# Multiplicity of ground states in quantum field models: applications of asymptotic fields

Fumio Hiroshima[*†]

February 27, 2004

**Abstract**

The ground states of an abstract model in quantum field theory are investigated. By means of the asymptotic field theory, we give a necessary and sufficient condition for that the expectation value of the number operator of ground states is finite, from which we obtain a wide-usable method to estimate an upper bound of the multiplicity of ground states. Ground states of massless GSB models and the Pauli-Fierz model with spin 1/2 are investigated as examples.

# Contents



[*]Department of Mathematics and Physics, Setsunan University, 572-8508, Osaka, Japan, e-mail: hiroshima@mpg.setsunan.ac.jp

[†]This work is partialy supported by Grant-in-Aid for Science Research (C) 1554019 from MEXT.

# 1 Preliminaries

## 1.1 Boson Fock spaces

Let $\mathcal{W}$ be a Hilbert space over $\mathbb{C}$ with a conjugation $^{-}$. The boson Fock space $\mathcal{F}_{\rm b}$ over $\mathcal{W}$ is defined by

$$\begin{aligned}\mathcal{F}_{\rm b} &= \mathcal{F}_{\rm b}(\mathcal{W}) := \bigoplus_{n=0}^{\infty}[\otimes_s^n \mathcal{W}] \\ &= \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \Psi^{(n)} \in \otimes_s^n \mathcal{W}, \|\Psi\|_{\mathcal{F}_{\rm b}}^2 := \sum_{n=0}^{\infty} \|\Psi^{(n)}\|_{\otimes^n \mathcal{W}}^2 < \infty\},\end{aligned}$$

where $\otimes_s^n \mathcal{W}$ denotes the $n$-fold symmetric tensor product of $\mathcal{W}$ with $\otimes_s^0 \mathcal{W} := \mathbb{C}$. In this paper $(f, g)_{\mathcal{K}}$ and $\|f\|_{\mathcal{K}}$ denote the scalar product and the norm on Hilbert space $\mathcal{K}$ over $\mathbb{C}$, respectively, where $(f, g)_{\mathcal{K}}$ is linear in $g$ and antilinear in $f$. Unless confusions arise we omit $\mathcal{K}$ of $(\cdot,\cdot)_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}}$. $D(T)$ denotes the domain of operator $T$. Moreover, for a bounded operator $S$, we denote its operator norm by $\|S\|$.

Fock vacuum $\Omega \in \mathcal{F}_{\rm b}$ is given by

$$\Omega = \{1, 0, 0, ...\}.$$

The finite particle subspace of $\mathcal{F}_{\rm b}$ is defined by

$$\mathcal{F}_{\rm fin} := \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm b} | \Psi^{(m)} = 0 \text{ for all } m \geq n \text{ with some } n\}.$$

It is known that $\mathcal{F}_{\rm fin}$ is dense in $\mathcal{F}_{\rm b}$. The creation operator $a^\dagger(f) : \mathcal{F}_{\rm b} \to \mathcal{F}_{\rm b}$ with test function $f \in \mathcal{W}$ is the densely defined linear operator in $\mathcal{F}_{\rm b}$ defined by

$$\begin{aligned}&(a^\dagger(f)\Psi)^{(0)} = 0, \\ &(a^\dagger(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1,\end{aligned}$$

where $S_n$ is the symmetrization operator on $\otimes^n \mathcal{W}$, i.e., $S_n[\otimes^n \mathcal{W}] = \otimes_s^n \mathcal{W}$. The annihilation operator $a(f), f \in \mathcal{W}$, is defined by

$$a(f) = (a^\dagger(\overline{f}))^* \lceil_{\mathcal{F}_{\rm fin}}.$$

Since it is seen that $a(f)$ and $a^\dagger(f)$ are closable operators, their closures are denoted by the same symbols, respectively. Note that $a^\sharp(f)$ ($a^\sharp = a$ or $a^\dagger$) is linear in $f$. On

$\mathcal{F}_{\text{fin}}$ the annihilation operator and the creation operator obey canonical commutation relations,

$$
\begin{aligned}
&[a(f), a^\dagger(g)] = (\overline{f}, g)_{\mathcal{W}}, \\
&[a(f), a(g)] = 0, \\
&[a^\dagger(f), a^\dagger(g)] = 0,
\end{aligned}
$$

where $[A, B] := AB - BA$. Define

$$\mathcal{F}_{\text{fin}}^{D} := \text{the linear hull of } \{a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega, \Omega | f_j \in D, j = 1, ..., , n \geq 1\}.$$

Let $S$ be a self-adjoint operator acting in $\mathcal{W}$. The second quantization of $S$,

$$d\Gamma(S) : \mathcal{F}_{\text{b}} \to \mathcal{F}_{\text{b}},$$

is defined by

$$d\Gamma(S) := \bigoplus_{n=0}^{\infty} \left( \sum_{j=1}^{n} \underbrace{1 \otimes \cdots \otimes \overset{j}{S} \otimes \cdots \otimes 1}_{n} \right),$$

with

$$D(d\Gamma(S)) := \mathcal{F}_{\text{fin}}^{D(S)}.$$

Here we define

$$(d\Gamma(S)\Psi)^{(0)} := 0.$$

In particular it follows that

$$d\Gamma(S)\Omega = 0. \tag{1.1}$$

Note that

$$d\Gamma(S)a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega = \sum_{j=1}^{n} a^\dagger(f_1)\cdots a^\dagger(Sf_j)\cdots a^\dagger(f_n)\Omega. \tag{1.2}$$

From (1.2) it follows that, for $f \in D(S)$,

$$[d\Gamma(S), a(f)] = -a(Sf), \tag{1.3}$$

$$[d\Gamma(S), a^\dagger(f)] = a^\dagger(Sf) \tag{1.4}$$

on $\mathcal{F}_{\text{fin}}^{D(S)}$. It is known that $d\Gamma(S)$ is essentially self-adjoint. The self-adjoint extension of $d\Gamma(S)$ is denoted by the same symbol $d\Gamma(S)$. It can be seen that unitary operator $e^{itd\Gamma(S)}$ acts as

$$e^{itd\Gamma(S)}a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega = a^\dagger(e^{itS}f_1)\cdots a^\dagger(e^{itS}f_n)\Omega.$$

Thus we see that that

$$e^{itd\Gamma(S)}a(f)e^{-itd\Gamma(S)} = a(e^{-itS}f), \tag{1.5}$$

$$e^{itd\Gamma(S)}a^\dagger(f)e^{-itd\Gamma(S)} = a^\dagger(e^{itS}f) \tag{1.6}$$

on $\mathcal{F}_{\rm fin}$. For a self-adjoint operator $T$, we write its spectrum (resp. essential spectrum, point spectrum) as $\sigma(T)$ (resp. $\sigma_{\rm ess}(T)$, $\sigma_{\rm p}(T)$). It is known that

$$\sigma(d\Gamma(S)) = \overline{\{0\}\bigcup\cup_{n=1}^\infty\left\{\sum_{j=1}^n \lambda_j \middle| \lambda_j \in \sigma(S), j=1,...,n\right\}}, \tag{1.7}$$

$$\sigma_{\rm p}(d\Gamma(S)) = \{0\}\bigcup\cup_{n=1}^\infty\left\{\sum_{j=1}^n \lambda_j \middle| \lambda_j \in \sigma_{\rm p}(S), j=1,...,n\right\}, \tag{1.8}$$

where $\overline{\{\cdots\}}$ denotes the closure of set $\{\cdots\}$. The second quantization of the identity operator 1 on $\mathcal{W}$, $d\Gamma(1)$, is referred to as the number operator, which is written as

$$N := d\Gamma(1).$$

We note that

$$D(N^k) = \{\Psi = \{\Psi^{(n)}\}_{n=0}^\infty | \sum_{n=0}^\infty n^{2k}\|\Psi^{(n)}\|^2 < \infty\}.$$

From (1.7) and (1.8) it follows that

$$\sigma(N) = \sigma_{\rm p}(N) = \mathbb{N}\cup\{0\}.$$

## 1.2 Abstract interaction systems

Let $\mathcal{H}$ be a Hilbert space. A Hilbert space for an abstract coupled system is given by

$$\mathcal{F} := \mathcal{H}\otimes\mathcal{F}_{\rm b},$$

and a decoupled Hamiltonian $H_0$ acting in $\mathcal{F}$ is of the form

$$H_0 = A\otimes 1 + 1\otimes d\Gamma(S).$$

Assumptions **(A1)** and **(A2)** are as follows.

**(A1)** Operator $A$ is a self-adjoint operator acting in $\mathcal{H}$, and bounded from below.

**(A2)** Operator $S$ is a nonnegative self-adjoint operator acting in $\mathcal{W}$.

Total Hamiltonians under consideration are of the form

$$H = H_0 + gH_{\rm I}, \tag{1.9}$$

where $g \in \mathbb{R}$ denotes a coupling constant and $H_{\rm I}$ a symmetric operator. Assumption **(A3)** is as follows.

**(A3)** $H_{\rm I}$ is $H_0$-bounded with

$$\|H_{\rm I}\Psi\| \leq a\|H_0\Psi\| + b\|\Psi\|, \quad \Psi \in D(H_0),$$

where $a$ and $b$ are nonnegative constants.

Under **(A3)**, by the Kato-Rellich theorem, $H$ is self-adjoint on $D(H_0)$ and bounded from below for $g$ with $|g| < 1/a$. Moreover $H$ is essentially self-adjoint on any core of $H_0$. The bottom of $\sigma(H)$ is denoted by

$$E(H) := \inf \sigma(H),$$

which is referred to as the ground state energy of $H$. If an eigenvector $\Psi$ associated with $E(H)$ exists, i.e.,

$$H\Psi = E(H)\Psi,$$

then $\Psi$ is called a ground state of $H$. Let $E_T(B)$ be the spectral projection of self-adjoint operator $T$ onto a Borel set $B \subset \mathbb{R}$. We set

$$P_T := E_T(\{E(T)\}).$$

Then $P_H$ denotes the projection onto the subspace spanned by ground states of $H$. The dimension of $P_H\mathcal{F}$ is called the multiplicity of ground states of $H$, and it is denoted by

$$\mathrm{m}(H) := \dim\, P_H\mathcal{F}.$$

If $\mathrm{m}(H) = 1$, then we call that the ground state of $H$ is unique.

## 1.3 Expectation values of the number operator

For Hamiltonians like as (1.9), the existence of a ground state $\varphi_{\mathrm{g}}$ such that

$$\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2}) \tag{1.10}$$

has been shown by many authors, e.g., [3, 8, 9, 14, 15, 23, 37]. Conversely, if $\varphi_{\mathrm{g}}$ exists, little attention, however, has been given to investigate whether (1.10) holds or not. Then the first task in this paper is to give a necessary and sufficient condition for

$$P_H \mathcal{F} \subset D(1 \otimes N^{1/2}). \tag{1.11}$$

As we will see later, to show (1.11) is also the primary problem in estimating an upper bound of $\mathrm{m}(H)$.

## 1.4 Massive and massless cases

Typical examples of Hilbert space $\mathcal{W}$ and nonnegative self-adjoint operator $S$ are

$$\mathcal{W} = L^2(\mathbb{R}^d), \tag{1.12}$$

$$S = \text{ the multiplication operator by } \omega_\nu(k) := \sqrt{|k|^2 + \nu^2}. \tag{1.13}$$

In the case of $\nu > 0$ (resp. $\nu = 0$), a model is referred to as a *massive* (resp. *massless*) model. Note that under **(A1)** and **(A3)**,

$$D(H) = D(H_0) = D(A \otimes 1) \cap D(1 \otimes d\Gamma(\omega_\nu)). \tag{1.14}$$

In a massive case, one can see that (1.11) is always satisfied. Actually in a massive case, we have $D(d\Gamma(\omega_\nu)) \subset D(N)$ and

$$\frac{1}{\nu}\|d\Gamma(\omega_\nu)\Psi\| \geq \|N\Psi\|, \quad \Psi \in D(d\Gamma(\omega_\nu)).$$

Together with (1.14) we obtain that

$$P_H \mathcal{F} \subset D(H) \subset D(1 \otimes d\Gamma(\omega_\nu)) \subset D(1 \otimes N) \subset D(1 \otimes N^{1/2}).$$

Hence (1.11) follows. Kernel $a(k)$ of $a(f)$, $f \in L^2(\mathbb{R}^d)$, is defined for each $k \in \mathbb{R}^d$ as

$$\left(a(k)\Psi\right)^{(n)}(k_1, ..., k_n) = \sqrt{n+1}\,\Psi^{(n+1)}(k, k_1, ..., k_n)$$

and

$$(a(f)\Psi)^{(n)} = \int f(k)(a(k)\Psi)^{(n)}dk$$

for $\Psi \in \mathcal{F}_{\mathrm{fin}}^{C_0^\infty(\mathbb{R}^d)}$, and it is directly seen that

$$\int_{\mathbb{R}^d} \|a(k)\Psi\|^2 dk = \|N^{1/2}\Psi\|^2, \quad \Psi \in \mathcal{F}_{\mathrm{fin}}^{C_0^\infty(\mathbb{R}^d)}. \tag{1.15}$$

From (1.15), $a(\cdot)\Psi$ for $\Psi \in D(N^{1/2})$ can be defined as an $\mathcal{F}_{\mathrm{b}}$-valued $L^2$ function on $\mathbb{R}^d$ by

$$a(\cdot)\Psi := s - \lim_{m\to\infty} a(\cdot)\Psi_m \quad in \quad L^2(\mathbb{R}^d; \mathcal{F}_{\mathrm{b}}),$$

where $s-\lim_{m\to\infty}$ denotes the strong limit in $L^2(\mathbb{R}^d; \mathcal{F}_{\mathrm{b}})$ and sequence $\Psi_m \in \mathcal{F}_{\mathrm{fin}}^{C_0^\infty(\mathbb{R}^d)}$ is such that $\Psi_m \to \Psi$ and $N^{1/2}\Psi_m \to N^{1/2}\Psi$ strongly as $m \to \infty$. By an *informal* calculation, it can be derived *pointwise* that

$$(1 \otimes a(k))\varphi_{\mathrm{g}} = g(H - E(H) + \omega(k))^{-1}[H_{\mathrm{I}}, 1 \otimes a(k)]\varphi_{\mathrm{g}}. \tag{1.16}$$

Note that at least we have to assume $\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2})$ for (1.16) to make a sense, and the right-hand side of (1.16) is also delicate. See e.g., [36, Lemma 2.6] and [12, p.170, *Conclusion* ] for this point. For massive cases, $(1 \otimes a(\cdot))\varphi_{\mathrm{g}}$ is well defined as an $\mathcal{F}$-valued $L^2$ function on $\mathbb{R}^d$, since $\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2})$, but of course it does not make sense pointwise. From (1.16) and (1.15) it follows that

$$\|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\|^2 = g^2 \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}[H_{\mathrm{I}}, 1 \otimes a(k)]\varphi_{\mathrm{g}}\|^2 dk. \tag{1.17}$$

We may say under some conditions that

$$\begin{aligned} &\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2}) \text{ and } \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}[H_{\mathrm{I}}, 1 \otimes a(k)]\varphi_{\mathrm{g}}\|^2 dk < \infty \\ &\Longrightarrow \|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\|^2 = g^2 \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}[H_{\mathrm{I}}, 1 \otimes a(k)]\varphi_{\mathrm{g}}\|^2 dk. \end{aligned}$$

Although (1.17) has been applied to study $\|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\|$ by many authors, it must be noted again that (1.17) is derived from *informal* formula (1.16).

We are most interested in analysis of ground states for massless cases. In this case $\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2})$ is not clear, and it is also not clear a priori that $(1 \otimes a(k))\varphi_{\mathrm{g}}$ makes a sense. Then it is uncertain that identity (1.16) holds true for massless cases in some sense. Furthermore the fact that the right-hand side of (1.17) is finite does not play a role in a criterion for whether $\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2})$ or not, since, at least, we have to assume $\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2})$ in (1.16).

Because of the tedious argument involved in establishing (1.16) pointwise, a quite different method is taken to show (1.17) in this paper. We will show under some conditions that

$$\varphi_{\rm g} \in D(1 \otimes N^{1/2}) \Longleftrightarrow \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}[H_{\rm I}, 1 \otimes a(k)]\varphi_{\rm g}\|^2 dk < \infty, \quad (1.18)$$

and (1.17) follows when the right or left-hand side of (1.18) holds. The method is an application of the fact that asymptotic annihilation operators vanish arbitrary ground states. See (1.23). As a result, (1.17) and (1.18) can be valid rigorously for both massive and massless cases without using (1.16). As far as we know, this method is new, cf., see [5, 6, 19, 20]. By means of (1.18) we can find a condition for $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$.

## 1.5 Multiplicity

Generally, in the case where $E(H)$ is discrete, the min-max principle [35] is available to estimate the multiplicity of ground states. Actually the ground state energy of a *massive* generalized-spin-boson (GSB) model with a sufficiently weak coupling is discrete. Hence the min-max principle can be applied for this model [3]. However, for some typical models, e.g., massless GSB models, the Pauli-Fierz model, and the Nelson model [31], etc., their ground state energy is the edge of the essential spectrum, namely it is not discrete. See also [2, 22]. Then the min-max principle does not work at all.

Instead of the min-max principle, we can apply an infinite dimensional version of the Perron-Frobenius theorem [16, 17] to show the uniqueness of its ground state. I.e., in a Schrödinger representation,

$$(\Psi, e^{-tH}\Phi) > 0, \quad \Psi \geq 0 \ (\not\equiv 0), \quad \Phi \geq 0 \ (\not\equiv 0), \qquad (1.19)$$

implies $\mathrm{m}(H) = 1$. Property (1.19) is called that $e^{-tH}$ is positivity-improving. The Perron-Frobenius theorem has been applied for some models, e.g., the Nelson model in [8], and the spinless Pauli-Fierz model in [24]. It is, however, for, e.g., the Pauli-Fierz model with spin 1/2, $H_{\rm PF}$, we can not apply the Perron-Frobenius theorem, since, as far as we know, a suitable representation for $e^{-tH_{\rm PF}}$ to be positivity-improving can not be constructed.

In this paper, applying the fact $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$, we establish a wide-usable method to estimate an upper bound of the multiplicity of ground states under some conditions.

## 1.6 Main results and strategies

The main results are (m1) and (m2).

**(m1)** We give a necessary and sufficient condition for $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$.

**(m2)** We prove $\mathrm{m}(H) \leq \mathrm{m}(A)$ under some conditions.

Strategies are as follows. It is proven that

$$\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2}) \Longleftrightarrow \sum_{m=1}^{\infty} \|(1 \otimes a(e_m))\varphi_{\mathrm{g}}\|^2 < \infty, \tag{1.20}$$

where $\{e_m\}_{m=1}^{\infty}$ is an arbitrary complete orthonormal system of $\mathcal{W}$. When the left or right-hand side of (1.20) holds, it follows that

$$\sum_{m=1}^{\infty} \|(1 \otimes a(e_m))\varphi_{\mathrm{g}}\|^2 = \|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\|^2. \tag{1.21}$$

Let us define an asymptotic annihilation operator by

$$a_+(f)\Psi := s - \lim_{t\to\infty} e^{-itH} e^{itH_0}(1 \otimes a(f))e^{-itH_0}e^{itH}\Psi. \tag{1.22}$$

Of course some conditions on $\Psi$ and $f$ are required to show the existence of $a_+(f)\Psi$. It is well known [1, 18], however, that (1.22) exists for an arbitrary ground state of $H$, $\Psi = \varphi_{\mathrm{g}}$, and $a_+(f)$ vanishes $\varphi_{\mathrm{g}}$, i.e.,

$$a_+(f)\varphi_{\mathrm{g}} = 0 \tag{1.23}$$

for some $f$. (1.23) is applied for (m1). We decompose $a_+(f)\Psi$ as

$$a_+(f)\Psi = (1 \otimes a(f))\Psi - gG(f)\Psi$$

with some operator $G(f) : \mathcal{F} \to \mathcal{F}$. From (1.23) it follows that

$$(1 \otimes a(f))\varphi_{\mathrm{g}} = gG(f)\varphi_{\mathrm{g}}. \tag{1.24}$$

We define the bounded operator $T_{\varphi_{\mathrm{g}}} : \mathcal{W} \to \mathcal{F}$ by

$$T_{\varphi_{\mathrm{g}}} f := G(f)\varphi_{\mathrm{g}}, \quad f \in \mathcal{W}. \tag{1.25}$$

I.e.,

$$(1 \otimes a(f))\varphi_{\mathrm{g}} = gT_{\varphi_{\mathrm{g}}} f. \tag{1.26}$$

It is seen that $T_{\varphi_{\rm g}}$ is an $\mathcal{F}$-valued integral operator such that

$$T_{\varphi_{\rm g}} f = \int_{\mathbb{R}^d} f(k)\kappa_{\varphi_{\rm g}}(k)dk$$

with some kernel $\kappa_{\varphi_{\rm g}}(k) \in \mathcal{F}$. See (2.29) for details. Note that

$$\sum_{m=1}^{\infty} \|T_{\varphi_{\rm g}} e_m\|^2 = \mathrm{Tr}(T^*_{\varphi_{\rm g}} T_{\varphi_{\rm g}}) = \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}}(k)\|^2 dk. \tag{1.27}$$

Using (1.20), (1.26) and (1.27), we see that

$$\varphi_{\rm g} \in D(1 \otimes N^{1/2}) \Longleftrightarrow g^2 \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}}(k)\|^2 dk < \infty,$$

and by (1.21),

$$\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = g^2 \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}}(k)\|^2 dk. \tag{1.28}$$

Thus we can obtain that

$$P_H\mathcal{F} \subset D(1 \otimes N^{1/2}) \Longleftrightarrow \int \|\kappa_{\varphi_{\rm g}}(k)\|^2 dk < \infty \text{ for all } \varphi_{\rm g} \in P_H\mathcal{F}.$$

To show (m2) we apply the method in [27], by which we can prove that

$$\dim(P_H\mathcal{F} \cap D(1 \otimes N^{1/2})) \leq \frac{1}{1-\delta(g)} \mathrm{m}(A),$$

where

$$\delta(g) = o(g) + \sup_{\varphi_{\rm g} \in P_H\mathcal{F} \cap D(1\otimes N^{1/2})} \frac{\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2}{\|\varphi_{\rm g}\|^2}.$$

By (1.28) and the fact

$$\lim_{g\to 0} \sup_{\varphi_{\rm g} \in P_H\mathcal{F}} \frac{\int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}}(k)\|^2 dk}{\|\varphi_{\rm g}\|^2} < \infty,$$

we see that $\lim_{g\to 0} \delta(g) = 0$. Hence for sufficiently small $g$,

$$\dim(P_H\mathcal{F} \cap D(1 \otimes N^{1/2})) \leq \mathrm{m}(A)$$

is proven. Together with the fact $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ under some conditions, we get

$$\mathrm{m}(H) = \dim P_H\mathcal{F} \leq \mathrm{m}(A).$$

We organize this paper as follows.

