# ON THE RELLICH INEQUALITY WITH MAGNETIC POTENTIALS

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## 1. Introduction

Rellich's inequality is that

$$\frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n}\frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^4}d\mathbf{x}\leq\int_{\mathbb{R}^n}|\Delta f(\mathbf{x})|^2d\mathbf{x} \tag{1.1}$$

for all $f\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ and $n\neq 2$, where the constant $n^2(n-4)^2/16$ is sharp. It first appeared in print in [9] but it had been given in 1953 in lectures at New York University which were published posthumously in [10]. In [2] an extension of Rellich's inequality to higher-order derivatives was derived through the application of a technique introduced in [8]. This technique, and Bennett's development, were the basis of the method used by Davies and Hinz in [4] in their comprehensive study of Rellich-type inequalities and their higher-order counterparts in $L^p(\Omega)$, for any $p\in[1,\infty)$ and $\Omega$ a complete Riemannian manifold with smooth boundary. The papers [2] and [4] were influenced by the work of Schmincke in [12]. In particular, Davies and Hinz obtain sharp constants $C$ for inequalities of the form

$$\int_{\mathbb{R}^n}\frac{|f(\mathbf{x})|^p}{|\mathbf{x}|^\beta}d\mathbf{x}\leq C\int_{\mathbb{R}^n}\frac{|\Delta^m f(\mathbf{x})|^p}{|\mathbf{x}|^\alpha}d\mathbf{x} \tag{1.2}$$

for suitable values of $\alpha,\beta,m,n,p$ and $f\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$.

In Rellich's original inequality there are two cases of special interest, and these have their counterparts in the work of Davies and Hinz. Firstly, when $n=2$ the inequality still holds but only for functions $f\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ which also satisfy

$$\int_0^{2\pi}f(r,\theta)\cos\theta d\theta=\int_0^{2\pi}f(r,\theta)\sin\theta d\theta=0.$$

Secondly, when $n=4$ there is no non-trivial inequality. Our primary concern will be with these special cases, but our general results have

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wider implications. Our approach has been influenced by the Laptev-Weidl inequality in [7] in which, *inter alia*, the Hardy-type inequality

$$\int_{\mathbb{R}^2} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^2} d\mathbf{x} \leq C \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} f(\mathbf{x})|^2 d\mathbf{x} \tag{1.3}$$

is established for $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, where $\nabla_{\mathbf{A}}$ is the *magnetic gradient*

$$\nabla_{\mathbf{A}} = \nabla - i\mathbf{A},$$

$\mathbf{A}$ is of the form (in polar co-ordinates)

$$\mathbf{A}(r,\theta) = \frac{\Psi(\theta)}{r}(-\sin\theta, \cos\theta), \quad \Psi \in L^\infty(0, 2\pi) \tag{1.4}$$

and the magnetic flux $\tilde{\Psi} := \frac{1}{2\pi}\int_0^{2\pi} \Psi(\theta) d\theta \notin \mathbb{Z}$. The magnetic field curl$\mathbf{A} = 0$ in $\mathbb{R}^2 \setminus \{0\}$ and is of Aharonov-Bohm type. The optimal constant in (1.3) is $C = \text{dist}(\tilde{\Psi}, \mathbb{Z})^{-2}$. Recall that when there is no magnetic potential $\mathbf{A}$ (or, equivalently, the magnetic flux $\tilde{\Psi} \in \mathbb{Z}$ by gauge invariance) there is no valid Hardy inequality. A method to establish an $L^p$-version of (1.3) is developed in [3] and this has also influenced the present work.

The $L^2$-version of our result is of the form

$$C \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^{\alpha+4}} d\mathbf{x} \leq \int_{\mathbb{R}^n} \frac{|\Delta_{\mathbf{A}} f(\mathbf{x})|^2}{|\mathbf{x}|^\alpha} d\mathbf{x}, \tag{1.5}$$

where $\Delta_{\mathbf{A}} = \nabla_{\mathbf{A}} \cdot \nabla_{\mathbf{A}}$ and $\mathbf{A}$ is a vector field in Poincaré (or transverse) gauge (i.e. $\mathbf{A}(\mathbf{x}) \cdot \mathbf{x} = 0$) which is an exact 1-form in the simply-connected domain $\mathbb{R}^n \setminus \mathcal{L}_n$, where in appropriate polar co-ordinates $\mathbf{x} = (r, \theta_1, \cdots, \theta_{n-1})$, $\mathcal{L}_n$ is the co-ordinate axis

$$\mathcal{L}_n = \{\mathbf{x} : r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} = 0\}.$$

As in the Laptev-Weidl inequality, our constant depends on the distance of the magnetic flux $\tilde{\Psi}$ to the set of integers $\mathbb{Z}$.

The $L^p$-version of our result takes a different form which emphasises the dependence of the inequalities on the behaviour of the functions $f$ in a transverse direction. For instance, when $n = 2$, we prove that

$$\|f/|\cdot|^2\|_{L^p(\mathbb{R}^2)} \leq B_p \|\frac{1}{r^2}\left(i\frac{\partial}{\partial\theta} + \Psi(\theta)\right) f\|_{L^p(\mathbb{R}^2)}, \tag{1.6}$$

where $B_p \leq C_p/d^2(\cos(\pi d))^{1/p'}$, $d := \text{dist}(\tilde{\Psi}, \mathbb{Z}) \neq 0, 1/2$ and there are similar inequalities for $n \geq 3$. Such an inequality clearly does not hold when $\Psi = 0$ since radial functions are excluded. When $p = 2$ there is a constant $C$ depending on $d$ such that

$$\|\frac{1}{r^2}\left(i\frac{\partial}{\partial\theta} + \Psi(\theta)\right) f\|_{L^2(\mathbb{R}^2)} \leq C\|\Delta_{\mathbf{A}}^2 f\|_{L^2(\mathbb{R}^2)}. \tag{1.7}$$

We also investigate an analogue of (1.7) for $n = 3$.

## 2. The main result in $L^2(\mathbb{R}^n)$.

Throughout this paper, polar coordinates in $\mathbb{R}^n$ will be denoted by $(r,\omega)$ with $r := |\mathbf{x}|$, $\omega = \mathbf{x}/|\mathbf{x}|$ for $\mathbf{x} \in \mathbb{R}^n$. The unit ball in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$ and $\mathbb{R}_+ := (0,\infty)$. We shall also denote the $L^2(\mathbb{R}^n)$ norm by $\|\cdot\|$.

**Theorem 1.** *Let $\Lambda_\omega$ be a non-negative self-adjoint operator in $L^2(\mathbb{S}^{n-1};d\omega)$ whose spectrum is discrete and consists of eigenvalues $\lambda_m, m \in \mathcal{I}$, where $\mathcal{I}$ is a countable index set. Let*

$$L_r := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} \tag{2.1}$$

*and define the operator $D := L_r + \frac{1}{r^2}\Lambda_\omega$ on the domain*

$$\begin{aligned}\mathcal{D}_0 := \{f : & f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}), f(r,\cdot) \in \mathcal{D}(\Lambda_\omega) \\ & \text{for } 0 < r < \infty, Df \in L^2(\mathbb{R}^n)\}\end{aligned} \tag{2.2}$$

*in $L^2(\mathbb{R}^n)$, where $\mathcal{D}(\Lambda_\omega)$ denotes the domain of $\Lambda_\omega$. Then, for all $f \in \mathcal{D}_0$ and $\alpha \geq -1$ such that $|\cdot|^{-\alpha/2}Df \in L^2(\mathbb{R}^n)$, we have that $|\cdot|^{-2-\alpha/2}f \in L^2(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} \frac{|Df(\mathbf{x})|^2}{|\mathbf{x}|^\alpha} d\mathbf{x} \geq C(n,\alpha) \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^{\alpha+4}} d\mathbf{x} \tag{2.3}$$

*where*

$$C(n,\alpha) = \inf_{m\in\mathcal{I}} \big\{\lambda_m + \big(\frac{n+\alpha}{2}\big)\big(\frac{n-\alpha-4}{2}\big)\big\}^2. \tag{2.4}$$

*Proof.* Since the spectrum of $\Lambda_\omega$ is assumed to be discrete, its normalised eigenvectors $u_m, m \in \mathcal{I}$ form an orthonormal basis of $L^2(\mathbb{S}^{n-1};d\omega)$: we assume that the eigenvalues are repeated according to multiplicity. For $f \in \mathcal{D}_0$, set

$$F_m(r) := \int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{u_m(\omega)}d\omega. \tag{2.5}$$

Then $F_m \in C_0^\infty(\mathbb{R}_+)$ and on using Parseval's identity we obtain

$$\begin{aligned}\sum_{m\in\mathcal{I}} \|L_r F_m\|^2_{L^2(\mathbb{R}_+,r^{n-1}dr)} &= \int_0^\infty \sum_{m\in\mathcal{I}} |L_r F_m(r)|^2 r^{n-1} dr \\ &= \|L_r f\|^2 < \infty,\end{aligned} \tag{2.6}$$

with

$$L_r F_m(r) = \int_{\mathbb{S}^{n-1}} L_r f(r,\omega)\overline{u_m(\omega)}d\omega.$$

Also, $L_r f, Df \in L^2(\mathbb{R}^n)$ imply that $|\cdot|^{-2}\Lambda_\omega f \in L^2(\mathbb{R}^n)$ and

$$\begin{aligned}\sum_{m\in\mathcal{I}} \lambda_m^2 \int_0^\infty |F_m(r)|^2 r^{n-5} dr &= \int_0^\infty \sum_{m\in\mathcal{I}} \lambda_m^2 |F_m(r)|^2 r^{n-5} dr \\ &= \||\cdot|^{-2}\Lambda_\omega f\|^2 < \infty.\end{aligned} \tag{2.7}$$

In fact if $|\cdot|^{-\alpha/2}Df \in L^2(\mathbb{R}^n)$ then $|\cdot|^{-2-\alpha/2}\Lambda_\omega f \in L^2(\mathbb{R}^n)$ and

$$\sum_{m\in\mathcal{I}}\int_0^\infty |F_m(r)|^2 r^{n-\alpha-5}dr = \||\cdot|^{-2-\alpha/2}f\|^2 < \infty, \tag{2.8}$$

$$\sum_{m\in\mathcal{I}}\int_0^\infty |L_rF_m(r)|^2 r^{n-\alpha-1}dr = \||\cdot|^{-\alpha/2}L_rf\|^2 < \infty, \tag{2.9}$$

$$\sum_{m\in\mathcal{I}}\lambda_m^2\int_0^\infty |F_m(r)|^2 r^{n-\alpha-5}dr = \||\cdot|^{-2-\alpha/2}\Lambda_\omega f\|^2 < \infty. \tag{2.10}$$

