A CONDITION OF BOSHERNITZAN AND UNIFORM CONVERGENCE IN THE MULTIPLICATIVE ERGODIC THEOREM

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Abstract. This paper is concerned with uniform convergence in the multiplicative ergodic theorem on aperiodic subshifts. If such a subshift satisfies a certain condition, originally introduced by Boshernitzan, every locally constant SL(2,$\mathbb{R}$)-valued cocycle is uniform. As a consequence, the corresponding Schrödinger operators exhibit Cantor spectrum of Lebesgue measure zero.

An investigation of Boshernitzan's condition then shows that these results cover all earlier results of this type and, moreover, provide various new ones. In particular, Boshernitzan's condition is shown to hold for almost all circle maps and almost all Arnoux-Rauzy subshifts.

1. Introduction

This paper is concerned with uniform convergence in the multiplicative ergodic theorem.

More precisely, let (Ω, T) be a topological dynamical system Thus, Ω is a compact metric space and $T : \Omega \longrightarrow \Omega$ is a homeomorphism. Assume furthermore that (Ω, T) is uniquely ergodic, that is, there exists a unique T-invariant probability measure μ on Ω.

As usual the dynamical system (Ω, T) is called minimal if every orbit $\{T^n\omega : n \in \mathbb{Z}\}$ is dense in Ω. It is called aperiodic if $T^n\omega \neq \omega$ for all $\omega \in \Omega$ and $n \neq 0$.

Let SL(2,$\mathbb{R}$) be the group of real valued 2×2-matrices with determinant equal to one equipped with the topology induced by the standard metric on 2×2 matrices.

To a continuous function $A : \Omega \longrightarrow$ SL(2,$\mathbb{R}$) we associate the cocycle

$$A(\cdot,\cdot) : \mathbb{Z} \times \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

defined by

$$A(n,\omega) \equiv \begin{cases} A(T^{n-1}\omega)\cdots A(\omega) & : \quad n > 0 \\ Id & : \quad n = 0 \\ A^{-1}(T^n\omega)\cdots A^{-1}(T^{-1}\omega) & : \quad n < 0. \end{cases}$$

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By the multiplicative ergodic theorem, there exists a $\Lambda(A) \in \mathbb{R}$ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\| \tag{1}$$

for μ-almost every $\omega \in \Omega$. Now, it is well known that unique ergodicity of (Ω, T) is equivalent to uniform convergence in the Birkhoff additive ergodic theorem when applied to continuous functions. Therefore, it is natural to investigate uniform convergence in (1). This motivates the following definition.

Definition 1.1. [39, 90]. *Let (Ω, T) be uniquely ergodic. The continuous function $A : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ is called uniform if the limit $\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$ exists for all $\omega \in \Omega$ and the convergence is uniform on Ω.*

Remark 1. For minimal topological dynamical systems, uniform existence of the limit in the definition implies uniform convergence. This was proven by Furstenberg and Weiss [40]. In fact, their result is even more general and applies to arbitrary real-valued continuous cocycles.

Various aspects of uniformity of cocycles have been considered in the past:

A first topic has been to provide examples of non-uniform cocycles. In fact, in [90] Walters asks the question whether every uniquely ergodic dynamical system with non-atomic measure μ admits a non-uniform cocycle. He presents a class of examples admitting non-uniform cocycles based on results of Veech [86]. He also gives another class of examples, namely suitable irrational rotations, for which non-uniformity was shown by Herman [45]. In general, however, Walters' question is still open.

A different line of study has been pursued by Furman in [39]. He characterizes uniformity of A on a given uniquely ergodic minimal (Ω, T) by a suitable hyperbolicity condition. The results of Furman can essentially be extended to uniquely ergodic systems (and, in fact, a strengthening of some sort can be obtained for minimal uniquely ergodic systems), as shown by Lenz in [65]. They also give that the corresponding results of [46] provide examples of non-uniform cocycles as discussed in [65].

Finally, somewhat complementary to Walters' original question, it is possible to study conditions on subshifts over finite alphabets which imply uniformity of locally constant cocycles. This topic and variants of it have been discussed at various places [23, 47, 62, 63, 64, 65]. It is the main focus of the present article. It is not only of intrinsic interest but also relevant in the study of spectral theory of certain Schrödinger operators, as recently shown by Lenz [63] (see below for details).

To elaborate on this and state our main results, we recall some further notions.

(Ω, T) is called a subshift over $\mathcal{A}$ if $\mathcal{A}$ is finite with discrete topology and Ω is a closed T-invariant subset of $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}^{\mathbb{Z}}$ carries the product topology and $T : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is given by $(Ts)(n) := s(n+1)$. A function F on Ω is called locally constant if there exists an $N \in \mathbb{N}$ with

$$F(\omega) = F(\rho) \ \text{ whenever } \ (\omega(-N), \dots, \omega(N)) = (\rho(-N), \dots, \rho(N)). \tag{2}$$

We will freely use notions from combinatorics on words (see, e.g., [67, 68]). In particular, the elements of $\mathcal{A}$ are called letters and the elements of the free monoid

$\mathcal{A}^*$ over $\mathcal{A}$ are called words. The length $|w|$ of a word w is the number of its letters. The number of occurrences of a word w in a word x is denoted by $\#_w(x)$.

Each subshift (Ω, T) over $\mathcal{A}$ gives rise to the associated set of words

$$\mathcal{W}(\Omega) := \{\omega(k)\cdots\omega(k+n-1) : k \in \mathbb{Z}, n \in \mathbb{N}, \omega \in \Omega\}. \tag{3}$$

For $w \in \mathcal{W}$, we define

$$V_w := \{\omega \in \Omega : \omega(1)\cdots\omega(|w|) = w\}.$$

Finally, if ν is a T-invariant probability measure on (Ω, T) and $n \in \mathbb{N}$, we set

$$\eta_\nu(n) := \min\{\nu(V_w) : w \in \mathcal{W}, |w| = n\}. \tag{4}$$

If (Ω, T) is uniquely ergodic with invariant probability measure μ , we set $\eta(n) := \eta_\mu(n)$.

Definition 1.2. *Let (Ω, T) be a subshift over $\mathcal{A}$. Then, (Ω, T) is said to satisfy condition* (B) *if there exists an ergodic probability measure ν on Ω with*

$$\limsup_{n\to\infty} n\,\eta_\nu(n) > 0.$$

Thus, (Ω, T) satisfies (B) *if and only if there exists an ergodic probability measure ν on Ω, a constant $C > 0$ and a sequence (l_n) in $\mathbb{N}$ with $l_n \to \infty$ for $n \to \infty$ such that $|w|\nu(V_w) \geq C$ whenever $w \in \mathcal{W}(\Omega)$ with $|w| = l_n$ for some $n \in \mathbb{N}$.*

This condition was introduced by Boshernitzan in [11] (also see [12] for related material). For minimal interval exchange transformations, it was shown to imply unique ergodicity by Veech in [89]. Finally, in [14], Boshernitzan showed that it implies unique ergodicity for arbitrary minimal subshifts.

Our main result is:

Theorem 1. *Let (Ω, T) be a minimal subshift which satisfies* (B). *Let $A : \Omega \longrightarrow$ SL(2,$\mathbb{R}$) be locally constant. Then, A is uniform.*

As discussed below, this result covers all earlier results of this form as given in [23, 47, 64, 65]. Moreover, as we will show below, it also applies to various new examples, including many circle maps and Arnoux-Rauzy subshifts. This point is worth emphasizing, as most circle maps and Arnoux-Rauzy subshifts seem to have been rather out of reach of earlier methods.

The proof of the main result is based on two steps. In the first step, we give various equivalent characterizations of condition (B). This is made precise in Theorem 5. This result may be of independent interest. In our context it shows that (B) implies uniform convergence on "many scales." In the second step, we use the so-called Avalanche Principle introduced by Goldstein and Schlag in [41] and extended by Bourgain and Jitomirskaya in [15] to conclude uniform convergence from uniform convergence on "many scales."

As a by-product of our proof, we obtain a simple combinatorial argument for unique ergodicity for subshifts satisfying (B). Unlike the proof given in [14], we do not need any apriori estimates on the number of invariant measures.

As mentioned already, our results are particularly relevant in the study of certain Schrödinger operators. This is discussed next:

To each bounded $V : \mathbb{Z} \longrightarrow \mathbb{R}$, we can associate the Schrödinger operator $H_V : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ acting by

$$(H_V u)(n) \equiv u(n+1) + u(n-1) + V(n)u(n).$$

The spectrum of H_V is denoted by $\sigma(H_V)$.

Now, let (Ω, T) be a subshift over $\mathcal{A}$ and assume without loss of generality that $\mathcal{A} \subset \mathbb{R}$. Then, (Ω, T) gives rise to the family $(H_\omega)_{\omega\in\Omega}$ of selfadjoint operators. These operators arise in the study of aperiodically ordered solids, so-called quasicrystals. They exhibit interesting spectral features such as Cantor spectrum of Lebesgue measure zero, purely singularly continuous spectrum and anomalous transport. They have attracted a lot of attention in recent years (see, e.g., the surveys [21, 85] and discussion below for details). Recently, Lenz has shown that uniformity of certain locally constant cocycles is intimately related to Cantor spectrum of Lebesgue measure zero for these operators [63]. This can be combined with our main result to give the following theorem (see below for details).

Theorem 2. *Let (Ω, T) be a minimal subshift which satisfies* (B). *If (Ω, T) is aperiodic, then there exists a Cantor set $\Sigma \subset \mathbb{R}$ of Lebesgue measure zero with $\sigma(H_\omega) = \Sigma$ for every $\omega \in \Omega$.*

This result covers all earlier results on Cantor spectrum of measure zero [1, 7, 8, 16, 24, 25, 63, 66, 73, 83, 84] as discussed below. More importantly, it gives various new ones. In particular, it covers almost all circle maps and Arnoux-Rauzy subshifts.

To give a flavor of these new examples, we mention the following theorem. Define for $\alpha, \theta, \beta \in (0,1)$ arbitrary, the function

$$V_{\alpha,\beta,\theta} : \mathbb{Z} \longrightarrow \{0,1\}, \quad \text{by } V_{\alpha,\beta,\theta}(n) := \chi_{[1-\beta,1)}(n\alpha + \theta \mod 1),$$

where χ_M denotes the characteristic function of the set M. These functions are called circle maps.

Theorem 3. *Let $\alpha \in (0,1)$ be irrational. Then, we have the following:*

(a) *For almost every $\beta \in (0,1)$, the spectrum $\sigma(H_{V_{\alpha,\beta,\theta}})$ is a Cantor set of Lebesgue measure zero for every $\theta \in (0,1)$.*

(b) *If α has bounded continued fraction expansion, then $\sigma(H_{V_{\alpha,\beta,\theta}})$ is a Cantor set of Lebesgue measure zero for every $\beta \in (0,1)$ and every $\theta \in (0,1)$.*

Remark 2. This result is particularly relevant as all earlier results on Cantor spectrum for circle maps [1, 8, 24, 83, 84] only cover a set of parameters (α, β) of Lebesgue measure zero in $(0,1) \times (0,1)$ (cf. Appendix A).

Finally, we mention the following by-product of our investigation. Details (and precise definitions) will be discussed in Section 8.

Theorem 4. *Let (Ω, T) be a minimal subshift which satisfies* (B) *and $(H_\omega)_{\omega\in\Omega}$ the associated family of operators. Then the Lyapunov exponent $\gamma : \mathbb{R} \longrightarrow [0, \infty)$ is continuous.*

The paper is organized as follows: In Section 2 we study condition (B) and show its equivalence to various other conditions. As a by-product this shows unique ergodicity of subshifts satisfying (B). Moreover, it is used in Section 3 to give a proof of our main result. Stability of the results under certain operations on the subshift is discussed in Section 4. In Section 5 we discuss examples for which (B) is known to hold. New examples, viz certain circle maps and Arnoux-Rauzy subshifts, are given in Sections 6 and 7. Finally, the application to Schrödinger operators is discussed in Section 8.

2. Boshernitzan's Condition (B)

In this section, we give various equivalent characterizations of (B). This is made precise in Theorem 5. Then, we provide a new proof of unique ergodicity for systems satisfying (B) in Theorem 6. Theorem 5 in some sense generalizes the main results of [62] and its proof heavily uses and extends ideas from there.

To state our result, we need some preparation. We start by introducing a variant of Boshernitzan's condition (B). Namely, if (Ω, T) is a subshift, we define for $w \in \mathcal{W}(\Omega)$ the set U_w by

$$U_w := \{\omega \in \Omega : \exists n \in \{0, 1, \ldots, |w| - 1\} \text{ such that } \omega(-n+1)\ldots\omega(-n+|w|) = w\}.$$

If ω belongs to U_w, we say that w occurs in ω around one.

Definition 2.1. *Let (Ω, T) be a subshift over $\mathcal{A}$. Then, (Ω, T) is said to satisfy condition* (B') *if there exists an ergodic probability measure ν on Ω, a constant $C' > 0$, and a sequence (l'_n) in $\mathbb{N}$ with $l'_n \to \infty$ for $n \to \infty$ such that $\nu(U_w) \geq C'$ whenever $w \in \mathcal{W}(\Omega)$ with $|w| = l'_n$ for some $n \in \mathbb{N}$.*

Next, we discuss a consequence of Kingman's ergodic theorem. Recall that $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ is called subadditive if it satisfies $F(xy) \leq F(x) + F(y)$ whenever $x, y, xy \in \mathcal{W}(\Omega)$, where (Ω, T) is an arbitrary subshift.

Proposition 2.2. *Let (Ω, T) be a uniquely ergodic subshift with invariant probability measure μ. Let $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ be subadditive, then there exists a number $\Lambda(F) \in \mathbb{R} \cup \{-\infty\}$ with*

$$\Lambda(F) = \lim_{n\to\infty} n^{-1} F(\omega(1)\cdots\omega(n))$$

for μ-almost every ω in Ω.

Proof. For $n \in \mathbb{N}$, we define the continuous function $f_n : \Omega \longrightarrow \mathbb{R}$, by

$$f_n(\omega) := F(\omega(1)\ldots\omega(n)).$$

As F is subadditive, (f_n) is a subadditive cocycle. Thus Kingman's subadditive theorem applies. This proves the statement. □

Theorem 5. *Let (Ω, T) be a minimal subshift over $\mathcal{A}$. Then the following conditions are equivalent:*

- (i) *(Ω, T) satisfies* (B).
- (ii) *(Ω, T) satisfies* (B').
- (iii) *(Ω, T) is uniquely ergodic and there exists a sequence (l'_n) in $\mathbb{N}$ with $l'_n \to \infty$ for $n \to \infty$ such that $\lim_{n\to\infty} |w_n|^{-1} F(w_n) = \Lambda(F)$ for every subadditive F and every sequence (w_n) in $\mathcal{W}(\Omega)$ with $|w_n| = l'_n$ for every $n \in \mathbb{N}$.*

The remainder of this section is devoted to a proof of this theorem. The proof will be split into several parts.

