A Mathematical Mechanism of Infrared Catastrophe

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Abstract

We derive a mathematical mechanism of infrared catastrophe from the idea for the proof in [24, 25] and generalize it so that we can apply it to several models in quantum field theory. We define the Carleman operator from the operator-theoretical pull-through formula on ground states. Characterizing infrared catastrophe by the Carleman operator, we study the mechanism of infrared catastrophe. And moreover, based on the mechanism, we show some proofs of absence of ground state.

1 Introduction

The mathematical structure on the infrared problem in quantum field theory has been gradually clarified by the former studies (see [1, 4, 5, 8, 9, 11, 14, 15, 17, 20, 22, 23, 24, 25, 26, 27, 30, 35] and the references in them). In particular, the typical model which represents the case that a ground state exists (resp. the case that any ground state does not exist) under the infrared condition [4, 5] is the Pauli-Fierz model [29] (resp. Nelson model [28]). When we argue the existence or absence of ground state, the decomposition in [15, Lemma 2.2] suggests that the dipole approximation is essential for the Pauli-Fierz model and Nelson model. Namely, whether a ground state exists or not is determined by the dipole approximation for the two. We give its overview here with symbolic notations. Let H_{full} be the full Hamiltonian of the Pauli-Fierz model or the Nelson model. In order to study whether a ground state for H_{full} exists or not, we first investigate whether the average of the number of bosons at the ground state (i.e., *soft bosons*) converges or not. By using the symbolic kernel $a(k)$ of the annihilation operator of bosons, if a ground sate Ψ_{g} exists, the average of the number of soft bosons is symbolically given by $\int_{\mathbb{R}^\nu} \|a(k)\Psi_{\mathrm{g}}\|_{\mathcal{F}}^2 dk$, where $\| \quad \|_{\mathcal{F}}$ denotes the norm of the state space $\mathcal{F}$. To estimate $\|a(k)\Psi_{\mathrm{g}}\|_{\mathcal{F}}$, we use the pull-through formula [16] on the ground state,

$$a(k)\Psi_{\mathrm{g}} = -\left(H_{\mathrm{full}} - E_0(H_{\mathrm{full}}) + |k|\right)^{-1}\left(\frac{\lambda(k)}{\sqrt{|k|}} e^{-ikx} B_{\mathrm{PT}}\right)\Psi_{\mathrm{g}},$$

which should be mathematically established in a certain sense as in [6, 13, 18, 25], where $E_0(H_{\mathrm{full}})$ is the ground state energy, B_{PT} an operator, and λ a ultraviolet cutoff function with the infrared singularity condition [4, 5], i.e.,

$$\lambda, \lambda/\sqrt{\omega} \in L^2(\mathbb{R}^\nu) \text{ but } \lambda/\omega \notin L^2(\mathbb{R}^\nu) \text{ for } \omega(k) = |k|.$$

By employing the decomposition, $e^{-ixk} = 1 + (e^{-ixk} - 1)$, as in [15, Lemma 2.2], we get

$$a(k)\Psi_{\mathrm{g}} = I_{\mathrm{dip}}(k) + I_{\mathrm{err}}(k), \tag{1.1}$$

where $I_{\rm dip}(k)$ is the dipole-approximated term, $I_{\rm err}(k)$ the error term between the original term and the dipole-approximated one, and they are given by

$$I_{\rm dip}(k) = -\left(H_{\rm full} - E_0(H_{\rm full}) + |k|\right)^{-1}\left(\frac{\lambda(k)}{\sqrt{|k|}}B_{\rm PT}\right)\Psi_{\rm g},$$

$$I_{\rm err}(k) = -\left(H_{\rm full} - E_0(H_{\rm full}) + |k|\right)^{-1}\left(\frac{\lambda(k)}{\sqrt{|k|}}(e^{-ikx}-1)B_{\rm PT}\right)\Psi_{\rm g}.$$

We note that

$$\int_{\mathbb{R}^\nu} \|I_{\rm err}(k)\|_{\mathcal{F}}^2 dk < \infty$$

by using $|e^{-ikx}-1| \le |x||k|$. So, whether the average of soft bosons converges or not is essentially determined by the dipole-approximated term $I_{\rm dip}(k)$. In the case of the Pauli-Fierz model, we get

$$\int_{\mathbb{R}^\nu} \|I_{\rm dip}(k)\|_{\mathcal{F}}^2 dk < \infty$$

(see the explanation [23] about [8, 20]). On the other hand, in the case of the Nelson model, we have

$$\int_{\mathbb{R}^\nu} \|I_{\rm dip}(k)\|_{\mathcal{F}}^2 dk = \infty$$

(see [15, Lemma 2.2]). So, as long as we know well the dipole-approximated model such as the Pauli-Fierz model or the Nelson model, we can realize the above decomposition (1.1) is very useful for the problem on the existence or absence of ground state.

In this paper we focus our mind on infrared catastrophe in the case that we are not sure whether a ground state exists for the dipole-approximated model such as the generalized spin-boson (GSB) model [3, 5]. Namely, we are interested in a mathematical proof that the divergence of the number of soft bosons implies absence of ground state for such models. Recently, the Carleman operator is used in [26] to describe the condition for existence of a ground state. Conversely, we also define the Carleman operator from the operator-theoretical pull-through formula on ground states [25] in order to argue our problem. Using the description by the Carleman operator and developing the way in [24] to prove absence of ground state, we characterize the mathematical mechanism of infrared catastrophe within a general framework in §2. We apply it to the modified GSB model in §3. Based on the mechanism, we show proofs of absence of ground state within the general framework with weak topology in §4 – §6 and with strong one in §7. After getting our results, we realize that several results on absence of ground state, known already in former works, are derived as corollaries of Theorems 2.4 and 2.7.

2 Main Results

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{F}_{\rm b}(L^2(S))$ the boson Fock space over $L^2(S)$, where S is arbitrary open subset of $\mathbb{R}^\nu$,

$$\mathcal{F}_{\rm b}(L^2(S)) := \bigoplus_{n=0}^{\infty} \otimes_{\rm s}^n L^2(S). \tag{2.1}$$

Here, for $n \in \mathbb{N}$, we denote the n-fold symmetric tensor product of $L^2(S)$ by $\otimes_{\rm s}^n L^2(S)$ with convention $\otimes_{\rm s}^0 L^2(S) := \mathbb{C}$. Set

$$\mathcal{F}_{\rm b} := \mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu))$$

for simplicity. We consider the Hilbert space

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\rm b},$$

which is the state space of several models in quantum field theory with a standard inner product $(\ \ ,\ \ )_{\mathcal{F}}$.

Let $h : S \longrightarrow [0\,,\,\infty)$ be Borel measurable such that $0 < h(k) < \infty$ for almost every (a.e.) $k \in S$ with respect to the ν-dimensional Lebesgue measure. We use the same symbol for the multiplication operator by h, acting in $L^2(S)$. We denote by $d\Gamma(h)$ the second quantization of h [32, Sect. X.7].

Let A be a self-adjoint operator acting in $\mathcal{H}$ bounded from below, $\mu > 0$ a constant, and $\omega : \mathbb{R}^\nu \longrightarrow [0\,,\,\infty)$ be continuous such that $0 < \omega(k) < \infty$ for every $k \in \mathbb{R}^\nu \setminus \{0\}$. The unperturbed Hamiltonian of our quantum filed model is defined by

$$H_0 := \frac{\mu}{2} A \otimes I + I \otimes d\Gamma(\omega) \tag{2.2}$$

with domain $D(H_0) := D(A \otimes I) \cap D(I \otimes d\Gamma(\omega)) \subset \mathcal{F}$, where I denotes identity operator and $D(T)$ the domain of an operator T. The operator H_0 is self-adjoint and bounded from below. For $\psi \in \mathcal{F}_{\rm b}(S)$, we introduce the following notation:

$$\psi = \oplus \sum_{n=0}^{\infty} \psi^{(n)}, \quad \psi^{(n)} \in \otimes_{\rm s}^n L^2(S) \ \ ; \ \ n \in \{0\} \cup \mathbb{N}. \tag{2.3}$$

We denote by $\mathcal{F}_{\rm b,0}(L^2(S))$ the finite particle space,

$$\mathcal{F}_{\rm b,0}(L^2(S)) := \left\{ \psi = \oplus \sum_{n=0}^{\infty} \psi^{(n)} \in \mathcal{F}_{\rm b}(L^2(S)) \,\middle|\, \exists n_0 \in \mathbb{N} \ \text{such that} \ \psi^{(n)} = 0, \ \forall n \geq n_0 \right\}. \tag{2.4}$$

We set

$$\mathcal{F}_{\rm b,0} := \mathcal{F}_{\rm b,0}(L^2(\mathbb{R}^\nu)).$$

We denote the smeared annihilation operators acting in $\mathcal{F}_{\rm b}(L^2(S))$ by $a(f)$ for every $f \in L^2(S)$, where we assume antilinearity in f [32, Sect. X.7]. On $\mathcal{F}_{\rm b,0}(L^2(S))$, we get

$$(a(f)\psi)^n (k_1, \cdots, k_n) := \sqrt{n+1} \int_S \overline{f(k)} \psi^{(n+1)}(k, k_1, \cdots, k_n) dk \ \ \in \otimes_{\rm s}^n L^2(S), \quad n \in \{0\} \cup \mathbb{N}. \tag{2.5}$$

Since $a(f)$ is closable on $\mathcal{F}_{\rm b,0}(L^2(S))$, we denote its closure by the same symbol. So, we always regard $a(f)$ as a closed operator.

