SCATTERING AMPLITUDE FOR THE SCHRÖDINGER EQUATION WITH STRONG MAGNETIC FIELD AND STRONG ELECTRIC POTENTIAL

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ABSTRACT. In this paper, we study the scattering amplitude for the Schrödinger equation with constant magnetic field. If the electric potential and the magnetic field have the same strength b, the behavior when $b \to +\infty$ is semi-classical. Under a non-trapping condition we approximate the scattering amplitude by a pseudo-differential operator.

1. INTRODUCTION

In this paper, we are interested in the Schrödinger equation with constant magnetic field

$$i\partial_t \psi = \mathrm{H}(b)\psi,$$

where $\mathrm{H}(b) = \mathrm{H}_0(b) + bV$ with $V \in L^\infty(\mathbb{R}^n)$ and

$$\mathrm{H}_0 = \left(i\frac{\partial}{\partial x} + \frac{b}{2}y\right)^2 + \left(i\frac{\partial}{\partial y} - \frac{b}{2}x\right)^2 - \Delta_z,$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}^{n-2}$ and Δ_z denotes the Laplacian on $\mathbb{R}_z^{n-2}$, with $n \geq 3$. Under suitable assumptions on V, it is well-known (see [1]) that the scattering operator $\mathbf{S} = \mathbf{S}(b)$ associated to the pair $(\mathrm{H}_0, \mathrm{H})$ is well defined. Our aim is to describe this operator when b goes to infinity. Notice that because of parameter b in front of V the limit under consideration deals with the case where the magnetic field and the electric potential have the same strength b. In all the paper, we will assume that V can be splitted into

$$V(x, y, z) = V^\infty(z) + W(x, y, z)$$

with $V^\infty \in C_0^\infty(\mathbb{R}^{n-2})$, $W \in C_0^\infty(\mathbb{R}^n)$ and $V, V^\infty \geq 0$. Under this assumption, the scattering matrix $S(E, b)$ was defined in [16] away from the point spectrum of $\mathrm{H}(b)$. The term "matrix" has to be understood differently according to the dimension n. In the case $n = 3$, $S(E, b)$ is a two by two matrix with coefficient in $\mathcal{L}(L^2(\mathbb{R}^2_{x,y}))$, whereas for $n \geq 4$ it is a unitary operator on $L^2(S^{n-3} \times \mathbb{R}^2_{x,y})$. Moreover it can be represented by mean of the Fourier transform and the resolvent as in the case of the Schrödinger equation. More precisely, let us denote

$$\widehat{\mathrm{H}}_0 = \left(i\frac{\partial}{\partial x} + \frac{b}{2}y\right)^2 + \left(i\frac{\partial}{\partial y} - \frac{b}{2}x\right)^2 \tag{1.1}$$

the Schrödinger operator with constant magnetic field on $L^2(\mathbb{R}^2)$. It is well-known that the spectrum of $\widehat{\mathrm{H}}_0$ is pure point [1] and given by the sequence of Landau levels $\mathbb{L} = \sigma(\widehat{\mathrm{H}}_0) = \sigma_{pp}(\widehat{\mathrm{H}}_0) = \{b\Lambda_q,\ q \in \mathbb{N}^*\}$, where $\Lambda_q = 2q - 1$, $\forall q \in \mathbb{N}^*$. Let

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us denote $\widetilde{\Pi}_q : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ the projector on the eigenspace associate to $b\Lambda_q$. We set $L^2_\alpha(\mathbb{R}^{n-2}_z) = \{f \in \mathcal{D}'(\mathbb{R}^{n-2}_z);\ \langle z\rangle^\alpha f \in L^2(\mathbb{R}^{n-2}_z)\}$ with $\langle z\rangle = (1+|z|^2)^{1/2}$ and we define $\mathcal{H}$ by $\mathcal{H} = \mathbb{R}^2$ if $n = 3$ and $\mathcal{H} = L^2(S^{n-3})$ if $n \geq 4$. For $E > 0$ we define $\widetilde{\mathcal{F}}_0(E) : L^2_\alpha(\mathbb{R}^{n-2}_z) \to \mathcal{H}$ by

$$\widetilde{\mathcal{F}}_0(E)\varphi = \frac{E^{-1/4}}{\sqrt{4\pi}} \int_{\mathbb{R}} (e^{-i\sqrt{E}\langle z,\xi\rangle}\varphi(z), e^{i\sqrt{E}\langle z,\xi\rangle}\varphi(z)dz$$

if $n = 3$ and

$$\widetilde{\mathcal{F}}_0(E)\varphi(\xi) = \frac{E^{\frac{n-4}{4}}}{\sqrt{2}(2\pi)^{\frac{n-2}{2}}} \int_{\mathbb{R}^{n-2}} e^{-i\sqrt{E}\langle z,\xi\rangle}\varphi(z)dz,$$

if $n \geq 4$. For $E > b\Lambda_1 = b$, we set

$$\begin{aligned} \mathcal{F}_0(E) :\ & L^2(\mathbb{R}^2_{x,y}, L^2_\alpha(\mathbb{R}^{n-2}_z)) \to L^2(\mathbb{R}^2_{x,y}, \mathcal{H}) \\ & \varphi \mapsto \sum_{b\Lambda_q \leq E} \widetilde{\Pi}_q \otimes \widetilde{\mathcal{F}}_0(E - b\Lambda_q)\varphi \end{aligned}$$

and in all dimensions we define

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^*_+, L^2(\mathbb{R}^2 \times S^{n-3}), dE)$$

by $\mathcal{F}\varphi(E) = \mathcal{F}_0(E)\varphi$. Remark that for $\varphi \in L^2(\mathbb{R}^n)$, $\mathcal{F}_0(E)\varphi$ makes sense only as an L^2 function with respect to E. The operator $\mathcal{F}$ is a unitary isomorphism and

- $\mathcal{F}\,\mathrm{H}_0\,\mathcal{F}^*$ is the multiplication by E on $L^2(\mathbb{R}^*_+, L^2(\mathbb{R}^2 \times S^{n-3}), dE)$
- for all $t > 0$, $\mathcal{F}S(b)\mathcal{F}^*$ and $e^{it\mathcal{F}\,\mathrm{H}_0\,\mathcal{F}^*}$ commute.

Hence (cf. [20], Theorem XIII.84), there exists a function $E \mapsto S(E, b)$ in the space $L^\infty(\mathbb{R}^*_+, \mathcal{L}(L^2(\mathbb{R}^2 \times S^{n-3})))$ such that

$$\forall \varphi \in L^2(\mathbb{R}^n),\ (\mathcal{F}S(b)\varphi)(E) = S(E, b)(\mathcal{F}\varphi)(E).$$

For $E > 0$, $S(E, b)$ is called the scattering matrix (it is a matrix only in the case $n = 3$). Under our assumptions on the potential V, this operator can be represented in a convenient way. Let K be a compact subset of $\mathbb{R}^{n-2}_z$ such that $\operatorname{supp} V \subset \mathbb{R}^2_{x,y} \times K$ and let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^{n-2})$ such that $\chi_2 = 1$ in a neighborhood of K and $\chi_1 = 1$ on $\operatorname{supp}\chi_2$. Then, it was shown in [16] that for all $E \in]b\Lambda_1, \infty \setminus (\mathbb{L} \cup \sigma_{pp}(H))$

$$T(E, b) := S(E, b) - Id = -2i\pi\mathcal{F}_0(E)[\Delta_z, \chi_1]R(E + i0)[\Delta_z, \chi_2]\mathcal{F}_0(E)^*. \tag{1.2}$$

In dimension $n = 3$ this formula will permit us to study the coefficients of the matrix $T(E, b)$ whereas in higher dimension it shows that $T(E, b)$ has a kernel

$$(\omega, \omega') \in S^{n-3} \times S^{n-3} \mapsto T(\omega, \omega', E, b) \in \mathcal{L}(L^2(\mathbb{R}^2_{x,y}))$$

which is smooth with respect to (ω, ω'). An important difference between the Schrodinger operator without magnetic field and the case we study in this article is the following. In the case $A = 0$, the scattering amplitude $T(., ., E, b)$ is a complex-valued function whereas in our case it takes its values in the space of bounded operators on $L^2(\mathbb{R}^2)$. The aim of this paper is to study this kernel for energies far from the Landau levels in the following sense. We take $\lambda \notin 2\mathbb{N}+1$ and we study the behavior of $T(\omega, \omega', \lambda b, b)$ when $b \to +\infty$. Remark that the distance between the energy λb and the set of Landau levels is proportional to b and so goes to infinity.

Throughout this paper, we will use the h-pseudo-differential calculus in a standard way (see [4], [11], [21]). Let us recall briefly the notations. For $m : \mathbb{R}^d \to$

$[0, +\infty[$ an order function (see Definition 7.5 in [4]) and $\delta \in [0,1]$, we say that $a(x,h) \in C^\infty(\mathbb{R}^d \times]0,1])$ belongs to the class $S^\delta(\mathbb{R}^d, m, h)$ if

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0,\ |\partial_x^\alpha a(x,h)| \leq C_\alpha h^{-\delta|\alpha|} m(x).$$

In the special case where $m = 1$, we write $S^\delta(\mathbb{R}^d, h)$ instead of $S^\delta(\mathbb{R}^d, 1, h)$ and if the semi-classical parameter h is unambiguous we write $S^\delta(\mathbb{R}^d)$. When the symbol a is an operator in $\mathcal{L}(F,G)$ (F, G being subspaces of a Hilbert space), the notation $|.|$ above has to be understood as a norm in $\mathcal{L}(F,G)$ and the associated class of symbols will be denoted by $S^\delta(\mathbb{R}^d, m, \mathcal{L}(F,G), h)$. We refer to the appendix of [2] for a a nice review of elementary properties of pseudo-differential operators with operator valued symbols. As in [2], for p in a suitable class of symbol, we will denote by $p^w(x, hD_x)$ the standard Weyl-quantization of p.

Before setting our theorems, we may discuss the elementary but instructive example where the potential V depends only on z. Then $V(z)$ and the projectors $\widetilde{\Pi}_q$ commute, so that

$$R(\lambda b + i0) = \sum_{q\in\mathbb{N}} (-\Delta_z + bV(z) - b(\lambda - \Lambda_q) - i0)^{-1} \otimes \widetilde{\Pi}_q.$$

Setting $h = b^{-1/2}$, it follows from (1.2) and the definition of $\mathcal{F}_0(E)$ that

$$T(\omega, \omega', \lambda b, b) = \sum_{\Lambda_q \leq \lambda} T_L(\omega, \omega', \lambda - \Lambda_q, h)\widetilde{\Pi}_q$$

where $T_L(\omega, \omega', \mathcal{E}, h)$ denotes the scattering amplitude for the pair $(-h^2\Delta_z, -h^2\Delta_z + V(z))$ at the energy $\mathcal{E}$. On the other hand, it is well-known since the works of Vainberg [23] and Robert-Tamura [22] that the asymptotics behavior of the scattering amplitude at the energy $\mathcal{E}$ depends on the nature of the classical trajectories on this energy surface. More precisely, when $\mathcal{E}$ is non-trapping (in the sense of [22]) they describe $T_L(\omega, \omega', \mathcal{E}, h)$ in terms of the classical particle scattered from the direction ω to the direction ω'. Therefore, in our case it is necessary that all the energy levels $\lambda - \Lambda_q$ (for $\Lambda_q < \Lambda$) are non-trapping for the symbol $p(z,\xi) = \xi^2 + V(z)$. This suggests the following assumptions in the general case and closes our parenthesis on this particular case.