Lemma 2.3. *Let (Ω, T) be a minimal subshift. Then, (Ω, T) satisfies* (B) *if and only if it satisfies* (B').

Proof. If (Ω, T) is periodic, validity of (B) and (B') is immediate. Thus, we can restrict our attention to aperiodic (Ω, T).

Apparently, $\nu(U_w) \leq |w|\nu(V_w)$ for all $w \in \mathcal{W}(\Omega)$ and all ergodic probability measures ν on Ω. Thus, (B') implies (B) (with the same ν, l_n, and C).

Conversely, assume that (Ω, T) satisfies (B). We will show that it satisfies (B') with $l'_n = [2l_n/3] + 1$, $n \in \mathbb{N}$, and $C' = C/9$. Here, for arbitrary $a \in \mathbb{R}$, we set $[a] := \sup\{n \in Z : n \le a\}$.

Consider $v \in \mathcal{W}(\Omega)$ with $|v| = l'_n$ for some $n \in \mathbb{N}$. Choose $w \in \mathcal{W}(\Omega)$ with $|w| = l_n$ such that v is a prefix of w. There are two cases:

Case 1. There exists a primitive $x \in \mathcal{W}(\Omega)$ and a prefix $\widetilde{x}$ of x such that $w = x^k \widetilde{x}$ for some $k \ge 6$.

As (Ω, T) is minimal and aperiodic, the word x does not occur with arbitrarily high powers. Thus, we can find $y \in \mathcal{W}(\Omega)$ such that

$$\widetilde{w} := x^{k-1} y \in \mathcal{W}(\Omega)$$

satisfies $|\widetilde{w}| = l_n$ but x^k is not a prefix of $\widetilde{w}$. Now, as x is primitive, it does not appear non-trivially in x^{k-1}. Therefore, different copies of $\widetilde{w}$ have distance at least $(k-2)|x|$. This gives

$$\nu(U_{\widetilde{w}}) \ge (k-2)|x|\nu(V_{\widetilde{w}}) \ge \frac{(k-2)|x|}{(k+2)|x|}|\widetilde{w}|\nu(V_{\widetilde{w}}) \ge \frac{1}{2}C.$$

Moreover, by construction, v is a subword of $\widetilde{w}$ (and even of x^{k-1}) with

$$\frac{|v|}{|\widetilde{w}|} \ge \frac{1}{2}.$$

Putting these estimates together, we infer

$$\nu(U_v) \ge \frac{1}{2}\nu(U_{\widetilde{w}}) \ge \frac{1}{2} \cdot \frac{1}{2} \cdot C = \frac{C}{4}.$$

Case 2. There does not exist a primitive x in $\mathcal{W}$ and a prefix $\widetilde{x}$ of x with $w = x^k \widetilde{x}$ for some $k \ge 6$.

In this case, different copies of w have distance at least $\frac{1}{6}|w|$. Therefore, we have

$$\nu(U_w) \ge \frac{1}{6}|w|\nu(V_w)$$

and this gives

$$\nu(U_v) \ge \frac{2}{3}\nu(U_w) \ge \frac{2}{3} \cdot \frac{1}{6} \cdot |w|\nu(V_w) \ge \frac{1}{9}C.$$

In both cases the desired estimates hold and the proof of the lemma is finished. □

We next give our proof of unique ergodicity for systems satisfying (B'). The proof proceeds in two steps. In the first step, we use (B') to show existence of the frequencies along certain sequences. In the second step, we show existence of the frequencies along all sequences. Let us emphasize that it is exactly this two-step procedure which is underlying the proof of our main result on locally constant matrices. However, in that case the details are more involved.

We need the following proposition.

Proposition 2.4. *Let (Ω, T) be a subshift with ergodic probability measure ν. Let $f : \Omega \longrightarrow \mathbb{R}$ be a bounded measurable function. Then,*

$$\lim_{n,m \ge 0, n+m \to \infty} \frac{1}{n+m} \sum_{k=-m}^{n} f(T^k \omega) = \nu(f)$$

for ν-almost every $\omega \in \Omega$.

Proof. By Birkhoff's ergodic theorem, we have both

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\omega) = \nu(f) \text{ and } \lim_{m\to\infty} \frac{1}{m} \sum_{k=-m}^{0} f(T^k\omega) = \nu(f)$$

for ν-almost every $\omega \in \Omega$. Now, for every sequence $(a_k)_{k\in\mathbb{Z}}$ with

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} a_k = \lim_{m\to\infty} \frac{1}{m} \sum_{k=-m}^{0} a_k = a,$$

one easily infers

$$\lim_{n,m\geq 0, n+m\to\infty} \frac{1}{n+m} \sum_{k=-m}^{n} a_k = a.$$

The statement follows immediately. □

Theorem 6. *If the subshift* (Ω, T) *satisfies* (B'), *it is uniquely ergodic and minimal.*

Proof. It suffices to show that the frequencies $\lim_{|x|\to\infty} \frac{\#_w(x)}{|x|}$ exist for every $w \in \mathcal{W}$. Then, the system is uniquely ergodic by standard reasoning. Moreover, in this case, the system is minimal as well as all frequencies are positive by (B').

Thus, let an arbitrary $w \in \mathcal{W}(\Omega)$ be given. We proceed in two steps.

Step 1. For all $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ with $\left|\frac{\#_w(x)}{|x|} - \nu(V_w)\right| \leq \varepsilon$ whenever $|x| = l'_n$ with $n \geq n_0$.

Step 2. For $\varepsilon > 0$, there exists an $N_0 = N_0(\varepsilon)$ with $\left|\frac{\#_w(x)}{|x|} - \nu(V_w)\right| \leq \varepsilon$ whenever $|x| \geq N_0$.

Here, Step 2 follows easily from Step 1 by partitioning long words x into pieces of length l'_n with sufficiently large $n \in \mathbb{N}$.

Thus, we are left with the task of proving Step 1. To do so, assume the contrary. Then, there exist $\delta > 0$, (x_n) in $\mathcal{W}$ and $(l'_{k(n)})$ in $\mathbb{N}$ with $|x_n| = l'_{k(n)}$, $k(n) \longrightarrow \infty$ and

$$\left|\frac{\#_w(x_n)}{|x_n|} - \nu(V_w)\right| \geq \delta \tag{5}$$

for every $n \in \mathbb{N}$. Consider

$$E := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} U_{x_k}.$$

By (B'), we have

$$\nu(E) = \lim_{n\to\infty} \nu(\cup_{k=n}^{\infty} U_{x_k}) \geq C' > 0.$$

Thus, by Proposition 2.4, we can find an ω in E with

$$\lim_{n,m\geq 0, n+m\to\infty} \frac{\#_w(\omega(-m)\ldots\omega(n))}{n+m} = \nu(V_w). \tag{6}$$

As ω belongs to E, there are infinitely many x_n occurring around one in ω. Now, if we calculate the occurrences of w along this sequence of x_n, we stay away from $\nu(V_w)$ by at least δ according to (5). On the other hand, by (6), we come arbitrarily close to $\nu(V_w)$ when calculating the frequency of w along any sequence of words occurring in ω around one. This contradiction proves Step 1 and therefore finishes the proof of the theorem by the discussion above. □

Our next task is to relate (B') and convergence in subadditive ergodic theorems. We need two auxiliary results.

Proposition 2.5. *Let (Ω, T) be a uniquely ergodic subshift and $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ be subadditive. Then,* $\limsup_{|x|\to\infty} |x|^{-1} F(x) \leq \Lambda(F)$.

Proof. Define f_n as in the proof of Proposition 2.2. Then, the statement is a direct consequence of Corollary 2 in [39]. □

Proposition 2.6. *Let (Ω, T) be a uniquely ergodic subshift with invariant probability measure μ. Let $w \in \mathcal{W}(\Omega)$ be arbitrary and denote by χ_{U_w} the characteristic function of U_w. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{U_w}(T^k\omega) = \mu(U_w)$$

uniformly in $\omega \in \Omega$.

Proof. As U_w is both closed and open, the characteristic function χ_{U_w} is continuous. Thus, the statement follows from unique ergodicity. □

Now, our result on subadditive ergodic theorems and (B') reads as follows.

Lemma 2.7. *Let (Ω, T) be a uniquely ergodic and minimal subshift. Let (w_n) be a sequence in $\mathcal{W}(\Omega)$ with $|w_n| \longrightarrow \infty$, $n \to \infty$. Then, the following assertions are equivalent:*

(i) $\lim_{n\to\infty} |w_n|^{-1} F(w_n) = \Lambda(F)$ *for every subadditive $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$.*
(ii) *There exists a $C' > 0$ with $\mu(U_{w_n}) \geq C'$ for every $n \in \mathbb{N}$.*

Proof. The proof can be thought of as an adaptation and extension of the proofs of Lemma 3.1 and Lemma 3.2 in [62] to our setting.

(i) $\Longrightarrow$ (ii). Assume the contrary. Then, the sequence $(\mu(U_{w_n}))$ is not bounded away from zero. By passing to a subsequence, we may then assume without loss of generality that

$$\sum_{n=1}^{\infty} \mu(U_{w_n}) < \frac{1}{2}. \tag{7}$$

As (Ω, T) is minimal, we have $\mu(U_{w_n}) > 0$ for every $n \in \mathbb{N}$. Moreover, by assumption, we have

$$|w_n| \longrightarrow \infty, n \longrightarrow \infty. \tag{8}$$

For $w, x \in \mathcal{W}(\Omega)$, we say that w occurs in x around $j \in \{1, \ldots, |x|\}$ if there exists $l \in \mathbb{N}$ with $l \leq j < l + |w| - 1$ and $x(l) \ldots x(l + |w| - 1) = w$.

Now, define for $n \in \mathbb{N}$, the function $F_n : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ by

$$F_n(x) := \#\{j \in \{1, \ldots, |x|\} : w_n \text{ occurs in } x \text{ around } j\}.$$

Here, $\#M$ denotes the cardinality of M. Thus $F_n(x)$ measures the amount of "space" covered in x by copies of w_n. Obviously, $-F_n$ is subadditive for every $n \in \mathbb{N}$.

The definition of F_n shows

$$F_n(\omega(1)\ldots\omega(m)) = \sum_{k=0}^{m-|w_n|-1} \chi_{U_{w_n}}(T^k\omega)$$

for arbitrary $\omega \in \Omega$ and $m \in \mathbb{N}$. Thus, by Proposition 2.6, we have

$$\lim_{|x|\to\infty} |x|^{-1} F_n(x) = \mu(U_{w_n})$$

for arbitrary but fixed $n \in \mathbb{N}$.

Invoking this equality and (7) and (8), we can choose inductively for every $k \in \mathbb{N}$ a number $n(k) \in \mathbb{N}$ with

$$\frac{|w_{n(k+1)}|}{2} > |w_{n(k)}|$$

and

$$\sum_{j=1}^{k} \frac{F_{n(j)}(x)}{|x|} < \frac{1}{2},$$

whenever $|x| \geq |w_{n(k+1)}|$. It is not hard to see that

$$F(x) := \sum_{j=1}^{\infty} F_{n(2j)}(x)$$

is finite for every $x \in \mathcal{W}(\Omega)$ and $-F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$, $x \mapsto -F(x)$, is subadditive. Therefore, by our assumption (i) the limit

$$-\Lambda(-F) = \lim_{n\to\infty} \frac{F(w_n)}{|w_n|}$$

exists. On the other hand, for every $k \in \mathbb{N}$, we have

$$\frac{F(w_{n(2k)})}{|w_{n(2k)}|} \geq \frac{F_{n(2k)}(w_{n(2k)})}{|w_{n(2k)}|} = 1$$

as well as

$$\frac{F(w_{n(2k+1)})}{|w_{n(2k+1)}|} = \frac{1}{|w_{n(2k+1)}|} \sum_{j=1}^{k} F_{n(2j)}(w_{n(2k+1)}) \leq \frac{1}{|w_{n(2k+1)}|} \sum_{j=1}^{2k} F_{n(j)}(w_{n(2k+1)}) < \frac{1}{2}.$$

This is a contradiction and the proof of this part of the lemma is finished.

(ii) $\Longrightarrow$ (i). Let $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ be subadditive. By Proposition 2.5, we have

$$\limsup_{|x|\to\infty} \frac{F(x)}{|x|} \leq \Lambda(F). \tag{9}$$

Thus, it remains to show

$$\Lambda(F) \leq \liminf_{n\to\infty} \frac{F(w_n)}{|w_n|}.$$

Assume the contrary. Then, $\Lambda(F) > -\infty$ and there exists a subsequence $(w_{n(k)})$ of (w_n) and $\delta > 0$ with

$$\frac{F(w_{n(k)})}{|w_{n(k)}|} \leq \Lambda(F) - \delta \tag{10}$$

for every $k \in \mathbb{N}$. For $w, x \in \mathcal{W}(\Omega)$, we define $\#^*_w(x)$ to be the maximal number of disjoint copies of w in x.