Let $H_{\rm QFT} = H_0 + H_{\rm I}$ be a self-adjoint operator acting in $\mathcal{F}$ and describing a quantum field model. If $H_{\rm QFT}$ has a (normalized) ground state, we denote it by $\Psi_{\rm QFT}$.

When the operator-theoretical pull-through formula on ground states holds in the same way as in [25] and it has the form of (2.6) below, we define an operator $B_{\rm PT}(k)$ by

$$a(f)\Psi_{\rm QFT} = - \int_{\mathbb{R}^\nu} \overline{f(k)} \left(H_{\rm QFT} - E_0(H_{\rm QFT}) + \omega(k) \right)^{-1} B_{\rm PT}(k) \Psi_{\rm QFT} dk, \quad \forall f \in C_0^\infty(\mathbb{R}^\nu \setminus \{0\}), \tag{2.6}$$

where $E_0(H_{\rm QFT})$ is the ground state energy of $H_{\rm QFT}$, i.e., $E_0(H_{\rm QFT}) := \inf \sigma(H_{\rm QFT})$ and $\sigma(T)$ denotes the spectrum of a closed operator T. We are preparing the work on the operator-theoretical pull-through formula for several models in quantum field theory [6].

We set

$$\widehat{H}_{\rm QFT} := H_{\rm QFT} - E_0(H_{\rm QFT}).$$

We assume the following conditions for $B_{\rm PT}(k)$:

(PT1) $B_{\rm PT}(k)$ is determined for every $k \in \mathbb{R}^\nu \setminus \{0\}$ as an operator acting in $\mathcal{F}$ and $B_{\rm PT}(\cdot)\Psi$ is measurable for every $\Psi \in D(H_0)$ (i.e., for every $\Phi \in \mathcal{F}$ and $\Psi \in D(H_0)$ $(\Phi\,,\, B_{\rm PT}(\cdot)\Psi)_{\mathcal{F}} : \mathbb{R}^\nu \longrightarrow \mathbb{C}$ is a measurable function).

(PT2) $B_{\rm PT}(k)\left(\widehat{H}_{\rm QFT} + \omega(k)\right)^{-1}$ is bounded for every $k \in \mathbb{R}^\nu \setminus \{0\}$.

For every $\varepsilon \geq 0$, we set

$$\mathbb{R}^\nu_{<\varepsilon} := \{k \in \mathbb{R}^\nu \,|\, |k| < \varepsilon\} \quad \text{and} \quad \mathbb{R}^\nu_{>\varepsilon} := \{k \in \mathbb{R}^\nu \,|\, |k| > \varepsilon\}\,. \tag{2.7}$$

Following this decomposition, for every $f \in L^2(\mathbb{R}^\nu)$ we define $f_{<\varepsilon} \in L^2(\mathbb{R}^\nu_{<\varepsilon})$ and $f_{>\varepsilon} \in L^2(\mathbb{R}^\nu_{>\varepsilon})$ by

$$f_{<\varepsilon} := \chi_{|k|<\varepsilon} f \quad \text{and} \quad f_{>\varepsilon} := \chi_{|k|>\varepsilon} f, \tag{2.8}$$

where $\chi_{|k|<\varepsilon}$ and $\chi_{|k|>\varepsilon}$ are characteristic functions defined by

$$\chi_{|k|<\varepsilon}(k) := \begin{cases} 1 & \text{if } |k| < \varepsilon, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \chi_{|k|>\varepsilon}(k) := \begin{cases} 1 & \text{if } |k| > \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

As is well known, there exists a unitary operator U_ε such that

$$U_\varepsilon \mathcal{F}_{\rm b} = \mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{<\varepsilon})) \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{>\varepsilon})), \tag{2.9}$$

$$U_\varepsilon d\Gamma(\omega) U_\varepsilon^* = d\Gamma(\omega_{<\varepsilon}) \otimes I + I \otimes d\Gamma(\omega_{>\varepsilon}), \tag{2.10}$$

$$U_\varepsilon d\Gamma(1) U_\varepsilon^* = d\Gamma(1_{<\varepsilon}) \otimes I + I \otimes d\Gamma(1_{>\varepsilon}). \tag{2.11}$$

We define the number operator N acting in $\mathcal{F}$ by

$$N := I \otimes d\Gamma(1). \tag{2.12}$$

Moreover, for every $\varepsilon > 0$, we define $N(\varepsilon)$ acting in $\mathcal{F}_{\rm b}$ by

$$N(\varepsilon) := U_\varepsilon^* \left(I \otimes d\Gamma(1_{>\varepsilon})\right) U_\varepsilon, \tag{2.13}$$

and an operator $N_{>\varepsilon}$ acting in $\mathcal{F}$ by

$$N_{>\varepsilon} := I \otimes N(\varepsilon). \tag{2.14}$$

We sometimes denote N by $N_{>0}$.

Lemma 2.1 *[19] If $\Psi \in D(H_0)$, then*

$$\Psi \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2}). \tag{2.15}$$

Proof: For every $\varepsilon > 0$, $d\Gamma(\omega) \cong d\Gamma(\omega_{<\varepsilon}) \otimes I + I \otimes d\Gamma(\omega_{>\varepsilon})$. Moreover, $D(I \otimes d\Gamma(\omega_{>\varepsilon})) \subset D(I \otimes d\Gamma(1_{>\varepsilon}))$. Thus, we get

$$\begin{aligned} \Psi \in D(H) \;&=\; D(H_0) = D(A \otimes I) \cap D(I \otimes d\Gamma(\omega)) \\ &\cong\; D(A \otimes I \otimes I) \cap D(I \otimes d\Gamma(\omega_{<\varepsilon}) \otimes I) \cap D(I \otimes I \otimes d\Gamma(\omega_{>\varepsilon})) \\ &\subset\; \mathcal{H} \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{<\varepsilon})) \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{>\varepsilon})) \cap D(I \otimes I \otimes d\Gamma(1_{>\varepsilon})) \\ &\cong\; D(N_{>\varepsilon}). \end{aligned}$$

Noting $D(N_{>\varepsilon}) \subset D(N_{>\varepsilon}^{1/2})$, we obtain (2.15). $\square$

Symbolically, the average of the number of bosons at the state $\Psi \in \mathcal{F}$ is give by $\int_{\mathbb{R}^\nu} \|a(k)\Psi\|_{\mathcal{F}}^2 dk$, of which justification is in §7. Then, we set

$$D_{\mathrm{CNB}} := \left\{ \Psi \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2}) \;\middle|\; \sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 < \infty \right\}. \tag{2.16}$$

By Lemma 7.2 in §7,

$$\text{if } \Psi \in D_{\mathrm{CNB}}, \quad \text{then } \int_{\mathbb{R}^\nu} \|a(k)\Psi\|_{\mathcal{F}}^2 dk = \sup_{\varepsilon>0} \int_{|k|>\varepsilon} \|a(k)\Psi\|_{\mathcal{F}}^2 dk < \infty \tag{2.17}$$

by Lebesgue's monotone convergence theorem. Thus, D_{CNB} is the state space such that the average of the number of bosons at the state in D_{CNB} converges.

We call $\sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2$ *the average of the number of soft bosons* if $\Psi_{\mathrm{QFT}} \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2})$. By Lemma 2.1, we can give a decision whether the average of the number of soft bosons is convergent or divergent if $D(H_{\mathrm{QFT}}) = D(H_0)$.

As is well known, $\mathcal{F}$ is unitarily equivalent to $\bigoplus_{n=0}^\infty L^2_{\mathrm{sym}}(\mathbb{R}^{\nu n}; \mathcal{H})$, where $L^2_{\mathrm{sym}}(\mathbb{R}^{\nu n}; \mathcal{H})$ is the Hilbert space of square integrable $\mathcal{H}$-valued, symmetric functions on $\mathbb{R}^{\nu n} = (\mathbb{R}^\nu)^n$ with convention $L^2_{\mathrm{sym}}(\mathbb{R}^{\nu n}; \mathcal{H})\lceil_{n=0} := \mathcal{H}$. Moreover, $L^2_{\mathrm{sym}}(\mathbb{R}^{\nu n}; \mathcal{H})$ is unitarily equivalent to $\mathcal{H} \otimes \left(\otimes_{\mathrm{s}}^n L^2(\mathbb{R}^\nu)\right)$. We often identify $\mathcal{F}$ as

$$\mathcal{F} = \bigoplus_{n=0}^\infty \mathcal{H} \otimes \left(\otimes_{\mathrm{s}}^n L^2(\mathbb{R}^\nu)\right) = \bigoplus_{n=0}^\infty L^2_{\mathrm{sym}}(\mathbb{R}^{\nu n}; \mathcal{H}) \tag{2.18}$$

in this paper. Then, through this identification we can denote all $\Psi \in \mathcal{F}$ by

$$\Psi = \oplus \sum_{n=0}^\infty \Psi^{(n)} = \Psi^{(0)} \oplus \Psi^{(1)} \oplus \cdots \oplus \Psi^{(n)} \oplus \cdots, \tag{2.19}$$

where

$$\Psi^{(n)} \in \mathcal{H} \otimes \left(\otimes_{\mathrm{s}}^n L^2(\mathbb{R}^\nu)\right) = L^2_{\mathrm{sym}}(\mathbb{R}^{\nu n}; \mathcal{H}). \tag{2.20}$$

For each $n_1, n_2 \in \mathbb{N}$, we introduce the following notation:

$$\oplus \sum_{n=n_1}^{n_2} \Psi^{(n)} = 0 \oplus \cdots 0 \oplus \Psi^{(n_1)} \oplus \cdots \oplus \Psi^{(n_2)} \oplus 0 \oplus \cdots \in \mathcal{H} \otimes \mathcal{F}_{\mathrm{b},0}. \tag{2.21}$$

Similarly, we use the following identification.

$$\mathcal{F} = \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^\nu_{<\varepsilon}; \mathcal{H})) \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^\nu_{>\varepsilon}; \mathcal{H})), \tag{2.22}$$

where $L^2(S; \mathcal{H})$ denotes the Hilbert space of square integrable $\mathcal{H}$-valued functions on an open set $\mathcal{S} \subset \mathbb{R}^\nu$.