Section 2 is devoted to show $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$. In Section 3, we estimate the multiplicity of ground states. In Sections 4, we give examples including massless GSB models and the Pauli-Fierz model. Finally in Section 5 we give appendixes.

## 2 Equivalent conditions to $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$

### 2.1 The number operator

Let $\{e_m\}_{m=1}^{\infty}$ be a complete orthonormal system of $\mathcal{W}$. We define

$$A_M, \quad M = 1, 2, ...,$$

by

$$A_M := (N+1)^{-1/2}\left(\sum_{m=1}^{M} a^\dagger(e_m)a(\overline{e_m})\right)(N+1)^{-1/2}.$$

**Lemma 2.1** *We have*

**(1)** *$A_M$ has a unique bounded operator extension $\overline{A_M}$,*

**(2)** *$\overline{A_M}$ is uniformly bounded in $M$ as $\|\overline{A_M}\| \le 1$,*

**(3)** *$s - \lim_{M\to\infty} A_M = N(N+1)^{-1}$.*

*Proof:* Let us define

$$\mathcal{F}_\omega := \left[\bigoplus_{n=0}^{\infty}\left\{\sum_{i_1\le\cdots\le i_n}^{\text{finite}} \alpha_{i_1,\dots,i_n} a^\dagger(e_{i_1})\cdots a^\dagger(e_{i_n})\Omega \,\middle|\, \alpha_{i_1,\dots,i_n} \in \mathbb{C}\right\}\right] \bigcap \mathcal{F}_{\text{fin}}.$$

Note that $\mathcal{F}_\omega$ is dense in $\mathcal{F}_{\text{b}}$. Let

$$\phi = a^\dagger(e_{i_1})\cdots a^\dagger(e_{i_n})\Omega, \quad i_1 \le \cdots \le i_n.$$

Then

$$\begin{aligned} A_M\phi &= \frac{1}{n+1}\sum_{j=1}^{n} a^\dagger(e_{i_1})\cdots a^\dagger(\sum_{m=1}^{M}(e_m, e_{i_j})e_m)\cdots a^\dagger(e_{i_n})\Omega \\ &= \beta_{i_1,\dots,i_n}(M)\phi, \end{aligned} \tag{2.1}$$

where

$$\beta_{i_1,\dots,i_n}(M) := \begin{cases} \frac{n}{n+1}, & i_n \le M, \\ \frac{n-1}{n+1}, & i_{n-1} \le M < i_n, \\ \vdots & \vdots \\ \frac{1}{n+1}, & i_1 \le M < i_2, \\ 0, & M < i_1. \end{cases}$$

Let $\Psi \in \mathcal{F}_\omega$ be such that

$$\Psi = \sum_{i_1 \le \cdots \le i_n}^{\text{finite}} \alpha_{i_1,\ldots,i_n} a^\dagger(e_{i_1}) \cdots a^\dagger(e_{i_n})\Omega. \tag{2.2}$$

We see that

$$\|\Psi\|^2 = \sum_{i_1 \le \cdots \le i_n}^{\text{finite}} |\alpha_{i_1,\ldots,i_n}|^2.$$

From (2.1) it follows that

$$A_M \Psi = \sum_{i_1 \le \cdots \le i_n}^{\text{finite}} \alpha_{i_1,\ldots,i_n} \beta_{i_1,\ldots,i_n}(M) a^\dagger(e_{i_1}) \cdots a^\dagger(e_{i_n}).$$

Then

$$\begin{aligned} 0 \ &\le \ \|A_M \Psi\|^2 = \sum_{i_1 \le \cdots \le i_n}^{\text{finite}} |\alpha_{i_1,\ldots,i_n}|^2 |\beta_{i_1,\ldots,i_n}(M)|^2 \\ &\le \ \sum_{i_1 \le \cdots \le i_n}^{\text{finite}} |\alpha_{i_1,\ldots,i_n}|^2 \left(\frac{n}{n+1}\right)^2 = \left(\frac{n}{n+1}\right)^2 \|\Psi\|^2. \end{aligned}$$

Note that $A_M$ leaves $\otimes_s^n \mathcal{W}$ invariant. Hence for an arbitrary $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_\omega$, we have

$$\|A_M\Psi\|^2 = \sum_{n=0}^\infty \|(A_M\Psi)^{(n)}\|^2 = \sum_{n=0}^\infty \|A_M \Psi^{(n)}\|^2 \le \sum_{n=0}^\infty \left(\frac{n}{n+1}\right)^2 \|\Psi^{(n)}\|^2 \le \|\Psi\|^2.$$

Since $\mathcal{F}_\omega$ is dense in $\mathcal{F}_{\mathrm{b}}$, (1) and (2) follow. Let $\Psi$ be as (2.2). We see that

$$s - \lim_{M\to\infty} A_M \Psi = \frac{n}{n+1}\Psi.$$

Hence for an arbitrary $\Phi \in \mathcal{F}_\omega$,

$$s - \lim_{M\to\infty} A_M \Phi = N(N+1)^{-1}\Phi.$$

For an arbitrary $\Phi \in \mathcal{F}_{\mathrm{b}}$ and an arbitrary $\epsilon > 0$, we can choose $\Phi_\epsilon \in \mathcal{F}_\omega$ such that

$$\|\Phi - \Phi_\epsilon\| < \epsilon.$$

Since $\|\overline{A_M}\| \le 1$, we obtain that

$$\begin{aligned} &\|\overline{A_M}\Phi - N(N+1)^{-1}\Phi\| \\ &\le \|\overline{A_M}\Phi - \overline{A_M}\Phi_\epsilon\| + \|\overline{A_M}\Phi_\epsilon - N(N+1)^{-1}\Phi_\epsilon\| + \|N(N+1)^{-1}(\Phi_\epsilon - \Phi)\| \\ &\le 2\epsilon + \|\overline{A_M}\Phi_\epsilon - N(N+1)^{-1}\Phi_\epsilon\|. \end{aligned}$$

Then

$$\lim_{M\to\infty} \|\overline{A_M}\Phi - N(N+1)^{-1}\Phi\| < 2\epsilon$$

for an arbitrary $\epsilon$. Thus (3) follows. $\square$

**Lemma 2.2** *Let $\{e_m\}_{m=1}^\infty$ be an arbitrary complete orthonormal system in $\mathcal{W}$. Then (1) and (2) are equivalent.*

**(1)** $\Psi \in D(N^{1/2})$.

**(2)** $\Psi \in \cap_{m=1}^\infty D(a(\overline{e_m}))$ *and*

$$\sum_{m=1}^\infty \|a(\overline{e_m})\Psi\|^2 < \infty. \tag{2.3}$$

*Moreover when (1) or (2) holds, it follows that*

$$\|N^{1/2}\Psi\|^2 = \sum_{m=1}^\infty \|a(\overline{e_m})\Psi\|^2.$$

*Proof:* (1) $\Rightarrow$ (2)

We see that $\mathcal{F}_\omega \subset D(N^{1/2})$ and

$$e^{-tN^{1/2}}\mathcal{F}_\omega \subset \mathcal{F}_\omega,$$

which implies that $\mathcal{F}_\omega$ is a core of $N^{1/2}$ by [33, X.49]. Then, for $\Psi \in D(N^{1/2})$, there exists a sequence $\Psi_\epsilon \in \mathcal{F}_\omega$ such that $s-\lim_{\epsilon\to 0}\Psi_\epsilon = \Psi$ and $s-\lim_{\epsilon\to 0} N^{1/2}\Psi_\epsilon = N^{1/2}\Psi$. It is well known that

$$\|a(f)\Phi\| \leq \|f\|\|N^{1/2}\Phi\|, \quad \Phi \in D(N^{1/2}).$$

Hence from the fact $\Psi \in D(N^{1/2})$, it follows that $\Psi \in D(a(\overline{e_m}))$. We have

$$\begin{aligned}
\sum_{m=1}^M \|a(\overline{e_m})\Psi_\epsilon\|^2 &= ((N+1)^{1/2}\Psi_\epsilon, A_M(N+1)^{1/2}\Psi_\epsilon) \\
&\leq \|(N+1)^{1/2}\Psi_\epsilon\|^2 \\
&\leq \|N^{1/2}\Psi_\epsilon\|^2 + \|\Psi_\epsilon\|^2.
\end{aligned} \tag{2.4}$$

From this it follows that $a(\overline{e_m})\Psi_\epsilon$ is a Cauchy sequence in $\epsilon$. Since $a(\overline{e_m})$ is a closed operator, $\lim_{\epsilon\to 0} a(\overline{e_m})\Psi_\epsilon = a(\overline{e_m})\Psi$ follows. Hence we obtain that, as $\epsilon \to 0$ on the both sides of (2.4),

$$\sum_{m=1}^M \|a(\overline{e_m})\Psi\|^2 \leq \|N^{1/2}\Psi\|^2 + \|\Psi\|^2.$$

Taking $M \to \infty$ on the both sides above, we have

$$\sum_{m=1}^{\infty} \|a(\overline{e_m})\Psi\|^2 \leq \|N^{1/2}\Psi\|^2 + \|\Psi\|^2.$$

Thus the desired results follow.

(2) $\Rightarrow$ (1) We see that

$$\begin{aligned} \sum_{m=1}^{\infty} \|a(\overline{e_m})\Psi\|^2 &= \lim_{M\to\infty} \sum_{m=1}^{M} \sum_{n=0}^{\infty} (a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)}) \\ &= \lim_{M\to\infty} \sum_{n=0}^{\infty} \sum_{m=1}^{M} (a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)}). \end{aligned}$$

Since $\sum_{m=1}^{M} (a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)})$ is monotonously increasing as $M \uparrow \infty$ and by (2.3),

$$\lim_{M\to\infty} \sum_{n=0}^{\infty} \sum_{m=1}^{M} (a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)}) < \infty,$$

we have by the Lebesgue monotone convergence theorem and (3) of Lemma 2.1,

$$\begin{aligned} &\lim_{M\to\infty} \sum_{n=0}^{\infty} \sum_{m=1}^{M} (a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)}) \\ &= \sum_{n=0}^{\infty} \lim_{M\to\infty} \sum_{m=1}^{M} (a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)}) \\ &= \sum_{n=0}^{\infty} \lim_{M\to\infty} ((N+1)^{1/2}\Psi^{(n)}, \overline{A_M}(N+1)^{1/2}\Psi^{(n)}) \\ &= \sum_{n=0}^{\infty} n(\Psi^{(n)}, \Psi^{(n)}) \\ &= \sum_{n=0}^{\infty} n\|\Psi^{(n)}\|^2 < \infty. \end{aligned}$$

This yields that $\Psi \in D(N^{1/2})$. $\square$

## 2.2 Weak commutators

In subsections 2.2-2.4, we consider the case where

$$\mathcal{W} = \oplus^D L^2(\mathbb{R}^d) \cong L^2(\mathbb{R}^d \times \{1, ..., D\}) \tag{2.5}$$

and

$$S = [\omega] \tag{2.6}$$

such that $[\omega] : \oplus^D L^2(\mathbb{R}^d) \to \oplus^D L^2(\mathbb{R}^d)$ is the multiplication operator defined by

$$[\omega](\oplus_{j=1}^D f_j) = \oplus_{j=1}^D \omega f_j, \tag{2.7}$$

with

$$\omega(\cdot) : \mathbb{R}^d \to [0, \infty), \qquad (\omega f)(k) = \omega(k) f(k).$$

The creation operator and the annihilation operator of $\mathcal{F}_{\mathrm{b}}(\mathcal{W})$ are denoted by

$$a^\sharp(f, j) := a(0 \oplus \cdots \oplus \overset{j}{f} \oplus \cdots \oplus 0), \quad f \in L^2(\mathbb{R}^d), \quad j = 1, ..., D,$$

which satisfy on $\mathcal{F}_{\mathrm{fin}}$,

$$\begin{aligned}
&[a(f, j), a^\dagger(g, j')] = (\bar{f}, g)\delta_{jj'}, \\
&[a^\dagger(f, j), a^\dagger(g, j')] = 0, \\
&[a(f, j), a(g, j')] = 0.
\end{aligned}$$

By (1.5)-(1.4),

$$e^{it(1\otimes d\Gamma([\omega]))}(1 \otimes a(f, j))e^{-it(1\otimes d\Gamma([\omega]))} = 1 \otimes a(e^{-it\omega} f, j), \tag{2.8}$$
$$e^{it(1\otimes d\Gamma([\omega]))}(1 \otimes a^\dagger(f, j))e^{-it(1\otimes d\Gamma([\omega]))} = 1 \otimes a^\dagger(e^{it\omega} f, j) \tag{2.9}$$

follow on $\mathcal{H} \otimes \mathcal{F}_{\mathrm{fin}}$, and

$$[1 \otimes d\Gamma([\omega]), 1 \otimes a(f, j)] = -1 \otimes a(\omega f, j), \tag{2.10}$$
$$[1 \otimes d\Gamma([\omega]), 1 \otimes a^\dagger(f, j)] = 1 \otimes a^\dagger(\omega f, j) \tag{2.11}$$

follow for $f \in D(\omega)$ on $\mathcal{H} \otimes \mathcal{F}_{\mathrm{fin}}^{D([\omega])}$. Let $S$ and $T$ be operators acting in a Hilbert space $\mathcal{K}$. We define a quadratic form $[S, T]_W^{\mathcal{D}}$ with a form domain

$$\mathcal{D} \subset D(S^*) \cap D(S) \cap D(T^*) \cap D(T)$$

by

$$[S, T]_W^D(\Psi, \Phi) := (S^*\Psi, T\Phi) - (T^*\Psi, S\Phi), \quad \Psi, \Phi \in \mathcal{D}.$$

**Proposition 2.3** *(1) Let $f, f/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. Then (2.8) and (2.9) can be extended on $D(1 \otimes d\Gamma([\omega]))$. (2) Let $\omega f, f/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. Then*

$$[1 \otimes d\Gamma([\omega]), 1 \otimes a(f, j)]_W^{D(1\otimes d\Gamma([\omega]))}(\Psi, \Phi) = (\Psi, -(1 \otimes a(\omega f, j))\Phi), \tag{2.12}$$
$$[1 \otimes d\Gamma([\omega]), 1 \otimes a^\dagger(f, j)]_W^{D(1\otimes d\Gamma([\omega]))}(\Psi, \Phi) = (\Psi, (1 \otimes a^\dagger(\omega f, j))\Phi). \tag{2.13}$$

*Proof:* Let $\Psi \in D(1 \otimes d\Gamma([\omega]))$. Since $\mathcal{H} \otimes \mathcal{F}_{\text{fin}}$ is a core of $1 \otimes d\Gamma([\omega])$, there exists a sequence $\Psi_\epsilon$ such that $\Psi_\epsilon \to \Psi$ and $(1 \otimes d\Gamma([\omega]))\Psi_\epsilon \to (1 \otimes d\Gamma([\omega]))\Psi$ strongly as $\epsilon \to 0$. It follows that

$$e^{it(1\otimes d\Gamma([\omega]))}(1 \otimes a(f,j))e^{-it(1\otimes d\Gamma([\omega]))}\Psi_\epsilon = 1 \otimes a(e^{-it\omega}f,j)\Psi_\epsilon, \tag{2.14}$$
$$e^{it(1\otimes d\Gamma([\omega]))}(1 \otimes a^\dagger(f,j))e^{-it(1\otimes d\Gamma([\omega]))}\Psi_\epsilon = 1 \otimes a^\dagger(e^{it\omega}f,j)\Psi_\epsilon. \tag{2.15}$$

Using well known inequalities

$$\begin{aligned}&\|(1 \otimes a(f,j))\Psi\| \le \|f/\sqrt{\omega}\|\|(1 \otimes d\Gamma([\omega])^{1/2})\Psi\|,\\ &\|(1 \otimes a^\dagger(f,j))\Psi\| \le \|f/\sqrt{\omega}\|\|(1 \otimes d\Gamma([\omega])^{1/2})\Psi\| + \|f\|\|\Psi\|,\end{aligned}$$

we see that the both hand sides of (2.14) and (2.15) converge strongly as $\epsilon \to 0$. Since $1 \otimes a^\sharp(f,j)$ is closed, (1) follows.

We shall prove (2). Let $\Psi, \Phi \in \mathcal{H} \otimes \mathcal{F}_{\text{fin}}^{D([\omega])}$. Then

$$\begin{aligned}((1 \otimes d\Gamma([\omega]))\Psi, (1 \otimes a(f,j))\Phi) - ((1 \otimes a^\dagger(\bar{f},j))\Psi, (1 \otimes d\Gamma([\omega]))\Phi)&\\ = (\Psi, -(1 \otimes a(f,j))\Phi).&\end{aligned}$$

Since $\mathcal{H} \otimes \mathcal{F}_{\text{fin}}^{D([\omega])}$ is a core of $1 \otimes d\Gamma([\omega])$, there exists a sequence $\Psi_\epsilon$ such that $\Psi_\epsilon \to \Psi$ and $(1 \otimes d\Gamma([\omega]))\Psi_\epsilon \to (1 \otimes d\Gamma([\omega]))\Psi$ strongly as $\epsilon \to 0$. By using the closedness of $1 \otimes a(f,j)$ and a similar limiting argument as that of (1), we obtain (2.12). (2.13) can be similarly proven. $\square$

## 2.3 Asymptotic fields

Define on $D(H)$,

$$\begin{aligned}a_t(f,j) &:= e^{-itH}e^{itH_0}(1 \otimes a(f,j))e^{-itH_0}e^{itH}\\ &= e^{-itH}(1 \otimes a(e^{-it\omega}f,j))e^{itH}.\end{aligned}$$

Note that

$$H_0 = A \otimes 1 + 1 \otimes d\Gamma([\omega]),$$

Assumption (**B1**) is as follows.