From now, we assume that $\lambda \in]2q_0 - 1, 2q_0 + 1[$, $q_0 \in \mathbb{N}^*$. For $(x,y) \in \mathbb{R}^2$, let us denote $p_{x,y}(z,\xi) = \xi^2 + V(x,y,z)$, $\forall (z,\xi) \in T^*\mathbb{R}^{n-2}$ and $t \mapsto \exp(tH_{p_{x,y}})(z,\xi)$ the solution of the Hamiltonian system

$$\dot{Z} = 2Z^*,\ \dot{Z}^* = -\nabla_z V(x,y,Z) \tag{1.3}$$

with initial condition (z,ξ).

In all the paper, we will denote $\lambda_q = \lambda - \Lambda_q$. We make the following non-trapping assumption

Assumption 1. *For all $q = 1, \ldots, q_0$ and all $(x,y) \in \mathbb{R}^2$,*

$$\lim_{|t|\to\infty} |\exp(tH_{p_{x,y}})(z,\xi)| = +\infty$$

for all $(z,\xi) \in T^*\mathbb{R}^{n-2}$ such that $\xi^2 + V(x,y,z) = \lambda - \Lambda_q$.

Remark that if the λ_q, $q = 1, \ldots, q_0$ are non-trapping for the symbol $\xi^2 + V^\infty(z)$ in the sense of [22] and $W, \nabla_z W$ are small enough in C^0 norm, then Assumption 1 is satistied. On the other hand, if we suppose that V^∞ is identically null, that is $V \in C_0^\infty(\mathbb{R}^n)$, then a necessary condition for Assumption 1 to be satisfied is

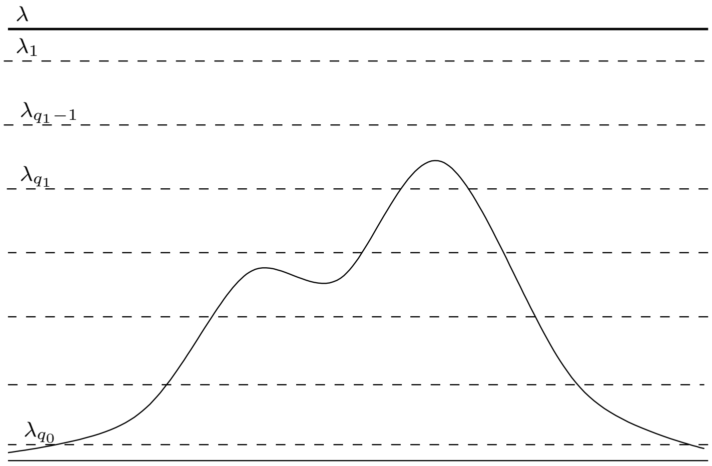

FIGURE 1. A potential satisfying Assumption 1

that $\sup_{\mathbb{R}^n} |V| < \lambda_{q_0}$. It is to avoid such restrictions that we chosed to study perturbations of potentials depending only on the variable z.

In dimension $n = 3$, the structure of the classical scattered trajectories is quiet simple and the preceding assumption is sufficient to state a theorem. Let us denote

$$S(E,b) = \begin{pmatrix} S_{11}(E,b) & S_{12}(E,b) \\ S_{21}(E,b) & S_{22}(E,b) \end{pmatrix}$$

the scattering matrix in dimension 3, with $S_{ij}(E,b) \in \mathcal{L}(L^2(\mathbb{R}))$. It is well known that $S_{11}(E,b) = S_{22}(E,b)$ and $S_{12}(E,b)S_{22}(E,b)^* = -S_{11}(E,b)S_{21}(E,b)^*$ so that we only have to study S_{11} and S_{12}. From the work of Knauf [10], we know that the trapping energies are essentialy critical points of the potential. More precisely, from Assumption 1, we deduce easily that there exists $q_1 \in \{1, \dots, q_0 + 1\}$ such that:

- for all $q \in \{1, \dots, q_1 - 1\}$ and all $(x,y) \in \mathbb{R}^2$, $\lambda_q > \sup_{z\in\mathbb{R}} V(x,y,z)$
- for all $q \in \{q_1, \dots, q_0\}$ and all $(x,y) \in \mathbb{R}^2$, the equation $V(x,y,z) = \lambda_q$ has exactly two solutions $\alpha_q(x,y) < \beta_q(x,y)$ and these solutions are non-critical points

Once again, we remark that α_q and β_q are smooth functions of the variable (x,y) and are independent on (x,y) when $|(x,y)| >> 1$.

Theorem 1. *Suppose that $n = 3$ and that Assumption 1 is fulfilled, then the diagonal coefficients are given by*

$$S_{11}(\lambda b, b) = \sum_{q=1}^{q_1-1} s^w_{d,q}(\lambda, y/2 - b^{-1}D_x, x/2 - b^{-1}D_y)\widetilde{\Pi}_q + \sum_{q=q_0+1}^{+\infty} \widetilde{\Pi}_q + \mathcal{O}(b^{-\infty})$$

in $\mathcal{L}(L^2(\mathbb{R}^2))$, with $s_{d,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$ and

$$s_{d,q}(\lambda, y, \eta) = \exp(ib^{1/2}\int_{-\infty}^{+\infty} \sqrt{\lambda_q - V(\eta,y,z)} - \sqrt{\lambda_q}dz) + b^{-1/2}r_{d,q}(\lambda,y,\eta),$$

where $r_{d,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$.

The off-diagonal coefficients satisfy

$$S_{21}(\lambda b, b) = \sum_{q=q_1}^{q_0} s^w_{a,q}(\lambda, y/2 - b^{-1}D_x, x/2 - b^{-1}D_y)\widetilde{\Pi}_q + \mathcal{O}(b^{-\infty})$$

in $\mathcal{L}(L^2(\mathbb{R}^2))$, *with* $s_{a,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$ *and*

$$s_{a,q}(\lambda, y, \eta) = i \exp(2ib^{1/2}(\sqrt{\lambda_q}\alpha_q(\eta, y) + \int_{-\infty}^{\alpha_q(\eta,y)} \sqrt{\lambda_q - V(\eta, y, z)} - \sqrt{\lambda_q} dz)) + b^{-1/2} r_{a,q}(\lambda, y, \eta),$$

where $r_{a,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$

and

$$S_{12}(\lambda b, b) = \sum_{q=q_1}^{q_0} s^w_{e,q}(\lambda, y/2 - b^{-1}D_x, x/2 - b^{-1}D_y)\widetilde{\Pi}_q + \mathcal{O}(b^{-\infty})$$

in $\mathcal{L}(L^2(\mathbb{R}^2))$, *with* $s_{e,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$ *and*

$$s_{e,q}(\lambda, y, \eta) = i \exp(2ib^{1/2}(\sqrt{\lambda_q}\beta_q(\eta, y) - \int_{\beta_q(\eta,y)}^{+\infty} \sqrt{\lambda_q - V(\eta, y, z)} - \sqrt{\lambda_q} dz)) + b^{-1/2} r_{e,q}(\lambda, y, \eta)$$

where $r_{e,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$.

Let us remark that the above result is proved under the only assumption that the λ_q are non-trapping. In the case where some of the energies are trapping, the problem may be treated by combining method used in this work with the paper of Ramond [19].

When $n \geq 4$, we need an additional assumption on the classical flow. Now, let us fix ω and ω' in S^{n-3} such that $\omega \neq \omega'$. We denote by Υ_ω the hyper-plane orthogonal to ω. We will denote by $\tilde{z} = (z_1, \ldots, z_{n-3})$ the variable in Υ_ω and we set $\hat{z} = (\tilde{z}, 0) \in \mathbb{R}^{n-2}$. As V is compactly supported in the variable z, out of a compact set the solutions of (1.3) are straight lines and it is easy to see that for all $q = 1, \ldots, q_0$ and $\tilde{z} \in \Upsilon_\omega$, there exists a unique solution $(Z_{q,\infty}(t, x, y, \tilde{z}, \omega), Z^*_{q,\infty}(t, x, y, \tilde{z}, \omega))$ of (1.3) such that for $-t > 0$ large enough

$$Z_{q,\infty}(t, x, y, \tilde{z}, \omega) = 2\sqrt{\lambda_q}\omega t + \hat{z}$$

Notice that $Z_{q,\infty}$ and $Z^*_{q,\infty}$ depend on (x, y). When it will be unambiguous, we will drop this dependence to lighten the notations.

Under Assumption 1, we can precise the behavior of these particles when t goes to $+\infty$. Indeed, V being compactly supported with respect to z, there exists $\theta_{q,\infty}(x, y, \tilde{z}, \omega) \in S^{n-3}$ and $r_{q,\infty}(x, y, \tilde{z}, \omega) \in \mathbb{R}^{n-2}$ such that for $t > 0$ large enough

$$Z_{q,\infty}(t, x, y, \tilde{z}, \omega) = 2\sqrt{\lambda_q}\theta_{q,\infty}(x, y, \tilde{z}, \omega)t + r_{q,\infty}(x, y, \tilde{z}, \omega) \tag{1.4}$$

$$Z^*_{q,\infty}(t, x, y, \tilde{z}, \omega) = \sqrt{\lambda_q}\theta_{q,\infty}(x, y, \tilde{z}, \omega). \tag{1.5}$$

For $\tilde{z} \in \Upsilon_\omega$, let us define the angular densities by

$$\widehat{\sigma}_q(x, y, \tilde{z}) = |\det(\theta_{q,\infty}, \partial_{z_1}\theta_{q,\infty}, \ldots, \partial_{z_{n-3}}\theta_{q,\infty})| \tag{1.6}$$

Assumption 2. *We say that the final direction* ω' *is regular with respect to* ω *if for all* $q \in \{1, \ldots, q_0\}$, $(x, y) \in \mathbb{R}^2$ *and all* $\tilde{z} \in \Upsilon_\omega$ *with* $\theta_{q,\infty}(x, y, \tilde{z}) = \omega'$, *we have* $\widehat{\sigma}_q(x, y, \tilde{z}) \neq 0$.

Once again, let us remark that if we suppose that W and $\nabla_z W$ are small enough, Assumption 2 will be satisfied as soos as for all $q \in \{1, \dots, q_0\}$, ω' is regular with respect to ω at the energy level λ_q for the symbol $\xi^2 + V^\infty(z)$ in the sense of [22]. This permits in particular to build potentials satisfying Assumptions 1 and 2.

