ON THE TUNNELING EFFECT FOR MAGNETIC SCHRÖDINGER OPERATORS IN ANTIDOT LATTICES

RUPERT L. FRANK

ABSTRACT. We study the Schrödinger operator $(h\mathbf{D}-\mathbf{A})^2$ with periodic magnetic field $B = \operatorname{curl}\mathbf{A}$ in an antidot lattice $\Omega_\infty = \mathbb{R}^2 \setminus \bigcup_{\alpha\in\Gamma}(U+\alpha)$. Neumann boundary conditions lead to spectrum below $h \inf B$. Under suitable assumptions on a "one-well problem" we prove that this spectrum is localized inside an exponentially small interval in the semi-classical limit $h \to 0$. For this purpose we construct a basis of the corresponding spectral subspace with natural localization and symmetry properties.

INTRODUCTION

0.1. In this paper we consider a magnetic Schrödinger operator

$$P_h := (h\mathbf{D} - \mathbf{A})^2 \tag{0.1}$$

in an antidot lattice, i.e., a connected domain $\Omega_\infty = \mathbb{R}^2 \setminus \bigcup_{\alpha\in\Gamma}(U + \alpha)$ which is periodic with respect to some lattice Γ. The "holes" $U + \alpha$, $\alpha \in \Gamma$, are compact sets with smooth boundary on which we impose Neumann boundary conditions. For simplicity we will assume in this introduction that $\mathbf{A}(x) = \frac{B}{2}(-x_2, x_1)^T$, $B > 0$, so the *magnetic field* $\operatorname{curl}\mathbf{A} = B$ is constant. More generally, our results are valid for $\mathbf{A}$ with Γ-periodic positive magnetic field and Γ-periodic circulations around the components of $\partial\Omega_\infty$. Our goal is to study the spectrum of P_h below the lowest Landau level hB in the semi-classical limit $h \to 0$. As we will see, the existence of this spectrum depends essentially on the choice of Neumann boundary conditions and its size is due to the interaction between the boundaries of the holes $U + \alpha$, $\alpha \in \Gamma$.

Before we state our main results let us consider other self-adjoint realizations of the differential expression (0.1). First we impose Dirichlet boundary conditions on $\partial\Omega_\infty$. Then the well-known inequality

$$hB \int_{\Omega_\infty} |u|^2\,dx \le \int_{\Omega_\infty} |(h\mathbf{D} - \mathbf{A})u|^2\,dx, \qquad u \in C_0^\infty(\Omega_\infty),$$

(see (1.12)) implies that the spectrum of the corresponding operator starts above hB. Therefore we can consider the spectrum of our original operator

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P_h below hB as a boundary phenomenon. In order to understand the effect of the boundary condition, we switch now the Dirichlet boundary condition to Neumann at the boundary of precisely one hole, say ∂U, and preserve Dirichlet at $\partial U + \alpha$ for $0 \neq \alpha \in \Gamma$. We denote the corresponding operator by $\tilde{P}_h$ and refer to it as "reference operator". For the study of this operator we can rely on the works and methods of [LuPa] and [HeMo]. In particular, it is known that $\tilde{P}_h$ has discrete spectrum below hB and that the lowest eigenvalue behaves like $\tilde{\lambda}(h) = \Theta_0 Bh + \mathcal{O}(h^{3/2})$ as $h \to 0$. Here $0 < \Theta_0 < 1$ is some universal constant.

Now assume that there are constants β_0, β_1 such that $\tilde{\lambda}(h)$ is the only eigenvalue of the reference operator $\tilde{P}_h$ in the interval

$$I(h) := [0, \Theta_0 Bh + \beta_0 B^{1/2} h^{3/2} + 2\beta_1 B^{1/4} h^{7/4}].$$

This is known to be true under "generic" assumption on ∂U, see [FoHe] and our discussion in Subsection 1.4.

Let us return to our original operator P_h. Our main result (Theorem 1.7 and Corollary 1.8) states that under the mentioned assumption the spectrum of the operator P_h in the interval $I(h)$ is non-empty and contained in an exponentially small (with respect to $\sqrt{h}$) neighbourhood of the reference eigenvalue $\tilde{\lambda}(h)$,

$$\sigma(P_h) \cap I(h) \subset [\tilde{\lambda}(h) - Ce^{-c/\sqrt{h}}, \tilde{\lambda}(h) + Ce^{-c/\sqrt{h}}]. \tag{0.2}$$

Moreover, we show that the spectral subspace of P_h corresponding to $I(h)$ is infinite-dimensional and we construct a basis of this space having natural localization and symmetry properties.

0.2. The basic idea behind our construction is that Neumann boundaries play the same role for the operator (0.1) as the minima of a potential V ("potential wells") do for the usual Schrödinger operator

$$-h^2\Delta + V. \tag{0.3}$$

Consequently, we will refer to the $\partial U + \alpha$ as "wells" in what follows.

To be more precise consider an eigenfunction $\tilde{\varphi}_h$ of the reference operator $\tilde{P}_h$ corresponding to the eigenvalue $\tilde{\lambda}(h)$. Then an Agmon-type estimate shows that $\tilde{\varphi}_h$ is semi-classically localized near ∂U (Proposition 2.4). We construct the basis of the spectral subspace of P_h corresponding to $I(h)$ essentially by (magnetically) translating the $\tilde{\varphi}_h$. Due to the interaction of the wells ("tunneling") the "one-well"-eigenvalue $\tilde{\lambda}(h)$ is "smeared". The decay estimates allow to prove that this smearing is exponentially small with respect to $\sqrt{h}$. For this we follow the approach to "multi-well"-problems developed by Helffer-Sjöstrand in a series of papers (see in particular [HeSj1], [HeSj2] and also [He], [DiSj] for textbook presentations). Their method was extended to the case of infinitely many wells in [Ou] and [Ca]. Our exposition is particularly influenced by the last mentioned paper.

Note that all these papers treat *electric* Schrödinger operators (0.3). The

case of an additional *magnetic* potential was consider in [HeSj3], [HeSj4]. This problem is more difficult and, in particular, the optimal "tunneling coefficient" c in (0.2) seems to be unknown. This is related to the decay of eigenfunctions of magnetic Schrödinger operators, see the discussion in Section 1.4. We would also like to mention [Na] for a similar problem.
Note that our result is valid without any assumption on the flux $\frac{B|Q|}{2\pi h}$. (Here Q is a fundamental domain of Γ.) However, the *nature* of the spectrum of P_h depends heavily on number-theoretic properties of the flux. In the rational case we expect absolutely continuous spectrum, whereas in the irrational case the spectrum might exhibit fractal structure and there might be analogies with Harper's equation (see, e.g., [HeSj4]).

0.3. Our problem is motivated by a question of superconductivity. Without going into details we mention that physical experiments suggest that the upper critical field $H_{C_3}(\kappa)$ is substantially increased in antidot lattices as compared to the case of a single antidot (see [MoBrLo]). Our results seem to indicate that this effect is only very small in the model and the regime that we are considering. We will discuss this question elsewhere.

0.4. The structure of this paper is as follows. In Section 1 we describe the setting of our problem (Subsection 1.1) and state the main results (Subsection 1.3). Note that our conditions there are wider and more abstract than those presented in this introduction. Examples are discussed in Subsection 1.4. Section 2 is devoted to prove decay estimates of Agmon-type. Finally in Section 3 we prove our main results.

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1. Setting of the problem. Main result

1.1. **Setting of the problem.**

1.1.1. *Definition of the operator P_h.* Let Γ be a two-dimensional lattice in $\mathbb{R}^2$ and $\Omega_\infty \subset \mathbb{R}^2$ an open, connected set with C^2-boundary which is periodic with respect to Γ, i.e., $\Omega_\infty + \alpha = \Omega_\infty$, $\alpha \in \Gamma$. Then we can write

$$\Omega_\infty = \mathbb{R}^2 \setminus \bigcup_{\alpha\in\Gamma} (U+\alpha)$$

with a compact set U disjoint from its Γ-translates and, if Q denotes a fundamental domain of Γ such that $U \subset Q$, we put $\Omega := \Omega_\infty \cap Q$.

Moreover, we assume that

$$U = \bigcup_{j=1}^{d} U_j$$

is the union of finitely many compact, connected and pairwise disjoint sets U_j, $1 \le j \le d$. We put

$$U_{j,\alpha} := U_j + \alpha, \qquad U_\alpha := U + \alpha, \qquad 1 \le j \le d,\ \alpha \in \Gamma,$$

and, as explained in the introduction, refer to their boundaries as *wells.*

Moreover, let $\mathbf{A} \in C^1(\overline{\Omega_\infty}, \mathbb{R}^2)$, the *magnetic vector potential*, and define the vector of *circulations* $\Phi := (\Phi_1, \dots, \Phi_d) \in \mathbb{R}^d$ and the *magnetic field* $B \in C(\overline{\Omega_\infty}, \mathbb{R})$ by

$$\Phi_j := \oint_{\partial U_j} \mathbf{A} \cdot dx, \qquad 1 \le j \le d, \tag{1.1}$$

$$B(x) := \operatorname{curl} \mathbf{A}(x), \qquad x \in \Omega_\infty. \tag{1.2}$$

Throughout the following we assume the periodicity conditions

$$\oint_{\partial U_{j,\alpha}} \mathbf{A} \cdot dx = \Phi_j \qquad 1 \le j \le d,\ \alpha \in \Gamma, \tag{1.3}$$

$$B(x+\alpha) = B(x), \qquad x \in \Omega_\infty,\ \alpha \in \Gamma, \tag{1.4}$$

and the positivity condition

$$b := \min_{x \in \overline{\Omega}} B(x) > 0. \tag{1.5}$$

The quadratic form

$$\begin{aligned} \mathcal{D}[q_h] &:= \{u \in L_2(\Omega_\infty) : (h\mathbf{D} - \mathbf{A})u \in L_2(\Omega_\infty, \mathbb{C}^2)\}, \\ q_h[u] &:= \int_{\Omega_\infty} |(h\mathbf{D} - \mathbf{A})u|^2 \, dx, \end{aligned} \tag{1.6}$$

generates a self-adjoint, non-negative operator P_h in the Hilbert space $L_2(\Omega_\infty)$. The functions $u \in \mathcal{D}(P_h)$ belong to $H^2_{loc}(\overline{\Omega_\infty})$ and satisfy the *magnetic Neumann condition*

$$\frac{h}{i}\frac{\partial u}{\partial \nu} - \langle \mathbf{A}, \nu \rangle \, u = 0 \qquad \text{on } \partial\Omega_\infty, \tag{1.7}$$

where ν is the exterior unit normal. We have

$$P_h u = ((hD_1 - A_1)^2 + (hD_2 - A_2)^2)u, \qquad u \in \mathcal{D}(P_h).$$

Due to gauge transformations the operators P_h are, up to unitary equivalence, completely determined by B and Φ, see Remark 1.4 below.

1.1.2. *Setting of the problem.* In the following we assume that h varies in a bounded set $J \subset (0, +\infty)$ having 0 as accumulation point. We fix a number b_0 such that

$$0 < b_0 < b \tag{1.8}$$

and a function $\beta : J \to [-b_0, +\infty)$ satisfying

$$\lim_{J \ni h \to 0} \beta(h) = 0.$$

Let $E_h(I(h))$ denote the spectral projection of P_h corresponding to the interval

$$I(h) := [0, h(b_0 + \beta(h))]. \tag{1.9}$$

Our goal is to study the restriction of the operator P_h to the subspace

$$\mathcal{F}_h := \mathcal{R}(E_h(I(h)))$$

for small $h \in J$. In particular, we are interested in the spectrum of P_h in $I(h)$.