It is not hard to see that

$$|w| \cdot \#_w^*(\omega(1)\dots\omega(m)) \geq \frac{1}{2} \sum_{k=0}^{m-|w|-1} \chi_{U_w}(T^k\omega)$$

for all $\omega \in \Omega$ and $m \in \mathbb{N}$. By Proposition 2.6, this implies

$$\liminf_{|x|\to\infty} \frac{\#_w^*(x)}{|x|}|w| \geq \frac{1}{2}\mu(U_w).$$

Combining this with our assumption (ii), we infer

$$\liminf_{|x|\to\infty} \frac{\#_{w_{n(k)}}^*(x)}{|x|}|w_{n(k)}| \geq \frac{C'}{2} \tag{11}$$

for every $k \in \mathbb{N}$. By (9), we can choose L_0 such that

$$\frac{F(x)}{|x|} \leq \Lambda(F) + \frac{C'}{16}\delta, \tag{12}$$

whenever $|x| \geq L_0$. Fix $k \in \mathbb{N}$ with $|w_{n(k)}| \geq L_0$. Using (11), we can now find an $L_1 \in \mathbb{R}$ such that every $x \in \mathcal{W}(\Omega)$ with $|x| \geq L_1$ can be written as $x = x_1 w_{n(k)} x_2 w_{n(k)} \dots x_l w_{n(k)} x_{l+1}$ with

$$\frac{l-2}{2} \geq \frac{C'}{8}\frac{|x|}{|w_{n(k)}|}. \tag{13}$$

Now, considering only every other copy of $w_{n(k)}$ in x, we can write x as $x = y_1 w_{n(k)} y_2 \dots y_r w_{n(k)} y_{r+1}$, with $|y_j| \geq |w_{n(k)}| \geq L_0$, $j = 1, \dots, r+1$, and by (13)

$$r \geq \frac{l-2}{2} \geq \frac{C'}{8}\frac{|x|}{|w_{n(k)}|}.$$

Using (12), (10) and this estimate, we can now calculate

$$\begin{aligned}
\frac{F(x)}{|x|} &\leq \sum_{j=1}^{r+1} \frac{F(y_j)}{|y_j|}\frac{|y_j|}{|x|} + \frac{F(w_{n(k)})}{|w_{n(k)}|}\frac{r|w_{n(k)}|}{|x|} \\
&\leq \sum_{j=1}^{r+1} (\Lambda(F) + \frac{C'}{16}\delta)\frac{|y_j|}{|x|} + (\Lambda(F) - \delta)\frac{r|w_{n(k)}|}{|x|} \\
&\leq \Lambda(F) + \frac{C'}{16}\delta - \frac{C'}{8}\frac{|x|}{|w_{n(k)}|}\frac{|w_{n(k)}|}{|x|}\delta \\
&\leq \Lambda(F) - \frac{C'}{16}\delta.
\end{aligned}$$

As this holds for arbitrary $x \in \mathcal{W}(\Omega)$ with $|x| \geq L_1$, we arrive at the obvious contradiction $\Lambda(F) \leq \Lambda(F) - \frac{C'}{16}\delta$. This finishes the proof. □

Proof of Theorem 5. Given the previous results, the proof is simple: The equivalence of (i) and (ii) is shown in Lemma 2.3. The implication (ii) $\Longrightarrow$ (iii) follows from Theorem 6 combined with Lemma 2.7. The implication (iii) $\Longrightarrow$ (ii) is immediate from Lemma 2.7. This finishes the proof of Theorem 5. □

3. Uniformity of Locally Constant Cocycles

In this section we provide a proof of our main result, Theorem 1. As mentioned already, the cornerstones of the proof are Theorem 5 and the so-called Avalanche Principle, introduced in [41] and later extended in [15].

We use the Avalanche Principle in the following form given in Lemma 5 of [15].

Lemma 3.1. *There exist constants $\lambda_0 > 0$ and $\kappa > 0$ such that*

$$\left| \log \|A_N \dots A_1\| + \sum_{j=2}^{N-1} \log \|A_j\| - \sum_{j=1}^{N-1} \log \|A_{j+1} A_j\| \right| \leq \frac{\kappa \cdot N}{\exp(\lambda)},$$

whenever $N = 3^P$ with $P \in \mathbb{N}$ and $A_1, \dots, A_N$ are elements of SL(2,$\mathbb{R}$) *such that*

- $\log \|A_j\| \geq \lambda \geq \lambda_0$ *for every $j = 1, \dots, N$;*
- $|\log \|A_j\| + \log \|A_{j+1}\| - \log \|A_j A_{j+1}\|| < \frac{1}{2}\lambda$ *for every $j = 1, \dots, N$.*

Remark 3. Actually, Lemma 5 in [15] is more general in that more general N are allowed.

Before we can give the proof of Theorem 1, we need one more auxiliary result.

Proposition 3.2. *Let (Ω, T) be an arbitrary subshift and $A : \Omega \longrightarrow$ SL(2,$\mathbb{R}$) a locally constant function. Then,*

$$0 = \lim_{n\to\infty} \sup \left\{ \frac{1}{n} |\log \|A(n,\omega)\| - \log \|A(n,\rho)\|| : \omega(1)\dots\omega(n) = \rho(1)\dots\rho(n) \right\}.$$

Proof. As A is locally constant, there exists an $N \in \mathbb{N}$ such that $A(\omega) = A(\rho)$, whenever $\omega(-N)\dots\omega(N) = \rho(-N)\dots\ \rho(N)$. Thus,

$$A(n - 2N, T^N\omega) = A(n - 2N, T^N\rho),$$

whenever $n \geq 2N$ and $\omega(1)\dots\omega(n) = \rho(1)\dots\rho(n)$. Moreover, for arbitrary matrices X, Y, Z in SL(2,$\mathbb{R}$), we have

$$\log \|Y\| - \log \|X\| - \log \|Z\| \leq \log \|XYZ\| \leq \log \|X\| + \log \|Y\| + \log \|Z\|,$$

where we used the triangle inequality as well as $\|M\| = \|M^{-1}\|$ for $M \in$ SL(2,$\mathbb{R}$). Finally, we have

$$A(n, \sigma) = A(N, T^{n-N}\sigma)A(n - 2N, T^N\sigma)A(N, \sigma).$$

Putting these three equations together, we arrive at the desired conclusion. □

Remark 4. Let us point out that the previous proposition is the only point in our considerations where local constancy of A enters. In particular, our main result holds for all A for which the conclusion of the proposition holds.

Proof of Theorem 1. Let (Ω, T) be a subshift satisfying (B) and let $A : \Omega \longrightarrow$ SL(2,$\mathbb{R}$) be locally constant. We have to show that A is uniform.

Case 1. $\Lambda(A) = 0$: As A takes values in SL(2,$\mathbb{R}$), we have $\|A(n,\omega)\| \geq 1$ and the estimate

$$0 \leq \liminf_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$$

holds uniformly in $\omega \in \Omega$. On the other hand, by Corollary 2 of [39], we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\| \leq \Lambda(A)$$

uniformly in $\omega \in \Omega$. This shows the desired uniformity in this case.

Case 2. $\Lambda(A) > 0$: Define $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ by

$$F(x) := \sup\{\log \|A(n,\omega)\| : \omega(1)\dots\omega(n) = x\}.$$

Apparently, F is subadditive. As discussed above, there exists then $\Lambda(F)$ with

$$\Lambda(F) = \lim_{n\to\infty} \frac{F(\omega(1)\dots\omega(n))}{n}$$

for μ-almost every $\omega \in \Omega$. On the other hand, by the multiplicative ergodic theorem, there also exists $\Lambda(A)$ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{\log \|A(n,\omega)\|}{n}$$

for μ-almost every $\omega \in \Omega$. By Proposition 3.2, we infer that $\Lambda(A) = \Lambda(F)$. Summarizing, we have

$$\Lambda(A) = \Lambda(F) > 0. \tag{14}$$

Combining this equation with Theorem 5, we infer

$$\lim_{n\to\infty} \frac{F(w_n)}{|w_n|} = \Lambda(A),$$

whenever (w_n) is a sequence with $|w_n| = l'_n$. Also, combining (14) with Proposition 2.5, we infer

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\| \le \limsup_{|x|\to\infty} \frac{F(x)}{|x|} \le \Lambda(A)$$

uniformly in $\omega \in \Omega$. It remains to show

$$\Lambda(A) \le \liminf_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$$

uniformly in $\omega \in \Omega$. To do so, let $\varepsilon > 0$ with $\varepsilon \le 1/12$ be given.

The preceding considerations and Proposition 3.2 give existence of $n_0 \in \mathbb{N}$ such that with

$$l := \frac{l'_{n_0}}{2},$$

the following holds:

(I) $\log \|A(n,\omega)\| \le \Lambda(A)(1+\varepsilon)n$ for all $\omega \in \Omega$ whenever $n \ge l$.

(II) $\log \|A(2l,\omega)\| \ge \Lambda(A)(1-\varepsilon)2l$ for all $\omega \in \Omega$.

(III) $\Lambda(A)(1-3\varepsilon)l \ge \lambda_0$.

(IV) $\frac{2\kappa}{l\exp(\lambda_0)} < \varepsilon\Lambda(A)$.

Here, λ_0 and κ are the constants from Lemma 3.1. Using (II), subadditivity and (I), we can calculate

$$\begin{aligned} \Lambda(A)(1-\varepsilon)2l &\le \log \|A(2l,\omega)\| \\ &\le \log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| \\ &\le \log \|A(l,\omega)\| + \Lambda(A)(1+\varepsilon)l. \end{aligned}$$

This implies $\Lambda(A)(1-3\varepsilon)l \le \log \|A(l,\omega)\|$ and therefore by (III),

$$\lambda_0 \le \Lambda(A)(1-3\varepsilon)l \le \log \|A(l,\omega)\| \tag{15}$$

for every $\omega \in \Omega$. Moreover, by subadditivity, (I) and (II), we have

$$\begin{aligned} & \left|\log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| - \log \|A(2l,\omega)\|\right| \\ = \;& \log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| - \log \|A(2l,\omega)\| \\ \leq \;& \Lambda(A)2l(1+\varepsilon) - \log \|A(2l,\omega)\| \\ \leq \;& \Lambda(A)2l(1+\varepsilon) - \Lambda(A)2l(1-\varepsilon) \\ = \;& \Lambda(A)4l\varepsilon \end{aligned}$$

for arbitrary $\omega \in \Omega$. Using the assumption $\varepsilon \leq 1/12$, we infer

$$\left|\log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| - \log \|A(l,\omega)\|\right| \leq \frac{1}{2}\Lambda(A)(1-3\varepsilon)l. \tag{16}$$

Equations (15) and (16) and (III) show that the Avalanche Principle, Lemma 3.1, with

$$\lambda = \Lambda(A)(1-3\varepsilon)l$$

can be applied to the matrices $A_1, \ldots, A_N$, where $N = 3^P$ with $P \in \mathbb{N}$ arbitrary and

$$A_j = A(l, T^{(j-1)l}\omega), \;\; j = 1, \ldots, N$$

with $\omega \in \Omega$ arbitrary. This gives

$$\left|\log \|A_N \ldots A_1\| + \sum_{j=2}^{N-1} \log \|A_j\| - \sum_{j=1}^{N-1} \log \|A_{j+1}A_j\|\right| \leq \frac{\kappa N}{\exp(\lambda)}.$$

This yields

$$\begin{aligned} \log \|A_N \ldots A_1\| \;\geq\;& \sum_{j=1}^{N-1} \log \|A_{j+1}A_j\| - \sum_{j=2}^{N-1} \log \|A_j\| - \frac{\kappa \cdot N}{\exp(\lambda)} \\ \geq\;& (N-1)\Lambda(A)(1-\varepsilon)2l - (N-2)\Lambda(A)(1+\varepsilon)l - \frac{\kappa \cdot N}{\exp(\lambda)} \\ =\;& \Lambda(A)Nl(1-3\varepsilon) + \Lambda(A)4\varepsilon l - \frac{\kappa \cdot N}{\exp(\lambda)} \\ \geq\;& \Lambda(A)Nl(1-3\varepsilon) - \frac{\kappa \cdot N}{\exp(\lambda)}. \end{aligned}$$

Here, we used (I) and (II) in the second step and positivity of $\Lambda(A)4\varepsilon l$ in the last step. Dividing by by $n := Nl$, and invoking (IV), we obtain

$$\Lambda(A)(1-4\varepsilon) \leq \frac{1}{n}\log \|A(n,\omega)\| \tag{17}$$

for all $\omega \in \Omega$ and all $n = 3^P \cdot l$ with $P \in \mathbb{N}$.

We finish the proof by showing that

$$\Lambda(A)(1-44\varepsilon) \leq \frac{1}{n}\log \|A(n,\omega)\| \tag{18}$$

for all $n \geq l$ and all $\omega \in \Omega$. As ε was arbitrary, this gives the desired statement.

To show (18), choose $\omega \in \Omega$ and and $n \geq l$. Let $P \in \mathbb{N} \cup \{0\}$ be such that

$$3^P \cdot l \leq n < 3^{P+1} \cdot l.$$

Then, by (17) and subadditivity we have

$$\begin{aligned}\Lambda(A)(1-4\varepsilon) &\leq \frac{1}{3^{P+2}l}\log\|A(3^{P+2}l,\omega)\| \\ &\leq \frac{1}{3^{P+2}l}\log\|A(n,\omega)\| + \frac{1}{3^{P+2}l}\ \log\|A(3^{P+2}l-n,T^n\omega)\| \\ &\leq \frac{1}{n}\log\|A(n,\omega)\|\cdot\frac{n}{3^{P+2}l} + \Lambda(A)(1+\varepsilon)(1-\frac{n}{3^{P+2}l}),\end{aligned}$$

where we could use (I) in the last estimate as, by assumption on n, $3^{P+2}l - n \geq 3^{P+1}2l > l$. Now, a direct calculation gives

$$\Lambda(A)\left(1+\varepsilon-5\varepsilon\frac{3^{P+2}l}{n}\right) \leq \frac{1}{n}\log\|A(n,\omega)\|.$$

As $3^{P+2}l/n \leq 9$ by the very choice of P, the desired equation (18) follows easily. This finishes the proof of our main theorem. $\square$

4. Stability of Uniform Convergence Under Substitutions

In the last section, we studied sufficient conditions on (Ω, T) to ensure property

$$\text{(P)}:\ \ \text{Every locally constant } A:\Omega\longrightarrow \mathrm{SL}(2,\mathbb{R}) \text{ is uniform.}$$

In this section, we consider "perturbations" $(\Omega(S), T)$ of (Ω, T) by substitutions S and study how validity of (P) for (Ω, T) is related to validity of (P) for $(\Omega(S), T)$.

We start with the necessary notation. Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. A map $S : \mathcal{A} \longrightarrow \mathcal{B}^*$ is called a substitution. Obviously, S can be extended to $\mathcal{A}^*$ in the obvious way. Moreover, for a two-sided infinite word $(\omega(n))_{n\in\mathbb{Z}}$ over $\mathcal{A}$, we can define $S(\omega)$ by

$$S(\omega) := \cdots S(\omega(-2))S(\omega(-1))|S(\omega(0))S(\omega(1))S(\omega(2))\cdots,$$

where $|$ denotes the position of zero. If (Ω, T) is a subshift over A and $S : \mathcal{A} \longrightarrow \mathcal{B}^*$ is a substitution, we define $\Omega(S)$ by

$$\Omega(S) := \{T^k S(\omega) : \omega\in\Omega, k\in\mathbb{Z}\}.$$

Then, $(\Omega(S), T)$ is a subshift over $\mathcal{B}$. It is not hard to see that $(\Omega(S), T)$ is minimal (uniquely ergodic) if Ω is minimal (uniquely ergodic).