Assume that the direction ω' is regular with respect to ω. For $q \in \{1, \dots, q_0\}$ and $(x, y) \in \mathbb{R}^2$ fixed, we deduce from the implicit function theorem that there exist $l_q = l_q(x, y) \in \mathbb{N}$ and $\tilde{z}_{q,1}(x, y), \dots, \tilde{z}_{q,l_q}(x, y) \in \Upsilon_\omega$ such that

$$\theta_{q,\infty}(x, y, z) = \omega' \Longleftrightarrow z \in \{\tilde{z}_{q,1}, \dots, \tilde{z}_{q,l_q}\}.$$

Moreover, working as in Proposition 1.9 of [14], we deduce from the local inverse theorem that the function $(x, y) \mapsto l_q(\text{x,y})$ is constant on $\mathbb{R}^2$. The inverse function theorem shows also that the functions $(x, y) \mapsto \tilde{z}_{q,j}(x, y)$ are C^∞ with respect to (x, y). On the other hand, W being compactly supported they are constant at infinity and therefore belongs to the class $S^0(\mathbb{R}^2)$ defined below.

Theorem 2. *Suppose that $n \geq 4$ and that $(\omega, \omega') \in S^{n-3} \times S^{n-3}$ with $\omega \neq \omega'$ satisfies Assumptions 1 and 2. Then $\lambda b \notin \sigma_{pp}(\mathrm{H}(b))$ and there exists a sequence $T_{q,j}(\omega, \omega', \lambda, b), q = 1, \dots, q_0,\ j \in \mathbb{N}$ of symbols in $S^{1/2}(\mathbb{R}^2, b^{-1})$ such that for all $N \in \mathbb{N}$,*

$$T(\omega, \omega', \lambda b, b) = b^{\frac{n-3}{4}} \sum_{q=1}^{q_0} \lambda_q^{\frac{n-3}{4}} T_q(\omega, \omega', \lambda b, b) \widetilde{\Pi}_q + \mathcal{O}(b^{\frac{n-3}{4} - N})$$

in $\mathcal{L}(L^2(\mathbb{R}^2))$, with

$$T_q(\omega, \omega', \lambda b, b) = \sum_{j=0}^{N} b^{-j/2} T^w_{q,j}(\omega, \omega', \lambda, b, y/2 - b^{-1} D_x, x/2 + b^{-1} D_y).$$

Moreover,

$$T_{q,0}(\omega, \omega', \lambda, b, y, \eta) = c(n) \sum_{l=1}^{l_q} \widehat{\sigma}_q(\eta, y, \tilde{z}_{q,l}(\eta, y))^{-1/2} e^{ib^{1/2}\mathbf{S_{q,l}}(y,\eta) - i\mu_{q,l}\pi/2}$$

where $c(n) = \frac{1}{2} e^{i(n-3)\pi/4} (2\pi)^{-(n-3)/2}$,

$$\begin{aligned}\mathbf{S_{q,l}}(y, \eta) = \int_{-\infty}^{+\infty} (|Z^*_{q,\infty}(t, \eta, y, \tilde{z}_{q,l}(\eta, y), \omega)|^2 \\ - V(\eta, y, Z_{q,\infty}(t, \eta, y, \tilde{z}_{q,l}(\eta, y), \omega)) - \lambda + \Lambda_q) dt \\ - r_{q,\infty}(\eta, y, \tilde{z}_{q,l}(\eta, y), \omega)\end{aligned}$$

and $\mu_{q,l}$ is the Maslov index of the trajectory $(Z_{q,\infty}, Z^*_{q,\infty})(t, \eta, y, \tilde{z}_{q,l}(\eta, y), \omega)$ on the Lagrangian manifold

$$\{(z, \xi) \in T^*\mathbb{R}^{n-2} \mid z = Z_{q,\infty}(t, \eta, y, \tilde{z}, \omega), \xi = Z^*_{q,\infty}(t, \eta, y, \tilde{z}, \omega)), \tilde{z} \in \Upsilon_q,\ t \in \mathbb{R}\}$$

and $\mu_{q,l}$ is independent on (y, η).

Remark that thanks to the special form of the symbol, the operators f_q and $\widetilde{\Pi}_q$ above commute. This will be more evident in the proof where, after unitary transformation, these operators have separate variables.

This result should be compared with the results obtained by Vainberg [23], Robert-Tamura [22] for the Schrödinger equation without magnetic field in the semi-classical setting. In their case, the only variable is z and the scattering amplitude is a function whose principal term is described by the underlying Hamiltonian system. Here we show that the scattering amplitude is a pseudo-differential operator in the variable (x, y) and roughly speaking, the principal term is given by the quantization (with respect to (x, y)) of the term of Vainberg, Robert-Tamura for the potential $z \mapsto V(x, y, z)$ where (x, y) is considered as a parameter. In the case where $V = V^\infty(z)$, this is coherent with the discussion preceding Theorem 1.

The plan of the paper is the following. In the next part we reduce our problem to a semi-classical problem for a matrix Schrödinger operator. In the third section, we obtain resolvent estimates under the non-trapping assumption and we show that up to small error terms this matrix can be taken to be diagonal. The fourth section is devoted to microlocal estimates of the resolvent and Egorov lemma for our problem. In section 5 we give an approximate representation of the evolution in terms of a pseudo-differential operator that we combine with stationary phase method in the sixth section to obtain Theorem 2. In section 7 we sketch the proof of Theorem 1 using the case $n \geq 4$.

2. Reduction by mean of an effective Hamiltonian

The aim of this section is to introduce the effective Hamiltonian associate to the operator $\mathrm{H}(b)$ and the parameter λ. As usual, this is done by considering a suitable Grushin problem. We refer to [3] and [24] for connected problems involving similar technics. First we introduce a well-known unitary operator that transforms $\widehat{\mathrm{H}}_0$ into the harmonic oscillator. For $f \in L^2(\mathbb{R}^n)$, let us set

$$\mathrm{U}\, f(x, y, z) = \frac{b^{3/4}}{2\pi} \int_{\mathbb{R}^2} e^{i\varphi_b(x,y,x',y')} f(x', y', z)dx'dy',$$

with $\varphi_b(x, y, x', y') = \frac{b}{2}xy + b^{1/2}(y' - y)x' - bxy'$. Then U is unitary from $L^2(\mathbb{R}^n)$ onto itself and it is well-known that

$$\mathrm{U}\,\mathrm{H}(b)\,\mathrm{U}^* = -\Delta_z + bN_x + bV^w(b^{-1}D_y + b^{-1/2}D_x, y - b^{-1/2}x, z), \tag{2.1}$$

where $N_x = -\frac{d^2}{dx^2} + x^2$ is the harmonic oscillator and we use the notation $D_x = \frac{1}{i}\frac{\partial}{\partial x}$. From [3], we know that U is in fact a metaplectic operator associate to the linear canonical transformation

$$\kappa(x, y, x^*, y^*) = (\frac{1}{\sqrt{b}}x^* + \frac{1}{b}y^*, y - \frac{x}{\sqrt{b}}, -\frac{\sqrt{b}}{2}x - \frac{b}{2}y, \frac{y^*}{2} - \frac{\sqrt{b}}{2}x^*),$$

whose it is easy to compute the inverse

$$\kappa^{-1}(x, y, x^*, y^*) = (-\frac{\sqrt{b}}{2}y - \frac{1}{\sqrt{b}}x^*, \frac{y}{2} - \frac{x^*}{b}, \frac{\sqrt{b}}{2}x - \frac{1}{\sqrt{b}}y^*, \frac{b}{2}x + y^*). \tag{2.2}$$

Moreover the right-hand side of (2.1) is self-adjoint with domain $\mathcal{D} = D(N_x) \otimes L^2(\mathbb{R}_y) \otimes H^2(\mathbb{R}_z^{n-2})$, where $D(N_x)$ is the domain of the harmonic oscillator. We refer to [2] and [18] for more details. Now, we introduce the semi-classical parameter $h = b^{-1/2} > 0$ which will tend to 0 as b goes to infinity. Hence, (2.1) takes the form

$$\mathrm{U}\,\mathrm{H}(b)\,\mathrm{U}^* = b(-h^2\Delta_z + N_x + V^w(h^2D_y + hD_x, y - hx, z)). \tag{2.3}$$

Let us introduce the eigenfunctions of the harmonic oscillator. For $j \in \mathbb{N}$, we denote by $\phi_j \in L^2(\mathbb{R}_x)$ the function such that $N_x\phi_j = \Lambda_j\phi_j$ and $\|\phi_j\|_{L^2(\mathbb{R})} = 1$. We define

R_- from $L^2(\mathbb{R}^{n-1}_{y,z})^{q_0}$ into $L^2(\mathbb{R}^n)$ by

$$R_-(\varphi_1,\dots,\varphi_{q_0})(x,y,z)=\sum_{q=1}^{q_0}\varphi_q(y,z)\phi_q(x),$$

and R_+ from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^{n-1}_{y,z})^{q_0}$ by

$$R_+\phi=(\langle\phi,\phi_1\rangle_{L^2(\mathbb{R}_x)},\dots,\langle\phi,\phi_{q_0}\rangle_{L^2(\mathbb{R}_x)}).$$

Let us denote by $\Pi_q=\mathrm{U}^*\,\widetilde{\Pi}_q\otimes\mathrm{Id}_z\,\mathrm{U}$ the projection onto $Span(\phi_q)\otimes L^2(\mathbb{R}^{n-1}_{y,z})$, $\Pi=\sum_{q=1}^{q_0}\Pi_q$ and $\widehat{\Pi}=\mathrm{Id}-\Pi$. Then we have

$$R_+R_-=\mathrm{Id}\ \text{ and }\ R_-R_+=\Pi \tag{2.4}$$

Let us introduce $F_{q_0}=\{s\in\mathbb{C},\operatorname{Im}s\geq 0\text{ and }\operatorname{Re}s=\lambda\}$. For $s\in F_{q_0}$, we consider the operator $\mathcal{P}(s):\ \mathcal{D}\times L^2(\mathbb{R}^{n-1}_{y,z})^{q_0}\to L^2(\mathbb{R}^n)\times\widetilde{\mathcal{D}}^{q_0}$ defined by

$$\mathcal{P}(s)=\begin{pmatrix}\widetilde{P}(h)-s & R_-\\ R_+ & 0\end{pmatrix}, \tag{2.5}$$

with $\widetilde{P}(h)=-h^2\Delta_z+N_x+V^w(h^2D_y+hD_x,y-hx,z)$ and $\widetilde{\mathcal{D}}=L^2(\mathbb{R}_y,H^2(\mathbb{R}^{n-2}_z))$. For $u\in\mathcal{D}$ and $v\in\widetilde{\mathcal{D}}$ we set

$$\|u\|_{\mathcal{D}}=\|N_xu\|_{L^2(\mathbb{R}^n)}+\sum_{j=0}^{2}\|(hD_z)^ju\|_{L^2(\mathbb{R}^n)}\ \text{ and }\ \|v\|_{\widetilde{\mathcal{D}}}=\sum_{j=0}^{2}\|(hD_z)^jv\|_{L^2(\mathbb{R}^{n-1}_{y,z})}.$$