1.1.3. *Definition of the reference operator $\tilde{P}_h$.* Together with the operator P_h we consider two other self-adjoint realizations $\tilde{P}_h$ and P_h^D of $(h\mathbf{D} - \mathbf{A})^2$ in $L_2(\Omega_\infty)$. The operator $\tilde{P}_h$ will serve as a *reference operator*, and functions in its domain satisfy the Neumann condition (1.7) only on ∂U and Dirichlet conditions

$$u = 0 \qquad \text{on } \partial U_\alpha,\ 0 \neq \alpha \in \Gamma. \tag{1.10}$$

The Dirichlet realization P_h^D will be used in several proofs below, and functions in its domain satisfy the Dirichlet condition (1.10) for all $\alpha \in \Gamma$.
More precisely, the operators $\tilde{P}_h$ and P_h^D are generated by quadratic forms $\tilde{q}_h$ and q_h^D given by the same expression as in (1.6) but with domains

(1.11)
$$\begin{aligned}
\mathcal{D}[\tilde{q}_h] &:= \{u \in L_2(\Omega_\infty) : (h\mathbf{D} - \mathbf{A})u \in L_2(\Omega_\infty, \mathbb{C}^2), u|_{\partial U_\alpha} = 0\, \forall \alpha \in \Gamma \setminus \{0\}\},\\
\mathcal{D}[q_h^D] &:= \{u \in L_2(\Omega_\infty) : (h\mathbf{D} - \mathbf{A})u \in L_2(\Omega_\infty, \mathbb{C}^2), u|_{\partial U_\alpha} = 0\, \forall \alpha \in \Gamma\}.
\end{aligned}$$

We recall the following inequality,

$$h \int_{\Omega_\infty} B|u|^2\,dx \leq \int_{\Omega_\infty} |(h\mathbf{D} - \mathbf{A})u|^2\,dx, \qquad u \in \mathcal{D}[q_h^D], \tag{1.12}$$

which follows from the (formal) commutation relation

$$[hD_1 - A_1, hD_2 - A_2] = ihB.$$

Therefore $\inf \sigma(P_h^D) \geq hb$, and it follows from a "magnetic" Persson's lemma (see, e.g., [Bo1]) that $\inf \sigma_{ess}(\tilde{P}_h) \geq hb$. The condition (1.8) implies that the spectrum of $\tilde{P}_h$ in $I(h)$ is discrete for sufficiently small h. We denote by $\tilde{E}_h(I(h))$ the spectral projection of $\tilde{P}_h$ corresponding to $I(h)$ and make the following assumptions.

Assumption 1.1. There exists a function $a : J \to \mathbb{R}_+$ satisfying

$$\liminf_{J \ni h \to 0} \sqrt{h} \log a(h) \geq 0 \tag{1.13}$$

such that for all sufficiently small $h \in J$

$$\sigma(\tilde{P}_h) \cap (h(b_0 + \beta(h)), h(b_0 + \beta(h)) + 2a(h)] = \emptyset.$$

Replacing $a(h)$ by $\min(a(h), h^2)$ if necessary, we can, and will, always assume that

$$\lim_{J \ni h \to 0} \frac{1}{h} a(h) = 0.$$

For simplicity of the exposition we will also assume

Assumption 1.2. For all sufficiently small $h \in J$ one has

$$\dim \mathcal{R}(\tilde{E}_h(I(h))) = 1.$$

Roughly speaking, we assume that the lowest eigenvalue of $\tilde{P}_h$ is simple and "well-separated" from the remaining part of the spectrum of $\tilde{P}_h$.
We note that there exists always an interval $I(h)$ such that Assumption 1.1 is satisfied. (This follows from standard spectral asymptotics.) However, for such an $I(h)$ Assumption 1.2 may not be fulfilled. In Remark 1.12 we comment briefly on the case when Assumption 1.2 is violated.
In Subsection 1.4 we will give examples where both assumptions are fulfilled.

1.2. **Auxiliary material.**

1.2.1. *Distances.* We denote by d the (degenerate) distance in $\mathbb{R}^2$ associated with the (degenerate) metric

$$(B(x) - b_0)\chi_{\overline{\Omega_\infty}}(x)\, dx^2,$$

where $\chi_{\overline{\Omega_\infty}}$ is the characteristic function of the set $\overline{\Omega_\infty}$, and put

$$d_\alpha(x) := \inf_{y \in U_\alpha} d(x, y), \qquad x \in \overline{\Omega_\infty},$$

for $\alpha \in \Gamma$. Then d_α is Lipschitz continuous in $\overline{\Omega_\infty}$ with

$$|\nabla d_\alpha| \leq \sqrt{B - b_0} \qquad \text{a.e.} \tag{1.14}$$

and by periodicity of B we have

$$d_\alpha(x - \beta) = d_{\alpha+\beta}(x), \qquad x \in \overline{\Omega_\infty},\ \alpha, \beta \in \Gamma. \tag{1.15}$$

Consequently, if we write $d(\alpha) := \inf_{x \in U_\alpha, y \in U} d(x, y)$, $\alpha \in \Gamma$, then

$$d(\alpha - \beta) = \inf_{x \in U_{\alpha-\beta}, y \in U} d(x, y) = \inf_{x \in U_\alpha, y \in U_\beta} d(x, y)$$

is the distance between U_α and U_β, and the minimal distance between the wells is

$$S_0 := \inf_{0 \neq \alpha \in \Gamma} d(\alpha).$$

We introduce the distance to the closest well by

$$d_\infty(x) := \inf_{\alpha\in\Gamma} d_\alpha(x) = \inf_{y\in\cup_{\alpha\in\Gamma}U_\alpha} d(x,y), \qquad x\in\overline{\Omega_\infty}.$$

In order to obtain precise decay estimates we put

$$d_\alpha^{(1)}(x) := \inf_{\alpha\neq\beta\in\Gamma}(d_\beta(x)+d(\beta-\alpha)), \qquad x\in\overline{\Omega_\infty},$$

for $\alpha\in\Gamma$ and note that properties analogous to (1.14) and (1.15) are satisfied. Moreover, we find

$$d^{(1)}(\alpha) := \inf_{x\in U_\alpha} d_0^{(1)}(x) = \begin{cases} 2S_0 & \text{if } \alpha=0,\\ d(\alpha) & \text{if } \alpha\neq 0.\end{cases}$$

Finally, we denote

$$d^{(2)}(\alpha) := \inf_{\alpha\neq\gamma,\tilde\gamma\neq 0}(d(\alpha-\gamma)+d(\gamma-\tilde\gamma)+d(\tilde\gamma)).$$

We outline now the construction of a cut-off function supported in a small neighbourhood of ∂U. We put

$$B(U,\delta) := \{x\in\Omega_\infty : d_0(x)<\delta\} \tag{1.16}$$

Lemma 1.3. *There exist $C,c,h_0>0$ and, for every $h\in(0,h_0]$, functions $\theta_h\in C_0^\infty(\overline{\Omega_\infty})$ such that*

$$0\le\theta_h\le 1, \qquad \theta_h=1 \text{ on } \overline{B(U,c\sqrt{h})}, \qquad \theta_h=0 \text{ on } \Omega_\infty\setminus B(U,\sqrt{h}),$$

$$\sqrt{h}|\nabla\theta_h|+h|\Delta\theta_h|\le C.$$

Proof. We can choose a neighbourhood V of ∂U in $\overline{\Omega_\infty}$ and a constant $C_1>0$ such that

$$\frac{1}{C_1}d_{0,eucl}(x)\le d_0(x)\le C_1 d_{0,eucl}(x), \qquad x\in V,$$

where $d_{0,eucl}(x)$ denotes the Euclidean distance between x and U. According to a theorem of Whitney (see [St]) there exists a regularization $\tilde d$ of $d_{0,eucl}$, i.e., a function $\tilde d\in C_0^\infty(\mathbb{R}^2\setminus U)$ such that for some constant $C_2>0$ and all $x\in\mathbb{R}^2\setminus U$

$$\frac{1}{C_2}d_{0,eucl}(x)\le\tilde d(x)\le C_2 d_{0,eucl}(x), \quad |\nabla\tilde d(x)|\le C_2, \quad |\Delta\tilde d(x)|\le C_2\frac{1}{\tilde d(x)}.$$

With a function $\eta\in C_0^\infty(\overline{\mathbb{R}_+})$, $0\le\eta\le 1$ such that $\eta(t)=1$ if $t\le(2C_1C_2)^{-1}$ and $\eta(t)=0$ if $t\ge(C_1C_2)^{-1}$ we set for sufficiently small h

$$\theta_h(x) := \eta(\tilde d(x)/\sqrt{h}),\ x\in V, \qquad \theta_h(x) := 0,\ x\in\Omega_\infty\setminus V,$$

and check easily the stated properties. ∎

1.2.2. *Magnetic translations.* Here we recall the definition of magnetic translations (see [Za], [DuNo]). We fix a basis $a^{(1)}, a^{(2)}$ of Γ and, with slight abuse of notation, denote the coordinates of $\alpha \in \Gamma$ with respect to this basis by $\alpha_1, \alpha_2 \in \mathbb{Z}$, i.e.,

$$\alpha = \alpha_1 a^{(1)} + \alpha_2 a^{(2)}, \qquad \alpha \in \Gamma, \alpha_1, \alpha_2 \in \mathbb{Z}.$$

Moreover, we denote by τ_α the operator of translation with respect to $\alpha \in \Gamma$,

$$(\tau_\alpha u)(x) := u(x-\alpha), \qquad x \in \Omega_\infty, \tag{1.17}$$

and put $\tau^{(j)} := \tau_{a^{(j)}}$ for $j = 1, 2$.

Let $j = 1, 2$. The periodicity conditions (1.3), (1.4) imply that $\operatorname{curl} \tau^{(j)} \mathbf{A} = \operatorname{curl} \mathbf{A}$ and that

$$\oint_\sigma \tau^{(j)} \mathbf{A} \cdot dx = \oint_\sigma \mathbf{A} \cdot dx$$

for every closed path σ in $\overline{\Omega_\infty}$, so there exists a function $\varphi^{(j)} \in C^2(\overline{\Omega_\infty}, \mathbb{R})$ such that

$$\mathbf{A} - \tau^{(j)} \mathbf{A} = \nabla \varphi^{(j)} \qquad \text{in } \overline{\Omega_\infty}.$$

Then the operator

$$T_h^{(j)} := e^{i\varphi^{(j)}/h} \tau^{(j)}$$

is unitary in $L_2(\Omega_\infty)$ and satisfies $T_h^{(j)} \mathcal{D}[q_h] = \mathcal{D}[q_h]$ and $q_h[T_h^{(j)} u] = q_h[u]$ for $u \in \mathcal{D}[q_h]$. It follows that $T_h^{(j)}$ commutes with P_h,

$$P_h T_h^{(j)} = T_h^{(j)} P_h. \tag{1.18}$$

However, $T_h^{(1)}$ and $T_h^{(2)}$ may not commute with each other. Indeed, one finds

$$T_h^{(1)} T_h^{(2)} = e^{iB_0/h} T_h^{(2)} T_h^{(1)},$$

where

$$B_0 := \int_\Omega B(x)\, dx + \sum_{j=1}^d \Phi_j \tag{1.19}$$

denotes the *total flux inside* Q. Finally, we set (suppressing the h-dependence in the notation)

$$T_\alpha := (T_h^{(1)})^{\alpha_1} (T_h^{(2)})^{\alpha_2}, \qquad \alpha \in \Gamma.$$

These are unitary operators which commute with P_h and satisfy

$$T_\alpha T_\beta = e^{-i\alpha_2\beta_1 B_0/h} T_{\alpha+\beta} = e^{i(\alpha_1\beta_2 - \alpha_2\beta_1)B_0/h} T_\beta T_\alpha, \qquad \alpha, \beta \in \Gamma. \tag{1.20}$$

