Stability of Latitudinal Vortex Rings with Polar Vortices

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1 Introduction

The motion of N point vortices of strengths $\Gamma_1, \ldots, \Gamma_N$ on a non-rotating sphere (center at the origin and radius 1) with position vectors $\mathbf{x}_1, \ldots, \mathbf{x}_N$ is governed by the equations [13].

$$\dot{\mathbf{x}}_k = \sum_{j=0}^{N} \Gamma_j \frac{\mathbf{x}_j \times \mathbf{x}_k}{1 - \mathbf{x}_j \cdot \mathbf{x}_k}, \quad (j \neq k) \tag{1}$$

where in the numerator we mean the cross product of the vectors $\mathbf{x}_j$ and $\mathbf{x}_k$ and in the denominator their Euclidean inner product.

We are interested in studying the stability properties of a class of relative equilibrium configurations of vortices: configurations of latitudinal rings with and without polar vortices. Relative equilibrium configurations are so called because the motion vanishes in an appropriate (rotating and/or translating) reference frame; for a review (and quite a number of results) on the subject we refer the reader to the quite exhaustive article by Aref and al. [1]. In this article our interest is focused on configurations that are m regular polygonal rings of n identical vortices, with or without central polar vortices (of arbitrary vorticity Γ_N and Γ_S). Such configurations undergo to a uniform rotational motion of angular speed ω and therefore, by viewing the dynamics in an appropriate co-rotating reference frame, we can treat them as equilibrium configurations. We then study their stability, linear and nonlinear, as a function of the number of vortices n on each ring, of the co-latitude θ_i, the vortex strengths Γ_j $(j = 1, ..., 2N+2)$. We recover previous results for vortices in the plane and on the sphere [5, 4, 13, 1] for the case of a single ring, with a simpler method, and we generalize them to the general case of m rings. Furthermore, we show why linear and nonlinear stability intervals coincide (a. e.).

The main considerations for our study are:

i) Make use of the Hamiltonian formalism for a theoretical study of the stability, linear and nonlinear.

ii) Change of reference frame: we view the dynamics in a frame corotating with the relative equilibrium configuration. The relative equilibrium becomes a equilibrium in the new reference system, and the standard techniques can be used to study its stability.

iii) It is preferable to work in the full system (as opposed to the reduced system). The "reduction" is made at the end, and actually we shall have it for "free". In fact, when the appropriate coordinates are used, the symmetries of the system are quite manifest and it is easy to identify the cyclic coordinates which are related to the symmetries and deduce the reduced Hamiltonian.

The article is organized as follows:

1) Introduction.

2) Equations of motion.

3) Relative equilibria.

4) Stability analysis.

   4.1) Linear stability & nonlinear stability of autonomous Hamiltonian systems.

   4.2) Stability of vortex rings: the main structure of the Hessian.

5) Particular cases.

   5.1) One ring.

   5.2) One ring with one polar vortex.

   5.3) One ring with two polar vortices.

   5.4) Two rings.

Before ending this section we would like to add that other people have been working on the same issues with different approaches: Cabral and Schmidt [6], and Laurent-Polz, Montaldi and Roberts [9]

2 The equations of motion & relative equilibria

The vortex equations (1) make use of Cartesian coordinates and are particularly useful when numerical simulations are made, as we shall discuss further on, in Sec. 4 when studying orbital stability. Nevertheless, for a theoretical understanding a Hamiltonian formulation of the vortex dynamics is more enlightening. Eq. (1) can be rewritten as

$$\Gamma_\alpha \mathbf{u}_\alpha = \mathbf{n} \times \nabla_\alpha H \tag{2}$$

where $\mathbf{u}_\alpha = (\dot{x}_\alpha, \dot{y}_\alpha, \dot{z}_\alpha)$; $\mathbf{n}$ is the unit-vector normal to the surface; $\nabla_\alpha = (\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial z_\alpha})$; H is the Hamiltonian describing the vortex dynamics. In fact, in the appropriate coordinates – reflecting the symmetry of the surface where the vortex dynamics takes place– Eq. (2) represents a system of Hamiltonian equations. For the sphere, in cylindrical coordinates $(x_\alpha, y_\alpha, z_\alpha) = (r_\alpha \cos\varphi_\alpha, r_\alpha \sin\varphi_\alpha, z_\alpha)$, where $r_\alpha = \sqrt{1 - z_\alpha^2}$ and $z_\alpha \in [-1, 1]$, Eq. (2) becomes

$$\Gamma_\alpha \dot{\varphi}_\alpha = -\frac{\partial H}{\partial z_\alpha}, \qquad \Gamma_\alpha \dot{z}_\alpha = \frac{\partial H}{\partial \varphi_\alpha}, \qquad \alpha = 1, ..., N, \tag{3}$$

where Γ_α and

$$(q_\alpha, p_\alpha) = (\varphi_\alpha, \Gamma_\alpha z_\alpha)$$

are respectively the vortex strength and a pair of conjugate variables associated to the αth vortex. In the variables (q_α, p_α), Eq. (3) takes the usual canonical form

$$\dot{q}_\alpha = -\frac{\partial H}{\partial p_\alpha}, \qquad \dot{p}_\alpha = \frac{\partial H}{\partial q_\alpha}, \qquad \alpha = 1, ..., N, \tag{4}$$

with

$$H = H(q_1, \ldots, q_N; p_1, \ldots, p_N) = \sum_{1 \le i < j \le N} \Gamma_i \Gamma_j \log(l_{ij}^2); \tag{5}$$

$$l_{ij}^2 = 2(1 - \mathbf{x}_i \cdot \mathbf{x}_j) = 2[1 - \frac{p_i p_j}{\Gamma_i \Gamma_j} - r_i r_j \cos(q_i - q_j)], \tag{6}$$

is the square of the Euclidean distance between the ith and jth vortices [1]; $r_j = \sqrt{1 - p_j^2/\Gamma_j^2}$ being the distance from $\mathbf{x}_j$ to the vertical axis (the line that goes through the poles of the sphere). It follows that a system of N point-vortices has N Hamiltonian degrees of freedom. Moreover, in addition to the Hamiltonian H, a system of point-vortices on a unit-sphere has the following integrals of motion:

$$L = \sum_{j=1}^{N} \Gamma_j \parallel \mathbf{x}_j \parallel^2$$

$$P_x = \sum_{j=1}^{N} \Gamma_j x_j, \quad P_y = \sum_{j=1}^{N} \Gamma_j y_j, \quad P_z = \sum_{j=1}^{N} \Gamma_j z_j,$$

[1] We can equivalently write the Hamiltonian in a more geometrical way by considering the geodesic distance on the sphere; this is beyond the purposes of this article and it is discussed in [8]

expressing respectively the conservation of angular momentum and linear momentum in the space. Notice that the integral of motion L simplifies to $L = \sum_{j=1}^{N} \Gamma_j$ and it is therefore redundant, while $P_z = \sum_{j=1}^{N} p_j$. Then, the moment of vorticity vector

$$\mathbf{M} = (P_x, P_y, P_z) \tag{7}$$

plays a special role in characterizing the dynamics of the vortex system. In fact, whenever $\mathbf{M} \neq 0$, we use the freedom in the axis orientations to select the reference system with the z-axis in the direction of $\mathbf{M}$. This choice gives $P_x = P_y = 0$.

Now, when considering configurations with some vortices near the poles, it is more appropriate to consider the following canonical coordinates for the "polar" vortices:

$$\begin{aligned} x_N &= \sqrt{2\Gamma_N(1-z_N)}\cos\varphi_N, \qquad y_N = \sqrt{2\Gamma_N(1-z_N)}\sin\varphi_N \\ x_S &= \sqrt{2\Gamma_S(1+z_S)}\cos\varphi_S, \qquad y_S = \sqrt{2\Gamma_S(1+z_S)}\sin\varphi_S, \end{aligned}$$

where Γ_N, (x_N, y_N) and Γ_S, (x_S, y_S) are respectively the vortex strength and the canonical variables for a northern and a southern polar vortex. In this section, to avoid confusion in notation, we shall use the indices $j = 1, ..., N$ for the vortices on the rings ($N = m \cdot n$), $j = N+1$ and $j = N+2$ respectively for the northern and southern polar vortices, so that

$$q_{N+1} = x_N, \quad p_{N+1} = y_N, \quad q_{N+2} = x_S, \quad p_{N+2} = y_S.$$

The vortex dynamics is still described by Eq. (4) which becomes

$$\dot{q}_j = -\frac{\partial H}{\partial p_j}, \quad \dot{p}_j = \frac{\partial H}{\partial q_j} \qquad (j = 1, \ldots, N+2) \tag{8}$$

with the function $H = H(q_1, \ldots, q_{N+2}; p_1, \ldots, p_{N+2})$ given by

$$H = H_o + H_1 + H_2, \tag{9}$$

where H_o is as H in Eq. (5) and

$$H_1 = \sum_{k=1}^{2} \Gamma_{N+k} \sum_{j=1}^{N} \Gamma_j \log(l_{j,N+k}^2) \text{and} \quad H_2 = \Gamma_{N+1}\Gamma_{N+2}\log(l_{N+1,N+2}^2),$$

with

$$l_{j,N+k}^2 = 2[1 - \frac{p_j}{\Gamma_j} z_{N+k} - r_j(q_{N+k}\cos(q_j) + p_{N+k}\sin(q_j))f_k], \tag{10}$$

$$l_{N+1,N+2}^2 = 2[1 - z_{N+1}z_{N+2} - (q_{N+1}q_{N+2} + p_{N+1}p_{N+2})f_1 f_2] \tag{11}$$

and

$$z_{N+1} = 1 - (q_{N+1}^2 + p_{N+1}^2)/2\Gamma_{N+1}, \quad z_{N+2} = -1 + (q_{N+2}^2 + p_{N+2}^2)/2\Gamma_{N+2},$$

$$f_k = \sqrt{(4\Gamma_{N+k} - q_{N+k}^2 - p_{N+k}^2)}/(2\Gamma_{N+k}), \qquad k = 1, 2.$$

Remark: As previously mentioned, whenever $M \neq 0$, we use the freedom of orienting the axes by choosing the z-axis in the direction of the momentum vector (7) $\mathbf{M} = (P_x, P_y, P_z) = (0, 0, M)$, which gives

$$P_x = \sum_{j=1}^{N} \Gamma_j r_j \cos(q_j) + \sum_{k=1}^{2} \Gamma_{N+k} q_{N+k} f_k = 0, \quad P_y = \sum_{j=1}^{N} \Gamma_j r_j \sin(q_j) + \sum_{k=1}^{2} \Gamma_{N+k} p_{N+k} f_k = 0, \tag{12}$$

and

$$M = \sum_{j=1}^{N} p_j + \Gamma_{N+1} - \Gamma_{N+2} - \frac{q_{N+1}^2 + p_{N+1}^2}{2} + \frac{q_{N+2}^2 + p_{N+2}^2}{2}. \tag{13}$$

Proposition 2.1

$$M_1 = \frac{q_{N+1}^2 + p_{N+1}^2 - q_{N+2}^2 - p_{N+2}^2}{2} \tag{14}$$

is a constant of motion.

Proof. By the Eq. (4)-(12) it is immediate to check that

$$\sum_{k=1}^{2} f_k q_{N+k} = -\sum_{j=1}^{N} \Gamma_j r_j \cos(q_j), \qquad \sum_{k=1}^{1} f_k p_{N+k} = -\sum_{j=1}^{N} \Gamma_j r_j \sin(q_j),$$

and therefore

$$\dot{M}_1 = q_{N+1}\dot{q}_{N+1} + p_{N+1}\dot{p}_{N+1} - q_{N+2}\dot{q}_{N+2} - p_{N+2}\dot{p}_{N+2} = 0.$$

Furthermore, Proposition 2.1 and Eq. (13) imply that

$$\tilde{M} = \sum_{j=1}^{N} p_j \tag{15}$$

is still a constant of motion.

3 Relative equilibria

We focus on configurations on $\mathbb{S}^2$ that are of m regular polygonal rings and a vortex at each pole. Each ring has with n identical vortices ($n = m/N$) and is at a vertical distance z_j from the equatorial plane. For vortices on the rings, we express the vertical distance z_j as a function of the colatitude variables θ_j, i.e. $z_j = \cos(\theta_j)$ and, therefore, $r_j = \sin(\theta_j)$. More specifically, we are considering vortices placed as follows:

- One vortex at $(0,0,1)^T$ with vorticity Γ_N and one vortex at $(0,0,-1)^T$ with vorticity Γ_S,
- m rings of n vortices each one, located at:

  a) (symmetrical configurations)
  $$(\sin(\theta_i)\cos(2\pi(j-1)/n), \sin(\theta_i)\sin(2\pi(j-1)/n), \cos(\theta_i))^T; \tag{16}$$
  or

  b) (staggered configurations)
  $$\begin{cases} (\sin(\theta_i)\cos(2\pi(j-1)/n), \sin(\theta_i)\sin(2\pi(j-1)/n), \cos(\theta_i))^T & \text{for } i \text{ odd} \\ (\sin(\theta_i)\cos(\pi(2j-1)/n), \sin(\theta_i)\sin(\pi(2j-1)/n), \cos(\theta_i))^T & \text{for } i \text{ even} \end{cases},$$

  for $j = 1, ..., n$ and vorticity of each one of the vortices in the i-th ring equal to Γ_i, for $i = 1, ..., m$.

Two two classes of (rotating) relative equilibria are possible (for specific values of $z_i = \cos(\theta_i), \quad i = 1, ..., m$): the symmetrical and the staggered configurations.

In this article we focus on the study of symmetrical configurations, since for the staggered a similar analysis applies. The conditions to have a (rotating)[1] relative equilibrium $Z^* = (q_1^*, ..., q_{N+2}^*, p_1^*, ..., p_{N+2}^*)$ are

$$\begin{aligned} \frac{dq_j}{dt}|_{Z^*} &= \omega, \quad \frac{dp_j}{dt}|_{Z^*} = 0, \qquad \text{for} \quad j = 1, ..., N, && (18) \\ \frac{dq_{N+k}}{dt}|_{Z^*} &= 0, \quad \frac{dp_{N+k}}{dt}|_{Z^*} = 0, \qquad \text{for} \quad k = 1, 2, && (19) \end{aligned}$$

with ω a constant, which gives

$$\begin{cases} q_j^*(t) = \omega t + q_j^*(0), \qquad z_j^*(t) = z_j^*(0), \qquad j = 1, ..., N, \\ q_{N+k}^*(t) = 0, \qquad p_{N+k}^*(t) = 0, \qquad k = 1, 2. \end{cases} \tag{20}$$

Then the conditions to have a relative equilibrium on m rings of n vortices each plus, eventually, vortices at the north and south pole of a sphere or at the origin in the plane can be simplified as follows:

[1] For a complete discussion about relative equilibria, rotational & translational, see the review article by Aref et al. [1].

Proposition 3.1 (symmetrical configurations)
There exists $\omega \in \mathbb{R}$ such that the following conditions hold.

a) In the spherical case we obtain

$$\omega = -\frac{1}{\sin^2(\theta_i)}\left[\Gamma_N(1+\cos(\theta_i)) - \Gamma_S(1-\cos(\theta_i)) + \Gamma_i(n-1)\cos(\theta_i) + \sum_{j=1, j\neq i}^{m} \Gamma_j n f_{ij}\right] \tag{21}$$

for $i = 1, ..., m$, where

$$f_{ij} = \cos(\theta_i) + \frac{x_{ij}^n + 1}{x_{ij}^n - 1}, \qquad x_{ij} = \frac{\sin\left(\frac{\theta_i+\theta_j}{2}\right) + \sin\left(\frac{\theta_i-\theta_j}{2}\right)}{\sin\left(\frac{\theta_i+\theta_j}{2}\right) - \sin\left(\frac{\theta_i-\theta_j}{2}\right)}.$$

b) The planar case is obtained from the spherical one by setting $\Gamma_S = 0$, putting $\theta_i = \epsilon\mu_i$ for $i = 1, ..., m$, letting ϵ go to zero and collecting the terms in the dominant power of ϵ. We obtain

$$\omega = -\frac{1}{\mu_i^2}\left[2\Gamma_N + \Gamma_i(n-1) + \sum_{j=1, j\neq i}^{m} \Gamma_j n \frac{2}{1-(\mu_j/\mu_i)^n}\right]$$

for $i = 1, ..., m$.

The proof uses the relation

$$\sum_{k=0}^{n-1} \frac{1}{1+x^2-2x\cos(2\pi k/n)} = \frac{n}{1-x^2}\frac{1+x^n}{1-x^n}$$

for $|x| < 1$, which is a particular case of Theorem 1 in Chapter 2 in the Ph.D. thesis of D.Bang. From this formula it is possible to obtain close expressions for sums of the form $\sum_{k=0}^{n-1}(a+b\cos(2\pi k/n))/(c+d\cos(2\pi k/n))$ if $|c| > |d|$. After several simplifications the result follows.

The Proposition 3.1 gives $m-1$ conditions to be satisfied by the colatitudes θ_i of the m rings, for fixed values of the vorticities. To better understand the conditions (21), let's consider few specific examples:

i) On the sphere: one ring ($m = 1$) with polar vortices (Fig. 3.*a*)). The Eq. (21) determines the angular velocity of the relative equilibrium:

$$\omega = -\Gamma_1 \frac{z_1}{r_1^2}(n-1) - \frac{\Gamma_N}{1-z_1} + \frac{\Gamma_S}{1+z_1}, \quad \text{for} \quad \forall z_1 \in (-1, 1); \tag{22}$$

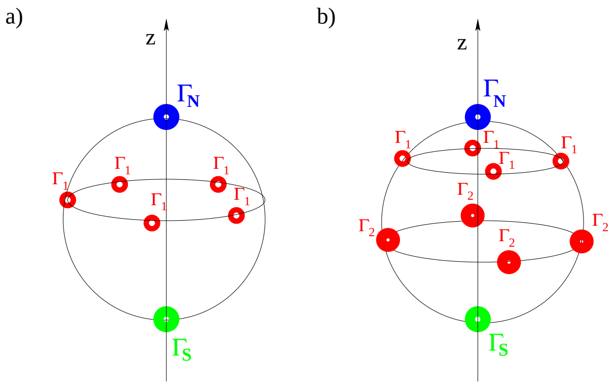

Figure 1: Configurations of rings and polar vortices: *a)* $m = 1$, $n = 5$; *b)* $m = 2$, $n = 4$.

ii) On the sphere: two rings ($m = 2$) with polar vortices (Fig. 3.*b*)). Eq. (21) become the system of equations

$$\begin{aligned}\Gamma_N \frac{1}{1-z_1} - \Gamma_S \frac{1}{1+z_1} \quad &+ \quad \Gamma_1 \frac{z_1}{r_1^2}(n-1) + \\ &\frac{\Gamma_2}{r_1} \sum_{j=1}^{n} \frac{r_1 z_2 - z_1 r_2 \cos(2\pi(j-1)/n))}{1 - z_1 z_2 - r_1 r_2 \cos(2\pi(j-1)/n)} + \omega = 0 \\ \Gamma_N \frac{1}{1-z_2} - \Gamma_S \frac{1}{1+z_2} \quad &+ \quad \Gamma_2 \frac{z_2}{r_2}(n-1) + \\ &\Gamma_1 \sum_{j=1}^{n} \frac{r_2 z_1 - z_2 r_1 \cos(2\pi(j-1)/n))}{1 - z_1 z_2 - r_1 r_2 \cos(2\pi(j-1)/n)} + \omega = 0\end{aligned}$$

where the angular velocity ω and $z_2 = \cos\theta_2$ are the unknowns, for given n, $z_1 = \cos\theta_1$, Γ_1, Γ_2, Γ_N and Γ_S.