**(B1)** $\omega$ satisfies (1) and (2).

**(1)** The Lebesgue measure of $K_\omega := \{k \in \mathbb{R}^d | \omega(k) = 0\}$ is zero.

**(2)** There exists a subset $K \subset \mathbb{R}^d$ with Lebesgue measure zero such that

$$\omega \in C^3(\mathbb{R}^d \setminus K)$$

and

$$\frac{\partial \omega}{\partial k_n}(k) \neq 0, \quad n = 1, ..., d, \quad k = (k_1, ..., k_d) \in \mathbb{R}^d \setminus K.$$

**Example 2.4** *A typical example of $\omega$ is $\omega(k) = |k|^p$ with $p > 0$. In this case*

$$K_\omega = \{0\}$$

*and*

$$K = \bigcup_{n=1}^{d} \{(k_1, ..., k_d) \in \mathbb{R}^d | k_n = 0\}.$$

**Lemma 2.5** *Suppose* **(2)** *of* **(B1)***. Then for $f \in C_0^2(\mathbb{R}^d \setminus K)$,*

$$\left| \int_{\mathbb{R}^d} e^{is\omega(k)} f(k) dk \right| \leq \frac{c}{s^2}$$

*with some constant $c$.*

*Proof:* We have, for $1 \leq m, n \leq d$,

$$e^{is\omega} = -\frac{1}{s^2} \left( \frac{\partial \omega}{\partial k_n} \right)^{-1} \frac{\partial}{\partial k_n} \left( \left( \frac{\partial \omega}{\partial k_m} \right)^{-1} \frac{\partial e^{is\omega}}{\partial k_m} \right)$$

on $\mathbb{R}^d \setminus K$. Hence it follows that by integration by parts,

$$\int_{\mathbb{R}^d} e^{is\omega(k)} f(k) dk = -\frac{1}{s^2} \int_{\mathbb{R}^d} e^{is\omega(k)} \frac{\partial}{\partial k_m} \left( \left( \frac{\partial \omega}{\partial k_m} \right)^{-1} \frac{\partial}{\partial k_n} \left( \left( \frac{\partial \omega}{\partial k_n} \right)^{-1} f(k) \right) \right) dk.$$

Thus we have

$$\left| \int_{\mathbb{R}^d} e^{is\omega(k)} f(k) dk \right| \leq \frac{1}{s^2} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial k_m} \left( \left( \frac{\partial \omega}{\partial k_m} \right)^{-1} \frac{\partial}{\partial k_n} \left( \left( \frac{\partial \omega}{\partial k_n} \right)^{-1} f(k) \right) \right) \right| dk.$$

Since the integrand of the right-hand side above is integrable, the lemma follows. □

**Proposition 2.6** *Suppose* **(B1)***. Let $f \in C^2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. Then*

$$s - \lim_{t \to \infty} a_t(f, j) \varphi_{\mathrm{g}} = 0, \quad j = 1, ..., D. \tag{2.16}$$

*Proof:* Note that it follows that

$$\|a_t(f,j)\Psi\| \le \|f/\sqrt{\omega}\| \|(1\otimes d\Gamma([\omega])^{1/2})e^{itH}\Psi\|, \quad j=1,...,D,$$

and by the closed graph theorem,

$$\|H_0\Psi\| \le c_1\|H\Psi\| + c_2\|\Psi\|, \quad \Psi \in D(H),$$

with some constants $c_1$ and $c_2$, and

$$\|(1\otimes d\Gamma([\omega])^{1/2})\Psi\| \le c_3\|(H_0+1)\Psi\|$$

with some constant $c_3$. Thus it follows that

$$\|a_t(f,j)\Psi\| \le c_4\|f/\sqrt{\omega}\| \|(H+1)\Psi\| \tag{2.17}$$

with some constant $c_4$. Let $\mathcal{D}$ be a core of $A$ and

$$\Psi = G\otimes a^\dagger(f_1,j_1)\cdots a^\dagger(f_n,j_n)\Omega, \tag{2.18}$$

where $G \in \mathcal{D}$ and $f_l \in C_0^\infty(\mathbb{R}^d \setminus K)$, $l=1,...,n$. We see that for an arbitrary $\delta \in \mathbb{R}$,

$$a(e^{-it(\omega-\delta)}f,j)\Psi = \sum_{l=1}^n (e^{it(\omega-\delta)}\bar{f}, f_l) G\otimes a^\dagger(f_1,j_1)\cdots a^\dagger\widehat{(f_l,j_l)}\cdots a^\dagger(f_n,j_n)\Omega,$$

where $\widehat{X}$ means neglecting $X$. Since $ff_l \in C_0^2(\mathbb{R}^d\setminus K)$, by Lemma 2.5 we see that

$$|(e^{it(\omega-\delta)}\bar{f}, f_l)| \le \frac{c_5}{|t|^2}$$

with some constant $c_5$. Hence

$$s-\lim_{t\to\infty} a(e^{it(\omega-\delta)}f,j)\Psi = 0$$

follows. Let $\mathcal{E}$ be the set of the linear hull of vectors such as (2.18), which is a core of $H_0$. Thus there exists $\Psi_\epsilon \in \mathcal{E}$ such that

$$\Psi_\epsilon \to \varphi_{\rm g}, \qquad H_0\Psi_\epsilon \to H_0\varphi_{\rm g}$$

strongly as $\epsilon \to 0$, which yields that

$$\lim_{\epsilon\to 0}\|(H_0+1)^{1/2}(\Psi_\epsilon - \varphi_{\rm g})\| = 0.$$

Let $\|(H_0+1)^{1/2}(\Psi_\epsilon - \varphi_{\mathrm{g}})\| < \epsilon$. We obtain that

$$\begin{aligned}
&\|a_t(f,j)\varphi_{\mathrm{g}}\| \\
&= \|(1\otimes a(e^{-it(\omega-E(H))}f,j))\varphi_{\mathrm{g}}\| \\
&\leq \|(1\otimes a(e^{-it(\omega-E(H))}f,j))\Psi_\epsilon\| + \|(1\otimes a(e^{-it(\omega-E(H))}f,j))(\Psi_\epsilon-\varphi_{\mathrm{g}})\| \\
&\leq \|(1\otimes a(e^{-it(\omega-E(H))}f,j))\Psi_\epsilon\| + C\|(H_0+1)^{1/2}(\Psi_\epsilon-\varphi_{\mathrm{g}})\| \\
&\leq \|(1\otimes a(e^{-it(\omega-E(H))}f,j))\Psi_\epsilon\| + C\epsilon.
\end{aligned}$$

Then

$$\lim_{t\to\infty}\|a_t(f,j)\varphi_{\mathrm{g}}\| < C\epsilon$$

for an arbitrary $\epsilon$. Then the proposition follows. □

In addition to **(B1)**, we introduce assumptions **(B2)-(B4)**.

**(B2)** There exists an operator

$$T_j(k):\mathcal{F}\to\mathcal{F},\quad k\in\mathbb{R}^d,\quad j=1,...,D,$$

such that

$$D(T_j(k))\supset D(H),\quad \text{a. e. } k\in\mathbb{R}^d,$$

and

$$[1\otimes a(f,j),H_{\mathrm{I}}]_W^{D(H)}(\Psi,\Phi)=\int_{\mathbb{R}^d} f(k)(\Psi,T_j(k)\Phi)dk.$$

**(B3)** Let $\Psi\in D(H)$ and $f\in C_0^2(\mathbb{R}^d\setminus\widetilde{K})$ with some measurable set $\widetilde{K}\subset\mathbb{R}^d$ such that $K\subset\widetilde{K}$ and its Lebesgue measure is zero. Then

$$\left|\int_{\mathbb{R}^d} dkf(k)(\Psi,e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_{\mathrm{g}})\right|\in L^1([0,\infty),ds).$$

**(B4)** $\|T_j(\cdot)\varphi_{\mathrm{g}}\|\in L^2(\mathbb{R}^d)$.

**Lemma 2.7** *Suppose* **(B1)-(B4)**. *Let $f,f/\sqrt{\omega}\in L^2(\mathbb{R}^d)$. Then it follows that*

$$\int_{\mathbb{R}^d}\|f(k)(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\mathrm{g}}\|dk<\infty \tag{2.19}$$

*and*

$$(1\otimes a(f,j))\varphi_{\mathrm{g}} = g\int_{\mathbb{R}^d} f(k)(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\mathrm{g}}dk. \tag{2.20}$$

*Proof:* Noting that

$$\|(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\rm g}\| \le \|T_j(k)\varphi_{\rm g}\|/\omega(k), \quad k \notin K_\omega,$$

we see that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \|f(k)(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\rm g}\|dk \\
&\le \left(\int_{|k|<1} \frac{|f(k)|^2}{\omega(k)}dk\right)^{1/2} \left(\int_{|k|<1} \omega(k)\|(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\rm g}\|^2 dk\right)^{1/2} \\
&+ \left(\int_{|k|\ge 1} |f(k)|^2 dk\right)^{1/2} \left(\int_{|k|\ge 1} \|(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\rm g}\|^2 dk\right)^{1/2} \\
&\le (\|f/\sqrt{\omega}\| + \|f\|)\|T_j(\cdot)\varphi_{\rm g}\| < \infty.
\end{aligned} \tag{2.21}$$

Then (2.19) follows. We divide a proof of (2.20) into three steps.
**(Step 1)** Let $f \in C_0^2(\mathbb{R}^d \setminus \widetilde{K})$, $f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$, and $\Psi, \Phi \in D(H)$. Then

$$(\Psi, (1\otimes a(f,j))\varphi_{\rm g}) = -ig\int_0^\infty \left(\int_{\mathbb{R}^d} (\Psi, f(k)e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_{\rm g})dk\right) ds. \tag{2.22}$$

*Proof of Step 1*
Let $\Psi, \Phi \in \mathcal{D} := C_0^\infty(\mathbb{R}^d) \otimes D(d\Gamma([\omega]))$. Note that $\mathcal{D}$ is a core of $H$. We see that by (2.12) of Proposition 2.3 and **(B2)**,

$$\begin{aligned}
&\frac{d}{dt}(\Psi, a_t(f,j)\Phi) \\
&= -i(He^{itH}\Psi, (1\otimes a(e^{-it\omega}f))e^{itH}\Phi) - i(e^{itH}\Psi, (1\otimes a(\omega e^{-it\omega}f))e^{itH}\Phi) \\
&\qquad\qquad\qquad + i((1\otimes a^\dagger(e^{it\omega}\bar{f}))e^{itH}\Psi, He^{itH}\Phi) \\
&= -ig(H_{\rm I}e^{itH}\Psi, (1\otimes a(e^{-it\omega}f))e^{itH}\Phi) + ig((1\otimes a^\dagger(e^{it\omega}\bar{f}))e^{itH}\Psi, H_{\rm I}e^{itH}\Phi) \\
&= ig[1\otimes a(e^{-it\omega}f), H_{\rm I}]_W^{D(H)}(e^{itH}\Psi, e^{itH}\Phi) \\
&= ig\int_{\mathbb{R}^d} f(k)e^{-it\omega(k)}(\Psi, e^{-itH}T_j(k)e^{itH}\Phi)dk.
\end{aligned}$$

Then we obtain that for $\Psi, \Phi \in \mathcal{D}$,

$$\begin{aligned}
&(\Psi, a_t(f,j)\Phi) \\
&= (\Psi, (1\otimes a(f,j))\Phi) + ig\int_0^t \left(\int_{\mathbb{R}^d} f(k)e^{-is\omega(k)}(\Psi, e^{-isH}T_j(k)e^{isH}\Phi)dk\right) ds.
\end{aligned} \tag{2.23}$$

Let $\Psi, \Phi \in D(H)$. There exist sequences $\Psi_m, \Phi_n \in \mathcal{D}$ such that $\lim_{m\to\infty}\Psi_m = \Psi$ and $\lim_{n\to\infty}\Phi_n = \Phi$ strongly. (2.23) holds true for $\Psi, \Phi$ replaced by $\Psi_m, \Phi_n$, respectively.

By a simple limiting argument as $m \to \infty$ and then $n \to \infty$, we get (2.23) for $\Psi, \Phi \in D(H)$. By Proposition 2.6 and (2.23) we have

$$\begin{aligned} 0 &= \lim_{t\to\infty} (\Psi, a_t(f,j)\varphi_{\mathrm{g}}) \\ &= (\Psi, (1 \otimes a(f,j))\varphi_{\mathrm{g}}) + ig \int_0^\infty \left( \int_{\mathbb{R}^d} (\Psi, f(k) e^{-is(H-E(H)+\omega(k))} T_j(k)\varphi_{\mathrm{g}}) dk \right) ds. \end{aligned}$$

Thus (2.22) follows. □

**(Step 2)** (2.20) holds true for $f$ such that $f \in C_0^2(\mathbb{R}^d \setminus \widetilde{K})$ and $f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$.

*Proof of Step 2*

By **(B3)** and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} &-ig \int_0^\infty \left( \int_{\mathbb{R}^d} (\Psi, f(k) e^{-is(H-E(H)+\omega(k))} T_j(k)\varphi_{\mathrm{g}}) dk \right) ds \\ &= -ig \lim_{\epsilon\to 0} \int_0^\infty ds e^{-\epsilon s} \left( \int_{\mathbb{R}^d} (\Psi, f(k) e^{-is(H-E(H)+\omega(k))} T_j(k)\varphi_{\mathrm{g}}) dk \right). \end{aligned}$$

By **(B4)**,

$$\begin{aligned} &\int_{\mathbb{R}^d} dk \int_0^\infty \left| e^{-s\epsilon} (\Psi, f(k) e^{-is(H-E(H)+\omega(k))} T_j(k)\varphi_{\mathrm{g}}) \right| ds \\ &\le \|\Psi\| \left( \int_{\mathbb{R}^d} |f(k)| \|T_j(k)\varphi_{\mathrm{g}}\| dk \right) \int_0^\infty e^{-s\epsilon} ds < \infty. \end{aligned}$$

Hence Fubini's theorem yields that $\int dk$ and $\int ds$ can be exchanged, i.e.,

$$\begin{aligned} &-ig \lim_{\epsilon\to 0} \int_0^\infty e^{-\epsilon s} \left( \int_{\mathbb{R}^d} (\Psi, f(k) e^{-is(H-E(H)+\omega(k))} T_j(k)\varphi_{\mathrm{g}}) dk \right) ds \\ &= -ig \lim_{\epsilon\to 0} \int_{\mathbb{R}^d} \left( \int_0^\infty (\Psi, f(k) e^{-is(H-E(H)+\omega(k)-i\epsilon)} T_j(k)\varphi_{\mathrm{g}}) ds \right) dk \\ &= g \lim_{\epsilon\to 0} \int_{\mathbb{R}^d} (\Psi, f(k)(H - E(H) + \omega(k) - i\epsilon)^{-1} T_j(k)\varphi_{\mathrm{g}}) dk. \end{aligned}$$

We can check that, for $k \notin K_\omega$,

$$|(\Psi, f(k)(H - E(H) + \omega(k) - i\epsilon)^{-1} T_j(k)\varphi_{\mathrm{g}})| \le \|\Psi\| |f(k)| \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\mathrm{g}}\|, \tag{2.24}$$

$$\int_{\mathbb{R}^d} |f(k)| \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\mathrm{g}}\| dk \le (\|f/\sqrt{\omega}\| + \|f\|) \|T_j(\cdot)\varphi_{\mathrm{g}}\| < \infty, \tag{2.25}$$

$$s - \lim_{\epsilon\to 0} (H - E(H) + \omega(k) - i\epsilon)^{-1}\varphi_{\mathrm{g}} = (H - E(H) + \omega(k))^{-1}\varphi_{\mathrm{g}}, \tag{2.26}$$

(2.24), (2.25) and (2.26) imply that by the Lebesgue dominated convergence theorem,

$$
\begin{aligned}
& g\lim_{\epsilon\to 0}\int_{\mathbb{R}^d}(\Psi, f(k)(H-E(H)+\omega(k)-i\epsilon)^{-1}T_j(k)\varphi_{\mathrm{g}})dk \\
&= g\int_{\mathbb{R}^d}(\Psi, f(k)(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\mathrm{g}})dk.
\end{aligned}
$$

Since, by (2.25) we have

$$
\begin{aligned}
(\Psi, a(f,j)\varphi_{\mathrm{g}}) &= g\int_{\mathbb{R}^d}(\Psi, f(k)(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\mathrm{g}})dk \\
&= (\Psi, g\int_{\mathbb{R}^d} f(k)(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\mathrm{g}}dk),
\end{aligned}
$$

we obtain (2.20). □

**(Step 3)** (2.20) holds true for $f$ such that $f, f/\sqrt{\omega}\in L^2(\mathbb{R}^d)$.

*Proof of Step 3*

Set

$$
g(k) := \begin{cases} f(k)/\sqrt{\omega(k)}, & |k|<1, \\ f(k), & |k|\geq 1. \end{cases}
$$

Since $g\in L^2(\mathbb{R}^d)$, there exists a sequence $g_\epsilon \in C_0^\infty(\mathbb{R}^d\setminus\widetilde{K})$ such that $g_\epsilon\to g$ strongly as $\epsilon\to 0$. Define

$$
f_\epsilon(k) := \begin{cases} \sqrt{\omega(k)}g_\epsilon(k), & |k|<1, \\ g_\epsilon(k), & |k|\geq 1. \end{cases}
$$

Hence $f_\epsilon \in C_0^3(\mathbb{R}^d\setminus\widetilde{K})$ by (2) of **(B1)**, and it follows that

$$
\int_{\mathbb{R}^d}|f(k)-f_\epsilon(k)|^2/\omega(k)dk \to 0, \tag{2.27}
$$

$$
\int_{|k|>1}|f(k)-f_\epsilon(k)|^2 dk \to 0, \tag{2.28}
$$

as $\epsilon\to 0$. We see that, by (2.27) and (2.28),

$$
\|(1\otimes a(f))\varphi_{\mathrm{g}} - (1\otimes a(f_\epsilon))\varphi_{\mathrm{g}}\| \leq \|(f-f_\epsilon)/\sqrt{\omega}\|\|(1\otimes d\Gamma([\omega])^{1/2})\varphi_{\mathrm{g}}\| \to 0
$$

and

$$
\begin{aligned}
&\left\|\int_{\mathbb{R}^d}(f(k)-f_\epsilon(k))(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\mathrm{g}}dk\right\| \\
&\leq \left\{\left(\int_{|k|<1}\frac{|f(k)-f_\epsilon(k)|^2}{\omega(k)}dk\right)^{1/2} + \left(\int_{|k|\geq 1}|f(k)-f_\epsilon(k)|^2 dk\right)^{1/2}\right\}\|T_j(\cdot)\varphi_{\mathrm{g}}\| \\
&\to 0
\end{aligned}
$$

as $\epsilon\to 0$. Then we can extend (2.20) to $f$ such that $f, f/\sqrt{\omega}\in L^2(\mathbb{R}^d)$. □

## 2.4 Main theorem I

Set

$$\kappa_{\varphi_{\rm g}j}(k) := (H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\rm g}, \quad k \notin K_\omega.$$

We define

$$T_{\varphi_{\rm g}j} : L^2(\mathbb{R}^d) \to \mathcal{F}, \quad j = 1, ..., D,$$

by

$$T_{\varphi_{\rm g}j} f := \int_{\mathbb{R}^d} f(k)\kappa_{\varphi_{\rm g}j}(k) dk,$$

where the integral is taken in the strong sense in $\mathcal{F}$. I.e., we have

$$(1 \otimes a(f, j))\varphi_{\rm g} = g T_{\varphi_{\rm g}j} f. \tag{2.29}$$

**Proposition 2.8** *(1) $T_{\varphi_{\rm g}j}$ is a Hilbert-Schmidt operator if and only if*

$$\int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}j}(k)\|^2 dk < \infty.$$

*(2) Suppose that $T_{\varphi_{\rm g}j}$ is a Hilbert-Schmidt operator, Then*

$$\sum_{m=1}^{\infty} \|T_{\varphi_{\rm g}j} e_m\|^2 = \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}j}(k)\|^2 dk$$

*for an arbitrary complete orthonormal system $\{e_m\}_{m=1}^\infty$ in $L^2(\mathbb{R}^d)$.*

*Proof:* The adjoint of $T_{\varphi_{\rm g}j}$,

$$T^*_{\varphi_{\rm g}j} : \mathcal{F} \to L^2(\mathbb{R}^d), \quad j = 1, ..., D,$$

is referred to as a Carleman operator (see e.g., [38, p.141]) with kernel $\kappa_{\varphi_{\rm g}j}$, i.e.,

$$T^*_{\varphi_{\rm g}j} \Phi(\cdot) := (\kappa_{\varphi_{\rm g}j}(\cdot), \Phi).$$

It is known [38, Theorem 6.12] that $T^*_{\varphi_{\rm g}j}$ is a Hilbert-Schmidt operator if and only if $\int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}j}(k)\|^2 dk < \infty$. Moreover suppose that $T^*_{\varphi_{\rm g}j}$ is a Hilbert-Schmidt operator. Then

$$\mathrm{Tr}(T_{\varphi_{\rm g}j} T^*_{\varphi_{\rm g}j}) = \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}j}(k)\|^2 dk$$

is also known, which implies that $T_{\varphi_{\rm g}j}$ is a Hilbert-Schmidt operator if and only if $\int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}j}(k)\|^2 dk < \infty$, and

$$\mathrm{Tr}(T^*_{\varphi_{\rm g}j} T_{\varphi_{\rm g}j}) = \mathrm{Tr}(T_{\varphi_{\rm g}j} T^*_{\varphi_{\rm g}j}) = \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}j}(k)\|^2 dk.$$

Thus the proposition follows. □

The main theorem in this section is as follows.

**Theorem 2.9** *Suppose* **(B1)-(B4)**. *Then (1), (2) and (3) are equivalent.*

**(1)** $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$,

**(2)** $T_{\varphi_{\rm g}j}$ *is a Hilbert-Schmidt operator for all* $j = 1, ..., D$ *and all* $\varphi_{\rm g} \in P_H\mathcal{F}$,

**(3)** $\int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_{\rm g}\|^2 dk < \infty$ *for all* $j = 1, ..., D$ *and all* $\varphi_{\rm g} \in P_H\mathcal{F}$.