Lemma 2.2 $\quad D_{\mathrm{CNB}} = D(N^{1/2})$ *and*

$$\sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 = \|N^{1/2}\Psi\|_{\mathcal{F}}^2 \tag{2.23}$$

for $\Psi \in D_{\mathrm{CNB}}$.

Proof: Let $\Psi \in D(N^{1/2})$. Then, we get $\Psi \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2})$ in the same way as the proof of Lemma 2.1, and moreoer, by Lemma 7.2 we get $a(\cdot)\Psi \in L^2(\mathbb{R}^\nu_{>\varepsilon};\mathcal{F})$ and

$$\|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 = \int_{|k|>\varepsilon} \|a(k)\Psi\|_{\mathcal{F}}^2 dk$$

for every $\varepsilon \geq 0$. Therefore, by Lebesgue's monotone convergence theorem we obtain

$$\sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 = \sup_{\varepsilon>0} \int_{|k|>\varepsilon} \|a(k)\Psi\|_{\mathcal{F}}^2 dk = \int_{\mathbb{R}^\nu} \|a(k)\Psi\|_{\mathcal{F}}^2 dk = \|N^{1/2}\Psi\|_{\mathcal{F}}^2 < \infty. \tag{2.24}$$

Conversely, let $\Psi = \oplus\sum_{n=0}^\infty \Psi^{(n)} \in D_{\mathrm{CNB}}$. Using the identification (2.22) and symmetry of $\Psi^{(n)}(k_1,\cdots,k_n)$, we get

$$\begin{aligned}
\|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 &\geq \sum_{n=1}^\infty \|N_{>\varepsilon}^{1/2}\Psi^{(n)}\|_{\otimes_s^n L^2(\mathbb{R}^\nu)}^2 \\
&= \sum_{n=1}^\infty \Bigg\{ n\int_{|k_1|,\cdots,|k_n|>\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n \\
&\qquad + \sum_{j=1}^{n-1} \binom{n}{j} j \int_{|k_1|,\cdots,|k_j|>\varepsilon\,;\,|k_{j+1}|,\cdots,|k_n|<\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n \\
&\qquad + \int_{|k_1|,\cdots,|k_n|<\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n \Bigg\} \\
&\geq \sum_{n=1}^\infty n\int_{|k_1|,\cdots,|k_n|>\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n =: \Theta(\varepsilon).
\end{aligned}$$

So, noting $\Theta(\varepsilon) \leq \Theta(\varepsilon')$ for $0 < \varepsilon' < \varepsilon$, we get

$$\infty > \sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 \geq \lim_{\varepsilon\to 0} \Theta(\varepsilon) \geq \lim_{\varepsilon\to 0} A_\nu(\varepsilon) = A_\nu \tag{2.25}$$

for each $\nu \in \mathbb{N}$, where

$$\begin{aligned}
A_\nu(\varepsilon) &:= \sum_{n=1}^\nu n\int_{|k_1|,\cdots,|k_n|>\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n \\
\to \quad &\sum_{n=1}^\nu n\int_{\mathbb{R}^\nu} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n =: A_\nu
\end{aligned}$$

as $\varepsilon \to 0$. By (2.25), $\{A_\nu\}_{\nu\in\mathbb{N}}$ is a bounded, monotone increasing sequence. Therefore, we have

$$\infty > \exists \lim_{\nu\to\infty} A_\nu = \sum_{n=1}^\infty n\int_{\mathbb{R}^\nu} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n,$$

which implies

$$\infty > \sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 \geq \|N^{1/2}\Psi\|_{\mathcal{F}}^2. \tag{2.26}$$

Thus, we have $\Psi \in D(N^{1/2})$ and (2.24) holds in the same way as above. □

Remark 2.1 *In the first version of this paper, the author tried to use Fatou's lemma to prove (2.26). But there was a mistake in his proof. In [12], Bruneau completes the author's idea. Namely, we have*

$$\|N_{>\varepsilon}^{1/2}\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} j \| \underbrace{(\overline{p}_\varepsilon \otimes \cdots \otimes \overline{p}_\varepsilon)}_{n-j} \otimes \underbrace{(p_\varepsilon \otimes \cdots \otimes p_\varepsilon)}_{j} \Psi^{(n)}\|_{\mathcal{F}}^2, \tag{2.27}$$

where p_ε is the orthogonal from $L^2(\mathbb{R}^\nu)$ onto $L^2(\mathbb{R}^\nu_{>\varepsilon})$ and $\overline{p}_\varepsilon := 1 - p_\varepsilon$. So, applying Fatou's lemma, we have also (2.26). We note that the limiting argument on (2.27) may imply Lemma 2.2 directly without the use of Lemma 7.2.

When a ground state Ψ_{QFT} of H_{QFT} exists, we can define an $\mathcal{F}$-valued function $K_{\mathrm{PT}} : \mathbb{R}^\nu \setminus \{0\} \longrightarrow \mathcal{F}$ by

$$K_{\mathrm{PT}}(k) := \left(\widehat{H}_{\mathrm{QFT}} + \omega(k)\right)^{-1} B_{\mathrm{PT}}(k)\Psi_{\mathrm{QFT}}, \quad k \in \mathbb{R}^\nu \setminus \{0\}, \tag{2.28}$$

since $\left(\widehat{H}_{\mathrm{QFT}} + \omega(k)\right)^{-1} B_{\mathrm{PT}}(k)$ is bounded for every $k \in \mathbb{R}^\nu \setminus \{0\}$ by (PT2). K_{PT} defined by (2.28) is measurable by (PT1). For the ground state Ψ_{QFT}, we define the *maximal Carleman operator $T_{\mathrm{PT}} : \mathcal{F} \to L^2(\mathbb{R}^\nu)$ for the ground state* by

$$D(T_{\mathrm{PT}}) := \left\{ \Phi \in \mathcal{F} \,\middle|\, (K_{\mathrm{PT}}(\cdot)\,,\, \Phi)_{\mathcal{F}} \in L^2(\mathbb{R}^\nu) \right\}, \tag{2.29}$$

$$(T_{\mathrm{PT}}\Phi)(k) := (K_{\mathrm{PT}}(k)\,,\,\Phi)_{\mathcal{F}}, \ \ \forall k \in \mathbb{R}^\nu \setminus \{0\}; \ \ \forall \Phi \in D(T_{\mathrm{PT}}). \tag{2.30}$$

Remark 2.2 *(i) If $K_{\mathrm{PT}} \in L^2(\mathbb{R}^\nu; \mathcal{F})$, then the maximal Carleman operator T_{PT} for the ground state is Hilbert-Schmidt [36, Theorems 6.12 and 6.13].*

(ii) If H_{QFT} is the Hamiltonian of the Pauli-Fierz model, T_{PT} is Hilbert-Schmidt.

Theorem 2.3 *Assume $D(H_{\mathrm{QFT}}) = D(H_0)$ and there exists a ground state Ψ_{QFT} of H_{QFT}. Then, $\Psi_{\mathrm{QFT}} \in D_{\mathrm{CNB}} = D(N^{1/2})$ if and only if T_{PT} is Hilbert-Schmidt.*

Proof: By Lemma 2.1, $\Psi_{\mathrm{QFT}} \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2})$. So, in the same way as in [25, Proposition 3.1],

$$\infty > \|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 = \sum_{\nu=0}^{\infty} \|a(f_\nu)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2, \tag{2.31}$$

where $\{f_\nu\}_{\nu=0}^{\infty}$ is a complete orthonormal system of $L^2(\mathbb{R}^\nu_{>\varepsilon})$. Here, in the same way as for T_{PT}, we define the maximal Carleman operator $T_\varepsilon : \mathcal{F} \to L^2(\mathbb{R}^\nu_{>\varepsilon})$ by

$$D(T_\varepsilon) := \left\{ \Phi \in \mathcal{F} \,\middle|\, (K_{\mathrm{PT}}(\cdot)\,,\, \Phi)_{\mathcal{F}} \in L^2(\mathbb{R}^\nu_{>\varepsilon}) \right\}, \tag{2.32}$$

$$(T_\varepsilon\Phi)(k) := (K_{\mathrm{PT}}(k)\,,\,\Phi)_{\mathcal{F}}, \ \ \forall k \in \mathbb{R}^\nu_{>\varepsilon}; \ \ \forall \Phi \in D(T_{\mathrm{PT}}). \tag{2.33}$$

By (2.6), (2.28), and (2.33), we have

$$a(f_\nu)\Psi_{\mathrm{QFT}} = -T_\varepsilon^*\overline{f}_\nu, \quad \nu \in \mathbb{N}. \tag{2.34}$$

Thus, by applying [31, Theorems VI.18 and VI.22] to (2.31) and (2.34), we get

$$\infty > \|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 = \sum_{\nu=0}^{\infty} \|T_\varepsilon^*\overline{f}_\nu\|_{\mathcal{F}}^2 = \sum_{\nu=0}^{\infty} \left(\overline{f}_\nu\,,\, T_\varepsilon T_\varepsilon^*\overline{f}_\nu\right)_{L^2(\mathbb{R}^\nu_{>\varepsilon})} = \mathrm{tr}(T_\varepsilon T_\varepsilon^*).$$