Figure 3 illustrates a couple of cases for $n = 5$.

iii) In the plane. The figure 3 illustrates, for the case of two rings and $n = 5$, the values of $\mu = \mu_2/\mu_1$ as a function of Γ_1, Γ_2 for $\Gamma_N = 1$, and also the values of μ for $\Gamma_1 = 1$ as a function of Γ_2 in the case of $\Gamma_N = 0$.

Remarks:

i) In Proposition 3.1 from Eq. 21 we can deduce that

$$\frac{1}{\sin^2(\theta_i)} \left[\Gamma_N(1+\cos(\theta_i)) - \Gamma_S(1-\cos(\theta_i)) + \Gamma_i(n-1)\cos(\theta_i) + \sum_{j=1, j\neq i}^{m} \Gamma_j n f_{ij} \right]$$

must be independent of i for $i = 1, ..., m$.

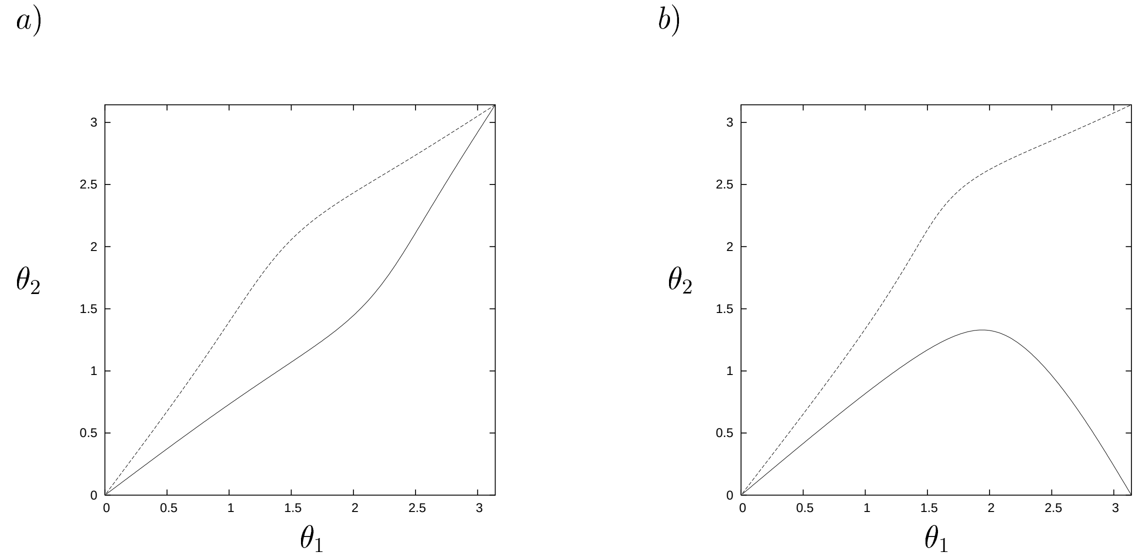

Figure 2: Case of two rings ($m = 2$), $n = 5$ and $\Gamma_N = 10$. *a*) $\Gamma_S = 3$, $\Gamma_1 = \Gamma_2 = 1$; *b*) $\Gamma_S = -3$, $\Gamma_1 = -1$, $\Gamma_2 = 2$

.

ii) A quite complete classification of relative equilibria on the sphere has been previously done by Lim, Montaldi and Roberts [10], where they find similar conditions (as in Proposition 3.1) for determining the relative equilibria.

4 Stability

Fig. 4 shows a numerical study of orbital stability for configurations of relative equilibria of a vortex ring with a northern polar vortex. On the vertical axes is the colatitude θ and on the horizontal axis the magnitude of the polar vortex strength Γ_N. For this case $m = 1$ and $N = n$. For a given N, the stability region is delimited by the two corresponding curves (labeled by the number N of the corresponding vortex configuration). The curves have been obtained by a direct numerical integration of Eq. (1) and the question arises: why the transition in stability occur at such points? Can we have a theoretical understanding of such curves?

In what follows we will study the linear and nonlinear stability of the family of symmetrical relative equilibria (16) and to do so it is easier to make use of the powerful tools of Hamiltonian dynamics and to consider the equations of motion in the form (26), that in a more compact form become

$$\dot{Z} = J\nabla H(Z) \tag{23}$$

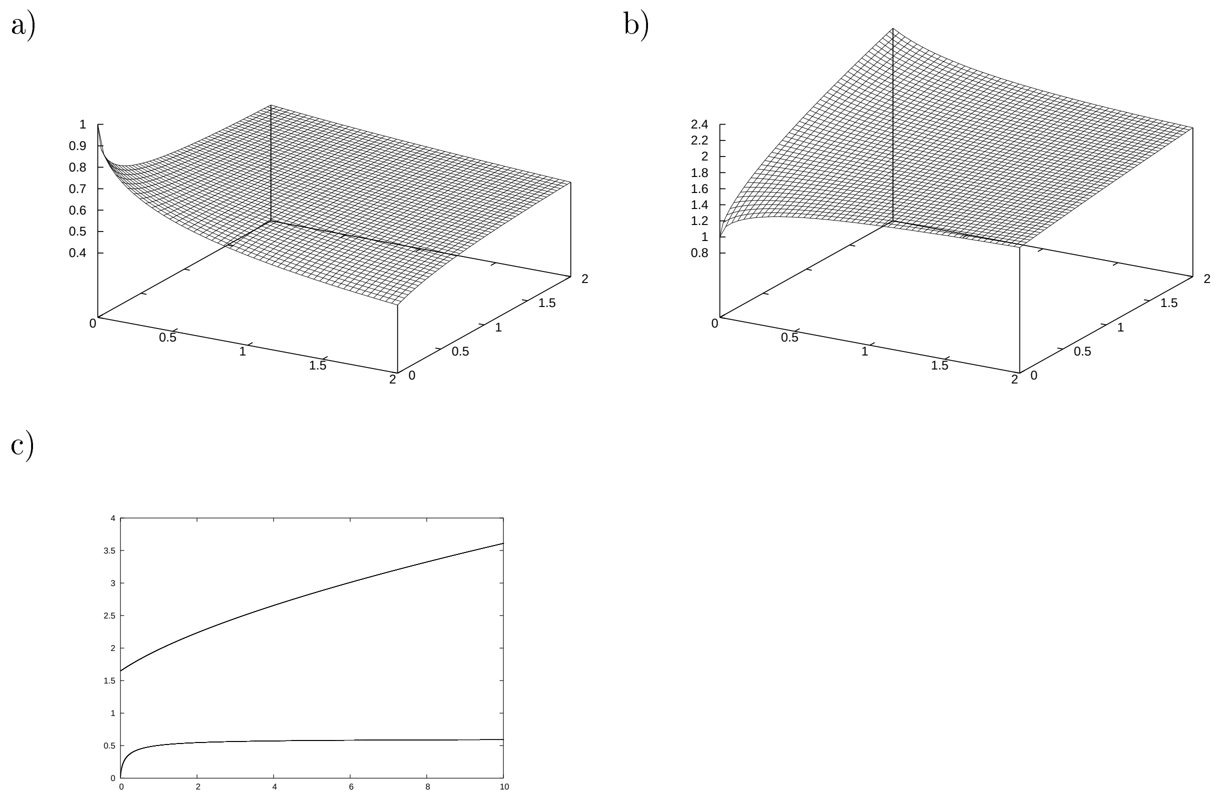

Figure 3: $n = 5$ and $\Gamma_N = 0$. a) μ as a function of Γ_1 and Γ_2, lower surface; b) μ as a function of Γ_1 and Γ_2, upper surface; c) μ as a function of Γ_2 for $\Gamma_1 = 1$.

where $Z = (\mathbf{q}, \mathbf{p})^T = (q_1, ..., q_{N+2}, p_1, ..., p_{N+2})^T$, J is

$$J = \left(\begin{array}{c|c} 0 & -\mathbb{I} \\ \hline \mathbb{I} & 0 \end{array} \right),$$

$\mathbb{I}$ is the $(2N \times 2N)$ identity matrix and ∇ is the gradient vector in the variables $(\mathbf{q}, \mathbf{p})$. Let Z^* denote a (symmetric) relative equilibrium of the family (16), representing a configuration of $m = N/n$ rings, each at a vertical distance z_i $(j = i, ..., m)$ from the equatorial plane, and two polar vortices, i.e. (16) for $2 < n \leq N$

$$\left\{ \begin{array}{ll} q_j^* = 2\pi(j-1)/n, \qquad p_j^* = \Gamma_i z_i, \qquad j = 1, ..., N, \quad i = \left[\frac{j-1}{n}\right] + 1, \\ q_{N+1}^* = 0, \quad p_{N+1}^* = 0, \quad z_{N+1} = 1, \qquad q_{N+2}^* = 0, \quad p_{N+2}^* = 0, \quad z_{N+2} = -1. \end{array} \right. \tag{24}$$

As stated in Proposition 3.1, the entire configuration is rigidly rotating with a frequency ω, solution of (21), i.e. (20)

$$\left\{ \begin{array}{lll} q_j(t) = \omega t + q_j^*, & z_j(t) = z_j^*, & j = 1, ..., N, \\ q_{N+k}(t) = q_{N+k}^*, & p_{N+k}(t) = p_{N+k}^*, & k = 1, 2. \end{array} \right.$$

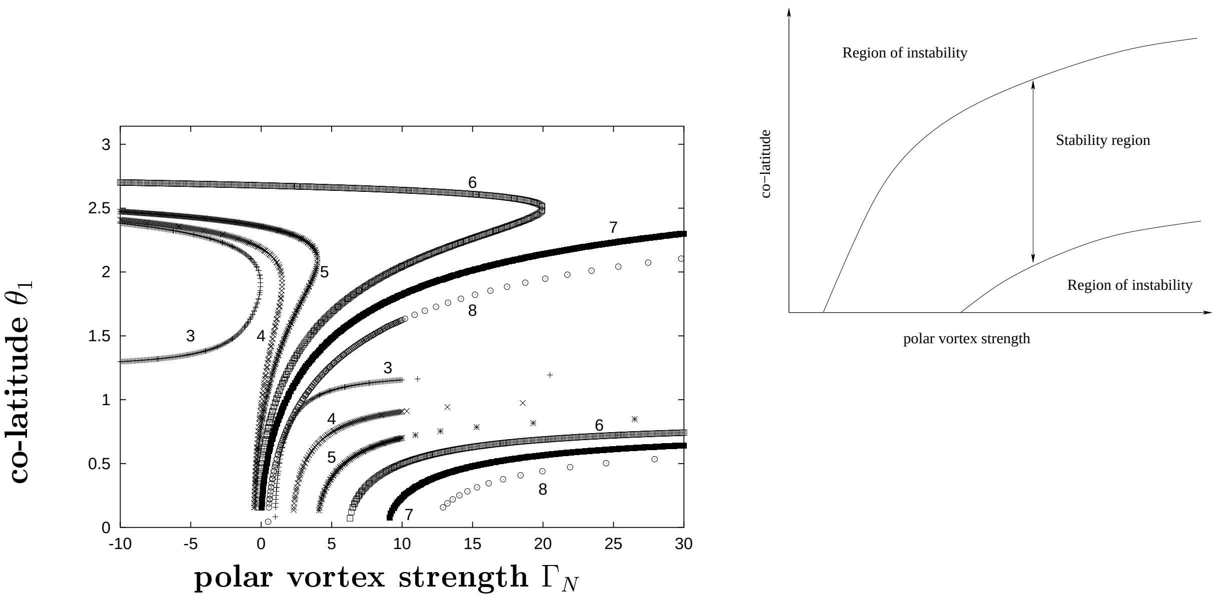

Figure 4: Orbital stability: a numerical study. Curves delimiting the orbital stability region as a function of the number of vortices N.

We want to study stability of its orbit. The only direct method is by numerical integration of Eq. (1), by perturbing the initial conditions and checking if the perturbed resulting orbit stays "near" the original unperturbed one (the relative equilibrium orbit) for sufficiently long time. Numerical methods are summarized in Sec. **??**. Since we want also to study the problem from a theoretical point of view, to be later on compared with our numerical predictions, we choose another approach. We observe that for every relative equilibrium, when viewing the dynamics from the reference frame in co-rotation with the vortex configuration, the vortices are "frozen" and in that frame Z^* is an equilibrium. This amounts to consider the coordinate transformation:

$$\tilde{q}_j = q_j - \omega t, \qquad \tilde{p}_j = p_j \qquad (j = 1, ..., N)$$
$$\tilde{q}_{N+k} = q_{N+k}\cos\omega t + p_{N+k}\sin\omega t, \quad \tilde{p}_{N+k} = -q_{N+k}\sin\omega t + p_{N+k}\cos\omega t, \quad k = 1, 2 \qquad .$$

The equations of motion (23) in the rotating coordinates become

$$\dot{\tilde{Z}} = J\tilde{\nabla}\tilde{H}(\tilde{Z}) \tag{25}$$

where $\tilde{\nabla}$ is the gradient with respect to the variables $\tilde{Z} = (\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) = (\tilde{q}_1, ..., \tilde{q}_{N+2}, \tilde{p}_1, ..., \tilde{p}_{N+2})$, $\tilde{H}(\tilde{Z}) = \tilde{H}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$,

$$\begin{aligned} \tilde{H}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) &= H(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) + \omega M, \\ &= H(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) \\ &+ \omega \sum_{j=1}^{N} \tilde{p}_j + \omega \left(\Gamma_{N+1} - \Gamma_{N+2} - \frac{\tilde{q}_{N+1}^2 + \tilde{p}_{N+1}^2}{2} + \frac{\tilde{q}_{N+2}^2 + \tilde{p}_{N+2}^2}{2} \right), \end{aligned} \tag{26}$$

$H(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ as H in Eq. (9), and M is the momentum magnitude defined in Eq. (13). In what follows, we drop the tilde on top of the variables regarding these nevertheless as coordinates in the rotating coordinate system.
Remarks:

1) $\tilde{H}$ is still a first integral of this system and that at the equilibrium Z^* the Hessian of $\tilde{H}$ is given by
$$\text{Hess } \tilde{H} = \text{Hess } H + \Omega, \tag{27}$$
where Ω is the square matrix of order $2N+4$ that differs from the zero matrix in that the only non-zero entries are:
$$\Omega_{N+1,N+1} = \Omega_{2N+2,2N+2} = -\omega, \quad \Omega_{N+2,N+2} = \Omega_{2N+4,2N+4} = \omega.$$

2) As anticipated, each relative equilibrium is an equilibrium when viewed in its own co-rotating reference system. Therefore, for a given (i.e. for specified m, number of rings, and n, number of vortices on each ring) configuration Z^* (21) we can study its stability as an equilibrium point of the autonomous system of equations (25).

4.1 Linear & Nonlinear stability of an autonomous Hamiltonian system

Let Z^* be an equilibrium (24) of the system (25). Near the equilibrium
$$Z = Z^* + \Delta Z,$$
where $\Delta Z = (\Delta q_1, ..., \Delta q_{N+2}, \Delta p_1, ..., \Delta p_{N+})^T$, and the Hamiltonian (26) is approximated by its quadratic form
$$\tilde{H}(Z^* + \Delta Z) = \tilde{H}(Z^*) \quad + \quad \frac{1}{2}\Delta Z^T \tilde{S} \Delta Z \quad + \quad h.o.t. \tag{28}$$
where $\tilde{S}$ is the Hessian of $\tilde{H}$ evaluated at the equilibrium Z^*, and $\tilde{H}(Z^*)$ is a constant which we shall then neglect in what follows.

i) **Linear stability**. It is deduced by studying the corresponding linearized system of equations,
$$\dot{\Delta Z} = J\tilde{S}\Delta Z. \tag{29}$$
By *linear stability* we mean spectral stability and it is inferred from the spectrum of the operator JS, which usually it is not so easy to compute! [1]

Then ideally, when possible, one way to proceed is following:
1) Compute the the eigenvalues λ_j and eigenvectors $\mathbf{v}_j$ ($j = 1, ..., 2N+2$) of $\tilde{S}$.

[1]Notice that while S is always symmetric, JS is almost never so.