*Suppose that one of (1), (2) and (3) holds, it follows that for an arbitrary ground state* $\varphi_{\rm g}$,

$$\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = g^2 \sum_{j=1}^{D} \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_{\rm g}\|^2 dk. \tag{2.30}$$

*Proof:* Let $\{e_m\}_{m=1}^\infty$ be a complete orthonormal system of $L^2(\mathbb{R}^d)$ such that $e_m/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. It is proven in Lemma 2.2 that $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ if and only if

$$\sum_{j=1}^{D}\sum_{m=1}^{\infty} \|(1 \otimes a(\overline{e_m}, j))\varphi_{\rm g}\|^2 < \infty \tag{2.31}$$

for an arbitrary $\varphi_{\rm g} \in P_H\mathcal{F}$. By (2.29),

$$(1 \otimes a(\overline{e_m}, j))\varphi_{\rm g} = gT_{\varphi_{\rm g}j}\overline{e_m}.$$

Hence $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ if and only if

$$g^2 \sum_{j=1}^{D}\sum_{m=1}^{\infty} \|T_{\varphi_{\rm g}j}\overline{e_m}\|^2 < \infty, \quad \varphi_{\rm g} \in P_H\mathcal{F}.$$

That is to say, $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ if and only if $T_{\varphi_{\rm g}j}$ is a Hilbert-Schmidt operator for all $j = 1, ..., D$, and all $\varphi_{\rm g} \in P_H\mathcal{F}$, i.e., by Proposition 2.8, $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ if and only if

$$g^2 \sum_{j=1}^{D} \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}j}(k)\|^2 dk < \infty, \quad \varphi_{\rm g} \in P_H\mathcal{F}.$$

Then the first half of the theorem is proven. Moreover by Lemma 2.2, when $\varphi_{\rm g} \in D(1 \otimes N^{1/2})$,

$$\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = \sum_{j=1}^{D}\sum_{m=1}^{\infty} \|(1 \otimes a(\overline{e_m}, j))\varphi_{\rm g}\|^2,$$

which yields that

$$\|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\|^2 = g^2 \sum_{j=1}^{D} \mathrm{Tr}(T^*_{\varphi_{\mathrm{g}} j} T_{\varphi_{\mathrm{g}} j}) = g^2 \sum_{j=1}^{D} \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\mathrm{g}} j}(k)\|^2 dk.$$

Thus the proof is complete. □

**Remark 2.10** *In [6] a more general formula than (2.30) is obtained.*

# 3 Proof of $\mathrm{m}(H) \leq \mathrm{m}(A)$

## 3.1 Quadratic forms

We revive $H = H_0 + gH_{\mathrm{I}}$, where $H_0 = A \otimes 1 + 1 \otimes d\Gamma(S)$, and $(\mathbf{A.1}) - (\mathbf{A.3})$ are assumed. Set

$$\overline{H}_0 := H_0 - E(H_0).$$

Actually

$$E(H_0) = E(A).$$

The quadratic form $\beta_0$ associated with $\overline{H}_0$ is defined by

$$\beta_0(\Psi, \Phi) := (\overline{H}_0^{1/2}\Psi, \overline{H}_0^{1/2}\Phi), \quad \Psi, \Phi \in D(\overline{H}_0^{1/2}).$$

Define a symmetric form by

$$\beta_{H_{\mathrm{I}}}(\Psi, \Phi) := (\Psi, H_{\mathrm{I}}\Phi), \quad \Psi, \Phi \in D(\overline{H}_0).$$

Since $\|H_{\mathrm{I}}\Psi\| \leq a\|H_0\Psi\| + b\|\Psi\|$, it follows that

$$\|H_{\mathrm{I}}\Psi\| \leq a\|\overline{H}_0\Psi\| + b'\|\Psi\|,$$

where $b' = b + a|E(H_0)|$. Then $H_{\mathrm{I}}(\overline{H}_0 + \mu)^{-1}$ and $(\overline{H}_0 + \mu)^{-1}H_{\mathrm{I}}$, $\mu > 0$, are bounded operators with

$$\|H_{\mathrm{I}}(\overline{H}_0 + \mu)^{-1}\| \leq a + b'/\mu, \quad \|(\overline{H}_0 + \mu)^{-1}H_{\mathrm{I}}\| \leq a + b'/\mu.$$

By an interpolation argument [33, Section IX], $(\overline{H}_0 + \mu)^{-1/2}H_{\mathrm{I}}(\overline{H}_0 + \mu)^{-1/2}$ is also a bounded operator with

$$\|(\overline{H}_0 + \mu)^{-1/2}H_{\mathrm{I}}(\overline{H}_0 + \mu)^{-1/2}\| \leq a + b'/\mu.$$

Then

$$|\beta_{H_{\rm I}}(\Psi,\Psi)| \leq (a+b'/\mu)\beta_0(\Psi,\Psi) + (a+b'/\mu)\|\Psi\|^2, \quad \Psi \in D(\overline{H}_0), \tag{3.1}$$

for an arbitrary $\mu > 0$. By (3.1), a polarization identity [32] and a limiting argument, $\beta_{H_{\rm I}}(\Psi,\Phi)$ can be extended to $\Psi,\Phi \in D(\overline{H}_0^{1/2})$. The extension of $\beta_{H_{\rm I}}$ is denoted by $\widetilde{\beta}_{H_{\rm I}}$, and which satisfies

$$|\widetilde{\beta}_{H_{\rm I}}(\Psi,\Psi)| \leq (a+b'/\mu)\beta_0(\Psi,\Psi) + (a+b'/\mu)\|\Psi\|^2, \quad \Psi \in D(\overline{H}_0^{1/2}). \tag{3.2}$$

Thus we see that, for a sufficiently small $g$,

$$\beta_H := \beta_0 + g\widetilde{\beta}_{H_{\rm I}}$$

is a semibounded closed quadratic form on $D(\overline{H}_0^{1/2}) \times D(\overline{H}_0^{1/2})$. Then by the representation theorem for forms [28, p.322, Theorem 2.1], there exists a unique self-adjoint operator $H'$ such that $D(H') \subset D(\overline{H}_0^{1/2})$ and

$$\beta_H(\Psi,\Phi) = (\Psi, H'\Phi), \quad \Psi \in D(\overline{H}_0^{1/2}), \quad \Phi \in D(H').$$

On the other hand, we can see directly that $D(H) \subset D(\overline{H}_0^{1/2})$ and

$$\beta_H(\Psi,\Phi) = (\Psi, H\Phi), \quad \Psi \in D(\overline{H}_0^{1/2}), \quad \Phi \in D(\overline{H}_0).$$

which yields that

$$H' = H.$$

I.e., $H$ is a unique self-adjoint operator associated with the quadratic form $\beta_H$. We generalize this fact in the next subsection.

## 3.2 Abstract results

As was seen in the previous subsection, self-adjoint operator $H = H_0 + gH_{\rm I}$ is defined through the quadratic form $\beta_H$. In this subsection, as a mathematical generalization, we define a total Hamiltonian $H_{\rm q}$ through an abstarct quadratic form, and estimate an upper bound of $\dim\left\{P_{H_{\rm q}}\mathcal{F} \cap D(1 \otimes N^{1/2})\right\}$.

**Remark 3.1** *Hamiltonians of the Nelson model* without *ultraviolet cutoffs are defined as the self-adjoint operator associated with a semibounded quadratic form. See [21, 31]. As far as we know, it can not be represented as the form* $H_0 + gH_{\rm I}$.

Let $\beta_{\rm int}$ be a symmetric quadratic form with form domain $D(\overline{H}_0^{1/2})$ such that

$$|\beta_{\rm int}(\Psi,\Psi)| \le a\beta_0(\Psi,\Psi) + b(\Psi,\Psi), \qquad \Psi \in D(\overline{H}_0^{1/2}), \tag{3.3}$$

with some nonnegative constants $a$ and $b$. Define the quadratic form $\beta$ on $D(\overline{H}_0^{1/2})$ by

$$\beta := \beta_0 + g\beta_{\rm int}.$$

**Proposition 3.2** *Let $|g| < 1/a$. Then there exists a unique self-adjoint operator $H_{\rm q}$ associated with $\beta$ such that its form domain is $D(\overline{H}_0^{1/2})$,*

$$\beta(\Psi,\Phi) = (\Psi, H_{\rm q}\Phi), \qquad \Psi \in D(\overline{H}_0^{1/2}), \Phi \in D(H_{\rm q}),$$

*and*

$$\beta(\Psi,\Phi) = (H_{{\rm q}+}^{1/2}\Psi, H_{{\rm q}+}^{1/2}\Phi) - (H_{{\rm q}-}^{1/2}\Psi, H_{{\rm q}-}^{1/2}\Phi), \qquad \Psi,\Phi \in D(\overline{H}_0^{1/2}),$$

*where*

$$H_{{\rm q}+} := H_{\rm q}E_{H_{\rm q}}((0,\infty)), \qquad H_{{\rm q}-} := -H_{\rm q}E_{H_{\rm q}}((-\infty,0]).$$

*Proof:* From (3.3) it follows that

$$|g\beta_{\rm int}(\Psi,\Psi)| \le |g|a\beta_0(\Psi,\Psi) + |g|b(\Psi,\Psi).$$

Hence by the KLMN theorem [33, Theorem X.17], the proposition follows. □

Assumptions **(Gap)** and **(N)** are as follows.

**(Gap)** $\inf \sigma_{\rm ess}(A) - E(A) > 0$.

**(N)** $\displaystyle\lim_{g\to 0} \sup_{\Psi \in (P_{H_{\rm q}}\mathcal{F})\cap D(1\otimes N^{1/2})} \frac{\|(1\otimes N^{1/2})\Psi\|}{\|\Psi\|} = 0.$

Suppose that $\sigma_{\rm p}(S) \not\ni 0$. Then by the facts that

$$\begin{aligned}
&\inf \sigma(d\Gamma(S)\lceil_{\oplus_{n=1}^\infty[\otimes_s^n\mathcal{W}]}) \ge 0,\\
&\sigma_{\rm p}(d\Gamma(S)\lceil_{\oplus_{n=1}^\infty[\otimes_s^n\mathcal{W}]}) \not\ni 0,\\
&\sigma(d\Gamma(S)\lceil_{\otimes_s^0\mathcal{W}}) = \sigma_{\rm p}(d\Gamma(S)\lceil_{\otimes_s^0\mathcal{W}}) = \{0\},
\end{aligned}$$

$d\Gamma(S)$ is nonnegative self-adjoint operator, and has a unique ground state $\Omega$ with eigenvalue 0. We have a lemma.

**Lemma 3.3** *Assume* **(A1)**, **(A2)**, **(Gap)**, **(N)** *and* $\sigma_{\mathrm{p}}(S) \not\ni 0$. *Then there exists* $\delta(g) > 0$ *such that*

$$\lim_{g \to 0} \delta(g) = 0$$

*and, for* $g$ *with* $\delta(g) < 1$,

$$\dim \left\{ (P_{H_{\mathrm{q}}} \mathcal{F}) \cap D(1 \otimes N^{1/2}) \right\} \leq \frac{1}{1 - \delta(g)} \mathrm{m}(A).$$

*Proof:* Let $\epsilon > 0$ be such that

$$[E(A), E(A) + \epsilon) \cap \sigma(A) = \{E(A)\}$$

and we set

$$\mathcal{P}_\epsilon := E_A([E(A), E(A) + \epsilon)), \qquad \mathcal{P}_\epsilon^\perp := 1 - \mathcal{P}_\epsilon.$$

Furthermore let

$$\mathcal{P}_\Omega := E_{d\Gamma(S)}(\{0\}).$$

We fix a $\varphi_{\mathrm{g}} \in (P_{H_{\mathrm{q}}} \mathcal{F}) \cap D(1 \otimes N^{1/2})$. Using the inequality

$$1 \otimes 1 \leq 1 \otimes N + 1 \otimes \mathcal{P}_\Omega$$

in the sense of form, we have

$$\begin{aligned} (\varphi_{\mathrm{g}}, \varphi_{\mathrm{g}}) &\leq ((1 \otimes N^{1/2})\varphi_{\mathrm{g}}, (1 \otimes N^{1/2})\varphi_{\mathrm{g}}) + (\varphi_{\mathrm{g}}, (1 \otimes \mathcal{P}_\Omega)\varphi_{\mathrm{g}}) \\ &\leq \|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\|^2 + \|(\mathcal{P}_\epsilon \otimes \mathcal{P}_\Omega)\varphi_{\mathrm{g}}\|^2 + \|(\mathcal{P}_\epsilon^\perp \otimes \mathcal{P}_\Omega)\varphi_{\mathrm{g}}\|^2. \end{aligned} \tag{3.4}$$

Let $Q := \mathcal{P}_\epsilon^\perp \otimes \mathcal{P}_\Omega$. In Proposition 5.1 we shall show that

$$\varphi_{\mathrm{g}} \in D(\overline{H}_0^{1/2}), \quad Q\varphi_{\mathrm{g}} \in D(\overline{H}_0^{1/2}) \tag{3.5}$$

and

$$\overline{H}_0^{1/2} Q \varphi_{\mathrm{g}} = Q \overline{H}_0^{1/2} \varphi_{\mathrm{g}}. \tag{3.6}$$

Hence we have

$$\begin{aligned} 0 &= (Q\varphi_{\mathrm{g}}, (H_{\mathrm{q}} - E(H_{\mathrm{q}}))\varphi_{\mathrm{g}}) \\ &= \beta(Q\varphi_{\mathrm{g}}, \varphi_{\mathrm{g}}) - E(H_{\mathrm{q}})(Q\varphi_{\mathrm{g}}, \varphi_{\mathrm{g}}) \\ &= \beta_0(Q\varphi_{\mathrm{g}}, \varphi_{\mathrm{g}}) + g\beta_{\mathrm{int}}(Q\varphi_{\mathrm{g}}, \varphi_{\mathrm{g}}) - E(H_{\mathrm{q}})(Q\varphi_{\mathrm{g}}, \varphi_{\mathrm{g}}). \end{aligned}$$

From this we have

$$-g\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g}) = (\overline{H}_0^{1/2}Q\varphi_{\rm g},\overline{H}_0^{1/2}\varphi_{\rm g}) - E(H_{\rm q})(Q\varphi_{\rm g},\varphi_{\rm g}). \tag{3.7}$$

Since we have by (3.6)

$$\begin{aligned}
&(\overline{H}_0^{1/2}Q\varphi_{\rm g},\overline{H}_0^{1/2}\varphi_{\rm g})\\
&= (\overline{H}_0^{1/2}Q\varphi_{\rm g},\overline{H}_0^{1/2}Q\varphi_{\rm g})\\
&= \int_{[E(A),\infty)\times[0,\infty)}(\lambda+\mu-E(A))d\|(E_A(\lambda)\otimes E_{d\Gamma(S)}(\mu))Q\varphi_{\rm g}\|^2\\
&= \int_{[E(A)+\epsilon,\infty)\times\{0\}}(\lambda+\mu-E(A))d\|(E_A(\lambda)\otimes E_{d\Gamma(S)}(\mu))Q\varphi_{\rm g}\|^2\\
&\ge \epsilon(\varphi_{\rm g},Q\varphi_{\rm g}),
\end{aligned}$$

then (3.7) implies that

$$-g\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g}) \ge (\epsilon - E(H_{\rm q}))(Q\varphi_{\rm g},\varphi_{\rm g}). \tag{3.8}$$

We shall estimate $|\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g})|$.

$$\begin{aligned}
\beta_0(\varphi_{\rm g},\varphi_{\rm g}) &= (\varphi_{\rm g},H_{\rm q}\varphi_{\rm g}) - g\beta_{\rm int}(\varphi_{\rm g},\varphi_{\rm g})\\
&\le E(H_{\rm q})\|\varphi_{\rm g}\|^2 + |g|\left(a\beta_0(\varphi_{\rm g},\varphi_{\rm g}) + b(\varphi_{\rm g}.\varphi_{\rm g})\right),
\end{aligned}$$

which yields that, since $|g|<1/a$,

$$\beta_0(\varphi_{\rm g},\varphi_{\rm g}) \le \frac{E(H_{\rm q})+|g|b}{1-a|g|}(\varphi_{\rm g},\varphi_{\rm g}).$$

Then we have

$$|\beta_{\rm int}(\varphi_{\rm g},\varphi_{\rm g})| \le (a\beta_0(\varphi_{\rm g},\varphi_{\rm g}) + b(\varphi_{\rm g},\varphi_{\rm g})) \le c_{\rm int}(\varphi_{\rm g},\varphi_{\rm g}),$$

where

$$c_{\rm int} := \frac{a(E(H_{\rm q})+|g|b)}{1-a|g|} + b.$$

From the polarization identity

$$\begin{aligned}
\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g}) = \frac{1}{4}&\left\{(\beta_{\rm int}((1+Q)\varphi_{\rm g},(1+Q)\varphi_{\rm g}) - \beta_{\rm int}((1-Q)\varphi_{\rm g},(1-Q)\varphi_{\rm g}))\right.\\
&\left.-i\left(\beta_{\rm int}((1+iQ)\varphi_{\rm g},(1+iQ)\varphi_{\rm g}) - \beta_{\rm int}((1-iQ)\varphi_{\rm g},(1-iQ)\varphi_{\rm g})\right)\right\},
\end{aligned}$$

it follows that

$$|\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g})| \le 2c_{\rm int}(\varphi_{\rm g},\varphi_{\rm g}). \tag{3.9}$$

Note that

$$|\beta(\Psi,\Psi)-\beta_0(\Psi,\Psi)| = |g||\beta_{\rm int}(\Psi,\Psi)| \leq |g|(a+b)\|(\overline{H}_0+1)^{1/2}\Psi\|^2.$$

Then

$$\lim_{g\to 0}\sup_{\Psi\in D(\overline{H}_0^{1/2})}\frac{|\beta(\Psi,\Psi)-\beta_0(\Psi,\Psi)|}{\|(\overline{H}_0+1)^{1/2}\Psi\|^2} \leq \lim_{g\to 0}|g|(a+b)=0,$$

which implies that for $z\in\mathbb{C}$ with $\Im z\neq 0$,

$$\lim_{g\to 0}\|(H_{\rm q}-z)^{-1}-(\overline{H}_0-z)^{-1}\|=0. \tag{3.10}$$

See Proposition 5.2 for a proof of (3.10). Thus it follows that

$$\lim_{g\to 0}E(H_{\rm q}) = E(\overline{H}_0)=0. \tag{3.11}$$

Then there exists a constant $c>0$ such that for all $g$ with $|g|<c$, it obeys that

$$\epsilon - E(H_{\rm q})>0.$$

Then by (3.8) and (3.9), for $g$ with $|g|<c$,

$$\|Q\varphi_{\rm g}\|^2 \leq |g|\frac{|\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g})|}{\epsilon-E(H_{\rm q})} \leq 2|g|\frac{c_{\rm int}}{\epsilon-E(H_{\rm q})}\|\varphi_{\rm g}\|^2.$$

Let

$$c(g) := \sup_{\Psi\in(P_{H_{\rm q}}\mathcal{F})\cap D(1\otimes N^{1/2})}\frac{\|(1\otimes N^{1/2})\Psi\|}{\|\Psi\|}.$$

Together with (3.4) we have

$$(\varphi_{\rm g},\varphi_{\rm g}) \leq c(g)^2\|\varphi_{\rm g}\|^2 + 2|g|\frac{c_{\rm int}}{\epsilon-E(H_{\rm q})}\|\varphi_{\rm g}\|^2 + \|(\mathcal{P}_\epsilon\otimes\mathcal{P}_\Omega)\varphi_{\rm g}\|^2. \tag{3.12}$$

Setting

$$\delta(g) := c(g)^2 + 2|g|\frac{c_{\rm int}}{\epsilon-E(H_{\rm q})},$$

we see that by (3.11) and **(N)**,

$$\lim_{g\to 0}\delta(g)=0.$$

Then by (3.12) there exists $g_*\leq c$ such that for $g$ with $|g|<g_*$,

$$(\varphi_{\rm g},\varphi_{\rm g}) \leq (1-\delta(g))^{-1}(\varphi_{\rm g},(\mathcal{P}_\epsilon\otimes\mathcal{P}_\Omega)\varphi_{\rm g}). \tag{3.13}$$

Let $\{\varphi_{\rm g}^j\}_{j=1}^M$, $M \leq \infty$, be a complete orthonormal system of $(P_{H_{\rm q}}\mathcal{F}) \cap D(1 \otimes N^{1/2})$. Then by (3.13),

$$(\varphi_{\rm g}^j, \varphi_{\rm g}^j) \leq (1-\delta(g))^{-1}(\varphi_{\rm g}^j, (\mathcal{P}_\epsilon \otimes \mathcal{P}_\Omega)\varphi_{\rm g}^j). \tag{3.14}$$

Summing up from $j = 1$ to $M$, we have

$$\dim\left\{(P_{H_{\rm q}}\mathcal{F}) \cap D(1 \otimes N^{1/2})\right\} \leq (1-\delta(g))^{-1}\sum_{j=1}^M(\varphi_{\rm g}^j, (\mathcal{P}_\epsilon \otimes \mathcal{P}_\Omega)\varphi_{\rm g}^j).$$

Since

$$\begin{aligned}\sum_{j=1}^M(\varphi_{\rm g}^j, (\mathcal{P}_\epsilon \otimes \mathcal{P}_\Omega)\varphi_{\rm g}^j) &= \sum_{j=1}^M(\varphi_{\rm g}^j, (P_A \otimes \mathcal{P}_\Omega)\varphi_{\rm g}^j) \\ &\leq \mathrm{Tr}(P_A \otimes \mathcal{P}_\Omega) \\ &= \mathrm{Tr}P_A \times \mathrm{Tr}\mathcal{P}_\Omega \\ &= \mathrm{m}(A),\end{aligned}$$

we obtain that

$$\dim\left\{(P_{H_{\rm q}}\mathcal{F}) \cap D(1 \otimes N^{1/2})\right\} \leq (1-\delta(g))^{-1}\mathrm{m}(A).$$

Thus the lemma is proven. □

From Lemma 3.3, corollaries immediately follow.