Theorem 7. *Let (Ω, T) be a minimal uniquely ergodic subshift over $\mathcal{A}$ that satisfies* (P)*. Let S be a substitution over $\mathcal{A}$. Then, $(\Omega(S), T)$ satisfies* (P) *as well.*

Proof. Let $B : \Omega(S) \longrightarrow \mathrm{SL}(2,\mathbb{R})$ be locally constant. Define

$$A:\Omega\longrightarrow \mathrm{SL}(2,\mathbb{R}) \ \text{ by } \ A(\omega) := B(|S(\omega(0))|, S(\omega)).$$

Then, A is locally constant as well and

$$A(n,\omega) = B(|S(\omega(0)\dots\omega(n-1))|, S(\omega)).$$

In particular, we have

$$\frac{\log\|B(|S(\omega(0)\dots\omega(n))|,S(\omega))\|}{|S(\omega(0)\dots\omega(n))|} = \frac{n+1}{|S(\omega(0)\dots\omega(n))|}\cdot\frac{\log\|A(n,\omega)\|}{n+1}. \tag{19}$$

By

$$|S(\omega(0)\dots\omega(n))| = \sum_{a\in\mathcal{A}} |S(a)|\#_a(\omega(0)\dots\omega(n))$$

and unique ergodicity of (Ω, T), the quotients

$$\frac{n+1}{|S(\omega(0)\dots\omega(n))|}$$

converge uniformly in $\omega \in \Omega$ towards a number ρ. From (19) and validity of (P) for (Ω, T) we infer that

$$\lim_{n\to\infty} \frac{\log \|B(|S(\omega(0)\dots\omega(n))|, S(\omega))\|}{|S(\omega(0)\dots\omega(n))|} = \rho \cdot \Lambda(A)$$

uniformly on Ω. As every $\sigma \in \Omega(S)$ has the form $\sigma = T^k S(\omega)$ with $|k| \leq \max\{|S(a)| : a \in \mathcal{A}\}$, uniform convergence of $\frac{1}{n}\log\|B(n,\sigma)\|$ follows. □

In certain cases, a converse of this theorem holds. To be more precise, let (Ω, T) be a subshift over $\mathcal{A}$ and S a substitution on $\mathcal{A}$. Then, S is called *recognizable* (with respect to (Ω, T)) if there exists a locally constant map

$$\widetilde{S} : \Omega(S) \longrightarrow \Omega \times \mathbb{Z}$$

with $\widetilde{S}(T^k S(\omega)) = (\omega, k)$, whenever $0 \leq k \leq |S(\omega(0))|$. Recognizability is known for various classes of substitutions that generate aperiodic subshifts, including all primitive substitutions [72] and all substitutions of constant length that are one-to-one [3] (cf. the discussion in [38]).

Theorem 8. *Let (Ω, T) be a uniquely ergodic minimal subshift over $\mathcal{A}$. Let S be a recognizable substitution over $\mathcal{A}$. If $(\Omega(S), T)$ satisfies* (P)*, then (Ω, T) satisfies* (P) *as well.*

Proof. Let $B : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ be locally constant. For $\sigma \in \Omega(S)$ define

$$A(\sigma) \equiv \begin{cases} B(\omega) & : \ \sigma = S(\omega) \\ id & : \ \text{otherwise.} \end{cases}$$

Note that $\sigma = S(\omega)$ if and only if the second component of $\widetilde{S}(\sigma)$ is 0. As $\widetilde{S}$ is locally constant, this shows that A is locally constant as well.

Moreover, by definition of A and recognizability of S, we have

$$A(|S(\omega(0)\dots\omega(n-1))|, S(\omega)) = B(n, \omega).$$

Now, the proof can be finished similarly to the proof of the previous theorem. □

There is an instance of the previous theorem that deserves special attention, viz subshifts derived by return words. Return words and the derived subshifts have been discussed by various authors since they were first introduced by Durand in [32]. We recall the necessary details next.

Let (Ω, T) be a minimal subshift and $w \in \mathcal{W}(\Omega)$ arbitrary. Then, $x \in \mathcal{W}(\Omega)$ is called a return word of w if xw satisfies the following three properties: it belongs to $\mathcal{W}(\Omega)$, it starts with w and it contains exactly two copies of w. We then introduce a new alphabet $\mathcal{A}_w$ consisting of the return words of w. Obviously, there is a natural map

$$S_w : \mathcal{A}_w \longrightarrow \mathcal{A}^*$$

which maps the return word x of w (which is a letter of $\mathcal{A}_w$) to the word x over $\mathcal{A}$. Partitioning every word $\omega \in \Omega$ according to occurrences of w, we obtain a unique two-sided infinite word ω_w over $\mathcal{A}_w$ with

$$T^{-k} S_w(\omega_w) = \omega$$

for $k \leq 0$ maximal with $\omega(k) \dots \omega(k + |w| - 1) = w$. We define

$$\Omega_w := \{\omega_w : \omega \in \Omega\}.$$

Then, (Ω_w, T) is a subshift, called the subshift derived from (Ω, T) with respect to w. It is not hard to see that (Ω_w, T) is minimal. Moreover, (Ω_w, T) is uniquely ergodic if (Ω, T) is uniquely ergodic. Clearly, S_w is recognizable and $(\Omega, T) = (\Omega_w(S_w), T)$ since the whole construction only depends on the (local) information of occurrences of w. Thus, we obtain the following corollary from the previous theorem.

Corollary 1. *Let (Ω, T) be a minimal uniquely ergodic subshift that satisfies* (P). *Let $w \in \mathcal{W}(\Omega)$ be arbitrary. Then, (Ω_w, T) satisfies* (P) *as well.*

The aim of this paper is to study (P). Given that (B) is a sufficient condition for (P), it is then natural to ask for stability properties of (B) as well. It turns out that (B) shares the stability features of (P).

Theorem 9. *Let (Ω, T) be a minimal uniquely ergodic subshift over $\mathcal{A}$. Let S be a substitution on $\mathcal{A}$ and $(\Omega(S), T)$ the corresponding subshift.*

(a) *If (Ω, T) satisfies* (B)*, so does $(\Omega(S), T)$.*

(b) *If $(\Omega(S), T)$ satisfies* (B) *and S is recognizable, then (Ω, T) satisfies* (B) *as well.*

Before we can give a proof, we note the following simple observation.

Proposition 4.1. *Let (Ω, T) be a minimal uniquely ergodic subshift satisfying* (B) *with length scales (l_n) and constant $C > 0$. Then,*

$$|w|\mu(V_w) \geq \frac{C}{N},$$

whenever $w \in \mathcal{W}(\Omega)$ satisfies $l_n/N \leq |w| \leq l_n$ for some $n \in \mathbb{N}$ and $N \in N$.

Proof. Every $w \in \mathcal{W}(\Omega)$ with $l_n/N \leq |w| \leq l_n$ is a prefix of a $v \in \mathcal{W}$ with $|v| = l_n$. Then, $V_v \subset V_w$ holds and (B) implies

$$|w|\mu(V_w) \geq \frac{|v|}{N}\mu(V_w) \geq \frac{|v|}{N}\mu(V_v) \geq \frac{C}{N}.$$

This finishes the proof of the proposition. $\square$

Proof of Theorem 9. Define $M := \{|S(a)| : a \in \mathcal{A}\}$ and denote the unique T-invariant probability measure on Ω (resp., $\Omega(S)$) by μ (resp., μ_S).

(a) We assume that (Ω, T) satisfies (B) with length scales (l_n) and constant $C > 0$. Let $w \in \mathcal{W}(\Omega(S))$ with $|w| = l_n$ for some $n \in \mathbb{N}$ be given. Then, there exists a word $v \in \mathcal{W}(\Omega)$ such that w is a subword of $S(v)$ and satisfies the estimate

$$\frac{|w|}{M} \leq |v| \leq |w|. \tag{20}$$

Choose $\omega \in \Omega$ arbitrary. Obviously,

$$\#_w(S(\omega(1) \dots \omega(k))) \geq \#_v(\omega(1) \dots \omega(k)).$$

Thus, counting occurrences of $w \in S(\omega)$ and occurrences of v in ω, we obtain by unique ergodicity

$$\begin{aligned} |w|\mu_S(V_w) &= |w| \lim_{n\to\infty} \frac{\#_w(S(\omega)(1) \dots S(\omega)(n))}{n} \geq |w| \lim_{k\to\infty} \frac{\#_v(\omega(1) \dots \omega(k))}{kM} \\ &= \frac{|w|}{M}\mu(V_v) \geq \frac{1}{M}|v|\mu(V_v) \geq \frac{1}{M^2}C, \end{aligned}$$

where we used (20) in the second-to-last step and Proposition 4.1 combined with (20) in the last step. This shows (B) for $(\Omega(S), T)$ along the same length scales (l_n) with new constant C/M^2.

(b) We assume that $(\Omega(S), T)$ satisfies (B) with constant $C > 0$ and length scales (l_n). By recognizability, there exists a map $\widetilde{S} : \Omega(S) \longrightarrow \Omega \times \mathbb{Z}$ and an $N \in \mathbb{N}$ with $\widetilde{S}(T^k S(\omega)) = (\omega, k)$, whenever $0 \leq k \leq |S(\omega(0))|$, and $\widetilde{S}(\omega) = \widetilde{S}(\rho)$, whenever $\omega(-N) \ldots \omega(N) = \rho(-N) \ldots \rho(N)$. Let n_0 be chosen such that

$$\left[\frac{l_n}{3M}\right] \geq N,$$

for all $n \geq n_0$.

Choose an arbitrary $v \in \mathcal{W}(\Omega)$ with $|v| = \left[\frac{l_n}{3M}\right]$ for some $n \geq n_0$.

Let $x, y \in \mathcal{W}(\Omega)$ be given with $|x| = |y| = |v|$ and $xvy \in \mathcal{W}(\Omega)$. By recognizability and our choice of the lengths of x, y and v, occurrences of $S(xvy)$ in $S(\omega)$ correspond to occurrences of v in ω for any $\omega \in \Omega$. Thus, we obtain

$$\#_v(\omega(1) \ldots \omega(n)) \geq \#_{S(xvy)}(S(\omega(1) \ldots \omega(n)))$$

for every $n \in \mathbb{N}$ and $\omega \in \Omega$. Therefore, a short calculation invoking unique ergodicity gives

$$\begin{aligned}
|v|\mu(V_v) &= |v| \lim_{n\to\infty} \frac{\#_v(\omega(1) \ldots \omega(n))}{n} \geq |v| \lim_{n\to\infty} \frac{\#_{S(xvy)}(S(\omega(1) \cdots \omega(n)))}{n} \\
&\geq \frac{|v|}{|S(xvy)|} \lim_{n\to\infty} \frac{|S(\omega(1) \cdots \omega(n))|}{n} |S(xvy)| \frac{\#_{S(xvy)}(S(\omega(1) \cdots \omega(n)))}{|S(\omega(1) \cdots \omega(n))|} \\
&\geq \frac{1}{3M} |S(xvy)| \mu_S(V_{S(xvy)}),
\end{aligned}$$

where we used the trivial bound $|S(x)|/|x| \geq 1$ in the second-to-last step. By construction, we have

$$\frac{l_n}{2M} \leq |xvy| \leq |S(xvy)| \leq l_n.$$

Thus, we can apply Proposition 4.1, and the assumption (B) on $\Omega(S)$, to our estimate on $|v|\mu(V_v)$ to obtain $|v|\mu(V_v) \geq \frac{C}{6M^2}$. As $v \in \mathcal{W}$ with $|v| = \left[\frac{l_n}{3M}\right]$ was arbitrary, we infer (B) with the new length scales $[l_n/3]$ for $n \geq n_0$ and new constant $C/(6M^2)$. □

5. Examples Known to Satisfy (B)

In this section we discuss the classes of subshifts for which the Boshernitzan condition is either known or a simple consequence of known results. In our discussion of the occurrence of zero-measure Cantor spectrum for Schrödinger operators in Section 8, this will be relevant since all the models for which this spectral property was previously known will be shown to satisfy (B). Hence we present a unified approach to all these results.

5.1. **Examples Satisfying (PW): Linearly Recurrent Subshifts and Subshifts Generated by Primitive Substitutions.** A subshift (Ω, T) over $\mathcal{A}$ satisfies the condition (PW) (for *positive weights*) if there exists a constant $C > 0$ such that

$$\liminf_{|x|\to\infty} \frac{\#_v(x)}{|x|} |v| \geq C \text{ for every } v \in \mathcal{W}(\Omega).$$

This condition was introduced by Lenz in [62]. There, it was shown that the class of subshifts satisfying (PW) is exactly the class of subshifts for which a uniform subadditive ergodic theorem holds. Moreover, (PW) implies minimality and unique ergodicity.

The following is obvious:

Proposition 5.1. *If the subshift* (Ω, T) *satisfies* (PW)*, then it satisfies* (B)*.*

The condition (PW) holds in many cases of interest. For example, it is easily seen to be satisfied for all linearly recurrent subshifts. Here, a subshift (Ω, T) is called *linearly recurrent* (or *linearly repetitive*) if there exists a constant K such that if $v, w \in \mathcal{W}(\Omega)$ with $|w| \geq K|v|$, then v is a subword of w.

We note:

Proposition 5.2. *If the subshift* (Ω, T) *is linearly recurrent, then it satisfies* (PW)*.*

The class of linearly recurrent subshifts was studied, for example, in [33, 34].

A popular way to generate linearly recurrent subshifts is via primitive substitutions. A substitution $S : \mathcal{A} \to \mathcal{A}^*$ is called *primitive* if there exists $k \in \mathbb{N}$ such that for every $a, b \in \mathcal{A}$, $S^k(a)$ contains b. Such a substitution generates a subshift (Ω, T) as follows. It is easy to see that there are $m \in \mathbb{N}$ and $a \in \mathcal{A}$ such that $S^m(a)$ begins with a. If we iterate S^m on the symbol a, we obtain a one-sided infinite limit, u, called a *substitution sequence.* Ω then consists of all two-sided sequences for which all subwords are also subwords of u. One can verify that this construction is in fact independent of the choice of u, and hence Ω is uniquely determined by S. Prominent examples are given by

$a \mapsto ab,\ b \mapsto a$	Fibonacci
$a \mapsto ab,\ b \mapsto ba$	Thue-Morse
$a \mapsto ab,\ b \mapsto aa$	Period Doubling
$a \mapsto ab,\ b \mapsto ac,\ c \mapsto db,\ d \mapsto dc$	Rudin-Shapiro

The following was shown in [34]:

Proposition 5.3. *If the subshift* (Ω, T) *is generated by a primitive substitution, then it is linearly recurrent.*

It may happen that a non-primitive substitution generates a linearly recurrent subshift. An example is given by $a \mapsto aaba$, $b \mapsto b$. In fact, the class of linearly recurrent subshifts generated by substitutions was characterized in [25]. In particular, it turns out that a subshift generated by a substitution is linearly recurrent if and only if it is minimal.