Since $\tilde{S}$ is a Hessian, it is symmetric and therefore its eigenvalues are real and it admits an orthonormal basis of eigenvectors $\mathbf{e}_j$.
2) Let $\mathcal{C}$ be the matrix

$$\mathcal{C} = \left(\ \mathbf{e}_1 \, \middle| \, \mathbf{e}_2 \, \middle| \ldots \middle| \, \mathbf{e}_{2N+2} \ \right). \tag{30}$$

If the change of variables

$$Z = \mathcal{C} Z' \tag{31}$$

is symplectic then, with respect to the new basis, the linearized system (29) becomes

$$\dot{\Delta Z'} = J \tilde{S}' \Delta Z' \tag{32}$$

and where

$$\tilde{S}' = \mathcal{C}^T \tilde{S} \mathcal{C} = \begin{pmatrix} \lambda_1 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & \ddots & 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \lambda_{N+1} & \ddots & & \vdots \\ \vdots & & \ddots & \lambda_{N+2} & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_{2N+2} \end{pmatrix}$$

and the Hamiltonian (29) becomes

$$\tilde{H} = \frac{1}{2} \sum_{j=1}^{N} (\lambda_j q'^2_j + \lambda_{j+N} p'^2_j) + h.o.t., \tag{33}$$

where $\lambda_j = \lambda_j(Z^*)$. We have reduced the problem to a system of independent harmonic oscillators! Then we have the following linear stability criterion

Theorem 4.1 (linear stability) *The equilibrium $Z^* = (\mathbf{q}', \mathbf{p}')$, with small oscillations Hamiltonian (33) is:*

1) *linearly stable if* $\forall j \quad \lambda_j \lambda_{j+N} > 0 \quad j = 1, ..., 2N+2$*;*
2) *linearly unstable if* $\exists j$ *such that* $\lambda_j \lambda_{j+N} < 0$.

Remarks:

a) If the case 1) is verified with the exception of some j for which $\lambda_j \lambda_{j+N} = 0$ then further analysis is necessary to infer about stability or instability. Nevertheless, in this case Z^* is commonly said to be *linearly neutrally stable.*

b) If the system (25)-(26) has some symmetries then those will manifest themselves as zero eigenvalues of S, which should not be taken into account when studying the stability. Therefore one has to work in the framework of the reduced system where the symmetries have been factored out and only the "essential" eigenvalues are taken in consideration. In our case we have a zero eigenvalue due to the invariance under rotations around the M-axis (which we have identified as the z-axis).

The following lemma helps us localizing such a zero eigenvalue:

Lemma 4.2 (single zero eigenvalue)
Let $Q = (q_{ij})$ be a symmetric $n \times n$ matrix satisfying $q_{kl} > 0$ for $k \neq l$, and $q_{kk} < 0$, $\sum_{l=1}^{N} q_{kl} = 0$ for each k. Then 0 is a simple eigenvalue of Q and all other eigenvalues are strictly negative.

A proof of this lemma can be found in the article by Montaldi, Soulière and Tokieda [11].

ii) **Nonlinear stability.** For autonomous Hamiltonian systems we are guaranteed to have at least one integral of motion, the Hamiltonian itself. Then we can make use of the following *sufficient* criterion to infer about (nonlinear) stability which was proved by Dirichlet [7, 14] for systems with integrals and subsequently generalized by Lyapunov in his direct method:

Theorem 4.3 (Dirichlet)
Let Z^* be an equilibrium of an autonomous system of ordinary differential equations

$$\frac{dZ}{dt} = f(Z), \qquad \Omega \subset \mathbb{R}^{2N}, \tag{34}$$

that is, $f(Z^*) = 0$.
If there exists a positive (or negative) definite integral $\mathbf{\Psi}$ of the system (34) in a neighborhood of the equilibrium Z^*, then Z^* is stable.

Then let Z^* be an equilibrium (24) and $\tilde{S}$ the Hessian of (26) evaluated at the equilibrium configuration. Since $\tilde{H}$ is a first integral of the system, Dirichlet's theorem implies that if at the equilibrium the quadratic form

$$\Delta Z^T \tilde{S} \Delta Z \tag{35}$$

is positive (or negative) definite, then the equilibrium is stable. But we have already discuss all this when dealing with linear stability: the definiteness of the quadratic form (108) is studied by diagonalizing the matrix $\tilde{S}$ and computing its eigenvalues. If the eigenvalues λ_j have all the same sign then $\tilde{S}$ is definite.

Then for autonomous Hamiltonian systems Dirichlet criterion is something more than a sufficient criterion for nonlinear stability:

Proposition 4.4 *If $\tilde{S}$ has non zero eigenvalues, the Dirichlet criterion is a necessary and sufficient condition for linear stability, and therefore for nonlinear stability, for Z^*.*

4.2 Stability of vortex rings: the main structure

We are now ready to study the stability (linear & nonlinear) of the relative equilibrium configurations Z^* of m rings of vortices with (or without) polar vortices, as defined in (16) (and in (24))

$$\begin{cases} q_j^* = 2\pi(j-1)/n, & p_j^* = \Gamma_i z_i, \quad j = 1, ..., N, \quad i = \left[\frac{j-1}{n}\right] + 1, \\ q_{N+1}^* = 0, \quad p_{N+1}^* = 0, \quad z_{N+1} = 1, & q_{N+2}^* = 0, \quad p_{N+2}^* = 0, \quad z_{N+2} = -1. \end{cases}$$

As discussed in the previous section, we begin by considering the coordinate change of variables which allow us to view the dynamics from a reference frame in corotation with the vortices. In such a frame the relative equilibrium become an equilibrium and we can then apply the techniques discussed in Sec. 4.1 to infer about linear and nonlinear stability. The main point of our analysis is to show that in our case the Hessian $\tilde{S}$ (29) is endowed with a structure that makes it easier to deduce its reduced form (i.e. to factor out the symmetries), and to compute its eigenvalues λ_j and its eigenvector $\mathbf{v}_j$ for all N. In fact, as summarized in Proposition 4.7, $\tilde{S}$ has a very particular block structure which is mainly made out of symmetric and skew-symmetric *circulant matrices*, i.e. $(n \times n)$ matrices of the form

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \dots & a_1 \end{pmatrix} \tag{36}$$

Circulant matrices are of special interest to us because we can easily compute their eigenvalues and eigenvectors for all n. In fact, it is immediate to show that:

Lemma 4.5 (circulant matrices) *All circulant matrices (36) have eigenvalues*

$$\lambda_j = \sum_{k=1}^{N} a_k r_j^{k-1}, \qquad j = 1, ..., n, \tag{37}$$

and corresponding eigenvectors

$$v_j = (1, r_j, ..., r_j^{n-1})^T, \qquad j = 1, ...n, \tag{38}$$

where $r_j = \exp(2\pi(j-1)/n)$ are solutions of $r^n = 1$.

Furthermore if a circulant matrix is symmetric (resp. skew-symmetric) we have that its eigenvalues are real (resp. imaginary).

Lemma 4.6 (symmetric circulant matrices)
Let A be a $(n \times n)$ symmetric circulant matrix; then it has n real eigenvalues of the form

$$\lambda_{Aj} = a_1 + \sum_{k=2}^{n} a_k \cos\left(\frac{2\pi(j-1)(k-1)}{n}\right), \qquad j = 1, ..., n. \tag{39}$$

The eigenvectors of $\lambda_1 = s$ form the one-dimensional subspace generated by $\mathbf{v}_1 = (1,1,...,1)^T$ while in the case n even, the eigenvectors of $\lambda_{n/2}$ form the subspace generated by the vector $\mathbf{v}_{n/2} = (1,-1,1,...,-1)^T$. The others eigenvectors come in pairs

$$\mathbf{v}_j = \begin{pmatrix} 1 \\ \cos\frac{2\pi(j-1)}{n} \\ \cos\frac{4\pi(j-1)}{n} \\ \vdots \\ \cos\frac{2(n-1)\pi(j-1)}{n} \end{pmatrix}, \quad \mathbf{v}'_j = \begin{pmatrix} 0 \\ \sin\frac{2\pi(j-1))}{n} \\ \sin\frac{4\pi(j-1)}{n} \\ \vdots \\ \sin\frac{2(n-1)\pi(j-1)}{n} \end{pmatrix},$$

and are associated to the double eigenvalue $\lambda_j = \lambda_{n-j+2}$ for $j = 2,...,\left[\frac{n-1}{2}\right]$.

For *skew-symmetric* circulant matrices a similar lemma holds with the eigenvalues given by

$$\lambda_{Aj} = a_1 + 2i\sum_{k=2}^{\left[\frac{n-1}{2}\right]} a_k \sin\left(\frac{2\pi(j-1)(k-1)}{n}\right), \qquad j = 1,...,n. \tag{40}$$

It is manifest that the eigenvectors $\mathbf{v}_j$ $(j = 1,...,n)$ of Lemma 4.6 are orthogonal to each other. To simplify the calculations, we shall consider the corresponding orthonormal basis

$$\mathbf{e}_1 = \sqrt{\frac{1}{n}}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}; \mathbf{e}_2 = \sqrt{\frac{2}{n}}\begin{pmatrix} 1 \\ \cos\left(\frac{2\pi}{n}\right)) \\ \vdots \\ \cos\left(\frac{2(n-1)\pi}{n}\right) \end{pmatrix}; \mathbf{e}_3 = \sqrt{\frac{2}{n}}\begin{pmatrix} 0 \\ \sin\left(\frac{2\pi}{n}\right) \\ \vdots \\ \sin\left(\frac{2(n-1)\pi}{n}\right) \end{pmatrix}; .... \tag{41}$$

and $\mathbf{e}_n = \sqrt{\frac{1}{n}}(1,-1,1,...,-1)^T$ if n is even. As we shall discuss more in detail in what follows the eigenvectors $\mathbf{e}_2$ and $\mathbf{e}_3$ play a special role in describing the interaction of each latitudinal ring with each of the polar vortices.

Proposition 4.7 (the structure of $\tilde{S}$)
Let's consider the symmetrical equilibrium vortex configuration Z^* (16) of m rings, with n vortex each, at vertical distances z_i $(i = 1,...,m)$ solutions of (20). Let $\tilde{S}$ be the Hessian of $\tilde{H}$ (26) evaluated at Z^*. Then $\tilde{S}$ has the following block structure

$$\tilde{S} = S + \Omega = \left(\begin{array}{c|c} \mathcal{Q} & \mathcal{G} \\ \hline \mathcal{G}^T & \mathcal{P} \end{array}\right) \tag{42}$$

where Ω has been defined in (27),

$$\mathcal{Q} = \left(\frac{\partial^2 \tilde{H}}{\partial q_\alpha \partial q_\beta}\right)_{|Z^*}, \qquad \mathcal{G} = \left(\frac{\partial^2 \tilde{H}}{\partial q_\alpha \partial p_\beta}\right)_{|Z^*} \qquad \mathcal{P} = \left(\frac{\partial^2 \tilde{H}}{\partial p_\alpha \partial p_\beta}\right)_{|Z^*}, \qquad \alpha\beta = 1, ..., N+2,$$

and

$$\mathcal{Q} = \left(\begin{array}{cccc|c|c} Q_1 & B_{12} & \dots & B_{1m} & \sqrt{\Gamma_N}\,\alpha_{1N} & \sqrt{\Gamma_S}\,\alpha_{1S} \\ B_{12}^T & Q_2 & \dots & B_{2m} & \sqrt{\Gamma_N}\,\alpha_{2N} & \sqrt{\Gamma_S}\,\alpha_{2S} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B_{1m}^T & B_{2m}^T & \dots & Q_m & \sqrt{\Gamma_N}\,\alpha_{mN} & \sqrt{\Gamma_S}\,\alpha_{mS} \\ \hline \sqrt{\Gamma_N}\,\alpha_{1N}^T & \sqrt{\Gamma_N}\,\alpha_{2N}^T & \dots & \sqrt{\Gamma_N}\,\alpha_{mN}^T & a_{11} & a_{12} \\ \hline \sqrt{\Gamma_S}\,\alpha_{1S}^T & \sqrt{\Gamma_S}\,\alpha_{2S}^T & \dots & \sqrt{\Gamma_S}\,\alpha_{mS}^T & a_{21} & a_{22} \end{array}\right) \tag{43}$$

$$\mathcal{P} = \left(\begin{array}{cccc|c|c} P_1 & D_{12} & \dots & D_{1m} & \sqrt{\Gamma_N}\,\delta_{1N} & \sqrt{\Gamma_S}\,\delta_{1S} \\ D_{12}^T & P_2 & \dots & D_{2m} & \sqrt{\Gamma_N}\,\delta_{2N} & \sqrt{\Gamma_S}\,\delta_{2S} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ D_{1m}^T & D_{2m}^T & \dots & P_m & \sqrt{\Gamma_N}\,\delta_{mN} & \sqrt{\Gamma_S}\,\delta_{mS} \\ \hline \sqrt{\Gamma_N}\,\delta_{1N}^T & \sqrt{\Gamma_N}\,\delta_{2N}^T & \dots & \sqrt{\Gamma_N}\,\delta_{mN}^T & b_{11} & b_{12} \\ \hline \sqrt{\Gamma_S}\,\delta_{1S}^T & \sqrt{\Gamma_S}\,\delta_{2S}^T & \dots & \sqrt{\Gamma_S}\,\delta_{mS}^T & b_{12} & b_{22} \end{array}\right) \tag{44}$$

$$\mathcal{G} = \left(\begin{array}{cccc|c|c} O & G_{12} & \dots & G_{1m} & \sqrt{\Gamma_N}\,\gamma_{1N} & \sqrt{\Gamma_S}\,\gamma_{1S} \\ F_{12} & O & \dots & G_{2m} & \sqrt{\Gamma_N}\,\gamma_{2N} & \sqrt{\Gamma_S}\,\gamma_{2S} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ F_{1m} & F_{2m} & \dots & O & \sqrt{\Gamma_N}\,\gamma_{mN} & \sqrt{\Gamma_S}\,\gamma_{mS} \\ \hline \sqrt{\Gamma_N}\beta_{1N}^T & \sqrt{\Gamma_N}\beta_{2N}^T & \dots & \sqrt{\Gamma_N}\beta_{mN}^T & 0 & 0 \\ \hline \sqrt{\Gamma_S}\,\beta_{1S}^T & \sqrt{\Gamma_S}\,\beta_{2S}^T & \dots & \sqrt{\Gamma_S}\,\beta_{mS}^T & 0 & 0 \end{array}\right) \tag{45}$$

where Q_i, P_i, B_{ik}, D_{ik} $(i, k = 1, ..., m)$ are all $(n \times n)$ symmetric circulant matrices while G_{ik} and F_{ik} are $(n \times n)$ skew-symmetric circulant matrices; β_{iN}, β_{iS}, γ_{iN} and γ_{is} are vectors proportional to $\mathbf{e}_2$ (41); α_{iN}, α_{iS}, δ_{iN}, δ_{iS}, are vectors proportional to $\mathbf{e}_3$ (41).

In fact, by direct computation we have the following:

1)

$$\begin{aligned} \alpha_{iN} &= \tilde{\alpha}_{iN}\mathbf{e}_3, \qquad \delta_{iN} = \tilde{\delta}_{iN}\mathbf{e}_3, \quad \gamma_{iN} = \tilde{\gamma}_{iN}\mathbf{e}_2, \quad \beta_{iN} = \tilde{\beta}_{iN}\mathbf{e}_2, & (46) \\ \alpha_{iS} &= \tilde{\alpha}_{iS}\mathbf{e}_3, \qquad \delta_{iS} = \tilde{\delta}_{iS}\mathbf{e}_3, \quad \gamma_{iS} = \tilde{\gamma}_{iS}\mathbf{e}_2, \quad \beta_{iS} = \tilde{\beta}_{iS}\mathbf{e}_2, & (47) \end{aligned}$$

where $i = 1, ..., m$ (m is the number of rings) and

$$\begin{aligned} \tilde{\alpha}_{iN} &= -\tilde{\gamma}_{iN} = \Gamma_i r_i \eta_i, \qquad \tilde{\delta}_{iN} = \tilde{\beta}_{iN} = -\frac{\eta_i}{r_i}, \qquad \eta_i = \frac{1}{1 - z_i} & (48) \\ \tilde{\alpha}_{iS} &= -\tilde{\gamma}_{iS} = \Gamma_i r_i \tilde{\eta}_i, \qquad \tilde{\delta}_{iS} = \tilde{\beta}_{iS} = -\frac{\tilde{\eta}_i}{r_i}, \qquad \tilde{\eta}_i = \frac{1}{1 + z_i}. & (49) \end{aligned}$$

2)

$$
\begin{aligned}
a_{11} &= \frac{\partial^2 \tilde{H}}{\partial q_{N+1}^2}_{|Z^*} = \sum_{i=1}^{m} \Gamma_i [z_i \eta_i n - r_i^2 \eta_i^2 \sum_{j=1}^{n} \cos^2(q_j)] - \frac{1}{2}\Gamma_{N+2} - \omega, \\
a_{22} &= \frac{\partial^2 \tilde{H}}{\partial q_{N+2}^2}_{|Z^*} = \sum_{i=1}^{m} \Gamma_i [-z_i \tilde{\eta}_i n - r_i^2 \tilde{\eta}_i^2 \sum_{j=1}^{n} \cos^2(q_j)] - \frac{1}{2}\Gamma_{N+1} + \omega, \\
b_{11} &= \frac{\partial^2 \tilde{H}}{\partial p_{N+1}^2}_{|Z^*} = \sum_{i=1}^{m} \Gamma_i [z_i \eta_i n - r_i^2 \eta_i^2 \sum_{j=1}^{n} \sin^2(q_j)] - \frac{1}{2}\Gamma_{N+2} - \omega, \\
b_{22} &= \frac{\partial^2 \tilde{H}}{\partial p_{N+2}^2}_{|Z^*} = \sum_{i=1}^{m} \Gamma_i [-z_i \tilde{\eta}_i n - r_i^2 \tilde{\eta}_i^2 \sum_{j=1}^{n} \sin^2(q_j)] - \frac{1}{2}\Gamma_{N+1} + \omega, \\
b_{12} &= b_{21} = \frac{\partial^2 \tilde{H}}{\partial p_{N+1} \partial p_{N+2}}_{|Z^*} = -\frac{1}{2}\sqrt{\Gamma_{N+1}\Gamma_{N+2}}, \qquad (50) \\
a_{12} &= a_{21} = \frac{\partial^2 \tilde{H}}{\partial q_{N+1} \partial q_{N+2}}_{|Z^*} = -\frac{1}{2}\sqrt{\Gamma_{N+1}\Gamma_{N+2}}. \qquad (51)
\end{aligned}
$$

Since for $n > 2$

$$
b_{11} - a_{11} = \sum_{i=1}^{m} [r_i^2 \eta_i^2 \sum_{j=1}^{n} \cos 2q_j^{(o)}] = 0, \quad b_{22} - a_{22} = \sum_{i=1}^{m} [r_i^2 \tilde{\eta}_i^2 \sum_{j=1}^{n} \cos 2q_j^{(o)}] = 0,
$$

we have

$$
a_{11} = b_{11} = -\frac{n}{2} \sum_{i=1}^{m} \Gamma_i - \Gamma_{N+2} - \omega, \quad a_{22} = b_{22} = -\frac{n}{2} \sum_{i=1}^{m} \Gamma_i - \Gamma_{N+1} + \omega. \qquad (52)
$$

From now on, we will consider only $n > 2$.