**Corollary 3.4** *We assume the same assumptions as in Lemma 3.3. Suppose that*

$$P_{H_{\rm q}}\mathcal{F} \subset D(1 \otimes N^{1/2}). \tag{3.15}$$

*Then*

$$\mathrm{m}(H_{\rm q}) \leq (1-\delta(g))^{-1}\mathrm{m}(A).$$

In addition, suppose that $g$ is such that $\delta(g) < 1/2$ and $\mathrm{m}(A) = 1$. Then $\mathrm{m}(H) = 1$.
*Proof:* Since $P_{H_{\rm q}}\mathcal{F} \cap D(1 \otimes N^{1/2}) = P_{H_{\rm q}}\mathcal{F}$, the corollary follows from Lemma 3.3. □

**Corollary 3.5 [Overlap]** *We assume the same assumptions as in Lemma 3.3 and (3.15). Let $g$ be such that $\delta(g) < 1$. Then for an arbitrary ground state $\varphi_{\rm g}$, it follows that*

$$(\varphi_{\rm g}, (P_A \otimes \mathcal{P}_\Omega)\varphi_{\rm g}) \neq 0.$$

*Proof:* By (3.13) it is seen that

$$0 < \|\varphi_{\rm g}\|^2 \leq (1-\delta(g))^{-1}(\varphi_{\rm g}, (\mathcal{P}_\epsilon \otimes \mathcal{P}_\Omega)\varphi_{\rm g}) = (1-\delta(g))^{-1}(\varphi_{\rm g}, (P_A \otimes \mathcal{P}_\Omega)\varphi_{\rm g}).$$

Hence the corollary follows. □

## 3.3 Main theorem II

We assume (2.5) and (2.6), i.e., $H = H_0 + gH_{\rm I}$ and $H_0 = A \otimes 1 + 1 \otimes d\Gamma([\omega])$. Now we are in the position to state the main theorem in this section.

**Theorem 3.6** *Supose that* **(B1)**-**(B4)**, **(A1)**, **(A3)**, **(Gap)**. *We assume that*

$$\int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\rm g}\|^2 dk < \infty,$$

*and*

$$\sup_{\varphi_{\rm g} \in P_H \mathcal{F}} \frac{\sum_{j=1}^D \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\rm g}\|^2}{\|\varphi_{\rm g}\|^2} dk < \infty. \tag{3.16}$$

*Then there exists a constant $g_*$ such that for $g$ with $|g| < g_*$, it follows that*

$$\mathrm{m}(H) \le \mathrm{m}(A).$$

*Proof:* By Theorem 2.9, it follows that $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ and

$$\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = g^2 \sum_{j=1}^D \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\rm g}\|^2 dk.$$

By (3.16) we have

$$\begin{aligned} &\lim_{g\to 0} \sup_{\Psi \in P_H\mathcal{F}} \frac{\|(1 \otimes N^{1/2})\varphi_{\rm g}\|}{\|\varphi_{\rm g}\|} \\ &= \lim_{g\to 0} |g| \sup_{\varphi_{\rm g} \in P_H\mathcal{F}} \left( \frac{\sum_{j=1}^D \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\rm g}\|^2 dk}{\|\varphi_{\rm g}\|^2} \right)^{1/2} = 0. \end{aligned}$$

From this and Corollary 3.4, the theorem follows. □

# 4 Examples

## 4.1 GSB models

GSB models are a generalization of the spin-boson model, which was introduced and investigated in [3]. Examples of GSB models are e.g., $N$-level systems coupled to a Bose field, lattice spin systems, the Pauli-Fierz model with the dipole approximation neglected $A^2$ term, a Fermi field coupled to a Bose field, etc. See [3, p. 457].

The Hilbert space on which GSB Hamiltonians act is

$$\mathcal{F}_{\rm GSB} := \mathcal{H} \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^d)),$$

where $\mathcal{H}$ is a Hilbert space. Let $a(f)$ and $a^\dagger(f), f \in L^2(\mathbb{R}^d)$, be the annihilation operator and the creation operator on $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$, respectively. We use the same notations $a(f)$ and $a^\dagger(f)$ as those of Subsection 1.1. We set

$$\phi(\lambda) := \frac{1}{\sqrt{2}}(a^\dagger(\bar{\lambda}) + a(\lambda)), \quad \lambda \in L^2(\mathbb{R}^d).$$

GSB Hamiltonians are defined by

$$H_{\mathrm{GSB}} := H_{\mathrm{GSB},0} + \alpha H_{\mathrm{GSB,I}}.$$

Here $\alpha \in \mathbb{R}$ is a coupling constant, and

$$H_{\mathrm{GSB},0} := \overline{A \otimes 1 + 1 \otimes d\Gamma(\omega_{\mathrm{GSB}})},$$
$$H_{\mathrm{GSB,I}} := \sum_{j=1}^{J} B_j \otimes \phi(\lambda_j),$$

where $\omega_{\mathrm{GSB}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a multiplication operator by $\omega_{\mathrm{GSB}}(k)$ such that

$$\omega_{\mathrm{GSB}}(\cdot) : \mathbb{R}^d \to [0, \infty)$$

and $\overline{X}$ denotes the closure of $X$. Assumption **(GSB1)**-**(GSB5)** are as follows.

**(GSB1)** Operator $A$ satisfies **(A1)**. Set $\overline{A} := A - E(A)$.

**(GSB2)** $\lambda_j, \lambda_j/\sqrt{\omega_{\mathrm{GSB}}} \in L^2(\mathbb{R}^d)$, $j = 1, ..., J$.

**(GSB3)** $B_j$, $j = 1, ..., J$, is a symmetric operator, $D(\overline{A}^{1/2}) \subset \cap_{j=1}^{J} D(B_j)$ and there exist constants $a_j$ and $b_j$ such that

$$\|B_j f\| \leq a_j \|\overline{A}^{1/2} f\| + b_j \|f\|, \quad f \in D(\overline{A}^{1/2}).$$

Moreover

$$|\alpha| < \left( \sum_{j=1}^{J} a_j \|\lambda_j/\sqrt{\omega_{\mathrm{GSB}}}\| \right)^{-1}.$$

**(GSB4)** $\omega_{\mathrm{GSB}}$ satisfies that

**(1)** $\omega_{\mathrm{GSB}}(\cdot)$ is continuous,

**(2)** $\lim_{|k| \to \infty} \omega_{\mathrm{GSB}}(k) = \infty$,

**(3)** there exist constants $C > 0$ and $\gamma > 0$ such that

$$|\omega_{\mathrm{GSB}}(k) - \omega_{\mathrm{GSB}}(k')| \leq C|k - k'|^{\gamma}(1 + \omega_{\mathrm{GSB}}(k) + \omega_{\mathrm{GSB}}(k')).$$

**(GSB5)** $\lambda_j$, $j = 1, ..., J$, is continuous.

**Proposition 4.1** *Assume* **(GSB1)-(GSB3)**. *Then $H_{\mathrm{GSB}}$ is self-adjoint on*

$$D(H_{\mathrm{GSB},0}) = D(A \otimes 1) \cap D(1 \otimes d\Gamma(\omega_{\mathrm{GSB}}))$$

*and bounded from below. Moreover it is essentially self-adjoint on any core of $H_{\mathrm{GSB},0}$.*

*Proof:* We can show that

$$\|H_{\mathrm{GSB,I}}\Psi\| \leq \left(\sum_{j=1}^{J} a_j \|\lambda_j / \sqrt{\omega_{\mathrm{GSB}}}\|\right) \|H_{\mathrm{GSB},0}\Psi\| + b\|\Psi\| \tag{4.1}$$

for $\Psi \in D(H_{\mathrm{GSB},0})$ with some constant $b$. Then by the Kato-Rellich theorem, the proposition follows. for details. $\square$

We introduce assumptions.

**(IR)** $\lambda_j / \omega_{\mathrm{GSB}} \in L^2(\mathbb{R}^d)$, $j = 1, ..., J$.

**(GSB6)** $\omega_{\mathrm{GSB}}$ satisfies **(B1)** with $\omega$ replaced by $\omega_{\mathrm{GSB}}$.

**(GSB7)** $\lambda_j \in C^2(\mathbb{R}^d)$, $j = 1, ..., J$.

**Proposition 4.2** *We assume* **(GSB1)-(GSB5)**, **(IR)** *and* **(Gap)**. *Then there exists a constant $\alpha_* > 0$ such that for $\alpha$ with $|\alpha| < \alpha_*$, $H_{\mathrm{GSB}}$ has a ground state $\varphi_{\mathrm{g}}$ such that $\|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\| < \infty$.*

*Proof:* See [3, Theorem 1.3]. $\square$

**Remark 4.3** *In [3, Theorem 1.3], it is actually supposed that self-adjoint operator $A$ has a compact resolvent, i.e., $\sigma(A) = \sigma_{\mathrm{p}}(A)$. However it can be extended to $A$ satisfying* **(Gap)**. *See [7, Appendix].*

Let $f \in C_0^2(\mathbb{R}^d \setminus K)$ and $\Psi, \Phi \in D(H_{\mathrm{GSB}})$. We have

$$
\begin{aligned}
[a(f), H_{\mathrm{GSB,I}}]_W^{D(H_{\mathrm{GSB}})}(\Psi, \Phi) &= (\Psi, \sum_{j=1}^{J} (\bar{f}, \lambda_j)(B_j \otimes 1)\Phi) \\
&= \int_{\mathbb{R}^d} f(k)(\Psi, \sum_{j=1}^{J} \lambda_j(k)(B_j \otimes 1)\Phi) dk \\
&= \int_{\mathbb{R}^d} f(k)(\Psi, T_{\mathrm{GSB}}(k)\Phi) dk,
\end{aligned}
$$

where

$$
T_{\mathrm{GSB}}(k) := \sum_{j=1}^{J} \lambda_j(k)(B_j \otimes 1).
$$

Our main theorem in this subsection is as follows.

**Theorem 4.4** *Suppose* **(GSB1)-(GSB3)**, **(IR)**, **(GSB 6)** *and* **(GSB 7)**. *Then it follows that*

$$
P_{H_{\mathrm{GSB}}} \mathcal{F}_{\mathrm{GSB}} \subset D(1 \otimes N^{1/2}), \tag{4.2}
$$

*and*

$$
\|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\|^2 = \alpha^2 \int_{\mathbb{R}^d} \|(H_{\mathrm{GSB}} - E(H_{\mathrm{GSB}}) + \omega_{\mathrm{GSB}}(k))^{-1} T_{\mathrm{GSB}}(k)\varphi_{\mathrm{g}}\|^2 dk. \tag{4.3}
$$

*In addition, suppose* **(Gap)**. *Then there exists $\alpha_{**}$ such that for $\alpha$ with $|\alpha| < \alpha_{**}$, it follows that*

$$
\mathrm{m}(H_{\mathrm{GSB}}) \le \mathrm{m}(A). \tag{4.4}
$$

*Proof:* We shall check assumptions **(B1)-(B4)** and (3) of Theorem 2.9 with the following identifications.

$$
\mathcal{F} = \mathcal{F}_{\mathrm{GSB}}, \quad H_0 = H_{\mathrm{GSB},0}, \quad H_{\mathrm{I}} = H_{\mathrm{GSB,I}}, \quad \omega = \omega_{\mathrm{GSB}}, \quad D = 1, \quad T_{j=1}(k) = T_{\mathrm{GSB}}(k).
$$

**(B1)** and **(B2)** have been already checked. We have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{\mathrm{GSB}} - E(H_{\mathrm{GSB}}) + \omega_{\mathrm{GSB}}(k))} T_{\mathrm{GSB}}(k)\varphi_{\mathrm{g}}) dk \\
&= \int_{\mathbb{R}^d} f(k) \sum_{j=1}^{J} \lambda_j(k) e^{-is\omega_{\mathrm{GSB}}(k)} (\Psi, e^{-is(H_{\mathrm{GSB}} - E(H_{\mathrm{GSB}}))}(B_j \otimes 1)\varphi_{\mathrm{g}}) dk \\
&= \sum_{j=1}^{J} (\Psi, e^{-is(H_{\mathrm{GSB}} - E(H_{\mathrm{GSB}}))}(B_j \otimes 1)\varphi_{\mathrm{g}}) \int_{\mathbb{R}^d} f(k)\lambda_j(k) e^{-is\omega_{\mathrm{GSB}}(k)} dk.
\end{aligned} \tag{4.5}
$$

Since $f\lambda_j \in C_0^2(\mathbb{R}^d \setminus K)$, we see that by Lemma 2.5,

$$\left| \int_{\mathbb{R}^d} f(k)\lambda_j(k) e^{-is\omega_{\mathrm{GSB}}(k)} dk \right| \in L^1([0,\infty), ds),$$

which implies, together with (4.5), that **(B3)** follows. We have

$$\int_{\mathbb{R}^d} \|T_{\mathrm{GSB}}(k)\varphi_{\mathrm{g}}\|^2 dk \leq J \sum_{j=1}^J \left( \int_{\mathbb{R}^d} |\lambda_j(k)|^2 dk \right) \|(B_j \otimes 1)\varphi_{\mathrm{g}}\|^2 < \infty,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d} \|(H_{\mathrm{GSB}} - E(H_{\mathrm{GSB}}) + \omega_{\mathrm{GSB}}(k))^{-1} T_{\mathrm{GSB}}(k)\varphi_{\mathrm{g}}\|^2 dk \\
&\leq \int_{\mathbb{R}^d} \frac{1}{\omega_{\mathrm{GSB}}(k)^2} \|T_{\mathrm{GSB}}(k)\varphi_{\mathrm{g}}\|^2 dk \\
&\leq J \sum_{j=1}^J \left( \int_{\mathbb{R}^d} \frac{|\lambda_j(k)|^2}{\omega_{\mathrm{GSB}}(k)^2} dk \right) \|(B_j \otimes 1)\varphi_{\mathrm{g}}\|^2 < \infty.
\end{aligned}$$

Thus **(B4)** and (3) of Theorem 2.9 follow. Hence (4.2) and (4.3) are proven. We check (3.16) in Theorem 3.6 to show (4.4). Note that

$$\|(B_j \otimes 1)\varphi_{\mathrm{g}}\| \leq a_j \|(\overline{A}^{1/2} \otimes 1)\varphi_{\mathrm{g}}\| + b_j \|\varphi_{\mathrm{g}}\| \tag{4.6}$$

and

$$\|(\overline{A}^{1/2} \otimes 1)\varphi_{\mathrm{g}}\| \leq \|\overline{H_{\mathrm{GSB},0}}^{1/2} \varphi_{\mathrm{g}}\|.$$

Since

$$\|\overline{H_{\mathrm{GSB},0}}^{1/2} \varphi_{\mathrm{g}}\|^2 = (\varphi_{\mathrm{g}}, \overline{H_{\mathrm{GSB},0}} \varphi_{\mathrm{g}}) \leq (c_1 E(H_{\mathrm{GSB}}) + c_2) \|\varphi_{\mathrm{g}}\|^2$$

with some constants $c_1$ and $c_2$, we have

$$\|(B_j \otimes 1)\varphi_{\mathrm{g}}\| \leq (a_j \left(c_1 E(H_{\mathrm{GSB}}) + c_2\right)^{1/2} + b_j) \|\varphi_{\mathrm{g}}\|.$$

Thus by (4.6),

$$\begin{aligned}
&\lim_{g \to 0} \sup_{\varphi_{\mathrm{g}} \in P_{H_{\mathrm{GSB}}} \mathcal{F}_{\mathrm{GSB}}} \frac{g^2 \int_{\mathbb{R}^d} \|(H_{\mathrm{GSB}} - E(H_{\mathrm{GSB}}) + \omega_{\mathrm{GSB}}(k))^{-1} T_{\mathrm{GSB}}(k)\varphi_{\mathrm{g}}\|^2 dk}{\|\varphi_{\mathrm{g}}\|^2} \\
&\leq \lim_{g \to 0} g^2 J \sum_{j=1}^J (a_j \left(c_1 E(H_{\mathrm{GSB}}) + c_2\right)^{1/2} + b_j)^2 \|\lambda_j / \omega_{\mathrm{GSB}}\|^2 = 0.
\end{aligned}$$

Then (4.4) follows from Theorem 3.6. □

**Corollary 4.5** *Assume* **(GSB1)**-**(GSB4)**, **(GSB6)**, **(GSB7)**, **(IR)** *and* **(Gap)**. *Then there exists* $\alpha_{***}$ *such that for* $\alpha$ *with* $|\alpha| < \alpha_{***}$, $H_{\rm GSB}$ *has a ground state and* $\mathrm{m}(H_{\rm GSB}) \le \mathrm{m}(A)$. *In particular in the case of* $\mathrm{m}(A) = 1$, $H_{\rm GSB}$ *has a unique ground state.*

*Proof:* It follows from Proposition 4.2 and Theorem 4.4. □

## 4.2 The Pauli-Fierz model

The Pauli-Fierz model describes a minimal interaction between electrons with spin 1/2 and a quantized radiation field quantized in the Coulomb gauge. The asymptotic field for $H_{\rm PF}$ is studied in e.g., [13, 22]. The Hilbert space for state vectors of the Pauli-Fierz Hamiltonian is given by

$$\mathcal{F}_{\rm PF} := L^2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathcal{F}_{\rm b}(L^2(\mathbb{R}^3\times\{1,2\})).$$

Formally the annihilation operator and the creation operator of $\mathcal{F}_{\rm b}(L^2(\mathbb{R}^3\times\{1,2\}))$ is denoted by

$$a^\sharp(f,j) = \int f(k)a^\sharp(k,j)dk.$$

The Pauli-Fierz Hamiltonian with ultraviolet cutoff $\hat{\varphi}$ is defined by

$$H_{\rm PF} := \frac{1}{2m}(p\otimes 1 - eA_{\hat{\varphi}})^2 + V\otimes 1 + 1\otimes H_{\rm f} - \frac{e}{2m}(\sigma\otimes 1)\cdot B_{\hat{\varphi}},$$

where $m>0$ and $e\in\mathbb{R}$ denote the mass of an electron and the charge of an electron, respectively. We regard $e$ as a coupling constant. $p$ denotes the momentum operator of an electron, i.e.,

$$p = (p_1,p_2,p_3) = (-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, -i\frac{\partial}{\partial x_3}),$$

and $V$ is an external potential. We identify $\mathcal{F}_{\rm PF}$ as

$$\mathcal{F}_{\rm PF} \cong \mathbb{C}^2\otimes\int_{\mathbb{R}^3}^{\oplus}\mathcal{F}_{\rm b}(L^2(\mathbb{R}^3\times\{1,2\}))dx, \tag{4.7}$$

where $\int_{\mathbb{R}^3}^{\oplus}\cdots dx$ denotes a constant fiber direct integral [35]. $A_{\hat{\varphi}}$ and $B_{\hat{\varphi}}$ denote a quantized radiation field and a quantized magnetic field with ultraviolet cutoff $\hat{\varphi}$, respectively, which are given by, under identification (4.7),

$$A_{\hat{\varphi}} := 1\otimes\int_{\mathbb{R}^3}^{\oplus}A_{\hat{\varphi}}(x)dx, \quad B_{\hat{\varphi}} := 1\otimes\int_{\mathbb{R}^3}^{\oplus}B_{\hat{\varphi}}(x)dx,$$

with

$$A_{\hat{\varphi}}(x) := \sum_{j=1,2} \int \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}} e(k,j) \left\{ e^{-ikx} a^\dagger(k,j) + e^{ikx} a(k,j) \right\} dk$$

and

$$\begin{aligned} B_{\hat{\varphi}}(x) &:= \mathrm{rot}_x A_{\hat{\varphi}}(x) \\ &= \sum_{j=1,2} \int \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}} (-ik \times e(k,j)) \left\{ e^{-ikx} a^\dagger(k,j) - e^{ikx} a(k,j) \right\} dk. \end{aligned}$$

Here

$$\omega_{\mathrm{PF}}(k) := |k|$$

and $\hat{\varphi}$ denotes an ultraviolet cutoff function.

$$H_{\mathrm{f}} := d\Gamma([\omega_{\mathrm{PF}}])$$

is the second quantization of the multiplication operator

$$[\omega_{\mathrm{PF}}] : L^2(\mathbb{R}^3 \times \{1,2\}) \to L^2(\mathbb{R}^3 \times \{1,2\})$$

such that

$$([\omega_{\mathrm{PF}}]f)(k,j) = \omega_{\mathrm{PF}}(k) f(k,j).$$

Vector

$$e(k,j) = (e_1(k,j), e_2(k,j), e_3(k,j)) \in \mathbb{R}^3, \quad j = 1,2,$$

denotes a polarization vector satisfying

$$e(k,1) \times e(k,2) = k/|k|, \quad |e(k,j)| = 1, \quad j = 1,2.$$

Note that

$$e(-k,1) = -e(k,1), \quad e(-k,2) = e(k,2).$$

Finally $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ denotes $2 \times 2$ Pauli matrices satisfying the anticommutation relations,

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}, \quad i,j = 1,2,3,$$

where $\{A, B\} := AB + BA$. Assumptions **(PF1)**- **(PF3)** are as follows.

**(PF1)**

**(1)** $\sqrt{\omega_{\mathrm{PF}}}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega_{\mathrm{PF}}}, \hat{\varphi}/\omega_{\mathrm{PF}} \in L^2(\mathbb{R}^3)$ and $\hat{\varphi}(k) = \hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$.

**(2)** $V$ is $\Delta$-bounded with a relative bound strictly less than one.

**(PF2)**

**(1)** $\hat{\varphi} \in C^\infty(\mathbb{R}^3)$.

**(2)** $e(\cdot, j) \in C^\infty(\mathbb{R}^3 \setminus \mathcal{Q})$, $j = 1, 2$, with some measurable set $\mathcal{Q}$ with its Lebesgue measure zero.

**(PF3)** The ground state energy of self-adjoint operator

$$h_\mathrm{p} := -\frac{1}{2m}\Delta + V$$

acting in $L^2(\mathbb{R}^3)$ is discrete.