Proposition 2.1. *For $s\in F_{q_0}$, the operator $\mathcal{P}(s)$ is invertible and its inverse $\mathcal{E}(s)$: $L^2(\mathbb{R}^n)\times\widetilde{\mathcal{D}}^{q_0}\to\mathcal{D}\times L^2(\mathbb{R}^{n-1}_{y,z})^{q_0}$ satisfies $\|\mathcal{E}(s)\|_{L^2(\mathbb{R}^n)\times\widetilde{\mathcal{D}}^{q_0},\mathcal{D}\times L^2(\mathbb{R}^{n-1}_{y,z})^{q_0}}=\mathcal{O}(1)$ uniformly with respect to $s\in F_{q_0}$ and $h\in]0,1]$.*

Proof. First we show that $\mathcal{P}(s)$ is injective. Suppose that $u\in\mathcal{D}$ and $u_-=(u_{1-},\dots,u_{q_0-})\in L^2(\mathbb{R}^{n-1}_{y,z})^{q_0}$ satisfy

$$\left\{\begin{array}{c}(\widetilde{P}(h)-s)u+R_-u_-=0\\ R_+u=0\end{array}\right. \tag{2.6}$$

From $\mathbb{R}_+u=0$ it follows that $u=\widehat{\Pi}u$. Hence, if we multiply the first equation of (2.6) by $\widehat{\Pi}$, we obtain

$$(\widehat{P}(h)-s)\widehat{\Pi}u=0,$$

where $\widehat{P}(h)=\widehat{\Pi}\widetilde{P}(h)\widehat{\Pi}:\ \widehat{\Pi}(\mathcal{D})\to\widehat{\Pi}(L^2(\mathbb{R}^n))$. As $V\geq 0$ it is clear that $\widehat{P}(h)$ is invertible for $s\in F_{q_0}$, so that $\widehat{\Pi}u=0$.

Now let us show that $\mathcal{P}(s)$ is surjective and take $(v,v_-)\in L^2(\mathbb{R}^n)\times\widetilde{\mathcal{D}}^{q_0}$. We have to solve

$$\left\{\begin{array}{c}(\widetilde{P}(h)-s)u+R_-u_-=v\\ R_+u=v_-\end{array}\right. \tag{2.7}$$

As $v_-\in\widetilde{\mathcal{D}}^{q_0}$,

$$u=R_-v_-+(\widehat{P}(h)-s)^{-1}(\widehat{\Pi}v-\widehat{\Pi}(\widetilde{P}(h)-s)R_-v_-)$$

is well-defined and belongs to $\mathcal{D}$. Hence we can set

$$u_-=R_+(v-(P(h)-s)u) \tag{2.8}$$

which belongs to $(L^2(\mathbb{R}^{n-1}_{y,z}))^{q_0}$. A simple calculation shows that (u, u_-) solves (2.7) so that $\mathcal{P}(s)$ is also surjective. Moreover, for $s \in F_{q_0}$ we know that $(\widehat{P}(h) - s)$ is invertible from $\widehat{\Pi}(\mathcal{D})$ onto $\widehat{\Pi}(L^2(\mathbb{R}^n))$ and using the fact that V is non negative, one shows easily that

$$\|(N_x - h^2\Delta_z)(\widehat{P}(h) - s)^{-1}u\|_{L^2} \leq C\|u\|_{L^2},$$

so that $\|(\widehat{P}(h) - s)^{-1}\|_{\widehat{\Pi}(L^2),\widehat{\Pi}(\mathcal{D})} = \mathcal{O}(1)$ uniformly with respect to $h \in]0, 1]$ and $s \in F_{q_0}$. Therefore,

(2.9)
$$\|u\|_{\mathcal{D}} \leq C(\|v_-\|_{\widetilde{\mathcal{D}}^{q_0}} + \|(\widehat{\Pi}v - \widehat{\Pi}(P(h) - s)R_-v_-\|_{L^2}) \leq C(\|v_-\|_{\widetilde{\mathcal{D}}^{q_0}} + \|v\|_{L^2})$$

where $C > 0$ denotes a constant that may change from line to line and is independant on s and h. Combining this estimate with (2.8), we obtain

$$\|u_-\|_{L^2(\mathbb{R}^{n-1})^{q_0}} \leq (\|v\|_{L^2} + \|u\|_{\mathcal{D}}) \leq C(\|v_-\|_{\widetilde{\mathcal{D}}^{q_0}} + \|v\|_{L^2}).$$

This equation together with (2.9) proves the estimate on $\mathcal{E}(s)$. □

As usual, we will use the Grushin problem to study the resolvent of $\widetilde{P}(h)$. For $s \in F_{q_0}$, we denote

$$\mathcal{E}(s) = \begin{pmatrix} E(s) & E_+(s) \\ E_-(s) & -E_\pm(s) \end{pmatrix}.$$

Then, for $s \in F_{q_0} \cap \{\operatorname{Im} s > 0\}$, we have the well-known relations $R_+E_+(s) = Id$, $E_-R_- = Id$, $\Pi E_+(s) = R_-$ and

$$(\widetilde{P}(h) - s)^{-1} = E(s) + E_+(s)E_\pm(s)^{-1}E_-(s), \tag{2.10}$$

where $E_\pm(s) = R_+(\widetilde{P}(h) - s)E_+$ is invertible for $s \in F_{q_0} \cap \{\operatorname{Im} s > 0\}$.

Proposition 2.2. *Assume that $s \in F_{q_0} \cap \{\operatorname{Im} s > 0\}$, then we have*

$$E(s) = (\widehat{P}(h) - s)^{-1}\widehat{\Pi}$$

and

$$E_\pm(s) = R_+(\widetilde{P}(h) - s)R_- - R_+\widetilde{P}(h)\widehat{\Pi}(\widehat{P}(h) - s)^{-1}\widehat{\Pi}\widetilde{P}(h)R_-. \tag{2.11}$$

Proof. Let $u \in \mathcal{L}^2(\mathbb{R}^n)$, then $\mathcal{E}(s)\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} E(s)u \\ E_-(s)u \end{pmatrix}$. Applying $\mathcal{P}(s)$, we obtain

$$\begin{cases} (\widetilde{P}(h) - s)E(s)u + R_-E_-(s)u = u \\ R_+E(s)u = 0 \end{cases}$$

The second equation yields $E(s)u = \widehat{\Pi}E(s)u$ so that the first one becomes

$$(\widetilde{P}(h) - s)\widehat{\Pi}E(s)u + R_-E_-(s)u = u.$$

Multiplying by $\widehat{\Pi}$ and using the fact that $\widehat{\Pi}R_- = 0$, we obtain $E(s)u = (\widehat{P}(h) - s)^{-1}\widehat{\Pi}u$. Now let us prove formula (2.11). Using the identities $\Pi E_+(s) = R_-$ and $R_+\widehat{\Pi} = \widehat{\Pi}R_- = 0$, it follows from the definition that,

$$E_\pm(s) = R_+(\widetilde{P}(h) - s)E_+(s) = R_+(\widetilde{P}(h) - s)R_- + R_+\widetilde{P}(h)\widehat{\Pi}E_+(s), \tag{2.12}$$

so that we have to compute $\widehat{\Pi}E_+(s)$. Let $u_- \in \widetilde{\mathcal{D}}^{q_0}$, then

$$\begin{pmatrix} 0 \\ u_- \end{pmatrix} = \mathcal{P}(s)\mathcal{E}(s)\begin{pmatrix} 0 \\ u_- \end{pmatrix} = \begin{pmatrix} (\widetilde{P}(h) - s)E_+(s)u_- - R_-E_\pm(s)u_- \\ R_+E_+(s)u_- \end{pmatrix}.$$

Therefore,

$$(\widetilde{P}(h)-s)\Pi E_+(s)u_- + (\widetilde{P}(h)-s)\widehat{\Pi}E_+(s)u_- - R_-E_\pm(s)u_- = 0$$

and multiplying again by $\widehat{\Pi}$, we obtain $\widehat{\Pi}E_+(s)u = -(\widehat{P}(h)-s)^{-1}\widehat{\Pi}(\widetilde{P}(h)-s)R_-u_-$. Combining this expression with (2.12), we obtain (2.11). □

For $s \in F_{q_0} \cap \{\operatorname{Im} s > 0\}$, we can multiply (2.10) by R_+ on the left and R_- on the right to get

$$R_+(\widetilde{P}(h)-s)^{-1}R_- = E_\pm(s)^{-1}. \tag{2.13}$$

Therefore, we would like to precise the form of the effective Hamiltonian $E_\pm(s)$. In the following lemma δ_{pq} denotes the Krönecker symbol.

Lemma 2.3. *Let $p,q \in \mathbb{N}^*$ and fix $N \in \mathbb{N}^*$, then there exists $W_{p,q,N} \in S^0(\mathbb{R}^n)$ such that*

$$\langle W^w(h^2D_y + hD_x, y - hx, z)\phi_p, \phi_q\rangle_{L^2(\mathbb{R}_x)} = W^w_{p,q,N}(y, h^2D_y, z, h) + R_N(h),$$

with $\|R_N\|_{L^2(\mathbb{R}^{n-1}_{y,z}),L^2(\mathbb{R}^{n-1}_{y,z})} = \mathcal{O}(h^N)$. Moreover, there exists a sequence $(q_{j,p,q})_{j\in\mathbb{N}}$ of symbols in $S^0(\mathbb{R}^n)$ such that $W_{p,q,N}(y,\eta,z,h) = \sum_{j=0}^N h^j q_{j,p,q}(y,\eta,z)$ with

$$\begin{aligned} q_{0,p,q}(y,\eta,z) &= \delta_{pq}W(\eta,y,z) \\ q_{1,p,q}(y,\eta,z) &= (1-\delta_{pq})\langle(\partial_\eta W(\eta,y,z)D_x - \partial_y W(\eta,y,z)x)\phi_p,\phi_q\rangle_{L^2(\mathbb{R}_x)} \\ q_{2,p,q}(y,\eta,z) &= \frac{\delta_{pq}}{4}\langle(\Delta_{x,y}W)(\eta,y,z)\phi_p,\phi_q\rangle_{L^2(\mathbb{R}_x)} + (1-\delta_{pq})\tilde{q}_{2,p,q}(y,\eta) \end{aligned}$$

with $\tilde{q}_{2,p,q} \in S^0(\mathbb{R}^n)$.