5.2. **Sturmian and Quasi-Sturmian Subshifts.** Consider a minimal subshift (Ω, T) over $\mathcal{A}$. Recall that the associated set of words is given by

$$\mathcal{W}(\Omega) := \{\omega(k) \cdots \omega(k+n-1) : k \in \mathbb{Z}, n \in \mathbb{N}, \omega \in \Omega\}.$$

The (factor) complexity function $p : \mathbb{N} \to \mathbb{N}$ is then defined by

$$p(n) = \#\mathcal{W}_n(\Omega), \tag{21}$$

where $\mathcal{W}_n(\Omega) = \mathcal{W}(\Omega) \cap \mathcal{A}^n$ and $\#$ denotes cardinality.

It is a fundamental result of Hedlund and Morse that periodicity can be characterized in terms of the complexity function [44]:

Theorem 10 (Hedlund-Morse)**.** *(Ω, T) is aperiodic if and only if $p(n) \geq n+1$ for every $n \in \mathbb{N}$.*

Aperiodic subshifts of minimal complexity, $p(n) = n+1$ for every $n \in \mathbb{N}$, exist and they are called Sturmian. If the complexity function satisfies $p(n) = n+k$ for $n \geq n_0$, $k, n_0 \in \mathbb{N}$, the subshift is called quasi-Sturmian. It is known that quasi-Sturmian subshifts are exactly those subshifts that are a morphic image of a Sturmian subshift; compare [18, 19, 74].

There are a large number of equivalent characterizations of Sturmian subshifts; compare [9]. We are mainly interested in their geometric description in terms of an irrational rotation. Let $\alpha \in (0,1)$ be irrational and consider the rotation by α on the circle,

$$R_\alpha : [0,1) \to [0,1), \;\; R_\alpha \theta = \{\theta + \alpha\},$$

where $\{x\}$ denotes the fractional part of x, $\{x\} = x \mod 1$. The coding of the rotation R_α according to a partition of the circle into two half-open intervals of length α and $1-\alpha$, respectively, is given by the sequences

$$v_n(\alpha, \theta) = \chi_{[0,\alpha)}(R_\alpha^n \theta).$$

We obtain a subshift

$$\begin{aligned} \Omega_\alpha &= \overline{\{v(\alpha,\theta) : \theta \in [0,1)\}} \\ &= \{v(\alpha,\theta) : \theta \in [0,1)\} \cup \{\tilde{v}^{(k)}(\alpha) : k \in \mathbb{Z}\} \subset \{0,1\}^{\mathbb{Z}} \end{aligned}$$

which can be shown to be Sturmian. Here, $\tilde{v}_n^{(k)}(\alpha) = \chi_{(0,\alpha]}(R_\alpha^{n+k} 0)$. Conversely, every Sturmian subshift is essentially of this form, that is, if Ω is minimal and has complexity function $p(n) = n+1$, then up to a one-to-one morphism, $\Omega = \Omega_\alpha$ for some irrational $\alpha \in (0,1)$.

By uniform distribution, the frequencies of factors of Ω are given by the Lebesgue measure of certain subsets of the torus. Explicitly, if we write $I_0 = [0,\alpha)$ and $I_1 = [\alpha, 1)$, then the word $w = w_1 \ldots w_n \in \{0,1\}^n$ occurs in $v(\alpha,\theta)$ at site $k+1$ if and only if

$$\{k\alpha + \theta\} \in I(w_1, \ldots, w_n) := \bigcap_{j=1}^{n} R_\alpha^{-j}(I_{w_j}).$$

This shows that the frequency of w is θ-independent and equal to the Lebesgue measure of $I(w_1, \ldots, w_n)$. It is not hard to see that $I(w_1, \ldots, w_n)$ is an interval whose boundary points are elements of the set

$$P_n(\alpha) := \{\{-j\alpha\} : 0 \leq j \leq n\}.$$

The $n+1$ points of $P_n(\alpha)$ partition the torus into $n+1$ subintervals and hence the length $h_n(\alpha)$ of the smallest of these intervals bounds the frequency of a factor of length n from below. It is therefore of interest to study $\limsup n h_n(\alpha)$.

To this end we recall the notion of a continued fraction expansion; compare [56, 79]. For every irrational $\alpha \in (0,1)$, there are uniquely determined $a_k \in \mathbb{N}$ such that

$$\alpha = [a_1, a_2, a_3, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}. \tag{22}$$

The associated rational approximants $\frac{p_k}{q_k}$ are defined by

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_k = a_k p_{k-1} + p_{k-2}, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2}. \end{aligned}$$

These rational numbers are best approximants to α in the following sense,

$$\min_{\substack{p,q\in\mathbb{N} \\ 0<q<q_{k+1}}} |q\alpha - p| = |q_k\alpha - p_k|\,, \tag{23}$$

and the quality of the approximation can be estimated according to

$$\frac{1}{q_k + q_{k+1}} < |q_k\alpha - p_k| < \frac{1}{q_{k+1}}. \tag{24}$$

By definition, we have

$$h_n(\alpha) = \min_{0<|q|\le n} \{q\alpha\}.$$

Notice that for $0 < q \le n$, we have $\min\{\{q\alpha\}, \{-q\alpha\}\} = \|q\alpha\|$, where we denote $\|x\| = \min_{p\in\mathbb{Z}} |x - p|$.

In particular,

$$h_n(\alpha) = \min_{0<q\le n} \|q\alpha\|. \tag{25}$$

As noted by Hartman [43], this shows that $h_n(\alpha)$ can be expressed in terms of the continued fraction approximants. Indeed, if we combine (23) and (25), we obtain:

Lemma 5.4 (Hartman)**.** *If $q_k \le n < q_{k+1}$, then*

$$h_n(\alpha) = |q_k\alpha - p_k|.$$

This allows us to show the following:

Theorem 11. *Every Sturmian subshift obeys the Boshernitzan condition* (B)*.*

Proof. We only need to show that

$$\limsup_{n\to\infty} n h_n(\alpha) \ge C > 0. \tag{26}$$

We shall verify this on the subsequence $n_k = q_{k+1} - 1$. Hartman's lemma together with (24) shows that

$$n_k h_{n_k}(\alpha) = (q_{k+1} - 1)|q_k\alpha - p_k| \ge \frac{q_{k+1} - 1}{q_k + q_{k+1}} = \frac{1 - q_{k+1}^{-1}}{1 + q_k q_{k+1}^{-1}}.$$

Thus (26) holds (with $C = 1/3$, say). □

Corollary 2. *Every quasi-Sturmian subshift obeys* (B)*.*

Proof. This follows from Theorem 11 along with the stability result, Theorem 9. □

5.3. **Interval Exchange Transformations.** Subshifts generated by interval exchange transformations (IET's) are natural generalizations of Sturmian subshifts. They were studied, for example, in [13, 35, 36, 37, 53, 54, 55, 69, 77, 87, 88, 89].

IET's are defined as follows. Given a probability vector $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_i > 0$ for $1 \le i \le m$, we let $\mu_0 = 0$, $\mu_i = \sum_{j=1}^{i} \lambda_j$, and $I_i = [\mu_{i-1}, \mu_i)$. Let τ be a permutation of $\mathcal{A}_m = \{1, \dots, m\}$, that is, $\tau \in S_m$, the symmetric group. Then $\lambda^\tau = (\lambda_{\tau^{-1}(1)}, \dots, \lambda_{\tau^{-1}(m)})$ is also a probability vector and we can form the corresponding μ_i^τ and I_i^τ. Denote the unit interval $[0, 1)$ by I. The (λ, τ) interval exchange transformation is then defined by

$$T : I \to I, \;\; T(x) = x - \mu_{i-1} + \mu^\tau_{\tau(i)-1} \text{ for } x \in I_i, \; 1 \le i \le m.$$

It exchanges the intervals I_i according to the permutation τ.

The transformation T is invertible and its inverse is given by the $(\lambda^\tau, \tau^{-1})$ interval exchange transformation.

The symbolic coding of $x \in I$ is $\omega_n(x) = i$ if $T^n(x) \in I_i$. This induces a subshift over the alphabet $\mathcal{A}_m$: $\Omega_{\lambda,\tau} = \overline{\{\omega(x) : x \in I\}}$.

Sturmian subshifts correspond to the case of two intervals as a first return map construction shows.

Keane [53] proved that if the orbits of the discontinuities μ_i of T are all infinite and pairwise distinct, then T is minimal. In this case, the coding is one-to-one and the subshift is minimal and aperiodic. This holds in particular if τ is irreducible and λ is irrational. Here, τ is called irreducible if $\tau(\{1, \dots, k\}) \ne (\{1, \dots, k\})$ for every $k < m$ and λ is called irrational if the λ_i are rationally independent.

Regarding property (B), Boshernitzan has proved two results. First, in [12] the following is shown:

Theorem 12 (Boshernitzan)**.** *For every irreducible $\tau \in S_m$ and for Lebesgue almost every λ, the subshift $\Omega_{\lambda,\tau}$ satisfies* (B)*.*

In fact, Boshernitzan shows that for every irreducible $\tau \in S_m$ and for Lebesgue almost every λ, the subshift $\Omega_{\lambda,\tau}$ satisfies a stronger condition where the sequence of n's for which $\eta(n)$ is large cannot be too sparse. This condition is easily seen to imply (B), and hence the theorem above.

Note that when combined with Keane's minimality result, Theorem 12 implies that almost every subshift arising from an interval exchange transformation is uniquely ergodic. The latter statement confirms a conjecture of Keane [53] and had earlier been proven by different methods by Masur [69] and Veech [88]. Keane had in fact conjectured that all minimal interval exchange transformations would give rise to a uniquely ergodic system. This was disproved by Keynes and Newton [55] using five intervals, and then by Keane [54] using four intervals (the smallest possible number). The conjecture was therefore modified in [54] and then ultimately proven by Masur and Veech.

In a different paper, [13], Boshernitzan singles out an explicit class of subshifts arising from interval exchange transformations that satisfy (B). The transformation T is said to be of (rational) rank k if the μ_i span a k-dimensional space over $\mathbb{Q}$ (the field of rational numbers).

Theorem 13 (Boshernitzan)**.** *If T has rank* 2*, the subshift $\Omega_{\lambda,\tau}$ satisfies* (B)*.*

6. Circle Maps

Let $\alpha \in (0,1)$ be irrational and $\beta \in (0,1)$ arbitrary. The coding of the rotation R_α according to a partition into two half-open intervals of length β and $1-\beta$, respectively, is given by the sequences

$$v_n(\alpha,\beta,\theta) = \chi_{[0,\beta)}(R_\alpha^n \theta).$$

We obtain a subshift

$$\Omega_{\alpha,\beta} = \overline{\{v(\alpha,\beta,\theta) : \theta \in [0,1)\}} \subset \{0,1\}^{\mathbb{Z}}. \tag{27}$$

Subshifts generated this way are usually called circle map subshifts or subshifts generated by the coding of a rotation. These natural generalizations of Sturmian subshifts were studied, for example, in [1, 2, 10, 28, 29, 30, 48, 52, 80].

To the best of our knowledge, the Boshernitzan condition for this class of subshifts has not been studied explicitly. It is, however, intimately related to classical results on inhomogeneous diophantine approximation problems. In this section we make this connection explicit and study the condition (B) for circle map subshifts.

To describe the relation of frequencies of finite words occurring in a subshift to the length of intervals on the circle, let us write, in analogy to the Sturmian case, $I_0 = [0,\beta)$ and $I_1 = [\beta,1)$. The word $w = w_1 \dots w_n \in \{0,1\}^n$ occurs in $v(\alpha,\beta,\theta)$ at site $k+1$ if and only if

$$R_\alpha^k(\theta) \in I(w_1,\dots,w_n) := \bigcap_{j=1}^{n} R_\alpha^{-j}(I_{w_j}).$$

Thus the frequency of w is θ-independent and equal to the Lebesgue measure of $I(w_1,\dots,w_n)$. Moreover, $I(w_1,\dots,w_n)$ is an interval whose boundary points are elements of the set

$$P_n(\alpha,\beta) := \{\{-j\alpha + k\beta\} : 1 \le j \le n,\ 0 \le k \le 1\}.$$

This shows in particular that $\Omega_{\alpha,\beta}$ is quasi-Sturmian when $\beta \in \mathbb{Z} + \alpha\mathbb{Z}$ as in this case $P_n(\alpha,\beta)$ splits the unit interval into $n+k$ subintervals for large n. On the other hand, when $\beta \notin \mathbb{Z} + \alpha\mathbb{Z}$, $P_n(\alpha,\beta)$ contains $2n$ elements and the complexity of $\Omega_{\alpha,\beta}$ is $p(n) = 2n$ for n large enough.

Again, the points of $P_n(\alpha,\beta)$ partition the torus into $2n$ (resp., $n+k$) subintervals and hence the length $h_n(\alpha,\beta)$ of the smallest of these intervals bounds the frequency of a factor of length n from below. Explicitly, we have

$$h_n(\alpha,\beta) = \min\left\{\|q\alpha + r\beta\| : 0 \le |q| \le n,\ 0 \le r \le 1,\ (q,r) \ne (0,0)\right\}.$$

Let us also define

$$\tilde{h}_n(\alpha,\beta) = \min\left\{\|q\alpha + \beta\| : 0 \le |q| \le n\right\}.$$

Then $h_n(\alpha,\beta) \le \tilde{h}_n(\alpha,\beta)$ and therefore

$$\limsup_{n\to\infty} n\tilde{h}_n(\alpha,\beta) = 0 \Rightarrow \limsup_{n\to\infty} nh_n(\alpha,\beta) = 0. \tag{28}$$

Since we saw in Theorem 11 above that the points of $P_n(\alpha)$ are nicely spaced for many values of n, the Boshernitzan condition can only fail for a circle map subshift $\Omega_{\alpha,\beta}$ if the orbit of the α-rotation comes too close to β. In other words, to prove such a negative result for a circle map subshift, it should be sufficient to study $\tilde{h}_n(\alpha,\beta)$, followed by an application of (28).

