In order to describe the analytic singularity, we use the flat FBI transform (or the Bargmann transform with parameters, or the Gaussian wave packet transform in physics literature). Let $h, \mu > 0$, and we set

$$T_{h,\mu} u(x, \xi) = c_{h,\mu} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h} e^{-\mu|x-y|^2/2h} u(y) dy \quad (2.2)$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, where $c_{h,\mu} = 2^{-n/2} \mu^{-n/4} (\pi h)^{-3n/4}$. It is well-known that $T_{h,\mu}$ is extended to an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. $T_{h,\mu}$ is also extended to a continuous linear map from $\mathcal{S}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^{2n})$. The analytic wave front set is defined as follows:

Definition 1. Let $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, and let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then, (x_0, ξ_0) is not in the analytic wave front set of u (i.e., $(x_0, \xi_0) \notin WF_a(u)$) if there is $\delta, \varepsilon > 0$ such that

$$\|T_{h,\mu} u\|_{L^2(B_\varepsilon((x_0, \xi_0)))} \leq C \exp(-\delta/h), \quad \text{for } 0 < h \leq 1.$$

Here we denote the open ball of radius $r > 0$ with the center at X by $B_r(X)$:

$$B_r(X) = \{Y \mid |X - Y| \leq r\},$$

and $\mu > 0$ is arbitrarily fixed.

Remark. $WF_a(u)$ is independent of the choice of $\mu > 0$ since $T_{h,\mu'} T_{h,\mu}^*$ is an integral operator on $L^2(\mathbb{R}^{2n})$ with a Gaussian kernel. Hence we may fix $\mu = 1$. There are equivalent definitions of the analytic wave front set (cf. [Sjs], [Ma2]). We choose this definition because it fits nicely to our argument.

We also use an analogue of the homogeneous wave front set introduced in [Na2].

Definition 2. Let $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, and let $u \in \mathcal{S}'(\mathbb{R}^n)$. (x_0, ξ_0) is not in the analytic homogeneous wave front set of u (i.e., $(x_0, \xi_0) \notin HWF_a(u)$) if there is $\delta > 0$ and a neighborhood Σ of (x_0, ξ_0) which is conic with respect to the group of variables $(x, \xi) \in \mathbb{R}^{2n}$, and such that

$$\|e^{\delta(|x|+|\xi|)} T_{1,\mu} u\|_{L^2(\Sigma)} < \infty,$$

where $\mu > 0$ is fixed.

Remark. The definition of $HWF_a(u)$ is independent of the choice of μ as well as $WF_a(u)$. Thus we may set $\mu = 1$, but it is often convenient to use various $\mu > 0$. We can unify the notion of WF_a and HWF_a using the Fourier-Bros-Iagolnitzer transform \mathbf{T} defined by,

$$\mathbf{T}u(x, \xi) := \int e^{i(x-y)\xi - \frac{\langle \xi \rangle}{\langle x \rangle} (x-y)^2} u(y) dy.$$

This can be seen by using the techniques of analytic microlocal analysis introduced in [Sjs] (in particular the proof of Proposition 6.2).

To state our main result, we recall notations of the Hamiltonian flow generated by the Riemannian metric $\{a_{jk}(x)\}$. Let

$$p(x, \xi) = \frac{1}{2} \sum_{j,k=1}^n a_{jk}(x) \xi_j \xi_k, \quad (2.3)$$

and let $\gamma = \{(y(t), \eta(t)) ; t \in \mathbb{R}\}$ be a corresponding integral curve of the Hamilton flow, that is, a solution of

$$\dot{y}(t) = \frac{\partial p}{\partial \xi}(y(t), \eta(t)), \quad \dot{\eta}(t) = -\frac{\partial p}{\partial x}(y(t), \eta(t)).$$

We say that γ is *backward nontrapping* if $|(y(t), \eta(t))| \rightarrow \infty$ as $t \rightarrow -\infty$. If γ is backward nontrapping, it is well-known that the asymptotic momentum,

$$\eta_- = \lim_{t \rightarrow -\infty} \eta(t)$$

exists, provided Assumption A is satisfied (cf., e.g., [CKS]). Our main result is the following one:

Theorem 2.1. *Suppose Assumption A, and suppose γ is backward nontrapping. Let η_- be the asymptotic momentum as $t \rightarrow -\infty$. Suppose there exists $t_0 > 0$ such that*

$$(-t_0\eta_-, \eta_-) \notin HWF_a(u_0), \quad (2.4)$$

then

$$((t - t_0)\eta_-, \eta_-) \notin HWF_a(u(t)) \quad \text{for } 0 < t < \min(t_0, T), \quad (2.5)$$

and moreover, if $t_0 < T$, then,

$$\gamma \cap WF_a(u(t)) = \emptyset, \quad (2.6)$$

for all t close enough to t_0 .

Remark. Actually, we prove a stronger result than (2.6). Namely, the statement holds true also for the uniform analytic wave front set as defined in [RoZu1, RoZu2].

Now we can recover and generalize the following two results of Robbiano and Zuily (we note $\sigma > 1$ and decaying condition on lower-order coefficients are assumed in [RoZu1, RoZu2]):

Corollary 2.2. *Assume Assumption A and γ is backward nontrapping. Moreover assume that $e^{\delta_0|x|}u_0 \in L^2(\Sigma_{\varepsilon_0})$ for some $\delta_0, \varepsilon_0 > 0$, where*

$$\Sigma_{\varepsilon_0} := \bigcup_{s \leq 0} \{x \mid |x - y(s)| \leq \varepsilon_0(1 + |s|)\}.$$

Then we have

$$\gamma \cap WF_a(u(t)) = \emptyset$$

for any $t > 0$.

Indeed, in this case (2.4) is satisfied for any $t_0 > 0$, since $|\xi| = O(|x|)$ on any small enough conic neighborhood of $(-t_0\eta_-, \eta_-)$.

Corollary 2.3. *Assume Assumption A and γ is backward nontrapping. Moreover, assume that $u_0(x) = a(x)e^{i\psi(x)}$ where $a \in L^2(\mathbb{R}^n)$; ψ is real-valued and analytic on \mathbb{R}^n ; and for some $m \geq 1$, a and ψ admit holomorphic extensions on a set of the form*

$$\tilde{\Sigma}_{\varepsilon_0} := \{x \in \mathbb{C}^n \mid \operatorname{Re} x \in \Sigma_{\varepsilon_0}, |\operatorname{Im} x| < \delta_0 |\operatorname{Re} x|^{\min(m-1, 1)}, |x| > R\}$$

and verify the following estimates on this set:

$$|\psi(x)| = O(|x|^m), \quad |a(x)| = O(|x|^M)$$

with some $M \geq 1$. In the case $m = 1$, assume also $|\nabla\psi(x)| = O(1)$ on the same set. Moreover, assume there exist $t_0 > 0$ and a positive constant C such that,

$$|\nabla\psi(-\lambda t_0 \eta_-) - \lambda \eta_-| \geq \frac{\lambda^{m-1}}{C} \quad (2.7)$$

for all $\lambda > 0$ sufficiently large. Then,

$$\gamma \cap WF_a(u(t)) = \emptyset$$

for any $t > 0$.

In this case (2.4) is satisfied with this value of t_0 (see Appendix A).