Remark 1.4. In Subsection 1.1 we started with $\mathbf{A}$ and introduced Φ and B in (1.1) and (1.2). In a certain sense it is also possible to start with Φ and B. Indeed, let $B \in C(\overline{\Omega_\infty}, \mathbb{R})$ be periodic and $\Phi \in \mathbb{R}^d$. Define B_0 by (1.19) and put $\mathbf{A}_0(x) := \frac{1}{2}(-x_2, x_1)^T$. Then there exists a periodic $\mathbf{A}_p \in C^1(\overline{\Omega_\infty}, \mathbb{R}^2)$ such that

$$\operatorname{curl}(\tfrac{B_0}{|Q|} \mathbf{A}_0 + \mathbf{A}_p) = B \qquad \text{in } \Omega_\infty,$$

$$\oint_{\partial U_{j,\alpha}} (\tfrac{B_0}{|Q|}\mathbf{A}_0 + \mathbf{A}_p) \cdot dx = \Phi_j, \qquad 1 \le j \le d,\ \alpha \in \Gamma.$$

(Indeed, $\mathbf{A}_p$ can be chosen as a periodic solution of $\operatorname{curl} \mathbf{A}_p = \tilde{B} - \frac{B_0}{|Q|}$ on $\mathbb{R}^2$, where $\tilde{B} \in C(\mathbb{R}^2, \mathbb{R})$ is a periodic extension of B satisfying $\int_{U_j} \tilde{B}\, dx = \Phi_j$.) Now if $\mathbf{A} \in C^1(\overline{\Omega_\infty}, \mathbb{R}^2)$ is another vector potential satisfying (1.2) and (1.3), then there exists a function $\varphi \in C^2(\overline{\Omega_\infty}, \mathbb{R})$ such that $\mathbf{A} - \frac{B_0}{|Q|}\mathbf{A}_0 - \mathbf{A}_p = \nabla\varphi$. In particular, the operators P_h with magnetic vector potential $\mathbf{A}$ and $P_{0,h}$ with magnetic vector potential $\frac{B_0}{|Q|}\mathbf{A}_0 + \mathbf{A}_p$ are unitarily equivalent by means of multiplication with $e^{i\varphi/h}$ (*gauge transformation*).

1.3. **Main result.** For all sufficiently small $h \in J$ we denote by $\tilde{\lambda}(h)$ the (according to Assumption 1.2) unique eigenvalue of $\tilde{P}_h$ in the interval $I(h)$ and by $\tilde{\varphi}_h$ a corresponding normalized eigenfunction.
As the first step in our construction, we choose (see Lemma 1.3) for all sufficiently small $h > 0$ a function $\theta_h \in C_0^\infty(\overline{\Omega_\infty})$ such that $0 \le \theta_h \le 1$, $\theta_h = 1$ on $\overline{B(U, c\sqrt{h})}$ and $\theta_h = 0$ on $\Omega_\infty \setminus B(U, \sqrt{h})$, where $B(U,\delta)$ is defined in (1.16). Then

$$\chi_h := 1 - \sum_{0 \ne \alpha \in \Gamma} \tau_\alpha \theta_h$$

(with the "usual" translations τ_α, see (1.17)) is a cut-off function which differs from 1 only in a small, h-dependent neighbourhood of the wells ∂U_α, $0 \ne \alpha \in \Gamma$. By our construction we can assume that there exists a constant C such that for all sufficiently small $h > 0$

$$\sqrt{h}|\nabla\chi_h| + h|\Delta\chi_h| \le C. \tag{1.21}$$

For the sake of simplicity we will from now on suppress in the notation the dependence of various objects on the parameter h.
We consider the functions

$$\psi_\alpha := T_\alpha(\chi\tilde{\varphi}), \qquad \alpha \in \Gamma. \tag{1.22}$$

Note that they belong to $\mathcal{D}(P)$ and that the remainder

$$r_\alpha := (P - \tilde{\lambda})\psi_\alpha, \qquad \alpha \in \Gamma, \tag{1.23}$$

is "small" in a suitable sense.
The next step in our construction is to project the ψ_α onto the spectral subspace $\mathcal{F}$,

$$v_\alpha := E(I)\psi_\alpha, \qquad \alpha \in \Gamma. \tag{1.24}$$

For these functions we will prove

Proposition 1.5. *There exists a $h_0 > 0$ such that for $h \in (0, h_0] \cap J$ the v_α, $\alpha \in \Gamma$, are linearly independent and complete in $\mathcal{F}$.*

Actually, we prove that v_α, $\alpha \in \Gamma$, is an "almost" orthonormal basis, see Lemmas 3.1 and 3.3.
In the remainder of this subsection we will assume that $h \in (0, h_0] \cap J$.
The last step in our construction is to orthonormalize the v_α,

$$e_\alpha := \sum_{\beta \in \Gamma} (\mathbf{V}^{-1/2})_{\alpha,\beta} v_\beta, \qquad \alpha \in \Gamma, \tag{1.25}$$

where the matrix $\mathbf{V} = (v_{\alpha,\beta})$ is defined by

$$v_{\alpha,\beta} := (v_\alpha, v_\beta), \qquad \alpha, \beta \in \Gamma. \tag{1.26}$$

We define the *interaction matrix* $\mathbf{W} = (w_{\alpha,\beta})$ by

$$w_{\alpha,\beta} := \frac{1}{2}((\psi_\alpha, r_\beta) + (r_\alpha, \psi_\beta)), \qquad \alpha, \beta \in \Gamma. \tag{1.27}$$

To simplify the statement of our main result we introduce the following notation. Note that we modify slightly the one used, e.g., in [He] in order to deal with uniformity.

Notation 1.6. If $(f_{\alpha,\beta}(h))$, $h \in (0, h_0] \cap J$, and $(c_{\alpha,\beta})$ are real matrices, we write

$$f_{\alpha,\beta}(h) = \tilde{\mathcal{O}}_{unif}(e^{c_{\alpha,\beta}/\sqrt{h}}) \tag{1.28}$$

if for every $0 < \epsilon \leq 1$ there exist $C_\epsilon, h_\epsilon > 0$ such that for all $h \in (0, h_\epsilon] \cap J$, $\alpha, \beta \in \Gamma$

$$|f_{\alpha,\beta}(h)| \leq C_\epsilon e^{(c_{\alpha,\beta} + \epsilon(|c_{\alpha,\beta}|+1))/\sqrt{h}}.$$

Now we can describe the matrix of P in the orthonormal basis e_α, $\alpha \in \Gamma$, of $\mathcal{F}$.

Theorem 1.7. *Under the above assumptions one has*

$$(Pe_\alpha, e_\beta) = \tilde{\lambda}\delta_{\alpha,\beta} + w_{\alpha,\beta} + \tilde{\mathcal{O}}_{unif}(e^{-d^{(2)}(\alpha-\beta)/\sqrt{h}}),$$
$$w_{\alpha,\beta} = \tilde{\mathcal{O}}_{unif}(e^{-d^{(1)}(\alpha-\beta)/\sqrt{h}}).$$

In particular it follows that the spectrum of P inside the interval I (whose length is of order h) is contained in an exponentially small neighbourhood of $\tilde{\lambda}$.

Corollary 1.8. *There exist $C, c, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ one has*

$$\sigma(P_h) \cap I(h) \subset [\tilde{\lambda}(h) - Ce^{-c/\sqrt{h}}, \tilde{\lambda}(h) + Ce^{-c/\sqrt{h}}].$$

Indeed, we will see that this is true for every $c < S_0$ (and some C, h_0 depending on c). However, this might not be optimal, see the discussion in Subsection 1.4.
An essential feature of our construction are the *localization* and *periodicity* properties of the basis e_α, $\alpha \in \Gamma$. An Agmon-type estimate proves that the function ψ_α is localized near ∂U_α (see Proposition 2.4 and Lemma 3.1), and this localization is preserved in each step of the construction. From this we

deduce that functions in the spectral subspace $\mathcal{F}$ are localized near $\partial\Omega_\infty$. More precisely, we show

Proposition 1.9. *For every $0 < \epsilon \leq 1$ there exist $C_\epsilon, h_\epsilon > 0$ such that for $h \in (0, h_\epsilon] \cap J$ one has*

$$\|e^{(1-\epsilon)d_\infty/\sqrt{h}}u\| \leq C_\epsilon \|u\|, \qquad u \in \mathcal{F}.$$

This confirms the intuition that the spectrum of P in I is the consequence of a *boundary effect* created by the Neumann condition, and that functions in $\mathcal{F}$ correspond to *edge states*.
Recall that the operator P commutes with the magnetic translations. This symmetry is reflected in our construction.

Proposition 1.10. *For $\alpha, \beta, \gamma \in \Gamma$ one has*

$$\begin{aligned} T_\alpha e_\beta &= e^{i\alpha_2\beta_1 B_0/\sqrt{h}} e_{\alpha+\beta}, \\ (Pe_{\alpha+\gamma}, e_{\beta+\gamma}) &= e^{i(\alpha_1-\beta_1)\gamma_2 B_0/h}(Pe_\alpha, e_\beta), \\ w_{\alpha+\gamma,\beta+\gamma} &= e^{i(\alpha_1-\beta_1)\gamma_2 B_0/h} w_{\alpha,\beta}. \end{aligned} \tag{1.29}$$

In particular, setting $p(\alpha) := (Pe_\alpha, e_0)$, we find that for $\alpha, \beta \in \Gamma$

$$(Pe_\alpha, e_\beta) = e^{i(\alpha_1-\beta_1)\beta_2 B_0/h} p(\alpha - \beta), \qquad p(-\alpha) = e^{i\alpha_1\alpha_2 B_0/h}\overline{p(\alpha)}. \tag{1.30}$$

So the matrix $((Pe_\alpha, e_\beta))$ acts as "twisted convolution" in $\ell_2(\Gamma)$.
As a final topic we will now discuss the case of *rational flux*, i.e., we assume that

$$J \subset J_q := \{\tfrac{B_0}{2\pi(\nu+q)} : \nu \in \mathbb{N}_0\} \tag{1.31}$$

for some $q \in (0, 1] \cap \mathbb{Q}$, which we write as

$$q = \tfrac{N}{M}, \qquad N, M \text{ relatively prime.} \tag{1.32}$$

For ease of notation we assume until the end of this subsection that $\Gamma = \mathbb{Z}^2$. We realize $\mathbb{T}^2$ as $[-\pi, \pi]^2$ with opposite points identified. For each $\theta \in \mathbb{T}^2$ we define the matrix $\mathbf{P}(\theta) = (p_{j,k}(\theta)) \in \mathbb{C}^{M\times M}$ by

$$p_{j,k}(\theta) := \sum_{n\in\mathbb{Z}^2} p(n_1, j - k + n_2 M) e^{2\pi i n_1 q k} e^{i\langle\theta,n\rangle}, \qquad 1 \leq j, k \leq M.$$

Here p is the function from (1.30), and the second relation there shows that $\mathbf{P}(\theta)$ is self-adjoint. Hence $\mathbf{P} := \int_{\mathbb{T}^2}^{\oplus} \mathbf{P}(\theta)\, d\theta$ is a self-adjoint operator in $\int_{\mathbb{T}^2}^{\oplus} \mathbb{C}^M\, d\theta$ and we can state

Proposition 1.11. *Assume* (1.31) *and* (1.32). *Then the restriction of P to the subspace $\mathcal{F}$ is unitarily equivalent to the operator $\mathbf{P}$ in $\int_{\mathbb{T}^2}^{\oplus} \mathbb{C}^M\, d\theta$. In particular, the spectrum of P in I consists of M (possibly overlapping, possibly degenerate) bands and has no singular continuous component.*

Thus in the special case where $q = 1$ we have $M = 1$ and $P|_{\mathcal{F}}$ is unitarily equivalent to multiplication in $L_2(\mathbb{T}^2)$ by the (real-valued) function

$$p_{1,1}(\theta) = \sum_{n\in\mathbb{Z}^2} p(n)e^{i\langle\theta,n\rangle}, \qquad \theta \in \mathbb{T}^2.$$

1.4. **Discussion of the results.**

1.4.1. *Examples.* First of all we would like to mention a simple extension of our main results.