3) Q_i and P_i have the structure:

$$
Q_i = \Gamma_i^2 \left[-(s + \tilde{s}_i)\mathbb{I} + A \right], \qquad P_i = \frac{\Gamma_i^2}{r_i^4}((t_i - \tilde{t}_i)\mathbb{I} - A), \qquad i = 1, ..., m. \qquad (53)
$$

where

$$
\begin{aligned}
s &= \sum_{k=2}^{n} a_k = \frac{n^2 - 1}{6}, \qquad \tilde{s}_i = \frac{r_i}{\Gamma_i} \sum_{k \neq i, k=1}^{m} \Gamma_k r_k \sum_{j=1}^{n} b_j^{(ik)}, \qquad (54) \\
t_i &= s - (n-1)(1 + z_i^2), \qquad \tilde{t}_i = \frac{r_i^4}{\Gamma_i^3}(\Gamma_{N+1}\eta_i^2, + \Gamma_{N+2}\tilde{\eta}_i^2) \qquad (55) \\
\eta_i &= \frac{1}{1 - z_i}, \qquad \tilde{\eta}_i = \frac{1}{1 + z_i} \qquad (56)
\end{aligned}
$$

$b_j^{(ik)}$ defined as in (64), and A is a $(n \times n)$ circulant matrix as well, whose elements verify

$$a_1 = 0, \quad \text{and} \quad a_j = \frac{1}{1-\cos(2\pi(j-1)/n)} = a_{n-j+2}, \quad j = 2, ..., n. \tag{57}$$

Therefore A is a symmetrical matrix and, by Lemma 4.5, it has n real eigenvalues of the form (39)

$$\lambda_{j\,A} = s - (j-1)(n-j+1), \qquad j = 1, ..., n. \tag{58}$$

In particular

$$\lambda_{1\,A} = s = \frac{n^2-1}{6}, \qquad \lambda_{min\,A} = -\frac{1}{24}[2n^2 + 1 + 3(-1)^n], \tag{59}$$

where $\lambda_{min\,A}$ is the minimum eigenvalue of A. The eigenvalues of the matrices Q_i and P_i are therefore

$$\begin{aligned} \lambda_{j\,Q_i} &= \Gamma_i^2(-s - \tilde{s}_i + \lambda_{A\,j}) = -\Gamma_i^2[(j-1)(n-j+1) + \tilde{s}_i], &(60)\\ \lambda_{j\,P_i} &= \frac{\Gamma_i^2}{r_i^4}(t_i - \tilde{t}_i - \lambda_{A\,j}) = -\frac{\Gamma_i^2}{r_i^4}((n-1)(1+z_i^2) - (j-1)(n-j+1)) \\ &\qquad -\frac{\Gamma_{N+1}}{\Gamma_i}\eta_i^2 - \frac{\Gamma_{N+2}}{\Gamma_i}\tilde{\eta}_i^2, &(61) \end{aligned}$$

where $j = 1, ..., n$ and $i = 1, ..., m$. In particular

$$\lambda_{1\,Q_i} = -\Gamma_i^2\,\tilde{s}_i, \qquad \lambda_{1\,P_i} = -\frac{\Gamma_i^2}{r_i^4}(n-1)(1+z_i^2) - \frac{\Gamma_{N+1}}{\Gamma_i}\eta_i^2 - \frac{\Gamma_{N+2}}{\Gamma_i}\tilde{\eta}_i^2. \tag{62}$$

4) The matrices

$$B_{ik} = \Gamma_i\Gamma_k r_i r_k \begin{pmatrix} b_1^{(ik)} & b_2^{(ik)} & \dots & b_n^{(ik)} \\ b_n^{(ik)} & b_1^{(ik)} & \dots & b_{n-1}^{(ik)} \\ \vdots & \vdots & \ddots & \vdots \\ b_2^{(ik)} & b_3^{(ik)} & \dots & b_1^{(ik)} \end{pmatrix}, \quad D_{ik} = \begin{pmatrix} d_1^{(ik)} & d_2^{(ik)} & \dots & d_n^{(ik)} \\ d_n^{(ik)} & d_1^{(ik)} & \dots & d_{n-1}^{(ik)} \\ \vdots & \vdots & \ddots & \vdots \\ d_2^{(ik)} & d_3^{(ik)} & \dots & d_1^{(ik)} \end{pmatrix}. \tag{63}$$

have elements

$$b_j^{(ik)} = \frac{r_i r_k - (1 - z_i z_k)\cos(2\pi(j-1)/n)}{[1 - z_i z_k - r_i r_k \cos(2\pi(j-1)/n)]^2} \tag{64}$$

$$d_j^{(ik)} = -\frac{1 - r_i r_k \cos(2\pi(j-1)/n)}{[1 - z_i z_k - r_i r_k \cos(2\pi(j-1)/n)]^2} < 0. \tag{65}$$

Proposition 4.8 *The $b_j^{(ik)}$ and $d_j^{(ik)}$ verify the properties:*

$$\begin{array}{ll} b_1^{(ik)} = -\frac{1}{1-z_iz_k-r_ir_k}, & b_j^{(ik)} = b_{n-j+2}^{(ik)}, \\ d_1^{(ik)} = -\frac{1-r_1r_k}{[1-z_iz_k-r_ir_k]^2} < 0, & d_j^{(ik)} = d_{n-j+2}^{(ik)}, \end{array} \tag{66}$$

Therefore B_{ik} and D_{ik} are symmetrical matrices and, by applying the Lemma 4.6, we have that their eigenvalues are (39):

$$\lambda_{j\,B_{ik}} = \Gamma_i\Gamma_k r_i r_k \left(b_1^{(ik)} + \sum_{\mu=2}^{n} b_\mu^{(ik)} \cos(2\pi(j-1)(\mu-1)/n) \right), \tag{67}$$

$$\lambda_{j\,D_{ik}} = d_1^{(ik)} + \sum_{\mu=2}^{n} d_\mu^{(ik)} \cos(2\pi(j-1)(\mu-1)/n), \tag{68}$$

for $j = 1, ..., n$, $i, k = 1, ..., m$. In particular, from Eq. (65) we can deduce

$$\lambda_{1\,D_{ik}} = \sum_{\mu=1}^{n} d_\mu^{(ik)} < 0, \quad \text{for all} \quad i, k = 1, ..., m. \tag{69}$$

5) The matrices G_{ik} and F_{ik} ($i, k = 1, ..., m$, $m \geq 2$) have the structure

$$G_{ik} = \Gamma_i^2\Gamma_k \frac{r_i}{r_k}(z_i - z_k)T_{ik}, \qquad F_{ik} = \Gamma_k^2\Gamma_i \frac{r_k}{r_i}(z_i - z_k)T_{ik}^T \tag{70}$$

where

$$T_{ik} = \begin{pmatrix} t_1^{(ik)} & t_2^{(ik)} & \dots & t_n^{(ik)} \\ t_n^{(ik)} & t_1^{(ik)} & \dots & t_{n-1}^{(ik)} \\ \vdots & \vdots & \ddots & \vdots \\ t_2^{(ik)} & t_3^{(ik)} & \dots & t_1^{(ik)} \end{pmatrix}.$$

and

$$t_j^{(ik)} = -\frac{\sin(2\pi(j-1)/n)}{[1 - z_iz_k - r_ir_k\cos(2\pi(j-1)/n)]^2}.$$

The $t_j^{(ik)}$ verify

$$t_1^{(ik)} = 0, \quad t_j^{(ik)} = -t_{n-j+2}^{(ik)}, \quad j = 2, ..., n.$$

Therefore T_{ik} are skew-symmetric matrices ($i, k = 1, ..., m$, $i \neq k$, $m \geq 2$) and therefore its eigenvalues are (40)

$$\lambda_{j\,T_{ik}} = 2i \sum_{m=2}^{\left[\frac{n-1}{2}\right]} t_m^{(ik)} \sin(2\pi(j-1)(m-1)/n).$$

Then the eigenvalues of G_{ik} and F_{ik} are

$$\lambda_{j\,G_{ik}} = \Gamma_i^2 \Gamma_k \frac{r_i}{r_k}(z_i - z_k)\lambda_{j\,T_{ik}}, \quad \lambda_{j\,F_{ik}} = -\Gamma_k^2 \Gamma_i \frac{r_k}{r_i}(z_i - z_k)\lambda_{j\,T_{ik}}, \tag{71}$$

and in particular

$$\lambda_{1\,G_{ik}} = \lambda_{1\,F_{ik}} = 0. \tag{72}$$

A special canonical transformation Consider the orthonormal basis of $\mathbf{R}^{N+2}$ (where $N = mn$) given by

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= (\mathbf{e}_1, \mathbf{O}, ..., \mathbf{O}, 0, 0), \ \ldots, \tilde{\mathbf{e}}_n = (\mathbf{e}_n, \mathbf{O}, ..., \mathbf{O}, 0, 0), \ \ldots, \tilde{\mathbf{e}}_{n+1} = (\mathbf{O}, \mathbf{e}_1, \mathbf{O}, ..., \mathbf{O}, 0, 0), \ \ldots, \\
&\ldots, \quad \tilde{\mathbf{e}}_{2n} = (\mathbf{O}, \mathbf{e}_n, \mathbf{O}, ..., \mathbf{O}, 0, 0), \quad \ldots \quad, \ \tilde{\mathbf{e}}_N = (\mathbf{O}, \mathbf{O}, ..., \mathbf{O}, \mathbf{e}_n, 0, 0), \\
\tilde{\mathbf{e}}_{N+1} &= (\mathbf{O}, \mathbf{O}, ..., \mathbf{O}, \mathbf{O}, 1, 0), \quad \tilde{\mathbf{e}}_{N+2} = (\mathbf{O}, \mathbf{O}, ..., \mathbf{O}, \mathbf{O}, 0, 1),
\end{aligned}$$

where $\mathbf{e}_j$ $(j = 1, ..., n)$ are the vectors of the orthonormal basis for A defined in Eq. (41). As in the previous section, Eq. (31)-(30), we can then perform the symplectic change of variables

$$Z = \mathcal{C}Z', \quad \text{where} \quad \mathcal{C} = \left(\begin{array}{c|c} \tilde{C} & \tilde{O} \\ \hline \tilde{O} & \tilde{C} \end{array} \right), \tag{73}$$

where $\tilde{O}$ is the $((N+2) \times (N+2))$ null-matrix; $\tilde{C}$ is the matrix that has for columns $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, ..., \tilde{\mathbf{e}}_{N+2}$ i.e.

$$\tilde{C} = \left(\ \tilde{\mathbf{e}}_1 \,\middle|\, \tilde{\mathbf{e}}_2 \,\middle|\, \ldots \,\middle|\, \tilde{\mathbf{e}}_{N+2} \ \right) = \left(\begin{array}{cccccc|cc}
C & \mathbf{O} & \ldots & \ldots & \mathbf{O} & & \mathbf{0} & \mathbf{0} \\
\mathbf{O} & C & \mathbf{O} & \ldots & \mathbf{O} & & \mathbf{0} & \mathbf{0} \\
\vdots & \ddots & \ddots & \ddots & \vdots & & \vdots & \vdots \\
\mathbf{O} & \ldots & \mathbf{O} & C & \mathbf{O} & & \mathbf{0} & \mathbf{0} \\
\mathbf{O} & \ldots & \ldots & \mathbf{O} & C & & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{0}^T & \ldots & \ldots & \mathbf{0}^T & \mathbf{0}^T & & 1 & 0 \\
\mathbf{0}^T & \ldots & \ldots & \mathbf{0}^T & \mathbf{0}^T & & 0 & 1
\end{array} \right), \tag{74}$$

where $\mathbf{O}$ is the $(n \times n)$ null-matrix, $\mathbf{0} \in \mathbb{R}^n$ is the null-vector of $\mathbb{R}^n$, C is the $(n \times n)$ orthonormal matrix

$$C = \left(\ \mathbf{e}_1 \,\middle|\, \mathbf{e}_2 \,\middle|\, \ldots \,\middle|\, \mathbf{e}_n \ \right) \tag{75}$$

of orthonormal eigenvectors (41) of a symmetric (or skew-symmetric) $(n \times n)$ matrix. In the variables $(\mathbf{q}, \mathbf{p})$ (73) becomes

$$\mathbf{q} = \tilde{C}\mathbf{q}' \qquad\qquad \mathbf{p} = \tilde{C}\mathbf{p}'. \tag{76}$$

Then the transformation (76) partially diagonalizes the Hessian $\tilde{S}$. In fact from (30)

$$\tilde{S}' = \mathcal{C}^T \tilde{S} \mathcal{C} = \left(\begin{array}{c|c} \tilde{C} & \tilde{O} \\ \hline \tilde{O} & \tilde{C} \end{array} \right) \left(\begin{array}{c|c} \mathcal{Q} & \mathcal{G} \\ \hline \mathcal{G}^T & \mathcal{P} \end{array} \right) \left(\begin{array}{c|c} \tilde{C} & \tilde{O} \\ \hline \tilde{O} & \tilde{C} \end{array} \right) = \left(\begin{array}{c|c} \tilde{C}\mathcal{Q}\tilde{C} & \tilde{C}\mathcal{G}\tilde{C} \\ \hline \tilde{C}\mathcal{G}^T\tilde{C} & \tilde{C}\mathcal{P}\tilde{C} \end{array} \right),$$

$$\tilde{C}\mathcal{Q}\tilde{C}=\left(\begin{array}{cccc|c|c}
CQ_1C & CB_{12}C & \dots & CB_{1m}C & \sqrt{\Gamma_N}\,\alpha_{1N} & \sqrt{\Gamma_S}\,\alpha_{1S}\\
CB_{12}^TC & CQ_2C & \dots & CB_{2m}C & \sqrt{\Gamma_N}\,\alpha_{2N} & \sqrt{\Gamma_S}\,\alpha_{2S}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
CB_{1m}^TC & CB_{2m}^TC & \dots & CQ_mC & \sqrt{\Gamma_N}\,\alpha_{mN} & \sqrt{\Gamma_S}\,\alpha_{mS}\\
\hline
\sqrt{\Gamma_N}\,\alpha_{1N}^T & \sqrt{\Gamma_N}\,\alpha_{2N}^T & \dots & \sqrt{\Gamma_N}\,\alpha_{mN}^T & a_{11} & a_{12}\\
\hline
\sqrt{\Gamma_S}\,\alpha_{1S}^T & \sqrt{\Gamma_S}\,\alpha_{2S}^T & \dots & \sqrt{\Gamma_S}\,\alpha_{mS}^T & a_{12} & a_{22}
\end{array}\right)$$

$$\tilde{C}\mathcal{P}\tilde{C}=\left(\begin{array}{cccc|c|c}
CP_1C & CD_{12}C & \dots & CD_{1m}C & \sqrt{\Gamma_N}\,\delta_{1N} & \sqrt{\Gamma_S}\,\delta_{1S}\\
CD_{12}^TC & CQ_2C & \dots & CD_{2m}C & \sqrt{\Gamma_N}\,\delta_{2N} & \sqrt{\Gamma_S}\,\delta_{2S}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
CD_{1m}^TC & CD_{2m}^TC & \dots & CP_mC & \sqrt{\Gamma_N}\,\delta_{mN} & \sqrt{\Gamma_S}\,\delta_{mS}\\
\hline
\sqrt{\Gamma_N}\,\alpha_{1N}^T & \sqrt{\Gamma_N}\,\delta_{2N}^T & \dots & \sqrt{\Gamma_N}\,\delta_{mN}^T & a_{11} & a_{12}\\
\hline
\sqrt{\Gamma_S}\,\alpha_{1S}^T & \sqrt{\Gamma_S}\,\delta_{2S}^T & \dots & \sqrt{\Gamma_S}\,\delta_{mS}^T & a_{12} & a_{22}
\end{array}\right)$$

$$\tilde{C}\mathcal{G}\tilde{C}=\left(\begin{array}{cccc|c|c}
O & CG_{12}C & \dots & CG_{1m}C & \sqrt{\Gamma_N}\,\gamma_{1N} & \sqrt{\Gamma_S}\,\gamma_{1S}\\
CF_{12}C & O & \dots & G_{2m} & \sqrt{\Gamma_N}\,\gamma_{2N} & \sqrt{\Gamma_S}\,\gamma_{2S}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
CF_{1m}C & CF_{2m}C & \dots & O & \sqrt{\Gamma_N}\,\gamma_{mN} & \sqrt{\Gamma_S}\,\gamma_{mS}\\
\hline
\sqrt{\Gamma_N}\beta_{1N}^T & \sqrt{\Gamma_N}\beta_{2N}^T & \dots & \sqrt{\Gamma_N}\beta_{mN}^T & 0 & 0\\
\hline
\sqrt{\Gamma_S}\,\beta_{1S}^T & \sqrt{\Gamma_S}\,\beta_{2S}^T & \dots & \sqrt{\Gamma_S}\,\beta_{mS}^T & 0 & 0
\end{array}\right),$$

and since the matrices Q_i, P_i, B_{ik}, D_{ik}, G_{ik} and F_{ik} are all circulant matrices, either symmetric or skew-symmetric, then the transformation induced by the matrix C diagonalizes all of them at once, i.e.

$$CQ_iC=\begin{pmatrix}
\lambda_{1\,Q_i} & 0 & \dots & \dots & 0\\
0 & \lambda_{2\,Q_i} & 0 & \dots & 0\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
0 & \dots & 0 & \lambda_{n-1\,Q_i} & 0\\
0 & \dots & 0 & 0 & \lambda_{n\,Q_i}
\end{pmatrix}.$$

Furthermore, with respect to the new basis $(\mathbf{e}_1,...,\mathbf{e}_n)$ each of the vectors α_{iN}, α_{iN}, α_{iN}, α_{iN}, α_{iN} is either proportional to $\mathbf{e}_2$ or $\mathbf{e}_3$, i.e.