3 Exponential weight estimates

Here we discuss a key estimate of the proof of Theorem 2.1. We assume $0 < \sigma \leq 1$ without loss of generality and, for simplicity, we write $T = T_{h, \mu}$. Let $\psi(x, \xi) \in C_0^\infty(\mathbb{R}^{2n})$ be an (h, μ) -dependent function such that there exists $C_1 > 1$ such that

$$\operatorname{supp}[\psi] \subset \left\{ (x, \xi) \mid \frac{1}{C_1} \leq |\xi| \leq C_1, \frac{1}{C_1 \mu} \leq \langle x \rangle \leq \frac{C_1}{\mu} \right\}$$

and that for any multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$,

$$|\partial_x^\alpha \partial_\xi^\beta \psi(x, \xi)| \leq C_{\alpha\beta} \mu^{|\alpha|}, \quad x, \xi \in \mathbb{R}^n,$$

uniformly with respect to $h, \mu \in (0, 1]$. We also suppose

$$\sup_{x, \xi} |\partial_x \psi(x, \xi)| < \nu, \quad \sup_{x, \xi} |\partial_\xi \psi(x, \xi)| < \nu \quad (3.1)$$

for any $h, \mu \in (0, 1]$. We let $f \in C_0^\infty(\mathbb{R}^{2n})$ such that

$$\operatorname{supp}[f] \subset \left\{ (x, \xi) \mid \frac{1}{C_2} \leq |\xi| \leq C_2, \frac{1}{C_2 \mu} \leq \langle x \rangle \leq \frac{C_2}{\mu} \right\}$$

with $C_2 > C_1$ and that

$$f = 1 \quad \text{on } \text{supp}[\psi].$$

Moreover, we suppose that for any $\alpha, \beta \in \mathbb{Z}_+^n$,

$$|\partial_x^\alpha \partial_\xi^\beta f(x, \xi)| \leq C'_{\alpha\beta} \mu^{|\alpha|}, \quad x, \xi \in \mathbb{R}^n,$$

and that $f \geq 0$, $\sqrt{f} \in C_0^\infty(\mathbb{R}^{2n})$ and \sqrt{f} satisfies the same estimates. We denote

$$\begin{aligned} \tilde{p}(x, \xi) &= h^{-2} \frac{1}{2} \sum_{j,k=1}^n a_{jk}(x) \xi_j \xi_k + h^{-1} \sum_{j=1}^n a_j(x) \xi_j + a_0(x), \\ \tilde{p}_\psi(x, \xi) &= \tilde{p}(x - \partial_\mu \psi(x, \xi), \xi + i\mu \partial_\mu \psi(x, \xi)), \end{aligned}$$

where

$$\partial_\mu = \mu^{-1} \partial_x + i \partial_\xi$$

Then we have:

Theorem 3.1. *Let ψ and f as above and suppose $0 < h/\mu \leq d$ with some $d > 0$. Then there exists $C > 0$ such that*

$$\begin{aligned} & |\langle e^{\psi/h} T u, f e^{\psi/h} T P u \rangle - \langle e^{\psi/h} T u, f \tilde{p}_\psi e^{\psi/h} T u \rangle | \\ & \leq C(h^{-1} \mu^{1+\sigma} + \mu^\sigma + h\mu^{\sigma-1}) \|\sqrt{f} e^{\psi/h} T u\|^2 + C(h^{-1} \mu + \mu^\sigma + h\mu^{\sigma-1}) \|u\|^2 \end{aligned}$$

for any $u \in L^2(\mathbb{R}^n)$.

By elementary computations, we can estimate $\text{Im } \tilde{p}_\psi$ as follows:

Lemma 3.2. *There exists $C > 0$ such that*

$$|\text{Im } \tilde{p}_\psi(x, \xi) - h^{-2} H_p \psi(x, \xi)| \leq C(h^{-2} \mu^2 + h^{-1} \mu^\sigma + \mu^{\sigma-1}),$$

where $H_p \psi = \frac{\partial p}{\partial \xi} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial p}{\partial x} \cdot \frac{\partial \psi}{\partial \xi}$, and p is the principal symbol of H defined by (2.3).

Combining these, we have the following key estimate in the proof of Theorem 2.1:

Corollary 3.3. *Under the same assumption as Theorem 3.1, there exists $C > 0$ such that*

$$\begin{aligned} & |\text{Im } \langle e^{\psi/h} T u, f e^{\psi/h} T P u \rangle - \langle e^{\psi/h} T u, f (h^{-2} H_p \psi) e^{\psi/h} T u \rangle | \\ & \leq C(h^{-2} \mu^2 + h^{-1} \mu^\sigma + \mu^{\sigma-1}) \|\sqrt{f} e^{\psi/h} T u\|^2 \\ & \quad + C(h^{-1} \mu + \mu^\sigma + h\mu^{\sigma-1}) \|u\|^2. \end{aligned}$$

We prove Theorem 3.1 in the remainder of this section. We follow the argument of [Ma2], [Na1], with an additional parameter μ and different symbol classes. We note μ may be considered as a scaling parameter with respect to x , whereas h is a scaling parameter with respect to ξ . We sometimes use the abbreviation: $T_\psi = e^{\psi/h}T$. In the following, we always assume

$$0 < h/\mu \leq d$$

with some constant $d > 0$. We denote

$$\begin{aligned} p_2(x, \xi) &= \frac{1}{2} \sum_{j,k=1}^n (a_{jk}(x) - \delta_{jk}) \xi_j \xi_k, \\ p_1(x, \xi) &= \sum_{j=1}^n a_j(x) \xi_j - \frac{1}{2i} \sum_{j,k=1}^n \partial_{x_j} a_{jk}(x) \xi_k, \\ p_0(x, \xi) &= p_0(x) = a_0(x) - \frac{1}{2i} \sum_{j=1}^n \partial_{x_j} a_j(x) - \frac{1}{8} \sum_{j,k=1}^n \partial_{x_j} \partial_{x_k} a_{jk}(x). \end{aligned}$$

Then we have

$$P = \frac{1}{2} h^{-2} (hD_x)^2 + h^{-2} p_2^W(x, hD_x) + h^{-1} p_1^W(x, hD_x) + p_0(x), \quad (3.2)$$

where $a^W(x, hD_x)$ denotes the Weyl-Hörmander quantization of a :

$$a^W(x, hD_x)u(x) = (2\pi h)^{-n} \int \int e^{(x-y)\cdot\xi/h} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ (cf. [Ho] Section 18.5). Note p_1, p_0 contain quantization error terms.

We set

$$q_j(x, \xi, x^*, \xi^*) = p_j(x - \xi^*, x^*), \quad x, \xi, x^*, \xi^* \in \mathbb{R}^n, \quad j = 0, 1, 2.$$

As in [Ma1], we have

$$TP_j = Q_jT,$$

where $P_j = p_j^W(x, hD_x)$ and $Q_j = q_j^W(x, \xi, hD_x, hD_\xi)$. We denote

$$\Phi(x, \xi, x^*, \xi^*) = \sqrt{1 + |x|^2 / \langle \xi^* \rangle^2} \quad \Psi(x, \xi, x^*, \xi^*) = \sqrt{1 + |\xi|^2 / \langle \xi - x^* \rangle^2},$$

and let \tilde{g} be a metric on \mathbb{R}^{4n} defined by

$$\tilde{g} = \frac{dx^2}{\Phi^2} + \frac{d\xi^2}{\Psi^2} + \frac{dx^{*2}}{\Psi^2} + \frac{d\xi^{*2}}{\Phi^2}.$$

We use the $S(m, g)$ symbol class notation of Hörmander [Ho] Section 18.5. In particular, $OPS(m, g)$ is the space of pseudodifferential operators with their symbol in $S(m, g)$.

The next two lemmas are fundamental in the pseudodifferential operator calculus on $\text{Ran}[e^{\psi/h}T]$. They imply that polynomials of $\langle hD_x - \xi \rangle$ and $\langle hD_\xi \rangle$ are uniformly bounded as quadratic forms on $\text{Ran}[e^{\psi/h}T]$. We denote by g_0 the flat metric on \mathbb{R}^{4n} , i.e., $g_0 = dx^2 + d\xi^2 + dx^{*2} + d\xi^{*2}$.

Lemma 3.4. *Suppose $Q \in OPS(\langle \xi - x^* \rangle^m \langle \xi^* \rangle^\ell, g_0)$ with some $m, \ell \in \mathbb{R}$. Then there exists $C > 0$ such that*

$$|\langle e^{\psi/h}Tu, Qe^{\psi/h}Tu \rangle| \leq C \|e^{\psi/h}Tu\|^2$$

for $u \in \mathcal{S}(\mathbb{R}^n)$.