To prove the theorem we start with

$$\int_{\mathbb{R}^n}|Df|^2\frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} = \int_{\mathbb{R}^n}|L_rf|^2\frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} + 2\Re e[\int_{\mathbb{R}^n}L_rf\overline{\Lambda_\omega f}\frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+2}}] + \int_{\mathbb{R}^n}|\Lambda_\omega f|^2\frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}}. \tag{2.11}$$

On integration by parts and using the Hardy inequality (see [5], II(9))

$$\int_0^\infty |F'(r)|^2 r^{t+2}dr \geq \left(\frac{t+1}{2}\right)^2\int_0^\infty |F(r)|^2 r^t dr, \tag{2.12}$$

we obtain

$$\begin{aligned}
&\int_0^\infty |L_rF_m(r)|^2 r^{n-\alpha-1}dr\\
&=\int_0^\infty (|F_m''(r)|^2 + \frac{2(n-1)}{r}\Re e[F_m''\overline{F_m'}]\\
&\quad + \frac{(n-1)^2}{r^2}|F_m'(r)|^2)r^{n-\alpha-1}dr\\
&\geq(\frac{n-\alpha-2}{2})^2\int_0^\infty |F_m'(r)|^2 r^{n-\alpha-3}dr\\
&\quad - (n-1)(n-\alpha-2)\int_0^\infty |F_m'(r)|^2 r^{n-\alpha-3}dr\\
&\quad + (n-1)^2\int_0^\infty |F_m'(r)|^2 r^{n-\alpha-3}dr\\
&=[(n-1)(\alpha+1) + (\frac{n-\alpha-2}{2})^2]\int_0^\infty |F_m'(r)|^2 r^{n-\alpha-3}dr\\
&\geq[(n-1)(\alpha+1) + (\frac{n-\alpha-2}{2})^2](\frac{n-\alpha-4}{2})^2\\
&\quad \times\int_0^\infty |F_m(r)|^2 r^{n-\alpha-5}dr.
\end{aligned}$$

Thus, from (2.8) and (2.9)

$$\int_{\mathbb{R}^n} \frac{|L_r f|^2}{|\mathbf{x}|^\alpha} d\mathbf{x} \geq \big\{(n-1)(\alpha+1) + \big(\frac{n-\alpha-2}{2}\big)^2\big\}\big(\frac{n-\alpha-4}{2}\big)^2 \times \int_{\mathbb{R}^n} \frac{|f|^2}{|\mathbf{x}|^{\alpha+4}} d\mathbf{x}. \tag{2.13}$$

Since

$$\int_{\mathbb{S}^{n-1}} L_r f(r,\omega)\overline{u_m(\omega)} d\omega = L_r F_m(r)$$

and

$$\int_{\mathbb{S}^{n-1}} \Lambda_\omega f(r,\omega)\overline{u_m(\omega)} d\omega = \lambda_m F_m(r),$$

it follows from Parseval's identity that in (2.11)

$$\int_{\mathbb{R}^n} \frac{L_r f\overline{\Lambda_\omega f}}{|\mathbf{x}|^{\alpha+2}} d\mathbf{x} = \int_0^\infty \sum_{m\in\mathcal{I}} \lambda_m \overline{F_m(r)} L_r F_m(r) r^{n-\alpha-3} dr. \tag{2.14}$$

Integration by parts yields

$$\begin{aligned}
&2\Re e\big[\int_0^\infty \overline{F_m(r)} L_r F_m(r) r^{n-\alpha-3} dr\big] \\
&= 2\Re e\big[\int_0^\infty F_m'\big(\overline{F_m'} r^{n-\alpha-3} + (n-\alpha-3)\overline{F_m} r^{n-\alpha-4}\big) dr\big] \\
&\quad + (n-1)(n-\alpha-4)\int_0^\infty |F_m|^2 r^{n-\alpha-5} dr \\
&= 2\int_0^\infty |F_m'|^2 r^{n-\alpha-3} dr \\
&\quad + \big\{-(n-\alpha-3)+(n-1)\big\}(n-\alpha-4)\big\} \\
&\quad \times \int_0^\infty |F_m|^2 r^{n-\alpha-5} dr \\
&\geq \big\{2\big(\frac{n-\alpha-4}{2}\big)^2 + (n-\alpha-4)(\alpha+2)\big\}\int_0^\infty |F_m|^2 r^{n-\alpha-5} dr
\end{aligned}$$

(by(2.12)),

$$= \frac{1}{2}(n-\alpha-4)(n+\alpha)\int_0^\infty |F_m|^2 r^{n-\alpha-5} dr.$$

This gives in (2.14)

$$2\Re e\big[\int_{\mathbb{R}^n} \frac{L_r f\overline{\Lambda_\omega f}}{|\mathbf{x}|^{\alpha+2}} d\mathbf{x}\big] \geq \frac{1}{2}(n-\alpha-4)(n+\alpha) \times \sum_{m\in\mathcal{I}} \lambda_m \int_0^\infty |F_m(r)|^2 r^{n-\alpha-5} dr. \tag{2.15}$$

Finally in (2.11), by (2.9),

$$\int_{\mathbb{R}^n} \frac{|\Lambda_\omega f|^2}{|\mathbf{x}|^{\alpha+4}} d\mathbf{x} = \sum_{m\in\mathcal{I}} \lambda_m^2 \int_0^\infty |F_m(r)|^2 r^{n-\alpha-5} dr. \tag{2.16}$$

The theorem follows on substituting (2.13), (2.15) and (2.16) in (2.11) and noting (2.8). □

**Corollary 1.** *Suppose that $n \geq 4$ or, when $n \leq 3$, that $C(n,0) > 0$. Then, there exists a positive constant $C(n)$ such that in the notation of Theorem 1*

$$\|| \cdot |^{-2} \Lambda_\omega f\| \leq C(n) \|Df\|. \tag{2.17}$$

*Proof.* From (2.13) and (2.15) with $\alpha = 0$, we have

$$\int_{\mathbb{R}^n} |L_r f|^2 d\mathbf{x} \geq \big[\frac{n(n-4)}{4}\big]^2 \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^4} d\mathbf{x}$$

and

$$\begin{aligned} 2\Re e\big[\int_{\mathbb{R}^n} L_r f \overline{\Lambda_\omega f} \frac{d\mathbf{x}}{|\mathbf{x}|^2}\big] &\geq \frac{1}{2} n(n-4) \int_{\mathbb{R}^n} \big(|\mathbf{x}|^{-2}\Lambda_\omega f\big)\big(|\mathbf{x}|^{-2}\overline{f}\big) d\mathbf{x} \\ &\geq -\frac{1}{4} n[n-4]_- \{\epsilon \|| \cdot |^{-2}\Lambda_\omega f\|^2 + \frac{1}{\epsilon} \|| \cdot |^{-2} f\|^2\}, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary and $[a]_- := \max\{0, -a\}$. Hence, from (2.11) with $\alpha = 0$,

$$\begin{aligned} \|Df\|^2 \geq & \frac{1}{4}[\frac{n(n-4)}{4}]^2 \|| \cdot |^{-2} f\|^2 \\ & - \frac{1}{4} n[n-4]_- \{\epsilon \|| \cdot |^{-2}\Lambda_\omega f\|^2 + \frac{1}{\epsilon} \|| \cdot |^{-2} f\|^2\} + \|| \cdot |^{-2}\Lambda_\omega f\|^2 \\ = & \{\frac{1}{4}[\frac{n(n-4)}{4}]^2 - \frac{1}{4\epsilon} n[n-4]_-\} \|| \cdot |^{-2} f\|^2 \\ & + \{1 - \frac{\epsilon}{4} n[n-4]_-\} \|| \cdot |^{-2}\Lambda_\omega f\|^2. \end{aligned}$$

When $n \geq 4$, this gives

$$\|Df\|^2 \geq \ \tfrac{1}{4}[\tfrac{n(n-4)}{4}]^2 \|| \cdot |^{-2} f\|^2 + \|| \cdot |^{-2}\Lambda_\omega f\|^2$$

whence (2.11) with $C(n) = 1$. For $n \leq 3$, we set $\epsilon = \frac{4}{n[n-4]_-}\delta$, $\delta \in (0,1)$, and use Theorem 1 to get

$$\begin{aligned} (1-\delta)\|| \cdot |^{-2}\Lambda_\omega f\|^2 \leq & \ \|Df\|^2 + [\tfrac{n(n-4)}{4}]^2 (\tfrac{1}{\delta} - \tfrac{1}{4}) \|| \cdot |^{-2} f\|^2 \\ \leq & \ \{1 + [\tfrac{n(n-4)}{4}]^2 C(n,0)^{-1} (\tfrac{1}{\delta} - \tfrac{1}{4})\} \|Df\|^2 \end{aligned}$$

which again yields (2.17). □

**Remark 1.** *There is a valid inequality (2.3) (i.e. $C(n,\alpha) > 0$) for all $n > \alpha + 4$.*

**Remark 2.** *It follows from the proof of Theorem 1 that for $f \in \mathcal{D}_0$,*

$$Df(r,\omega) = \sum_{m\in\mathcal{I}} \big(L_r + r^{-2}\lambda_m\big)F_m(r)u_m(\omega)$$

*in the $L^2(\mathbb{S}^{n-1})$ sense and $F_m \in C_0^\infty(\mathbb{R}_+)$. Hence, on identifying $L^2(\mathbb{R}^n)$ with $L^2(\mathbb{R}_+; r^{n-1}dr) \bigotimes L^2(\mathbb{S}^{n-1}; d\omega)$, we have that*

$$\mathcal{D}_0 = \bigoplus_{m\in\mathcal{I}} \big\{C_0^\infty(\mathbb{R}_+) \bigotimes \{u_m\}\big\}$$

*and*

$$D = \bigoplus_{m\in\mathcal{I}} \big\{\big(L_r + \lambda_m r^{-2}\big) \bigotimes I_m,$$

*where $\bigoplus$ denotes orthogonal sum and $I_m$ is the identity on $\{u_m\}$.*