$$\alpha_{iN}=\tilde{\alpha}_{iN}(0,0,1,0,...,0)^T,\quad \beta_{iN}=\tilde{\beta}_{iN}(0,1,0,0,...,0)^T,\quad \delta_{iN}=\tilde{\beta}_{iN}(0,0,1,0,...,0)^T,$$
$$\gamma_{iN}=-\tilde{\alpha}_{iN}(0,1,0,0,...,0)^T\alpha_{iS}=\tilde{\alpha}_{iS}(0,0,1,0,...,0)^T,\quad \beta_{iS}=\tilde{\beta}_{iS}(0,1,0,0,...,0)^T,$$
$$\delta_{iS}=\tilde{\beta}_{iS}(0,0,1,0,...,0)^T,\quad \gamma_{iS}=-\tilde{\alpha}_{iS}(0,1,0,0,...,0)^T$$

5 Particular cases

5.1 One ring

In the case of relative equilibrium configuration of one latitudinal ring of N identical vortices of vorticity Γ_1, we have $m = 1$, $n = N$ and, from Proposition 3.1, the angular velocity ω of the equilibrium configuration is (22)

$$\omega \;=\; -\Gamma_1 \frac{z_1}{r_1^2}(N-1), \quad \forall z_1 \in (-1,1).$$

From Eq.(27),(60),(61), the Hessian $\tilde{S}$ has the structure

$$\tilde{S} = \left(\begin{array}{c|c} Q & \mathbf{O} \\ \hline \mathbf{O} & P \end{array}\right) \tag{77}$$

where, since we have only one ring and one vortex, to simplify the notation, from Eq. (53), (54), (55), we have denoted

$$Q = Q_1 = \Gamma_1^2(-s\mathbb{I}+A), \quad P = P_1 = \frac{\Gamma_1^2}{r_1^4}(t_1\mathbb{I}-A), \quad s = \frac{N^2-1}{6}, \quad t_1 = s-(N-1)(1+z_1^2). \tag{78}$$

As described in the previous section we consider the symplectic change of variables

$$Z = \mathcal{C}Z', \quad \text{i.e.} \quad \mathbf{q} = \tilde{C}\mathbf{q}', \qquad \mathbf{p} = \tilde{C}\mathbf{p}',$$

where $Z = (\mathbf{q},\mathbf{p})^T$, $\mathbf{q} = (q_1,...,q_N)$, $\mathbf{p} = (p_1,...,p_N)$. Then from Eq. (74)

$$\tilde{S}' = \mathcal{C}^T\tilde{S}\mathcal{C} = \left(\begin{array}{cccccc|cccccc} \lambda_{Q\,1} &&&&&&&&&&& \\ & \lambda_{Q\,2} &&&&&&&&&& \\ && \lambda_{Q\,2} &&&&&&&&& \\ &&& \lambda_{Q\,3} &&&&&&&& \\ &&&& \ddots &&&&&&& \\ &&&&& \lambda_{Q\,[N/2]} &&&&&& \\ \hline &&&&&& \lambda_{P\,1} &&&&& \\ &&&&&&& \lambda_{P\,2} &&&& \\ &&&&&&&& \lambda_{P\,2} &&& \\ &&&&&&&&& \lambda_{P\,3} && \\ &&&&&&&&&& \ddots & \\ &&&&&&&&&&& \lambda_{P\,[N/2]} \end{array}\right), \tag{79}$$

where

$$\begin{cases} \lambda_{j\,Q} = -\Gamma_1^2(j-1)(N-j+1) \le 0 \\ \lambda_{j\,P} = \frac{\Gamma_1^2}{r_1^4}[(1-N)(1+z_1^2)+(j-1)(N-j+1)], \end{cases} \quad j = 1,...,\left[\frac{N}{2}\right]. \tag{80}$$

Furthermore, since Q (Eq. (78)) verifies the hypothesis of Lemma 4.2, zero is a simple eigenvalue ($\lambda_{1\,Q} = 0$) and the remaining eigenvalues have all the same sign (in the case of the sphere, negative). The zero eigenvalue is due to the invariance of the Hessian with respect to rotations. The Hessian (27), modulo the rotation-invariance, is therefore definite if P is definite as Q, i.e. if all its eigenvalues have the same sign as the $\lambda_{j\,Q}$ ($j = 2, ..., [\frac{N}{2}]$).

Remarks:

1) Notice that the by the transformation (31)-(75) the momentum M (7) changes into

$$M = \sum_{j=1}^{N} p_j = \sum_{j=1}^{N} \sum_{\alpha=0}^{N-1} \tilde{p}_\alpha C_{\alpha j} = \sum_{\alpha=0}^{N-1} \tilde{p}_\alpha \sum_{j=1}^{N} C_{\alpha j} = \sum_{\alpha=0}^{N-1} \tilde{p}_\alpha \delta_{\alpha\,0} = \tilde{p}_1.$$

Therefore with respect to the new variables, $(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$, the momentum M is the variable conjugate to $\tilde{q}_1$, which is cyclic. It follows that:

Proposition 5.1 *(***Reduced Hamiltonian***)*
Near the equilibrium Z_*, the reduced Hamiltonian takes the form:

$$\tilde{H}_r = \frac{1}{2} \sum_{j=2}^{N} (\lambda_{jQ}\,\tilde{q}_j^2 + \lambda_{jP}\,\tilde{p}_j^2) + h.o.t.,$$

Proposition 5.1 tells us that for $\lambda_{j\,Q}\lambda_{j\,P} > 0$ ($j = 1, ..., [N/2]$), the vortex dynamics is the one of $N - 1$ harmonic oscillators of frequencies

$$\omega_j = \sqrt{\lambda_{j\,Q}\lambda_{j\,P}}. \tag{81}$$

2) From (81) it follows immediately from Theorem 4.1, Theorem 4.3 and Lemma 4.2 the following result about linear and nonlinear stability:

Theorem 5.2 (linear and nonlinear stability)
If the Hessian S has the block-structure (108); P and Q have the structure (78); P has non-zero $\lambda_{1\,P}, ..., \lambda_{[N/2]\,P}$ eigenvalues then Dirichlet criterion is a necessary and sufficient condition for linear stability, and therefore as well for non-linear stability, for Z^*.

Remark: if P is indefinite the linear system is obviously unstable, and if P is semidefinite, linear (neutral) stability follows if the Hessian S_{Z^*} (27) is semidefinite.

Then it follows from Theorem 5.2:

Theorem 5.3 **(spherical case ($m = 1,\ n = N$))**
The equilibrium (16) is (linearly and nonlinearly) stable if

$$\lambda_{max\,P} = \frac{\Gamma_1^2}{r_1^4}(t_1 - \lambda_{min\,A}) < 0,$$

$$\textit{i.e. if} \qquad -4(N-1)(11-N) + 24(N-1)r_1^2 + 2N^2 + 1 + 3(-1)^N < 0.$$

and it is unstable if the inequality is reversed.

Therefore, for a given colatitude θ , stability is guaranteed for $2 \leq N \leq N_*$, where

$$N_* = 7 - 4r_1^2 \qquad \text{if } N_* \text{ is odd} \tag{82}$$

$$N_* = 2\left(2 - r_1^2 + \sqrt{2 - 3r_1^2 + r_1^4}\right) \qquad \text{if } N_* \text{ is even} \tag{83}$$

or, equivalently, for a given N we have stability for $0^o \leq \theta \leq \theta^*$ and $180^o - \theta^* \leq \theta \leq 180^o$, where $\theta^* = \arcsin(r_1^*)$ and

$$r_1^{*2} = \frac{7-N}{4} \qquad \text{for } N \text{ odd,}$$

$$r_1^{*2} < -\frac{N^2 - 8N + 8}{4(N-1)} \qquad \text{for } N \text{ even}$$

where $r_1^* = \sqrt{1 - z_1^{*2}}$.

Furthermore, the same kind of stability analysis easily generalizes to vortex dynamics in the plane. In fact, consider the polygonal vortex configurations

$$p_j^{(o)} = p_1 \quad \text{and} \quad q_j^{(o)} = \frac{2\pi(j-1)}{N}, \quad j = 1, ..., N, \tag{84}$$

where ($r_j = \sqrt{2p_j}, \varphi_j = q_j$) are the usual polar coordinates for the jth vortex in the plane and there is no loss of generality in choosing $p_1 = 1$. It is immediate to prove that (84) is a relative equilibrium configuration and that the stability analysis is similar to the one on the sphere with

$$Q = \Gamma_1^2(s\mathbb{I} - A) \quad \text{and} \quad P = \Gamma_1^2(-t_p\mathbb{I} + A),$$

where $t_p = \frac{1}{6}(N-1)(11-N)$. Then as before we can view the dynamics in the frame co-rotating with the vortices and we find easily the stability result [13]:

Theorem 5.4 **(planar case ($m = 1, \quad n = N$))**
The equilibrium (84) is (linearly and nonlinearly) stable if

$$\lambda_{min\,P} = \Gamma_1^2(t_p + \lambda_{min\,A}) > 0,$$

$$\textit{i.e.} \qquad 4(N-1)(11-N) - 2N^2 - 1 - 3(-1)^N > 0,$$

from which stability is guaranteed for $N < 7$.

Notice that in Eq. (82) when we approach the north or the south (i.e. $\theta \to 0^o$ or $\theta \to 180^o$ in Eq. (78)) we recover the planar case, $N_* \to 7$, as expected by the planar approximation being a valid approximation of the spherical shell in a neighborhood of a pole.

5.2 One ring with a polar vortex

In the case of relative equilibrium configuration of one latitudinal ring of N identical vortices of vorticity Γ_1 and one polar vortex, say at the north pole, of vorticity Γ_N, we have $m = 1$, $n = N$ and, from Proposition 3.1, the angular velocity ω of the equilibrium configuration is (22)

$$\omega \quad = \quad -\Gamma_1 \frac{z_1}{r_1^2}(N-1) - \frac{\Gamma_N}{1-z_1}, \qquad \forall z_1 \in (-1,1).$$

From Eq.(27),(60),(61), the Hessian $\tilde{S}$ has the structure

$$\tilde{S} = \left(\begin{array}{c|c|c|c} Q & \sqrt{\Gamma_N}\alpha_N & \mathbf{O} & \sqrt{\Gamma_N}\gamma_N \\ \hline \sqrt{\Gamma_N}\alpha_N^T & a & \sqrt{\Gamma_N}\beta_N^T & 0 \\ \hline \mathbf{O} & \sqrt{\Gamma_N}\beta_N & \tilde{P} & \sqrt{\Gamma_N}\delta_N \\ \hline \sqrt{\Gamma_N}\gamma_N & 0 & \sqrt{\Gamma_N}\delta_N^T & a \end{array}\right) \tag{85}$$

which is mostly the Hessian of one ring of vortices (108) implemented of two columns and two rows: the vectors α_N, β_N, δ_N, $\gamma_N \in \mathbb{R}^N$ describing the interaction of the northern polar ring with the vortex ring. From Eq. (46), (48), (52), (53), (55), (78) we have denoted

$$Q = \Gamma_1^2(-s\mathbb{I} + A), \quad \tilde{P} = P - \frac{\Gamma_1^2}{r_1^4}\tilde{t}_1\mathbb{I} = \frac{\Gamma_1^2}{r_1^4}((t_1 - \tilde{t}_1)\mathbb{I} - A), \tag{86}$$

$$\tilde{t}_i = \frac{r_1^4}{\Gamma_1^3}\Gamma_N\eta^2, \quad \eta = \frac{1}{1-z_1}, \quad a = a_{11} = -\frac{N}{2}\Gamma_1 - \omega,$$

$$\begin{aligned} &\alpha_N = \alpha_{1N}, \quad \tilde{\alpha}_N = \tilde{\alpha}_{1N} \quad \beta_N = \beta_{1N}, \quad \tilde{\beta}_N = \tilde{\beta}_{1N}, \\ &\delta_N = \delta_{1N}, \quad \tilde{\delta}_N = \tilde{\delta}_{1N}, \quad \gamma_N = \gamma_{1N}, \quad \tilde{\gamma}_N = \tilde{\gamma}_{1N}. \end{aligned}$$

As described in the previous section we consider the symplectic change of variables

$$Z = \mathcal{C}Z', \quad \text{i.e.} \quad \mathbf{q} = \tilde{C}\mathbf{q}', \qquad \mathbf{p} = \tilde{C}\mathbf{p}',$$

where $Z = (\mathbf{q}, \mathbf{p})^T$, $\mathbf{q} = (q_1, ..., q_N, q_{N+1})$, $\mathbf{p} = (p_1, ..., p_N, p_{N+1})$. Then from Eq. (74)

$$
\tilde{S}' = \mathcal{C}^T \tilde{S} \mathcal{C}
$$

$$
= \left(\begin{array}{cccccc|c|cccccc|c}
\lambda_{1\,Q} & 0 & \dots & \dots & \dots & 0 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\
0 & \lambda_{2\,Q} & 0 & \dots & \dots & 0 & 0 & 0 & & \dots & & & 0 & -\sqrt{\Gamma}\tilde{\alpha} \\
0 & 0 & \lambda_{2\,Q} & 0 & \dots & 0 & \sqrt{\Gamma_N}\tilde{\alpha} & & & & & & & 0 \\
 & & & \lambda_{3\,Q} & & & 0 & & & & & & & 0 \\
 & & & & \ddots & & \vdots & & & & & & & \vdots \\
 & & & & & \lambda_{[N/2]\,Q} & 0 & & & & & & & 0 \\
\hline
0 & 0 & \sqrt{\Gamma_N}\tilde{\alpha} & 0 & \dots & 0 & a & 0 & \sqrt{\Gamma_N}\tilde{\beta} & 0 & & \dots & 0 & 0 \\
\hline
 & & & & & & 0 & \lambda_{1\,\tilde{P}} & & & & & & 0 \\
 & & & & & & \sqrt{\Gamma_N}\tilde{\beta} & & \lambda_{2\,\tilde{P}} & & & & & 0 \\
 & & & & & & 0 & & & \lambda_{2\,\tilde{P}} & & & & \sqrt{\Gamma_N}\tilde{\beta} \\
 & & & & & & 0 & & & & \lambda_{3\,\tilde{P}} & & & 0 \\
 & & & & & & \vdots & & & & & \ddots & & \vdots \\
 & & & & & & 0 & & & & & & \lambda_{[N/2]\,\tilde{P}} & 0 \\
\hline
0 & -\sqrt{\Gamma_N}\tilde{\alpha} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \sqrt{\Gamma_N}\tilde{\beta} & 0 & \dots & 0 & a
\end{array}\right) \tag{87}
$$

where

$$
\lambda_{j\,\tilde{P}} = \lambda_{j\,P} - \frac{\Gamma_N}{\Gamma_1}\eta^2, \quad \tilde{\alpha}_N = r_1 \eta \sqrt{\frac{N}{2}} > 0, \quad \tilde{\beta}_N = -\frac{\tilde{\alpha}_N}{r_1^2} < 0 \quad (\text{for } r_1 \neq 0), \tag{88}
$$

$\lambda_{j\,Q}$ and $\lambda_{j\,P}$ have been defined in Eq. (80). From now on we set $\Gamma_1 = 1$ (which amounts to express Γ_N in units of Γ_1) to deal only with the relevant parameters: Γ_N, N, z_1 or r_1 (since $r_1 = \sin\theta_1$ and $z_1 = \cos\theta_1$).

Remarks:

1) By the transformation (76)-(74) the momentum M (13)

$$
\begin{aligned}
M &= \sum_{j=1}^{N} p_j + \Gamma_N - \frac{q_{N+1}^2 + p_{N+1}^2}{2} = \sum_{j=1}^{N}\sum_{\alpha=0}^{N-1} C_{\alpha j}\tilde{p}_j + \Gamma_N - \frac{q_{N+1}^2 + p_{N+1}^2}{2} \\
&= \tilde{p}_1 + \Gamma_N - \frac{q_{N+1}^2 + p_{N+1}^2}{2}
\end{aligned}
$$

and therefore, by Proposition 2.1, $\tilde{p}_1$ is a conserved quantity. It follows that:

Proposition 5.5 (**Reduced Hamiltonian for N+1 vortices**)
Near the equilibrium Z^* (16) the reduced Hamiltonian takes the form

$$
\tilde{H}_r = \frac{1}{2}\sum_{j=4}^{N}(\lambda_{jQ}\tilde{q}_j^2 + \lambda_{j\tilde{P}}\tilde{p}_j^2) + \frac{1}{2}B_2(\tilde{q}_2, \tilde{q}_3, \tilde{q}_{N+1}, \tilde{p}_2, \tilde{p}_3, \tilde{p}_{N+1}) + h.o.t.
$$

where B_2 is the quadratic form

$$
\begin{aligned}
B_2 &= \lambda_{2\,Q}(\tilde{q}_2^2 + \tilde{q}_3^2) + \lambda_{2\,\tilde{P}}(\tilde{p}_2^2 + \tilde{p}_3^2) + a(q_{N+1}^2 + p_{N+1}^2) \\
&\quad + 2\sqrt{\Gamma_N}\alpha_N(q_{N+1}q_3 - p_{N+1}q_2) + 2\sqrt{\Gamma_N}\beta_N(q_{N+1}p_2 + p_{N+1}p_3).
\end{aligned}
$$

2) From Eq. (111) it is manifest that the value $\Gamma_N = 0$ is a particular value. In fact in this case the $(N+1)$-vortex system decouples into a N vortex system (the latitudinal ring) and a test particle (or passive tracer) at the pole. The dynamics of the N vortex system is no influenced by the passive tracer so that the effective phase-space reduces to the one of the N vortices, already studied in the previous section, plus a two dimensional subspace, describing the test-particle dynamics.

3) For all $\Gamma_N \neq 0$, the eigenvalues of the reduced Hessian are

$$\lambda_{3\,Q},\ \ldots,\ \lambda_{[N/2]\,Q},\ \lambda_{3\,\tilde{P}},\ \ldots,\ \lambda_{[N/2]\,\tilde{P}} \tag{89}$$

together with the eigenvalues of the matrix

$$G_1 = \left(\begin{array}{ccc|ccc} \lambda_{2\,Q} & 0 & 0 & 0 & 0 & -\sqrt{\Gamma_N}\tilde{\alpha}_N \\ 0 & \lambda_{2\,Q} & \sqrt{\Gamma_N}\tilde{\beta}_N & 0 & 0 & 0 \\ 0 & \sqrt{\Gamma_N}\tilde{\alpha}_N & a & \sqrt{\Gamma_N}\tilde{\beta}_N & 0 & 0 \\ \hline 0 & 0 & \sqrt{\Gamma_N}\tilde{\beta}_N & \lambda_{2\,\tilde{P}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{2\,\tilde{P}} & \sqrt{\Gamma_N}\tilde{\beta}_N \\ -\sqrt{\Gamma_N}\tilde{\alpha}_N & 0 & 0 & 0 & \sqrt{\Gamma_N}\tilde{\beta}_N & a \end{array}\right) \tag{90}$$

Notice that $\lambda_{j\,Q}$ and $\lambda_{j\,\tilde{P}} = \lambda_{j\,P} - \Gamma_N \eta^2$, $j = 2, \ldots, [N/2]-1$ have multiplicity 2 while, if N is even, $\lambda_{[N/2]\,Q}$ and $\lambda_{[N/2]\,\tilde{Q}}$ have multiplicity one.