Remark 1.12. Analogues of the results in Subsection 1.3 are valid when Assumption 1.2 is replaced by the weaker assumption that

$$1 \leq \dim \mathcal{R}(\tilde{E}_h(I(h))) =: m(h) < \infty \tag{1.33}$$

for all sufficiently small $h > 0$ (and Assumption 1.1 still holds). In particular, the spectrum of P is contained in exponentially small neighbourhoods of the eigenvalues $\tilde{\lambda}_1, ..., \tilde{\lambda}_m$ of $\tilde{P}$ in I. This extension is rather standard and its proof will be omitted.
We recall (see the end of Subsection 1.1) that, whenever $\tilde{P}$ has an eigenvalue below hb for all sufficiently small $h > 0$, it is possible to find an interval I such that Assumption 1.1 and (1.33) are fulfilled.

Let us give now examples when (1.33) or Assumptions 1.1 and 1.2 hold. Assume that $\partial U \in C^3$, $\mathbf{A} \in C^3(\overline{\Omega_\infty}, \mathbb{R}^2)$ and that

$$\Theta_0 b' < b, \tag{1.34}$$

where $0 < \Theta_0 < 1$ is a universal constant defined, e.g., in [HeMo],

$$b' := \min_{x\in\partial U} B(x)$$

and b has been introduced in (1.5). Then the results of [HeMo] (that Ω_∞ is unbounded is not essential here) show that for small $h > 0$ the operator $\tilde{P}$ has an eigenvalue below hb and that the lowest eigenvalue $\tilde{\lambda}(h)$ satisfies

$$-Ch^{5/4} \leq \tilde{\lambda}(h) - \Theta_0 b' h \leq Ch^{3/2} \tag{1.35}$$

for some constant $C > 0$. In particular, (1.33) is true with $I(h)$ defined by (1.9) and

$$b_0 := \Theta_0 b', \qquad \beta(h) := Ch^{1/2}.$$

In order to give now sufficient conditions for Assumptions 1.1 and 1.2 we assume now that the magnetic field $B \equiv b > 0$ is constant (so that (1.34) holds) and that $\partial U \in C^\infty$. (Then the inequality (1.35) can be considerably strengthened, see [HeMo].) We concentrate on two special cases.

Example 1.13. Assume that the curvature $\kappa : \partial U \to \mathbb{R}$ has a unique minimum (with the convention that $\kappa \geq 0$ if U is convex), which is non-degenerate. Then the methods of [FoHe] show that the eigenvalues $\tilde{\lambda}_n(h)$ of

$\tilde{P}_h$ below hB admit a complete asymptotic expansion with respect to $h^{1/8}$ whose first terms are

$$\tilde{\lambda}_n(h) = \Theta_0 Bh + \beta_0 B^{1/2}h^{3/2} + \beta_1(2n-1)B^{1/4}h^{7/4} + \mathcal{O}_n(h^{15/8}),$$

with some constants β_0, β_1 depending on ∂U. In this case Assumptions 1.1 and 1.2 are fulfilled if we put

$$b_0 := \Theta_0 B, \qquad \beta(h) := \beta_0 B^{1/2}h^{1/2} + 2\beta_1 B^{1/4}h^{3/4}.$$

This is the situation we referred to in the introduction.

Example 1.14. Assume that U is a disc and that $\Phi_1 = B|U|$. Then it is possible to extend $\mathbf{A}$ to $\mathbb{R}^2 \setminus U$ with curl $\mathbf{A} \equiv B$ and to consider the operator $\tilde{\tilde{P}}_h := (h\mathbf{D} - \mathbf{A})^2$ in $L_2(\mathbb{R}^2 \setminus U)$. This leads to the "exterior" version of the problem studied in [BaPhTa], and for the splitting between the first two eigenvalues $\tilde{\tilde{\lambda}}_1, \tilde{\tilde{\lambda}}_2$ it is expected (see [FoHe]) that

$$0 = \liminf_{h\to 0} \frac{\tilde{\tilde{\lambda}}_2 - \tilde{\tilde{\lambda}}_1}{h^2} < \limsup_{h\to 0} \frac{\tilde{\tilde{\lambda}}_2 - \tilde{\tilde{\lambda}}_1}{h^2}. \tag{1.36}$$

Note that the lim inf is attained, i.e., there exists a sequence of h for which the lowest eigenvalue of $\tilde{\tilde{P}}_h$ is degenerate. By standard arguments the relation (1.36) can be carried over to the eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2$ of our reference operator $\tilde{P}$. From $\limsup_{h\to 0} \frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{h^2} > 0$ we deduce that Assumptions 1.1 and 1.2 are fulfilled for a suitably chosen set J of parameters h.

Our results are also meaningful in the case of *piecewise smooth domains.* If ∂U is only Lipschitz continuous then functions in $\mathcal{D}(P)$ and $\mathcal{D}(\tilde{P})$ may not be of class H^2 up to the boundary, and the boundary condition (1.7) holds only on smooth parts of the boundary. However, this does not affect our proof, and all our results remain valid in this situation.
Now assume that ∂U is piecewise C^3 with corners $S_1, \cdots, S_n$ and corresponding angles $\alpha_1, \cdots, \alpha_n \in (0, 2\pi)$.Then the methods of [Bo1] (see also [Bo2], [Bo3]) show that there exists a "universal" (i.e. independent of U and B) function $\mu : (0, 2\pi) \to (0, \Theta_0]$ such that if

$$b_0 := \min\{b'\Theta_0, B(S_1)\mu(\alpha_1), \cdots, B(S_n)\mu(\alpha_n)\} < b$$

then $\tilde{P}$ has an eigenvalue below hb and that in this case the lowest eigenvalue $\tilde{\lambda}(h)$ satisfies

$$-Ch^{5/4} \leq \tilde{\lambda}(h) - b_0 h \leq Ch^{9/8}$$

with a constant $C > 0$. So (1.33) is fulfilled. For sufficient conditions that $\tilde{\lambda}$ is simple and "well-separated" from the second eigenvalue we refer to [Bo1] and [BoDa].

1.4.2. *Open problems.* As we have already mentioned, we believe that our error estimates are not optimal. Indeed, we expect that the decay of the reference eigenfunction $\tilde{\varphi}$ is faster than $e^{-d_0/\sqrt{h}}$ (see Proposition 2.4). Then the weights $d^{(1)}, d^{(2)}$ in Theorem 1.7 could be increased and, in particular, c could be chosen larger than S_0 in Corollary 1.8. We believe however, that $\mathbf{W}$ is the main term of $((Pe_\alpha, e_\beta)) - \tilde{\lambda}\mathbf{I}$ as $h \to 0$. A lower bound on the terms $w_{\alpha,\beta}$ with α, β such that $d(\alpha - \beta) = S_0$ requires presumably rather precise knowledge of the function $\tilde{\varphi}$, see [HeSj1], [He] for the non-magnetic case.
A problem which has certain similarities with the semi-classical limit considered here is that of the *tight binding approximation*, see, e.g., [HePa], [Da], [Co]. Also in this problem a lower bound (on an analoguous quantity) seems to be unknown in the presence of a magnetic field. It would be interesting to study the case when U is a disc in more detail, relying on the explicit form of the reference eigenfunctions.
We would also like to mention that the approach of [HeSj4] to Harper's equation applies to our problem in the case of irrational flux, *provided* the error terms in Theorem 1.7 can be bounded from below. In the opposite case of rational flux it might be interesting to discuss the band structure in more detail (absolute continuity, existence and number of gaps). Note that in the case (1.31) the magnetic translations $T^{(1)}, (T^{(2)})^M$ commute, so one can apply Floquet theory. (This is essentially what we do in the proof of Proposition 1.11.) This can be used to give an alternative proof of our main results following [Ou].

2. Decay estimates

2.1. **Weights.** As we have explained in the introduction, the main idea behind our proof is that in the semi-classical limit the interaction between the wells ∂U_α becomes small. Technically this is the consequence of decay estimates for eigenfunctions and resolvents, which are the subject of this section. We follow the ideas of [Ag] and their semi-classical version in [HeSj1]. See also [HeMo] for the case of magnetic Schrödinger operators with Neumann boundary conditions.
We introduce the following class of weights,

$$\mathcal{W} := \{\omega \in C^{0,1}(\overline{\Omega_\infty}, \mathbb{R}) : \text{ess-}\inf_{x\in\Omega_\infty} (B(x) - b_0 - |\nabla\omega(x)|^2) > 0\}.$$

Here $C^{0,1}(\overline{\Omega_\infty}, \mathbb{R})$ denotes the class of uniformly Lipschitz continuous, real-valued functions on $\overline{\Omega_\infty}$, and it is well-known that functions in this class have a.e. derivatives of first order.
For $\omega \in \mathcal{W}$ and $h > 0$ we define the Hilbert space

$$L_2(\omega/\sqrt{h}) := \{u \in L_{2,loc}(\Omega_\infty) : e^{\omega/\sqrt{h}} u \in L_2(\Omega_\infty)\}$$

with norm

$$\|u\|_{\omega/\sqrt{h}} := \|e^{\omega/\sqrt{h}} u\|, \qquad u \in L_2(\omega/\sqrt{h}).$$

By $\|.\|_{\omega/\sqrt{h}}$ we will also denote the norm of a bounded operator on $L_2(\omega/\sqrt{h})$. Integration by parts gives us the following important identity.

Lemma 2.1. *Let $\omega \in \mathcal{W}$ and let $P^\#$ be one of the operators P, $\tilde{P}$, P^D with quadratic form $q^\#$. Then for all $h > 0$ and all $u \in \mathcal{D}(P^\#)$ such that $u, P^\# u \in L_2(\omega/\sqrt{h})$ one has*

$$\mathrm{Re}(P^\# u, e^{2\omega/\sqrt{h}}u) = q^\#[e^{\omega/\sqrt{h}}u] - h\||\nabla\omega|e^{\omega/\sqrt{h}}u\|^2. \tag{2.1}$$

In particular, we are interested in weights that are constant on $\partial\Omega_\infty$. We introduce the notations

$$\mathrm{osc}_\alpha(\omega) := \sup_{x\in\partial U_\alpha} \omega(x) - \inf_{x\in\partial U_\alpha} \omega(x), \qquad \alpha \in \Gamma,$$

and

$$\mathcal{W}_0 := \{\omega \in \mathcal{W} : \mathrm{osc}_\alpha(\omega) = 0,\, \alpha \in \Gamma\}.$$

Examples of functions in this class are the functions introduced in Subsection 1.2.1,

$$(1-\epsilon)d_\alpha, (1-\epsilon)d_\alpha^{(1)}, (1-\epsilon)d_\infty \in \mathcal{W}_0, \qquad 0 < \epsilon \le 1, \alpha \in \Gamma.$$

2.2. **The Dirichlet realization.** Due to the inequality (1.12) it is very easy to estimate the resolvent of the operator P^D. We have

Proposition 2.2. *Let $\omega \in \mathcal{W}$. Then there exist $C_0, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ and z with $\mathrm{Re}z \le h(b_0 + \beta(h)) + a(h)$ the operator $(P^D - z)^{-1}$ maps $L_2(\omega/\sqrt{h})$ into itself with norm*

$$\|(P^D - z)^{-1}\|_{\omega/\sqrt{h}} \le \frac{C_0}{h}.$$

Proof. Since ω may not be bounded from above, we replace it by $\omega_N := \min(\omega, N)$, $N \in \mathbb{N}$. Let $v \in L_2(\omega/\sqrt{h})$ and put $u := (P^D - z)^{-1}v$. Then (2.1) and (1.12) yield

$$\begin{aligned} &h\int_{\Omega_\infty} (B(x) - |\nabla\omega_N|^2 - \frac{\mathrm{Re}z}{h})e^{2\omega_N/\sqrt{h}}|u|^2\,dx \le \\ &\le \mathrm{Re}((P^D - z)u, e^{2\omega_N/\sqrt{h}}u) \le \|(P^D - z)u\|_{\omega_N/\sqrt{h}}\|u\|_{\omega_N/\sqrt{h}}. \end{aligned} \tag{2.2}$$