*The map $J : F \mapsto r^{(n-1)/2}F : L^2(\mathbb{R}_+; r^{n-1}dr) \to L^2(\mathbb{R}_+)$is an isometry and if $D_m := L_r + \lambda_m r^{-2}$ on $C_0^\infty(\mathbb{R}_+)$, then $D_m = J^{-1}T_mJ$, where $T_m$ also has domain $C_0^\infty(\mathbb{R}_+)$ and*

$$T_m g = -\frac{d^2g}{dr^2} + r^{-2}\big[\lambda_m + \frac{(n-1)(n-3)}{4}\big]g.$$

*By Hardy's inequality, for $g \in C_0^\infty(\mathbb{R}_+)$,*

$$-\int_0^\infty \overline{g}g''dr = \int_0^\infty |g'|^2dr \geq \frac{1}{4}\int_0^\infty r^{-2}|g|^2dr$$

*and so*

$$\int_0^\infty \overline{g}T_m g dr \geq \big[\lambda_m + \big(\frac{n-2}{2}\big)^2\big]\frac{1}{4}\int_0^\infty r^{-2}|g|^2dr.$$

*Thus $T_m$ and $D_m$ are non-negative.*

## 3. Applications of Theorem 1

We shall apply Theorem 1 to the operator $D = -\Delta_{\mathbf{A}}$, the magnetic Laplacian associated with a magnetic potential $\mathbf{A}$ of Aharonov-Bohm type when $n = 2$ and which has analogous characteristics for other values of $n$. Although the analysis continues to be effective for unrestricted values of $n$, we shall concentrate on the cases $n = 2, 4$, which are the singular cases when $D = -\Delta$ and (2.3) is the standard Rellich inequality. To handle the case $n = 4$, we shall also need to discuss the case $n = 3$.

3.1. **The case $n = 2$.** A magnetic field $B = \mathrm{curl}\mathbf{A}$ of Aharonov-Bohm type vanishes in $\mathbb{R}^2\backslash\{0\}$ and in the simply connected domain $\mathbb{R}^2\backslash[0,\infty)$ the 1-form associated with $\mathbf{A}$ is exact. Hence there exists a potential $P$ such that $\mathbf{A} = \nabla P$. If we assume, as we can without loss of generality, that $\mathbf{A}\cdot\mathbf{x} = 0$ (Poincaré gauge) (see [13], Section 8.4.2), then in polar coordinates $\mathbf{x} = r(\cos\theta, \sin\theta)$,

$$\mathbf{A}(r,\theta) := \frac{1}{r}\Psi(\theta)\mathbf{e}_\theta, \quad \Psi \in L^\infty(\mathbb{S}^1), \qquad \Psi(0) = \Psi(2\pi),$$

where $\mathbf{e}_\theta := (-\sin\theta, \cos\theta)$. We have, with $\mathbf{e}_r := (\cos\theta, \sin\theta)$,

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial_r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}$$

and define

$$\nabla_{\mathbf{A}} := \nabla - i\mathbf{A} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\left(\frac{\partial}{\partial\theta} - i\Psi(\theta)\right) \tag{3.1}$$

and

$$-\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(i\frac{\partial}{\partial\theta} + \Psi(\theta)\right)^2. \tag{3.2}$$

Thus, in the notation of Theorem 1, $\Lambda_\omega = \Lambda_\theta$ is the non-negative self-adjoint operator in $L^2(0, 2\pi)$ defined by $\Lambda_\theta = K_\theta^2$, where

$$K_\theta u(\theta) = i\frac{du}{d\theta} + \Psi(\theta)u(\theta) \tag{3.3}$$

with domain

$$\big\{u : u \in C^1[0, 2\pi], u(0) = u(2\pi)\big\}.$$

It is readily seen that $K_\theta$ has eigenvalues $m + \widetilde{\Psi}, m \in \mathbb{Z}$, where $\widetilde{\Psi} = (1/2\pi)\int_0^{2\pi}\Psi(\theta)d\theta$ is the magnetic flux, and the corresponding normalised eigenfunctions are

$$u_m(\theta) = \frac{1}{\sqrt{2\pi}}\exp[-i(\theta[m + \tilde{\Psi}] - \int_0^\theta \Psi(\eta)d\eta)] \tag{3.4}$$

which constitute an orthonormal basis of $L^2(\mathbb{S}^1)$.

For $m \in \mathbb{Z}, U : f \mapsto e^{-im\theta}f$ is unitary on $L^2(\mathbb{R}^2)$ and

$$U^{-1}\nabla_{\mathbf{A}}U = \nabla_{\widetilde{\mathbf{A}}},$$

where $\widetilde{\mathbf{A}} = \frac{(\Psi+m)}{r}\mathbf{e}_\theta$. Since $\mathrm{curl}\mathbf{A} = \mathrm{curl}\widetilde{\mathbf{A}}$, the operators $\nabla_{\mathbf{A}}, \nabla_{\widetilde{\mathbf{A}}}$, and also $\Delta_{\mathbf{A}}, \Delta_{\widetilde{\mathbf{A}}}$ are gauge equivalent. Furthermore the associated magnetic fluxes differ by $m$. Therefore we may assume that $\tilde{\Psi} \in [0, 1)$.

Since $\mathcal{D}_0 = C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ we have from Theorem 1

**Corollary 2.** *For all $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and $\alpha \geq -1$,*

$$\int_{\mathbb{R}^2} |\Delta_{\mathbf{A}}f(x)|^2 \frac{dx}{|x|^\alpha} \geq C(2, \alpha)\int_{\mathbb{R}^2} |f(x)|^2 \frac{dx}{|x|^{\alpha+4}}, \tag{3.5}$$

*where*

$$C(2, \alpha) = \inf_{m\in\mathbb{Z}}\{(m + \tilde{\Psi})^2 - (\frac{\alpha+2}{2})^2\}^2. \tag{3.6}$$

*If $\tilde{\Psi} \notin \mathbb{Z}$ ($\tilde{\Psi} \in [0, 1)$ without loss of generality), we have*

$$\begin{aligned} C(2, 0) = &\ \min\{(\tilde{\Psi}^2 - 1)^2, \tilde{\Psi}^2(\tilde{\Psi} - 2)^2\} \\ = &\ \begin{cases} (\tilde{\Psi}^2 - 1)^2 & \text{if } \ \tilde{\Psi} \in [\frac{1}{2}, 1), \\ \tilde{\Psi}^2(\tilde{\Psi} - 2)^2 & \text{if } \ \tilde{\Psi} \in [0, \frac{1}{2}). \end{cases} \end{aligned} \tag{3.7}$$

**Remark 3.** *If $\tilde{\Psi} \in \mathbb{Z}$, then $C(2,0) = 0$. However, if $F_1 = F_{-1} = 0$ (see (2.5)), the infimum in (3.6) is over $m \in \mathbb{Z} \setminus \{-1, 1\}$ and this gives $C(2,0) = 1$. Hence, Rellich's result in ([10], p.100) for $n = 2$ is recovered.*

3.2. **The case $n = 3$.** In spherical polar coordinates, we define the orthonormal vectors

$$\begin{aligned} \mathbf{e}_0 &:= \tfrac{\mathbf{x}}{|\mathbf{x}|} = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2), \\ \mathbf{e}_1 &:= (-\sin\theta_1, \cos\theta_1\cos\theta_2, \cos\theta_1\sin\theta_2), \\ \mathbf{e}_2 &:= (0, -\sin\theta_2, \cos\theta_2), \end{aligned}$$

where $r = |\mathbf{x}| \in (0,\infty)$, $\theta_1 \in (0,\pi)$, and $\theta_2 \in (0, 2\pi)$. We now take

$$\mathbf{A} := \frac{1}{r\sin\theta_1}\Psi(\theta_2)\mathbf{e}_2, \qquad \Psi \in L^\infty(0,2\pi), \qquad \Psi(0) = \Psi(2\pi),$$

on $\mathbb{R}^3 \setminus \mathcal{L}_3$, where $\mathcal{L}_3 = \{(r,\theta_1,\theta_2) : r\sin\theta_1 = 0\}$. It is in Poincaré gauge and $\mathrm{curl}\mathbf{A} = 0$ in $\mathbb{R}^3 \setminus \mathcal{L}_3$.

Then

$$\nabla_{\mathbf{A}} = \nabla - i\mathbf{A} = \mathbf{e}_0\frac{\partial}{\partial r} + \mathbf{e}_1\frac{1}{r}\frac{\partial}{\partial\theta_1} + \mathbf{e}_2\frac{1}{r\sin\theta_1}[\frac{\partial}{\partial\theta_2} - i\Psi(\theta_2)] \tag{3.8}$$

and

$$-\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda(\theta_1,\theta_2) \tag{3.9}$$

where

$$\Lambda(\theta_1,\theta_2) = -\frac{\partial^2}{\partial\theta_1^2} - \cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}K_{\theta_2}^2 \tag{3.10}$$

and

$$K_{\theta_2} = i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2). \tag{3.11}$$

As above for (3.3), the self-adjoint operator $K_{\theta_2}$ in $L^2(\mathbb{S}^1)$ has eigenvalues $k + \tilde{\Psi}$, $k \in \mathbb{Z}$, with $\tilde{\Psi} := \frac{1}{2\pi}\int_0^{2\pi}\Psi(\theta_2)d\theta_2$, and eigenvectors $u_k(\theta_2)$ which form an orthonormal basis of $L^2(\mathbb{S}^1)$. Identifying $L^2(\mathbb{S}^2)$ with $\bigoplus_{k\in\mathbb{Z}}\left(L^2(0,\pi;\sin\theta_1 d\theta_1)\bigotimes\{u_k\}\right)$ we shall take the operator $\Lambda_\omega$ of Theorem 1 to be

$$\Lambda_\omega = \bigoplus_{k\in\mathbb{Z}}\left(\Lambda_k(\theta_1)\bigotimes I_k\right), \tag{3.12}$$

where $I_k$ denotes the identity on $\{u_k\}$ and $\Lambda_k(\theta_1)$ is a self-adjoint realisation of the operator $\Lambda_k^0(\theta_1)$ defined on $C_0^\infty(0,\pi)$ by

$$\Lambda_k^0(\theta_1)u = \big(-\frac{d^2}{d\theta_1^2} - \cot\theta_1\frac{d}{d\theta_1} + \frac{(k+\tilde{\Psi})^2}{\sin^2\theta_1}\big)u. \tag{3.13}$$

Before we are able to apply Theorem 1 we must first make a suitable choice of the operators $\Lambda_k(\theta_1)$ for all $k \in \mathbb{Z}$ and determine their eigenvalues. The information required is contained in the next two lemmas.