Linear stability:

We study the stability of the linear system (32),

$$\dot{\Delta Z'} = J\tilde{S}'\Delta Z', \quad \text{or equivalently} \quad J\dot{\Delta Z'} = -\tilde{S}'\Delta Z', \tag{91}$$

where $\Delta Z' = (\Delta q_2, \ldots, \Delta q_N, \Delta q_{N+1}, \Delta p_2, \ldots, \Delta p_N, \Delta p_{N+1})$. Following Proposition 5.5, the analysis is divided in two parts:

i) A subsystem of $N-4$ harmonic oscillators

$$\dot{\Delta q_j} = -\lambda_{j\,Q}\lambda_{j\,\tilde{P}}\Delta q_j, \qquad j = 4, ..., N.$$

ii) A remaining subsystem of six degrees of freedom (91):

$$\dot{\Delta y} = J_6 G_1 \Delta y \tag{92}$$

where $\Delta y = (\Delta q_2, \Delta q_3, \Delta q_{N+1}, \Delta p_2, \Delta p_3, \Delta p_{N+1})$, G_1 as in Eq. (90) and J_6 is the (6×6)

$$J_6 = \left(\begin{array}{c|c} 0 & -\mathbb{I}_3 \\ \hline \mathbb{I}_3 & 0 \end{array}\right)$$

and $\mathbb{I}_3$ is the (3×3) identity matrix.

Then from Theorem 4.1 and Eq. (80), Proposition 5.5, it follows:

Corollary 5.6 *The configuration (16) of polar vortex plus a ring of N identical vortices is linearly stable if the two conditions are simultaneously verified:*

$$\begin{aligned} &1) \qquad && \lambda_{max\,\tilde{P}} = \frac{1}{r_1^4}(t_1 - \lambda_{min\,A}) - \Gamma_N \eta^2 < 0; \\ &\quad i.e. && \Gamma_N > \Gamma_{min} = \frac{t_1 - \lambda_{min\,A}}{(1+z_1)^2} \qquad (93) \\ &2) && \text{the six eigenvalues of} \quad J_6 G_1 \quad \text{are all purely imaginary} \end{aligned}$$

Remarks:

i) If $N = 3$ only the second condition is required for stability (since the first condition concerns the variables q_j and p_j with $j = 4, ..., N$).

ii) Concerning the second condition, since we are considering a linearized system associated to a Hamiltonian one, the corresponding sixth-degree characteristic equation (for the eigenvalues λ) has only even powers

$$\lambda^6 + \mathtt{A}\lambda^4 + \mathtt{B}\lambda^2 + \mathtt{C} = 0$$

and therefore it can be reduced to a lower third-degree equation by introducing the variable $w = \lambda^2$,

$$w^3 + \mathtt{A}w^2 + \mathtt{B}w + \mathtt{C} = 0, \qquad (94)$$

where

$$\mathtt{A} = 2\lambda_{2Q}\lambda_{2\tilde{P}} + a^2 - 4\Gamma_N \frac{\tilde{\alpha}_N^2}{r_1^2}, \quad \mathtt{C} = (\Gamma_N \lambda_{2Q} \tilde{\beta}_N^2 - a\lambda_{2Q}\lambda_{2\tilde{P}} + \Gamma_N \lambda_{2\tilde{P}} \tilde{\alpha}_N^2)^2 \geq 0 \qquad (95)$$

$$\begin{aligned} \mathtt{B} = {} & 4\Gamma_N \frac{\tilde{\alpha}_N^4}{r_1^4} + 2\lambda_{2Q}\lambda_{2\tilde{P}} a^2 - 2a\Gamma_N \frac{\tilde{\alpha}_N^2}{r_1^4}(\lambda_{2Q} + r_1^4 \lambda_{2\tilde{P}}) \\ & -4\Gamma_N \lambda_{2Q}\lambda_{2\tilde{P}} \frac{\tilde{\alpha}_N^2}{r_1^2} + \lambda_{2Q}^2 \lambda_{2\tilde{P}}^2. \end{aligned} \qquad (96)$$

Then condition 2) of Corollary 5.6 is equivalent to seek under what conditions (i.e. values of Γ_N, z_1 and N) we have three negative roots w_1, w_2 and w_3. Notice that, by Eq. (94) and (95),

$$\mathtt{C} = -w_1 w_2 w_3 \geq 0.$$

Then whenever $\mathtt{C} > 0$, one root, say w_1, is negative while the other two could be:
I) $w_2 < w_3 < 0$; *II)* $w_2 = w_3^*$ (with $\Re e(w_2) < 0$); or *III)* $w_2 > w_3 > 0$.
For our system only case *I)* and *II)* are verified as explained in the appendix A. For a given N, by changing the parameters Γ_N and z_1, we pass by having three negative real

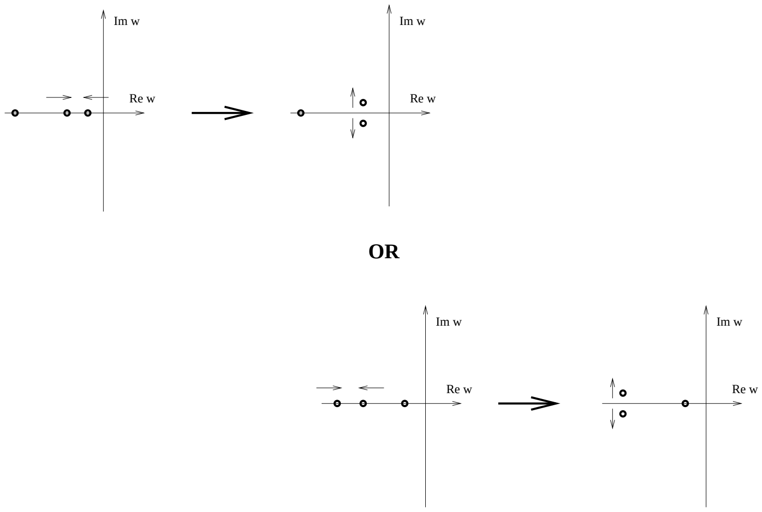

Figure 5: Hopf bifurcation.

roots to having one negative real and two complex conjugate roots (as shown in Fig. 5.2). This is a typical example of Hamiltonian Hopf bifurcation. Then the transition takes place along the line of double root (i.e. when $w_2 = w_3$).

Proposition 5.7 (double root) *Given N, in the plane Γ_N-θ_1 double roots of Eq. (94) are points of the curves $\Gamma_N = \Gamma_N(\theta_1)$ solutions of*

$$27\mathtt{C}'^2 + 4\mathtt{B}'^3 = 0, \quad \text{with} \quad \mathtt{B}' = \mathtt{B} - \mathtt{A}^2/3, \quad \mathtt{C}' = \mathtt{C} - \frac{\mathtt{AB}}{3} + \frac{2\mathtt{A}^3}{27}. \tag{97}$$

Proof: Given the Eq. (94) $w^3 + \mathtt{A}w^2 + \mathtt{B}w + \mathtt{C} = 0$, consider the change of variable $y = w + \mathtt{A}/3$ which transforms (94) into

$$y^3 + \mathtt{B}'y + \mathtt{C}' = 0, \quad \text{with} \quad \mathtt{B}' = \mathtt{B} - \mathtt{A}^2/3, \quad \mathtt{C}' = \mathtt{C} - \frac{\mathtt{AB}}{3} + \frac{2\mathtt{A}^3}{27} \tag{98}$$

Then a double root of (98) is a solution to the system of equations

$$\begin{cases} y^3 + \mathtt{B}'y + \mathtt{C}' = 0 \\ 3y^2 + \mathtt{B}' = 0 \end{cases},$$

from which (97) follows.

Figure 5.2 shows the stability regions for $N = 3, ..., 8$. The solid lines represent the theoretical curves of linear stability while the star-like points represent the numerical curves of orbital stability. For the case $N = 3$, in Figure 5.2.a the solid lines represent the curves of Eq. (97) and the stability region is between the two curves. It corresponds to verifying the second condition of Corollary 5.6. For the case $N = 4, ..., 8$, in Figure 5.2.b-f the second condition of Corollary 5.6 corresponds to the region to the right of the first solid curve on the left ($\Gamma_N = \tilde{\Gamma}_1$) which for $1 > z_1 > (N-2)/N$ (or $\arccos((N-2)/N)) > \theta_1 > 0$) is bounded by the last solid curve on the right ($\Gamma_N = \tilde{\Gamma}_2$), let's call this region $\mathcal{S}_2$; while the first condition of Corollary 5.6 corresponds to the region $\mathcal{S}_1$ to the right of the central curve (which represents the line $\Gamma_N = \Gamma_{min}$). Since for stability we need both conditions to be fulfilled, we identify the stability region as the intersection of the two, $\mathcal{S}_1 \cap \mathcal{S}_2$, i.e. the region to the right of the curve $\Gamma_N = \Gamma_{min}$,which for $1 > z_1 > (N-2)/N$ (or $\arccos((N-2)/N)) > \theta_1 > 0$) is bounded by the last solid curve on the right ($\Gamma_N = \tilde{\Gamma}_2$). We would like to stress the perfect agreement between the numerical (orbital stability) and the theoretical results (linear stability).

Theorem 5.8 (linear stability $N+1$) *For $N \geq 4$, the equilibrium is linearly stable if*

$$\begin{aligned} \Gamma_{min} < \Gamma < \tilde{\Gamma}_2 \qquad &\text{for} \qquad \frac{N-2}{N} < z_1 < 1, \\ \Gamma > \Gamma_{min} \qquad &\text{for} \qquad -1 < z_1 < \frac{N-2}{N}, \end{aligned}$$

where

$$\tilde{\Gamma}_2 = \frac{(N-1)z_1[Nz_1^2 - 2(N-1)z_1 - N]}{(1+z_1)^2[(N-2) - Nz_1]}$$

and

$$\Gamma_{min} = \frac{(1-N)(1+z_1^2) + m(N-m)}{(1+z_1)^2}, \qquad m = \left[\frac{N}{2}\right].$$

Nonlinear stability:

Details are given in the appendix B, here we summarize the main result about nonlinear stability.

Theorem 5.9 *i) For $N \geq 7$, the equilibrium is nonlinearly stable if*

$$\begin{aligned} \Gamma_{min} < \Gamma < \tilde{\Gamma}_2 \qquad &\text{for} \qquad \frac{N-2}{N} < z_1 < 1, \\ \Gamma > \Gamma_{min} \qquad &\text{for} \qquad -1 < z_1 < \frac{N-2}{N}. \end{aligned}$$

ii) For $4 \leq N \leq 6$, the equilibrium is nonlinearly stable if

$$\begin{aligned} \Gamma_{min} < \Gamma < \tilde{\Gamma}_2 \qquad &\text{for} \qquad \frac{N-2}{N} < z_1 < 1, \\ \Gamma > \Gamma_{min} \qquad &\text{for} \qquad \frac{N-2}{N} > z_1 > \begin{cases} -\frac{(N-2)^2}{4(N-1)} & \text{if} \quad N \text{ even} \\ -\frac{(N-3)}{4} & \text{if} \quad N \text{ odd.} \end{cases}, \end{aligned}$$

5.3 One ring with two polar vortices

In the case of relative equilibrium configuration of one latitudinal ring of N identical vortices of vorticity Γ_1 and two polar vortices, one at the north pole, of vorticity Γ_N, and one at the south pole, of vorticity Γ_S, we have $m = 1$, $n = N$ and, from Proposition 3.1, the angular velocity ω of the equilibrium configuration is (22)

$$\omega \;=\; -\Gamma_1 \frac{z_1}{r_1^2}(N-1) - \frac{\Gamma_N}{1-z_1} + \frac{\Gamma_S}{1+z_1}, \quad \forall z_1 \in (-1,1).$$

From Eq.(27),(60),(61), the Hessian $\tilde{S}$ has the structure

$$\tilde{S} = \left(\begin{array}{c|c|c|c|c|c}
Q & \sqrt{\Gamma_N}\alpha_N & \sqrt{\Gamma_S}\alpha_S & \mathbf{O} & \sqrt{\Gamma_N}\gamma_N & \sqrt{\Gamma_S}\gamma_S \\
\hline
\sqrt{\Gamma_N}\alpha_N^T & a_{11} & a_{12} & \sqrt{\Gamma_N}\beta_N^T & 0 & 0 \\
\hline
\sqrt{\Gamma_S}\alpha_S^T & a_{21} & a_{22} & \sqrt{\Gamma_N}\beta_N^T & 0 & 0 \\
\hline
\mathbf{O} & \sqrt{\Gamma_N}\beta_N & \sqrt{\Gamma_S}\beta_S & \tilde{P} & \sqrt{\Gamma_N}\delta_N & \sqrt{\Gamma_S}\delta_S \\
\hline
\sqrt{\Gamma_N}\gamma_N^T & 0 & 0 & \sqrt{\Gamma_N}\delta_N^T & a_1 1 & a_1 2 \\
\hline
\sqrt{\Gamma_S}\gamma_S^T & 0 & 0 & \sqrt{\Gamma_S}\delta_S^T & a_2 1 & a_2 2
\end{array}\right) \tag{99}$$

which is mostly the Hessian of one ring of vortices (85) implemented of two columns and two rows to include the interaction of the ring with the southern polar vortex as well, by means of the vectors α_S, β_S, δ_S, $\gamma_N \in \mathbb{R}^N$. From Eq. (46),(48),(53),(55), (78), (86) we have denoted

$$\tilde{t}_i = \frac{r_1^4}{\Gamma_1^3}(\Gamma_N \eta^2 + \Gamma_S \tilde{\eta}^2), \quad \eta = \frac{1}{1-z_1}, \quad \tilde{\eta} = \frac{1}{1+z_1}$$

$$\begin{array}{llll}
\alpha_S = \alpha_{1S}, & \tilde{\alpha}_N = \tilde{\alpha}_{1S} & \beta_S = \beta_{1S}, & \tilde{\beta}_N = \tilde{\beta}_{1N}, \\
\delta_S = \delta_{1N}, & \tilde{\delta}_N = \tilde{\delta}_{1S}, & \gamma_S = \gamma_{1S}, & \tilde{\gamma}_S = \tilde{\gamma}_{1S}.
\end{array}$$

As described in the previous section we consider the symplectic change of variables

$$Z = \mathcal{C}Z', \quad \text{i.e.} \quad \mathbf{q} = \tilde{C}\mathbf{q}', \qquad \mathbf{p} = \tilde{C}\mathbf{p}',$$

where $Z = (\mathbf{q}, \mathbf{p})^T$, $\mathbf{q} = (q_1, ..., q_N, q_{N+1}, q_{N+2})$, $\mathbf{p} = (p_1, ..., p_N, p_{N+1}, p_{N+2})$. Then from Eq. (74)

$$\tilde{S}' = \mathcal{C}^T \tilde{S} \mathcal{C} = \left(\begin{array}{c|c}
\mathcal{Q} & \mathcal{G} \\
\hline
\mathcal{G} & \mathcal{P}
\end{array}\right)$$

where

$$
\mathcal{Q} = \left(\begin{array}{cccccc|c|c}
\lambda_{1\,Q} & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\
0 & \lambda_{2\,Q} & 0 & \dots & \dots & 0 & 0 & 0 \\
0 & 0 & \lambda_{2\,Q} & 0 & \dots & 0 & \sqrt{\Gamma_N}\tilde{\alpha}_N & \sqrt{\Gamma_S}\tilde{\alpha}_S \\
 & & & \lambda_{3\,Q} & & & 0 & 0 \\
 & & & & \ddots & & \vdots & \vdots \\
 & & & & & \lambda_{[N/2]\,Q} & 0 & 0 \\
\hline
0 & 0 & \sqrt{\Gamma_N}\tilde{\alpha}_N & 0 & \dots & 0 & a_{11} & a_{12} \\
\hline
0 & 0 & \sqrt{\Gamma_S}\tilde{\alpha}_S & 0 & \dots & 0 & a_{22} & a_{21}
\end{array}\right) \tag{100}
$$

$$
\mathcal{P} = \left(\begin{array}{cccccc|c|c}
\lambda_{1\,\tilde{P}} & 0 & \dots & & & & 0 & 0 \\
0 & \lambda_{2\,\tilde{P}} & 0 & \dots & & & 0 & 0 \\
0 & 0 & \lambda_{2\,\tilde{P}} & 0 & \dots & 0 & \sqrt{\Gamma_N}\tilde{\delta}_N & \sqrt{\Gamma_S}\tilde{\delta}_S \\
 & & & \lambda_{3\,\tilde{P}} & & & 0 & 0 \\
 & & & & \ddots & & \vdots & \vdots \\
 & & & & & \lambda_{[N/2]\,\tilde{P}} & 0 & 0 \\
\hline
0 & 0 & \sqrt{\Gamma_N}\tilde{\delta}_N & 0 & \dots & 0 & a_{11} & a_{12} \\
\hline
0 & 0 & \sqrt{\Gamma_S}\tilde{\delta}_S & 0 & \dots & 0 & a_{21} & a_{22}
\end{array}\right), \tag{101}
$$

where

$$
\lambda_{j\,\tilde{P}} = \lambda_{j\,P} - \frac{\Gamma_N}{\Gamma_1}\eta^2 - \frac{\Gamma_S}{\Gamma_1}\tilde{\eta}^2, \quad \tilde{\alpha}_N = r_1\eta\sqrt{\frac{N}{2}}, \quad \tilde{\beta}_N = -\frac{\tilde{\alpha}_N}{r_1^2}, \quad \tilde{\alpha}_S = r_1\tilde{\eta}\sqrt{\frac{N}{2}}, \quad \tilde{\beta}_S = -\frac{\tilde{\alpha}_S}{r_1^2}, \tag{102}
$$

$\lambda_{j\,Q}$ and $\lambda_{j\,P}$ have been defined in Eq. (80);

$$
a_{11} = -\omega - \frac{N}{2}\Gamma_1 - \Gamma_S, \quad a_{22} = +\omega - \frac{N}{2}\Gamma_1 - \Gamma_N, \quad a_{12} = a_{21} = -\frac{1}{2}\sqrt{\Gamma_N\Gamma_S},
$$

as in (52) for $m = 1$. From now on we set $\Gamma_1 = 1$ (which amounts to express Γ_N and Γ_S in units of Γ_1) to deal only with the relevant parameters: Γ_N, N, z_1 or r_1 (since $r_1 = \sin\theta_1$ and $z_1 = \cos\theta_1$).