Motivated by Hardy and Littlewood [42], Morimoto [70, 71] carried out an in-depth analysis of the asymptotic behavior of the numbers $\tilde{h}_n(\alpha, \beta)$. Morimoto's results and related ones were summarized in [57]. While it is possible to deduce consequences regarding the Boshernitzan condition from these papers, we choose to give direct and elementary proofs of our positive results below and make reference to a specific theorem of Morimoto only for a complementary negative result.

Our first result shows that the Boshernitzan condition holds in almost all cases.

Theorem 14. *Let $\alpha \in (0,1)$ be irrational. Then the subshift $\Omega_{\alpha,\beta}$ satisfies* (B) *for Lebesgue almost every $\beta \in (0,1)$.*

Proof. Denote the set of β's for which the Boshernitzan condition fails by $N(\alpha)$,

$$N(\alpha) = \{\beta \in (0,1) : \Omega_{\alpha,\beta} \text{ does not satisfy (B)}\}.$$

By (26) and Theorem 11, there exists a sequence $n_k \to \infty$ such that

$$\liminf_{k\to\infty} n_k h_{n_k}(\alpha) = C > 0.$$

Let $\varepsilon > 0$ with $\varepsilon < C$ be given and denote the $\frac{\varepsilon}{2n}$-neighborhood of the set $\{\{q\alpha\} : 0 < |q| \leq n\}$ by $U(\varepsilon, n)$. Clearly, every $\beta \in N(\alpha)$ belongs to $U(\varepsilon, n_k)$ for $k \geq k_0(\beta)$. Therefore,

$$N(\alpha) \subseteq \liminf_{k\to\infty} U(\varepsilon, n_k) = \bigcup_{m=1}^{\infty} \bigcap_{k\geq m} U(\varepsilon, n_k). \tag{29}$$

The sets

$$S_m = \bigcap_{k\geq m} U(\varepsilon, n_k)$$

obey $S_m \subseteq S_{m+1}$ and $|S_m| \leq \varepsilon$ for every m; $|\cdot|$ denoting Lebesgue measure. Hence,

$$\left| \liminf_{k\to\infty} U(\varepsilon, n_k) \right| \leq \varepsilon.$$

It follows that $N(\alpha)$ has zero Lebesgue measure. □

The next result concerns a subclass of α's for which the Boshernitzan condition holds for all β's.

Theorem 15. *Let $\alpha \in (0,1)$ be irrational with bounded continued fraction coefficients, that is, $a_n \leq C$. Then, $\Omega_{\alpha,\beta}$ satisfies* (B) *for every $\beta \in (0,1)$.*

Proof. By Lemma 5.4 and (24), we have

$$h_n(\alpha) > \frac{1}{q_k + q_{k+1}},$$

where k is chosen such that $q_k \leq n < q_{k+1}$. Thus, for every n, we have

$$nh_n(\alpha) > \frac{n}{q_k + q_{k+1}} \geq \frac{q_k}{(a_{k+1}+2)q_k} \geq \frac{1}{C+2}. \tag{30}$$

Now assume there exists $\beta \in (0,1)$ such that $\Omega_{\alpha,\beta}$ does not satisfy (B). Let $\varepsilon = (7C+14)^{-1}$. As $\limsup_{n\to\infty} n\varepsilon(n) = 0$, we have $n\varepsilon(n) < \epsilon$ for every sufficiently large n. Thus, for each such n we can find a word of length n with frequency less than ε/n. Now, each such word corresponds to an interval with length less than ε/n with boundary points in $P_n(\alpha, \beta)$. Moreover, invoking (30) and the fact that $\epsilon < 1/(C+2)$, we infer that the length of the interval has the form $|m_n\alpha - \beta - k_n|$

with $|m_n| \le n$. To summarize, we see that for every n large enough there exist k_n, m_n with $|m_n| \le n$ such that

$$|m_n\alpha - \beta - k_n| \le \frac{\varepsilon}{n}.$$

Clearly, the mapping $n \mapsto m_n$ can take on each value only finitely many times. Therefore, there exists a sequence $n_j \to \infty$ such that $m_{n_j} \neq m_{n_j+1}$. This implies

$$\begin{aligned}\left|\left(m_{n_j+1} - m_{n_j}\right)\alpha - \left(k_{n_j+1} - k_{n_j}\right)\right| &\le \left|m_{n_j+1}\alpha - \beta - k_{n_j+1}\right| + \left|m_{n_j}\alpha - \beta - k_{n_j}\right| \\ &\le \frac{\varepsilon}{n_j+1} + \frac{\varepsilon}{n_j} \\ &\le \frac{2\varepsilon}{n_j}.\end{aligned}$$

Since $0 < \left|m_{n_j+1} - m_{n_j}\right| \le 2(n_j+1) \le 3n_j =: \tilde{n}_j$, we obtain $\tilde{n}_j h_{\tilde{n}_j}(\alpha) \le 6\varepsilon < (C+2)^{-1}$, which contradicts (30). ☐

This raises the question whether $\Omega_{\alpha,\beta}$ satisfies (B) for every β also in the case where α has unbounded coefficients a_n. It is a consequence of a result of Morimoto [71] that this is not the case.

Theorem 16 (Morimoto)**.** *Let $\alpha \in (0,1)$ be irrational with unbounded continued fraction coefficients. Then, there exists $\beta \in (0,1)$ such that*

$$\limsup_{n\to\infty} n\tilde{h}_n(\alpha,\beta) = 0.$$

Corollary 3. *Let $\alpha \in (0,1)$ be irrational with unbounded continued fraction coefficients. Then, there exists $\beta \in (0,1)$ such that $\Omega_{\alpha,\beta}$ does not satisfy* (B).

Proof. This is an immediate consequence of Theorem 16 and (28). ☐

We close this section with a brief discussion of the case where the circle is partitioned into a finite number of half-open intervals. To be specific, let $0 < \beta_1 < \cdots < \beta_{p-1} < 1$ and associate the intervals of the induced partition with p symbols: Let $\beta_p = \beta_0 = 0$ and

$$v_n(\theta) = k \Leftrightarrow R_\alpha^n(\theta) \in [\beta_k, \beta_{k+1}).$$

We obtain a subshift over the alphabet $\{0,1,\ldots,p-1\}$,

$$\Omega_\beta = \overline{\{v(\theta) : \theta \in [0,1)\}}.$$

Again, the word $w = w_1 \ldots w_n \in \{0,1\}^n$ occurs in $v(\theta)$ at site $k+1$ if and only if

$$R_\alpha^k\theta \in I(w_1,\ldots,w_n) := \bigcap_{j=1}^n R_\alpha^{-j}(I_{w_j})$$

and the connected components of the sets $I(w_1,\ldots,w_n)$ are bounded by the points

$$\{-j\alpha + \beta_k : 1 \le j \le n, 0 \le k \le p-1\}. \tag{31}$$

Recall that $\limsup_{n\to\infty} n\tilde{h}_n(\alpha,\beta)$ is an important quantity in the case of a partition of the circle into two intervals. In fact, we showed that this quantity being positive is a necessary condition for (B) to hold. When there are three or more intervals, however, we will need to require a much stronger condition as the β_i's may now "take turns" in being well approximated by the α-orbit. Indeed, we shall now be interested in studying $\liminf_{n\to\infty} n\tilde{h}_n(\alpha,\gamma)$ (for certain values of γ, associated with the β_i's). More precisely, define the following quantity:

$$M(\alpha,\gamma) = \liminf_{|n|\to\infty} |n| \cdot \|n\alpha - \gamma\|.$$

Let

$$P(\alpha) = \{\gamma : M(\alpha,\gamma) > 0\}.$$

Then, we have the following result:

Theorem 17. *Let $\alpha \in (0,1)$ be irrational. Suppose that $0 = \beta_0 < \beta_1 < \cdots < \beta_{p-1} < \beta_p = 1$ are such that*

$$\beta_k - \beta_l \in P(\alpha) \text{ for } 0 \le k \ne l \le p-1.$$

Then the subshift (Ω_β, T) satisfies the Boshernitzan condition (B).

Remarks. 1. This gives a finite number of conditions whose combination is a sufficient condition for (B) to hold.

2. The set $P(\alpha)$ is non-empty for every irrational α. In fact, for every irrational α there exists a suitable γ such that $M(\alpha,\gamma) > 1/32$; compare [79, Theorem IV.9.3].

3. We discuss in Appendix B how $M(\alpha,\gamma)$ can be computed with the help of the so-called *negative continued fraction expansion* of α and the *α-expansion* of γ.

Proof. By (31), all frequencies of words of length n are bounded from below by

$$\hat{h}_n(\alpha,\beta) = \min\left\{\|q\alpha + \beta_k - \beta_l\| : 0 \le |q| \le n,\ 0 \le k,l \le p-1,\ (q, k-l) \ne (0,0)\right\}.$$

As in our considerations above, we choose a sequence $n_k \to \infty$ such that

$$\liminf_{k\to\infty} n_k h_{n_k}(\alpha) = C > 0.$$

By assumption, we have

$$D = \min\{M(\alpha, \beta_k - \beta_l) : 0 \le k \ne l \le p-1\} > 0.$$

Notice that with these choices of C and D, frequencies of words of length n_k are bounded from below by

$$\hat{h}_{n_k}(\alpha,\beta) \ge \min\left\{\frac{C - o(1)}{n_k}, \frac{D - o(1)}{n_k}\right\}.$$

Putting everything together, we obtain

$$\limsup_{n\to\infty} n \cdot \eta(n) \ge \liminf_{k\to\infty} n_k \cdot \eta(n_k) \ge \min\{C, D\} > 0,$$

and hence (B) is satisfied. $\square$

7. Arnoux-Rauzy Subshifts and Episturmian Subshifts

In this section we consider another natural generalization of Sturmian subshifts, namely, Arnoux-Rauzy subshifts and, more generally, episturmian subshifts. These subshifts were studied, for example, in [4, 27, 31, 50, 51, 78, 91]. They share with Sturmian subshifts the fact that, for each n, there is a unique subword of length n that has multiple extensions to the right. Our main results will show that, similarly to the circle map case, the Boshernitzan condition is almost always satisfied, but not always.

Let us consider a minimal subshift (Ω, T) over the alphabet $\mathcal{A}_m = \{1, 2, \ldots, m\}$, where $m \ge 2$. Recall that the set of subwords of length n occurring in elements of Ω is denoted by $\mathcal{W}_n(\Omega)$ (cf. (3)) and that the complexity function p is defined by $p(n) = \sharp\mathcal{W}_n(\Omega)$ (cf. (21)). A word $w \in \mathcal{W}(\Omega)$ is called *right-special* (resp., *left-special*) if there are distinct symbols $a, b \in \mathcal{A}_m$ such that $wa, wb \in \mathcal{W}(\Omega)$ (resp., $aw, bw \in \mathcal{W}(\Omega)$). A word that is both right-special and left-special is called *bispecial*.

For later use, let us recall the *Rauzy graphs* that are associated with $\mathcal{W}(\Omega)$. For each n, we consider the directed graph $\mathcal{R}_n = (V_n, A_n)$, where the vertex set is given by $V_n = \mathcal{W}_n(\Omega)$, and A_n contains the arc from aw to wb, $a, b \in \mathcal{A}_m$, $|w| = n-1$, if and only if $awb \in \mathcal{W}_{n+1}(\Omega)$. That is, $|V_n| = p(n)$ and $|A_n| = p(n+1)$. Moreover, a word is right-special (resp., left-special) if and only if its out-degree (resp., in-degree) is ≥ 2.

Note that the complexity function of a Sturmian subshift obeys $p(n+1)-p(n) = 1$ for every n and hence for every length, there is a unique right-special factor and a unique left-special factor, each having exactly two extensions. This property is clearly characteristic for a Sturmian subshift.

Arnoux-Rauzy subshifts and episturmian subshifts relax this restriction on the possible extensions somewhat, and they are defined as follows: Ω is called an *Arnoux-Rauzy subshift* if for every n, there is a unique right-special word $r_n \in \mathcal{W}(\Omega)$ and a unique left-special word $l_n \in \mathcal{W}(\Omega)$, both having exactly m extensions. This implies in particular that $p(1) = m$ and hence

$$p(n) = (m-1)n + 1.$$

Arnoux-Rauzy subshifts over $\mathcal{A}_2$ are exactly the Sturmian subshifts.

On the other hand, Ω is called *episturmian* if $\mathcal{W}(\Omega)$ is closed under reversal (i.e., for every $w = w_1 \ldots w_n \in \mathcal{W}(\Omega)$, we have $w^R = w_n \ldots w_1 \in \mathcal{W}(\Omega)$) and for every n, there is exactly one right-special word $r_n \in \mathcal{W}(\Omega)$.

It is easy to see that every Arnoux-Rauzy subshift is episturmian. On the other hand, every episturmian subshift turns out to be a morphic image of some Arnoux-Rauzy subshift. We shall explain this connection below. Since we are interested in studying the Boshernitzan condition, this fact is important and allows us to limit our attention to the Arnoux-Rauzy case.

Risley and Zamboni [78] found two useful descriptions of a given Arnoux-Rauzy subshift, namely, in terms of the recursive structure of the bispecial words and in terms of an S-adic system.

Let ϵ be the empty word and let $\{\epsilon = w_1, w_2, \ldots\}$ be the set of all bispecial words in $\mathcal{W}(\Omega)$, ordered so that $0 = |w_1| < |w_2| < \cdots$. Let $I = \{i_n\}$ be the sequence of elements i_n of $\mathcal{A}_m$ so that $w_n i_n$ is left-special. The sequence I is called the *index sequence* associated with Ω. Risley and Zamboni prove that, for every n, w_{n+1} is the *palindromic closure* $(w_n i_n)^+$ of $w_n i_n$, that is, the shortest palindrome that has $w_n i_n$ as a prefix. Conversely, given any sequence I, one can associate a subshift Ω as follows: Start with $w_1 = \epsilon$ and define w_n inductively by $w_{n+1} = (w_n i_n)^+$. The sequence of words $\{w_n\}$ has a unique one-sided infinite limit $w_\infty \in \mathcal{A}_m^{\mathbb{N}}$, called the *characteristic sequence*, which then gives rise to the subshift $(\Omega(I), T)$ in the standard way; $\Omega(I)$ consists of all two-sided infinite sequences whose subwords occur in w. Risley and Zamboni prove the following characterization.