**Lemma 1.** *For $\mu \in [0, \infty)$, the associated Legendre equation*

$$\frac{d^2u}{d\theta^2} + \cot\theta \frac{du}{d\theta} + \left(\lambda - \frac{\mu^2}{\sin^2\theta}\right) u = 0, \quad \lambda \in \mathbb{C}, \tag{3.14}$$

*is in the limit-circle case at 0 and $\pi$ if $\mu \in [0, 1)$ and in the limit-point case at 0 and $\pi$ otherwise.*

*Let*

$$\mathcal{D}_\mu := \big\{u : u, \sin\theta \frac{du}{d\theta} \in AC_{loc}(0, \pi), u, L_\mu u \in L^2(0, \pi; \sin\theta d\theta)\big\}, \tag{3.15}$$

*where*

$$L_\mu := -\frac{d^2}{d\theta^2} - \cot\theta \frac{d}{d\theta} + \frac{\mu^2}{\sin^2\theta}$$

*and denote the restriction of $L_\mu$ to $C_0^\infty(0, \pi)$ by $\Lambda_\mu^0$. Then $\Lambda_\mu^0$ is non-negative. It is essentially-self-adjoint if and only if $\mu \in [1, \infty)$ and for $\mu \in [0, 1)$ its Friedrichs extension $\Lambda_\mu$ is the realisation of $L_\mu$ on the following domains:*

- *if $\mu \in (0, 1)$,*

$$\mathcal{D}(\Lambda_\mu) = \big\{u : u \in \mathcal{D}_\mu, \sin^\mu\theta \; u(\theta) \to 0 \; \text{as} \; \theta \to 0, \pi\big\}; \tag{3.16}$$

- *if $\mu = 0$,*

$$\mathcal{D}(\Lambda_\mu) = \big\{u : u \in \mathcal{D}_\mu, u(\theta)/\ln(\cot\theta) \to 0 \quad \text{as} \; \theta \to 0, \pi\big\}; \tag{3.17}$$

- *if $\mu \in [1, \infty)$, $\Lambda_\mu$ is the closure of $\Lambda_\mu^0$ and*

$$\mathcal{D}(\Lambda_\mu) = \mathcal{D}_\mu. \tag{3.18}$$

*For $\mu \in [0, 1)$, $\Lambda_\mu \geq \mu(\mu + 1)$.*

*Proof.* On substituting $x = \cos\theta$, (3.14) becomes

$$\tau_\mu u := -\frac{d}{dx}\big\{(1 - x^2)\frac{du}{dx}\big\} + \frac{\mu^2}{1 - x^2} u = \lambda u, \quad x \in (-1, 1). \tag{3.19}$$

Denote the restriction of $\tau_\mu$ to $C_0^\infty(-1, 1)$ by $T_\mu^0$. Clearly $T_\mu^0 \geq 0$.

Define the functions

$$\begin{aligned} a(x) &= (1 - x^2)^{\mu/2}, \\ b(x) &= a(x)h(x), \quad h(x) = |\int_0^x (1 - t^2)^{-1-\mu} dt|. \end{aligned} \tag{3.20}$$

Note that

$$h(x) = \begin{cases} = & \frac{1}{2}\left|\ln\left(\frac{1+x}{1-x}\right)\right| \quad \text{for} \quad \mu = 0, \\ \approx & (1 - x^2)^{-\mu} \quad \text{for} \quad \mu > 0, \end{cases} \tag{3.21}$$

where $f \approx g$ means that $f(x)/g(x)$ is bounded above and below by positive constants. We also have

$$\begin{aligned} & a(x)/b(x) \to 0 \text{ as } x \to \pm 1; \\ & \int_0^1 \frac{dt}{(1-t^2)a^2(t)} = \int_{-1}^0 \frac{dt}{(1-t^2)a^2(t)} = \infty; \\ & \int_\delta^1 \frac{dt}{(1-t^2)b^2(t)} < \infty, \quad |\int_{-1}^{-\delta} \frac{dt}{(1-t^2)b^2(t)}| < \infty, \end{aligned}$$

for any $\delta > 0$. Furthermore

$$\begin{aligned} \tau_\mu a &= \mu(\mu+1)a, \\ \tau_\mu b &= \mu(\mu+1)b. \end{aligned} \tag{3.22}$$

Hence $a, b$ are respectively *principal* and *non-principal* solutions of $\tau_\mu u = \mu(\mu+1)u$. For $\mu \in [0,1), a, b \in L^2(-1,1)$ and so (3.19) is in the limit-circle case at $-1$ and $+1$. If $\mu \geq 1, b$ is neither in $L^2(-1,0)$ nor $L^2(0,1)$ and hence (3.19) is limit-point at both the end points $\pm 1$. Thus $T_\mu^0$ is essentially-self-adjoint for $\mu \geq 1$. To characterise the Friedrichs extension $T_\mu$ of $T_\mu^0$ when $\mu \in [0,1)$ we apply Rosenberger's Theorem 3 in [11] to get

$$\mathcal{D}(T_\mu) = \big\{u : u \in \mathcal{D}_\mu, \lim_{x \to -1} u(x)/b(x) = \lim_{x \to 1} u(x)/b(x) = 0\big\}, \tag{3.23}$$

where

$$\mathcal{D}_\mu = \big\{u : u, (1-x^2)u' \in AC_{loc}(-1,1), u, \tau_\mu u \in L^2(-1,1)\big\}.$$

For $\mu \geq 1$ we have seen that $T_\mu^0$ is essentially-self-adjoint and so its unique self-adjoint extension has the domain of its adjoint, namely $\mathcal{D}_\mu$.

To prove that $T_\mu \geq \mu(\mu+1)$ when $\mu \in [0,1)$, we appeal to results of Kalf in [6]. In Remark 3 he gives the following characterisation of $T_\mu$:

$$\begin{aligned} \mathcal{D}(T_\mu) = \big\{u :& u \in \mathcal{D}_\mu, \int^x (1-t^2)b^2(t)\big|\big(\frac{u}{b}\big)'\big|^2 dt < \infty \text{ at } \pm 1 \\ & \text{and } \lim_{x \to \pm 1} |u(x)/b(x)| = 0\big\}. \end{aligned} \tag{3.24}$$

On using Jacobi's factorisation identity (5) in [6], it follows from (3.22) (cf [6] (6)) that for $u \in \mathcal{D}_\mu$ and $-1 < X < Y < 1$

$$\begin{aligned} \int_X^Y \overline{u}\tau_\mu u dx =& [u,b](x)\overline{u(x)}/b(x)\big|_X^Y \\ & + \int_X^Y \Big\{(1-x^2)b^2\big|\big(\frac{u}{b}\big)'\big|^2 + \mu(\mu+1)|u|^2\Big\} dx, \end{aligned} \tag{3.25}$$

where

$$[u,b](x) := (1-x^2)\{u(x)b'(x) - u'(x)b(x)\}.$$

Since $b \in \mathcal{D}_\mu$ for $\mu \in [0,1)$ we know from Green's identity

$$\int_X^Y \big(u\tau_\mu b - b\tau_\mu u\big)dx = [u,b](x)\big|_X^Y$$

that $\lim_{x\to\pm 1}[u,b](x)$ exists. Hence, from (3.23) and (3.24), for $u \in \mathcal{D}(T_\mu)$,

$$\begin{aligned}\int_{-1}^1 \overline{u}T_\mu u dx &= \int_{-1}^1 \big\{(1-x^2)b^2\big|\big(\frac{u}{b}\big)'\big|^2 + \mu(\mu+1)|u|^2\big\}dx \\ &\geq \mu(\mu+1)\|u\|^2. \end{aligned} \tag{3.26}$$

The lemma follows on reverting to the $\theta$-co-ordinates. □

**Lemma 2.** *The operator $T_\mu$ in Lemma 1 has a discrete spectrum consisting of eigenvalues*

$$\nu_j(\mu) = \big(j-\mu\big)\big(j+1-\mu\big), \quad j \in \mathbb{Z}\setminus\{0\}. \tag{3.27}$$

*Proof.* From Bateman [1] (§3.4, p.143), the associated Legendre polynomials $P_\nu^\mu(x)$ and $P_\nu^\mu(-x)$ are linearly independent solutions of (3.19) with $\lambda = \nu(\nu+1) \in \mathbb{C}\setminus\mathbb{R}$ and as $x \to \pm 1$,

$$P_\nu^\mu(x) \approx (x \mp 1)^{-\mu/2};$$

see Bateman [1], pp.163-64. We now use an argument similar to that of Titchmarsh in [14], §§4.5, 4.6. Let $\phi_0(x) := P_\nu^\mu(x)$ and

$$\begin{aligned}\theta(x,\lambda) &= \tfrac{\phi_0(x)+\phi_0(-x)}{2\phi_0(0)} \\ \phi(x,\lambda) &= \tfrac{\phi_0(-x)-\phi_0(x)}{2\phi_0'(0)}.\end{aligned} \tag{3.28}$$

Note that $\phi_0(0) \neq 0$ and $\phi_0'(0) \neq 0$ if $\nu+\mu \notin \mathbb{Z}$ from [1], p 145, (20) and (23). We therefore have

$$\theta(0,\lambda) = 1,\ \theta'(0,\lambda) = 0;\ \phi(0,\lambda) = 0,\ \phi'(0,\lambda) = -1,$$

so that $\theta$ and $\phi$ satisfy the initial conditions (2.1.4) in [14] with $\alpha = 0$.