Proof. The proof is the same as in [2] and it suffices to make a Taylor expansion with respect to hD_x and hx. It follows that $W_{p,q,N}(y,\eta,z,h) = \sum_{j=0}^N h^j q_{j,p,q}(y,\eta,z)$ where the $q_{j,p,q},\ j=0,1,2$ are given by the above formula with

$$\tilde{q}_{2,p,q}(y,\eta,z) = \frac{1}{2}\langle\big((\partial_y^2 - \partial_\eta^2)W(\eta,y,z)x^2 - \partial_y\partial_\eta W(\eta,y,z)(x.D_x + D_x.x)\big)\,\phi_p,\phi_q\rangle,$$

and $R_N(h)$ is given by

$$R_N(h) = h^N\sum_{m=0}^N\int_0^1(1-t)^{N-1}\langle\widetilde{W}_l(\eta + htD_x, y - htx, z)D_x^l x^{N-l}\phi_p,\phi_q\rangle dt,$$

with $\widetilde{W}_l \in C_0^\infty(\mathbb{R}^n)$ As $\widetilde{W}_l$ is compactly supported and the ϕ_p decay exponentially at infinity as well as their derivatives, it is clear that $\|R_N\|_{L^2(\mathbb{R}^{n-1}_{y,z}),L^2(\mathbb{R}^{n-1}_{y,z})} = \mathcal{O}(h^N)$ and the proof is complete. □

Now we are in position to prove the main result of this section.

Proposition 2.4. *Suppose that $s \in F_{q_0}$. Then, there exists a sequence of matrix valued symbols $(E_j(y,\eta,z,\xi,s))_{j\in\mathbb{N}}$ such that $E_0 \in S^0(\mathbb{R}^{2n-2}, \langle\xi\rangle^2, \mathcal{L}(\mathbb{R}^{q_0}))$, $E_j \in S^0(\mathbb{R}^{2n-2}, 1, \mathcal{L}(\mathbb{R}^{q_0})), \forall j \geq 1$ and for all $N \in \mathbb{N}^*$,*

$$E_\pm(s) = \sum_{j=0}^N h^j E_j^w(y, h^2D_y, z, hD_z, s) + R_N(h,s),$$

with $\|R_N(h,s)\|_{L^2(\mathbb{R}^{n-1}_{y,z}),L^2(\mathbb{R}^{n-1}_{y,z})} = \mathcal{O}(h^N)$ uniformly with respect to $s \in F_{q_0}$. Moreover, we have

$$E_0(y,\eta,z,\xi,s) = \operatorname{diag}((\xi^2 + V(\eta,y,z) + \Lambda_j - s)_{j=1,\dots,q_0}),$$

E_1 is off-diagonal and for all $j \geq 1$ the seminorms of E_j are bounded uniformly with respect to $s \in F_{q_0}$.

Proof. We start from the formula giving $E_{\pm}(s)$ in Proposition 2.2. From Lemma 2.3, we deduce immediately that

$$R_+(\widetilde{P}(h) - s)R_- = \sum_{j=0}^{N} h^j \widetilde{E}_j^w(y, h^2 D_y, z, hD_z, s) + \widetilde{R}_N(h, s)$$

where $\widetilde{E}_0(y, \eta, z, \xi, s) = \mathrm{diag}((\xi^2 + V(\eta, y, z) + \Lambda_q - s)_{q=1,\dots,q_0})$, $\widetilde{E}_1$ is off-diagonal, $\forall j \geq 1, \widetilde{E}_j$ is independent on s, and $\|\widetilde{R}_N(h, s)\| = \mathcal{O}(h^N)$. Hence it remains to study the second term of the right-hand side of (2.11)

$$G(s) = R_+\widetilde{P}(h)\widehat{\Pi}(\widehat{P}(h) - s)^{-1}\widehat{\Pi}\widetilde{P}(h)R_-. \tag{2.14}$$

We start by building a parametrix for $\widehat{\Pi}(\widehat{P}(h) - s)^{-1}\widehat{\Pi}$. Let us denote $F = Span\{\phi_q,\ q \geq q_0 + 1\}$. Then $(\widehat{P}(h) - s) :\ \mathrm{Im}\,\widehat{\Pi} \to \mathrm{Im}\,\widehat{\Pi}$ satisfies

$$(\widehat{P}(h) - s) = P_0^w(y, h^2 D_y, z, hD_z, h)$$

with $P_0(y, \eta, z, \xi, h) = \widehat{\Pi}(\xi^2 + N_x + V^w(\eta + hD_x, y - hx, z) - s)\widehat{\Pi} \in S^0(\langle \xi^2 \rangle, \mathcal{L}(F))$. For $s \in F_{q_0}$, let us set $Q_0(y, \eta, z, \xi, h) = \widehat{\Pi}(\xi^2 + N_x + V(\eta + hD_x, y - hx, z) - s)^{-1}\widehat{\Pi}$. Using the fact that $V \geq 0$, it is clear that

$$\|(\xi^2 + N_x + V(\eta + hD_x, y - hx, z) - s)^{-1}\|_{\mathcal{L}(F)} = \mathcal{O}(\langle \xi \rangle^{-2})$$

uniformly with respect to $s \in F_{q_0}$, and hence $Q_0 \in S^0(\langle \xi \rangle^{-2}, \mathcal{L}(F))$. Therefore, the symbolic calculus yields

$$\begin{aligned}(\widehat{P}(h) - s)Q_0^w(y, h^2 D_y, z, hD_z, h) &= \widehat{\Pi} + h\widehat{\Pi}\widetilde{Q}_1^w(y, h^2 D_y, z, hD_z, h)\widehat{\Pi} \\ &\quad - \widehat{\Pi}[W^w(y, h^2 D_y, z, hD_z), \Pi]\widetilde{Q}_0^w(y, h^2 D_y, z, hD_z, h)\widehat{\Pi}\end{aligned} \tag{2.15}$$

where $\widetilde{Q}_0(y, \eta, z, \xi, h) = (\xi^2 + N_x + V(\eta + hD_0, y - hx, z) - s)^{-1}\widehat{\Pi}$ belongs to $S^0(\langle \xi \rangle^{-2}, \mathcal{L}(F, L^2(\mathbb{R}_x))$ and $\widetilde{Q}_1$ is in $S^0(\mathcal{L}(F))$. Now, let us examine the above commutator. Let $\phi \in L^2(\mathbb{R}^n)$ and denote $\mathcal{W} = W^w(y, h^2 D_y, z, hD_z)$, then

$$[\mathcal{W}, \Pi]\phi = \sum_{q=1}^{q_0} \left(\mathcal{W}(\langle \phi, \phi_q \rangle_{L^2(\mathbb{R}_x)}\phi_q) - \langle \mathcal{W}\phi, \phi_q \rangle \phi_q\right).$$

Working as in Lemma 2.3, it is not hard to see that

$$\begin{aligned}\mathcal{W}\Pi\phi = \sum_{q=1}^{q_0} \mathcal{W}(\langle \phi, \phi_q \rangle \phi_q) &= W^w(h^2 D_y, y, z)\Pi\phi \\ &\quad + h\sum_{q=1}^{q_0} \int_0^1 \partial_x W^w(h^2 D_y + thD_x, y - thx, z)dt \langle \phi, \phi_q \rangle D_x\phi_q \\ &\quad - h\sum_{q=1}^{q_0} \int_0^1 \partial_y W^w(h^2 D_y + thD_x, y - thx, z)dt \langle \phi, \phi_q \rangle x\phi_q \\ &= \Pi W^w(h^2 D_y, y, z)\Pi\phi + hW_1^w(y, h^2 D_y, z, hD_z)\phi\end{aligned}$$

where for $\varphi = \sum_{q=1}^{\infty} \lambda_q \phi_q,\ \lambda_1, \dots, \lambda_q \in \mathbb{R}$,

$$W_1(y,\eta,z,\xi)\varphi = \sum_{q=1}^{q_0} \lambda_q \int_0^1 \partial_x W^w(\eta + thD_x, y - thx, z) dt D_x \phi_q$$
$$- \sum_{q=1}^{q_0} \lambda_q \int_0^1 \partial_y W^w(\eta + thD_x, y - thx, z) dt x \phi_q.$$

Using again the exponential decay of the ϕ_q and the fact that W is compactly supported, we show that $W_1 \in S^0(1, \mathcal{L}(L^2(\mathbb{R}_x)))$. On the other hand, using the fact $\Pi\mathcal{W} = (\mathcal{W}\Pi)^*$ it is clear that $\Pi\mathcal{W} = \Pi W^w(h^2 D_y, y, z)\Pi + hW_2^w(y, h^2 D_y, z, hD_z)$ with $W_2 \in S^0(1, \mathcal{L}(L^2(\mathbb{R}_x)))$. Therefore, one can find $W_3 \in S^0(1, \mathcal{L}(L^2(\mathbb{R}_x)))$ such that $[\mathcal{W}, \Pi] = hW_3(y, h^2 D_y, z, hD_z)$. Combining this result with (2.15), we deduce from the symbolic calculus that

$$(\widehat{P}(h) - s) Q_0^w(y, h^2 D_y, z, hD_z, h) = \widehat{\Pi} + h\widehat{\Pi}\widetilde{\widetilde{Q}}_1^w(y, h^2 D_y, z, hD_z, h)\widehat{\Pi}, \tag{2.16}$$

with $\widetilde{\widetilde{Q}}_1 \in S^0(\mathcal{L}(F))$. Hence, we can iterate the process to get a parametrix at order N

$$(\widehat{P}(h) - s)^{-1} = \sum_{j=1}^{N} h^j Q_j^w(y, h^2 D_y, z, hD_z, h) + R_N(h) \tag{2.17}$$

with $\|R_N\| = \mathcal{O}(h^N)$. Now let us remark that

$$\widehat{\Pi}\widetilde{P}(h)R_- = [\widehat{\Pi}, \mathcal{W}]R_- = hW_3(y, h^2 D_y, z, hD_z)R_- \tag{2.18}$$

and

$$R_+\widetilde{P}(h)\widehat{\Pi} = hR_+ W_4(y, h^2 D_y, z, hD_z) \tag{2.19}$$

with $W_4 \in S^0(1, \mathcal{L}(L^2(\mathbb{R}_x)))$. Combining equations (2.14), (2.17), (2.18) and (2.19) it follows that $G(s)$ has the required form. □

3. Resolvent estimates

In this section, we prove that under the non-trapping condition on λ, λb is not an eigenvalue of H(b) and we establish a resolvent estimate. As usual, the main ingredient is the existence of a global escape function.