Lemma 3.5. *Suppose $Q \in OPS(\langle \xi \rangle^a \langle x \rangle^b \langle \xi - x^* \rangle^m \langle \xi^* \rangle^\ell, \tilde{g})$ with some $a, b, m, \ell \in \mathbb{R}$. Then, for any $N > 0$, there exists $C > 0$ such that*

$$|\langle e^{\psi/h}Tu, fQe^{\psi/h}Tu \rangle| \leq C \left(\mu^{-b} \|\sqrt{f}e^{\psi/h}Tu\|^2 + \mu^{-b} \|u\|^2 \right) \quad (3.3)$$

for $u \in \mathcal{S}(\mathbb{R}^n)$.

Given the above two lemmas, it is convenient to consider the symbols q_j 's in the following symbol classes.

Lemma 3.6. *For $j = 0, 1, 2$, one has,*

$$q_j \in S(m_j, \tilde{g}),$$

with $m_j = \langle \xi \rangle^j \langle x \rangle^{2-j-\sigma} (\langle \xi - x^* \rangle^j \langle \xi^* \rangle^{|2-j-\sigma|})$.

Proof. We recall

$$p_j \in S(\langle \xi \rangle^j \langle x \rangle^{2-j-\sigma}, g),$$

where $g = dx^2/\langle x \rangle^2 + d\xi^2/\langle \xi \rangle^2$. Hence, by the definition of q_j , we have

$$q_j \in S\left(\langle x^* \rangle^j \langle x - \xi^* \rangle^{2-j-\sigma}, \frac{dx^2}{\langle x - \xi^* \rangle^2} + \frac{d\xi^2}{\langle x^* \rangle^2} + \frac{dx^{*2}}{\langle x^* \rangle^2} + \frac{d\xi^{*2}}{\langle x - \xi^* \rangle^2}\right)$$

for $j = 0, 1, 2$. On the other hand, it is easy to see

$$\begin{aligned} \max(1, C^{-1} \langle \xi \rangle \langle \xi - x^* \rangle^{-1}) &\leq \langle x^* \rangle \leq C \langle \xi \rangle \langle \xi - x^* \rangle, \\ \max(1, C^{-1} \langle x \rangle \langle \xi^* \rangle^{-1}) &\leq \langle x - \xi^* \rangle \leq C \langle x \rangle \langle \xi^* \rangle \end{aligned}$$

with some $C > 0$. Combining these, we conclude the assertion. \square

Thanks to Assumption A and (3.1), we have the standard result,

$$R_j := e^{\psi/h}Q_j e^{-\psi/h} - Q_{j\psi} \in OPS(h^2 \Phi^{-2} \Psi^{-2} m_j, \tilde{g})$$

and thus

$$R_j \in OPS(h^2 \langle \xi \rangle^{j-2} \langle x \rangle^{-\sigma-j} (\langle \xi - x^* \rangle^{j+2} \langle \xi^* \rangle^{|\sigma-2+j|+2}), \tilde{g}), \quad (3.4)$$

where

$$q_{j\psi}(x, \xi, x^*, \xi^*) = q_j(x, \xi, x^* + i\partial_x\psi, \xi^* + i\partial_\xi\psi) = p_j(x - \xi^* - i\partial_\xi\psi, x^* + i\partial_x\psi),$$

and $Q_{j\psi} = q_{j\psi}^W(x, \xi, hD_x, hD_\xi)$.

Applying Lemma 3.5 to R_j , we have

$$|\langle e^{\psi/h}Tu, fR_je^{\psi/h}Tu \rangle| \leq Ch^2\mu^{\sigma+j}(\|\sqrt{f}e^{\psi/h}Tu\|^2 + \|u\|^2). \quad (3.5)$$

It remains only to estimate $\langle e^{\psi/h}Tu, fQ_{j\psi}e^{\psi/h}Tu \rangle$ for $j = 0, 1, 2$. We denote

$$p_{j\psi}(x, \xi) = q_{j\psi}(x, \xi, \xi - \mu\partial_\xi\psi, \mu^{-1}\partial_\xi\psi) = p_j(x - \partial_\mu\psi, \xi + i\mu\partial_\mu\psi)$$

for $j = 0, 1, 2$.

Lemma 3.7. *There exists $C > 0$ such that*

(i)

$$\begin{aligned} & |\langle e^{\psi/h}Tu, fe^{\psi/h}\{\frac{1}{2}(hD_x)^2 - \frac{1}{2}(\xi + i\mu\partial_\mu\psi(x, \xi))^2 + \frac{n}{4}\mu h\}Tu \rangle| \\ & \leq Ch\mu(\mu\|\sqrt{f}e^{\psi/h}Tu\|^2 + \|u\|^2), \end{aligned}$$

(ii)

$$\begin{aligned} & |\langle e^{\psi/h}Tu, f\{Q_{j\psi} - p_{j\psi}(x, \xi)\}e^{\psi/h}Tu \rangle| \\ & \leq Ch\mu^{j-1+\sigma}(\|\sqrt{f}e^{\psi/h}Tu\|^2 + \|u\|^2), \end{aligned}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, $j = 0, 1, 2$, and $h, \mu \in (0, 1]$ such that $h/\mu \leq d$.

Proof. (i) By Lemma C.1, we have

$$\begin{aligned} & \langle e^{\psi/h}Tu, fe^{\psi/h}(hD_x)^2Tu \rangle \\ & = \langle e^{\psi/h}Tu, [(\xi + i\mu\partial_\mu\psi + \frac{i}{2}h\mu\partial_\mu)^2 f(x, \xi)]e^{\psi/h}Tu \rangle \\ & = \langle e^{\psi/h}Tu, [(\xi + i\mu\partial_\mu\psi)^2 f - n\frac{h\mu}{2}f \\ & \quad + ih\mu(\xi + i\mu\partial_\mu\psi)\partial_\mu f - \frac{1}{2}h\mu^2(\partial_\mu^2\psi)f - \frac{1}{4}h^2\mu^2\partial_\mu^2 f]e^{\psi/h}Tu \rangle, \end{aligned}$$

and the claim follows immediately.

(ii) We consider the case $j = 2$ only. The claim for the other cases can be shown similarly. We write

$$\begin{aligned} & q_{2\psi}(x, \xi, x^*, \xi^*) - q_{2\psi}(x, \xi, \xi - \mu\partial_\xi\psi, \mu^{-1}\partial_\xi\psi) \\ & = q_{2\psi}^{(1)}(x, \xi, x^*, \xi^*)(x^* - \xi + \mu\partial_\xi\psi) + q_{2\psi}^{(2)}(x, \xi, x^*, \xi^*)(\xi^* - \mu^{-1}\partial_\xi\psi), \end{aligned}$$

where

$$\begin{aligned}
& q_{2\psi}^{(1)}(x, \xi, x^*, \xi^*) \\
&= \int_0^1 \frac{\partial q_{2\psi}}{\partial x^*}(x, \xi, sx^* + (1-s)(\xi - \mu\partial_\xi\psi), s\xi^* + (1-s)\mu^{-1}\partial_x\psi) ds, \\
& q_{2\psi}^{(2)}(x, \xi, x^*, \xi^*) \\
&= \int_0^1 \frac{\partial q_{2\psi}}{\partial \xi^*}(x, \xi, sx^* + (1-s)(\xi - \mu\partial_\xi\psi), s\xi^* + (1-s)\mu^{-1}\partial_x\psi) ds.
\end{aligned}$$

It is easy to check

$$\begin{aligned}
q_{2\psi}^{(1)} &\in S(\langle \xi \rangle \langle x \rangle^{-\sigma} (\langle \xi - x^* \rangle^3 \langle \xi^* \rangle^\sigma), \tilde{g}), \\
q_{2\psi}^{(2)} &\in S(\langle \xi \rangle^2 \langle x \rangle^{-1-\sigma} (\langle \xi - x^* \rangle^2 \langle \xi^* \rangle^{1+\sigma}), \tilde{g}).
\end{aligned}$$