In view of (3.23), a Titchmarsh-Weyl function for the interval $(0,1)$ is determined as

$$m_2(\lambda) = \lim_{X\to 1} l_2(\lambda, X)$$

where

$$\lim_{X\to 1}\big\{\big[\theta(X,\lambda) + l_2(\lambda,X)\phi(X,\lambda)\big]/b(X)\big\} = 0. \tag{3.29}$$

If $\lambda = \nu(\nu+1) \notin \mathbb{R}$, then $\lim_{X\to 1}\phi(X,\lambda)/b(X) \neq 0$ since otherwise $\phi$ would be an eigenfunction of the self-adjoint operator generated by $\tau_\mu$ on the interval $[0,1)$ with boundary conditions $u(0) = 0$, $\lim_{x\to 1} u(x)/b(x) = 0$. Hence (3.29) yields

$$m_2(\lambda) = -\lim_{X\to 1}\big\{\theta(X,\lambda)/\phi(X,\lambda)\big\}. \tag{3.30}$$

Of course, this corresponds to a point on the limit-circle when $\mu \in [0,1)$ but is unique otherwise. Hence, on substituting (3.28),

$$m_2(\lambda) = \lim_{b\to 1-} \left\{ \frac{\phi_0(b)+\phi_0(-b)}{\phi_0(b)-\phi_0(-b)} \right\} \frac{\phi_0'(0)}{\phi_0(0)}. \tag{3.31}$$

From Bateman [1], p.163, since $\mu \notin \mathbb{N}$, as $x \to 1$

$$\begin{aligned} \phi_0(x) &\approx 2^{\mu/2}(1-x)^{-\mu/2}/\Gamma(1-\mu), \\ \phi_0(-x) &\approx -2^{\mu/2}\pi^{-1}(1-x)^{-\mu/2}\sin(\pi\nu)\Gamma(\mu). \end{aligned}$$

Thus, on recalling that $\Gamma(\mu)\Gamma(1-\mu) = \pi \csc(\pi\mu)$ (see [1], p.3),

$$m_2(\lambda) = K(\nu,\mu)\frac{\phi_0'(0)}{\phi_0(0)}, \qquad \lambda = \nu(\nu+1) \in \mathbb{C}\setminus\mathbb{R}$$

where

$$K(\nu,\mu) = \tan\frac{\pi}{2}(\mu-\nu)\cot\frac{\pi}{2}(\mu+\nu).$$

The corresponding Titchmarsh-Weyl solution of (3.19) on $(0,1)$ is

$$\begin{aligned} \psi_2(x,\lambda) &= \theta(x,\lambda) + m_2(\lambda)\phi(x,\lambda) \\ &= \tfrac{1}{2\phi_0(0)}\{[1-K(\nu,\mu)]\phi_0(x) + [1+K(\nu,\mu)]\phi_0(-x)\}. \end{aligned} \tag{3.32}$$

In the interval $(-1,0)$, we put $x=-y$, $y\in(0,1)$, to get the Titchmarsh-Weyl solution

$$\psi_1(x,\lambda) = \theta(x,\lambda) + m_1(\lambda)\phi(x,\lambda)$$

where

$$\begin{aligned} m_1(\lambda) &= \lim_{y\to -1}\left\{\tfrac{\phi_0(y)+\phi_0(-y)}{\phi_0(-y)-\phi_0(y)}\right\}\tfrac{\phi_0'(0)}{\phi_0(0)} \\ &= -K(\nu,\mu)\tfrac{\phi_0'(0)}{\phi_0(0)} \end{aligned}$$

and so

$$\psi_1(x,\lambda) = \frac{1}{2\phi_0(0)}\left\{[1-K(\nu,\mu)]\phi_0(-x) + [1+K(\nu,\mu)]\phi_0(x)\right\}. \tag{3.33}$$

For $x<z$, the Green's function $G(x,z;\lambda)$ of (3.19) on $(-1,1)$ is given by

$$G(x,z;\lambda) = \frac{\psi_1(x,\lambda)\psi_2(z,\lambda)}{W(\psi_1,\psi_2)}$$

where

$$W(\psi_1,\psi_2) = (\psi_2\psi_1' - \psi_1\psi_2')(0) = m_2(\lambda) - m_1(\lambda) = 2K(\nu,\mu)\frac{\phi_0'(0)}{\phi_0(0)}. \tag{3.34}$$

It follows that $T_\mu$ has a discrete spectrum with eigenvalues located at the zeros of

$$X(\nu,\mu) := \sin\frac{\pi}{2}(\mu-\nu)\cos\frac{\pi}{2}(\mu+\nu)\phi_0(0)\phi_0'(0).$$

From Bateman [1], p.145 ((20) & (23)),

$$\begin{aligned}\phi_0(0) &= 2^{\mu}\pi^{-\frac{1}{2}}\cos\tfrac{\pi}{2}(\mu+\nu)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)\\ \phi_0'(0) &= 2^{\mu+1}\pi^{-\frac{1}{2}}\sin\tfrac{\pi}{2}(\mu+\nu)\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu).\end{aligned}$$

The zeros of $X(\nu,\mu)$ are therefore given by

$$1+\frac{1}{2}\nu-\frac{1}{2}\mu=-m,\ m\in\mathbb{N}_0;\ \mu+\nu=2n+1,\ n\in\mathbb{Z};$$
$$\text{and}\ \ \mu-\nu=2n,\ n\in\mathbb{Z}.$$

When $1+\frac{1}{2}\nu-\frac{1}{2}\mu=-m$,

$$\lambda=\nu(\nu+1)=(2m+1-\mu)(2m+1-\mu+1),\qquad m\in\mathbb{N}_0.$$

When $\mu+\nu=2n+1$,

$$\lambda=(2n+1-\mu)(2n+1-\mu+1),\qquad n\in\mathbb{Z},$$

and $\mu-\nu=2n$ yields

$$\lambda=(2n-1-\mu)(2n-1-\mu+1),\ n\in\mathbb{Z}.$$

The lemma is therefore proved. □

**Corollary 3.** *For all $f\in C_0^\infty(\mathbb{R}^3\setminus\{0\})$, we have*

$$\int_{\mathbb{R}^3}|\Delta_{\mathbf{A}}f(x)|^2\frac{dx}{|x|^\alpha}\geq C(3,\alpha)\int_{\mathbb{R}^3}|f(x)|^2\frac{dx}{|x|^{\alpha+4}}\tag{3.35}$$

*where*

$$C(3,\alpha):=\inf_{m\in\mathbb{Z}}\{(m-\tilde{\Psi})(m-\tilde{\Psi}+1)-\frac{(3+\alpha)(1+\alpha)}{4}\}^2.\tag{3.36}$$

*Proof.* We may clearly suppose that $|\cdot|^{-\alpha/2}\Delta_{\mathbf{A}}f\in L^2(\mathbb{R}^3)$. Then $\Lambda_\omega f=\Lambda(\theta_1,\theta_2)f\in L^2(\mathbb{R}^3)$ and for all $r\in(0,\infty),\Lambda(\theta_2,\theta_2)f\in L^2(\mathbb{S}^2)$. If $\Lambda_k(\theta_1)$ denotes provisionally the formal operator in (3.13) and

$$F_k(r,\theta_1):=\int_0^{2\pi}f(r,\theta_1,\theta_2)\overline{u_k(\theta_2)}d\theta_2,$$

where the $u_k$ are the functions in (3.4), then we have

$$\int_0^{2\pi}|\Lambda(\theta_1,\theta_2)f(r,\theta_1,\theta_2)|^2d\theta_2=\sum_{k\in\mathbb{Z}}|\Lambda_k(\theta_1)F_k(r,\theta_1)|^2.$$

Hence, for any $k\in\mathbb{Z}$ and $r\in(0,\infty),\Lambda_k(\theta_1)F_k(r,\theta_1)\in L^2(0,\pi;\sin\theta_1 d\theta_1)$. Since $F_k(r,\cdot)\in L^2(0,\pi;\sin\theta_1 d\theta_1)$ and the boundary conditions in Lemma 1 are satisfied, it follows that $F_k(r,\cdot)$ lies in the domain of the operator $\Lambda_k(\theta_1)$ and $f(r,\cdot)\in\mathcal{D}(\Lambda_\omega)$.

From Lemmas 1 and 2 the eigenvalues of the operator $\Lambda_k$ in (3.12) are

$$\nu_j(k)=(j-|k+\tilde{\Psi}|)(j+1-|k+\tilde{\Psi}|),\quad j\in\mathbb{Z}\setminus\{0\}.\tag{3.37}$$

Denote the corresponding normalised eigenvectors by $P_{j,k}(\theta_1;\tilde{\Psi})$. Then

$$Y_{j,k;\tilde{\Psi}}(\theta_1,\theta_2) := P_{j,k}(\theta_1;\tilde{\Psi})u_k(\theta_2), \quad j,k\in\mathbb{Z}$$

are the eigenvectors of $\Lambda_\omega$ corresponding to the eigenvalues $\nu_j(k)$ and form an orthonormal basis of $L^2(\mathbb{S}^2)$. The corollary follows from Theorem 1. □

**Remark 4.** *When $\alpha = 0$ and $\tilde{\Psi} = 0$ in (3.36), $C(3,0) = \frac{9}{16}$, which is the constant obtained by Rellich* [10].

3.3. **The case $n = 4$.** In this case we define the orthonormal vectors

$$\begin{aligned}
\mathbf{e}_0 &= \tfrac{\mathbf{x}}{|\mathbf{x}|} = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \sin\theta_1\sin\theta_2\sin\theta_3)\\
\mathbf{e}_1 &= (-\sin\theta_1, \cos\theta_1\cos\theta_2, \cos\theta_1\sin\theta_2\cos\theta_3, \cos\theta_1\sin\theta_2\sin\theta_3)\\
\mathbf{e}_2 &= (0, -\sin\theta_2, \cos\theta_2\cos\theta_3, \cos\theta_2\sin\theta_3)\\
\mathbf{e}_3 &= (0,0,-\sin\theta_3,\cos\theta_3),
\end{aligned}$$

where $\theta_1,\theta_2\in(0,\pi), \theta_3\in(0,2\pi)$. In this case,

$$\nabla = \mathbf{e}_0\frac{\partial}{\partial r} + \mathbf{e}_1(\frac{1}{r}\frac{\partial}{\partial\theta_1}) + \mathbf{e}_2(\frac{1}{r\sin\theta_1}\frac{\partial}{\partial\theta_2}) + \mathbf{e}_3(\frac{1}{r\sin\theta_1\sin\theta_2}\frac{\partial}{\partial\theta_3}).$$