Namely, T_ε is a Hilbert-Schmidt operator from $\mathcal{F}$ to $L^2(\mathbb{R}^\nu_{>\varepsilon})$. So, in the same way as in the proof of [26, Proposition 2.8], we get

$$\|N_{>\varepsilon}^{1/2}\Psi_{\rm QFT}\|^2_{\mathcal{F}} = {\rm tr}(T_\varepsilon T_\varepsilon^*) = \int_{|k|>\varepsilon} \|K_{\rm PT}(k)\|^2_{\mathcal{F}} dk.$$

Thus, our theorem follows from this equality by applying Lebesgue's monotone convergence theorem, [36, Theorem 6.12], and Lemma 2.2. □

Theorem 2.4 *Suppose that $D(H_{\rm QFT}) = D(H_0)$. If a ground state $\Psi_{\rm QFT}$ of $H_{\rm QFT}$ exists, then*

$$D(T_{\rm PT}) \supset D_{\rm CNB} = D(N^{1/2}). \tag{2.35}$$

We prove this theorem in §4. As corollaries of Theorem 2.4, we can prove Dereziński and Gérard's lemma [15, Lemma 2.6] in the weak topology and we obtain a generalization of [5, Theorem 3.4].

Corollary 2.5 *(i) Assume that $D(H_{\rm QFT}) = D(H_0)$. In addition, assume there exist a measurable function $g : \mathbb{R}^\nu \longrightarrow \mathbb{C}$ and an operator $J_{\rm err}(k)$ acting in $\mathcal{F}$ for every $k \in \mathbb{R}^\nu \setminus \{0\}$ such that*

$$B_{\rm PT}(k) = g(k)(I \otimes I) + J_{\rm err}(k) \tag{2.36}$$

for $k \in \mathbb{R}^\nu \setminus \{0\}$ with $g/\omega \notin L^2(\mathbb{R}^\nu)$. If a ground state $\Psi_{\rm QFT}$ exists, then $\omega(\cdot)^{-1}J_{\rm err}(\cdot)\Psi_{\rm QFT} \notin L^2(\mathbb{R}^\nu;\mathcal{F})$.

(ii) Assume $B_{\rm PT}(k)$ is independent of all $k \in \mathbb{R}^\nu\setminus\{0\}$, i.e., $B_{\rm PT}(k) = B_{\rm PT}$. Moreover, assume $H_{\rm QFT}$ and $B_{\rm PT}$ are strongly commutable. If a ground state $\Psi_{\rm QFT}$ exists, then $B_{\rm PT}\Psi_{\rm QFT} = 0$.

Proof: (i) We use the reduction of absurdity. So, we suppose that $\omega(\cdot)^{-1}J_{\rm err}(\cdot)\Psi_{\rm QFT} \in L^2(\mathbb{R}^\nu;\mathcal{F})$. By Theorem 2.4, $(K_{\rm PT}(\cdot)\,,\,\Phi)_{\mathcal{F}} \in L^2(\mathbb{R}^\nu)$ for all $\Phi \in D(N^{1/2})$. Since $g(\cdot)\Psi_{\rm QFT} = B_{\rm PT}(\cdot)\Psi_{\rm QFT} - J_{\rm err}(\cdot)\Psi_{\rm QFT}$, we get

$$\begin{aligned}
&|\,(\Psi_{\rm QFT}\,,\,\Phi)\,|^2 \int_{|k|>\varepsilon} \frac{|g(k)|^2}{\omega(k)^2}dk \\
\le\ & 2\int_{\mathbb{R}^\nu} |\,(K_{\rm PT}(k)\,,\,\Phi)_{\mathcal{F}}\,|^2 dk + 2\int_{\mathbb{R}^\nu} |\,\Big((\widehat{H}_{\rm QFT} + \omega(k))^{-1}J_{\rm err}(k)\Psi_{\rm QFT}\,,\,\Phi\Big)_{\mathcal{F}}\,|^2 dk \\
\le\ & 2\int_{\mathbb{R}^\nu} |\,(K_{\rm PT}(k)\,,\,\Phi)_{\mathcal{F}}\,|^2 dk + 2\|\Phi\|^2_{\mathcal{F}} \int_{\mathbb{R}^\nu} \|\omega(k)^{-1}J_{\rm err}(k)\Psi_{\rm QFT}\|^2_{\mathcal{F}} dk < \infty.
\end{aligned}$$

So, by Lebesgue's monotone convergence theorem and taking $\varepsilon \to 0$, we have

$$|\,(\Psi_{\rm QFT}\,,\,\Phi)\,|^2 \int_{\mathbb{R}^\nu} \frac{|g(k)|^2}{\omega(k)^2}dk < \infty.$$

Thus, $(\Psi_{\rm QFT}\,,\,\Phi)$ has to be zero, i.e., $(\Psi_{\rm QFT}\,,\,\Phi) = 0$. Since $D(N^{1/2})$ is dense in $\mathcal{F}$, we obtain $\Psi_{\rm QFT} = 0$, which contradicts $\Psi_{\rm QFT}$ is the ground state.

(ii) can be proven similarly to (i). □

Corollary 2.6 *(i) Assume $D(H_{\rm QFT}) = D(H_0)$ and that the decomposition (2.36) in Corollary 2.5 holds. Then, there is no ground state $\Psi_{\rm QFT}$ satisfying $\omega(\cdot)^{-1} J_{\rm err}(\cdot)\Psi_{\rm QFT} \in L^2(\mathbb{R}^\nu; \mathcal{F})$.*

(ii) Assume $B_{\rm PT}(k)$ is independent of all $k \in \mathbb{R}^\nu \setminus \{0\}$, i.e., $B_{\rm PT}(k) = B_{\rm PT}$. If $H_{\rm QFT}$ and $B_{\rm PT}$ are strongly commutable, then there is no ground state $\Psi_{\rm QFT}$ satisfying $B_{\rm PT}\Psi_{\rm QFT} \neq 0$.

Theorem 2.7 *Assume $D(H_{\rm QFT}) = D(H_0)$. In addition, assume the following conditions:*

(Ass1) *A ground state $\Psi_{\rm QFT}$ exists.*

(Ass2) *There exist a function $\lambda_{\rm IR} \in L^2(\mathbb{R}^\nu)$ and an operator $B_{\rm IR}(k)$ acting in $\mathcal{F}$ for every $k \in \mathbb{R}^\nu$ such that*

(Ass2-1) $B_{\rm PT}(k) = \lambda_{\rm IR}(k) B_{\rm IR}(k)$ on $D(B_{\rm PT}(k)) = D(B_{\rm IR}(k))$ for $k \neq 0$,

(Ass2-2) $\lambda_{\rm IR}/\omega \notin L^2(K)$ for all neighborhoods K of $k = 0$,

(Ass2-3) $B_{\rm IR}(k)\Psi \longrightarrow B_{\rm IR}(0)\Psi$ in $\mathcal{F}$ for every $\Psi \in D(H_0)$ as $k \to 0$.

If $\Phi \in D(T_{\rm PT})$ satisfies

$$\frac{1}{\omega(\cdot)}\left(\Phi\,,\,(\widehat{H}_{\rm QFT} + \omega(\cdot))^{-1}\widehat{H}_{\rm QFT} B_{\rm PT}(\cdot)\Psi_{\rm QFT}\right)_{\mathcal{F}} \in L^2(\mathbb{R}^\nu),$$

then $(\Phi\,,\,B_{\rm IR}(0)\Psi_{\rm QFT})_{\mathcal{F}} = 0$.

We prove this theorem in §5. From this theorem, we obtain the following generalization of the method in [24].

Corollary 2.8 *Assume $D(H_{\rm QFT}) = D(H_0)$ and (Ass2). Then, there is no ground state $\Psi_{\rm QFT}$ in $\mathcal{F}$ such that*

$$\frac{1}{\omega(\cdot)}\left(\Phi\,,\,(\widehat{H}_{\rm QFT} + \omega(\cdot))^{-1}\widehat{H}_{\rm QFT} B_{\rm PT}(\cdot)\Psi_{\rm QFT}\right)_{\mathcal{F}} \in L^2(\mathbb{R}^\nu), \quad \forall\Phi \in D(N^{1/2}), \tag{2.37}$$

and $B_{\rm IR}(0)\Psi_{\rm QFT} \neq 0$.