Proposition 7.1 (Risley-Zamboni)**.** *For every Arnoux-Rauzy subshift (Ω, T) over $\mathcal{A}_m$, every $a \in \mathcal{A}_m$ occurs in the index sequence $\{i_n\}$ infinitely many times and $\Omega = \Omega(I)$. Conversely, for every sequence $\{i_n\} \in \mathcal{A}_m^{\mathbb{N}}$ such that every $a \in \mathcal{A}_m$*

occurs in $\{i_n\}$ *infinitely many times,* $(\Omega(I), T)$ *is an Arnoux-Rauzy subshift and* $\{i_n\}$ *is its index sequence.*

The S-adic description of an Arnoux-Rauzy subshift, that is, involving iterated morphisms chosen from a finite set, found in [78] reads as follows.

Proposition 7.2 (Risley-Zamboni)**.** *Let* (Ω, T) *be an Arnoux-Rauzy subshift over* $\mathcal{A}_m$ *and* $\{i_n\}$ *the associated index sequence. For each* $a \in \mathcal{A}_m$*, define the morphism* τ_a *by*

$$\tau_a(a) = a \text{ and } \tau_a(b) = ab \text{ for } b \in \mathcal{A}_m \setminus \{a\}.$$

Then for every $a \in \mathcal{A}_m$*, the characteristic sequence is given by*

$$\lim_{m\to\infty} \tau_{i_1} \circ \cdots \circ \tau_{i_m}(a).$$

We can now state our positive result regarding the Boshernitzan condition for Arnoux-Rauzy subshifts.

Theorem 18. *Let* (Ω, T) *be an Arnoux-Rauzy subshift over* $\mathcal{A}_m$ *and* $\{i_n\}$ *the associated index sequence. Suppose there is* $N \in \mathbb{N}$ *such that for a sequence* $k_j \to \infty$*, each of the words* $i_{k_j} \dots i_{k_j+N-1}$ *contains all symbols from* $\mathcal{A}_m$*. Then the Boshernitzan condition* (B) *holds.*

This result is similar to Theorem 14 in the sense that if we put any probability measure ν on $\mathcal{A}_m$ assigning positive weight to each symbol, then almost all sequences $\{i_n\}$ with respect to the product measure $\nu^{\mathbb{N}}$ correspond to Arnoux-Rauzy subshifts that satisfy the assumption of Theorem 18.

Before proving this theorem, we state our negative result, which is an analog of Corollary 3.

Theorem 19. *For every* $m \geq 3$*, there exists an Arnoux-Rauzy subshift over* $\mathcal{A}_m$ *that does not satisfy the Boshernitzan condition* (B)*.*

Remark 5. The assumption $m \geq 3$ is of course necessary since the case $m = 1$ is trivial and the case $m = 2$ corresponds to the Sturmian case, where the Boshernitzan condition always holds; compare Theorem 11.

The Arnoux-Rauzy subshifts are uniquely ergodic and we set

$$d(w) \equiv \mu(V_w),\ w \in \mathcal{W}(\Omega),$$

where, as usual, the unique invariant probability measure is denoted by μ.

Proof of Theorem 18. This proof employs the description of the subshift in terms of the bispecial words; compare Proposition 7.1.

Observe that there is some k_0 such that $|w_k| \leq 2|w_{k-1}|$ for every $k \geq k_0$. Essentially, we need that $i_1, \dots, i_{k_0-1}$ contains all symbols from $\mathcal{A}_m$.

Now consider a value of $k \geq k_0$ such that $i_k \dots i_{k+N-1}$ contains all symbols from $\mathcal{A}_m$. We claim that

$$|w_k| \cdot \eta(|w_k|) \geq 2^{-N}. \tag{32}$$

By the assumption, this implies

$$\limsup_{n\to\infty} n \cdot \eta(n) \geq 2^{-N}$$

and hence the Boshernitzan condition (B).

The Rauzy graph $\mathcal{R}_{|w_k|}$ has one vertex (namely, w_k) with in-degree and out-degree m, while all other vertices have in-degree and out-degree 1. Thus, the graph splits up into m loops that all contain w_k and are pairwise disjoint otherwise. These loops can be indexed in an obvious way by the elements of the alphabet $\mathcal{A}_m$.

Since $w_{k+1} = (w_k i_k)^+$, w_{k+1} begins and ends with w_k and, moreover, w_{k+1} contains all words that correspond to the loop in $\mathcal{R}_{|w_k|}$ indexed by i_k. Iterating this argument, we see that w_{k+N} contains the words from all loops and hence all words from $\mathcal{W}_{|w_k|}(\Omega)$. This implies

$$\min_{w \in \mathcal{W}_{|w_k|}(\Omega)} d(w) \geq d(w_{k+N}) \geq \frac{1}{|w_{k+N}|} \geq \frac{1}{2^N |w_k|}$$

and hence (32), finishing the proof. □

Proof of Theorem 19. This proof employs the description of the subshift in terms of an S-adic structure; compare Proposition 7.2.

We shall construct an index sequence $\{i_n\}$ over three symbols (i.e., over the alphabet $\mathcal{A}_3$) such that the corresponding Arnoux-Rauzy subshift does not satisfy the Boshernitzan condition (B). It is easy to verify that the same idea can be used to construct such a subshift over $\mathcal{A}_m$ for any $m \geq 3$.

The index sequence will have the form

$$i_1 i_2 i_3 \ldots = 1^{a_1} 2^{a_2} 3^{a_3} 1^{a_4} 2^{a_5} 3^{a_6} 1^{a_7} \ldots \tag{33}$$

with a rapidly increasing sequence of integers, $\{a_n\}$.

By the special form of the Rauzy graph, the words $w_k a$ label all the frequencies of words in $\mathcal{W}_{|w_k|+1}(\Omega)$ since words corresponding to arcs on a given loop in $\mathcal{R}_{|w_k|}$ must have the same frequency. Put differently,

$$\eta(|w_k| + 1) = \min_{a \in \mathcal{A}_3} d_{w_\infty}(w_k a). \tag{34}$$

Here, we make the dependence of the frequency on w_∞ explicit.

Moreover, it is sufficient to control $\eta(n)$ for these special values of n since every subword u that is not bispecial has a unique extension to either the left or the right, and this extension must have the same frequency. This shows

$$\eta(|w_k| + 1) \geq \eta(n) \text{ for } |w_k| + 1 \leq n \leq |w_{k+1}|. \tag{35}$$

Now write $\mu_{k,m} = \tau_{i_k} \circ \cdots \circ \tau_{i_{k+m-1}}$. Proposition 7.2 says that the characteristic sequence is given by the limit

$$w = \lim_{m \to \infty} \mu_{1,m}(a) \text{ for every } a \in \mathcal{A}_3.$$

We also define

$$w^{(k)} = \lim_{m \to \infty} \mu_{k,m}(a) = (\mu_{1,k-1})^{-1}(w).$$

By [50], $w^{(k)}$ is the derived sequence labeling the return words of w_k in w_∞. In particular, $w^{(k)}$ labels the occurrences of $w_k a$, $a \in \mathcal{A}_3$, in w. Moreover,

$$d_{w_\infty}(w_k a) = \frac{d_{w^{(k)}}(a)}{\sum_{b \in \mathcal{A}_3} d_{w^{(k)}}(b) |\mu_{1,k-1}(b)|} \leq \frac{d_{w^{(k)}}(a)}{\min_{b \in \mathcal{A}_3} |\mu_{1,k-1}(b)|}. \tag{36}$$

Combining (34) and (36), we obtain

$$(|w_k| + 1) \cdot \eta(|w_k| + 1) \leq \frac{|w_k| + 1}{\min_{b \in \mathcal{A}_3} |\mu_{1,k-1}(b)|} \cdot \min_{a \in \mathcal{A}_3} d_{w^{(k)}}(a). \tag{37}$$

Notice that $(|w_k| + 1)(\min_{b\in\mathcal{A}_3} |\mu_{1,k-1}(b)|)^{-1}$ only depends on $i_1, \ldots, i_{k-1}$ and $\min_{a\in\mathcal{A}_3} d_{w^{(k)}}(a)$ only depends on $i_k, i_{k+1}, \ldots$. Thus, if we choose a rapidly increasing sequence $\{a_n\}$ in (33), we can arrange for

$$\lim_{k\to\infty} (|w_k| + 1) \cdot \eta(|w_k| + 1) = 0. \tag{38}$$

This together with (35) implies

$$\lim_{n\to\infty} n \cdot \eta(n) = 0,$$

proving the theorem.

Let us briefly comment on (38). Choose a monotonically decreasing sequence $e_k \to 0$. Assign any value ≥ 1 to a_1. Then, a_2 should be chosen large enough so that for $1 \leq k \leq a_1$, (37) yields

$$(|w_k| + 1) \cdot \eta(|w_k| + 1) \leq e_k. \tag{39}$$

Here we use that between consecutive 3's in $w^{(k)}$, there must be at least a_2 2's. Next, we choose a_3 so large that (39) holds for $a_1 + 1 \leq k \leq a_2$. Here we use that between consecutive 1's in $w^{(k)}$, there must be at least a_3 3's. We can continue in this fashion, thereby generating a sequence $\{a_n\}$ such that (39) holds for all k. This shows in particular that $(|w_k| + 1) \cdot \eta(|w_k| + 1)$ can go to zero arbitrarily fast. $\square$

One may wonder what sequences are generated by the procedures described before Propositions 7.1 and 7.2 if one starts with an index sequence that does not necessarily satisfy the assumption above, namely, that all symbols occur infinitely often. It was shown by Droubay, Justin and Pirillo [31, 50] that one obtains episturmian subshifts and, conversely, every episturmian subshift can be generated in this way.

Proposition 7.3 (Droubay, Justin, Pirillo)**.** *For every episturmian subshift (Ω, T) over $\mathcal{A}_m$, there exists an index sequence $\{i_n\}$ such that $\Omega = \Omega(I)$. Conversely, for every sequence $\{i_n\} \in \mathcal{A}_m^{\mathbb{N}}$, $(\Omega(I), T)$ is an episturmian subshift and $\{i_n\}$ is its index sequence. For every $a \in \mathcal{A}_m$, the characteristic sequence is given by*

$$\lim_{m\to\infty} \tau_{i_1} \circ \cdots \circ \tau_{i_m}(a).$$

We can now quickly deduce results concerning (B) (and hence (P)) for episturmian subshifts. If (Ω, T) is an episturmian subshift over $\mathcal{A}_m$, denote by $\mathcal{A} \subseteq \mathcal{A}_m$ the set of all symbols that occur in its index sequence infinitely many times. Fix k such that $i_k, i_{k+1}, \ldots$ only contains symbols from $\mathcal{A}$. Thus this tail sequence corresponds to an Arnoux-Rauzy subshift over $|\mathcal{A}|$ symbols and the given episturmian subshift is a morphic image (under $\mu_{1,k-1}$) of it. (Note that $|\mathcal{A}| \geq 2$ since (Ω, T) is aperiodic.) If the associated Arnoux-Rauzy subshift satisfies (B) (if, e.g., Theorem 18 applies), then (Ω, T) satisfies (B) by Theorem 9. On the other hand, since every Arnoux-Rauzy subshift is episturmian, Theorem 19 shows that not all episturmian subshifts satisfy (B). In this context, it is interesting to note that Justin and Pirillo showed that all episturmian subshifts are uniquely ergodic [50].

8. Application to Schrödinger Operators

In this section we discuss applications of our previous study to spectral theory of Schrödinger operators. This is based on methods introduced in [63] by Lenz.

Let (Ω, T) be a minimal uniquely ergodic subshift over the finite set $\mathcal{A}$ and assume $\mathcal{A} \subset \mathbb{R}$. As discussed in the introduction, (Ω, T) gives rise to the family $(H_\omega)_{\omega\in\Omega}$ of selfadjoint operators $H_\omega : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ acting by

$$(H_V u)(n) \equiv u(n+1) + u(n-1) + \omega(n)u(n).$$

As (Ω, T) is minimal, there exists a set $\Sigma \equiv \Sigma((\Omega, T)) \subset \mathbb{R}$ with

$$\sigma(H_\omega) = \Sigma \text{ for all } \omega \in \Omega$$

(see, e.g., [8]). We will assume furthermore that (Ω, T) is aperiodic. Such subshifts have attracted a lot of attention in recent years for both physical and mathematical reasons:

These subshifts can serve as models for a special class of solids discovered in 1984 by Shechtman et al. [81]. These solids, later called quasicrystals, have very special mechanical, electrical, and diffraction properties [49, 82]. In the quantum mechanical description of electrical (i.e., conductance) properties of these solids, one is led to the operators (H_ω) above. These operators in turn have a tendency to display intriguing mathematical features. These features include:

($\mathcal{Z}$) Cantor spectrum of Lebesgue measure zero, that is, Σ is a Cantor set of Lebesgue measure zero.
($\mathcal{SC}$) Purely singular continuous spectrum, that is, absence of both point spectrum and absolutely continuous spectrum.
($\mathcal{AT}$) Anomalous transport.

By now, absence of absolutely continuous spectrum is completely established for all relevant subshifts due to results of Last and Simon [61] in combination with earlier results of Kotani [59]. The other spectral features have been investigated for large, but special, classes of examples. Here, our focus is on ($\mathcal{Z}$). As for the other properties, we refer the reader to the survey articles [21, 83].

The property ($\mathcal{Z}$) has been investigated for several models: For the period-doubling substitution and the Thue-Morse substitution, it was shown to hold by Bellissard et al. in [7] (cf. earlier work of Bellissard [6] as well). A more general result for primitive substitutions has then been obtained by Bovier and Ghez [16]. Recently, proofs of ($\mathcal{Z}$) for all primitive substitutions were obtained by Liu et al. [66] and, independently, by Lenz [63]. For special examples of on-primitive substitutions, ($\mathcal{Z}$) has recently been investigated by de Oliveira and Lima [73]. Their results were extended by Damanik and Lenz [25].

For Sturmian operators, ($\mathcal{Z}$) has been proven by Sütő in the golden mean case (= Fibonacci substitution) [83, 84]. The general case was then treated by Bellissard et al. [8]. A different approach to ($\mathcal{Z}$) in the Sturmian case has been developed in [26] by Damanik and Lenz. A suitably modified version of this approach can also be used to study ($\mathcal{Z}$) for a certain class of substitutions as shown by Damanik [22].

For quasi-Sturmian operators, ($\mathcal{Z}$) was shown in [24]. Later a different proof was given in [64].