We now take

$$\mathbf{A} := \frac{1}{r\sin\theta_1\sin\theta_2}\Psi(\theta_3)\mathbf{e}_3, \qquad \Psi\in L^\infty(0,2\pi), \qquad \Psi(0)=\Psi(2\pi), \tag{3.38}$$

in $\mathbb{R}^4\setminus\mathcal{L}_4$, where $\mathcal{L}_4 := \{\mathbf{x} = (r,\theta_1,\cdots,\theta_3) : r\sin\theta_1\sin\theta_2 = 0\}$. We then have

$$-\Delta_{\mathbf{A}} = L_r + \frac{1}{r^2}\Lambda$$

for $L_r$ defined in (2.1), where

$$\Lambda = -\frac{\partial^2}{\partial\theta_1^2} - 2\cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}\{-\frac{\partial^2}{\partial\theta_2^2} - \cot\theta_2\frac{\partial}{\partial\theta_2} + \frac{K_{\theta_3}^2}{\sin^2\theta_2}\} \tag{3.39}$$

with

$$K_{\theta_3} := i\frac{\partial}{\partial\theta_3} + \Psi(\theta_3).$$

On repeating the procedure in Section 3.1 and using the notation of Section 3.1, we take $\Lambda_\omega$ to be

$$\Lambda_\omega = \bigoplus_{j,k\in\mathbb{Z}}\left(\Lambda_{j,k}(\theta_1)\bigotimes I_{j,k}\right), \tag{3.40}$$

where $\Lambda_{j,k}(\theta_1)$ is the self-adjoint operator generated as in Section 3.1 by

$$\Lambda_{j,k}(\theta_1) = -\frac{\partial^2}{\partial\theta_1^2} - 2\cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{\nu_j(k)}{\sin^2\theta_1} \tag{3.41}$$

in $L^2((0,\pi);\sin^2\theta_1\sin\theta_2 d\theta_1)$. As in the analogous situation in Section 3.1, the operator $\Lambda_{j,k}(\theta_1)$ is chosen to be the Friedrichs extension of the

operator defined on $C_0^\infty(0,\pi)$ or its unique self-adjoint extension when there is one. To apply Theorem 1 we need

**Lemma 3.** *The Friedrichs extension of* $\Lambda_{j,k}(\theta_1)\big|_{C_0^\infty(0,\pi)}$ *in* $L^2((0,\pi);\sin^2\theta_1 d\theta_1)$ *has eigenvalues*

$$\gamma_\ell(j,k) = (\ell - [\nu_j(k)+1/4]^{\frac{1}{2}})(\ell - [\nu_j(k)+1/4]^{\frac{1}{2}}+1) - \frac{3}{4}, \qquad \ell \in \mathbb{Z}' \tag{3.42}$$

*where* $\nu_j(k)$ *is given by (3.37) and* $\mathbb{Z}' := \{\ell \in \mathbb{Z} : \gamma_\ell(j,k) \geq 0\}$.

*Proof.* On substituting $x = \cos\theta_1$, the equation $\Lambda_{j,k}(\theta_1)u = \lambda u$ becomes

$$(1-x^2)\frac{d^2u}{dx^2} - 3x\frac{du}{dx} + (\lambda - \frac{\nu_j(k)}{1-x^2})u = 0.$$

Further, set $w = (1-x^2)^{\frac{1}{4}}u$ to obtain

$$(1-x^2)\frac{d^2w}{dx^2} - 2x\frac{dw}{dx} + (\lambda + \frac{3}{4} - \frac{\nu_j(k)+\frac{1}{4}}{1-x^2})w = 0 \tag{3.43}$$

with $L^2(-1,1)$ for the underlying Hilbert space. The problem is therefore reduced to that for (3.19) with $\lambda + \frac{3}{4}$ instead of $\lambda$ and $\nu_j(k) + \frac{1}{4}$ for $\mu^2$, and the lemma follows from Lemma 2. □

From (3.37), $\nu_j(k) = (j - |k+\tilde{\Psi}|)(j+1-|k+\tilde{\Psi}|)$, which implies that

$$\nu_j(k) + \frac{1}{4} = \left(j - |k+\tilde{\Psi}| + \frac{1}{2}\right)^2.$$

Thus, from (3.42), $\gamma_\ell(j,k) = (\ell - j + |k+\tilde{\Psi}|)^2 - 1$ if $j - |k+\tilde{\Psi}| + 1/2 \geq 0$ and $\gamma_\ell(j,k) = (\ell + j - |k+\tilde{\Psi}|+1)^2 - 1$ otherwise. These can be enumerated as

$$\lambda_m = (m+\tilde{\Psi})^2 - 1, \qquad m \in \mathbb{Z}'$$

where $\mathbb{Z}' := \{m : (m+\tilde{\Psi})^2 \geq 1\}$.

It follows by an argument similar to that in the proof of Corollary 3 that for any $f \in C_0^\infty(\mathbb{R}^4 \setminus \{0\})$ with $\Delta_{\mathbf{A}} f \in L^2(\mathbb{R}^4)$ we have $f(r,\cdot) \in \mathcal{D}(\Lambda_\omega)$. Hence, from Theorem 1,

**Corollary 4.** *Let* $f \in C_0^\infty(\mathbb{R}^4 \setminus \{0\})$. *Then*

$$\int_{\mathbb{R}^4} |\Delta_{\mathbf{A}} f(x)|^2 \frac{dx}{|x|^\alpha} \geq C(4,\alpha) \int_{R^4} |f(x)|^2 \frac{dx}{|x|^{\alpha+4}}$$

*where*

$$C(4,\alpha) := \inf_{m\in\mathbb{Z}'} \{[(m+\tilde{\Psi})^2 - 1 - \frac{\alpha(\alpha+4)}{4}]^2\}.$$

*In particular, when* $\alpha = 0$

$$C(4,0) = \min\{(-1+\tilde{\Psi})^2 - 1, (-2+\tilde{\Psi})^2 - 1\}^2 > 0.$$

When $\tilde{\Psi} = 0, C(4,0) = 0$ and the inequality is trivial. If $F_1 = F_2 = 0$ (see (2.5)), then the infimum is attained for $m = \pm 2$ giving $C(4,0) = 9$.

## 4. Rellich-type inequalities in $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

We shall now apply the method used in [3] to prove the following theorem.

**Theorem 2.** *For all $p \in (1, \infty)$, $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and*

$$d := \inf_{k \in \mathbb{Z}} \{|k - \tilde{\Psi}|\} \neq 0, \frac{1}{2},$$

*where $\tilde{\Psi} = \frac{1}{2\pi} \int_0^{2\pi} \Psi(\theta) d\theta$ and $\Psi(0) = \Psi(2\pi)$, we have*

$$\|f/|\cdot|^2\|_{L^p(\mathbb{R}^2)} \leq B_p \| \frac{1}{r^2} (i \frac{\partial}{\partial \theta} + \Psi(\theta))^2 f \|_{L^p(\mathbb{R}^2)} \tag{4.1}$$

*where*

$$B_p \leq \frac{C_p}{d^2 (\cos \pi d)^{1/p'}}, \qquad C_p := \frac{(2\pi)^{\frac{1}{p}} \pi^{1+\frac{2}{p'}}}{8}, \qquad p' = \frac{p}{p-1}.$$

*When $n \geq 3$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \mathcal{L}_n), \mathcal{L}_n = \{(r, \theta_1, \cdots, \theta_{n-1} : \sin \theta_1 \cdots \sin \theta_{n-2} = 0\}$,*

$$\begin{aligned} &\|f/r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}\|_{L^p(\mathbb{R}^n)} \\ &\qquad \leq B_p \| \frac{1}{r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}} (i \frac{\partial}{\partial \theta_{n-1}} + \Psi(\theta_{n-1}))^2 f \|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{4.2}$$

Before embarking on the proof of this theorem we need some preliminaries. Let

$$u(r, \theta) = e^{-2\pi i \Phi(\theta)} f(r, \theta), \qquad \Phi(\theta) := \frac{1}{2\pi} \int_0^\theta \Psi(t) dt. \tag{4.3}$$

Then

$$\frac{\partial u}{\partial \theta} = e^{-2\pi i \Phi(\theta)} (\frac{\partial}{\partial \theta} - i\Psi(\theta)) f(r, \theta) \tag{4.4}$$

and for $f \in C^1(\mathbb{R}^2)$ and $\delta = e^{-2\pi i \tilde{\Psi}}$

$$\begin{aligned} u(r, 2\pi) &= \delta\ u(r, 0), \\ \tfrac{\partial u}{\partial \theta}(r, 2\pi) &= \delta \tfrac{\partial u}{\partial \theta}(r, 0). \end{aligned} \tag{4.5}$$

We now define

$$W_\delta^p \equiv W_\delta^p[0, 2\pi] := \{v \in C^1[0, 2\pi] : v(2\pi) = \delta v(0)\}. \tag{4.6}$$

Hence, as a function of $\theta$, $u = u(r, \cdot)$ and $u' \equiv \frac{\partial u}{\partial \theta}(r, \cdot)$ satisfy

$$u, u' \in W_\delta^p.$$

Consequently, from Theorem 1 of [3], for $x \in [0, 2\pi]$

$$\begin{aligned} u(x) &= \int_0^{2\pi} u'(y) \overline{h_y'}(x, y) dy \\ u'(x) &= \int_0^{2\pi} u''(y) \overline{h_y'}(x, y) dy \end{aligned}$$

where $h(x,y) = (y-x)\{\chi_{_{[0,x)}}(y) - \frac{\delta}{\delta-1}\} - \frac{2\pi\delta}{(\delta-1)^2}$ in which $\chi$ denotes the characteristic function. It follows that

$$u(x) = \int_0^{2\pi} H(x,z)u''(z)dz \tag{4.7}$$

where

$$\begin{aligned} H(x,z) = & \int_0^{2\pi} \overline{h'_y(x,y)h'_z(y,z)}dy \\ = & \begin{cases} \frac{z-x}{1-\delta} + \frac{2\pi\delta}{(1-\delta)^2}, & x \le z \\ \frac{\delta(z-x)}{1-\delta} + \frac{2\pi\delta}{(1-\delta)^2}, & x > z. \end{cases} \end{aligned} \tag{4.8}$$