By the same arguments given in the previous section we can prove the following proposition

Proposition 5.10 (Reduced Hamiltonian for N+2 vortices)
Near the equilibrium Z^* (16) the reduced Hamiltonian takes the form

$$
\tilde{H}_r = \frac{1}{2}\sum_{j=4}^{N}(\lambda_{jQ}\tilde{q}_j^2 + \lambda_{j\tilde{P}}\tilde{p}_j^2) + \frac{1}{2}\tilde{B}_2(\tilde{q}_2, \tilde{q}_3, \tilde{q}_{N+1}, \tilde{q}_{N+2}, \tilde{p}_2, \tilde{p}_3, \tilde{p}_{N+1}, \tilde{p}_{N+2}) + h.o.t.
$$

where $\tilde{B}_2$ is the quadratic form

$$
\begin{aligned}
\tilde{B}_2 \;=\;& \lambda_{2\,Q}(\tilde{q}_2^2 + \tilde{q}_3^2) + \lambda_{2\,\tilde{P}}(\tilde{p}_2^2 + \tilde{p}_3^2) + a_{11}(\tilde{q}_{N+1}^2 + \tilde{p}_{N+1}^2) + a_{22}(\tilde{q}_{N+2}^2 + \tilde{p}_{N+2}^2) \\
& + 2(\sqrt{\Gamma_S}\alpha_S + \sqrt{\Gamma_N}\alpha_N)(\tilde{q}_{N+1}\tilde{q}_3 - \tilde{p}_{N+1}\tilde{q}_2) \\
& + 2(\sqrt{\Gamma_S}\beta_S + \sqrt{\Gamma_N}\beta_N)(\tilde{q}_{N+1}\tilde{p}_2 + \tilde{p}_{N+1}\tilde{p}_3).
\end{aligned}
$$

From Proposition 5.10 it follows that for all $\Gamma_N \neq 0$ and $\Gamma_S \neq 0$, the eigenvalues of the reduced Hessian are

$$\lambda_{3\,Q},\ \ldots,\ \lambda_{[N/2]\,Q},\ \lambda_{3\,\tilde{P}},\ \ldots,\ \lambda_{[N/2]\,\tilde{P}}$$

together with the eigenvalues of the matrix

$$G_2 = \left(\begin{array}{cccc|cccc}
\lambda_{2\,Q} & 0 & 0 & 0 & 0 & 0 & -\sqrt{\Gamma_N}\tilde{\alpha}_N & -\sqrt{\Gamma_S}\tilde{\alpha}_S \\
0 & \lambda_{2\,Q} & \sqrt{\Gamma_N}\tilde{\beta}_N & \sqrt{\Gamma_S}\tilde{\beta}_S & 0 & 0 & 0 & 0 \\
0 & \sqrt{\Gamma_N}\tilde{\alpha}_N & a_{11} & a_{12} & \sqrt{\Gamma_N}\tilde{\beta}_N & 0 & 0 & 0 \\
0 & \sqrt{\Gamma_S}\tilde{\alpha}_S & a_{12} & a_{22} & \sqrt{\Gamma_S}\tilde{\beta}_S & 0 & 0 & 0 \\
\hline
0 & 0 & \sqrt{\Gamma_N}\tilde{\beta}_N & \sqrt{\Gamma_S}\tilde{\beta}_S & \lambda_{2\,\tilde{P}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_{2\,\tilde{P}} & \sqrt{\Gamma_N}\tilde{\beta}_N & \sqrt{\Gamma_S}\tilde{\beta}_S \\
-\sqrt{\Gamma_N}\tilde{\alpha}_N & 0 & 0 & 0 & 0 & \sqrt{\Gamma_N}\tilde{\beta}_N & a_{11} & a_{12} \\
-\sqrt{\Gamma_S}\tilde{\alpha}_S & 0 & 0 & 0 & 0 & \sqrt{\Gamma_S}\tilde{\beta}_S & a_{12} & a_{22}
\end{array}\right) \tag{103}$$

Then the linear stability analysis is very similar to the one done in the previous section for one ring and one polar vortex.

Linear stability:

We study the stability of the linear system (32),

$$\dot{\Delta Z'} = J\tilde{S}'\Delta Z', \quad \text{or equivalently} \quad J\dot{\Delta Z'} = -\tilde{S}'\Delta Z', \tag{104}$$

where $\Delta Z' = (\Delta q_2, \ldots, \Delta q_N, \Delta q_{N+1}, \Delta q_{N+12}, \Delta p_2, \ldots, \Delta p_N, \Delta p_{N+1}, \Delta p_{N+2})$. Following Proposition 5.10, as in the previous section, the analysis is divided in two parts:

i) A subsystem of $N-4$ harmonic oscillators

$$\dot{\Delta q_j} = -\lambda_{j\,Q}\lambda_{j\,\tilde{P}}\Delta q_j, \qquad j = 4, ..., N.$$

ii) A remaining subsystem of eight degrees of freedom (91):

$$\dot{\Delta y} = J_8\tilde{G}_1\Delta y \tag{105}$$

where $\Delta y = (\Delta q_2, \Delta q_3, \Delta q_{N+1}, \Delta q_{N+2}, \Delta p_2, \Delta p_3, \Delta p_{N+1}, \Delta p_{N+2})$, $\tilde{G}_1$ as in Eq. (90) and J_8 is the (8×8)

$$J_8 = \left(\begin{array}{c|c} 0 & -\mathbb{I}_4 \\ \hline \mathbb{I}_4 & 0 \end{array}\right)$$

and $\mathbb{I}_4$ is the (4×4) identity matrix.

Then from Theorem 4.1 and Eq. (80), Proposition 5.10, it follows:

Corollary 5.11 *The configuration (16) of two polar vortices plus a ring of N identical vortices is linearly stable if the two conditions are simultaneously verified:*

$$\begin{aligned}
&1) \qquad && \lambda_{max\,\tilde{P}} = \frac{1}{r_1^4}(t_1 - \lambda_{min\,A}) - \Gamma_N\eta^2 - \Gamma_S\tilde{\eta}^2 < 0; \\
&\textit{i.e.} \qquad && \Gamma_N > \tilde{\Gamma}_{min} = \frac{t_1 - \lambda_{min\,A}}{(1+z_1)^2} - \Gamma_S\frac{(1-z_1)^2}{(1+z_1)^2} \qquad (106) \\
&2) \qquad && \text{the eight eigenvalues of} \quad J_8\tilde{G}_1 \quad \text{are all purely imaginary}
\end{aligned}$$

5.4 Two rings

Let's consider the case of relative equilibrium configurations of two latitudinal rings ($m = 2$) of n vortices each: a first ring at a vertical distance $z^{(1)}$, with vortices of vorticity Γ_1, and a second ring of a vertical distance $z^{(2)}$, with vortices of vorticity Γ_2. From Proposition 3.1 for a given $z^{(1)}$ the angular velocity ω of the equilibrium configuration and $z^{(2)}$ are solutions of the system of equations (22). There is no loss in generality in setting $\Gamma_1 = 1$. As already discussed in the previous sections we view the dynamics from a system in co-rotation with the vortex configuration. In the new reference system, the relative equilibrium configuration is then an equilibrium configuration (Z^*):

$$\begin{array}{llll} \varphi_\alpha &= \frac{2\pi(\alpha-1)}{n}, & \varphi_{\alpha+n} = \frac{2\pi(\alpha-1)}{n}, & \alpha = 1, ..., n, \\ z_\alpha &= z^{(1)}, & z_{\alpha+n} = z^{(2)}. & \end{array} \tag{107}$$

From now to simplify the notation we will use

$$z_1 = z^{(1)}, \qquad z_2 = z^{(2)}.$$

From Eq.(27), Proposition 4.7 and Eq. (60), (61), when evaluated at the equilibrium Z^* the Hessian $\tilde{S}$ has the structure

$$S_{Z^*} = \left(\begin{array}{c|c} \mathcal{Q} & \mathcal{G} \\ \hline \mathcal{G}^t & \mathcal{P} \end{array}\right) \tag{108}$$

The matrices $\mathcal{Q}$, $\mathcal{G}$ and $\mathcal{P}$ have the following block-structure

$$\mathcal{Q} = \left(\frac{\partial^2 H}{\partial q_\alpha \partial q_\beta}\right)_{|equ.} = \left(\begin{array}{c|c} Q_1 & B \\ \hline B^t & Q_2 \end{array}\right), \qquad \mathcal{G} = \left(\frac{\partial^2 H}{\partial q_\alpha \partial p_\beta}\right)_{|equ.} = \left(\begin{array}{c|c} O & G \\ \hline F & O \end{array}\right), \tag{109}$$

$$\text{and} \qquad \mathcal{P} = \left(\frac{\partial^2 H}{\partial p_\alpha \partial p_\beta}\right)_{|equ.} = \left(\begin{array}{c|c} P_1 & D \\ \hline D^t & P_2 \end{array}\right) \qquad \alpha\beta = 1, ..., 2N. \tag{110}$$

From Eq. (63),(70),(53),(55), (78) we have denoted

$$Q_i = \Gamma_i^2(-(s+\tilde{s}_i)\mathbb{I} + A), \quad P_i = \frac{\Gamma_i^2}{r_i^2}(t_i\mathbb{I} - A), \quad i = 1, 2,$$

$$b_j = \frac{r_1 r_2 - (1 - z_1 z_2)\cos(2\pi(j-1)/n)}{[1 - z_1 z_2 - r_1 r_2 \cos(2\pi(j-1)/n)]^2}, \quad d_j = -\frac{1 - r_1 r_2 \cos(2\pi(j-1)/n)}{[1 - z_1 z_2 - r_1 r_2 \cos(2\pi(j-1)/n)]^2} < 0,$$

where $r_i = \sqrt{1 - z_i^2}$ and

$$\tilde{s}_1 = r_1 r_2 \Gamma_2 \sum_{j=1}^n b_j, \quad \tilde{s}_2 = r_1 r_2 \frac{1}{\Gamma_2} \sum_{j=1}^n b_j, \quad s = \frac{n^2-1}{6}, \quad t_i = s - (n-1)(1 + z_i^2),$$

and

$$B = B_{12}, \quad D = D_{12}, \quad G = G_{12}, \quad F = F_{12},$$

which describe the interaction between the first ring and the second ring. From Proposition 4.7 we know that B and D are symmetrical circulant matrices, with real eigenvalues (67)-(68)

$$\lambda_{j\,B} = \Gamma_2 r_1 r_2 \left(b_1 + \sum_{\mu=2}^{n} b_\mu \cos(2\pi(j-1)(\mu-1)/n) \right),$$

$$\lambda_{j\,D} = d_1 + \sum_{\mu=2}^{n} d_\mu \cos(2\pi(j-1)(\mu-1)/n),$$

while G and F are skew-symmetric circulant matrices with imaginary eigenvalues (71)

$$\lambda_{j\,G} = \Gamma_2 \frac{r_1}{r_2}(z_1 - z_2)\lambda_{j\,T}, \quad \lambda_{j\,F} = -\Gamma_2^2 \frac{r_2}{r_1}(z_1 - z_2)\lambda_{j\,T}.$$

As described in the previous sections we consider the symplectic change (75) of variables

$$Z = \mathcal{C}Z', \quad \text{i.e.} \quad \mathbf{q} = \tilde{C}\mathbf{q}', \qquad \mathbf{p} = \tilde{C}\mathbf{p}',$$

where $Z = (\mathbf{q}, \mathbf{p})^T$, $\mathbf{q} = (q_1, ..., q_n, q_{n+1}, ..., q_{2n})$, $\mathbf{p} = (p_1, ..., p_n, p_{n+1}, ..., p_{2n})$, $N = 2n$. Then from Eq. (74)

$$\tilde{S}' = \mathcal{C}^T \tilde{S} \mathcal{C} = \left(\begin{array}{ccc|ccc|ccc|ccc}
\lambda_{1\,Q_1} & & & \lambda_{1\,B} & & & 0 & & & \lambda_{1\,G} & & \\
& \ddots & & & \ddots & & & \ddots & & & \ddots & \\
& & \lambda_{n\,Q_1} & & & \lambda_{n\,B} & & & 0 & & & \lambda_{n\,G} \\
\hline
\lambda_{1\,B} & & & \lambda_{1\,Q_2} & & & \lambda_{1\,F} & & & 0 & & \\
& \ddots & & & \ddots & & & \ddots & & & \ddots & \\
& & \lambda_{n\,B} & & & \lambda_{n\,Q_2} & & & \lambda_{1\,F} & & & 0 \\
\hline
0 & & & \lambda_{1\,F} & & & \lambda_{1\,P_1} & & & \lambda_{1\,D} & & \\
& \ddots & & & \ddots & & & \ddots & & & \ddots & \\
& & 0 & & & \lambda_{n\,F} & & & \lambda_{1\,P_1} & & & \lambda_{n\,D} \\
\hline
\lambda_{1\,G} & & & 0 & & & \lambda_{1\,D} & & & \lambda_{1\,P_2} & & \\
& \ddots & & & \ddots & & & \ddots & & & \ddots & \\
& & \lambda_{n\,G} & & & 0 & & & \lambda_{1\,D} & & & \lambda_{n\,P_2}
\end{array} \right),. \tag{111}$$

Notice that in $\tilde{S}'$ only the variables q_j, q_{j+1}, p_j and p_{j+1} are coupled (for $j = 1, ..., n$) so that in order to diagonalize $\tilde{S}'$ we simply have to diagonalize the (4×4) sub-matrices

$$\tilde{S}_j' = \left(\begin{array}{c|c} \mathcal{Q}_j & \mathcal{G}_j \\ \hline \mathcal{F}_j & \mathcal{P}_j \end{array} \right), = \left(\begin{array}{cc|cc} \lambda_{j\,Q_1} & \lambda_{j\,B} & 0 & \lambda_{j\,G} \\ \lambda_{j\,B} & \lambda_{j\,Q_2} & \lambda_{j\,F} & 0 \\ \hline 0 & \lambda_{j\,F} & \lambda_{j\,P_1} & \lambda_{j\,D} \\ \lambda_{j\,G} & 0 & \lambda_{j\,D} & \lambda_{j\,P_2} \end{array} \right), \qquad j = 1, ..., n. \tag{112}$$

Linear stability:

Let's at first consider the linear system (32),

$$\dot{\Delta Z}' = J\tilde{S}'\Delta Z', \quad \text{or equivalently} \quad J\dot{\Delta Z}' = -\tilde{S}'\Delta Z', \tag{113}$$

where $\Delta Z' = (\Delta q_1, \ldots, \Delta q_n, \ldots, \Delta q_{2n}, \Delta p_1, \ldots, \Delta p_n, \ldots, \Delta p_{2n})$. The analysis is divided in two parts:

i) A sub-system of four degrees of freedom

$$\dot{\Delta y_1} = J_4 \tilde{S}_1' \Delta y_1 \tag{114}$$

where $\Delta y_1 = (\Delta q_1, \Delta q_{n+1}, \Delta p_1, \Delta p_{n+1})$, $\tilde{S}_1'$ as in Eq. (112) and J_4 is the (4×4)

$$J_4 = \left(\begin{array}{c|c} 0 & -\mathbb{I}_2 \\ \hline \mathbb{I}_2 & 0 \end{array}\right)$$

and $\mathbb{I}_2$ is the (2×2) identity matrix. As stated in Proposition 5.12 this sub-system contains the zero eigenvalue.

ii) Remaining $n-1$ sub-systems of four degrees of freedom (91):

$$\dot{\Delta y_j} = J_4 \tilde{S}_j' \Delta y_j, \qquad j = 2, ..., n, \tag{115}$$

where $\Delta y_j = (\Delta q_j, \Delta q_{j+n}, \Delta p_j, \Delta p_{j+n})$, $\tilde{S}_j'$ as in Eq. (112).