All approaches to ($\mathcal{Z}$) are based on a fundamental result of Kotani [59]. To discuss this result, we need some preparation.

Spectral properties of the operators (H_ω) are intimately linked to behavior of solutions of the difference equation

$$u(n+1) + u(n-1) + (\omega(n) - E)u(n) = 0 \tag{40}$$

for $E \in \mathbb{R}$. To study this behavior, we define, for $E \in \mathbb{R}$, the locally constant function $M^E : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ by

$$M^E(\omega) \equiv \begin{pmatrix} E - \omega(1) & -1 \\ 1 & 0 \end{pmatrix}. \tag{41}$$

Then, it is easy to see that a sequence u is a solution of the difference equation (40) if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M^E(n,\omega) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix},\ n \in \mathbb{Z}. \tag{42}$$

The rate of exponential growth of solutions of (40) is then measured by the so-called Lyapunov exponent $\gamma(E) \equiv \Lambda(M^E)$. The fundamental result of Kotani, mentioned above, says that (due to aperiodicity)

$$|\{E \in \mathbb{R} : \gamma(E) = 0\}| = 0, \tag{43}$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}$. By general principles, it is clear that $\{E \in \mathbb{R} : \gamma(E) = 0\} \subset \Sigma$ [17]. The overall strategy to prove $(\mathcal{Z})$ is then to show

$$\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}. \tag{44}$$

Given (44), Σ can not contain an interval by (43). Moreover, Σ is a closed set, as the spectrum of an operator always is. Finally, Σ does not contain isolated points, again by general principles on random operators [17]. Hence, Σ is a Cantor set of measure zero if (44) holds.

The standard approach to (44) used to rely on trace maps. Trace maps are a powerful tool in the study of spectral properties. In particular, they can be used not only to study $(\mathcal{Z})$, but also to investigate $(\mathcal{SC})$ and $(\mathcal{AT})$. However, trace maps do not seem to be available as soon as the dynamical systems get more complicated. This difficulty is avoided in a new approach to $(\mathcal{Z})$ introduced in [63]. There, validity of (44) is related to certain ergodic properties of the underlying dynamical system. More precisely, the abstract cornerstone of this new approach is the following result.

Theorem 20. [63] *Let (Ω, T) be a minimal uniquely ergodic subshift over $\mathcal{A} \subset \mathbb{R}$. Then, $\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}$ if and only if M^E is uniform for every $E \in \mathbb{R}$. In this case, the map $\gamma : \mathbb{R} \longrightarrow [0,\infty)$ is continuous.*

Given this theorem, it becomes possible to show (44) by studying uniformity of the functions M^E. In fact, as shown in [63], uniformity of M^E holds for all systems satisfying (PW) and, in particular, for all linearly repetitive systems (see [62] as well). In [63], this was used to prove $(\mathcal{Z})$ for all primitive substitutions. Later $(\mathcal{Z})$ has been established for various further systems by showing linear repetitivity [1, 25, 73].

Proof of Theorem 2. Given Theorem 20 and Kotanis result (43), the assertion follows easily from our main result: By (B) and Theorem 1, the function M^E is uniform for every $E \in \mathbb{R}$. By Theorem 20, this implies $\Sigma = \{E : \gamma(E) = 0\}$. By (43), this gives that Σ is a Cantor set of Lebesgue measure zero, as discussed above. $\square$

Proof of Theorem 3. The result follows from Theorem 2 and the results regarding the validity of (B) for circle map subshifts of Section 6. $\square$

Proof of Theorem 4. The assertion is immediate from Theorem 20 and our main result, Theorem 1. □

As becomes clear from the discussion in Section 5, Theorem 2 generalizes all earlier results on ($\mathcal{Z}$). Moreover, it gives various new ones. One of these new results on ($\mathcal{Z}$) is Theorem 3. Similarly, combining Theorem 2 and the results of Subsection 5.3, we obtain another new result on ($\mathcal{Z}$) for subshifts associated with interval exchange transformations. To the best of our knowledge this is the first result on ($\mathcal{Z}$) for operators associated to interval exchange transformations (not counting those which are Sturmian or linearly repetitive).

Theorem 21. *Let $\tau \in S_m$ be irreducible. Then, for Lebesgue almost every λ, $\Sigma = \Sigma(\Omega_{\lambda,\tau})$ is a Cantor set of Lebesgue measure zero.*

Proof. As discussed in Subsection 5.3, if $\tau \in S_m$ is irreducible, $\Omega_{\lambda,\tau}$ is minimal, aperiodic, and satisfies (B) for almost every λ. This, combined with Theorem 2, yields the assertion. □

Finally, we also mention the following result for Arnoux-Rauzy subshifts, which follows from Theorem 2 and the discussion in Section 7

Theorem 22. *Let (Ω, T) be an aperiodic Arnoux-Rauzy subshift over $\mathcal{A}_m$ and $\{i_n\}$ the associated index sequence. Suppose there is $N \in \mathbb{N}$ such that for a sequence $k_j \to \infty$, each of the words $i_{k_j} \dots i_{k_j+N-1}$ contains all symbols from $\mathcal{A}_m$. Then, Σ is a Cantor set of measure zero.*

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Appendix A. Almost Every Circle Map Subshift Has Infinite Index

In this section, we show that the previous results on zero-measure Cantor spectrum for Schrödinger operators associated with circle map subshifts only cover a zero-measure set in parameter space. This should be seen in connection with Theorem 3, where this spectral result is established for almost all parameter values.

Recall that for every $(\alpha, \beta) \in (0,1) \times (0,1)$, we may define a subshift $\Omega_{\alpha,\beta}$ as in (27).

Proofs of zero-measure spectrum for the associated operators based on trace map dynamics were given in [8, 24, 83, 84]. They cover the case of arbitrary irrational $\alpha \in (0,1)$ and β's in $(0,1)$ of the form $\beta = m\alpha + n$. This is clearly a zero-measure set in $(0,1) \times (0,1)$.

The paper [1] applies the results of [63] and shows zero-measure spectrum for a class of circle map subshifts that is characterized by means of a generalized continued fraction algorithm. Essentially, [1] characterizes the pairs (α, β) for which the associated subshifts are linearly recurrent. We want to show that these, too, form a set of measure zero.

To this end, we note that every aperiodic linearly recurrent subshift Ω has finite index in the sense that there is $N < \infty$ such that its set of finite subwords, $\mathcal{W}(\Omega)$, contains no word of the form w^N. (This is immediate from the definition.)

We say that a subshift Ω has infinite index if for every $n \geq 1$, there is a word w such that $w^n \in \mathcal{W}(\Omega)$ and prove the following:

Proposition A.1. *For almost every $(\alpha, \beta) \in (0,1) \times (0,1)$, the subshift $\Omega_{\alpha,\beta}$ has infinite index.*

Remarks. (a) This implies that for almost every $(\alpha, \beta) \in (0,1) \times (0,1)$, $\Omega_{\alpha,\beta}$ is not linearly recurrent.

(b) Our proof is an extension of arguments from [28, 52].

Proof. It suffices to show that for each fixed $\beta \in (0,1)$, $\Omega_{\alpha,\beta}$ has infinite index for almost every $\alpha \in (0,1)$.

For a sequence $l_k \to \infty$ with

$$\sum_{k=1}^{\infty} l_k^{-1} = \infty \tag{45}$$

(e.g., $l_k = k$), we define the sets $G_{\alpha,\beta}(k) \subseteq [0,1)$ by

$$G_{\alpha,\beta}(k) = \{\theta \in [0,1) : V_\theta(mq_k + j) = V_\theta(j),\ -2l_k + 1 \leq m \leq 2l_k - 1, 1 \leq j \leq q_k\},$$

where

$$V_\theta(n) = \chi_{[0,\beta)}(R_\alpha^n \theta).$$

It is clearly sufficient to show that for each $\beta \in (0,1)$ fixed (and $|\cdot|$ denoting Lebesgue measure),

$$\left| \limsup_{k\to\infty} G_{\alpha,\beta}(k) \right| > 0 \text{ for almost every } \alpha.$$

Since $|\limsup G_{\alpha,\beta}(k)| \geq \limsup |G_{\alpha,\beta}(k)|$, this will follow from

$$\limsup_{k\to\infty} |G_{\alpha,\beta}(k)| > 0 \text{ for almost every } \alpha. \tag{46}$$

Define

$$G^{(1)}_{\alpha,\beta}(k) = \left\{ \theta : \min_{(-2l_k+2)q_k+1 \leq m \leq (2l_k-1)q_k} \|m\alpha + \theta\| > |q_k\alpha - p_k| \right\},$$

$$G^{(2)}_{\alpha,\beta}(k) = \left\{ \theta : \min_{(-2l_k+2)q_k+1 \leq m \leq (2l_k-1)q_k} \|m\alpha + \theta - \beta\| > |q_k\alpha - p_k| \right\}.$$

It follows from (23) that

$$\|((m \pm q_k)\alpha + \theta) - (m\alpha + \theta)\| = |q_k\alpha - p_k|.$$

This in turn implies

$$G_{\alpha,\beta}(k) \subseteq G^{(1)}_{\alpha,\beta}(k) \cap G^{(2)}_{\alpha,\beta}(k) \tag{47}$$

On the other hand, we have

$$G^{(1)}_{\alpha,\beta}(k)^c = \bigcup_{m=(-2l_k+2)q_k+1}^{(2l_k-1)q_k} \{\theta : \|m\alpha + \theta\| \leq |q_k\alpha - p_k|\},$$

$$G^{(2)}_{\alpha,\beta}(k)^c = \bigcup_{m=(-2l_k+2)q_k+1}^{(2l_k-1)q_k} \{\theta : \|m\alpha + \theta - \beta\| \leq |q_k\alpha - p_k|\},$$

which, by (24), gives for $i = 1, 2$,

$$|G^{(i)}_{\alpha,\beta}(k)^c| \le 2q_k(4l_k - 3)|q_k\alpha - p_k| \le (8l_k - 6)\frac{q_k}{q_{k+1}} \le \frac{8l_k}{a_{k+1}}. \tag{48}$$

Combining (47) and (48), we get

$$\limsup_{k\to\infty} |G_{\alpha,\beta}(k)| \ge 1 - \liminf_{k\to\infty} \frac{16l_k}{a_{k+1}}.$$

By our assumption (45), we have that $\liminf_{k\to\infty} \frac{16l_k}{a_{k+1}}$ is less than $1/2$, say, for almost every α [56, Theorem 30]. This shows (46) and hence concludes the proof. □

Appendix B. Some Remarks on Inhomogeneous Diophantine Approximation

Let $\alpha \in (0,1)$ be irrational and let $\gamma \in [0,1)$. The *two-sided inhomogeneous approximation constant* $M(\alpha,\gamma)$ is given by

$$M(\alpha,\gamma) = \liminf_{|n|\to\infty} |n| \cdot \|n\alpha - \gamma\|,$$

where $\|\cdot\|$ denotes the distance from the closest integer. The number $M(\alpha,\gamma)$ turned out to be important in our study of the Boshernitzan condition for circle map subshifts corresponding to partitions of the unit circle into at least three intervals; compare Section 6. In this appendix we sketch a way to compute $M(\alpha,\gamma)$ which was proposed by Pinner. For background information, we refer the reader to the excellent texts by Khinchin [56] and Rockett and Szüsz [79]. We shall present results from [75]. Related work can be found in Cusick et al. [20] and Komatsu [58].

The *negative continued fraction expansion* of α is given by

$$\alpha = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}}} =: [0; a_1, a_2, a_3, \ldots]^-,$$

where the integers $a_i \ge 2$ are generated as follows:

$$\alpha_0 := \{\alpha\},\ a_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil,\ \alpha_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil - \frac{1}{\alpha_n}.$$

The corresponding *convergents* $p_n/q_n = [0; a_1, a_2, \ldots, a_n]^-$ are given by

$$\begin{aligned} p_{-1} &= -1, \quad p_0 = 0, \quad p_{n+1} = a_{n+1}p_n - p_{n-1},\\ q_{-1} &= 0, \quad\ \ q_0 = 1, \quad q_{n+1} = a_{n+1}q_n - q_{n-1}. \end{aligned}$$

There is a simple way to switch back and forth between regular and negative continued fraction expansion; see [75].

Write

$$\overline{\alpha}_i := [0; a_i, a_{i-1}, \ldots, a_1]^-,\ \ \alpha_i := [0; a_{i+1}, a_{i+2}, \ldots]^-.$$

Then

$$D_i := q_i\alpha - p_i = \alpha_0 \cdots \alpha_i,\ \ q_i = (\overline{\alpha}_1 \cdots \overline{\alpha}_i)^{-1}.$$

The α-*expansion* of γ is now obtained as follows: Let

$$\gamma_0 := \{\gamma\}, \quad b_{n+1} := \left\lfloor \frac{\gamma_n}{\alpha_n} \right\rfloor, \quad \gamma_{n+1} := \left\{ \frac{\gamma_n}{\alpha_n} \right\},$$

so that

$$\{\gamma\} = \sum_{i=1}^{\infty} b_i D_{i-1}.$$

Finally, with $t_k := 2b_k - a_k + 2$, let

$$d_k^- := \sum_{j=1}^{k} \frac{t_j\, q_{j-1}}{q_k}, \quad d_k^+ := \sum_{j=k+1}^{\infty} \frac{t_j\, D_{j-1}}{D_{k-1}}$$

and

$$\begin{aligned} s_1(k) &:= \tfrac{1}{4}(1 - \overline{\alpha}_k + d_k^-)(1 - \alpha_k + d_k^+) q_k D_{k-1}, \\ s_2(k) &:= \tfrac{1}{4}(1 + \overline{\alpha}_k + d_k^-)(1 + \alpha_k + d_k^+) q_k D_{k-1}, \\ s_3(k) &:= \tfrac{1}{4}|1 - \overline{\alpha}_k - d_k^-|\, |1 - \alpha_k - d_k^+| q_k D_{k-1}, \\ s_4(k) &:= \tfrac{1}{4}(1 + \overline{\alpha}_k - d_k^-)(1 + \alpha_k + d_k^+) q_k D_{k-1}. \end{aligned}$$

We have the following result [75]:

Theorem 23 (Pinner)**.** *Suppose that $\gamma \notin \mathbb{Z}\alpha + \mathbb{Z}$ and that its α-expansion has $b_i = a_i - 1$ at most finitely many times. Then*

$$M(\alpha, \gamma) = \liminf_{k \to \infty} \min\{s_1(k), s_2(k), s_3(k), s_4(k)\}.$$

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