Hence, for all $u \in W^p_\delta$ and $x \in [0, 2\pi]$

$$|u(x)| \le \|H(x,\cdot)\|_{L^{p'}(0,2\pi)}\|u''\|_{L^p(0,2\pi)}, \tag{4.9}$$

when $1 < p < \infty$ and $p' = \infty$ if $p = 1$. We have from (4.8), for $1 < p < \infty$,

$$\int_0^x |H(x,z)|^{p'}dz = \frac{1}{|1-\delta|^{p'}}\int_0^x |x - z - \frac{2\pi}{1-\delta}|^{p'}dz =: \frac{1}{|1-\delta|^{p'}}I_1(x)$$

say. Let $\delta = e^{2\pi id}$. (Note that $\delta = e^{\pm 2\pi id}$.) Then

$$\frac{1}{1-\delta} = \frac{1}{2}[1 + i\cot(\pi d)]$$

and

$$\begin{aligned} I_1(x) = & \int_0^x [(x-z-\pi)^2 + \pi^2\cot^2(\pi d)]^{p'/2}dz \\ = & [\pi\cot(\pi d)]^{p'+1}\int_{\frac{-(x-\pi)}{\pi\cot(\pi d)}}^{\pi/(\pi\cot(\pi d))}[w^2+1]^{p'/2}dw \end{aligned}$$

on substituting $z = (x-\pi) + \pi\cot(\pi d)w$. Similarly

$$\int_x^{2\pi} |H(x,z)|^{p'}dz = \frac{1}{|1-\delta|^{p'}}I_2(x)$$

and

$$I_2(x) = \int_x^{2\pi} [(z-x-\pi)^2 + \pi^2\cot^2(\pi d)]^{p'/2}dz.$$

Also

$$\begin{aligned} \int_0^{2\pi} |H(x,z)|^{p'}dz = & \frac{[\pi\cot(\pi d)]^{p'+1}}{|1-\delta|^{p'}}\int_{-\tan(\pi d)}^{\tan(\pi d)}(w^2+1)^{p'/2}dw \\ = & \frac{[\pi\cos(\pi d)]^{p'+1}}{2^{p'-1}[\sin(\pi d)]^{2p'+1}}\int_0^{\pi d}[\sec\theta]^{p'+2}d\theta \end{aligned} \tag{4.10}$$

on substituting $w = \tan\theta$ and noting that $|1-\delta| = 2\sin(\pi d)$. In $[0, \pi d]$, $\cos\theta \ge \cos(\pi d)$ and $\sin(\pi d) \ge 2d$, since $d \le \frac{1}{2}$. Hence, if $0 < d < \frac{1}{2}$,

$$\int_0^{2\pi} |H(x,z)|^{p'}dz \le \frac{\pi^{2+p'}}{8^{p'}}\frac{1}{\cos(\pi d)d^{2p'}}. \tag{4.11}$$

The same holds if $\delta = e^{-2\pi id}$. Hence, from (4.9) we have the following lemma.

**Lemma 4.** *For all $u$ for which $u, u' \in W^p_\delta$, $1 < p < \infty$, and $x \in [0, 2\pi]$, if $0 < d < \frac{1}{2}$,*

$$|u(x)| \leq A_p \frac{1}{[\cos(\pi d)]^{1/p'}} \frac{1}{d^2} \|u''\|_{L^p(0,2\pi)}, \tag{4.12}$$

*where*

$$A_p \leq \frac{\pi^{1+2/p'}}{8}.$$

If $p = 2$ and $d = \frac{1}{2}$, then in (4.10) the last integral becomes

$$\int_0^{\frac{\pi}{2}} \sec^4 \theta d\theta = \infty.$$

From (4.10), a more accurate version of (4.12) is

$$|u(x)| \leq \left\{ \frac{[\pi \cos(\pi d)]^{p'+1}}{2^{p'-1}[\sin(\pi d)]^{2p'+1}} \int_0^{\pi d} [\sec \theta]^{p'+2} d\theta \right\}^{1/p'} \|u''\|_{L^p(0,2\pi)}. \tag{4.13}$$

*Proof.* of *Theorem 2.* On reverting to the functions defined in (4.3), by Lemma 4 we have that

$$|u(r,\theta)| \leq A_p \frac{1}{[\cos(\pi d)]^{1/p'}} \frac{1}{d^2} \|\frac{\partial^2}{\partial \theta^2} u(r,\cdot)\|_{L^p(0,2\pi)}.$$

This yields (4.1) and (4.2) on noting (4.4). □

**Remark 5.** *If $\tilde{\Psi} \in \mathbb{Z}$, (4.1) does not hold for any $p \in (1, \infty)$ since the inequality*

$$\|\frac{u}{|\cdot|^2}\|_{L^p(\mathbb{R}^2)} \leq const. \|\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}\|_{L^p(\mathbb{R}^2)}$$

*fails for any radial function $u$.*

**Remark 6.** *From Corollary 1 we have that*

$$\|\frac{1}{r^2}(i\frac{\partial}{\partial \theta} + \Psi(\theta))^2 f\|_{L^2(\mathbb{R}^2)} \leq C(2) \|\Delta_{\mathbf{A}} f\|_{L^2(\mathbb{R}^2)}$$

*for all $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. This fact and Theorem 2 indicate that the introduction of the vector field $\mathbf{A}$ affects the transverse rather than the radial part of the inequality (at least when $p = 2$), as for the Laptev-Weidl* [7] *inequality.*

**Remark 7.** *For $n = 3$, it is a consequence of Theorem 3 below that*

$$\|\frac{1}{r^2 \sin^2 \theta_1} \left( i\frac{\partial}{\partial \theta_2} + \Psi(\theta_2) \right)^2 f\|_{L^2(\mathbb{R}^3)} \leq C \|\Delta_{\mathbf{A}} f\|_{L^2(\mathbb{R}^3)},$$

*but only for functions $f \in C_0^\infty(\mathbb{R}^3 \setminus \{\mathcal{L}_3\})$ whose Fourier coefficients $F_m(r,\cdot)$ vanish for specified values of $m$. This is reminiscent of Rellich's requirement for his inequality when $n = 2$. If $\rho = r \sin \theta_1$, the distance of $x \in \mathbb{R}^3$ from the $x_3 -$ axis, $\mathcal{L}_3$, we therefore have an inequality*

$$\|f/\rho^2\|_{L^2(\mathbb{R}^3)} \leq C \|\Delta_{\mathbf{A}} f\|_{L^2(\mathbb{R}^3)}$$

*for a restricted set of functions $f \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ supported away from $\mathcal{L}_3$. This is effectively a two-dimensional result as the vector field $\mathbf{A}$ is singular on $\mathcal{L}_3$ and the use of cylindrical co-ordinates would reduce the problem to one in $\mathbb{R}^2$ with $\mathbf{A}$ of Aharonov-Bohm type.*

To prove the theorem we shall need

**Lemma 5.** *Let $u \in C_0^2(-1,1)$. Then*

$$\int_{-1}^{1} \frac{|u(x)|^2}{(1-x^2)^2} dx \leq c^2 \int_{-1}^{1} |u'(x)|^2 dx \tag{4.14}$$

$$\int_{-1}^{1} |u'(x)|^2 dx \leq c^2 \int_{-1}^{1} \left| \frac{d}{dx} \left( [1-x^2] u'(x) \right) \right|^2 dx \tag{4.15}$$

*where*

$$\ln\left(\frac{c+1}{c-1}\right) = 2c. \tag{4.16}$$

*Proof.* On integration by parts,

$$\begin{aligned} \int_{-1}^{1} \frac{|u(x)|^2}{(1-x^2)^2} dx &= -2\Re e \left[ \int_{-1}^{1} \overline{u} u' \left( \int_0^x \frac{dt}{(1-t^2)^2} \right) dx \right] \\ &= -2\Re e \left[ \int_{-1}^{1} \frac{\overline{u(x)}}{(1-x^2)} u'(x) \varphi(x) dx \right], \end{aligned}$$

where

$$\varphi(x) := (1-x^2) \int_0^x \frac{dt}{(1-t^2)^2}.$$

Hence, by the Cauchy-Schwarz inequality,

$$\int_{-1}^{1} \frac{|u(x)|^2}{(1-x^2)^2} dx \leq c \left( \int_{-1}^{1} \frac{|u(x)|^2}{(1-x^2)^2} dx \right)^{1/2} \left( \int_{-1}^{1} |u'|^2 dx \right)^{1/2},$$

whence (4.14) with $c = 2 \sup_{x \in (-1,1)} |\varphi(x)|$. The function $\varphi$ is skew-symmetric, is given by

$$\varphi(x) = \frac{x}{2} + \frac{1}{4}(1-x^2) \ln\left(\frac{1+x}{1-x}\right)$$

for $x \geq 0$ and attains its maximum value of $c/2$ at the point $x = 1/c$ satisfying (4.16). This yields (4.14).

For (4.15) we have similarly

$$\begin{aligned} \int_{-1}^{1} |u'|^2 dx &= \int |(1-x^2)u'(x)|^2 \frac{dx}{(1-x^2)^2} \\ &= -2\Re e \left[ \int_{-1}^{1} (1-x^2) \overline{u'(x)} \{ \frac{d}{dx} (1-x^2) u'(x) \} \int_0^x \frac{dt}{(1-t^2)^2} dx \right] \\ &= -2\Re e \left[ \int_{-1}^{1} u'(x) \{ \frac{d}{dx} (1-x^2) u'(x) \} \varphi(x) dx \right] \end{aligned}$$

and (4.15) follows as before. □

**Theorem 3.** *For all $f \in C_0^\infty(\mathbb{S}^2 \setminus \mathcal{L}_3)$ there exist positive constants $K_1, K_2$ such that*

$$\left\| \sin^{-2}\theta_1 \left( i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2) \right)^2 f \right\|^2_{L^2(\mathbb{S}^2)} \leq K_1 \left\| \Lambda(\theta_1,\theta_2) f \right\|^2_{L^2(\mathbb{S}^2)} + K_2 \left\| \sin^{-2}\theta_1 f \right\|^2_{L^2(\mathbb{S}^2)}, \quad (4.17)$$

*where $\Lambda(\theta_1,\theta_2)$ is given in (3.10), i.e.*

$$\Lambda(\theta_1,\theta_2) = -\frac{\partial^2}{\partial^2\theta_1} - \cot\theta_1 \frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}\left( i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2) \right)^2.$$

*Let $\tilde{\Psi} \in [0,1)$ and let $f$ be such that*

$$\begin{aligned} F_m(\theta_1) &:= \int_0^{2\pi} f(\theta_1,\theta_2)\overline{u_m(\theta_2)} d\theta_2 \\ &= 0 \quad \text{for } \mathrm{m} \in \mathcal{J}(\tilde{\Psi}), \end{aligned} \quad (4.18)$$