Let $H_{\mathrm{PF},0}$ be $H_\mathrm{PF}$ with $e = 0$, i.e.,

$$H_{\mathrm{PF},0} := H_\mathrm{p} \otimes 1 + 1 \otimes H_\mathrm{f},$$

where

$$H_\mathrm{p} := \begin{pmatrix} h_\mathrm{p} & 0 \\ 0 & h_\mathrm{p} \end{pmatrix} \text{ acting in } L^2(\mathbb{R}^3; \mathbb{C}^2) \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3).$$

In what follows, simply we write $T \otimes 1$ for $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \otimes 1$ unless confusions arise. We note that $H_{\mathrm{PF},0}$ is self-adjoint on

$$D(H_{\mathrm{PF},0}) = D(\Delta \otimes 1) \cap D(1 \otimes H_\mathrm{f}).$$

Note that

$$(p \otimes 1) \cdot A_{\hat{\varphi}} = A_{\hat{\varphi}} \cdot (p \otimes 1) \tag{4.8}$$

on $D(H_{\mathrm{PF},0})$. We set

$$H_\mathrm{PF} = H_{\mathrm{PF},0} + eH_{\mathrm{PF,I}},$$

where, by (4.8),

$$H_{\mathrm{PF,I}} := -\frac{1}{m}(p \otimes 1) \cdot A_{\hat{\varphi}} + \frac{e}{2m} A_{\hat{\varphi}} \cdot A_{\hat{\varphi}} - \frac{1}{2m}(\sigma \otimes 1) \cdot B_{\hat{\varphi}}.$$

**Proposition 4.6** *Assume* **(PF1)**, **(PF2)**. *Then $H_\mathrm{PF}$ is self-adjoint on $D(H_{\mathrm{PF},0})$ and bounded from below. Moreover it is essentially self-adjoint on any core of $H_{\mathrm{PF},0}$.*

*Proof:* See [25, 26]. □

**Proposition 4.7** *Suppose* **(PF1)** *and* **(PF3)**. *Then there exists a constant* $e_* \le \infty$ *such that for* $e$ *with* $|e| \le e_*$, $H_{\rm PF}$ *has a ground state such that* $\varphi_{\rm g} \in D(1 \otimes N^{1/2})$.

*Proof:* See e.g., [9, 10, 15, 29, 30, 23]. □

**Remark 4.8** *For some* $V$, *we can take* $e_* = \infty$. *See [15, 29].*

**Remark 4.9** *Spinless Pauli-Fierz Hamiltonians are defined by*

$$H_{\rm PF}^{\rm sl} := \frac{1}{2m}(p \otimes 1 - eA_{\hat\varphi})^2 + V \otimes 1 + 1 \otimes H_{\rm f},$$

*which acts in* $\mathcal{F} = L^2(\mathbb{R}^3) \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^3 \times \{1,2\}))$. *It can be proven that* $H_{\rm PF}^{\rm sl}$ *has a ground state* $\varphi_{\rm g}$ *such that* $\varphi_{\rm g} \in D(1 \otimes N^{1/2})$, *and it is unique [24]. Then it follows that* $P_{H_{\rm PF}^{\rm sl}} \mathcal{F} \subset D(1 \otimes N^{1/2})$.

We have

$$\begin{aligned}
&[1 \otimes a(f,j), H_{\rm PF,I}]_W^{D(H_{\rm PF})}(\Psi, \Phi) \\
&= (\Psi, \left\{ -\frac{1}{m} K^{(1)}(x,j) \cdot (p \otimes 1 - eA_{\hat\varphi}) - \frac{1}{2m} K^{(2)}(x,j) \cdot (\sigma \otimes 1) \right\} \Phi),
\end{aligned}$$

where

$$K^{(1)}(x,j) := (\overline{f}, \lambda_j^{(1)}(x)), \quad K^{(2)}(x,j) := (\overline{f}, \lambda_j^{(2)}(x)),$$

and

$$\begin{aligned}
\lambda_j^{(1)}(x,k) &:= \frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}} e^{-ikx} e(k,j), \\
\lambda_j^{(2)}(x,k) &:= \frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}} e^{-ikx} (-ik \times e(k,j)).
\end{aligned}$$

Hence we have

$$[1 \otimes a(f,j), H_{\rm PF,I}]_W^{D(H_{\rm PF})}(\Psi, \Phi) = \int f(k)(\Psi, T_{{\rm PF}j}(k)\Psi) dk,$$

where

$$\begin{aligned}
&T_{{\rm PF}j}(k) := T_{{\rm PF}j}^{(1)}(k) + T_{{\rm PF}j}^{(2)}(k), \\
&T_{{\rm PF}j}^{(1)}(k) := -\frac{1}{m} \frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}} e^{-ikx} e(k,j) \cdot (p \otimes 1 - eA_{\hat\varphi}), \\
&T_{{\rm PF}j}^{(2)}(k) := -\frac{1}{2m} \frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}} e^{-ikx} (-ik \times e(k,j)) \cdot (\sigma \otimes 1).
\end{aligned}$$

Let $H_{\mathrm{PF}}^0$ be $H_{\mathrm{PF}}$ with $V=0$. Then the binding energy is defined by

$$E_{\mathrm{bin}} := E(H_{\mathrm{PF}}^0) - E(H_{\mathrm{PF}}).$$

**Proposition 4.10** *Assume that $h_{\mathrm{p}}$ has a ground state in $L^2(\mathbb{R}^3)$. Then*

$$E_{\mathrm{bin}} \geq -E(h_{\mathrm{p}}).$$

*Proof:* See Appendix 5.3. □

Assumption **(V)** is as follows.

**(V)** Potential $V = V_+ - V_-$ ($V_+(x) = \max\{0, V(x)\}$, $V_-(x) = \min\{0, V(x)\}$) satisfies that (1) $\lim_{|x|\to\infty} V_-(x) = V_\infty < \infty$, (2) $|x|^2 V_- \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$, (3) $E_{\mathrm{bin}} > V_\infty$.

A typical example of $V$ in **(V)** is Coulomb potential $-e^2/|x|$.

**Lemma 4.11** *Suppose* **(V)***. Then it follows that*

$$\sup_{\varphi_{\mathrm{g}} \in P_H \mathcal{F}} \frac{\|(|x| \otimes 1)\varphi_{\mathrm{g}}\|}{\|\varphi_{\mathrm{g}}\|} < c_{\mathrm{exp}}$$

*with some constant $c_{\mathrm{exp}}$.*

*Proof:* See Appendix 5.4. □

The main theorem in this subsection is as follows.

**Theorem 4.12** *Assume* **(PF1)***,* **(PF2)***, and* **(V)***. Then it follows that*

$$P_{H_{\mathrm{PF}}} \mathcal{F}_{\mathrm{PF}} \subset D(1 \otimes N^{1/2}), \tag{4.9}$$

*and*

$$\|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\|^2 = e^2 \sum_{j=1,2} \int_{\mathbb{R}^3} \|(H_{\mathrm{PF}} - E(H_{\mathrm{PF}}) + \omega_{\mathrm{PF}}(k))^{-1} T_{\mathrm{PF}j}(k)\varphi_{\mathrm{g}}\|^2 dk. \tag{4.10}$$

*In addition, assume* **(PF3)***, then there exists a constant $e_{**}$ such that for $e$ with $|e| < e_{**}$,*

$$\mathrm{m}(H_{\mathrm{PF}}) \leq \mathrm{m}(H_{\mathrm{p}}). \tag{4.11}$$

**Remark 4.13** *Suppose that e.g., $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $V_-$ is infinitesimally small with respect to $\Delta$. Then, by a Feynman-Kac formula, it is shown that $e^{-t(h_{\mathrm{p}} - E(h_{\mathrm{p}}))}$ is positivity improving in $L^2(\mathbb{R}^3)$. Hence $h_{\mathrm{p}}$ has a unique ground state in $L^2(\mathbb{R}^3)$. Then* $\mathrm{m}(H_{\mathrm{p}}) = 2$.

To prove Theorem 4.12 it is sufficient to check (**B1**)-(**B4**) and (3) of Theorem 2.9 with the following identifications.

$$\mathcal{F} = \mathcal{F}_{\rm PF}, \quad H_0 = H_{\rm PF,0}, \quad H_{\rm I} = H_{\rm PF,I}, \quad \omega = \omega_{\rm PF}, \quad D = 2, \quad T_j(k) = T_{{\rm PF}j}(k).$$

Let

$$K := \bigcup_{n=1}^{3} \{(k_1, k_2, k_3) \in \mathbb{R}^3 | k_n = 0\}$$

and

$$\widetilde{K} := K \cup \mathcal{Q} \cup \{0\}.$$

**Lemma 4.14** *Assume* **(PF1)** *and* **(PF2)**. *Then for* $f \in C_0^2(\mathbb{R}^3 \setminus \widetilde{K})$ *and* $\Psi \in D(H_{\rm PF})$,

$$\left| \int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))} {T_{\rm PF}}_j^{(l)}(k)\varphi_{\rm g}) dk \right| \in L^1([0,\infty), ds), \quad l = 1, 2.$$

*Proof:* Note that

$$\sum_{\mu=1,2,3} [K_\mu^{(1)}(x,j), (p \otimes 1 - eA_{\hat{\varphi}})_\mu] = 0$$

on $D(H_{\rm PF})$. We see that

$$\begin{aligned}
&\int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))} {T_{\rm PF}}_j^{(1)}(k)\varphi_{\rm g}) dk \\
&= -\frac{1}{m} \sum_{\mu=1,2,3} ((p \otimes 1 - eA_{\hat{\varphi}})_\mu e^{-is(H_{\rm PF} - E(H_{\rm PF}))}\Psi, K_\mu^{(1)}(s,x,j)\varphi_{\rm g}),
\end{aligned}$$

where

$$K^{(1)}(s,x,j) := (e^{is\omega_{\rm PF}} \bar{f}, \lambda_j^{(1)}(x)).$$

Since

$$e^{is\omega_{\rm PF}} = \frac{\omega_{\rm PF}(k)}{k_\mu} \frac{1}{is} \frac{\partial}{\partial k_\mu} e^{is\omega_{\rm PF}}, \quad \mu = 1, 2, 3,$$

we have

$$K^{(1)}(s,x,j) = \frac{1}{s}(K_1^{(1)}(s,x,j) + x_\mu K_2^{(1)}(s,x,j)),$$

where

$$\begin{aligned}
K_1^{(1)}(s,x,j) &:= -i \int_{\mathbb{R}^3} e^{-i(s\omega_{\rm PF}(k) + kx)} \frac{\partial}{\partial k_\mu} \left( \frac{\omega_{\rm PF}(k)}{k_\mu} \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\rm PF}(k)}} f(k) e(k,j) \right) dk, \\
K_2^{(1)}(s,x,j) &:= \int_{\mathbb{R}^3} e^{-i(s\omega_{\rm PF}(k) + kx)} \frac{\partial}{\partial k_\mu} \left( \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\rm PF}(k)}} f(k) e(k,j) \right) dk.
\end{aligned}$$

From the fact that $\hat{\varphi} \in C^\infty(\mathbb{R}^3)$ and $f \in C_0^2(\mathbb{R}^3 \setminus \widetilde{K})$, it follows that for $\nu = 1, 2, 3$,

$$\frac{\partial}{\partial k_\mu}\left(\frac{\omega_{\mathrm{PF}}(k)}{k_\mu}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}f(k)e_\nu(k,j)\right) \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}),$$
$$\frac{\partial}{\partial k_\mu}\left(\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}f(k)e_\nu(k,j)\right) \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}).$$

Thus by [34, Theorem XI.19 (c)] there exist constants $c_1$ and $c_2$ such that

$$\sup_x |K_{l,\mu}^{(1)}(s,x,j)| \leq \frac{c_l}{1+s}, \quad l = 1, 2, \quad \mu = 1, 2, 3. \tag{4.12}$$

By this we have

$$\|K_\mu^{(1)}(s,x,j)\varphi_{\mathrm{g}}\| \leq \frac{1}{s(s+1)}(c_1\|\varphi_{\mathrm{g}}\| + c_2\|(|x| \otimes 1)\varphi_{\mathrm{g}}\|). \tag{4.13}$$

Since

$$\|(p \otimes 1 - eA_{\hat{\varphi}})_\mu\Psi\| \leq c_1'\|(H_{\mathrm{PF}} - E(H_{\mathrm{PF}}))\Psi\| + c_2'\|\Psi\| \tag{4.14}$$

with some constants $c_1'$ and $c_2'$, we conclude that

$$\left|-\frac{1}{m}\sum_{\mu=1,2,3}((p \otimes 1 - eA_{\hat{\varphi}})_\mu e^{-is(H_{\mathrm{PF}} - E(H_{\mathrm{PF}}))}\Psi, K_\mu(s,x,j)\varphi_{\mathrm{g}})\right|$$
$$\leq \frac{3}{m}(c_1'\|(H_{\mathrm{PF}} - E(H_{\mathrm{PF}}))\Psi\| + c_2'\|\Psi\|)(c_1\|\varphi_{\mathrm{g}}\| + c_2\|(|x| \otimes 1)\varphi_{\mathrm{g}}\|)\frac{1}{s(1+s)}. \tag{4.15}$$

From this it follows that

$$-\frac{1}{m}\sum_{\mu=1,2,3}((p \otimes 1 - eA_{\hat{\varphi}})_\mu e^{-is(H_{\mathrm{PF}} - E(H_{\mathrm{PF}}))}\Psi, K_\mu(s,x,j)\varphi_{\mathrm{g}}) \in L^1([0,\infty), ds).$$

Similarly we can estimate

$$\int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{\mathrm{PF}} - E(H_{\mathrm{PF}}) + \omega_{\mathrm{PF}}(k))}{T_{\mathrm{PF}}}_j^{(2)}(k)\varphi_{\mathrm{g}})dk$$
$$= -\frac{1}{2m}\sum_{\mu=1,2,3}((\sigma_\mu \otimes 1)e^{-is(H_{\mathrm{PF}} - E(H_{\mathrm{PF}}))}\Psi, K_\mu^{(2)}(s,x,j)\varphi_{\mathrm{g}}),$$

where

$$K^{(2)}(s,x,j) := (e^{is\omega_{\mathrm{PF}}}\bar{f}, \lambda_j^{(2)}(x)).$$

We have

$$K^{(2)}(s,x,j) = \frac{1}{s}(K_1^{(2)}(s,x,j) + x_\mu K_2^{(2)}(s,x,j)),$$

where

$$
\begin{aligned}
&K_1^{(2)}(s,x,j)\\
&:= -i\int_{\mathbb{R}^3} e^{-i(s\omega_{\mathrm{PF}}(k)+kx)}\frac{\partial}{\partial k_\mu}\left(\frac{\omega_{\mathrm{PF}}(k)}{k_\mu}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}f(k)(-ik\times e(k,j))\right)dk,\\
&K_2^{(2)}(s,x,j)\\
&:= \int_{\mathbb{R}^3} e^{-i(s\omega_{\mathrm{PF}}(k)+kx)}\frac{\partial}{\partial k_\mu}\left(\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}f(k)(-ik\times e(k,j))\right)dk.
\end{aligned}
$$

Since for $\mu,\nu=1,2,3$,

$$
\begin{aligned}
&\frac{\partial}{\partial k_\mu}\left(\frac{\omega_{\mathrm{PF}}(k)}{k_\mu}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}f(k)(-ik\times e(k,j))_\nu\right)\in C_0^\infty(\mathbb{R}^d\setminus\{0\}),\\
&\frac{\partial}{\partial k_\mu}\left(\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}f(k)(-ik\times e(k,j))_\nu\right)\in C_0^\infty(\mathbb{R}^d\setminus\{0\}),
\end{aligned}
$$

we can see that there exist constants $\widetilde{c}_1$ and $\widetilde{c}_2$ such that

$$
\sup_x |K_{l,\mu}^{(2)}(s,x,j)|\le \frac{\widetilde{c}_l}{1+s},\quad l=1,2,\quad \mu=1,2,3. \tag{4.16}
$$

Then we conclude that

$$
\begin{aligned}
&\left|-\frac{1}{2m}\sum_{\mu=1,2,3}((\sigma_\mu\otimes 1)e^{-is(H_{\mathrm{PF}}-E(H_{\mathrm{PF}}))}\Psi, K_\mu^{(2)}(s,x,j)\varphi_{\mathrm{g}})\right|\\
&\le \frac{3}{2m}\|\Psi\|\left(\widetilde{c}_1\|\varphi_{\mathrm{g}}\|+\widetilde{c}_2\|(|x|\otimes 1)\varphi_{\mathrm{g}}\|\right)\frac{1}{s(1+s)}.
\end{aligned}
$$

Thus

$$
-\frac{1}{2m}\sum_{\mu=1,2,3}((\sigma_\mu\otimes 1)e^{-is(H_{\mathrm{PF}}-E(H_{\mathrm{PF}}))}\Psi, K_\mu^{(2)}(s,x,j)\varphi_{\mathrm{g}})\in L^1([0,\infty),ds).
$$

Hence the lemma is proven. □

**Lemma 4.15** *We have*

$$
\|T_{\mathrm{PF}}{}_j^{(l)}(\cdot)\varphi_{\mathrm{g}}\|\in L^2(\mathbb{R}^3),\quad l=1,2.
$$

*Proof:* It follows that

$$\begin{aligned}\|T_{\mathrm{PF}}{}_j^{(1)}(k)\varphi_{\mathrm{g}}\| &\leq \frac{1}{m}\sum_{\mu=1,2,3}\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\mathrm{PF}}(k)}}|e(k,j)_\mu|\|(p\otimes 1-eA_{\hat{\varphi}})_\mu\varphi_{\mathrm{g}}\| \\ &\leq \sum_{\mu=1}^{3}\frac{1}{m}\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\mathrm{PF}}(k)}}\|(p\otimes 1-eA_{\hat{\varphi}})_\mu\varphi_{\mathrm{g}}\|,\end{aligned}$$

and

$$\begin{aligned}\|T_{\mathrm{PF}}{}_j^{(2)}(k)\varphi_{\mathrm{g}}\| &\leq \frac{1}{2m}\sum_{\mu=1,2,3}\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\mathrm{PF}}(k)}}|(k\times e(k,j))_\mu|\|(\sigma_\mu\otimes 1)\varphi_{\mathrm{g}}\| \\ &\leq \frac{3}{2m}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}|k|\|\varphi_{\mathrm{g}}\|.\end{aligned}$$

Since $\sqrt{\omega_{\mathrm{PF}}}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega_{\mathrm{PF}}}\in L^2(\mathbb{R}^3)$, the lemma follows. □

**Lemma 4.16** *Suppose that* $\Psi\in D(H_{\mathrm{PF},0})\cap D(|x|\otimes 1)$ *and* $H_{\mathrm{PF}}\Psi\in D(|x|\otimes 1)$. *Then for* $\mu=1,2,3$,

**(1)** $(x_\mu\otimes 1)\Psi\in D(H_{\mathrm{PF}})$,

**(2)** $[x_\mu\otimes 1, H_{\mathrm{PF}}]\Psi=\dfrac{i}{m}(p\otimes 1-eA_{\hat{\varphi}})_\mu\Psi$.

*In particular, it follows that* $(x_\mu\otimes 1)\varphi_{\mathrm{g}}\in H_{\mathrm{PF}}$ *with*

$$\frac{i}{m}(p\otimes 1-eA_{\hat{\varphi}})_\mu\varphi_{\mathrm{g}}=[x_\mu\otimes 1,H_{\mathrm{PF}}]\varphi_{\mathrm{g}}=(H_{\mathrm{PF}}-E(H_{\mathrm{PF}}))(x_\mu\otimes 1)\varphi_{\mathrm{g}}.$$

*Proof:* See Appendix 5.5. □

**Lemma 4.17** *Assume* **(PF1)**. *Then*

$$\int_{\mathbb{R}^3}\|(H_{\mathrm{PF}}-E(H_{\mathrm{PF}})+\omega_{\mathrm{PF}}(k))^{-1}T_{\mathrm{PF}}{}_j^{(l)}(k)\varphi_{\mathrm{g}}\|^2dk<\infty,\quad l=1,2.$$

*Proof:* Note that by Lemma 4.16

$$\begin{aligned}&T_{\mathrm{PF}}{}_j^{(1)}(k)\varphi_{\mathrm{g}}\\ &=-\frac{1}{m}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}e^{-ikx}e(k,j)\cdot(p\otimes 1-eA_{\hat{\varphi}})\varphi_{\mathrm{g}}\\ &=-\frac{1}{m}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}e^{-ikx}e(k,j)\cdot(-im)[x\otimes 1,H_{\mathrm{PF}}]\varphi_{\mathrm{g}}\\ &=-\frac{1}{m}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\mathrm{PF}}(k)}}e^{-ikx}e(k,j)\cdot(-im)(H_{\mathrm{PF}}-E(H_{\mathrm{PF}}))(x\otimes 1)\varphi_{\mathrm{g}}.\end{aligned}$$