We put $\delta := \text{ess-}\inf_{x\in\Omega_\infty}(B(x) - b_0 - |\nabla\omega(x)|^2) > 0$ and choose $h_0 > 0$ such that

$$\beta(h) + \frac{a(h)}{h} \le \frac{\delta}{2}, \qquad h \in (0, h_0] \cap J.$$

Then (2.2) implies

$$\|u\|_{\omega_N/\sqrt{h}} \le \frac{2}{\delta h}\|(P^D - z)u\|_{\omega_N/\sqrt{h}}.$$

Since $(P^D - z)u = v \in L_2(\omega/\sqrt{h})$ we can pass to the limit $N \to \infty$ and find $(P^D - z)^{-1}v = u \in L_2(\omega/\sqrt{h})$ and the estimate as claimed. □

Combining Proposition 2.2 with Lemma 2.1 we obtain immediately

Corollary 2.3. *Let $\omega \in \mathcal{W}$. Then there exist $C_0, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ and z with* $\mathrm{Re} z \leq h(b_0 + \beta(h)) + a(h)$ *one has*

$$q^D[e^{\omega/\sqrt{h}}(P^D - z)^{-1}v] + h\|(P^D - z)^{-1}v\|^2_{\omega/\sqrt{h}} \leq \frac{C_0}{h}\|v\|^2_{\omega/\sqrt{h}}, \quad v \in L_2(\omega/\sqrt{h}).$$

2.3. **The reference operator.** We turn now to the study of the operator $\tilde{P}$. We will establish estimates in weighted spaces for its eigenfunction $\tilde{\varphi}$ in Subsection 2.3.1 and for its resolvent in Subsection 2.3.2.

2.3.1. *Estimates of eigenfunctions.* Recall that $\tilde{\varphi}$ is a normalized eigenfunction of $\tilde{P}$ corresponding to the eigenvalue $\tilde{\lambda}$ in the interval I.

Proposition 2.4. *Let $\omega \in \mathcal{W}$ such that $\omega = 0$ on ∂U. Then there exist $C_0, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ one has $\tilde{\varphi} \in L_2(\omega/\sqrt{h})$ and*

$$\|\tilde{\varphi}\|_{\omega/\sqrt{h}} \leq C_0.$$

Proof. We assume that ω is bounded from above, otherwise we proceed as in the proof of Proposition 2.2. For every $\epsilon > 0$ we choose $\chi_{0,\epsilon}, \chi_{1,\epsilon} \in C^\infty(\overline{\Omega_\infty})$ such that $\chi_{0,\epsilon}^2 + \chi_{1,\epsilon}^2 = 1$ and such that $\chi_{1,\epsilon}$ satisfies $\chi_{1,\epsilon} = 1$ in a neighbourhood of ∂U and $\omega \leq \frac{\epsilon}{2}$ on $\mathrm{supp}(\chi_{1,\epsilon})$. Since $\omega \in C^{0,1}(\overline{\Omega_\infty})$ we can assume

$$\sum_{j=0}^{1} |\nabla \chi_{j,\epsilon}|^2 \leq \frac{C_1}{\epsilon^2} \tag{2.3}$$

with C_1 independent of ϵ. (For the construction of this function one can proceed as in Lemma 1.3.) We have

$$\tilde{q}[u] = \sum_{j=0}^{1} (\tilde{q}[\chi_{j,\epsilon} u] - h^2 \||\nabla \chi_{j,\epsilon}| u\|^2), \qquad u \in \mathcal{D}[\tilde{q}].$$

Combining this (with $u = e^{\omega/\sqrt{h}}\tilde{\varphi}$) and (2.1) (with $u = \tilde{\varphi}$) we find

(2.4)

$$\tilde{q}[\chi_{0,\epsilon} e^{\omega/\sqrt{h}}\tilde{\varphi}] - h\int_{\Omega_\infty} (\frac{\tilde{\lambda}}{h} + |\nabla\omega|^2 + h\sum_{j=0}^{1}|\nabla\chi_{j,\epsilon}|^2)\chi_{0,\epsilon}^2 e^{2\omega/\sqrt{h}}|\tilde{\varphi}|^2\, dx =$$

$$= -\tilde{q}[\chi_{1,\epsilon} e^{\omega/\sqrt{h}}\tilde{\varphi}] + h\int_{\Omega_\infty} (\frac{\tilde{\lambda}}{h} + |\nabla\omega|^2 + h\sum_{j=0}^{1}|\nabla\chi_{j,\epsilon}|^2)\chi_{1,\epsilon}^2 e^{2\omega/\sqrt{h}}|\tilde{\varphi}|^2\, dx.$$

Using the inequaltity (1.12), the fact that $\tilde{\lambda} \in I$ and (2.3), we can estimate the LHS of (2.4) from below by

$$h\int_{\Omega_\infty} (B - \frac{\tilde{\lambda}}{h} - |\nabla\omega|^2 - h\sum_{j=0}^{1}|\nabla\chi_{j,\epsilon}|^2)\chi_{0,\epsilon}^2 e^{2\omega/\sqrt{h}}|\tilde{\varphi}|^2\, dx \geq$$

$$\geq h\int_{\Omega_\infty} (\delta - \beta(h) - \frac{hC_1}{\epsilon^2})\chi_{0,\epsilon}^2 e^{2\omega/\sqrt{h}}|\tilde{\varphi}|^2\, dx,$$

where $\delta := \text{ess-}\inf_{x\in\Omega_\infty}(B(x) - b_0 - |\nabla\omega(x)|^2) > 0$. We choose $\epsilon = C_2\sqrt{h}$ with C_2 sufficiently large and then $h_0 > 0$ such that

$$\beta(h) + \frac{hC_1}{\epsilon^2} = \beta(h) + \frac{C_1}{C_2^2} \leq \frac{\delta}{2}, \qquad h \in (0, h_0] \cap J.$$

Then the LHS of (2.4) can be bounded from below by

$$\frac{h\delta}{2}\int_{\Omega_\infty} \chi^2_{0,C_2\sqrt{h}} e^{2\omega/\sqrt{h}}|\tilde{\varphi}|^2\, dx.$$

Since the RHS of (2.4) can be estimated from above by

$$hC_3\int_{\Omega_\infty} \chi^2_{1,C_2\sqrt{h}} e^{2\omega/\sqrt{h}}|\tilde{\varphi}|^2\, dx$$

with a constant $C_3 > 0$ independent of $h \in (0, h_0] \cap J$, we obtain

$$\|\tilde{\varphi}\|^2_{\omega/\sqrt{h}} \leq \frac{2}{\delta}(C_3 + \frac{\delta}{2})\int_{\Omega_\infty} \chi^2_{1,C_2\sqrt{h}} e^{2\omega/\sqrt{h}}|\tilde{\varphi}|^2\, dx,$$

and the assertion follows if we note that $e^{2\omega/\sqrt{h}} \leq e^{2C_2}$ on $\text{supp}(\chi_{1,C_2\sqrt{h}})$. □

Combining Proposition 2.4 with Lemma 2.1 we obtain immediately

Corollary 2.5. *Let $\omega \in \mathcal{W}$ such that $\omega = 0$ on ∂U. Then there exist $C_0, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ one has*

$$\tilde{q}[e^{\omega/\sqrt{h}}\tilde{\varphi}] + h\|\tilde{\varphi}\|^2_{\omega/\sqrt{h}} \leq C_0 h.$$

2.3.2. *Resolvent estimates.* Assumption 1.2 implies that the curve

$$\gamma_h := \{z \in \mathbb{C} : \text{dist}(z, I(h)) = a(h)\} \tag{2.5}$$

(dist referring to the Euclidean distance, of course) belongs to the resolvent set of $\tilde{P}$ and encloses the interval $I(h)$. Moreover, we have the estimate

$$\|(\tilde{P} - z)^{-1}\| \leq \frac{1}{a(h)}, \qquad z \in \gamma. \tag{2.6}$$

The analogue in weighted spaces is

Proposition 2.6. *Let $\omega \in \mathcal{W}$, then there exist $C_0, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ and $z \in \gamma$ the operator $(\tilde{P} - z)^{-1}$ maps $L_2(\omega/\sqrt{h})$ into itself with norm*

$$\|(\tilde{P} - z)^{-1}\|_{\omega/\sqrt{h}} \leq \frac{C_0}{a(h)} e^{\text{osc}_0(\omega)/\sqrt{h}}.$$

Proof. The first part of the proof is similar to the one of Proposition 2.4. We assume that ω is bounded from above and choose for every $\epsilon > 0$ functions $\chi_{0,\epsilon}, \chi_{1,\epsilon} \in C^\infty(\overline{\Omega_\infty})$ such that $\chi^2_{0,\epsilon} + \chi^2_{1,\epsilon} = 1$ and such that $\chi_{1,\epsilon}$ satisfies $\chi_{1,\epsilon} = 1$ in a neighbourhood of ∂U and

$$\inf_{\partial U} \omega - \epsilon \leq \omega \leq \sup_{\partial U} \omega + \epsilon \qquad \text{on } \text{supp}(\chi_{1,\epsilon}). \tag{2.7}$$

Again we can assume that (2.3) holds.
Let $v \in L_2(\omega/\sqrt{h})$ and put $u := (\tilde{P} - z)^{-1}v$. Instead of equation (2.4) we have now

$$\tilde{q}[\chi_{0,\epsilon}e^{\omega/\sqrt{h}}u] - h\int_{\Omega_\infty}(\frac{\mathrm{Re}\ z}{h} + |\nabla\omega|^2 + h\sum_{j=0}^{1}|\nabla\chi_{j,\epsilon}|^2)\chi_{0,\epsilon}^2 e^{2\omega/\sqrt{h}}|u|^2\,dx =$$
$$= \mathrm{Re}\ ((\tilde{P} - z)u, e^{2\omega/\sqrt{h}}u) - \tilde{q}[\chi_{1,\epsilon}e^{\omega/\sqrt{h}}u] +$$
$$+ h\int_{\Omega_\infty}(\frac{\mathrm{Re}\ z}{h} + |\nabla\omega|^2 + h\sum_{j=0}^{1}|\nabla\chi_{j,\epsilon}|^2)\chi_{1,\epsilon}^2 e^{2\omega/\sqrt{h}}|u|^2\,dx.$$

Again we take $\epsilon = C_2\sqrt{h}$ and, proceeding as above, we find that there are $C_3, h_1 > 0$ such that for $h \in (0, h_1] \cap J$ and $\mathrm{Re}\ z \le h(b_0 + \beta(h)) + a(h)$ one has

$$\|u\|_{\omega/\sqrt{h}}^2 \le C_3(\frac{1}{h}\mathrm{Re}\ ((\tilde{P} - z)u, e^{2\omega/\sqrt{h}}u) + \|\chi_{1,C_2\sqrt{h}}u\|_{\omega/\sqrt{h}}^2).$$

It is clear now that the assertion will follow if we prove that there are $C_4, h_2 > 0$ such that for $h \in (0, h_2] \cap J$ and $z \in \gamma$ one has

$$\|\chi_{1,C_2\sqrt{h}}u\|_{\omega/\sqrt{h}} \le \frac{C_4}{a(h)}e^{\mathrm{osc}_0(\omega)/\sqrt{h}}\|(\tilde{P} - z)u\|_{\omega/\sqrt{h}}. \tag{2.8}$$