We denote $Q_{2\psi}^{(j)} = q_{2\psi}^{(j)W}(x, \xi, hD_x, hD_\xi)$. Then by the symbol calculus, we have

$$\begin{aligned}
R^{(2)} &:= Q_{2\psi} - q_{2\psi}(x, \xi, \xi - \mu\partial_\xi\psi, \mu^{-1}\partial_x\psi) \\
&\quad - \frac{1}{2} \{AQ_{2\psi}^{(1)} + Q_{2\psi}^{(1)}A + BQ_{2\psi}^{(2)} + Q_{2\psi}^{(2)}B\} \\
&\in OPS(h^2 \langle x \rangle^{-2-\sigma} (\langle \xi - x^* \rangle^4 \langle \xi^* \rangle^{\sigma+2}), \tilde{g}). \tag{3.6}
\end{aligned}$$

Then, using the equation (B.2), we compute

$$\begin{aligned}
& T_\psi^* f \{AQ_{2\psi}^{(1)} + Q_{2\psi}^{(1)}A + BQ_{2\psi}^{(2)} + Q_{2\psi}^{(2)}B\} T_\psi \\
&= T_\psi^* f \{i\mu[B, Q_{2\psi}^{(1)}] + (i\mu)^{-1}[A, Q_{2\psi}^{(2)}]\} T_\psi \\
&\quad + T_\psi^* \{([f, A] + i\mu[B, f])Q_{2\psi}^{(1)} + ([f, B] + (i\mu)^{-1}[A, f])Q_{2\psi}^{(2)}\} T_\psi \\
&=: T_\psi^* L_1 T_\psi + T_\psi^* L_2 T_\psi.
\end{aligned}$$

We note

$$\begin{aligned}
[B, Q_{2\psi}^{(1)}] &\in OPS(h \langle x \rangle^{-\sigma} (\langle \xi - x^* \rangle^4 \langle \xi^* \rangle^\sigma), \tilde{g}), \\
[A, Q_{2\psi}^{(2)}] &\in OPS(h \langle \xi \rangle^2 \langle x \rangle^{-2-\sigma} (\langle \xi - x^* \rangle^3 \langle \xi^* \rangle^{\sigma+4}), \tilde{g}).
\end{aligned}$$

Hence, by applying Lemma 3.5, we have

$$|\langle e^{\psi/h} T u, L_1 e^{\psi/h} T u \rangle| \leq Ch\mu^{1+\sigma} (\|\sqrt{f} e^{\psi/h} T u\|^2 + \|u\|^2).$$

On the other hand, by applying Lemma B.1, we also have

$$|\langle e^{\psi/h} T u, L_2 e^{\psi/h} T u \rangle| \leq Ch\mu^{1+\sigma} (\|\sqrt{f} e^{\psi/h} T u\|^2 + \|u\|^2).$$

Finally, by (3.6) and Lemma 3.5, we learn

$$|\langle e^{\psi/h} T u, f R^{(2)} e^{\psi/h} T u \rangle| \leq Ch^2 \mu^{2+\sigma} (\|\sqrt{f} e^{\psi/h} T u\|^2 + \|u\|^2)$$

Combining these we conclude the assertion (ii) for $j = 2$. \square

Now we recall (3.2). Combining Lemma 3.7 with (3.5), we conclude Theorem 3.1. \square

4 Proof of Theorem 2.1

We suppose $\gamma = \{(y(t), \eta(t)) \mid t \in \mathbb{R}\}$ is backward nontrapping, $t_0 > 0$, and let $\eta_- = \lim_{t \rightarrow -\infty} \eta(t)$ as in the assumptions of Theorem 2.1. We suppose $(-t_0\eta_-, \eta_-) \notin HWF_a(u_0)$, hence we can find a conic neighborhood of $(-t_0\eta_-, \eta_-) : \Gamma \subset \mathbb{R}^{2n}$ and $\delta_1 > 0$ such that

$$\|e^{\delta_1(|x|+|\xi|)}T_{1,1}u_0\|_{L^2(\Gamma)} < \infty.$$

For $(x', \xi') \in \mathbb{R}^{2n}$ and $a_1, a_2 > 0$, we denote a neighborhood of (x', ξ') of size (a_1, a_2) by

$$B(x', \xi'; a_1, a_2) = \{(x, \xi) \in \mathbb{R}^{2n} \mid |x - x'| < a_1, |\xi - \xi'| < a_2\}.$$

Then, for sufficiently small $\delta \in (0, |\eta_-|)$, we have

$$\|T_{1,1}u_0\|_{L^2(B(-h^{-1}t_0\eta_-, h^{-1}\eta_-; h^{-1}\delta t_0, h^{-1}\delta))} \leq Ce^{-\delta/h}$$

for $h \in (0, 1]$. By a change of integration variables, it is equivalent to

$$\|T_{h,h}u_0\|_{L^2(B(-h^{-1}t_0\eta_-, \eta_-; h^{-1}\delta t_0, \delta))} \leq Ce^{-\delta/h}.$$

We use the following weight function. Let $\chi \in C_0^\infty(\mathbb{R}_+)$ such that $\chi(r) = 1$ if $r \leq 1/2$, $\chi(r) = 0$ if $r \geq 1$, and that $\chi'(r) \leq 0$ for $r > 0$. We set $\delta_1 = \delta/4$ and,

$$\varphi(t, x, \xi) := \chi\left(\frac{|x - t\xi|}{\delta_1|t|}\right) \chi\left(\frac{|\xi - \eta_-|}{\delta_1}\right), \quad x, \xi \in \mathbb{R}^n, t < 0.$$

Since $(\partial_t + H_{\xi^2/2})|x - t\xi| = 0$, $H_p = H_{\xi^2/2} + O(\langle x \rangle^{-\sigma}|\xi|)\partial_x + O(\langle x \rangle^{-\sigma-1}|\xi|^2)\partial_\xi$, and $|x| \geq \delta|t|/2$ while $|\xi| = O(1)$ on the support of φ , we immediately obtain,

$$\frac{\partial \varphi}{\partial t} + H_p \varphi = \frac{|x - t\xi|}{\delta_1|t|^2} \chi'\left(\frac{|x - t\xi|}{\delta_1|t|}\right) \chi\left(\frac{|\xi - \eta_-|}{\delta_1}\right) + O(|t|^{-1-\sigma}),$$

and thus,

$$\frac{\partial \varphi}{\partial t} + H_p \varphi \leq C|t|^{-1-\sigma}, \quad (4.1)$$

for some constant $C > 0$. Now we set

$$\psi(t, x, \xi) = \delta \varphi(h^{-1}t, x, \xi),$$

and we note,

$$\text{supp}[\psi(t, \cdot, \cdot)] \subset B(h^{-1}t\eta_-, \eta_-; h^{-1}\delta|t|/2, \delta/2).$$

Hence, we have

$$\|e^{\psi(t,x,\xi)/h}T_{h,h}u_0\| \leq C < \infty \quad (4.2)$$

for $t \in [-t_0 - \varepsilon_1, -t_0 + \varepsilon_1]$ with sufficiently small $\varepsilon_1 > 0$. We then set

$$\mu(t) = \frac{-t_0}{t} h$$

for $t < 0$. We note $\mu(t)$ is monotone increasing and $\mu(t) \geq h$ for $t \in [-t_0, 0)$. It is elementary computation to confirm that $\psi(t, x, \xi)$ satisfies the conditions of Section 3, since $\langle x \rangle^{-1} = O(h|t|^{-1}) = O(\mu)$ on the support of $\psi(t, \cdot, \cdot)$.