Proposition 5.12 (Reduced Hamiltonian for $m = 2$) *Near the equilibrium Z^* (107) the reduced Hamiltonian takes the form*

$$\tilde{H}_r = (\lambda_{2\,\tilde{S}_1'} \tilde{q}_{n+1}^2 + \lambda_{4\,\tilde{S}_1'} \tilde{p}_{n+1}^2) + \sum_{j=2}^{n} (\lambda_{1\,\tilde{S}_j'} \tilde{q}_j^2 + \lambda_{3\,\tilde{S}_j'} \tilde{p}_j^2) + (\lambda_{2\,\tilde{S}_j'} \tilde{q}_{j+n}^2 + \lambda_{4\,\tilde{S}_j'} \tilde{p}_{j+n}^2). \tag{116}$$

In Proposition 5.12 we have considered the symplectic change of variables

$$Z' = \tilde{\mathcal{C}} \tilde{Z}, \quad \text{where} \quad Z_j' = \tilde{\mathcal{C}}_j \tilde{Z}_j, \quad j = 1, ..., n,$$

$Z_j = (q_j, q_{j+n}, p_j, p_{j+n})$ and $\tilde{\mathcal{C}}_j$ is a (4×4) matrix which diagonalizes $\tilde{S}_j'$, i.e.

$$\tilde{\mathcal{C}}_j^T \tilde{S}_j' \tilde{\mathcal{C}}_j = \left(\begin{array}{cc|cc} \lambda_{1\,\tilde{S}_j'} & 0 & 0 & 0 \\ 0 & \lambda_{2\,\tilde{S}_j'} & 0 & 0 \\ \hline 0 & 0 & \lambda_{3\,\tilde{S}_j'} & 0 \\ 0 & 0 & 0 & \lambda_{4\,\tilde{S}_j'} \end{array}\right)$$

Then from Theorem 4.1, we have the following corollary for linear stability

Corollary 5.13 (Linear stability) *The equilibrium Z^* (107) is linearly stable if*

i) $L_1 = J_4 \tilde{S}_1'$ has two purely imaginary eigenvalues, in addition to the double zero eigenvalue;

ii) the eigenvalues of $L_j = J_4\tilde{S}_j'$ $(j = 2, ..., n)$ are purely imaginary

We have then to compute the eigenvalues of L_j , and this is done in the following proposition:

Proposition 5.14 *The eigenvalues of the matrices $L_j = J_4\tilde{S}_j'$ $(j = 1, ..., n)$ are*

$$\begin{aligned} \lambda_{1\,L_j} &= +\sqrt{-\mathtt{B}_j + \sqrt{\mathtt{B}_j^2 - 4\mathtt{C}_j}}, \quad \lambda_{2\,L_j} = +\sqrt{-\mathtt{B}_j - \sqrt{\mathtt{B}_j^2 - 4\mathtt{C}_j}}, \\ \lambda_{3\,L_j} &= -\sqrt{-\mathtt{B}_j + \sqrt{\mathtt{B}_j^2 - 4\mathtt{C}_j}}, \quad \lambda_{4\,L_j} = -\sqrt{-\mathtt{B}_j - \sqrt{\mathtt{B}_j^2 - 4\mathtt{C}_j}}. \end{aligned} \tag{117}$$

where

$$\mathtt{B}_j = \lambda_{j\,Q_1}\lambda_{j\,P_1} + \lambda_{j\,Q_2}\lambda_{j\,P_2} + 2(\lambda_{j\,B}\lambda_{j\,D} - \lambda_{j\,G}\lambda_{j\,F}), \tag{118}$$

$$\begin{aligned} \mathtt{C}_j = {} & (\lambda_{j\,G}^2 - \lambda_{j\,Q_1}\lambda_{j\,P_2})\lambda_{j\,F}^2 - \lambda_{j\,Q_2}\lambda_{j\,P_1}\lambda_{j\,G}^2 \\ & -2\lambda_{j\,G}\lambda_{j\,F}\lambda_{j\,B}\lambda_{j\,D} + (\lambda_{j\,Q_1}\lambda_{j\,Q_2} - \lambda_{j\,B}^2)(\lambda_{j\,P_1}\lambda_{j\,P_2} - \lambda_{j\,D}^2). \end{aligned} \tag{119}$$

Remarks:

i) It follows from Proposition 5.12, and, in particular from Eq. (117), that if

$$\mathtt{B}_j > 0 \quad \text{and} \quad \frac{\mathtt{B}_j^2}{4} > \mathtt{C}_j > 0,$$

then the eigenvalues of $L_j = J\tilde{S}_j'$ $(j = 1, ..., n)$ are purely imaginary.

ii) From Eq. (67)-(68)

$$\lambda_{j\,G}^2 < 0, \quad \lambda_{j\,F}^2 < 0, \quad sgn(\lambda_{j\,G}\lambda_{j\,F}) = sgn(\Gamma_2^3(z_1 - z_2)) \quad j = 1, ..., n,$$

and therefore if

$$\lambda_{j\,P_i}, \quad \lambda_{j\,Q_i}, \quad (i = 1, 2,\ j = 1, ..., n) \quad \text{have all the same sign}$$

then the first part respectively of b, in (118), and c, in (119), are positive.

From Eq. (71)-(72), Theorem 5.3 we have:

Proposition 5.15 *The eigenvalues of the matrices Q_i, P_i, B, D, F and G have the following properties:*

i) for $j=1$:

$$\lambda_{1\,P_i} = \frac{\Gamma_i^2}{r_i^4}(1-n)(1+z_i^2) < 0, \quad \lambda_{1\,Q_1} = \lambda_{1\,Q_2} = -\Gamma_2 r_1 r_2 \sum_{\mu=1}^{n} b_\mu \quad i=1,2,$$

$$\lambda_{1\,D} = \sum_{\mu=1}^{n} d_\mu < 0, \quad \lambda_{1\,B} = -\lambda_{1\,Q_1}, \quad \lambda_{1\,F} = \lambda_{1\,G} = 0;$$

therefore

$$\mathtt{B}_1 = \lambda_{1\,Q_1}(\lambda_{1\,P_i} + \lambda_{1\,P_i} - 2\lambda_{1\,D}), \quad \mathtt{C}_1 = 0,$$

and

$$\lambda_{1\,L_1} = \lambda_{3\,L_1} = 0, \quad \lambda_{2\,L_1} = -\lambda_{4\,L_1} = \sqrt{-2\mathtt{B}_1};$$

ii) for $j \geq 2$:

$$\lambda_{max\,P_i} < 0 \quad \Leftrightarrow \quad 2 < n \leq n_* \quad \text{where} \quad \begin{cases} n_* = 7 - 4r_i^2 & \text{if } n \text{ is odd} \\ n_* = 2(2 - r_i^2 + \sqrt{2 - 3r_i^2 + r_i^4}) & \text{if } n \text{ is even} \end{cases}$$

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Appendix

A The cubic equation

B Nonlinear Analysis for $N+1$ and $N+2$ vortices

B.1 $N+1$ vortices

Let's consider the (6×6) minor (90) of the Hessian matrix $\tilde{S}'$

$$\tilde{G}_1 = \left(\begin{array}{ccc|ccc} \lambda_{2\,Q} & 0 & -\sqrt{\Gamma_N}\tilde{\alpha}_N & 0 & 0 & 0 \\ 0 & \lambda_{2\,\tilde{P}} & \sqrt{\Gamma_N}\tilde{\beta}_N & 0 & 0 & 0 \\ -\sqrt{\Gamma_N}\tilde{\alpha}_N & \sqrt{\Gamma_N}\tilde{\beta}_N & a & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_{2\,Q} & 0 & \sqrt{\Gamma_N}\tilde{\alpha}_N \\ 0 & 0 & 0 & 0 & \lambda_{2\,\tilde{P}} & \sqrt{\Gamma_N}\tilde{\beta}_N \\ 0 & 0 & 0 & \sqrt{\Gamma_N}\tilde{\alpha}_N & \sqrt{\Gamma_N}\tilde{\beta}_N & a \end{array}\right)$$

where, with respect to Eq. (90), we hare rearranged the variables as $(q_3, p_2, q_{N+1}, q_2, p3, p_{N+1})$. Then from Theorem 4.3 and Proposition 5.5 we have the following

Corollary B.1 *If the two conditions are simultaneously verified*

$$1) \qquad \lambda_{max\,\tilde{P}} = \frac{1}{r_1^4}(t_1 - \lambda_{min\,A}) - \Gamma_N \eta^2 < 0;$$

$$i.e. \qquad \Gamma_N > \Gamma_{min} = \frac{t_1 - \lambda_{min\,A}}{(1+z_1)^2} \tag{120}$$

$$2) \qquad \tilde{G}_1 \quad \text{is negative definite,} \tag{121}$$

then the configuration (16) of polar vortex plus a ring of N identical vortices is nonlinearly stable

Remarks:

i) In Corollary B.1 the first condition is the same as in Corollary 5.6.

ii) As for the linear case, for $N = 3$ only the second condition is required.

iii) Concerning the second condition, due to the block-diagonal structure of $\tilde{G}_1$, it is equivalent to proving that the (3×3) matrix

$$\tilde{M}_1 = \begin{pmatrix} \lambda_{2\,Q} & 0 & -\sqrt{\Gamma_N}\tilde{\alpha}_N \\ 0 & \lambda_{2\,\tilde{P}} & \sqrt{\Gamma_N}\tilde{\beta}_N \\ -\sqrt{\Gamma_N}\tilde{\alpha}_N & \sqrt{\Gamma_N}\tilde{\beta}_N & a \end{pmatrix} \tag{122}$$

is negative definite. Since $\lambda_{2\,Q} = 1 - N < 0$ $(\forall N)$, and by Corollary B.1 if the first condition, Eq. (125), $\lambda_{2\,\tilde{P}} < 0$, then $\tilde{M}_1$ is negative definite if its determinant is negative, i.e. if

$$\Delta \equiv \det(\tilde{M}_1) = \lambda_{2\,Q}(\lambda_{2\,\tilde{P}}a - \Gamma_N \beta_N^2) - \Gamma_N \alpha_N^2 \lambda_{2\,\tilde{P}} < 0 \tag{123}$$

Therefore, from the above discussion we can consider the following proposition already proved.

Proposition B.2 *If*

$$\lambda_{max\,\tilde{P}} < 0 \quad \text{and} \quad \Delta < 0.$$

then the relative equilibrium (16) of one polar vortex with a ring of N identical vortices is nonlinearly stable

B.2 $N+2$ vortices

Let's consider the (6×6) minor (112) of the Hessian matrix $\tilde{S}'$

$$\tilde{G}_2 = \left(\begin{array}{cccc|cccc} \lambda_{2\,Q} & 0 & -\sqrt{\Gamma_N}\tilde{\alpha}_N & -\sqrt{\Gamma_S}\tilde{\alpha}_S & 0 & 0 & 0 & 0 \\ 0 & \lambda_{2\,\tilde{P}} & \sqrt{\Gamma_N}\tilde{\beta}_N & \sqrt{\Gamma_S}\tilde{\beta}_S & 0 & 0 & 0 & 0 \\ -\sqrt{\Gamma_N}\tilde{\alpha}_N & \sqrt{\Gamma_N}\tilde{\beta}_N & a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ -\sqrt{\Gamma_S}\tilde{\alpha}_S & \sqrt{\Gamma_S}\tilde{\beta}_S & a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_{2\,Q} & 0 & \sqrt{\Gamma_N}\tilde{\alpha}_N & \sqrt{\Gamma_S}\tilde{\alpha}_S \\ 0 & 0 & 0 & 0 & 0 & \lambda_{2\,\tilde{P}} & \sqrt{\Gamma_N}\tilde{\beta}_N & \sqrt{\Gamma_S}\tilde{\beta}_S \\ 0 & 0 & 0 & 0 & \sqrt{\Gamma_N}\tilde{\alpha}_N & \sqrt{\Gamma_N}\tilde{\beta}_N & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & \sqrt{\Gamma_S}\tilde{\alpha}_S & \sqrt{\Gamma_S}\tilde{\beta}_S & a_{21} & a_{22} \end{array}\right) \tag{124}$$

where, with respect to Eq. (112), we hare rearranged the variables as

$$(q_3, p_2, q_{N+1}, q_{N+2}, q_2, p3, p_{N+1}, p_{N+2}).$$

Then from Theorem 4.3 and Proposition 5.10 we have the following

Corollary B.3 *If the two conditions are simultaneously verified*

$$1) \qquad \lambda_{max\,\tilde{P}} = \frac{1}{r_1^4}(t_1 - \lambda_{min\,A}) - \Gamma_N \eta^2 - \Gamma_S \tilde{\eta}^2 < 0;$$

$$i.e. \qquad \Gamma_N > \tilde{\Gamma}_{min} = \frac{t_1 - \lambda_{min\,A}}{(1+z_1)^2} - \Gamma_S \frac{(1-z_1)^2}{(1+z_1)^2} \tag{125}$$

$$2) \qquad \tilde{G}_2 \quad \text{is negative definite,} \tag{126}$$

then the configuration (16) of two polar vortices plus a ring of N identical vortices is nonlinearly stable

Remarks:

i) In Corollary B.3 the first condition is the same as in Corollary 5.11.

ii) As for the linear case, for $N = 3$ only the second condition is required.

iii) Concerning the second condition, due to the block-diagonal structure of $\tilde{G}_2$, it is equivalent to proving that the (4×4) matrix

$$\tilde{M}_2 = \begin{pmatrix} \lambda_{2\,Q} & 0 & -\sqrt{\Gamma_N}\tilde{\alpha}_N & -\sqrt{\Gamma_S}\tilde{\alpha}_S \\ 0 & \lambda_{2\,\tilde{P}} & \sqrt{\Gamma_N}\tilde{\beta}_N & \sqrt{\Gamma_S}\tilde{\beta}_S \\ -\sqrt{\Gamma_N}\tilde{\alpha}_N & \sqrt{\Gamma_N}\tilde{\beta}_N & a_{11} & a_{12} \\ -\sqrt{\Gamma_S}\tilde{\alpha}_S & \sqrt{\Gamma_S}\tilde{\beta}_S & a_{12} & a_{22} \end{pmatrix} \tag{127}$$

is negative definite. Since $\lambda_{2\,Q} = 1 - N < 0$ $(\forall N)$, and by Corollary B.3 if the first condition, Eq. (125), $\lambda_{2\,\tilde{P}} < 0$, then $\tilde{M}_2$ is negative definite if its principal minors of order three and four are respectively negative and positive, i.e. if

$$\begin{aligned}\Delta_3 &= \lambda_{2\,Q}(\lambda_{2\,\tilde{P}}a_{11} - \Gamma_N\tilde{\beta}_N^2) - \Gamma_N\tilde{\alpha}_N^2\lambda_{2\,\tilde{P}} < 0, \qquad (128)\\ \Delta_4 &\equiv \det(\tilde{M}_1) = \Gamma_S[(\tilde{\alpha}_N\tilde{\beta}_S - \tilde{\alpha}_S\tilde{\beta}_N)^2\Gamma_N - a_{11}(\lambda_{2\,Q}\tilde{\beta}_S^2 + \lambda_{2\,\tilde{P}}\tilde{\alpha}_S^2)]\\ &\quad -\Gamma_N\Gamma_S(\tilde{\beta}_N\tilde{\beta}_S\lambda_{2\,Q} + \tilde{\alpha}_S\tilde{\alpha}_N\lambda_{2\,\tilde{P}}) - \Gamma_N a_{22}(\lambda_{2\,Q}\tilde{\beta}_N^2 + \lambda_{2\,\tilde{P}}\tilde{\alpha}_N^2)\\ &\quad +\lambda_{2\,Q}\lambda_{2\,\tilde{P}}\left(a_{11}a_{22} - \frac{1}{4}\Gamma_S\Gamma_N\right) > 0 \qquad (129)\end{aligned}$$

Therefore, from the above discussion we can consider the following proposition already proved.

Proposition B.4 *If*

$$\lambda_{max\,\tilde{P}} < 0, \quad \Delta_3 < 0 \quad \text{and} \quad \Delta_4 > 0.$$

then the relative equilibrium (16) of two polar vortices with a ring of N identical vortices is nonlinearly stable

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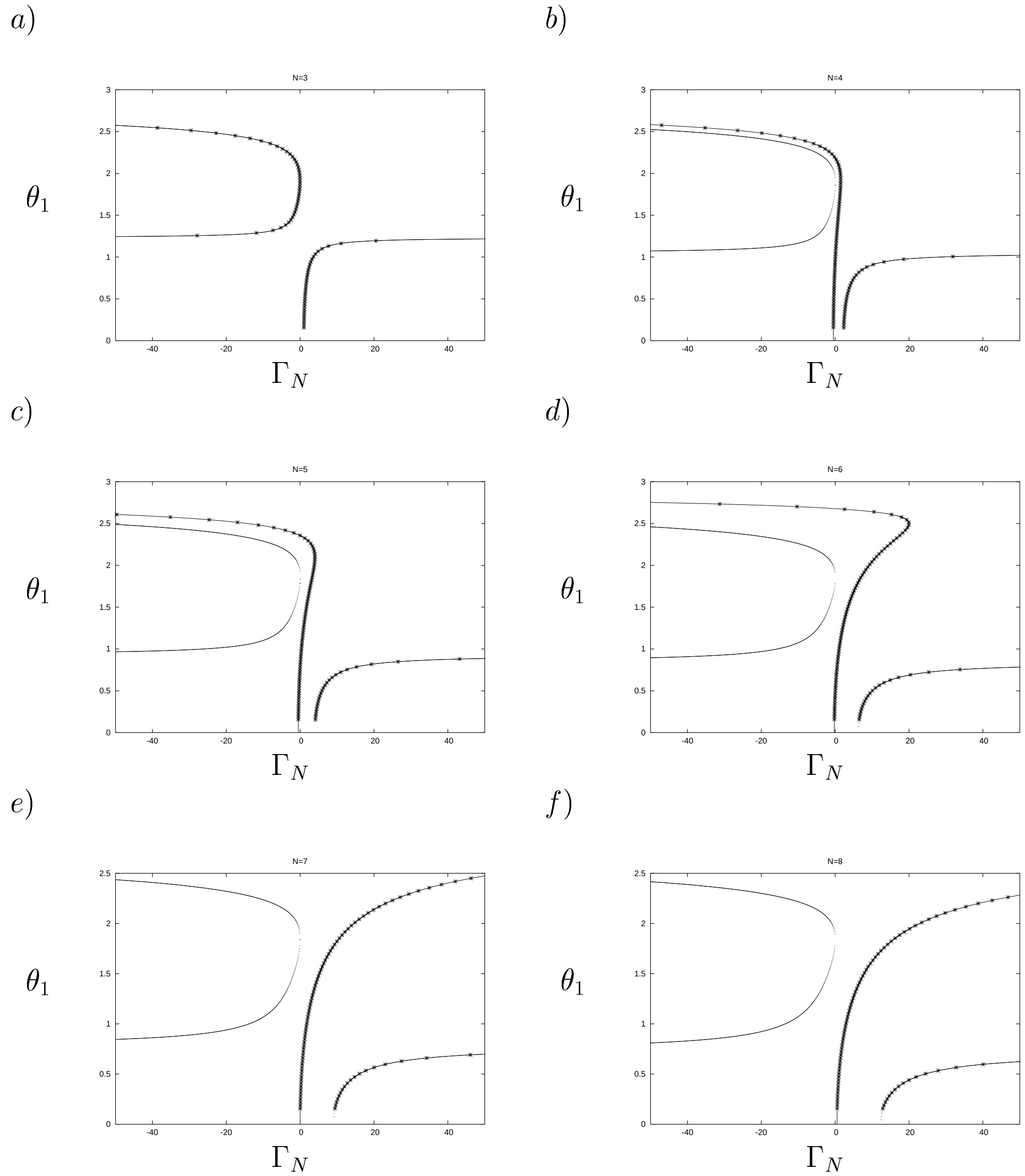

Figure 6: Theoretical (solid line) and numerical (star-like points) stability curves. a) $N = 3$; b) $N = 4$; c) $N = 5$; d) $N = 6$; e) $N = 7$; f) $N = 8$.