For this we choose $\chi_{2,\epsilon} \in C_0^\infty(\overline{\Omega_\infty})$ such that $0 \le \chi_{2,\epsilon} \le 1$, $\chi_{2,\epsilon} = 1$ in a neighbourhood of ∂U and $\chi_{1,\epsilon} = 1$ on $\mathrm{supp}(\chi_{2,\epsilon})$. Moreover, we can assume that

$$\epsilon|\nabla\chi_{2,\epsilon}| + \epsilon^2|\Delta\chi_{2,\epsilon}| \le \tilde{C}_1 \tag{2.9}$$

holds with $\tilde{C}_1$ independent of ϵ. Then we can write

$$\begin{aligned} u = (1 - \chi_{2,\epsilon})(P^D - z)^{-1}(1 - \chi_{1,\epsilon})v + (\tilde{P} - z)^{-1}\chi_{1,\epsilon}v + \\ + (\tilde{P} - z)^{-1}\chi_{1,\epsilon}[\tilde{P}, \chi_{2,\epsilon}](P^D - z)^{-1}(1 - \chi_{1,\epsilon})v. \end{aligned} \tag{2.10}$$

As before we choose $\epsilon = C_2\sqrt{h}$ and for simplicity we write χ_1, χ_2 instead of $\chi_{1,C_2\sqrt{h}}, \chi_{2,C_2\sqrt{h}}$.
We consider the three terms in (2.10) separately. For the first one we use Proposition 2.2 and obtain

$$\|\chi_1(1 - \chi_2)(P^D - z)^{-1}(1 - \chi_1)v\|_{\omega/\sqrt{h}} \le \frac{C_5}{h}\|v\|_{\omega/\sqrt{h}}.$$

For the second term we use (2.6), (2.7) and obtain

$$\|\chi_1(\tilde{P} - z)^{-1}\chi_1 v\|_{\omega/\sqrt{h}} \le \frac{1}{a(h)}e^{\mathrm{osc}_0(\omega)/\sqrt{h}}e^{2C_2}\|v\|_{\omega/\sqrt{h}}.$$

For the third term we put $w := (P^D - z)^{-1}(1 - \chi_1)v$, so that again by (2.6), (2.7)

$$\|\chi_1(\tilde{P} - z)^{-1}\chi_1[\tilde{P}, \chi_2]w\|_{\omega/\sqrt{h}} \le \frac{1}{a(h)}e^{\mathrm{osc}_0(\omega)/\sqrt{h}}e^{2C_2}\|[\tilde{P}, \chi_2]w\|_{\omega/\sqrt{h}}.$$

We find easily (taking into account (2.9)) that

$$\|[\tilde{P},\chi_2]w\|^2_{\omega/\sqrt{h}} \leq C_6(hq^D[e^{\omega/\sqrt{h}}w] + h^2\|w\|^2_{\omega/\sqrt{h}}), \tag{2.11}$$

so by Corollary 2.3 we obtain for the third term

$$\|\chi_1(\tilde{P}-z)^{-1}\chi_1[\tilde{P},\chi_2]w\|_{\omega/\sqrt{h}} \leq \frac{C_7}{a(h)}e^{\operatorname{osc}_0(\omega)/\sqrt{h}}\|v\|_{\omega/\sqrt{h}}.$$

and (2.8) follows by adding the estimates of the three terms. □

Combining Proposition 2.6 with Lemma 2.1 we obtain immediately

Corollary 2.7. *Let $\omega \in \mathcal{W}$, then there exist $C_0, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ and $z \in \gamma$ one has*

$$\tilde{q}[e^{\omega/\sqrt{h}}(\tilde{P}-z)^{-1}v] + h\|(\tilde{P}-z)^{-1}v\|^2_{\omega/\sqrt{h}} \leq \frac{C_0 h}{a(h)^2}e^{2\operatorname{osc}_0(\omega)/\sqrt{h}}\|v\|^2_{\omega/\sqrt{h}},$$
$$v \in L_2(\omega/\sqrt{h}).$$

2.4. **The operator P.** We want to derive estimates for the resolvent of P in weighted spaces from the estimates we have established in the previous subsection for the reference operator. For this we construct an approximate resolvent in Subsection 2.4.1. This will allow us also to define an approximate spectral projection in Subsection 2.4.2.

2.4.1. *An approximate resolvent.* Recall that γ was defined in (2.5). We will prove

Proposition 2.8. *Let $\omega \in \mathcal{W}_0$, then there exist $C_0, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ and $z \in \gamma$ the operator $(P-z)^{-1}$ maps $L_2(\omega/\sqrt{h})$ into itself with norm*

$$\|(P-z)^{-1}\|_{\omega/\sqrt{h}} \leq \frac{C_0}{a(h)}.$$

For the proof we choose an (h-independent) function $\tilde{\chi} \in C_0^\infty(\overline{\Omega_\infty})$ such that $0 \leq \tilde{\chi} \leq 1$, $\tilde{\chi} = 1$ in a neighbourhood of ∂U, $\tilde{\chi} = 0$ outside a neighbourhood of Ω and $\sum_{\alpha\in\Gamma}\tau_\alpha\tilde{\chi} = 1$ in $\overline{\Omega_\infty}$. We recall the (h-dependent) function χ introduced in Subsection 1.3. In the following we assume that h is so small that

$$\chi = 1 \qquad \text{in a neighbourhood of } \operatorname{supp}(\tilde{\chi}). \tag{2.12}$$

Now we define an approximate resolvent by

$$R^{app}(z) := \sum_{\alpha\in\Gamma}\tilde{R}^{app}_\alpha(z), \qquad z \in \rho(\tilde{P}), \tag{2.13}$$

where

$$\tilde{R}^{app}_\alpha(z) := T_\alpha\chi(\tilde{P}-z)^{-1}\tilde{\chi}T^*_\alpha, \qquad z \in \rho(\tilde{P}), \alpha \in \Gamma.$$

From the resolvent estimates for the reference operator $\tilde{P}$ in Proposition 2.6 we obtain

Lemma 2.9. *Let $\omega \in \mathcal{W}_0$, then there exist $C_0, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ and $z \in \gamma$ the operator $R^{app}(z)$ maps $L_2(\omega/\sqrt{h})$ into itself with norm*

$$\|R^{app}(z)\|_{\omega/\sqrt{h}} \leq \frac{C_0}{a(h)}.$$

Proof. For $v \in L_2(\omega/\sqrt{h})$ Proposition 2.6 yields that

$$\begin{aligned}\|\tilde{R}^{app}_\alpha(z)v\|_{\omega/\sqrt{h}} &= \|\chi(\tilde{P}-z)^{-1}\tilde{\chi}T^*_\alpha v\|_{\tau_{-\alpha}\omega/\sqrt{h}} \leq \\ &\leq \frac{C_1}{a(h)}\|\tilde{\chi}T^*_\alpha v\|_{\tau_{-\alpha}\omega/\sqrt{h}} = \frac{C_1}{a(h)}\|(\tau_\alpha\tilde{\chi})v\|_{\omega/\sqrt{h}}\end{aligned}$$

for $h \in (0, h_0] \cap J$. The proof of that proposition shows that C_1, h_1 can be chosen independently of $\alpha \in \Gamma$. Using that $(\tau_\alpha\chi)(\tau_\beta\chi) = 0$ if $|\alpha - \beta| \geq C_2$ and applying Schur's lemma we find that

$$\begin{aligned}\|\tilde{R}^{app}(z)v\|^2_{\omega/\sqrt{h}} &\leq \sum_\alpha \sum_{|\alpha-\beta|<C_2} \|\tilde{R}^{app}_\alpha(z)v\|_{\omega/\sqrt{h}}\|\tilde{R}^{app}_\beta(z)v\|_{\omega/\sqrt{h}} \leq \\ &\leq C_3 \sum_\alpha \|\tilde{R}^{app}_\alpha(z)v\|^2_{\omega/\sqrt{h}} \leq \frac{C_3C_1^2}{a(h)^2}\|v\|^2_{\omega/\sqrt{h}},\end{aligned}$$

which establishes the assertion. ☐

Now we note that $\mathcal{R}(R^{app}(z)) \subset \mathcal{D}(P)$ and that, using (2.12),

$$(P-z)R^{app}(z) = I + K(z), \qquad z \in \rho(\tilde{P}) \tag{2.14}$$

where

$$K(z) := \sum_{\alpha\in\Gamma} \tilde{K}_\alpha(z), \qquad z \in \rho(\tilde{P}),$$

$$\tilde{K}_\alpha(z) := T_\alpha[P,\chi](\tilde{P}-z)^{-1}\tilde{\chi}T^*_\alpha, \qquad z \in \rho(\tilde{P}), \alpha \in \Gamma.$$

The remainder $K(z)$ is small, more precisely,

Lemma 2.10. *Let $\omega \in \mathcal{W}_0$, then there exist $C_0, c_0, h_0 > 0$ such that for $h \in (0, h_0] \cap J$ and $z \in \gamma$ the operator $K(z)$ maps $L_2(\omega/\sqrt{h})$ into itself with norm*

$$\|K(z)v\|_{\omega/\sqrt{h}} \leq C_0 e^{-c_0/\sqrt{h}}.$$

Proof. We first note that an estimate similar to (2.11) (taking into account (1.21)) combined with Corollary 2.7 yields

$$\begin{aligned}&\|[P,\chi](\tilde{P}-z)^{-1}v\|^2_{\omega'/\sqrt{h}} \leq \\ &\leq C_1\left(h\tilde{q}[e^{\omega'/\sqrt{h}}(\tilde{P}-z)^{-1}v] + h^2\|(\tilde{P}-z)^{-1}v\|^2_{\omega'/\sqrt{h}}\right) \leq \\ &\leq \frac{C_2h^2}{a(h)^2}\|v\|^2_{\omega'/\sqrt{h}}\end{aligned} \tag{2.15}$$

for any $\omega' \in \mathcal{W}_0$. Now (2.12) allows us to choose $\tilde{\omega} \in \mathcal{W}$ such that $\tilde{\omega} = 0$ on $\operatorname{supp}(\tilde{\chi})$ and $c_1 := \min_{\operatorname{supp}(|\nabla\chi|)} \tilde{\omega} > 0$. Multiplying $\tilde{\omega}$ by a small positive constant if necessary, we may assume that $\tilde{\omega}+\tau_\alpha\omega \in \mathcal{W}_0$ for all $\alpha \in \Gamma$. Then (2.15) yields

$$\begin{aligned}
\|\tilde{K}_\alpha(z)v\|_{\omega/\sqrt{h}} &= \|[P,\chi](\tilde{P}-z)^{-1}\tilde{\chi}T_\alpha^* v\|_{\tau_{-\alpha}\omega/\sqrt{h}} \le \\
&\le e^{-c_1/\sqrt{h}}\|[P,\chi](\tilde{P}-z)^{-1}\tilde{\chi}T_\alpha^* v\|_{(\tilde{\omega}+\tau_{-\alpha}\omega)/\sqrt{h}} \le \\
&\le \frac{\sqrt{C_2}h}{a(h)}e^{-c_1/\sqrt{h}}\|\tilde{\chi}T_\alpha^* v\|_{(\tilde{\omega}+\tau_{-\alpha}\omega)/\sqrt{h}} = \frac{\sqrt{C_2}h}{a(h)}e^{-c_1/\sqrt{h}}\|(\tau_\alpha\tilde{\chi})v\|_{\omega/\sqrt{h}}.
\end{aligned}$$

It follows from the proofs of Proposition 2.6 and Corollary 2.7 that C_2 can be chosen independently of $\alpha \in \Gamma$. Using the assumption (1.13) and proceeding as at the end of the previous proof we obtain the assertion. □

The proof of Proposition 2.8 is now easy.

Proof of Proposition 2.8. It follows from Lemma 2.10 that for sufficiently small h the operator $I+K(z)$ is boundedly invertible in $L_2(\omega/\sqrt{h})$ and that the norm of the inverse can be bounded uniformly in h. Since

$$(P-z)^{-1} = R^{app}(z)(I+K(z))^{-1},$$

the assertion follows from Lemma 2.9. □

2.4.2. *An approximate projection.* Now we would like to construct an approximation to the spectral projection $E(I) = -\frac{1}{2\pi i}\oint_\gamma (P-z)^{-1}\,dz$, γ being defined in (2.5). In view of the approximate resolvent (2.13) it is natural to consider

$$E^{app}(I) := -\frac{1}{2\pi i}\oint_\gamma R^{app}(z)\,dz.$$

We remark

Lemma 2.11. $\mathcal{R}(E^{app}(I)) \subset \overline{\operatorname{span}\{\psi_\alpha : \alpha \in \Gamma\}}$.