Then, we choose $f \in C^\infty(\mathbb{R}_-; C_0^\infty(\mathbb{R}^{2n}))$ of the form,

$$f(t, x, \xi) = \chi_1(|\mu(t)x|)\chi_1(|\xi|),$$

where $\chi_1 \in C_0^\infty((2A)^{-1}; 2A)$, $\chi_1 = 1$ on $[A^{-1}, A]$, with $A > 0$ sufficiently large in order to have $f = 1$ on the support of ψ . In particular, f satisfies to the conditions of Section 3.

Now, we set

$$F(t) = \left\| \sqrt{f(t, \cdot, \cdot)} e^{\psi(t, \cdot, \cdot)/h} T_{h, \mu(t)} u(t + t_0) \right\|^2$$

for $t \in [-t_0, 0)$, and we are find a differential inequality satisfied by $F(t)$. Setting $\tilde{u}(t) = u(t + t_0)$, and writing T instead of $T_{h, \mu(t)}$, we compute,

$$\begin{aligned} \frac{d}{dt} F(t) &= \langle e^{\psi/h} T(-iP)\tilde{u}, f e^{\psi/h} T\tilde{u} \rangle + \langle e^{\psi/h} T\tilde{u}, f e^{\psi/h} T(-iP)\tilde{u} \rangle \\ &\quad + \langle e^{\psi/h} \tilde{u}, 2h^{-1} f \left(\frac{\partial}{\partial t} \psi \right) e^{\psi/h} T\tilde{u} \rangle \\ &\quad + \langle e^{\psi/h} \left[\frac{\partial}{\partial t}, T \right] \tilde{u}, f e^{\psi/h} T\tilde{u} \rangle + \langle e^{\psi/h} T\tilde{u}, f e^{\psi/h} \left[\frac{\partial}{\partial t}, T \right] \tilde{u} \rangle \\ &\quad + \langle e^{\psi/h} T\tilde{u}, \left(\frac{\partial f}{\partial t} \right) e^{\psi/h} T\tilde{u} \rangle \\ &= F_1 + F_2 + F_3 + F_4. \end{aligned} \tag{4.3}$$

At first, we have

$$\begin{aligned} F_1 &= 2\text{Im} \left[\langle e^{\psi/h} T\tilde{u}, f e^{\psi/h} T P \tilde{u} \rangle \right] \\ &= 2 \langle e^{\psi/h} T\tilde{u}, f (h^{-2} H_p \psi) e^{\psi/h} T\tilde{u} \rangle + r(t), \end{aligned}$$

where $r(t)$ is estimated by Corollary 3.3. Hence we learn

$$\begin{aligned} F_1 + F_2 &= 2h^{-2} \langle e^{\psi/h} T\tilde{u}, f (h \frac{\partial}{\partial t} \psi + H_p \psi) e^{\psi/h} T\tilde{u} \rangle + r(t) \\ &\leq C h^{-1+\sigma} |t|^{-1-\sigma} F(t) + r(t) \end{aligned} \tag{4.4}$$

using the inequality (4.1).

Lemma 4.1. *There exists $C > 0$ such that*

$$F_3(t) \leq C \left\{ h^{-1} \mu'(t) F(t) + \mu'(t) \|\tilde{u}\|^2 \right\}.$$

Proof. We note

$$\begin{aligned}
\left[\frac{\partial}{\partial t}, T\right]\tilde{u} &= c_{h,\mu} \int \left(-\frac{\mu'(t)}{2h}\right) |x-y|^2 e^{i(x-y)\cdot\xi/h-\mu|x-y|^2/2h} \tilde{u}(y) dy \\
&= (2h)^{-1} \mu'(t) \int h^2 \Delta_\xi e^{i(x-y)\cdot\xi/h-\mu|x-y|^2/2h} \tilde{u}(y) dy \\
&= \frac{1}{2} h \mu'(t) \Delta_\xi T \tilde{u}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
F_3 &= \frac{1}{2} h \mu'(t) \langle T \tilde{u}, (\Delta_\xi f e^{2\psi/h} + f e^{\psi/h} \Delta_\xi) T \tilde{u} \rangle \\
&= h \mu'(t) \langle T \tilde{u}, \{ \sqrt{f} e^{\psi/h} \Delta_\xi \sqrt{f} e^{\psi/h} + |\nabla_\xi(\sqrt{f} e^{\psi/h})|^2 \} T \tilde{u} \rangle \\
&\leq h \left\{ h^{-2} \langle \sqrt{f} e^{\psi/h} T \tilde{u}, |\nabla_\xi \psi|^2 \sqrt{f} e^{\psi/h} T \tilde{u} \rangle \right. \\
&\quad \left. + \langle T \tilde{u}, \left(2 \nabla_\xi \sqrt{f} \cdot h^{-1} \nabla_\xi \psi + |\nabla_\xi \sqrt{f}|^2 \right) T \tilde{u} \rangle \right\} \\
&\leq C h^{-1} \mu'(t) \|\sqrt{f} e^{\psi/h} T \tilde{u}\|^2 + C \mu'(t) \|\tilde{u}\|^2,
\end{aligned}$$

since $\nabla_\xi \sqrt{f}$ is supported away from $\text{supp}[\psi]$. \square

For the last term, we easily have

$$F_4 = \langle T \tilde{u}, \left(\frac{\partial f}{\partial t}\right) T \tilde{u} \rangle = O(|t|^{-1}).$$

We note

$$\mu'(t) = t_0 h t^{-2} = O(h|t|^{-2}).$$

Combining Lemma 4.1 with (4.3) and (4.4), we obtain the following estimate:

Lemma 4.2. *There exists $C > 0$ such that*

$$\frac{d}{dt} F(t) \leq A(t) F(t) + B(t) \|\tilde{u}\|^2,$$

where

$$A(t) = C(h^{-1+\sigma}|t|^{-1-\sigma} + |t|^{-2}), \quad B(t) = C|t|^{-1}.$$

By Gronwall's inequality, we have

$$\begin{aligned}
F(t) &\leq e^{\int_{-t_0}^t A(s) ds} \left(F(-t_0) + \int_{-t_0}^t e^{-\int_{-t_0}^s A(\tau) d\tau} B(s) \|\tilde{u}\|^2 ds \right) \\
&\leq e^{\int_{-t_0}^t A(s) ds} \left(F(-t_0) + \int_{-t_0}^t B(s) ds \cdot \|\tilde{u}\|^2 \right).
\end{aligned}$$

for $t < 0$. For each $t \in (-t_0, 0)$, this immediately implies

$$F(t) = \|\sqrt{f} e^{\psi(t,x,\xi)/h} T_{h,|t_0|/t/h} \tilde{u}(t)\|^2 = O(e^{C(t)h^{-1+\sigma}})$$

as $h \rightarrow 0$, with some $C(t) > 0$. Since

$$\psi(t, x, \xi) = \delta \quad \text{on} \quad B(h^{-1}t\eta_-, \eta_-; h^{-1}\delta_1|t|/4, \delta_1/4)$$

if h is sufficiently small, this implies $(t\eta_-, \eta_-) \notin HWF_a(\tilde{u}(t))$. This proves the first claim of Theorem 2.1.