Proof: We prove our corollary by the reduction of absurdity. So, we suppose that there is a ground state satisfying (2.37) and $B_{\rm IR}(0)\Psi_{\rm QFT} \neq 0$. By Theorems 2.4 and 2.7, we get $(\Phi\,,\,B_{\rm IR}(0)\Psi_{\rm QFT})_{\mathcal{F}} = 0$ for all $\Phi \in D(N^{1/2})$. Since $D(N^{1/2})$ is dense in $\mathcal{F}$, we have $B_{\rm IR}(0)\Psi_{\rm QFT} = 0$, but it contradicts the assumption $B_{\rm IR}(0)\Psi_{\rm QFT} \neq 0$. □

Corollary 2.9 *Assume $D(H_{\rm QFT}) = D(H_0)$ and (Ass2). Then, there is no ground state $\Psi_{\rm QFT}$ in $D(T_{\rm PT})$ such that $(\Psi_{\rm QFT}\,,\,B_{\rm IR}(0)\Psi_{\rm QFT})_{\mathcal{F}} \neq 0$.*

Proof: We get easily

$$\frac{1}{\omega(\cdot)}\left(\Psi_{\rm QFT}\,,\,(\widehat{H}_{\rm QFT} + \omega(\cdot))^{-1}\widehat{H}_{\rm QFT} B_{\rm PT}(\cdot)\Psi_{\rm QFT}\right)_{\mathcal{F}} = 0.$$

Thus, corollary follows from Theorem 2.7. □

Remark 2.3 *For the Nelson model, $B_{\rm IR}(0) = I \otimes I$, so that $B_{\rm IR}(0)\Psi_{\rm QFT} \neq 0$, if a ground state $\Psi_{\rm QFT}$ exists. And moreover, by [24, 25],*

$$\frac{1}{\omega(\cdot)}(\widehat{H}_{\rm QFT} + \omega(\cdot))^{-1}\widehat{H}_{\rm QFT} B_{\rm PT}(\cdot)\Psi_{\rm QFT} \in L^2(\mathbb{R}^\nu; \mathcal{F}),$$

provided $\Psi_{\rm QFT}$ exists. So, (2.37) holds. Thus, since Corollary 2.8 works, the Nelson model has no ground state in $\mathcal{F}$.

3 An Example (GSB Model)

Let g be a bounded continuous function on $\mathbb{R}\times\mathbb{R}^\nu$, $\lambda \in L^2(\mathbb{R}^\nu)$, B a symmetric operator acting in $\mathcal{H}$, and C a self-adjoint operator acting in $\mathcal{H}$. We set

$$\rho(s,k) := g(s,k)\lambda(k), \quad \forall s \in \mathbb{R} \text{ and almost every } k \in \mathbb{R}^\nu. \tag{3.1}$$

We assume the following (A.0), (A.1), and (A.2):

(A.0) For all $(s,k) \in \mathbb{R}\times\mathbb{R}^\nu$, $|g(s,k)| \le 1$ and

$$\lim_{k\to 0} g(s,k) = 1, \quad \forall s \in \mathbb{R}.$$

(A.1) $\lambda, \lambda/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$.

(A.2) Let $\widetilde{A} := \mu\left(A - \inf\sigma(A)\right)/2$. Then, $D(\widetilde{A}^{1/2}) \subset D(B)$, and there exist constants $a_1 \ge 0$, $b_1 \ge 0$ such that for all $u \in D(\widetilde{A}^{1/2})$

$$\begin{aligned} &\|Bu\|_{\mathcal{H}} \le a_1\|\widetilde{A}^{1/2}\|_{\mathcal{H}} + b_1\|u\|_{\mathcal{H}}, \\ &|\alpha| < \left(a_1\|\lambda/\sqrt{\omega}\|_{L^2(\mathbb{R}^\nu)}\right)^{-1}. \end{aligned}$$

For all $\Psi \in \mathcal{H}\otimes\mathcal{F}_{\mathrm{b},0}$, we define $a(\rho(C,\cdot))$ by

$$\begin{aligned} &(a(\rho(C,\cdot))\Psi)^n(k_1,\cdots,k_n) \\ &:= \sqrt{n+1}\int_{\mathbb{R}^\nu} \rho(C,k)^*\Psi^{(n+1)}(k,k_1,\cdots,k_n)dk \in L^2_{\mathrm{sym}}(\mathbb{R}^{\nu n};\mathcal{H}), \\ &\qquad\qquad \forall n \in \{0\}\cup\mathbb{N}. \end{aligned} \tag{3.2}$$

Then, it is easy to check that

$$\|a(\rho(C,\cdot))\Psi\|_{\mathcal{F}} \le \|\lambda\|_{L^2(\mathbb{R}^\nu)}\|N^{1/2}\Psi\|_{\mathcal{F}}, \qquad \forall \Psi \in D(N^{1/2}). \tag{3.3}$$

We can check that for $\Psi \in \mathcal{H}\otimes\mathcal{F}_{\mathrm{b},0}$

$$(a(\rho(C,\cdot))^*\Psi)^{(0)} = 0, \tag{3.4}$$

$$\begin{aligned} &(a(\rho(C,\cdot))^*\Psi)^{(n)}(k_1,\cdots,k_n) \\ &:= \frac{1}{\sqrt{n}}\sum_{j=1}^n \rho(C,k_j)\Psi^{(n-1)}(k_1,\cdots,\cancel{k_j}\cdots,k_n)dk \in L^2_{\mathrm{sym}}(\mathbb{R}^{\nu n};\mathcal{H}), \\ &\qquad\qquad \forall n \in \mathbb{N}, \end{aligned} \tag{3.5}$$

so that $D(a(\rho(C,\cdot))^*)$ is dense in $\mathcal{F}$, which implies $a(\rho(C,\cdot))$ is closable. Thus, we denote its closure by the same symbol.

Then, for $\alpha \in \mathbb{R}\setminus\{0\}$ we define the interaction by

$$H_{\mathrm{I}} := \frac{\alpha}{\sqrt{2}}\Big\{(B\otimes I)a(\rho(C,\cdot)) + a(\rho(C,\cdot))^*(B\otimes I)\Big\}. \tag{3.6}$$

We modify the original GSB model in [3] so that it includes models such as in [11]. Then, the total Hamiltonian H of GSB model is defined by

$$H := H_0 + H_{\mathrm{I}} \tag{3.7}$$

acting in $\mathcal{F}$.

In the same way as in [3, Proposition 1.1], we obtain

Proposition 3.1 *Assume (A.0), (A.1), and (A.2). Then, H is self-adjoint on $D(H) = D(H_0)$ and bounded from below.*

We set

$$E_0(H) := \inf \sigma(H), \tag{3.8}$$

which is called the ground state energy, and

$$\widehat{H} := H - E_0(H).$$

Here we add the following additional assumptions:

(A.3) $\lambda \in C^2(\mathbb{R}^\nu \setminus \{0\})$.

(A.4) $\omega \in C^3(\mathbb{R}^\nu \setminus \{0\})$ and $\partial\omega(k)/\partial k_n \neq 0$ on $\mathbb{R}^\nu \setminus \{0\}$, $n = 1, \cdots, \nu$.

We are interested in the situation with the following condition:

(IRSC) $\lambda/\omega \notin L^2(\mathbb{R}^\nu)$,

which is called the *infrared singularity condition* in [4].

Remark 3.1 *Actually, (A.3) is a technical assumption. We can extend our results so that $\lambda = \chi_{|k|<\Lambda}$ for a fixed constant $\Lambda > 0$, a ultraviolet cutoff.*

Example 3.2 *The function $\omega(k) = |k|^p$ on $\mathbb{R}^\nu$ with a constant $p > 0$ satisfies (A.4).*

Theorem 3.3 (pull-through formula on ground states) *For all $f \in C_0^\infty(\mathbb{R}^\nu \setminus \{0\})$,*

$$a(f)\Psi_{\mathrm{QFT}} = -\frac{\alpha}{\sqrt{2}} \int_{\mathbb{R}^\nu} \overline{f(k)}\lambda(k) \left(H - E_0(H) + \omega(k)\right)^{-1} g(C,k)(B \otimes I)\Psi_{\mathrm{QFT}} dk. \tag{3.9}$$

Therefore, for H

$$B_{\mathrm{PT}}(k) = \frac{\alpha}{\sqrt{2}}\lambda(k)g(C,k)(B \otimes I). \tag{3.10}$$

The following corollary follows from this theorem, which is a revision of [5, Lemma 5.1] with a simple proof.

Corollary 3.4 *For every $f \in C_0^\infty(\mathbb{R}^\nu \setminus \{0\})$,*

$$\begin{aligned} & a(f)\Psi_{\mathrm{QFT}} \\ = \; & -\frac{\alpha}{\sqrt{2}} \int_{\mathbb{R}^\nu} \overline{f(k)}\frac{\lambda(k)}{\omega(k)} g(C,k)\left(B \otimes I\right)\Psi_{\mathrm{QFT}} dk \\ & + \frac{\alpha}{\sqrt{2}} \int_{\mathbb{R}^\nu} \overline{f(k)}\frac{\lambda(k)}{\omega(k)} \left(\widehat{H} + \omega(k)\right)^{-1} \widehat{H} g(C,k)\left(B \otimes I\right)\Psi_{\mathrm{QFT}} dk \\ = \; & -\frac{\alpha}{\sqrt{2}} \int_{\mathbb{R}^\nu} \overline{f(k)}\frac{\lambda(k)}{\omega(k)} g(C,k)\left(B \otimes I\right)\Psi_{QFT} dk \\ & + \frac{\alpha}{\sqrt{2}} \int_{\mathbb{R}^\nu} \overline{f(k)}\frac{\lambda(k)}{\omega(k)} \left(\widehat{H} + \omega(k)\right)^{-1} \widehat{H} Q_0 g(C,k)\left(B \otimes I\right)\Psi_{QFT} dk, \end{aligned} \tag{3.11}$$

where $Q_0 := 1 - P_0$ and P_0 is the orthogonal projection on $\ker\widehat{H}$.

By Proposition 3.1, we can define $m(\Psi_{\text{QFT}})$ by

$$m(\Psi_{\text{QFT}}) := (\Psi_{\text{QFT}}\, , \, (B \otimes I)\Psi_{\text{QFT}})_{\mathcal{F}}\, . \tag{3.12}$$

Then, we have the following characterization of $m(\Psi_{\text{QFT}})$.