Proof. The Assumption 1.1 implies $\tilde{E}(I) = (.,\tilde{\varphi})\tilde{\varphi}$, so

$$-\frac{1}{2\pi i}\oint_\gamma \tilde{R}_\alpha^{app}(z)\,dz = T_\alpha\chi\tilde{E}(I)\tilde{\chi}T_\alpha^* = (.,T_\alpha(\tilde{\chi}\tilde{\varphi}))\psi_\alpha,$$

and the assertion follows after summation over $\alpha \in \Gamma$. □

Now we show that $E^{app}(I)$ is a good approximation for $E(I)$.

Proposition 2.12. *Let $\omega \in \mathcal{W}_0$, then there exist $C_0, c_0, h_0 > 0$ such that for $h \in (0,h_0] \cap J$ and $z \in \gamma$ the operator $E^{app}(I) - E(I)$ maps $L_2(\omega/\sqrt{h})$ into itself with norm*

$$\|E^{app}(I) - E(I)\|_{\omega/\sqrt{h}} \le C_0 e^{-c_0/\sqrt{h}}. \tag{2.16}$$

Proof. It follows from (2.14) that

$$(P-z)^{-1} = R^{app}(z) - R^{app}(z)K(z)(I+K(z))^{-1},$$

and therefore

$$E^{app}(I) - E(I) = -\frac{1}{2\pi i}\oint_\gamma R^{app}(z)K(z)(I+K(z))^{-1}\,dz.$$

The assertion follows now from Lemmas 2.9, 2.10 and assumption (1.13). $\square$

3. Proof of the main results

3.1. Symmetry relations. Proof of Proposition 1.10. This subsection is concerned with the "algebraic" part of our construction. Namely, we will note several symmetry relations that will be used later. We will also deduce Proposition 1.10, ignoring for the moment the definition of $\mathbf{V}^{-1/2}$ and the convergence of the series (1.25). This will be treated in Subsections 3.2 and 3.3 below.

Recall the definition of the functions $\psi_\alpha, r_\alpha, v_\alpha$ and of the matrices $\mathbf{V}, \mathbf{W}$ from Subsection 1.3. The relation (1.20) gives us

$$\begin{aligned} T_\alpha \psi_\beta &= e^{-i\alpha_2\beta_1 B_0/h}\psi_{\alpha+\beta}, \\ T_\alpha r_\beta &= e^{-i\alpha_2\beta_1 B_0/h} r_{\alpha+\beta}, \\ T_\alpha v_\beta &= e^{-i\alpha_2\beta_1 B_0/h} v_{\alpha+\beta}, \end{aligned} \tag{3.1}$$

and therefore also

$$\begin{aligned} v_{\alpha,\beta} &= e^{-i(\alpha_1-\beta_1)\gamma_2 B_0/h} v_{\alpha+\gamma,\beta+\gamma}, \\ w_{\alpha,\beta} &= e^{-i(\alpha_1-\beta_1)\gamma_2 B_0/h} w_{\alpha+\gamma,\beta+\gamma} \end{aligned}$$

for $\alpha,\beta,\gamma\in\Gamma$. We note that the last two relations are equivalent to the commutation of the matrices $\mathbf{V},\mathbf{W}$ with the family of matrices $\mathbf{E}_\gamma$, $\gamma\in\Gamma$, where

$$(\mathbf{E}_\gamma)_{\alpha,\beta} := e^{-i\alpha_1\gamma_2 B_0/h}\delta_{\alpha-\beta,-\gamma}, \qquad \alpha,\beta\in\Gamma.$$

Proof of Proposition 1.10. Together with $\mathbf{V}$ also $\mathbf{V}^{-1/2}$ commutes with $\mathbf{E}_\gamma$, $\gamma\in\Gamma$. This proves the second relation in (1.29). Combining this with the third relation in (3.1) yields the first relation in (1.29). $\square$

3.2. Proof of Proposition 1.5. We recall the Notation 1.6.

Lemma 3.1. *One has*

$$(\psi_\alpha,\psi_\beta) = \delta_{\alpha,\beta} + \tilde{\mathcal{O}}_{unif}(e^{-d^{(1)}(\alpha-\beta)/\sqrt{h}}).$$

Proof. If $\alpha=\beta$ we have, using $d_0 \geq S_0 - \sqrt{h}$ on $\operatorname{supp}(1-\chi^2)$,

$$0 \leq 1 - \|\psi_\alpha\|^2 = \|\sqrt{1-\chi^2}\tilde\varphi\|^2 \leq e^{2(1-\epsilon)}e^{-2(1-\epsilon)S_0/\sqrt{h}}\|\tilde\varphi\|^2_{(1-\epsilon)d_0/\sqrt{h}}.$$

If $\alpha\neq\beta$ we have

$$\begin{aligned} |(\psi_\alpha,\psi_\beta)| &\leq e^{(1-\epsilon)d(\alpha-\beta)/\sqrt{h}}\|\psi_\alpha\|_{(1-\epsilon)d_\alpha/\sqrt{h}}\|\psi_\beta\|_{(1-\epsilon)d_\beta/\sqrt{h}} \leq \\ &\leq e^{(1-\epsilon)d(\alpha-\beta)/\sqrt{h}}\|\tilde\varphi\|^2_{(1-\epsilon)d_0/\sqrt{h}}. \end{aligned}$$

Thus in both cases the assertion follows from Proposition 2.4 with $\omega = (1-\epsilon)d_0$. $\square$

To carry this result over to the v_α we need the following estimate on the functions r_α defined in (1.23).

Lemma 3.2. *For every $0 < \epsilon \leq 1$ there exist $C_\epsilon, c_\epsilon, h_\epsilon > 0$ such that for $h \in (0, h_\epsilon] \cap J$ and $\alpha \in \Gamma$*

$$\|r_\alpha\|_{(1-\epsilon)d_\alpha^{(1)}/\sqrt{h}} \leq C_\epsilon e^{-c_\epsilon/\sqrt{h}}.$$

Proof. Since $r_\alpha = T_\alpha r_0$ it suffices to consider the case $\alpha = 0$. Then

$$r_0 = -2ih\nabla\chi \cdot (h\mathbf{D} - \mathbf{A})\tilde{\varphi} - h^2(\Delta\chi)\tilde{\varphi}$$

and $d_0^{(1)} \leq d_0 + 2\sqrt{h}$ on $\operatorname{supp}(r_0)$. Therefore it suffices to prove

$$\|r_0\|_{(1-\epsilon)d_0/\sqrt{h}} \leq \tilde{C}_\epsilon e^{-c_\epsilon/\sqrt{h}}. \tag{3.2}$$

For this we note that, using (1.21) and Corollary 2.5, we find easily

$$\|r_0\|^2_{\omega/\sqrt{h}} \leq C_1(h\tilde{q}[e^{\omega/\sqrt{h}}\tilde{\varphi}] + h^2\|\tilde{\varphi}\|^2_{\omega/\sqrt{h}}) \leq C_2 h^2$$

for any $\omega \in \mathcal{W}$ such that $\omega = 0$ on ∂U. Now choose $\tilde{\omega} \in \mathcal{W}$ (independent of h) such that $\tilde{\omega} = 0$ on ∂U and $\inf_{\operatorname{supp}(|\nabla\chi_h|)} \tilde{\omega} \geq c > 0$ for all sufficiently small h. Multiplying $\tilde{\omega}$ by a (possibly ϵ-dependent) constant if necessary, we can assume that $(1-\epsilon)d_0 + \tilde{\omega} \in \mathcal{W}$. Then

$$\|r_0\|_{(1-\epsilon)d_0/\sqrt{h}} \leq e^{-c/\sqrt{h}}\|r_0\|_{((1-\epsilon)d_0+\tilde{\omega})/\sqrt{h}} \leq \sqrt{C_2}he^{-c/\sqrt{h}},$$

as asserted. $\square$

Now we are able to prove that the matrix of the v_α is close to the matrix of the ψ_α.

Lemma 3.3. *One has*

$$(v_\alpha, v_\beta) = (\psi_\alpha, \psi_\beta) + \tilde{\mathcal{O}}_{unif}(e^{-d^{(2)}(\alpha-\beta)/\sqrt{h}}).$$

Proof. Using $v_\alpha - \psi_\alpha \in \mathcal{F}^\perp$ and $d_\alpha^{(1)} + d_\beta^{(1)} \geq d^{(2)}(\alpha - \beta)$ we find

$$\begin{aligned}
|(v_\alpha, v_\beta) - (\psi_\alpha, \psi_\beta)| &= |(v_\alpha - \psi_\alpha, v_\beta - \psi_\beta)| \leq \\
&\leq e^{-(1-\epsilon)d^{(2)}(\alpha-\beta)/\sqrt{h}}\|v_\alpha - \psi_\alpha\|_{(1-\epsilon)d_\alpha^{(1)}/\sqrt{h}}\|v_\beta - \psi_\beta\|_{(1-\epsilon)d_\beta^{(1)}/\sqrt{h}} = \\
&= e^{-(1-\epsilon)d^{(2)}(\alpha-\beta)/\sqrt{h}}\|v_0 - \psi_0\|^2_{(1-\epsilon)d_0^{(1)}/\sqrt{h}}.
\end{aligned}$$

Therefore the assertion will follow if we prove that $\|v_0 - \psi_0\|_{(1-\epsilon)d_0^{(1)}/\sqrt{h}}$ is uniformly bounded in h. We actually show that there exist $C_\epsilon, c_\epsilon, h_\epsilon > 0$ such that for $h \in (0, h_\epsilon] \cap J$

$$\|v_0 - \psi_0\|_{(1-\epsilon)d_0^{(1)}/\sqrt{h}} \leq C_\epsilon e^{-c_\epsilon/\sqrt{h}}. \tag{3.3}$$

To establish this we note that

$$(P-z)^{-1}\psi_0 = \frac{1}{\tilde\lambda - z}\psi_0 - \frac{1}{\tilde\lambda - z}(P-z)^{-1} r_0,$$

and therefore with γ as in (2.5)

$$v_0 - \psi_0 = -\frac{1}{2\pi i}\oint_\gamma (P-z)^{-1}\psi_0 \, dz - \psi_0 = \frac{1}{2\pi i}\oint_\gamma \frac{1}{\tilde\lambda - z}(P-z)^{-1} r_0 \, dz.$$

The estimate (3.3) follows now from Lemma 3.2, Proposition 2.8 with $\omega = (1-\epsilon)d_0^{(1)}$ and assumption (1.13). □

We can give now the

Proof of Proposition 1.5. From the estimate $d(\alpha) \geq c|\alpha|$, $\alpha \in \Gamma$, we obtain

$$\sum_{\beta\neq\alpha} e^{-d^{(1)}(\alpha-\beta)/2\sqrt{h}} \leq C_1 e^{-c_1/\sqrt{h}}$$

for $h \in (0,1]$, $\alpha \in \Gamma$, and for some $C_1, c_1 > 0$. Lemmas 3.1 and 3.3 together with Schur's lemma imply now

(3.4)

$$(1 - C_0 e^{-c_0/\sqrt{h}})\|a\|^2 \leq \|\sum_{\alpha\in\Gamma} a_\alpha v_\alpha\|^2 \leq (1 + C_0 e^{-c_0/\sqrt{h}})\|a\|^2, \quad a \in \ell_2(\Gamma).$$

for $h \in (0, h_0]$ and some $C_0, c_0, h_0 > 0$. In particular, the v_α, $\alpha \in \Gamma$, are linearly independent for sufficiently small h.