Moreover, by elementary computation, we obtain

$$\int_{-t_0}^{-Rh} A(t)dt \leq C'h^{-1}R^{-\sigma}$$

for $R \geq 1$. We fix $R > 0$ so that

$$0 < C'R^{-\sigma} \leq \delta/2,$$

and hence

$$\int_{-t_0}^{-Rh} A(t)dt \leq \delta h^{-1}/2 \quad \text{as } h \rightarrow 0.$$

Similarly, we have

$$\int_{-t_0}^{-Rh} B(t)dt = O(|\log h|) \quad \text{as } h \rightarrow 0.$$

Combining these estimates, we learn $F(-Rh) = O(e^{\delta/h})$ as $h \rightarrow 0$, and hence,

$$e^{-\delta/h}F(-Rh) = \left\| \sqrt{f}e^{(\psi(-Rh, x, \xi) - \delta/2)/h} T_{h, \mu(-Rh)} \tilde{u}(-Rh) \right\|^2 \leq C < \infty.$$

We recall $\mu(-Rh) = t_0/R$ and

$$\psi(-Rh, x, \xi) = \delta\varphi(-R, x, \xi) = \delta\chi\left(\frac{|x + R\xi|}{\delta_1 R}\right)\chi\left(\frac{|\xi - \eta_-|}{\delta_1}\right).$$

By standard estimates on the flow, we also know that $|-R\eta_- - y(-R)| = O(R^{1-\sigma})$ and $|\eta_- - \eta(-R)| = O(R^{-\sigma})$. Thus, increasing R if necessary, we obtain

$$\left\| T_{h, t_0/R} \tilde{u}(-Rh) \right\|_{L^2(B(\gamma(-R); \delta_1 R/8, \delta_1/8))}^2 \leq Ce^{-\delta/h}.$$

Moreover, in the above argument, we may replace $-t_0$ by t in a small neighborhood of $-t_0$, e.g., $t \in [-t_0 - \varepsilon_1, -t_0 + \varepsilon_1]$ with some $\varepsilon_1 > 0$. Then, we learn

$$\left\| T_{h, t_0/R} \tilde{u}(s - Rh) \right\|_{L^2(\gamma(-R); \delta_1 R/8, \delta_1/8)}^2 \leq Ce^{-\delta/h} \quad (4.5)$$

uniformly for $s \in [-\varepsilon_1, \varepsilon_1]$. In particular, this implies $\gamma(-R) \notin WF_a(\tilde{u}(s))$ for $s \in [-\varepsilon_1/2, \varepsilon_1/2]$. Then, we use the following variation of the *propagation of the microsupport theorem*:

Lemma 4.3. *Let $u(t)$ be a solution to the Schrödinger equation for t in a neighborhood of 0. Then, if the following condition:*

$$\exists \delta, \varepsilon > 0 \text{ such that } \|T_{h,1}u(ht)\|_{L^2(B(\gamma(t);\delta,\delta))} = O(e^{-\varepsilon/h}) \text{ as } h \rightarrow 0 \quad (4.6)$$

holds for some t , it holds for all $t \in \mathbb{R}$.

This lemma is an immediate consequence of [Ma2], Chapter 4, Exercise 9, after a scaling of the variable $t: t \mapsto ht$. We give a concise proof in Appendix D for completeness. Now the estimate (4.5) implies that for any $t \in \mathbb{R}$, $s \in [-\varepsilon_0, \varepsilon_0]$, there is $\delta, \varepsilon > 0$ such that

$$\|T_{h,1}\tilde{u}(s - hR + ht)\|_{L^2(B(\gamma(t-R);\delta,\delta))} = O(e^{-\varepsilon/h}).$$

In particular, setting $t' = t - R$ and $s' = s + ht'$, we learn $\gamma(t') \notin WF_a(\tilde{u}(s'))$ for any $t' \in \mathbb{R}$ and $s' \in [-\varepsilon_0/2, \varepsilon_0/2]$. This completes the proof of Theorem 2.1. \square

A Proof of Corollary 2.3

It is enough to show that $T_{1,1}u_0 = O(e^{-\delta\langle \xi \rangle})$ for some $\delta > 0$, uniformly for (x, ξ) in a conic neighborhood of $(-t_0\eta_-, \eta_-)$, where

$$T_{1,1}u_0(x, \xi) = c \int e^{i(x-y)\cdot\xi + i\psi(y) - |x-y|^2/2} a(y) dy, \quad (A.1)$$

(with $c := 2^{-n/2}\pi^{-3n/4}$).

We first assume $m \geq 2$. For $\lambda \geq 1$, we set $(x_\lambda, \xi_\lambda) := (-\lambda t_0\eta_-, \lambda\eta_-)$ and $\eta_\lambda := \nabla\psi(x_\lambda)$. Then, for $y \in \mathbb{C}^n$ such that $|y - x_\lambda| \leq 2\varepsilon\lambda$ (with $\varepsilon > 0$ small enough), using the analyticity of ψ , standard Cauchy estimates, and the fact that ψ is real on the real, by a second-order Taylor expansion at x_λ , we see that,

$$\text{Im } \psi(y) = \text{Im } (y \cdot \eta_\lambda) + O(\varepsilon\lambda^{m-1}|t \text{ Im } y|)$$

uniformly for $\lambda \geq 1$ and $\varepsilon > 0$ small enough.

Now, we apply the following change of contour of integration in (A.1):

$$\mathbb{R}^n \ni y \mapsto z := y - i\delta\lambda \frac{\xi - \eta_\lambda}{|\xi - \eta_\lambda|} \chi(\lambda^{-1}|y - x_\lambda|),$$

(where $\chi \in C_0^\infty([0, 2\varepsilon])$, $\chi = 1$ on $[0, \varepsilon]$, and $\varepsilon \geq \delta > 0$ are small enough). On this contour, we have,

$$\begin{aligned} & \text{Re } (i(x-z)\xi + i\psi(z) - |x-z|^2/2) \\ &= \text{Im } z \cdot (\xi - \eta_\lambda) - |x-y|^2/2 + |\text{Im } z|^2/2 + O(\varepsilon\lambda^{m-1}|\text{Im } z|) \\ &= -\delta\lambda|\xi - \eta_\lambda|\chi - |x-y|^2/2 + \delta^2\lambda^2\chi^2 + O(\varepsilon\delta\lambda^m\chi). \end{aligned}$$

In particular, using (2.7), we obtain for $\lambda^{-1}\xi$ close enough to η_- ,

$$\begin{aligned} & \operatorname{Re} (i(x-z) \cdot \xi + i\psi(z) - |x-z|^2/2) \\ & \leq -\delta\lambda^m\chi/(2C) - |x-y|^2/2 + \delta^2\lambda^2\chi^2 + O(\varepsilon\delta\lambda^m\chi) \end{aligned}$$

and thus, by choosing first ε small enough, and then δ small enough (and since $m \geq 2$),

$$\operatorname{Re} (i(x-z) \cdot \xi + i\psi(z) - |x-z|^2/2) \leq -\delta\lambda^m\chi/(4C) - |x-y|^2/2. \quad (\text{A.2})$$

As a consequence, for $\lambda^{-1}|x-x_\lambda| \leq \varepsilon/2$, we obtain,

$$\begin{aligned} & |T_{1,1}u_0(x, \xi)| \\ & = O\left(\int_{|y-x_\lambda| \leq \varepsilon\lambda} \langle y \rangle^M e^{-\delta\lambda^m/(4C)} dy + \int_{|y-x_\lambda| \geq \varepsilon\lambda} \langle y \rangle^M e^{-\varepsilon^2\lambda^2/16 - |x-y|^2/4} dy\right) \\ & = O\left(\lambda^{M+n} e^{-\delta\lambda^m/(4C)} + \lambda^M e^{-\varepsilon\lambda^2/16}\right) \end{aligned}$$

and the result for $m = 2$ follows.