Proposition 3.5 *Assume $g(s,k) = 1$ for all $(s,k) \in \mathbb{R} \times \mathbb{R}^\nu$. For the ground state Ψ_{QFT},*

$$\alpha m(\Psi_{\text{QFT}}) > 0 \ \textit{(resp. } < 0\textit{) iff} \qquad (\Psi_{\text{QFT}}\, , \, a(\lambda)\Psi_{\text{QFT}})_{\mathcal{F}} < 0 \quad \textit{(resp. } > 0\textit{)}. \tag{3.13}$$

Proof: The proposition follows from Corollary 3.4. □

Remark 3.2 *Let $\alpha = 1$. For the dipole-approximated Pauli-Fierz model, since we know there exists its ground state, we have $(\Psi_{\text{QFT}}\, , \, a(f)\Psi_{\text{QFT}})_{\mathcal{F}} = 0$ for every $f \in L^2(\mathbb{R})$. For the dipole-approximated Nelson model, if there exists its ground state, then $(\Psi_{\text{QFT}}\, , \, a(\lambda)\Psi_{\text{QFT}})_{\mathcal{F}} < 0$. Thus, considering $(\Psi_{\text{QFT}}\, , \, a(\lambda)\Psi_{\text{QFT}})_{\mathcal{F}}$ instead of L.H.S. of (3.13) is useful when we compare GSB model with other models.*

Theorem 3.6 *Let $\omega(k) = |k|$ for simplicity. Assume (A.1)-(A.4), (IRSC), and $g(s,k) = 1$ for all $(s,k) \in \mathbb{R} \times \mathbb{R}^\nu$. H has no ground state Ψ_{QFT} in $D(T_{\text{PT}})$ such that $(\Psi_{\text{QFT}}\, , \, a(\lambda)\Psi_{\text{QFT}})_{\mathcal{F}} \neq 0$.*

Proof: This theorem follows from Corollary 2.9. □

Theorem 3.7 *Assume (A.1)-(A.4), and (IRSC). Moreover, assume there exist $\delta > 2 - \nu/2$, $C_{\text{GSB}} > 0$, and $k_0 > 0$ such that*

$$\begin{aligned} &\| \, [H\, , \, B_{\text{PT}}(k)] \, \Psi \|_{\mathcal{F}} \leq C_{\text{GSB}} |k|^\delta \|\Psi\|_{\mathcal{F}} \\ &\qquad\qquad \text{for } \Psi \in D(HB_{\text{PT}}(k)) \cap D(B_{\text{PT}}(k)H) \text{ for } |k| < k_0. \end{aligned} \tag{3.14}$$

Then, H has no ground state Ψ_{QFT} in $\mathcal{F}$ satisfying $B_{\text{IR}}(0)\Psi_{\text{QFT}} \neq 0$.

Proof: We set

$$B_{\text{IR}}(k) = \frac{\alpha}{\sqrt{2}} g(C,k)(B \otimes I).$$

Then, (Ass2) is satisfied. Noting

$$\begin{aligned} &\frac{1}{\omega(\cdot)} \left(\Phi\, , \, (\widehat{H} + \omega(\cdot))^{-1} \widehat{H} B_{\text{PT}}(\cdot) \Psi_{\text{QFT}} \right)_{\mathcal{F}} \\ = \ &\left(\Phi\, , \, (\widehat{H} + \omega(\cdot))^{-1} \left[\widehat{H}\, , \, B_{\text{PT}}(\cdot) \right] (\widehat{H} + \omega(\cdot))^{-1} \Psi_{\text{QFT}} \right)_{\mathcal{F}} \end{aligned}$$

for every $\Phi \in D(N^{1/2})$ and $|k|^{2\delta+\nu-5}\lambda(k)^2 \leq k_0^{2\delta+\nu-4}|k|^{-1}\lambda(k)^2$ for $|k| < k_0$, we have

$$\frac{1}{\omega(\cdot)} \left(\Phi\, , \, (\widehat{H} + \omega(\cdot))^{-1} \widehat{H} B_{\text{PT}}(\cdot) \Psi_{\text{QFT}} \right)_{\mathcal{F}} \in L^2(\mathbb{R}^\nu)$$

by (3.14). By Proposition 3.5, $B_{\text{IR}}(0)\Psi_{\text{QFT}} \neq 0$. Thus, by Corollary 2.8 we obtain our theorem. □

4 Proof of Theorem 2.4

Let $\Phi \in D_{\mathrm{CNB}}$. We define a functional $F_\Phi(f) : C_0^\infty(\mathbb{R}^\nu_{>\varepsilon}) \to \mathbb{C}$ by

$$F_\Phi(f) := (a(f)\Psi_{\mathrm{QFT}}\,,\,\Phi)_{\mathcal{F}}\,, \quad \forall f \in C_0^\infty(\mathbb{R}^\nu_{>\varepsilon}).$$

Since $\Phi \in D(N_{>\varepsilon}^{1/2})$ for every $\varepsilon > 0$ and Ψ_{QFT} is normalized, we get

$$|F_\Phi(f)| \le \|f\|_{L^2(\mathbb{R}^\nu_{>\varepsilon})}\|(N_{>\varepsilon}+I)^{1/2}\Phi\|_{\mathcal{F}}.$$

Since $C_0^\infty(\mathbb{R}^\nu_{>\varepsilon})$ is dense in $L^2(\mathbb{R}^\nu_{>\varepsilon})$, we get a bounden functional from $L^2(\mathbb{R}^\nu_{>\varepsilon})$ to $\mathbb{C}$ as the extension of F_Φ. We denote it by the same symbol, i.e., $F_\Phi(f) : L^2(\mathbb{R}^\nu_{>\varepsilon}) \to \mathbb{C}$. By Riesz's lemma, there exists $u_\Phi \in L^2(\mathbb{R}^\nu_{>\varepsilon})$ such that

$$F_\Phi(f) = (u_\Phi\,,\,f)_{L^2(\mathbb{R}^\nu_{>\varepsilon})}\,, \tag{4.1}$$

$$\|u_\Phi\|_{L^2(\mathbb{R}^\nu_{>\varepsilon})} = \|F_\Phi\|_{L^2(\mathbb{R}^\nu_{>\varepsilon})^*} \le \|(N_{>\varepsilon}+I)^{1/2}\Phi\|_{\mathcal{F}}. \tag{4.2}$$

By (4.1), for $f \in C_0^\infty(\mathbb{R}^\nu_{>\varepsilon})$

$$\begin{aligned} & (f\,,\,u_\Phi)_{L^2(\mathbb{R}^\nu_{>\varepsilon})} = \overline{F_\Phi(f)} = (\Phi\,,\,a(f)\Psi_{\mathrm{QFT}})_{\mathcal{F}} \\ = & \; -\int_{|k|>\varepsilon} \overline{f(k)}\Big(\Phi\,,\,\Big(\widehat{H}_{\mathrm{QFT}}+\omega(k)\Big)^{-1}B_{\mathrm{PT}}(k)\Psi_{\mathrm{QFT}}\Big)_{\mathcal{F}}\,dk. \end{aligned}$$

Since $C_0^\infty(\mathbb{R}^\nu_{>\varepsilon})$ is dense in $L^2(\mathbb{R}^\nu_{>\varepsilon})$,

$$L^2(\mathbb{R}^\nu_{>\varepsilon}) \ni u_\Phi = -\Big(\Phi\,,\,\Big(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot)\Big)^{-1}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\Big)_{\mathcal{F}}.$$

By this and (4.2), we get

$$\begin{aligned} & \int_{|k|>\varepsilon}\Big|\Big(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\Big)_{\mathcal{F}}\Big|^2 = \|u_\Phi\|^2_{L^2(\mathbb{R}^\nu_{>\varepsilon})} \\ \le & \;\; \|(N_{>\varepsilon}+I)^{1/2}\Phi\|^2_{\mathcal{F}} \le \sup_{\varepsilon>0}\|N_{>\varepsilon}^{1/2}\Phi\|_{\mathcal{F}}^{1/2} + \|\Phi\|_{\mathcal{F}}^{1/2} < \infty, \end{aligned}$$

since $\Phi \in D_{\mathrm{CNB}}$. So, by Lebesgue's monotone convergence theorem,

$$\begin{aligned} & \int_{\mathbb{R}^\nu}\Big|\Big(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\Big)_{\mathcal{F}}\Big|^2 \\ = & \;\; \lim_{\varepsilon\to 0}\int_{|k|>\varepsilon}\Big|\Big(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\Big)_{\mathcal{F}}\Big|^2 \\ < & \;\; \sup_{\varepsilon>0}\|N_{>\varepsilon}^{1/2}\Phi\|_{\mathcal{F}}^{1/2} + \|\Phi\|_{\mathcal{F}}^{1/2} < \infty. \end{aligned}$$

Thus, $(K_{\mathrm{PT}}(\cdot)\,,\,\Phi) \in L^2(\mathbb{R}^\nu)$, which means $\Phi \in D(T_{\mathrm{PT}})$. Then, Lemma 2.2 completes our theorem.