Hence we have

$$
\begin{aligned}
&(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}T_{{\rm PF}_j}^{(1)}(k)\varphi_{\rm g}\\
&=i\frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}}e(k,j)(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}e^{-ikx}(H_{\rm PF}-E(H_{\rm PF}))(x\otimes 1)\varphi_{\rm g}\\
&=i\frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}}e(k,j)(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}\\
&\qquad\qquad\qquad\times(H_{\rm PF}(k)-E(H_{\rm PF}))e^{-ikx}(x\otimes 1)\varphi_{\rm g},
\end{aligned}
$$

where we used that $e^{ikx}$ maps $D(H_{\rm PF,0})$ onto itself (see Appendix 5.5) and on $D(H_{\rm PF})$,

$$
\begin{aligned}
H_{\rm PF}(k)&:=e^{-ikx}H_{\rm PF}e^{ikx}\\
&=\frac{1}{2m}(p\otimes 1+k-eA_{\hat\varphi})^2+V\otimes 1+1\otimes H_{\rm f}-\frac{1}{2m}(\sigma\otimes 1)\cdot B_{\hat\varphi}\\
&=H_{\rm PF}+\frac{1}{m}(p\otimes 1-eA_{\hat\varphi})\cdot k+\frac{1}{2m}|k|^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}\left(H_{\rm PF}(k)-E(H_{\rm PF})\right)e^{-ikx}(x_\mu\otimes 1)\varphi_{\rm g}\|\\
&\le\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}(H_{\rm PF}-E(H_{\rm PF}))e^{-ikx}(x_\mu\otimes 1)\varphi_{\rm g}\| \quad (4.17)\\
&+\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}\frac{1}{m}(p\otimes 1-eA_{\hat\varphi})\cdot ke^{-ikx}(x_\mu\otimes 1)\varphi_{\rm g}\| \quad (4.18)\\
&+\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}\frac{1}{2m}|k|^2e^{-ikx}(x_\mu\otimes 1)\varphi_{\rm g}\|. \quad (4.19)
\end{aligned}
$$

We have

$$
|(4.17)|\le\|(|x|\otimes 1)\varphi_{\rm g}\| \tag{4.20}
$$

and

$$
|(4.19)|\le\frac{1}{2m}\frac{|k|^2}{\omega_{\rm PF}(k)}\|(|x|\otimes 1)\varphi_{\rm g}\|=\frac{1}{2m}|k|\|(|x|\otimes 1)\varphi_{\rm g}\|. \tag{4.21}
$$

Note that by (4.14),

$$
\begin{aligned}
&\|(p\otimes 1-eA_{\hat\varphi})_\mu(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}\varphi_{\rm g}\|\\
&\le c_1'\|(H_{\rm PF}-E(H_{\rm PF}))(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}\varphi_{\rm g}\|\\
&\qquad+c_2'\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}\varphi_{\rm g}\|\\
&\le c_1'\|\varphi_{\rm g}\|+\frac{c_2'}{\omega_{\rm PF}(k)}\|\varphi_{\rm g}\|.
\end{aligned}
$$

Then

$$|(4.18)| \leq \frac{3}{m}|k|(c_1' + \frac{c_2'}{\omega_{\rm PF}(k)})\|(|x|\otimes 1)\varphi_{\rm g}\| = \frac{3}{m}(c_1'|k| + c_2')\|(|x|\otimes 1)\varphi_{\rm g}\|. \quad (4.22)$$

Together with (4.20), (4.21), (4.22), we have

$$\begin{aligned}&\|(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))^{-1} T_{{\rm PF}_j}^{(1)}(k)\varphi_{\rm g}\| \\ &\leq 3\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}}\left(1 + \frac{|k|}{2m} + \frac{3}{m}(c_1'|k| + c_2')\right)\|(|x|\otimes 1)\varphi_{\rm g}\|. \end{aligned} \quad (4.23)$$

Since $\sqrt{\omega_{\rm PF}}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega_{\rm PF}} \in L^2(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \|(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))^{-1} T_{{\rm PF}_j}^{(1)}(k)\varphi_{\rm g}\|^2 dk < \infty \quad (4.24)$$

follows. Moreover we have

$$\begin{aligned}&\|(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))^{-1} T_{{\rm PF}_j}^{(2)}(k)\varphi_{\rm g}\| \\ &\leq \frac{3}{2m}\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}}|k|\frac{1}{\omega_{\rm PF}(k)}\|\varphi_{\rm g}\| \\ &= \frac{3}{2m}\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}}\|\varphi_{\rm g}\|. \end{aligned} \quad (4.25)$$

Hence

$$\int_{\mathbb{R}^3} \|(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))^{-1} T_{{\rm PF}_j}^{(2)}(k)\varphi_{\rm g}\|^2 dk < \infty \quad (4.26)$$

follows. Thus by (4.24) and (4.26), we get the desired results. □

*Proof of Theorem 4.12*

Lemmas 4.14, 4.15 and 4.17 correspond to assumptions **(B3)**, **(B4)** and (3) of Theorem 2.9, respectively. Then (4.9) and (4.10) follow from Theorem 2.9. By (4.23) and (4.25), we have

$$\begin{aligned}&\frac{\|(1\otimes N^{1/2})\varphi_{\rm g}\|^2}{\|\varphi_{\rm g}\|^2} \\ &= e^2\frac{\sum_{j=1,2}\int_{\mathbb{R}^3}\|(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))^{-1}T_{\rm PF}(k)^{(j)}\varphi_{\rm g}\|^2 dk}{\|\varphi_{\rm g}\|^2} \\ &\leq 6e^2\int\left\{3\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}}\left(1 + \frac{|k|}{2m} + \frac{3}{m}(c_1'|k| + c_2')\right)\right\}^2 dk\frac{\|(|x|\otimes 1)\varphi_{\rm g}\|^2}{\|\varphi_{\rm g}\|^2} \\ &\qquad +6e^2\int\left\{\frac{3}{2m}\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}}\right\}^2 dk.\end{aligned}$$

Since, by Lemma 5.3,

$$\frac{\|(|x|\otimes 1)\varphi_{\rm g}\|^2}{\|\varphi_{\rm g}\|^2}\leq c_{\rm exp},$$

we obtain

$$\lim_{e\to 0}\sup_{\varphi_{\rm g}\in P_{H_{\rm PF}}\mathcal{F}_{\rm PF}} e^2\frac{\sum_{j=1,2}\int_{\mathbb{R}^3}\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}T_{\rm PF}(k)^{(j)}\varphi_{\rm g}\|^2dk}{\|\varphi_{\rm g}\|^2}=0.$$

Thus (4.11) follows from Theorem 3.6. □

**Remark 4.18** *Although, in [27], formula (4.10):*

$$\|(1\otimes N^{1/2})\varphi_{\rm g}\|^2=e^2\sum_{j=1,2}\int_{\mathbb{R}^3}\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}T_{{\rm PF}j}(k)\|^2dk$$

*has been used to show* $\mathrm{m}(H_{\rm p})\leq 2$*, there is no exact proof to derive this formula in it. See Subsection 1.3.*

In [27] it has been also proven that $2\leq \mathrm{m}(H_{\rm PF})$ under some conditions on $V$. We state a theorem.

**Theorem 4.19** *In addition to* $(\mathbf{PF1})-(\mathbf{PF3})$ *and* $(\mathbf{V})$*, we assume* $\mathrm{m}(H_{\rm p})=2$ *and* $V(x)=V(-x)$*. Then there exists a constant* $e_{***}$ *such that for* $e$ *with* $|e|<e_{***}$*,* $\mathrm{m}(H_{\rm PF})=2$*.*

*Proof:* $\mathrm{m}(H_{\rm PF})\leq 2$ follows from Theorem 4.12 and $2\leq \mathrm{m}(H_{\rm PF})$ from [27]. We omit details. □

## 4.3 Concluding remarks

We can apply the method stated in this paper to a wide class of interaction Hamiltonians in quantum field models. Hamiltonian $H_{\rm CD}$ of the Coulomb-Dirac system is defined as an operator acting in

$$\mathcal{F}_{\rm CD}=\mathcal{F}_{\rm f}(\oplus^4L^2(\mathbb{R}^3))\otimes\mathcal{F}_{\rm b}(L^2(\mathbb{R}^3{\times}\{1,2\})),$$

where $\mathcal{F}_{\rm f}(\oplus^4L^2(\mathbb{R}^3))$ denotes a fermion Fock space over $\oplus^4L^2(\mathbb{R}^3)$. The Coulomb-Dirac system describes an interaction of positrons and relativistic electrons through photons in the Coulomb gauge. Operator $H_{\rm CD}$ is of the form

$$H_{\rm CD}=H_{\rm f}\otimes 1+1\otimes d\Gamma(\omega_{\rm CD})+eH_{\rm rad}+e^2H_{\rm Coulomb},$$

where $H_{\rm f}$ denotes a free Hamiltonian of $\mathcal{F}_{\rm f}(\oplus^4 L^2(\mathbb{R}^3))$, $\omega_{\rm CD}$ the multiplication operator by $\omega_{\rm CD}(k)=|k|$, and $H_{\rm rad}$, $H_{\rm Coulomb}$ interaction terms. $H_{\rm CD}$ has been investigated in [11], where the self-adjointness and the existence of a ground state are proven under some conditions. It is known that $\mathrm{m}(H_{\rm f})=1$. Then using the method in this paper we can also show

$$\mathrm{m}(H_{\rm CD}) \le \mathrm{m}(H_{\rm f}) = 1,$$

i.e., the ground state of $H_{\rm CD}$ is unique for a sufficiently small $e$. We omit details.

# 5 Appendix

## 5.1 Proofs of (3.5) and (3.6)

**Proposition 5.1 ((3.5) and (3.6))** *We have* $\varphi_{\rm g} \in D(\overline{H}_0^{1/2})$, $Q\varphi_{\rm g} \in D(\overline{H}_0^{1/2})$ *and*

$$\overline{H}_0^{1/2} Q\varphi_{\rm g} = Q\overline{H}_0^{1/2}\varphi_{\rm g}. \tag{5.1}$$

*Proof:* From the fact that the form domain $Q(H)$ of $H$ satisfies

$$D(H) \subset Q(H) = D(\overline{H}_0^{1/2}),$$

it follows that $\varphi_{\rm g} \in D(\overline{H}_0^{1/2})$. Since $[e^{-t\overline{H}_0^{1/2}}, Q]=0$, we have

$$\frac{e^{-t\overline{H}_0^{1/2}}-1}{t} Q\varphi_{\rm g} = Q\frac{e^{-t\overline{H}_0^{1/2}}-1}{t}\varphi_{\rm g}, \quad t>0. \tag{5.2}$$

Since $\varphi_{\rm g} \in D(\overline{H}_0^{1/2})$, we see that

$$s-\lim_{t\to 0} Q\frac{e^{-t\overline{H}_0^{1/2}}-1}{t}\varphi_{\rm g} = -Q\overline{H}_0^{1/2}\varphi_{\rm g}.$$

Hence the left-hand side of (5.2) converges, which implies that $Q\varphi_{\rm g} \in D(\overline{H}_0^{1/2})$ and (5.1) follows. $\square$

## 5.2 Proof of (3.10)

**Proposition 5.2 ((3.10))** *Assume that*

$$\lim_{g\to 0} \sup_{\Psi \in D(\overline{H}_0^{1/2})} \frac{|\beta(\Psi,\Psi)-\beta_0(\Psi,\Psi)|}{\|(\overline{H}_0+1)^{1/2}\Psi\ |^2} = 0. \tag{5.3}$$

*Then for $z \in \mathbb{C}$ with $\Im z \neq 0$,*

$$\lim_{g\to 0} \|(H_{\mathrm{q}} - z)^{-1} - (\overline{H}_0 - z)^{-1}\| = 0.$$

*Proof:* Set

$$K_0 := (\overline{H}_0 + 1)^{-1/2}.$$

We have

$$\begin{aligned}
&(\Psi, K_0 H_{\mathrm{q}} K_0 \Psi) \\
&= (\Psi, K_0 (H_{\mathrm{q}+}^{1/2} H_{\mathrm{q}+}^{1/2} - H_{\mathrm{q}-}^{1/2} H_{\mathrm{q}-}^{1/2}) K_0 \Psi) \\
&= (\Psi, K_0 (H_{\mathrm{q}+}^{1/2} H_{\mathrm{q}+}^{1/2}) K_0 \Psi) - (\Psi, K_0 (H_{\mathrm{q}-}^{1/2} H_{\mathrm{q}-}^{1/2}) K_0 \Psi) \\
&= (H_{\mathrm{q}+}^{1/2} K_0 \Psi, H_{\mathrm{q}+}^{1/2} K_0 \Psi) - (H_{\mathrm{q}-}^{1/2} K_0 \Psi, H_{\mathrm{q}-}^{1/2} K_0 \Psi) \\
&= \beta(K_0\Psi, K_0\Psi) \qquad (5.4) \\
&\leq (1 + |g|a)\beta_0(K_0\Psi, K_0\Psi) + |g| b (K_0\Psi, K_0\Psi) \\
&\leq C(g)\|\Psi\|^2,
\end{aligned}$$

where

$$C(g) := 1 + |g|(a + b).$$

Then $K_0 H_{\mathrm{q}} K_0$ is a bounded operator and

$$\|K_0 H_{\mathrm{q}} K_0\| < C(g).$$

Since the range of $K_0$ equals to $D(\overline{H}_0^{1/2})$, we have from (5.3)

$$\lim_{g\to 0} \sup_{\Psi \in \mathcal{F}} \frac{|\beta(K_0\Psi, K_0\Psi) - \beta_0(K_0\Psi, K_0\Psi)|}{\|\Psi\|^2} = 0. \qquad (5.5)$$

By (5.4), i.e.,

$$\beta(K_0\Psi, K_0\Psi) = (\Psi, K_0 H_{\mathrm{q}} K_0 \Psi)$$

and

$$\beta_0(K_0\Psi, K_0\Psi) = (\Psi, K_0 \overline{H}_0 K_0 \Psi),$$

(5.5) implies that

$$\lim_{g\to 0} \sup_{\Psi \in \mathcal{F}} \frac{|(\Psi, K_0(H_{\mathrm{q}} - \overline{H}_0) K_0 \Psi)|}{\|\Psi\|^2} = 0.$$

Hence we obtain that

$$\lim_{g\to 0} \|K_0 (H_{\mathrm{q}} - \overline{H}_0) K_0\| = 0. \qquad (5.6)$$

Moreover, for $z \in \mathbb{C}$ with $\Im z \neq 0$,

$$
\begin{aligned}
&|\beta_0((H_{\rm q}-z)^{-1}\Psi,(H_{\rm q}-z)^{-1}\Psi)| \\
&\le |\beta((H_{\rm q}-z)^{-1}\Psi,(H_{\rm q}-z)^{-1}\Psi)| + |g\beta_{\rm int}((H_{\rm q}-z)^{-1}\Psi,(H_{\rm q}-z)^{-1}\Psi)| \\
&\le |((H_{\rm q}-z)^{-1}\Psi, H_{\rm q}(H_{\rm q}-z)^{-1}\Psi)| + a|g|\beta_0((H_{\rm q}-z)^{-1}\Psi,(H_{\rm q}-z)^{-1}\Psi) \\
&\qquad\qquad +b|g|((H_{\rm q}-z)^{-1}\Psi,(H_{\rm q}-z)^{-1}\Psi).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|\overline{H}_0^{1/2}(H_{\rm q}-z)^{-1}\Psi\|^2 \\
&= \beta_0((H_{\rm q}-z)^{-1}\Psi,(H_{\rm q}-z)^{-1}\Psi) \\
&\le \frac{1}{1-a|g|}\Big\{\Big((H_{\rm q}-z)^{-1}\Psi, H_{\rm q}(H_{\rm q}-z)^{-1}\Psi\Big) \\
&\qquad\qquad\qquad +b|g|\Big((H_{\rm q}-z)^{-1}\Psi,(H_{\rm q}-z)^{-1}\Psi\Big)\Big\} \\
&= \int_{\inf\sigma(H_{\rm q})}^{\infty}\left(\frac{1}{1-|g|a}\frac{\lambda+|g|b}{(\lambda-z)^2}\right) d\|E_{H_{\rm q}}(\lambda)\Psi\|^2 \\
&\le d(g)\|\Psi\|^2,
\end{aligned}
$$

where

$$
d(g) := \sup_{\lambda\in\mathbb{R}}\left|\frac{1}{1-|g|a}\frac{\lambda+|g|b}{(\lambda-z)^2}\right|.
$$

Thus we see that

$$
\|K_0^{-1}(H_{\rm q}-z)^{-1}\| < \|\overline{H}_0^{1/2}(H_{\rm q}-z)^{-1}\| + \|(H_{\rm q}-z)^{-1}\| \le \sqrt{d(g)} + \frac{1}{|\Im z|} := D(g).
$$

Directly we have

$$
\begin{aligned}
&\|(H_{\rm q}-z)^{-1} - (\overline{H}_0-z)^{-1}\| \\
&\le \|(H_{\rm q}-z)^{-1}K_0^{-1}\|\|K_0(H_{\rm q}-\overline{H}_0)K_0\|\|K_0^{-1}(\overline{H}_0-z)^{-1}\| \\
&\le D(g)D(0)\|K_0(H_{\rm q}-\overline{H}_0)K_0\|.
\end{aligned}
$$

By (5.6) and the fact

$$
\lim_{g\to 0} D(g) < \infty,
$$

we obtain that

$$
\lim_{g\to 0}\|(H_{\rm q}-z)^{-1} - (\overline{H}_0-z)^{-1}\| = \lim_{g\to 0} D(g)D(0)\|K_0(H_{\rm q}-\overline{H}_0)K_0\| = 0.
$$

Thus the proposition is proven. □

## 5.3 Proof of Proposition 4.10

*Proof of Proposition 4.10*

We see an outline of a proof. See [15, 21] for details. It is enough to show that for an arbitrary $\epsilon > 0$, there exists a vector $\Psi_\epsilon \in D(H_{\mathrm{PF}})$ such that

$$(\Psi_\epsilon, (H_{\mathrm{PF}} - (E(H^0_{\mathrm{PF}}) + \epsilon + E(h_{\mathrm{p}}))\Psi_\epsilon) < 0. \tag{5.7}$$

We identify $\mathcal{F}_{\mathrm{PF}}$ with $L^2(\mathbb{R}^3; \oplus^2\mathcal{F}_{\mathrm{b}})$, i.e., $\oplus^2\mathcal{F}_{\mathrm{b}}$-valued $L^2$-function over $\mathbb{R}^3$. Let $f$ be a ground state of $h_{\mathrm{p}}$. Assume that $f$ is real and $\|f\| = 1$. For $\epsilon > 0$, let

$$\Phi_\epsilon = \sum_m^{\text{finite}} f^\epsilon_m \otimes \Phi^\epsilon_m \tag{5.8}$$

be such that $f^\epsilon_m \in C^\infty_0(\mathbb{R}^3)$, $\Phi^\epsilon_m \in \oplus^2\mathcal{F}_{\mathrm{fin}}$ and

$$(\Phi_\epsilon, H^0_{\mathrm{PF}}\Phi_\epsilon) \le E(H^0_{\mathrm{PF}}) + \epsilon. \tag{5.9}$$

Actually, since the linear hull of vectors such as (5.8) is a core of $H_{\mathrm{PF}}$, there exists $\Phi_\epsilon$ such as (5.8) satisfying (5.9). Note that $f\Phi_\epsilon \in \mathcal{F}$. Let

$$U_y := e^{iy\cdot p} \otimes 1, \quad y \in \mathbb{R}^3.$$

We can see that $fU_y\Phi_\epsilon \in D(H_{\mathrm{PF}})$ and

$$\begin{aligned}
\Omega_y &:= (fU_y\Phi_\epsilon, (H_{\mathrm{PF}} - (E(H^0_{\mathrm{PF}}) + \epsilon + E(h_{\mathrm{p}}))fU_y\Phi_\epsilon)_{\mathcal{F}_{\mathrm{PF}}} \\
&= \int_{\mathbb{R}^3} \left\{ (\Phi_\epsilon(x), (H^0_{\mathrm{PF}}\Phi_\epsilon)(x))_{\oplus^2\mathcal{F}_{\mathrm{b}}} - (E(H^0_{\mathrm{PF}}) + \epsilon)\|\Phi_\epsilon(x)\|^2_{\oplus^2\mathcal{F}_{\mathrm{b}}} \right\} f(x-y)^2 dx \\
&\quad + \int_{\mathbb{R}^3} (\Phi_\epsilon(x), ((p\otimes 1)\Phi_\epsilon)(x))_{\oplus^2\mathcal{F}_{\mathrm{b}}} f(x-y)(pf)(x-y)dx.
\end{aligned}$$

Note that $fU_y\Phi_\epsilon$ and $H_{\mathrm{PF},0}fU_y\Phi_\epsilon$ are strongly continuous in $y$. Then $\Omega_y$ is continuous in $y$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Omega_y dy \\
&= \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dx \left\{ (\Phi_\epsilon(x), (H^0_{\mathrm{PF}}\Phi_\epsilon)(x))_{\oplus^2\mathcal{F}_{\mathrm{b}}} - (E(H^0_{\mathrm{PF}}) + \epsilon)\|\Phi_\epsilon(x)\|^2_{\oplus^2\mathcal{F}_{\mathrm{b}}} \right\} f(x-y)^2 \\
&= \int_{\mathbb{R}^3} dx \left\{ (\Phi_\epsilon(x), (H^0_{\mathrm{PF}}\Phi_\epsilon)(x))_{\oplus^2\mathcal{F}_{\mathrm{b}}} - (E(H^0_{\mathrm{PF}}) + \epsilon)\|\Phi_\epsilon(x)\|^2_{\oplus^2\mathcal{F}_{\mathrm{b}}} \right\} \\
&= (\Phi_\epsilon, (H^0_{\mathrm{PF}} - (E(H^0_{\mathrm{PF}}) + \epsilon)\Phi_\epsilon) < 0.
\end{aligned}$$

Here we used that $\|f\| = 1$, and by the fact that $f$ is real,

$$\int_{\mathbb{R}^3} f(x-y)(p_\mu f)(x-y)dy = (f, p_\mu f) = 0, \quad \mu = 1, 2, 3.$$

Thus we conclude that there exists $y_0 \in \mathbb{R}^3$ such that $\Omega_{y_0} < 0$, which yields that

$$(fU_{y_0}\Phi_\epsilon, (H_{\rm PF} - (E(H_{\rm PF}^0) + \epsilon + E(h_{\rm p}))fU_{y_0}\Phi_\epsilon) < 0.$$

The proof is complete. □

## 5.4 Proof of Lemma 4.11

We show a more general lemma than Lemma 4.11.