3.1. Construction of an escape function.

Proposition 3.1. *Suppose that λ satisfies Assumption 1 and let $q \in \{1, \dots, q_0\}$. Let K be a compact subset of $\cup_{(x,y)\in\mathbb{R}^2} p_{x,y}^{-1}([\lambda_q - \epsilon, \lambda_q + \epsilon])$. Then, one can find a function $G_q(x,y,z,\xi) \in C_b^\infty(\mathbb{R}^2 \times T^*\mathbb{R}^{n-2})$ such that $H_{p_{x,y}} G_q \geq 0$ and*

$$\forall (x,y) \in \mathbb{R}^2, \forall (z,\xi) \in K,\ H_{p_{x,y}} G_q \geq 1$$

Proof. The proof is based on the existence of an escape function for non-trapping energies [7], [8]. We only have to take care with the dependence on (x,y). Let $R > 0$ be such that $\operatorname{supp}_{x,y} W \subset B_{\mathbb{R}^2}(0,R)$ and $R' > 0$ such that $K \subset \{|z| \leq R'\}$. For all $(x,y) \in \mathbb{R}^2$, using assumption 1, we deduce from [7] that there exists $\widetilde{G}_{x,y} \in C_b^\infty(T^*\mathbb{R}^{n-2})$ such that $H_{p_{x,y}} \widetilde{G}_{x,y}(z,\xi) \geq 0$ and

$$H_{p_{x,y}} \widetilde{G}_{x,y}(z,\xi) \geq 1$$

for all (z,ξ) such that $|z| \leq R'$ and $p_{x,y}(z,\xi) \in [\lambda_q - \epsilon, \lambda_q + \epsilon]$. Recall that in [7], $\widetilde{G}_{x,y}$ is defined as the integral of a cut-off function along the flow $\exp(tH_{p_{x,y}})$. Hence, it follows from regularity theorems of solutions of ordinary differential equations with respect to parameters, that $\widetilde{G}_{x,y}(z,\xi)$ is C^∞ with respect to (x,y). Moreover, for $|(x,y)| \geq R$, $\widetilde{G}_{x,y}$ is independent on (x,y). Therefore, $(x,y,z,\xi) \mapsto G_q(x,y,z,\xi) := \widetilde{G}_{x,y}(z,\xi)$ is in $C_b^\infty(\mathbb{R}^2 \times T^*\mathbb{R}^{n-2})$ and the proof is complete. $\square$

Corollary 3.2. *Suppose that λ satisfies Assumption 1, then for b large enough, $\lambda b \notin \sigma_{pp}(\mathrm{H}(b))$.*

Proof. The proof is very classic. By unitary conjugation, it suffices to prove that λ is not an eigenvalue of $\widetilde{P}(h)$ for $h > 0$ small enough. Moreover, from the relation

$$\mathcal{E}(\lambda)\mathcal{P}(\lambda)\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} E_+R_+u \\ -E_\pm R_+u \end{pmatrix},$$

it suffices to prove that the kernel of $E_\pm(\lambda)$ is reduced to zero.

Making use of the escape function, we build a conjugate operator for $E_\pm(\lambda)$. For $q = 1,\dots,q_0$, let $\widetilde{g}_q \in C_0^\infty(\mathbb{R})$ be equal to 1 near λ_q and with disjoint supports. Then we can build $g_1,\dots,g_{q_0} \in C_0^\infty(\mathbb{R})$ with disjoint supports such that $g_q = 1$ on $\operatorname{supp}\widetilde{g}_q$. Let $\chi \in C_0^\infty(\mathbb{R}^{n-2})$ be equal to 1 on the support of V and let $\mu > 0$. We define the conjugate operator by

$$\begin{aligned} A = \chi(\mu z) &\begin{pmatrix} A_1(y,z,h^2D_y,hD_z) & & \\ & \ddots & \\ & & A_{q_0}(y,z,h^2D_y,hD_z) \end{pmatrix} \\ + (1-\chi)(\mu z) &\begin{pmatrix} g_1(h^2D_z^2)(z.hD_z + hD_z.z) & & \\ & \ddots & \\ & & g_{q_0}(h^2D_z^2)(z.hD_z + hD_z.z) \end{pmatrix}, \end{aligned}$$

with $A_q(y,z,\eta,\xi) = g_q(p_{\eta,y}(z,\xi))G_q(\eta,y,z,\xi)$. Thanks to the energy cut-off, A is a bounded operator on $L^2(\mathbb{R}^{n-1}_{y,z}))^{q_0}$ and maps $H^2(\mathbb{R}^{n-1}_{y,z})^{q_0}$ into itself. Hence, the commutator $[A, E_\pm(\lambda)]$ is well defined with domain $H^2(\mathbb{R}^{n-1}_{y,z})^{q_0}$. From Proposition 2.4 and symbolic calculus in $S^0(T^*\mathbb{R}^{n-1}_{y,z})$, we deduce that

$$[A, E_\pm(\lambda)] = \operatorname{diag}((C_q), q = 1,\dots,q_0) + h^2K(h)$$

with

$$C_q = [\widetilde{A}_q^w(y,z,h^2D_y,hD_z), h^2D_z^2 + V^w(h^2D_y,y,z)],$$

where $\widetilde{A}_q(y,z,\eta,\xi) = \chi(\mu z)A_q(y,z,\eta,\xi) + (1-\chi)(\mu z)z\xi$ and $K(h)$ is a bounded operator on $L^2(\mathbb{R}^{n-1})$ with norm $\|K(h)\| = \mathcal{O}(1)$. From the symbolic calculus in $S^0(\mathbb{R}^2_{y,\eta}, 1, \mathcal{L}(H^2(\mathbb{R}^{n-2}_z), L^2(\mathbb{R}^{n-2}_z)))$, we deduce that $C_q = B_q^w(y,h^2D_y) + \mathcal{O}(h^2)$ with

$$B_q(y,\eta) = [\widetilde{A}_q^w(y,z,\eta,hD_z), h^2D_z^2 + V(\eta,y,z)].$$

Therefore, the symbolic calculus in $S^0(T^*\mathbb{R}^{n-2}_z)$ yields $B_q(y,\eta) = D_q^w(y,\eta,z,hD_z)$ with $D_q(y,\eta,z,\xi) = -ihH_{p_{\eta,y}}\widetilde{A}_q(y,\eta,z,\xi) + \mathcal{O}(h^2)$. From Proposition 3.1, we deduce as in [12] that if we fix $\mu > 0$ small enough, then for all $h > 0$, $iD_q(y,\eta,z,\xi) \geq Ch\widetilde{g}_q(p_{\eta,y}(z,\xi))$. Hence,

$$i[A, E_\pm(\lambda)] \geq Ch\ \operatorname{diag}(k_q(y,z,h^2D_y,hD_z))_{q=1,\dots,q_0}),$$

with $k_q(y,z,\eta,\xi) = \widetilde{g}_q(p_{\eta,y}(z,\xi))$. Let $u = (u_1,\dots,u_q)$ in $\ker E_\pm(\lambda)$. For $q = 1,\dots,q_0$,

$$u_q = k_q(y,z,h^2D_y,hD_z)u_q + (1-k_q)(y,z,h^2D_y,hD_z)u_q$$

and it is easy to prove that $(1-k_q)(y,z,h^2D_y,hD_z)u_q = \mathcal{O}(h)\|u_q\|$. Hence, $0 = i\langle[A,E_\pm(\lambda)]u,u\rangle \geq Ch\|u\|^2$ for $h > 0$ small enough and the proof is complete. □

Using this result it is possible to obtain a more convenient expression for the scattering matrix.

3.2. **Reduced form of the scattering matrix.** From the preceding corollary, we know that λb is not an eigenvalue of $\mathrm{H}(b)$. Therefore, the scattering matrix at the energy λb is well defined and satisfies

$$S(\lambda b,b) - Id = -2i\pi\mathcal{F}_0(\lambda b)[\Delta_z,\chi_1]R(\lambda b+i0)[\Delta_z,\chi_2]\mathcal{F}_0(\lambda b)^*, \tag{3.1}$$

where $R(\lambda b+i0) = \lim_{\mu\to0^+}(\mathrm{H}(b)-\lambda b-i\mu)^{-1}$ exists in the space of bounded operators $\mathcal{L}(L^2(\mathbb{R}^n,\langle z\rangle^\alpha dxdydz),L^2(\mathbb{R}^n,\langle z\rangle^{-\alpha}dxdydz))$ for all $\alpha > 1/2$ (cf. Proposition 2.1 of [16]). Hence, it follows from (2.3) that

$$R(\lambda b+i0) = h^2\,\mathrm{U}^*(\widetilde{P}(h)-\lambda-i0)^{-1}\,\mathrm{U}$$

where $(\widetilde{P}(h)-\lambda-i0)^{-1} = \lim_{\mu\to0^+}(\widetilde{P}(h)-\lambda-i\mu)^{-1}$ exists in $\mathcal{L}(L^2_\alpha,L^2_{-\alpha})$ for all $\alpha > 1/2$. Plugging this expression into (3.1), setting $\mathcal{T}(\lambda,h) = T(\lambda b,b)$ and using the fact that U acts only on the variables (x,y), we obtain

$$\mathcal{T}(\lambda,h) = -2i\pi h^2\,\mathrm{U}^*\,\mathcal{F}_0(h^{-2}\lambda)[\Delta_z,\chi_1](\widetilde{P}(h)-\lambda b-i0)^{-1}[\Delta_z,\chi_2]\mathcal{F}_0(h^{-2}\lambda)^*\,\mathrm{U}\,.$$

Moreover, by definition of $\mathcal{F}_0$, we know that $\mathcal{F}_0(h^{-2}\lambda) = \mathcal{F}_0(h^{-2}\lambda)\Pi$ so that

$$\begin{aligned}\mathcal{T}(\lambda,h) = -2i\pi h^2\,\mathrm{U}^*\mathcal{F}_0(h^{-2}\lambda)R_-[\Delta_z,\chi_1]&\\ R_+(\widetilde{P}(h)-\lambda b-i0)^{-1}R_-[\Delta_z,\chi_2]R_+\mathcal{F}_0(h^{-2}\lambda)^*\,\mathrm{U}\,.&\end{aligned}$$

Combined with (2.13), this equation yields

$$\mathcal{T}(\lambda,h) = -2i\pi h^2\,\mathrm{U}^*\,\mathcal{F}_h(\lambda)[\Delta_z,\chi_1]E_\pm(\lambda)^{-1}[\Delta_z,\chi_2]\mathcal{F}_h(\lambda)^*\,\mathrm{U}, \tag{3.2}$$

where $E_\pm(\lambda) = \lim_{\mu\to0^+}E_\pm(\lambda+i\mu)$ exists in $\mathcal{L}((L^2_\alpha)^{q_0},(L^2_{-\alpha})^{q_0})$ for $\alpha > 1/2$ and $\mathcal{F}_h(\lambda):\ L^2_\alpha(\mathbb{R}^{n-1}_{y,z})^{q_0} \to L^2(\mathbb{R}^2_{x,y})$ is defined by $\mathcal{F}_h(\lambda) = \mathcal{F}_0(h^{-2}\lambda)R_-$, that is

$$\mathcal{F}_h(\lambda)(\varphi_1,\dots,\varphi_{q_0})(x,y,\omega) = \sum_{q=1}^{q_0}\phi_q(x)(\widetilde{\mathcal{F}}_0(h^{-2}(\lambda_q))\varphi_q(y,.))(\omega).$$

3.3. **Diagonalization of $E_\pm(\lambda)$.** The starting point of this section is the following

Proposition 3.3. *Suppose that Assumption 1 is satisfied, then*

$$\|E_\pm(\lambda)^{-1}\|_{L^2_\alpha(\mathbb{R}^{n-1}_{y,z})^{q_0},L^2_{-\alpha}(\mathbb{R}^{n-1}_{y,z})^{q_0}} = \mathcal{O}(h^{-1})$$

for all $\alpha > 1/2$.