5 Proof of Theorem 2.7

Suppose $\Phi \in D(T_{\mathrm{PT}})$ satisfies

$$\frac{1}{\omega(\cdot)}\Big(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}\widehat{H}_{\mathrm{QFT}}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\Big)_{\mathcal{F}} \in L^2(\mathbb{R}^\nu)$$

now. Since $\Phi \in D(T_{\mathrm{PT}})$, we get

$$\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}} \in L^2(\mathbb{R}^\nu)$$

by (2.29). So, we can define $F \in L^2(\mathbb{R}^\nu)$ by

$$\begin{aligned} F(\cdot) \;:=\; & \left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}} \\ & + \frac{\lambda_{\mathrm{IR}}(\cdot)}{\omega(\cdot)}\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}\widehat{H}_{\mathrm{QFT}}B_{\mathrm{IR}}(\cdot)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}}. \end{aligned}$$

In the same way as in [24, 25], for $k \neq 0$

$$\begin{aligned} & B_{\mathrm{PT}}(k)(\widehat{H}_{\mathrm{QFT}}+\omega(k))^{-1} - (\widehat{H}_{\mathrm{QFT}}+\omega(k))^{-1}B_{\mathrm{PT}}(k) \\ =\; & (\widehat{H}_{\mathrm{QFT}}+\omega(k))^{-1}\left[\widehat{H}_{\mathrm{QFT}}\,,\,B_{\mathrm{PT}}(k)\right](\widehat{H}_{\mathrm{QFT}}+\omega(k))^{-1}, \end{aligned} \tag{5.1}$$

where we note that $(\widehat{H}_{\mathrm{QFT}}+\omega(k))^{-1}\widehat{H}_{\mathrm{QFT}}$ and $(\widehat{H}_{\mathrm{QFT}}+\omega(k))^{-1}B_{\mathrm{PT}}(k)$ are bounded for $k \neq 0$ by (PT2). So, for every $\varepsilon > 0$, as an identity on $L^2(\mathbb{R}^\nu_{>\varepsilon})$, by (5.1) we get

$$F(\cdot) = \frac{\lambda_{\mathrm{IR}}(\cdot)}{\omega(\cdot)}\,(\Phi\,,\,B_{\mathrm{IR}}(\cdot)\Psi_{\mathrm{QFT}})_{\mathcal{F}}\,.$$

So, we get

$$\int_{|k|>\varepsilon}\frac{|\lambda_{\mathrm{IR}}(k)|^2}{\omega(k)^2}\left|(\Phi\,,\,B_{\mathrm{IR}}(k)\Psi_{\mathrm{IR}})_{\mathcal{F}}\right|^2 dk = \int_{|k|>\varepsilon}|F(k)|^2dk.$$

Since $F \in L^2(\mathbb{R}^\nu)$, by Lebesgue's monotone convergence theorem, we have

$$\frac{\lambda_{\mathrm{IR}}(\cdot)}{\omega(\cdot)}\,(\Phi\,,\,B_{\mathrm{IR}}(\cdot)\Psi_{\mathrm{QFT}})_{\mathcal{F}} \in L^2(\mathbb{R}^\nu). \tag{5.2}$$

Since $|\,(\Phi\,,\,B_{\mathrm{IR}}(k)\Psi_{\mathrm{QFT}})_{\mathcal{F}}\,|^2 \to |\,(\Phi\,,\,B_{\mathrm{IR}}(0)\Psi_{\mathrm{QFT}})_{\mathcal{F}}\,|^2$ as $k \to 0$ by (Ass2-3), for every $0 < \varepsilon < |\,(\Phi\,,\,B_{\mathrm{IR}}(0)\Psi_{\mathrm{QFT}})\,|^2$, there exists a positive number $\delta_\Phi(\varepsilon) > 0$ such that

$$|\,(\Phi\,,\,B_{\mathrm{IR}}(0)\Psi_{\mathrm{QFT}})_{\mathcal{F}}\,|^2 - \varepsilon \leq |\,(\Phi\,,\,B_{\mathrm{IR}}(k)\Psi_{\mathrm{QFT}})_{\mathcal{F}}\,|^2 \quad \text{for } |k| < \delta_\Phi(\varepsilon).$$

Thus, we have

$$\begin{aligned} 0 \;<\; & \int_{|k|<\delta_\Phi(\varepsilon)}\frac{|\lambda_{\mathrm{IR}}(k)|^2}{\omega(k)^2}dk\left\{\left|(\Phi\,,\,B_{\mathrm{IR}}(0)\Psi_{\mathrm{QFT}})_{\mathcal{F}}\right|^2 - \varepsilon\right\} \\ \leq\; & \int_{\mathbb{R}^\nu}\frac{|\lambda_{\mathrm{IR}}(k)|^2}{\omega(k)^2}\left|(\Phi\,,\,B_{\mathrm{IR}}(k)\Psi_{\mathrm{QFT}})_{\mathcal{F}}\right|^2 dk < \infty \end{aligned}$$

by (5.2). So, by (Ass2-2), we have $\left|(\Phi\,,\,B_{\mathrm{IR}}(0)\Psi_{\mathrm{QFT}})_{\mathcal{F}}\right|^2 = \varepsilon$, which implies $(\Phi\,,\,B_{\mathrm{IR}}(0)\Psi_{\mathrm{QFT}})_{\mathcal{F}} = 0$ taking $\varepsilon \to 0$.

6 Proofs of Theorem 3.3 and Its Corollaries

Proof of Theorem 3.3: Here we prove Theorem 3.3 with the same way as in [25]. By [5, (5.7)], we have

$$I \otimes a(f)\Psi_{\text{QFT}} = i\frac{\alpha}{\sqrt{2}}\int_0^\infty \left(\int_{\mathbb{R}^\nu} \overline{f(k)} e^{it\omega(k)}\lambda(k)e^{it\widehat{H}}g(C,k)\,(B\otimes I)\,\Psi_{\text{QFT}}dk\right)dt. \tag{6.1}$$

For $\Psi \in D(H)$, by using Fubini's theorem and Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
&(\Psi\,,\,(I\otimes a(f))\,\Psi_{\text{QFT}})_{\mathcal{F}} \\
=\;& \left(\Psi\,,\,i\frac{\alpha}{\sqrt{2}}\int_{\mathbb{R}^\nu}\overline{f(k)}\left(\int_0^\infty e^{it\left(\widehat{H}+\omega(k)\right)}\lambda(k)g(C,k)(B\otimes I)\Psi_{\text{QFT}}dt\right)dk\right)_{\mathcal{F}}.
\end{aligned} \tag{6.2}$$

Moreover, for every $k \in \mathbb{R}^\nu \setminus \{0\}$

$$\begin{aligned}
&\int_0^\infty e^{it(\widehat{H}+\omega(k))}\lambda(k)g(C,k)(B\otimes I)\Psi_{\text{QFT}}dt \\
=\;& i\left(\widehat{H}+\omega(k)\right)^{-1}\lambda(k)g(C,k)(B\otimes I)\Psi_{\text{QFT}}.
\end{aligned} \tag{6.3}$$

Then, (3.9) follows from (6.2) and (6.3).

Proof of Corollary 3.4: We have only to use the first step to derive the decomposition in [23, (17)] (see also [24, 25]). Namely, since $\widehat{H}(\widehat{H}+\omega(k))^{-1}$ and $(B\otimes I)(\widehat{H}+\omega(k))^{-1}$ are bounded, we have

$$\begin{aligned}
&(B\otimes I)\left(\widehat{H}+\omega(k)\right)^{-1} - \left(\widehat{H}+\omega(k)\right)^{-1}(B\otimes I) \\
=\;& \left(\widehat{H}+\omega(k)\right)^{-1}\left[\left(\widehat{H}+\omega(k)\right)\,,\,(B\otimes I)\right]\left(\widehat{H}+\omega(k)\right)^{-1} \\
=\;& \left(\widehat{H}+\omega(k)\right)^{-1}[H\,,\,(B\otimes I)]\left(\widehat{H}+\omega(k)\right)^{-1},
\end{aligned}$$

on $D(H_0)$, (3.11) follows from this equality and (3.9).

Proof of Proposition 3.5: Since $g(s,k) = 1$ for $(s,k) \in \mathbb{R}\times\mathbb{R}^\nu$, $g(C,k) = I$ for all $k \in \mathbb{R}^\nu$. By Corollary 3.4, for $f \in C_0^\infty(\mathbb{R}^\nu \setminus \{0\})$ we have

$$(\Psi_{\text{QFT}}\,,\,a(\lambda)\Psi_{\text{QFT}})_{\mathcal{F}} \;=\; -\frac{\alpha}{\sqrt{2}}\int_{\mathbb{R}^\nu}\frac{|\lambda(k)|^2}{\omega(k)}(\Psi_{\text{QFT}}\,,\,(B\otimes I)\Psi_{\text{QFT}}).$$

Proposition 3.5. follows from this equality and (A.0).

7 Appendix: L^2-Theoretical Method

In this appendix, we show a proof of infrared catastrophe for the Hamiltonian H of GSB model defined by (3.7) in strong topology, which is direct application of [15, Lemma 2.6].

When $\Psi_{\text{QFT}} \in D(B\otimes I)$, we set

$$K(k) := \frac{\alpha\lambda(k)}{\sqrt{2}\omega(k)}\left(\widehat{H}+\omega(k)\right)^{-1}\widehat{H}\,(B\otimes I)\,\Psi_{\text{QFT}},\quad \forall k\in\mathbb{R}^\nu\setminus\{0\}.$$

We prove the following theorem:

Theorem 7.1 *Assume (A.1)-(A.4), (IRSC), and $g(s,k) = 1$ for all $(s,k) \in \mathbb{R} \times \mathbb{R}^\nu$. H has no ground state $\Psi_{\rm QFT}$ in $\mathcal{F}$ such that $K \in L^2(\mathbb{R}^\nu; \mathcal{F})$ and $(\Psi_{\rm QFT}\,, a(\lambda)\Psi_{\rm QFT})_{\mathcal{F}} \neq 0$.*

To prove this theorem, it is convenient to introduce the kernel $a(k)$ of the annihilation operator. Fix $k \in \mathbb{R}^\nu$ arbitrarily. Then, the kernel $a(k)$ of the annihilation operator is given by

$$\left(a(k)\psi\right)^{(n)}(k_1, \cdots, k_n) := \sqrt{n+1}\psi^{(n+1)}(k, k_1, \cdots, k_n) \tag{7.1}$$

for $n = 0, 1, 2, \cdots$ and $\psi \in D_{\mathcal{S}} := \left\{\psi = \oplus\sum_{n=0}^\infty \psi^{(n)} \in \mathcal{F}_{\rm b,0} \,|\, \psi^{(n)} \in \mathcal{S}(\mathbb{R}^\nu); n \in \mathbb{N}\right\}$, where $\mathcal{S}(\mathbb{R}^\nu)$ is the set of all functions in the Schwartz class. It is well known that $a(k)^*$ is not densely defined [32, Sect. X.7]; indeed, $a(k)^*$ is trivial [2, Proposition 8.2], i.e., $D(a(k)^*) = \{0\}$, so that $a(k)$ is *not closable* by [31, Theorem VIII.1(b)].