To prove that they are complete in $\mathcal{F}$ for sufficiently small h it suffices to prove this for the ψ_α, $\alpha \in \Gamma$. So let $u \in \mathcal{F}$ such that $(u, \psi_\alpha) = 0$ for all $\alpha \in \Gamma$. We have to prove $u = 0$. It follows from Lemma 2.11 that

$$v := E^{app}(I)u \in \overline{\operatorname{span}\{\psi_\alpha : \alpha \in \Gamma\}},$$

and so by Proposition 2.12 with $\omega = 0$

$$\|u\|^2 + \|v\|^2 = \|u - v\|^2 \leq C^2 e^{-2c/\sqrt{h}}\|u\|^2.$$

For sufficiently small h (independent of u) this implies $u = 0$, which completes the proof. □

Now we can also prove the localization near $\partial\Omega_\infty$ of functions in $\mathcal{F}$.

Proof of Proposition 1.9. Put $m_{\alpha,\epsilon} := (e^{(1-\epsilon)d_\infty/\sqrt{h}} v_\alpha, e^{(1-\epsilon)d_\infty/\sqrt{h}} v_0)$. Below we shall prove that there exist $C_\epsilon, h_\epsilon > 0$ such that for $h \in (0, h_\epsilon] \cap J$

$$\sum_{\alpha\in\Gamma} |m_{\alpha,\epsilon}| \leq C_\epsilon < \infty. \tag{3.5}$$

This implies that for $a \in \ell_2(\Gamma)$

$$\|\sum_{\alpha\in\Gamma} a_\alpha v_\alpha\|^2_{(1-\epsilon)d_\infty/\sqrt{h}} \leq \sum_{\alpha,\beta\in\Gamma} |(e^{(1-\epsilon)d_\infty/\sqrt{h}} v_\alpha, e^{(1-\epsilon)d_\infty/\sqrt{h}} v_\beta)||a_\alpha||a_\beta| =$$

$$= \sum_{\alpha,\beta\in\Gamma} |m_{\alpha-\beta}||a_\alpha||a_\beta| \leq C_\epsilon \|a\|^2,$$

and therefore Proposition 1.5 and (3.4) yield the assertion.
It remains to prove (3.5). First of all note that by Lemma 3.1 and equation (3.3) there exist $\tilde{C}_\epsilon, h_\epsilon > 0$ such that for $h \in (0, h_\epsilon] \cap J$ and $\alpha \in \Gamma$

$$\|v_\alpha\|_{(1-\epsilon)d_\infty/\sqrt{h}} = \|v_0\|_{(1-\epsilon)d_\infty/\sqrt{h}} = \|v_0\|_{(1-\epsilon)d_0/\sqrt{h}} \le \tilde{C}_\epsilon.$$

Now consider the set $M_\alpha := \{x \in \Omega_\infty : d_\alpha(x) \le d_0(x)\}$. It is not difficult to prove that $\inf_{x\in M_\alpha} d_0(x) \ge C|\alpha|$, $\alpha \in \Gamma$, and therefore

$$\inf_{x\in M_\alpha} (d_0(x) - d_\infty(x)) \ge C(|\alpha| - c)_+, \qquad \alpha \in \Gamma,$$

for some $C, c > 0$. We split

$$m_{\alpha,\epsilon} = \int_{M_\alpha} e^{2(1-\epsilon)d_\infty/\sqrt{h}} v_\alpha \overline{v_0}\, dx + \int_{\Omega_\infty \setminus M_\alpha} e^{2(1-\epsilon)d_\infty/\sqrt{h}} v_\alpha \overline{v_0}\, dx$$

and consider the first summand. The above estimates yield

$$\begin{aligned} &|\int_{M_\alpha} e^{2(1-\epsilon)d_\infty/\sqrt{h}} v_\alpha \overline{v_0}\, dx|^2 \le \\ &\le \sup_{x\in M_\alpha} e^{-(1-\epsilon)(d_0(x)-d_\infty(x))/\sqrt{h}} \|v_\alpha\|_{(1-\epsilon)d_\infty/\sqrt{h}} \|v_0\|_{(1-\epsilon)d_0/\sqrt{h}} \le \\ &\le e^{-(1-\epsilon)C(|\alpha|-c)_+/\sqrt{h}} \tilde{C}_\epsilon^2. \end{aligned}$$

A similar estimate can be obtained for the second summand. This proves (3.5). $\square$

3.3. **Proof of Theorem 1.7.** We begin by noting that

$$(P\psi_\alpha, \psi_\beta) = \tilde{\lambda}(\delta_{\alpha,\beta}\|\psi_\alpha\|^2 + t_{\alpha,\beta}) + w_{\alpha,\beta}, \qquad \alpha, \beta \in \Gamma,$$

where the matrix $\mathbf{T} = (t_{\alpha,\beta})$ is defined by

$$t_{\alpha,\beta} := (1 - \delta_{\alpha,\beta})(\psi_\alpha, \psi_\beta), \qquad \alpha, \beta \in \Gamma,$$

and the interaction matrix $\mathbf{W} = (w_{\alpha,\beta})$ has been defined in (1.27). It follows from Lemma 3.1 that

$$(P\psi_\alpha, \psi_\beta) = \tilde{\lambda}(\delta_{\alpha,\beta} + t_{\alpha,\beta}) + w_{\alpha,\beta} + \tilde{\mathcal{O}}_{unif}(e^{-d^{(2)}(\alpha-\beta)/\sqrt{h}}), \tag{3.6}$$

$$t_{\alpha,\beta} = \tilde{\mathcal{O}}_{unif}(e^{-d^{(1)}(\alpha-\beta)/\sqrt{h}}), \tag{3.7}$$

and from Proposition 2.4 and (3.2) that

$$w_{\alpha,\beta} = \tilde{\mathcal{O}}_{unif}(e^{-d^{(1)}(\alpha-\beta)/\sqrt{h}}). \tag{3.8}$$

Concerning the v_α we have

Lemma 3.4. *One has*

$$(Pv_\alpha, v_\beta) = (P\psi_\alpha, \psi_\beta) + \tilde{\mathcal{O}}_{unif}(e^{-d^{(2)}(\alpha-\beta)/\sqrt{h}}).$$

Proof. We proceed as in the proof of Lemma 3.3 and write

$$P(v_0 - \psi_0) = -r_0 + \frac{1}{2\pi i} \oint \frac{z}{\tilde{\lambda} - z} (P - z)^{-1} r_0 \, dz.$$

As above, the assertion follows now from Proposition 2.8 and Lemma 3.2. $\square$

Combining this lemma with (3.6) we find

$$(Pv_\alpha, v_\beta) = \tilde{\lambda}(\delta_{\alpha,\beta} + t_{\alpha,\beta}) + w_{\alpha,\beta} + \tilde{\mathcal{O}}_{unif}(e^{-d^{(2)}(\alpha-\beta)/\sqrt{h}}). \tag{3.9}$$

The proof of Theorem 1.7 is now easy.

Proof of Theorem 1.7. Recall that the matrix $\mathbf{V}$ has been defined in (1.26) and satisfies, according to Lemmas 3.1 and 3.3, the estimate

$$v_{\alpha,\beta} = \delta_{\alpha,\beta} + t_{\alpha,\beta} + \tilde{\mathcal{O}}_{unif}(e^{-d^{(2)}(\alpha-\beta)/\sqrt{h}}).$$

It follows from Appendix B in [Ca] that this implies

$$(\mathbf{V}^{-1/2})_{\alpha,\beta} = \delta_{\alpha,\beta} - \frac{1}{2} t_{\alpha,\beta} + \tilde{\mathcal{O}}_{unif}(e^{-d^{(2)}(\alpha-\beta)/\sqrt{h}}).$$

Combining this with (3.9) and using (3.7), (3.8) we conclude again with [Ca] that

$$(Pe_\alpha, e_\beta) = \tilde{\lambda}\delta_{\alpha,\beta} + w_{\alpha,\beta} + \tilde{\mathcal{O}}_{unif}(e^{-d^{(2)}(\alpha-\beta)/\sqrt{h}}),$$

as claimed. $\square$

Proof of Corollary 1.8. It follows from Theorem 1.7 and Schur's lemma that for every $c < S_0$ the operator $e^{c/\sqrt{h}}((Pe_\alpha, e_\beta) - \tilde{\lambda}\delta_{\alpha,\beta})$ in $\ell_2(\Gamma)$ is uniformly bounded in h. $\square$

3.4. **The case of rational flux.** We assume now (1.31) and (1.32) and give the

Proof of Proposition 1.11. The operator $\mathcal{U} : \ell_2(\mathbb{Z}^2) \to \int_{\mathbb{T}^2}^{\oplus} \mathbb{C}^M \, d\theta$,

$$(\mathcal{U}a)_j(\theta) := \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} a_{(n_1, j + n_2 M)} e^{i\langle n, \theta \rangle}, \theta \in \mathbb{T}^2, 1 \le j \le M,$$

is unitary and satisfies (recall (1.30))

$$
\begin{aligned}
(\mathcal{U}((Pe_n, e_m))a)_j(\theta) &= \frac{1}{2\pi} \sum_{n,m\in\mathbb{Z}^2} (Pe_{n_1, j+n_2M}, e_m) a_m e^{i\langle n,\theta\rangle} = \\
&= \frac{1}{2\pi} \sum_{n,m\in\mathbb{Z}^2} e^{i(n_1-m_1)m_2B_0/h} p(n_1-m_1, j+n_2M-m_2) a_m e^{i\langle n,\theta\rangle} = \\
&= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}^2} \sum_{n_2\in\mathbb{Z}} a_m e^{i(m_1\theta_1+n_2\theta_2)} \times \\
&\qquad \times \left(\sum_{n_1\in\mathbb{Z}} p(n_1, j+n_2M-m_2) e^{in_1(\theta_1+m_2B_0/h)} \right) = \\
&= \frac{1}{2\pi} \sum_{k=1}^{M} \sum_{m\in\mathbb{Z}^2} \sum_{n_2\in\mathbb{Z}} a_{(m_1, k+m_2M)} e^{i(m_1\theta_1+n_2\theta_2)} \times \\
&\qquad \times \left(\sum_{n_1\in\mathbb{Z}} p(n_1, j-k+(n_2-m_2)M) e^{in_1(\theta_1+2\pi kq)} \right) = \\
&= \frac{1}{2\pi} \sum_{k=1}^{M} \sum_{m\in\mathbb{Z}^2} p_{j,k}(\theta) a_{(m_1,k+m_2M)} e^{i\langle m,\theta\rangle} = (\mathbf{P}\mathcal{U}a)_j(\theta).
\end{aligned}
$$

This proves the first assertion of the proposition.
Now if $\mu_1(\theta) \leq ... \leq \mu_M(\theta)$ denote the eigenvalues of $\mathbf{P}(\theta)$ counting multiplicities, we obtain

$$\sigma(P) \cap I = \sigma(\mathbf{P}) = \bigcup_{j=1}^{M} \mu_j(\mathbb{T}^2).$$

It follows from the exponential decay of the sequence $p(n)$, $n \in \mathbb{Z}^2$, that the matrices $\mathbf{P}(\theta)$ depend real analytically on θ. Hence $\det(\mathbf{P}(\theta) - \lambda)$ is a real analytic function of $\theta \in \mathbb{T}^2, \lambda \in \mathbb{R}$. This implies (see, e.g., [HemHer]) that the Borel measure on $\mathbb{R}$ defined by

$$\mu(\Lambda) := |\{\theta \in \mathbb{T}^2 : \det(\mathbf{P}(\theta) - \lambda) = 0 \text{ for some } \lambda \in \Lambda\}|, \qquad \Lambda \subset \mathbb{R},$$

has no singular continuous part and that therefore the spectrum of $\mathbf{P}$ has no singular continuous part. This completes the proof of the proposition. ☐

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Royal Institute of Technology, Department of Mathematics, Lindstedtsvägen 25, 10044 Stockholm, Sweden

E-mail address: rupert@math.kth.se