**Lemma 5.3** *Assume the following facts.*

**(1)** $\lim_{|x|\to\infty} V_-(x) = V_\infty < \infty$.

**(2)** *A positive function $G$ satisfies that*

- **(i)** $G \in C^1(\mathbb{R}^3 \setminus \mathcal{N}) \cap C(\mathbb{R}^3)$ *with some compact set $\mathcal{N}$,*
- **(ii)** $\sup_{x\in\mathbb{R}^3\setminus\mathcal{N}} |\nabla G(x)| < \infty$,
- **(iii)** $G^2 V_- \in L^\infty_{\rm loc}(\mathbb{R}^3)$,
- **(iv)** $\lim_{|x|\to\infty} |G(x)||x|^{-l} = d < \infty$ *with some $l \geq 0$ and $d > 0$.*

**(3)** $E_{\rm bin} > V_\infty$.

*Then there exists a constant $c_{\rm exp}$ such that*

$$\sup_{\varphi_{\rm g}\in P_{H_{\rm PF}}\mathcal{F}} \frac{\|(G\otimes 1)\varphi_{\rm g}\|}{\|\varphi_{\rm g}\|} \leq c_{\rm exp}. \tag{5.10}$$

*Proof:* This proof is a generalization of [15]. Let $\chi_{|x|>R} \in C^\infty(\mathbb{R}^3)$ be a function such that $\chi_{|x|>R}(x) = 0$ for $|x| < R$, and $\chi_{|x|>R}(x) = 1$ for $|x| > R+1$. Since $\mathcal{N}$ is compact, $\chi_{|x|>R}G \in C^1(\mathbb{R}^3)$ for a sufficiently large $R$. Since $(1-\chi_{|x|>R})G$ is a bounded operator, it is enough to prove (5.10) with $G$ replaced by $\chi_{|x|>R}G$. We reset $\chi_{|x|>R}G$ as $G$.

(1) Suppose that $G \in C_0^\infty(\mathbb{R}^3)$. Then we have

$$(G\varphi_{\rm g}, (H_{\rm PF} - E(H_{\rm PF}))G\varphi_{\rm g}) = \frac{1}{2m}(\varphi_{\rm g}, |\nabla G|^2\varphi_{\rm g}).$$

The left-hand side above is

$$\begin{aligned}(G\varphi_{\rm g}, (H_{\rm PF} - E(H_{\rm PF}))G\varphi_{\rm g}) &= (G\varphi_{\rm g}, \left\{(H_{\rm PF}^0 - E(H_{\rm PF})) + V\right\}G\varphi_{\rm g}) \\ &\geq E_{\rm bin}\|G\varphi_{\rm g}\|^2 + (G\varphi_{\rm g}, VG\varphi_{\rm g}) \\ &\geq E_{\rm bin}\|G\varphi_{\rm g}\|^2 + (G\varphi_{\rm g}, -V_-G\varphi_{\rm g})..\end{aligned}$$

Hence

$$E_{\rm bin}\|G\varphi_{\rm g}\|^2 \le \frac{1}{2m}(\varphi_{\rm g}, |\nabla G|^2\varphi_{\rm g}) - (G\varphi_{\rm g}, VG\varphi_{\rm g}). \tag{5.11}$$

Let $\epsilon$ satisfy

$$E_{\rm bin} - V_\infty - \epsilon > 0.$$

There exists $R$ such that

$$|V_-(x) - V_\infty| \le \epsilon, \quad |x| > R.$$

Let

$$a := \sup_{|x|<R} G^2(x)V_-(x), \quad b := \frac{1}{2m}\sup_{x\in\mathbb{R}^3}|\nabla G(x)|^2.$$

Then

$$E_{\rm bin}\|G\varphi_{\rm g}\|^2 \le b\|\varphi_{\rm g}\|^2 + a\|\varphi_{\rm g}\|^2 + (V_\infty+\epsilon)\|G\varphi_{\rm g}\|^2,$$

and

$$\|G\varphi_{\rm g}\|^2 \le \frac{a+b}{E_{\rm bin} - V_\infty - \epsilon}\|\varphi_{\rm g}\|^2. \tag{5.12}$$

(2) Suppose that $G \in C^\infty(\mathbb{R}^3)$. Let $\chi \in C^\infty(\mathbb{R}^3)$ be

$$\chi(x) := \begin{cases} 1, & |x| \le 1, \\ \theta(x), & 1 < |x| < 2, \\ 0, & |x| \ge 2, \end{cases}$$

such that $\partial\theta(x)/\partial x_\mu \le 0$, $\mu = 1,2,3$, $\sup_{x\in\mathbb{R}^3}|\theta(x)| \le 1$ and $\sup_{x\in\mathbb{R}^3}|\nabla\chi(x)| \le c$ with some constant $c$. Define $\chi_n(x) := \chi(x/n^l)$ and set

$$G_n := \chi_n G.$$

Note that

$$|\nabla\chi_n(x)| = \begin{cases} 0, & |x| < n^l, \\ \le cn^{-l}, & n^l \le |x| \le 2n^l, \\ 0, & |x| > 2n^l. \end{cases} \tag{5.13}$$

It follows that by (5.11),

$$E_{\rm bin}\|G_n\varphi_{\rm g}\|^2 \le \frac{1}{2m}(\varphi_{\rm g}, |\nabla G_n|^2\varphi_{\rm g}) + (G_n\varphi_{\rm g}, V_-G_n\varphi_{\rm g}).$$

Since

$$|\nabla G_n|^2 = |\nabla\chi_n \cdot G_n|^2 + |\chi_n\cdot\nabla G_n|^2 + 2(\nabla\chi_n\cdot G_n)\cdot(\chi_n\cdot\nabla G_n),$$

we have by (5.13) and assumption (iv) of (2), for sufficiently large $n$,

$$\begin{aligned}
&\frac{1}{2m}(\varphi_{\rm g}, |\nabla G_n|^2\varphi_{\rm g})\\
&\leq \frac{1}{2m}\frac{c^2}{n^{2l}}\int_{n^l\leq|x|\leq 2n^l}|G(x)|^2\|\varphi_{\rm g}(x)\|^2_{\mathcal{F}_{\rm b}}dx+\frac{1}{2m}(\varphi_{\rm g}, |\nabla G|^2\varphi_{\rm g})\\
&+\frac{1}{m}\left(\frac{c^2}{n^{2l}}\int_{n^l\leq|x|\leq 2n^l}|G(x)|^2\|\varphi_{\rm g}(x)\|^2_{\mathcal{F}_{\rm b}}dx\right)^{1/2}(\varphi_{\rm g}, |\nabla G|^2\varphi_{\rm g})^{1/2}\\
&\leq \left((2c^2d/m)+b+\sqrt{8c^2db/m}\right)\|\varphi_{\rm g}\|^2. \qquad (5.14)
\end{aligned}$$

Moreover

$$\sup_{|x|<R} G_n^2(x)|V(x)| \leq \sup_{|x|<R} G^2(x)|V(x)| = a.$$

Thus we obtain that

$$(E_{\rm bin}-V_\infty-\epsilon)\|G_n\varphi_{\rm g}\|^2 \leq \left(b+a+(2c^2d/m)+\sqrt{8c^2db/m}\right)\|\varphi_{\rm g}\|^2. \qquad (5.15)$$

Since $G_n$ is monotonously increasing in $n$ and $\lim_{n\to\infty} G_n(x) = G(x)$, by (5.15) and the Lebesgue monotone convergence theorem, we see that $G\|\varphi_{\rm g}(\cdot)\|_{\mathcal{F}_{\rm b}} \in L^2(\mathbb{R}^3)$ and

$$\lim_{n\to\infty}\|G_n\varphi_{\rm g}\|^2 = \lim_{n\to\infty}\int_{\mathbb{R}^3} G_n(x)^2\|\varphi_{\rm g}(x)\|^2_{\mathcal{F}_{\rm b}}dx = \|G\varphi_{\rm g}\|^2.$$

In particular by (5.14),

$$\lim_{n\to\infty}\frac{1}{2m}(\varphi_{\rm g}, |\nabla G_n|^2\varphi_{\rm g}) \leq b\|\varphi_{\rm g}\|^2.$$

Thus (5.12) follows for $G \in C^\infty$.

(3) Suppose that $G \in C^1(\mathbb{R}^3)$. Let $\rho \in C_0^\infty(\mathbb{R}^3)$ and $\rho > 0$ such that $\int_{\mathbb{R}^3}\rho(x)dx = 1$. Set $\rho_\epsilon = \rho(\cdot/\epsilon)\epsilon^{-3}$. Define

$$G_\epsilon := \rho_\epsilon * G.$$

Since $G_\epsilon \in C^\infty(\mathbb{R}^3)$, we have

$$\|G_\epsilon\varphi_{\rm g}\|^2 \leq \frac{a_\epsilon+b_\epsilon}{E_{\rm bin}-V_\infty-\epsilon}\|\varphi_{\rm g}\|^2,$$

where $a_\epsilon$ and $b_\epsilon$ are $a$ and $b$ with $G$ replaced by $G_\epsilon$, respectively. Note that

$$\sup_{|x|<R} G_\epsilon(x) = \sup_{|x|<R}\int_{\mathbb{R}^3}\rho(z)G(x-\epsilon z)dz \leq \sup_{|x|<R'} G(x),$$

with some $R'$, and

$$\sup_{x\in\mathbb{R}^3}|\nabla G_\epsilon(x)| = \sup_{x\in\mathbb{R}^3}\left|\int_{\mathbb{R}^3}\rho(z)\nabla G(x-\epsilon z)dz\right| \leq \sup_{x\in\mathbb{R}^3}|\nabla G(x)|,$$

which yields that

$$a_\epsilon \leq \sup_{|x|<R'} G^2(x)|V(x)| := a', \quad b_\epsilon \leq b.$$

Hence we obtain that

$$\|G_\epsilon\varphi_{\rm g}\|^2 \leq \left(\frac{a'+b}{E_{\rm bin}-V_\infty-\epsilon}\right)\|\varphi_{\rm g}\|^2.$$

From this,

$$\liminf_{\epsilon\to 0}\|G_\epsilon\varphi_{\rm g}\|^2 < \infty$$

follows. Since

$$\liminf_{\epsilon\to 0} G_\epsilon^2(x)\|\varphi_{\rm g}(x)\|^2_{\mathcal{F}_{\rm b}} = G^2(x)\|\varphi_{\rm g}(x)\|^2_{\mathcal{F}_{\rm b}},$$

Fatou's lemma yields that $G\|\varphi_{\rm g}(\cdot)\|_{\mathcal{F}_{\rm b}} \in L^2(\mathbb{R}^3)$ and

$$\|G\varphi_{\rm g}\|^2 \leq \liminf_{\epsilon\to 0}\|G_\epsilon\varphi_{\rm g}\|^2 \leq \left(\frac{a'+b}{E_{\rm bin}-V_\infty-\epsilon}\right)\|\varphi_{\rm g}\|^2.$$

Then

$$\frac{\|G\varphi_{\rm g}\|^2}{\|\varphi_{\rm g}\|^2} \leq \frac{a'+b}{E_{\rm bin}-V_\infty-\epsilon}$$

follows. The proof is complete. $\square$

**Remark 5.4** *It can be also proven that $\varphi_{\rm g} \in D(e^{\beta|x|}\otimes 1)$ for some $\beta$ under some conditions on $V$, e.g., $V(x) = -e^2/|x|$. See [15, 21].*

## 5.5 Proof of Lemma 4.16

*Proof of Lemma 4.16*

We prove the lemma for $\mu = 1$. Proofs for $\mu = 2,3$ are the same as that of $\mu = 1$. Let $\epsilon = (a,0,0) \in \mathbb{R}^3$. We see that $e^{i\epsilon x}$ maps $D(H_{\rm PF,0})$ onto itself. Thus

$$e^{i\epsilon x}H_{\rm PF}\Psi = H_{\rm PF}e^{i\epsilon x}\Psi + (H_{\rm PF}(\epsilon) - H_{\rm PF})e^{i\epsilon x}\Psi, \quad \Psi \in D(H_{\rm PF,0}),$$

where

$$\begin{aligned} H_{\rm PF}(\epsilon) &:= \frac{1}{2m}(p\otimes 1 + \epsilon - eA_{\hat\varphi}) + V\otimes 1 + 1\otimes H_{\rm f} - \frac{1}{2m}(\sigma\otimes 1)\cdot B_{\hat\varphi} \\ &= H_{\rm PF} + \frac{1}{m}(p\otimes 1 - eA_{\hat\varphi})\cdot\epsilon + \frac{1}{2m}|\epsilon|^2. \end{aligned}$$

We have

$$\left(\frac{e^{i\epsilon x}-1}{a}\right)H_{\rm PF}\Psi = H_{\rm PF}\left(\frac{e^{i\epsilon x}-1}{a}\right)\Psi + \frac{1}{a}(H_{\rm PF}(\epsilon)-H_{\rm PF})e^{i\epsilon x}\Psi.$$

By the assumption we see that

$$s-\lim_{a\to 0}\left(\frac{e^{i\epsilon x}-1}{a}\right)H_{\rm PF}\Psi = i(x_1\otimes 1)H_{\rm PF}\Psi. \tag{5.16}$$

Directly we see that

$$\begin{aligned}\frac{1}{a}(H_{\rm PF}(\epsilon)-H_{\rm PF})e^{i\epsilon x}\Psi &= \left(\frac{1}{m}(p\otimes 1-eA_{\hat\varphi})_1+\frac{1}{2m}a\right)e^{i\epsilon x}\Psi \\ &= e^{i\epsilon x}\left(\frac{1}{m}(p\otimes 1-\epsilon-eA_{\hat\varphi})_1+\frac{1}{2m}a\right)\Psi \\ &\to \frac{1}{m}(p\otimes 1-eA_{\hat\varphi})_1\Psi \end{aligned} \tag{5.17}$$

strongly as $a\to 0$. From (5.16) and (5.17) it follows that

$$s-\lim_{\epsilon\to 0}H_{\rm PF}\left(\frac{e^{i\epsilon x}-1}{a}\right)\Psi = i(x_1\otimes 1)\Psi - \frac{1}{m}(p\otimes 1-eA_{\hat\varphi})_1\Psi.$$

Since $H_{\rm PF}$ is closed and

$$s-\lim_{\epsilon\to 0}\left(\frac{e^{i\epsilon x}-1}{a}\right)\Psi = i(x_1\otimes 1)\Psi,$$

it follows that $(x_1\otimes 1)\Psi\in D(H_{\rm PF})$. Thus (1) of Lemma 4.16 is proven. Let

$$\Psi\in\mathcal{S}:=C_0^\infty(\mathbb{R}^3;\mathbb{C}^2)\otimes[\bigoplus_{n=0}^\infty\otimes_s^n C_0^\infty(\mathbb{R}^3)].$$

Then we have

$$[(x_1\otimes 1),H_{\rm PF}]\Psi = \frac{i}{m}(p\otimes 1-eA_{\hat\varphi})_1\Psi. \tag{5.18}$$

Hence

$$\|[(x_1\otimes 1),H_{\rm PF}]\Psi\|\le c\|(H_0+1)^{1/2}\Psi\|$$

follows with some constant $c$. Then the closure $\overline{[(x_1\otimes 1),H_{\rm PF}]\lceil_{\mathcal{S}}}$ is well defined on $D(H_0^{1/2})$ and satisfies

$$\overline{[(x_1\otimes 1),H_{\rm PF}]}\Psi = \frac{i}{m}(p\otimes 1-eA_{\hat\varphi})_1\Psi$$

for $\Psi\in D(H_0^{1/2})$. In particular,

$$[(x_1\otimes 1),H_{\rm PF}]\varphi_{\rm g} = \overline{[(x_1\otimes 1),H_{\rm PF}]\lceil_{\mathcal{S}}}\varphi_{\rm g} = \frac{i}{m}(p\otimes 1-eA_{\hat\varphi})_1\varphi_{\rm g}$$

follows, since $\varphi_{\rm g}\in D((x_1\otimes 1)H_{\rm PF})\cap D(H_{\rm PF}(x_1\otimes 1))$. □

**Acknowledgements** I am very grateful to A. Arai and M. Hirokawa for useful comments and discussions.

# References

[1] S. Albeverio, An introduction to some mathematical aspects of scattering theory in models of quantum fields, *Scattering theory in mathematical physics* ed. by J. A LaVita and J. P. Marchand (1974), 299–381.

[2] A. Arai, Essential spectrum of a self-adjoint operator on an abstract Hilbert space of Fock type and applications to quantum field Hamiltonians, *J. Math. Anal. Appl.* **246** (2000), 189–216.

[3] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* **151** (1997), 455–503.

[4] A. Arai and M. Hirokawa, Ground states of a general class of quantum field Hamiltonians, *Rev. Math. Phys.* **12** (2000), 1085–1135.

[5] A. Arai, M. Hirokawa, and F. Hiroshima, On the absence of eigenvectors of Hamiltonians in a class of massless quantum field models without infrared cutoff, *J. Funct. Anal.* **168** (1999), 470–497.

[6] A. Arai, M. Hirokawa, and F. Hiroshima, Regularity of ground states in quantum field models, in preparation.

[7] A. Arai and H. Kawano, Enhanced binding in a general class of quantum field models, *Rev. Math. Phys.* **15** (2003), 387–423.

[8] V. Bach, J. Fröhlich, I. M. Sigal, Quantum electrodynamics of confined nonrelativistic particles, *Adv. Math.* **137** (1998), 299–395.

[9] V. Bach, J. Fröhlich, I. M. Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field, *Commun. Math. Phys.* **207** (1999), 249–290.

[10] J. M. Barbaroux, T. Chen and S. Vugalter, Binding conditions for atomic N-electron systems in non-relativistic QED arXiv math-ph/0304019, preprint, 2003.

[11] J. M. Barbaroux, M. Dimassi and J. C. Guillot, Quantum electrodynamics of relativistic bound states with cutoffs, arXiv:math-ph/0312011, preprint 2003.

[12] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschritte der Physik* **22** (1974), 159–198.

[13] J. Fröhlich, M. Griesemer and B. Schlein, Asymptotic electromagnetic fields in a mode of quantum-mechanical matter interacting with the quantum radiation field, *Adv. in Math.* **164** (2001), 349–398.

[14] C. Gérard, On the existence of ground states for massless Pauli-Fierz Hamiltonians, *Ann. Henri Poincaré* **1** (2000), 443–459.

[15] M. Griesemer, E. Lieb and M. Loss, Ground states in non-relativistic quantum electrodynamics, *Invent. Math.* **145** (2001), 557–595.

[16] L. Gross, A noncommutative extension of the Perron-Frobenius theorem, *Bull. Amer. Math. Soc.* **77** (1971), 343–347.

[17] L. Gross, Existence and uniqueness of physical ground states, *J. Funct. Anal.* **10** (1972), 52–109.

[18] R. Høegh-Krohn, Asymptotic fields in some models of quantum field theory I, *J. Math. Phys.* **9** (1968), 2075–2080.

[19] M. Hirokawa Infrared Catastrophe for Nelson's Model, mp-arc 03-512, preprint, 2003.

[20] M. Hirokawa, Mathematical Addendum for "Infrared Catastrophe for Nelson's Model" (mp-arc 03-512), mp-arc,03-551, preprint, 2003.

[21] M. Hirokawa, F. Hiroshima, and H. Spohn, Ground state for point particles interacting through a massless scalar Bose field, arXiv:math-ph/0211050, preprint, 2002, to be published in *Adv. Math.*

[22] F. Hiroshima, Ground states and spectrum of quantum electrodynamics of non-relativistic particles, *Trans. Amer. Math. Soc.* **353** (2001), 4497–4528.

[23] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics I, *J. Math. Phys.* **40** (1999), 6209–6222.

[24] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics II, *J. Math. Phys.* **41** (2000), 661–674.

[25] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants, *Commun. Math. Phys.* **211** (2000), 585–613.

[26] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants, *Ann. Henri Poincaré*, **3** (2002), 171–201.

[27] F. Hiroshima and H. Spohn, Ground state degeneracy of the Pauli-Fierz model with spin, *Adv. Theor. Math. Phys.* **5** (2001), 1091–1104.

[28] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.

[29] E. Lieb and M. Loss, Existence of atoms and molecules in non-relativistic quantum electrodynamics, *Adv. Theor. Math. Phys.* **7** (2003), 667-710.

[30] E. Lieb and M. Loss, A note on polarization vectors in quantum electrodynamics, arXiv:math-ph/0401016, preprint, 2004.

[31] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

[32] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I*, Academic Press, 1980.

[33] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II*, Academic Press, 1975.

[34] M. Reed and B. Simon, *Methods of Modern Mathematical Physics III*, Academic Press, 1979.

[35] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV*, Academic Press, 1978.

[36] A. D. Sloan, The polaron without cutoffs in two space dimensions, *J. Math. Phys.* **15** (1974), 190–201.

[37] H. Spohn, Ground state of quantum particle coupled to a scalar boson field, *Lett. Math. Phys.* **44** (1998), 9–16.

[38] J. Weidmann, *Linear Operators in Hilbert Spaces*, GTM 68, Springer-Verlag, 1980.