Proof. Thanks to the existence of a global escape function, we have an exact conjugate operator and the above estimate is a consequence of the works of Mourre [17]. More precisely, for a semi-classical version of Mourre's results we refer to the paper of C. Gérard [6] □

Using Proposition 3.3, we can localize with respect to energy in $\mathcal{T}(\lambda,h)$. More precisely, for $\mu,\nu\in C_0^\infty(\mathbb{R})$ we will say that $\mu\prec\nu$ if $\nu=1$ on $\operatorname{supp}\mu$. Let $\nu_0,\nu_2\in C_0^\infty(\mathbb{R})$ be such that ν_0 is equal to 1 on $[-2\sup|V|, q_0+2\sup|V|]$ and $\nu_0\prec 1_{[-R,R]}\prec\nu_2$ with $R>0$ to be chosen large enough. Then $(1-\nu_0)(h^2D_z^2)\mathcal{F}_h(\lambda)=\mathcal{O}(h^\infty)$ and (3.2) yields

$$\mathcal{T}(\lambda,h)=-2i\pi h^2\,\mathrm{U}^*\,\mathcal{F}_h(\lambda)\nu_0(h^2D_z^2)[\Delta_z,\chi_1]E_\pm(\lambda)^{-1}[\Delta_z,\chi_2]\nu_0(h^2D_z^2)\mathcal{F}_h(\lambda)^*\,\mathrm{U}+\mathcal{O}(h^\infty)\tag{3.3}$$

in $\mathcal{L}(L^2(\mathbb{R}^2_{x,y}))$. Denoting

$$\widetilde{E}_\pm(s)=E_0^w(y,h^2D_y,z,hD_z)+\nu_2(h^2D_z^2)\left(E_\pm(s)-E_0^w(y,h^2D_y,z,hD_z)\right),$$

it is easy to show that $(\widetilde{E}_\pm(\lambda+i\mu))^{-1}$ exists for $\mu\geq 0$ as an operator from $(L^2_\alpha)^{q_0}$ into $(L^2_{-\alpha})^{q_0}$, $\alpha>1/2$. Using the escape function built precedently, one can also show that $\|\widetilde{E}_\pm(\lambda)^{-1}\|_{\alpha,-\alpha}=\mathcal{O}(h^{-1})$. Moreover, the resolvent formula and Proposition 2.4 yield

$$\begin{aligned}\left(\widetilde{E}_\pm(\lambda)^{-1}-E_\pm(\lambda)^{-1}\right)\nu_0(h^2D_z^2)=&\\ \widetilde{E}_\pm(\lambda)^{-1}(1-\nu_2)(h^2D_z^2)&\left(E_>(h)+R_N(h,\lambda)\right)E_\pm(\lambda)^{-1}\nu_0(h^2D_z^2),\end{aligned}\tag{3.4}$$

with $E_>(h)=\sum_{j=1}^N h^jE_j^w(y,h^2D_y,z,hD_z,\lambda)$ and $\|R_N\|=\mathcal{O}(h^N)$. Moreover, it follows from the functional calculus for $E_\pm(\lambda)$ that for $R>0$ large enough, $(1-\nu_2)(h^2D_z^2)E_\pm(\lambda)^{-1}\nu_0(h^2D_z^2)=\mathcal{O}(h^\infty)$ in $\mathcal{L}((L^2_\alpha)^{q_0},(L^2_{-\alpha})^{q_0})$. Combining this estimate with (3.3) and (3.4) and the estimates of $E_\pm(\lambda)^{-1}$, $\widetilde{E}_\pm(\lambda)^{-1}$, we obtain

$$\mathcal{T}(\lambda,h)=-2i\pi h^2\,\mathrm{U}^*\,\mathcal{F}_h(\lambda)\nu_0(h^2D_z^2)[\Delta_z,\chi_1]\widetilde{E}_\pm(\lambda)^{-1}[\Delta_z,\chi_2]\nu_0(h^2D_z^2)\mathcal{F}_h(\lambda)^*\,\mathrm{U}+\mathcal{O}(h^{N-2}).\tag{3.5}$$

This formula shows that the main object in our problem is $\widetilde{E}_\pm(\lambda)^{-1}$. From Proposition 2.4, we know that $\widetilde{E}_\pm(\lambda)$ is equal to a matrix-diagonal Schrödinger operator plus terms of lower order. The following proposition says that it can be diagonalized at all order.

Proposition 3.4. *For all $N_0\in\mathbb{N}^*$, there exists a unitary transformation U_{N_0} on $L^2(\mathbb{R}^{n-1}_{y,z})$ such that*

$$U_{N_0}\widetilde{E}_\pm(\lambda)U_{N_0}^*=\mathcal{P}_{N_0}(\lambda)+h^{N_0}R_{N_0}(h),$$

with $\|R_{N_0}(h)\|_{L^2,L^2}=\mathcal{O}(1)$ and $\mathcal{P}_{N_0}(\lambda)=\operatorname{diag}((p_q^w(y,z,h^2D_y,hD_z,N_0)-\lambda_q)_{q=1,\dots,q_0})$ with $p_q(.,N_0)\in S^0(T^*\mathbb{R}^{n-1},\langle\xi\rangle^2)$. Moreover, $p_q(y,z,\eta,\xi,N_0)=\displaystyle\sum_{m=0}^{N_0}h^mp_{q,m}(y,z,\eta,\xi)$ with $p_{q,0}=\xi^2+V(\eta,y,z)$, $p_{q,m}\in S^0(T^*\mathbb{R}^{n-1})$ for $m\geq 1$ and $p_{q,m}$ compactly supported with respect to z.

Proof. We prove the slightly stronger following result: for $N\in\{1,\dots,N_0\}$ we show that there exists a unitary transformation U_N on $L^2(\mathbb{R}^{n-1}_{y,z})$ such that

$$U_N\widetilde{E}_\pm(\lambda)U_N^*=\left(\mathcal{P}_N(\lambda)+h^N\mathcal{K}_N\right)+h^{N_0}R_{N_0}$$

with

i) $\mathcal{P}_N(\lambda) = \mathrm{diag}((p_q^w(y,z,h^2D_y,hD_z,N) - \lambda_q)_{q=1,\dots,q_0})$ with $p_q(.,N)$ belonging to $S^0(T^*\mathbb{R}^{n-1},\langle\xi\rangle^2)$.
ii) $p_q(y,z,\eta,\xi,N_0) = \sum_{m=0}^{N_0} h^m p_{q,m}(y,z,\eta,\xi)$ with $p_{q,0} = \xi^2 + V(\eta,y,z)$ and $p_{q,m} \in S^0(T^*\mathbb{R}^{n-1})$ for $m \geq 1$.
iii) $\|R_{N_0}\| = \mathcal{O}(1)$
iv) $\mathcal{K}_N = (k_{N,i,j}^w(y,z,h^2D_y,hD_z,h))_{i,j=1,\dots,q_0}$ with $r_{N,i,j} \in S^0(T^*\mathbb{R}^{n-1})$
v) for all $i,j = 1,\dots,q_0$, $k_{N,i,i} = 0$, $k_{N,i,j} = k_{N,j,i}$ and $k_{N,i,j}$ is compactly supported with respect to ξ.

We show this result by induction on $N \in \{1,\dots,N_0\}$. The initial case $N = 1$ follows directly from Proposition 2.4 with $U_1 = Id$ and formula (3.5).

Suppose that the property is true at rank N. As it was suggested in [5], we look for U_{N+1} under the form

$$U_{N+1} = exp(h^N u_{N+1}^w(y,z,h^2D_y,hD_z))$$

with $u_{N+1} = (u_{N+1,i,j})_{i,j=1,\dots,q_0}$ such that $u_{N+1,i,j} \in S^0(T^*\mathbb{R}^{n-1})$ is real valued and $u_{N+1,i,j} = -u_{N+1,j,i}$ for all i,j. Under these assumptions, U_{N+1} is clearly unitary and $U_{N+1}^* = \exp(-h^N u_{N+1}^w(y,z,h^2D_y,hD_z))$. On the other hand, denoting

$$C_N = U_{N+1}\left(\mathcal{P}_N + h^N\mathcal{K}_N(h)\right)U_{N+1}^*,$$

and $\mathbf{u_{N+1}} = U_{N+1}^w(y,z,h^2D_y,hD_z)$, we have

$$C_N = \mathcal{P}_N + h^N\left(\mathcal{K}_N(h) + \mathbf{u_{N+1}}\mathcal{P}_N - \mathcal{P}_N\mathbf{u_{N+1}}\right) + W_N \tag{3.6}$$

with

$$\begin{aligned} W_N &= h^{2N}\left(\mathbf{u_{N+1}}\mathcal{K}_N(h) - \mathcal{K}_N(h)\mathbf{u_{N+1}}\right)\\ &\quad + \sum_{l,k\geq 1}\frac{(h^N\mathbf{u_{N+1}})^k}{k!}(\mathcal{P}_N + h^N\mathcal{K}_N(h))\frac{(-h^N\mathbf{u_{N+1}})^l}{l!}. \end{aligned} \tag{3.7}$$

Hence, we have to find $\mathbf{u_{N+1}}$ such that

$$\mathcal{K}_N(h) + \mathbf{u_{N+1}}\mathcal{P}_N - \mathcal{P}_N\mathbf{u_{N+1}} = 0.$$

A simple matrix calculus shows that we must solve

$$p_{i,0}^w(\dots)u_{N+1,i,j}^w(\dots) - u_{N+1,i,j}^w(\dots)p_{j,0}^w(\dots) = k_{N,i,j}^w(\dots)$$

for all $i < j = 1,\dots,q_0$, where $(\dots) = (y,z,h^2D_y,hD_z)$. Let $\nu_3 \in C_0^\infty(\mathbb{R})$ be such that $\nu_3 = 1$ on $\mathrm{supp}_\xi\, k_{N,i,j}$ and set

$$u_{N+1,i,j}(y,z,\eta,\xi) = \frac{1}{\Lambda_i - \Lambda_j}\nu_3(\xi^2)k_{N,i,j}(y,z,\eta,\xi)$$

for $i \neq j$ and $u_{N+1,i,i} = 0$ for $i = 1,\dots,q_0$. It follows from the symbolic calculus that

$$C_N = \mathcal{P}_N + h^{N+1}\mathcal{K}_{N+1} + W_N + h^{N_0}R_{N_0}$$

with $R_{N_0} = \mathcal{O}(1)$, $\mathcal{K}_N$ satisfying *iv*), and for all i,j, $k_{N+1,i,j} = \nu_3(\xi^2)k_{N+1,i,j}$. Moreover, C_N and W_N being symmetric, it follows that $\mathcal{K}_{N+1}$ satisfies *v*).