*where the $u_m$ are given by (3.4) and*

$$\mathcal{J}(\tilde{\Psi}) = \begin{cases} \{-1,1\}, & \text{when } \tilde{\Psi} \in [0, 0.447566), \\ \{-1,-2,0,1\}, & \text{when } \tilde{\Psi} \in (0.447566, 0.552434), \\ \{-2,0\}, & \text{when } \tilde{\Psi} \in (0.552434, 1). \end{cases}$$

*Then*

$$\left\| \Lambda(\theta_1,\theta_2) f \right\|^2_{L^2(\mathbb{S}^2)} \geq C \left\| \sin^{-2}\theta_1 f \right\|^2_{L^2(\mathbb{S}^2)}, \quad (4.19)$$

*where*

$$C \leq \min_{\mathrm{m}\notin\mathcal{J}(\tilde{\Psi})} \left\{ \mathrm{c}^{-4} + (\mathrm{m}+\tilde{\Psi})^4 - 2(2-\mathrm{c}^{-2})(\mathrm{m}+\tilde{\Psi})^2 \right\} \quad (4.20)$$

*and $c$ is the positive solution of*

$$c = \frac{1}{2}\ln\left( \frac{c+1}{c-1} \right). \quad (4.21)$$

*Thus, for all $f \in C_0^\infty(\mathbb{R}^3 \setminus \mathcal{L}_3)$ satisfying (4.18) we have*

$$\left\| \sin^{-2}\theta_1 \left( i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2) \right)^2 f \right\|^2_{L^2(\mathbb{S}^2)} \leq K \left\| \Lambda(\theta_1,\theta_2) f \right\|^2_{L^2(\mathbb{S}^2)} \quad (4.22)$$

*and so, from Corollary 1,*

$$\left\| (r\sin\theta_1)^{-2} \left( i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2) \right)^2 f \right\|^2_{L^2(\mathbb{R}^3)} \leq \|Df\|^2_{L^2(\mathbb{R}^3)}. \quad (4.23)$$

*Proof.* Set

$$M_{\theta_1} := -\frac{\partial^2}{\partial\theta_1^2} - \cot\theta_1 \frac{\partial}{\partial\theta_1}$$

and

$$\Lambda_{\theta_2} := K_{\theta_2}^2,$$

where $K_{\theta_2}$ is the self-adjoint operator defined by (3.11) in $L^2(\mathbb{S}^1)$; $\Lambda_{\theta_2}$ has a discrete spectrum of eigenvalues $\lambda_m = (m+\tilde{\Psi})^2$ and eigenvectors $u_m, m \in \mathbb{Z}$ which form an orthonormal basis of $L^2(0, 2\pi)$. Hence

$$\Lambda(\theta_1, \theta_2) = M_{\theta_1} + \frac{1}{\sin^2\theta_1}\Lambda_{\theta_2}.$$

Let

$$F_m(\theta_1) := \int_0^{2\pi} f(\theta_1, \theta_2)\overline{u_m(\theta_2)}d\theta_2. \tag{4.24}$$

Then, by Parseval's identity,

$$\begin{aligned}
M_{\theta_1}F_m(\theta_1) &= \int_0^{2\pi} M_{\theta_1} f(\theta_1, \theta_2)\overline{u_m(\theta_2)}d\theta_2, \\
\|M_{\theta_1} f\|^2_{L^2(\mathbb{S}^2)} &= \sum_{m\in\mathbb{Z}} \int_0^{\pi} |M_{\theta_1}F_m(\theta_1)|^2 \sin\theta_1 d\theta_1, \\
\int_{\mathbb{S}^2} (M_{\theta_1} f)\left(\frac{\Lambda_{\theta_2} f}{\sin^2\theta_1}\right) d\omega &= \sum_{m\in\mathbb{Z}} \lambda_m \int_0^{\pi} M_{\theta_1}F_m(\theta_1)\frac{\overline{F_m(\theta_1)}}{\sin^2\theta_1}\sin\theta_1 d\theta_1, \\
\int_{\mathbb{S}^2} \left|\frac{\Lambda_{\theta_2} f}{\sin^2\theta_1}\right|^2 d\omega &= \sum_{m\in\mathbb{Z}} \lambda_m^2 \int_0^{\pi} \left|\frac{F_m(\theta_1)}{\sin^2(\theta_1)}\right|^2 \sin\theta_1 d\theta_1.
\end{aligned} \tag{4.25}$$

Therefore, with $\|\cdot\|, (\cdot,\cdot)$ denoting the $L^2(\mathbb{S}^2)$ norm and inner product,

$$\begin{aligned}
\|\Lambda(\theta_1, \theta_2) f\|^2 =& \|M_{\theta_1} f\|^2 + 2\Re e(M_{\theta_1} f, \frac{\Lambda_{\theta_2}}{\sin^2\theta_1} f) + \left\|\frac{\Lambda_{\theta_2}}{\sin^2\theta_1} f\right\|^2 \\
=& \sum_{m\in\mathbb{Z}} \{ \int_0^{\pi} |M_{\theta_1}F_m(\theta_1)|^2 \sin\theta_1 d\theta_1 \\
&+ 2\lambda_m \Re e \int_0^{\pi} M_{\theta_1}F_m(\theta_1)\frac{\overline{F_m(\theta_1)}}{\sin^2(\theta_1)}\sin\theta_1 d\theta_1 \\
&+ \lambda_m^2 \int_0^{\pi} \left|\frac{F_m(\theta_1)}{\sin^2(\theta_1)}\right|^2 \sin\theta_1 d\theta_1 \}.
\end{aligned} \tag{4.26}$$

Let $x = \cos\theta_1$ and write $F_m(x)$ for $F_m(\theta_1)$. Then, on using Lemma 5 and integrating by parts, (4.26) becomes

$$\begin{aligned}
\|\Lambda(\theta_1,\theta_2)f\|^2 &= \sum_{m\in\mathbb{Z}}\{\int_{-1}^{1}\left|-\frac{d}{dx}\{(1-x^2)F_m'(x)\}\right|^2 dx \\
&\quad + 2\Re e[\lambda_m\int_{-1}^{1}\left(-\frac{d}{dx}\{(1-x^2)F_m'(x)\}\right)\frac{\overline{F_m(x)}}{(1-x^2)}dx] \\
&\quad + \lambda_m^2\int_{-1}^{1}\left|\frac{F_m(x)}{(1-x^2)}\right|^2 dx\} \\
&\geq \sum_{m\in\mathbb{Z}}\left[c^{-4}+\lambda_m^2\right]\int_{-1}^{1}\left|\frac{F_m(x)}{(1-x^2)}\right|^2 dx \\
&\quad + 2\Re e[\lambda_m\int_{-1}^{1}(1-x^2)F_m'\left\{\frac{\overline{F_m'(x)}}{(1-x^2)}+\frac{2x}{(1-x^2)^2}\overline{F_m}\right\}dx] \\
&\geq \sum_{m\in\mathbb{Z}}\{\left[c^{-4}+\lambda_m^2+\frac{2\lambda_m}{c^2}\right]\int_{-1}^{1}\left|\frac{F_m(x)}{(1-x^2)}\right|^2 dx \\
&\quad - 2\lambda_m\int_{-1}^{1}\left|\frac{F_m(x)}{(1-x^2)}\right|^2(1+x^2)dx\} \qquad (4.27)
\end{aligned}$$

by (4.15) and since

$$\begin{aligned}
\Re e\int_{-1}^{1}F_m'\overline{F_m}\frac{2x}{(1-x^2)}dx &= -\int_{-1}^{1}|F_m|^2\left[\frac{1}{(1-x^2)}+\frac{2x^2}{(1-x^2)^2}dx\right. \\
&= -\int_{-1}^{1}|F_m|^2\left(\frac{1+x^2}{(1-x^2)^2}\right)dx,
\end{aligned}$$

on integration by parts again. From (4.27) it follows that

$$\begin{aligned}
\|\Lambda(\theta_1,\theta_2)f\|^2 \geq & c^{-4}\sum_{m\in\mathbb{Z}}\int_{-1}^{1}\left|\frac{F_m}{1-x^2}\right|^2 dx+\sum_{m\in\mathbb{Z}}\lambda_m^2\int_{-1}^{1}\left|\frac{F_m}{1-x^2}\right|^2 dx \\
& - 2(2-c^{-2})\sum_{m\in\mathbb{Z}}\lambda_m\int_{-1}^{1}\left|\frac{F_m}{1-x^2}\right|^2 dx \\
= & c^{-4}\left\|\frac{f}{\sin^2\theta_1}\right\|^2+\left\|\frac{\Lambda_{\theta_2}f}{\sin^2\theta_1}\right\|^2 \\
& - 2(2-c^{-2})\left(\frac{\Lambda_{\theta_2}f}{\sin^2\theta_1},\frac{f}{\sin^2\theta_1}\right) \\
\geq & \left\{c^{-4}-\varepsilon^{-1}(2-c^{-2})\right\}\left\|\frac{f}{\sin^2\theta_1}\right\|^2 \\
& + \left\{1-\varepsilon(2-c^{-2})\right\}\left\|\frac{\Lambda_{\theta_2}f}{\sin^2\theta_1}\right\|^2
\end{aligned}$$

for any $\varepsilon > 0$. The inequality (4.17) follows on choosing $\varepsilon$ sufficiently small.

We also have from (4.27)

$$\|\Lambda(\theta_1, \theta_2) f\|^2 \geq \sum_{m \in \mathbb{Z}} K_m \int_{-1}^{1} \left| \frac{F_m}{(1-x^2)} \right|^2 dx, \tag{4.28}$$

where

$$K_m = \left( c^{-4} + \lambda_m^2 - 2(2 - c^{-2})\lambda_m \right).$$

A calculation of the constant $c$ gives

$$c = 1.19967864$$

to 8 decimal places. On recalling that $\lambda_m = (m + \tilde{\Psi})^2$, it follows that the $K_m$ are positive except for the following exceptions:

- if $0 \leq \tilde{\Psi} < 0.447566, K_1, K_{-1} < 0$;
- if $0.447566 < \tilde{\Psi} < 0.552434, K_{-2}, K_{-1}, K_0, K_1 < 0$;
- if $0.552434 < \tilde{\Psi} < 1, K_0, K_{-2} < 0$.

This completes the proof. □

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