In the case $m = 1$, we observe that $\nabla\psi(y) = O(1)$ and $\operatorname{Hess}\psi(y) = O(1)$ uniformly on $\{y \in \mathbb{C}^n \mid |\operatorname{Re} y - x_\lambda| \leq \varepsilon\lambda, |\operatorname{Im} y| \leq \varepsilon\}$, and thus, by a first-order Taylor expansion, $\operatorname{Im} \psi(y) = O((1+\varepsilon\lambda)|\operatorname{Im} y|)$ on this set. Therefore, applying in (A.1) the following change of contour of integration:

$$\mathbb{R}^n \ni y \mapsto z := y - i\delta \frac{\xi}{|\xi|} \chi(\lambda^{-1}|y-x_\lambda|),$$

(with $\delta \leq \varepsilon$ and χ as before), this time we obtain,

$$\operatorname{Re} (i(x-z) \cdot \xi + i\psi(z) - |x-z|^2/2) = -\delta|\xi|\chi - |x-y|^2/2 + O(1 + \varepsilon\delta\lambda\chi) \quad (\text{A.3})$$

and thus, for $\lambda^{-1}(x, \xi)$ close enough to $(-t_0\eta_-, \eta_-)$, we obtain as before,

$$|T_{1,1}u_0(x, \xi)| = O\left(\lambda^{M+n} e^{-\delta\lambda/2} + \lambda^M e^{-\varepsilon\lambda^2/16}\right),$$

and the result follows. \square

Remark. Actually, in the case $m \geq 2$, we have indeed proved that $T_1u_0 = O(e^{-\delta(\xi)^2})$ for some $\delta > 0$, uniformly for (x, ξ) in a \mathbb{R}^{2n} -conic neighborhood of $(-t_0\eta_-, \eta_-)$.

B Proof of Lemmas 3.4 and 3.5

Proof of Lemma 3.4. We prove the lemma by induction in m and ℓ . The claim is obvious if $m = \ell = 0$. We suppose the claim holds for (m, ℓ) and prove the claim for $(m+1, \ell)$ and $(m, \ell+1)$. We write

$$A = hD_x - \xi + \mu\partial_\xi\psi(x, \xi), \quad B = hD_\xi - \mu^{-1}\partial_x\psi(x, \xi). \quad (\text{B.1})$$

We recall (cf. [Ma2], [Na1]):

$$(A - i\mu B)e^{\psi/h}Tu = 0 \quad \text{for } u \in \mathcal{S}(\mathbb{R}^n), \quad (\text{B.2})$$

which can be shown by straightforward computation. If $Q \in OPS(\langle \xi - x^* \rangle^{m+1} \langle \xi^* \rangle^\ell, g_0)$ we write

$$Q = \tilde{Q}(A + i), \quad \tilde{Q} = Q(A + i)^{-1} \in OPS(\langle \xi - x^* \rangle^m \langle \xi^* \rangle^\ell, g_0).$$

Then we have

$$\begin{aligned} T_\psi^* \tilde{Q} A T_\psi &= \frac{1}{2} T_\psi^* \{ \tilde{Q} A + A \tilde{Q} + [\tilde{Q}, A] \} T_\psi \\ &= \frac{i\mu}{2} T_\psi^* [\tilde{Q}, B] T_\psi + \frac{1}{2} T_\psi^* [\tilde{Q}, A] T_\psi. \end{aligned}$$

It is easy to check $[\tilde{Q}, A], [\tilde{Q}, B] \in OPS(h \langle \xi - x^* \rangle^m \langle \xi^* \rangle^\ell, g_0)$. By the induction hypothesis, we learn $\langle T_\psi u, \tilde{Q} A T_\psi u \rangle \leq C \|T_\psi u\|^2$.

Similarly, if $Q \in OPS(\langle \xi - x^* \rangle^m \langle \xi^* \rangle^{\ell+1}, g_0)$, then we write

$$Q = \tilde{Q}(B + i), \quad \tilde{Q} = Q(B + i)^{-1} \in OPS(\langle \xi - x^* \rangle^m \langle \xi^* \rangle^\ell, g_0),$$

and we compute

$$T_\psi^* \tilde{Q} B T_\psi = \frac{1}{2i\mu} T_\psi^* [\tilde{Q}, A] T_\psi + \frac{1}{2} T_\psi^* [\tilde{Q}, B] T_\psi.$$

We note $[\tilde{Q}, A] \in OPS(h \langle \xi - x^* \rangle^m \langle \xi^* \rangle^\ell, g_0)$, and since $h/\mu \leq d$, we have

$$\langle T_\psi u, \tilde{Q} B T_\psi u \rangle \leq C(h/\mu + 1) \|T_\psi u\|^2 \leq C' \|T_\psi u\|^2$$

again using the induction hypothesis. \square

Proof of Lemma 3.5. Since $\langle \xi \rangle = O(1)$ and $\langle x \rangle = O(\mu^{-1})$ on the support of f , we may assume $a = b = 0$ without loss of generality. Analogously to the above lemma, we prove the assertion by induction in m and ℓ . At first, we suppose $Q \in S(1, \tilde{g})$. Then

$$\langle T_\psi u, f Q T_\psi u \rangle = \langle \sqrt{f} T_\psi u, Q \sqrt{f} T_\psi u \rangle + \langle T_\psi u, [Q, \sqrt{f}] \sqrt{f} T_\psi u \rangle.$$

The first term in the right hand side is bounded by $C \|\sqrt{f} T_\psi u\|^2$ since Q is bounded in $L^2(\mathbb{R}^{2n})$. The second term is estimated using the following lemma:

Lemma B.1. *Let $Q \in OPS(\langle \xi \rangle^a \langle x \rangle^b \langle \xi - x^* \rangle^m \langle \xi^* \rangle^\ell, \tilde{g})$, and suppose the symbol of Q has an asymptotic expansion supported in $\text{supp}[\nabla f]$. Then for any $N > 0$, there exists $C > 0$ such that*

$$|\langle e^{\psi/h} T u, Q e^{\psi/h} T u \rangle| \leq C (h^N \mu^N \|\sqrt{f} e^{\psi/h} T u\|^2 + \mu^{-b} \|u\|^2) \quad (\text{B.3})$$

for $u \in \mathcal{S}(\mathbb{R}^n)$.

We postpone the proof of Lemma B.1, and proceed with the proof of Lemma 3.5. Since the symbol of $[Q, \sqrt{f}]$ has an asymptotic expansion supported in $\text{supp}[\nabla f]$, we can apply Lemma B.1 to learn

$$|\langle T_\psi u, [Q, \sqrt{f}] T_\psi u \rangle| \leq C(h^N \mu^N \|\sqrt{f} T_\psi u\|^2 + \|u\|^2).$$

Thus the claim holds if $m = \ell = 0$.

Now suppose $Q \in OPS(\langle \xi - x^* \rangle^{m+1} \langle \xi^* \rangle^\ell, \tilde{g})$. Denoting $\tilde{Q} = Q(A+i)^{-1} \in OPS(\langle \xi - x^* \rangle^m \langle \xi^* \rangle^\ell, \tilde{g})$, we have

$$\begin{aligned} T_\psi^* f Q T_\psi &= T_\psi^* f \tilde{Q} A T_\psi + i T_\psi^* f \tilde{Q} T_\psi \\ &= \frac{i\mu}{2} T_\psi^* f [\tilde{Q}, B] T_\psi + \frac{1}{2} T_\psi^* f [\tilde{Q}, A] T_\psi + \frac{1}{2} T_\psi^* [f, A] \tilde{Q} T_\psi \\ &\quad + \frac{i\mu}{2} T_\psi^* [B, f] \tilde{Q} T_\psi + i T_\psi^* f \tilde{Q} T_\psi. \end{aligned}$$

The first, the second and the last terms in the right hand side are estimated by the induction hypothesis, and the third and the fourth terms are estimated by Lemma B.1 to obtain the assertion for $(m+1, \ell)$. The argument for the case $Q \in OPS(\langle \xi - x^* \rangle^m \langle \xi^* \rangle^{\ell+1}, \tilde{g})$ is similar, and we omit the detail. \square

Proof of Lemma B.1. By the assumption on f , we can find $\tilde{f} \in C_0^\infty(\mathbb{R}^{2n})$ such that (i) $\tilde{f} = 1$ on $\text{supp}[\psi]$; (ii) $f = 1$ on $\text{supp}[\tilde{f}]$; (iii) for any $\alpha, \beta \in \mathbb{Z}_+^n$, $|\partial_x^\alpha \partial_\xi^\beta \tilde{f}(x, \xi)| \leq C_{\alpha\beta} \mu^{|\alpha|}$ ($x, \xi \in \mathbb{R}^n$). We write

$$Q = (1 - \tilde{f})Q(1 - \tilde{f}) + \{\tilde{f}Q + Q\tilde{f} - \tilde{f}Q\tilde{f}\} = Q_1 + Q_2.$$