In order to extend the domain of $a(k)$, we introduce the facts we use in this paper following Gross' work [21] and Sloan's[33, 34].

Lemma 7.2 *Fix $\varepsilon \geq 0$ and $\psi \in D(N(\varepsilon)^{1/2})$ arbitrarily. Then,*

$$\left(a(k)\psi\right)^{(n)} = \sqrt{n+1}\psi^{(n+1)}(k, \underbrace{\cdot, \cdots, \cdot}_{n}) \in \otimes_{\rm s}^n L^2(\mathbb{R}^\nu), \quad \text{a. e. } k \in \mathbb{R}^\nu_{>\varepsilon};\, n \in \{0\} \cup \mathbb{N}, \tag{7.2}$$

and moreover,

$$a(\cdot)\psi = \oplus\sum_{n=0}^\infty \left(a(\cdot)\psi\right)^n \in L^2(\mathbb{R}^\nu_{>\varepsilon}; \mathcal{F}_{\rm b}), \tag{7.3}$$

$$\|N_{>\varepsilon}^{1/2}\psi\|^2_{\mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{>\varepsilon}))} = \int_{|k|>\varepsilon} \|a(k)\psi\|^2_{\mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{>\varepsilon}))} dk. \tag{7.4}$$

Proof: We fix $\varepsilon \geq 0$ and $\psi \in D(N(\varepsilon)^{1/2})$ arbitrarily. Since $D_{\mathcal{S}}$ is a core for $N(\varepsilon)^{1/2}$, there exists a sequence $\{\psi_\nu\}_{\nu=1}^\infty \subset D_{\mathcal{S}}$ such that $\psi_\nu \to \psi$ and $N(\varepsilon)^{1/2}\psi_\nu \to N(\varepsilon)^{1/2}\psi$ as $\nu \to \infty$. So, since $\psi_\nu, \psi_{\nu'} \in D_{\mathcal{S}}$, we get

$$\int_{|k|>\varepsilon} \|a(k)(\psi_\nu - \psi_{\nu'})\|^2_{\mathcal{F}_{\rm b}} dk = \|N(\varepsilon)^{1/2}(\psi_\nu - \psi_{\nu'})\|^2_{\mathcal{F}_{\rm b}} \to 0$$

as $\nu, \nu' \to \infty$, which implies

$$a(\cdot)\psi := \exists \text{ s - }\lim_{\nu\to\infty} a(\cdot)\psi_\nu \in L^2(\mathbb{R}^\nu_{>\varepsilon}; \mathcal{F}_{\rm b}). \tag{7.5}$$

So, we have

$$\begin{aligned}
&\|N_{>\varepsilon}^{1/2}\psi_\nu\|^2_{\mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{>\varepsilon}))} = \int_{|k|>\varepsilon} \|a(k)\psi_\nu\|^2_{\mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{>\varepsilon}))} dk \\
\to\ &\|N_{>\varepsilon}^{1/2}\psi\|^2_{\mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{>\varepsilon}))} = \int_{|k|>\varepsilon} \|a(k)\psi\|^2_{\mathcal{F}_{\rm b}(L^2(\mathbb{R}^\nu_{>\varepsilon}))} dk
\end{aligned}$$

as $\nu \to \infty$. For every $f \in L^2(\mathbb{R}^\nu_{>\varepsilon})$, we get

$$\|a(f)(\psi_\nu - \psi)\|_{\mathcal{F}_{\rm f}} \leq \|f\|_{L^2(\mathbb{R}^\nu_{>\varepsilon})} \|N(\varepsilon)^{1/2}(\psi_\nu - \psi)\|^2_{\mathcal{F}_{\rm b}}\| \to 0$$

as $\nu \to \infty$, which implies

$$\text{s - }\lim_{\nu\to\infty} a(f)\psi_\nu = a(f)\psi, \qquad \forall f \in L^2(\mathbb{R}^\nu_{>\varepsilon}; \mathcal{F}_{\rm b}). \tag{7.6}$$

By (7.5) and (7.6), for every $f \in L^2(\mathbb{R}^\nu_{>\varepsilon})$

$$a(f)\psi = \mathrm{s}\text{-}\lim_{\nu\to\infty} a(f)\psi_\nu = \lim_{\nu\to\infty}\int_{|k|>\varepsilon}\overline{f(k)}a(k)\psi_\nu dk = \int_{|k|>\varepsilon}\overline{f(k)}a(k)\psi dk,$$

while

$$a(f)\psi = \oplus\sum_{n=0}^{\infty}\sqrt{n+1}\int_{|k|>\varepsilon}\overline{f(k)}\psi^{(n+1)}(k, \underbrace{\cdot,\cdots,\cdot}_{n}).$$

Therefore, we have our lemma. $\square$

Proof of Theorem 7.1: We use the reduction of absurdity. So, suppose there is a ground state Ψ_{QFT} such that $K \in L^2(\mathbb{R}^\nu;\mathcal{F})$ and $(\Psi_{\mathrm{QFT}}, a(\lambda)\Psi_{\mathrm{QFT}})_{\mathcal{F}} \neq 0$. Then, by Lemmas 2.1, 7.2, and Corollary 3.4, we have

$$K(\cdot) = \left\{a(\cdot) + \frac{\alpha\lambda(\cdot)}{\sqrt{2}\omega(\cdot)}(B\otimes I)\right\}\Psi_{\mathrm{QFT}} \in L^2(\mathbb{R}^\nu_{>\varepsilon};\mathcal{F}), \quad \forall\varepsilon>0.$$

Since $K(\cdot) \in L^2(\mathbb{R}^\nu;\mathcal{F}) = L^2(\mathbb{R}^\nu_{>0};\mathcal{F})$, by taking $\varepsilon\to 0$ with Lebesgue's monotone convergence theorem,

$$\left\{a(\cdot) + \frac{\alpha\lambda(\cdot)}{\sqrt{2}\omega(\cdot)}(B\otimes I)\right\}\Psi_{\mathrm{QFT}} \in L^2(\mathbb{R}^\nu;\mathcal{F}). \tag{7.7}$$

For $\Psi \in \mathcal{H}\otimes\mathcal{F}_{\mathrm{b},0}$, we set

$$N(\Psi) := \max\left\{n \in \{0\}\cup\mathbb{N}\,\middle|\, \Psi^{(n)} \neq 0\right\} \tag{7.8}$$

and define $G(\cdot) : \mathbb{R}^\nu \longrightarrow \mathcal{F}$ by

$$G(k)\Psi := -\oplus\sum_{n=0}^{N(\Psi)-1}\frac{\alpha\lambda(k)}{\sqrt{2}\omega(k)}(B\otimes I)\Psi^{(n)}, \quad k\neq 0. \tag{7.9}$$

For Ψ_{QFT} and each $\nu\in\mathbb{N}$, we set

$$\Psi_{\mathrm{QFT}}(\nu) := \oplus\sum_{n=0}^{\nu}{\Psi_{\mathrm{QFT}}}^{(n)} \in D(N^{1/2}). \tag{7.10}$$

By (7.7), we get

$$\{a(\cdot) - G(\cdot)\}\Psi_{\mathrm{QFT}}(\nu) \in L^2(\mathbb{R}^\nu;\mathcal{F}). \tag{7.11}$$

Since $\Psi_{\mathrm{QFT}}(\nu) \in D(N^{1/2})$, by Lemma 7.2 we get

$$a(\cdot)\Psi_{\mathrm{QFT}}(\nu) \in L^2(\mathbb{R}^\nu;\mathcal{F}). \tag{7.12}$$

Thus, since we have $m(\Psi_{\mathrm{QFT}}) \neq 0$ by Proposition 3.5, (7.11) and (7.12) imply

$$L^2(\mathbb{R}^\nu;\mathcal{F}) \ni a(\cdot)\Psi_{\mathrm{QFT}}(\nu) - \{a(\cdot)-G(\cdot)\}\Psi_{\mathrm{QFT}}(\nu) = G(\cdot)\Psi_{\mathrm{QFT}}(\nu) \notin L^2(\mathbb{R}^\nu;\mathcal{F}),$$

provided that $\Psi_{\mathrm{QFT}}(\nu) \neq 0$. Therefore, $\Psi_{\mathrm{QFT}}(\nu)$ has to be the zero, i.e., $\Psi_{\mathrm{QFT}}(\nu) = 0$. This means

$$\Psi_{\mathrm{QFT}} = \mathrm{s}\text{-}\lim_{\nu\to\infty}\Psi_{\mathrm{QFT}}(\nu) = 0,$$

which contradicts that Ψ_{QFT} is a ground state.

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