The symbol of Q_2 has vanishing asymptotic expansion since $\text{supp}[\nabla f] \cap \text{supp}[\tilde{f}] = \emptyset$. In particular, since $\mu^{-N} \langle x \rangle^{-N} \tilde{f}$ is uniformly bounded together with all its derivatives, we can deduce,

$$Q_2 \in OPS(h^N \mu^{N-b} \langle \xi - x^* \rangle^{m+N} \langle \xi^* \rangle^{\ell+N}, \tilde{g})$$

for any N . Hence, by Lemma 3.4, we have

$$\begin{aligned} |\langle e^{\psi/h} T u, Q_2 e^{\psi/h} T u \rangle| &\leq C h^N \langle x \rangle^{b-N} \langle e^{\psi/h} T u, e^{\psi/h} T u \rangle \\ &\leq C h^N \mu^{N-b} \{ \langle e^{\psi/h} T u, f e^{\psi/h} T u \rangle + \langle e^{\psi/h} T u, (1-f) e^{\psi/h} T u \rangle \} \\ &\leq C h^N \mu^{N-b} (\|\sqrt{f} e^{\psi/h} T u\|^2 + \|u\|^2). \end{aligned}$$

On the other hand, we note $Q_1 \in OPS((\mu^{-b} + h^N) \langle \xi - x^* \rangle^{m+|a|} \langle \xi^* \rangle^{\ell+|b|}, \tilde{g})$ for all N (where h^N comes from the remainder term of the asymptotic expansion of the symbol), and $e^{\psi/h} Q_1 e^{\psi/h} = Q_1$. Hence, for $h \leq d\mu$, we have

$$|\langle e^{\psi/h} T u, Q_1 e^{\psi/h} T u \rangle| = |\langle T u, Q_1 T u \rangle| \leq C \mu^{-b} \|u\|^2$$

again by Lemma 3.4. Combining these, we obtain the assertion. \square

C Exponential weight formula for differential operators

Here we prove an exponential weight formula for differential operators, which was introduced in Introduction of [Ma1]. We reproduce the proof for the reader's convenience.

Lemma C.1. *Suppose $f \in C_0^\infty(\mathbb{R}^{2n})$, and we write $T = T_{h,\mu}$. Then for any $\alpha \in \mathbb{Z}_+^n$,*

$$\langle e^{\psi/h}Tv, fe^{\psi/h}T(hD_x)^\alpha u \rangle = \frac{1}{2} \langle e^{\psi/h}Tv, [(\xi + i\mu\partial_\mu\psi + \frac{i}{2}h\mu\partial_\mu)^\alpha f]e^{\psi/h}Tu \rangle$$

for $u, v \in \mathcal{S}(\mathbb{R}^n)$.

Proof. At first, we note

$$\begin{aligned} e^{\psi/h}T(hD_x) &= e^{\psi/h}(hD_x)T = (hD_x + i\partial_x\psi)e^{\psi/h}T \\ &= (A + \xi + i\mu\partial_\mu\psi)e^{\psi/h}T \end{aligned}$$

where A is defined in (B.1). On the other hand, by virtue of (B.2), we have

$$\begin{aligned} &\langle e^{\psi/h}Tv, fAe^{\psi/h}Tu \rangle \\ &= \frac{1}{2} \langle e^{\psi/h}Tv, (Af + fA)e^{\psi/h}Tu \rangle + \frac{1}{2} \langle e^{\psi/h}Tv, [f, A]e^{\psi/h}Tu \rangle \\ &= \frac{1}{2} \langle e^{\psi/h}Tv, (-i\mu Bf + i\mu fB)e^{\psi/h}Tu \rangle + \frac{1}{2} \langle e^{\psi/h}Tv, [f, A]e^{\psi/h}Tu \rangle \\ &= \frac{1}{2} \langle e^{\psi/h}Tv, [f, A + i\mu B]e^{\psi/h}Tu \rangle \\ &= \frac{1}{2} \langle e^{\psi/h}Tv, (ih\mu\partial_\mu f)e^{\psi/h}Tu \rangle. \end{aligned}$$

Combining these, we learn

$$\langle e^{\psi/h}Tv, fe^{\psi/h}T(hD_x)u \rangle = \langle e^{\psi/h}Tv, [(\xi + i\mu\partial_\mu\psi + \frac{i}{2}h\mu\partial_\mu)f]e^{\psi/h}Tu \rangle.$$

Iterating this procedure, we conclude the assertion. \square

D Proof of Lemma 4.3

Let $t_0 < t_1$, and we suppose (4.6) is satisfied for $t = t_0$. We choose $\varphi \in C_0^\infty(B(\gamma(t_0); \delta, \delta))$ such that $\varphi = 1$ on $B(\gamma(t_0); \delta/2, \delta/2)$. By the assumption, we have

$$\|e^{\varepsilon\varphi(x,\xi)/h}T_{h,1}u(ht_0)\| \leq C < \infty \quad \text{as } h \rightarrow 0,$$

for sufficiently small $\varepsilon > 0$. We then set

$$\psi(t, x, \xi) = \varphi(\exp(t_0 - t)H_p(x, \xi))$$

for $t \in [t_0, t_1]$, $x, \xi \in \mathbb{R}^n$. We also choose $f \in C_0^\infty(\mathbb{R}^{2n})$ so that

$$f(x, \xi) = 1 \quad \text{on} \quad \bigcup_{t_0 \leq t \leq t_1} \text{supp}[\psi(t, \cdot, \cdot)].$$

Then we apply Theorem 3.1 with $\mu = 1$ to obtain

$$\begin{aligned} & \left| \langle e^{\varepsilon\psi(t)/h} T_{h,1} u(ht), f e^{\varepsilon\psi(t)/h} T_{h,1} P u(ht) \rangle \right. \\ & \quad \left. - \langle e^{\varepsilon\psi(t)/h} T_{h,1} u(ht), \tilde{p}_{\varepsilon\psi} e^{\varepsilon\psi(t)/h} T_{h,1} u(ht) \rangle \right| \\ & \leq Ch^{-1} (\|e^{\varepsilon\psi(t)/h} T_{h,1} u(ht)\|^2 + \|u(ht)\|^2) \end{aligned}$$

uniformly for $t \in [t_0, t_1]$, $\varepsilon \in (0, 1]$. Analogously to Lemma 3.2, we have

$$|\text{Im } \tilde{p}_{\varepsilon\psi}(x, \xi) - h^{-2}\varepsilon H_p \psi| \leq Ch^{-2}\varepsilon^2.$$

We note $\frac{\partial}{\partial t} \psi + H_p \psi = 0$. Hence, analogously to Lemma 4.1, we have

$$h^{-1} \frac{d}{dt} F(t) \leq C(h^{-1} + h^{-2}\varepsilon^2) F(t) + Ch^{-1} \|u(ht)\|^2$$

for $t \in [t_0, t_1]$, where $F(t) = \|\sqrt{f} e^{\varepsilon\psi(t)/h} T_{h,1} u(ht)\|^2$. Hence we learn

$$F(t_1) \leq C_1 \varepsilon^{-C_2} e^{C_3 \varepsilon^2/h}$$

where $C_1, C_2, C_3 > 0$ are independent of h and ε . Thus we obtain

$$\|T_{h,1} u(ht_1)\|_{L^2(B(\gamma(t_1); \delta', \delta'))}^2 \leq C_1 \varepsilon^{-C_2} e^{-(2\varepsilon - C_3 \varepsilon^2)/h}$$

with sufficiently small $\delta' > 0$. By choosing ε sufficiently small, we conclude (4.6) for $t = t_1$. Proof for the case $t_1 < t_0$ is similar, and we omit the detail. \square

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