Now, let us study the term W_N. Notice that the definition of u_{N+1} insures that the series in (3.7) converges in the space of bounded operators. From the composition theorem of pseudo-differentials operators and thanks to the cut-off in energy, it is clear that $W_N = (h^{2N} w^w_{N,i,j}(y,z,h^2D_y,hD_z,h))_{i,j=1,\dots,q_0}$ with $w_{N,i,j} \in S^0(T^*\mathbb{R}^{n-1})$. Therefore, we can write

$$C_N = \widetilde{\mathcal{P}}_N + h^{N+1}\widetilde{\mathcal{K}}_N + h^{N_0}R_{N_0}$$

with $\widetilde{\mathcal{P}}_N = \mathcal{P}_N + h^{2N}\operatorname{diag}(w^w_{N,i,i}(y,z,h^2D_y,hD_z,h))_{i=1,\dots,q_0})$ and $\tilde{\mathcal{K}}_{N+1}$ satisfying *iv*). Moreover, combining equation (3.7) with the fact that $\mathcal{K}_{N+1}$ verifies *v*), it is clear that $\tilde{\mathcal{K}}_{N+1}$ satisfies also *v*). □

Using this proposition, $\mathcal{T}(\lambda,h)$ can be written as

$$\mathcal{T}(\lambda,h) = -2i\pi h^{-2}\,\mathrm{U}^*\,\mathcal{F}_h(\lambda)\mathcal{A}_{N_0}\mathcal{P}_{N_0}(\lambda)^{-1}B_{N_0}\mathcal{F}_h(\lambda)^*\,\mathrm{U} + \mathcal{O}(h^{N_0-2}), \tag{3.8}$$

with $\mathcal{A}_{N_0} = \nu_0(h^2D_z^2)[h^2\Delta_z,\chi_1]U_{N_0}$ and $\mathcal{B}_{N_0} = U^*_{N_0}[h^2\Delta_z,\chi_2]\nu_0(h^2D_z^2)$. Moreover, we have the following

Proposition 3.5. *Let $N \in \mathbb{N}$, then there exists some symbols $A_{N_0}(y,z,\eta,\xi,h)$ and $B_{N_0}(y,z,\eta,\xi,h)$ in $S^0(T^*\mathbb{R}^{n-1},\mathcal{L}(\mathbb{R}^{q_0}),1)$ such that*

$$\mathcal{A}_{N_0} = A^w_{N_0}(y,z,h^2D_y,hD_z,h) + \mathcal{O}(h^N)$$
$$\mathcal{B}_{N_0} = B^w_{N_0}(y,z,h^2D_y,hD_z,h) + \mathcal{O}(h^N).$$

Proof. It is a straightforward consequence of the proof of Proposition 3.4. Indeed, up to pseudo-differential operators, $\mathcal{A}_{N_0}$ is a product of $exp(h^k\mathbf{u_{k+1}})$, with $\mathbf{u_{k+1}}$ pseudo-differential. Expanding the exponential and using the fact that $\mathbf{u_{k+1}}$ is bounded, we obtain the result. □

4. Spatial and finite-time localization

In this section, we prove microlocal resolvent estimates similar to those of [9], [22] and use it to microlocalize near the classical trajectories in the formula giving the scattering matrix.

4.1. Microlocal resolvent estimates.

Before stating the main result of this section, we need to introduce the incoming and outgoing subspaces of the phase space. For $R > 0, d > 1$ and $\sigma \in]-1,1[$ we set

$$\Gamma_\pm(R,d,\sigma) = \{(y,z,\eta,\xi) \in T^*\mathbb{R}^{n-1};\ |z| \geq R,\ d^{-1} < |\xi| < d,\ \pm\cos(z,\xi) > \pm\sigma\},$$

where $\cos(z,\xi) = \frac{\langle z,\xi\rangle}{|z||\xi|}$.

Lemma 4.1. *Let $\omega_\pm \in S^0(T^*\mathbb{R}^{n-1})$ be supported in $\Gamma_\pm(R,d,\sigma_\pm)$ with $R > 0$ large enough. Then*

i)
$$\|\mathcal{P}_{N_0}(\lambda)^{-1}\omega_+(y,z,h^2D_y,hD_z)\|_{-\alpha+\delta,-\alpha} = \mathcal{O}(h^{-1})$$
for all $\delta > 1,\ \alpha > 1/2$.

ii) *If $\sigma_+ > \sigma_-$, then*
$$\|\omega_-(y,z,h^2D_y,hD_z)\mathcal{P}_{N_0}(\lambda)^{-1}\omega_+(y,z,h^2D_y,hD_z)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty)$$
for all $\alpha >> 1$.

iii) *If $\omega \in S^0(T^*\mathbb{R}^{n-1})$ satisfies $\operatorname{supp}_z(\omega) \subset B(0,R/2)$ then*
$$\|\omega(y,z,h^2D_y,hD_z)\mathcal{P}_{N_0}(\lambda)^{-1}\omega_+(y,z,h^2D_y,hD_z)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty)$$
for all $\alpha >> 1$.

Proof. From the symbolic calculus in $S^0(T^*\mathbb{R}^{n-1})$ it suffices to prove the lemma for symbols depending only on (z,ξ). Moreover, $\mathcal{P}_{N_0}(\lambda)$ being diagonal, it suffices to prove microlocal resolvent estimates for

$$P_q(h) = -h^2\Delta_z + V^w(h^2D_y, y, z) + \Lambda_q + h^2Q^w(y, z, h^2D_y, hD_z, h)$$

with $Q \in S^0(T^*\mathbb{R}^{n-1})$. The proof is the same as in [22] except that we have to take care with the variable y. Let us introduce $\widetilde{P}_q(h) = -h^2\Delta_z + V^w(h^2D_y, y, z) + \Lambda_q$, $R_q(\lambda) = (P_q(h) - \lambda - i0)^{-1}$ and $\widetilde{R}_q(\lambda) = (\widetilde{P}_q(h) - \lambda - i0)^{-1}$. We show first that for $\omega_\pm \in S^0(T^*\mathbb{R}^{n-2}_z)$ we have

i’)
$$\|\widetilde{R}_q(\lambda)\omega_+(z, hD_z)\|_{-\alpha+\delta,-\alpha} = \mathcal{O}(h^{-1})$$
for all $\delta > 1,\ \alpha > 1/2$.

ii’) If $\sigma_+ > \sigma_-$, then
$$\|\omega_-(z, hD_z)\widetilde{R}_q(\lambda)\omega_+(y, z, h^2D_y, hD_z)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty)$$
for all $\alpha >> 1$.

iii’) If $\omega \in S^0(T^*\mathbb{R}^{n-2})$ satisfies $\mathrm{supp}_z(\omega) \subset B(0, R/2)$ then
$$\|\omega(z, hD_z)\widetilde{R}_q(\lambda)\omega_+(z, hD_z)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty)$$
for all $\alpha >> 1$.

First, let us notice that for λ satisfying Assumption 1, the proof of Proposition 3.3 shows that

$$\|\widetilde{R}_q(\lambda)\|_{\alpha,-\alpha} + \|R_q(\lambda)\|_{\alpha,-\alpha} = \mathcal{O}(h^{-1}),\ \forall \alpha > 1/2. \tag{4.1}$$

Next, we construct a parametrix in the outgoing area. The potential V being compactly supported, the construction is simpler than in [22]. For $t > 0$, let us define $\Omega(t) : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ by

$$\Omega(t)f(y, z) = \int e^{ih^{-1}((z-z')\xi - t\xi^2)}\omega_+(\frac{z + z'}{2}, \xi)f(y, z')dz'd\xi. \tag{4.2}$$

The Duhamel’s principle yields

$$e^{-ih^{-1}t\widetilde{P}_q(h)}\omega_+(z, hD_z) = \Omega(t) - ih^{-1}\int_0^t e^{ih^{-1}(s-t)\widetilde{P}_q(h)}R(s)ds \tag{4.3}$$

with $R(s) = V^w(h^2D_y, y, z)\Omega(s)$. Mimicking the proof of [22], we show by integration by parts that for $\delta > 1$,

$$\|\Omega(t)\|_{-\alpha+\delta,-\alpha} = \mathcal{O}(\langle t\rangle^{-\delta}) \tag{4.4}$$
$$\|R(t)\|_{-\alpha+\delta,-\alpha} = \mathcal{O}(h^N\langle t\rangle^{-\delta}),\ \forall N \in \mathbb{N}. \tag{4.5}$$

From this formula, it is easy to deduce the point $i')$. Indeed, we have

$$\begin{aligned}\widetilde{R}_q(\lambda)\omega_+(z, hD_z) &= ih^{-1}\int_0^{+\infty} e^{-ih^{-1}t(\widetilde{P}_q(h)-\lambda)}\omega_+(z, hD_z)dt\\ &= ih^{-1}\int_0^{+\infty} e^{ih^{-1}t\lambda}\Omega(t)dt + \widetilde{R}_q(\lambda)\int_0^{+\infty} e^{ih^{-1}t\lambda}R(t)dt\end{aligned}$$

Combining this equation with (4.1), (4.4), (4.5) and the integrability of $\langle t\rangle^{-\delta}$ for $\delta > 1$, we obtain $i')$. To prove $ii')$ and $iii')$, we proceed as in [22]. For instance, to prove $ii')$ we show that

$$\|\omega_-(z, hD_z)\Omega(t)\|_{-\alpha+\delta,-\alpha} = \mathcal{O}(h^\infty\langle t\rangle^{-\infty})$$
$$\|\omega_-(z, hD_z)R(t)\|_{-\beta+\delta,\beta-\delta} = \mathcal{O}(h^N\langle t\rangle^{-\delta}),\ \forall N \in \mathbb{N}.$$

for any $\beta < N/2$.

Now, let us explain how we deduce $i), ii)$ and $iii)$ from $i'), ii')$ and $iii')$. By definition, Q being compactly supported with respect to z,

$$R_q(\lambda)\omega_+(z, hD_z) = \widetilde{R}_q(\lambda)(Id + h^2Q^w(y, z, h^2D_y, hD_z)\widetilde{R}_q(\lambda))^{-1}\omega_+(z, hD_z).$$

Thanks to estimate (4.1), for $M \in \mathbb{N}$ we can expand this formula to obtain (4.6)

$$R_q(\lambda)\omega_+(z, hD_z) = \sum_{k=0}^{M}(-1)^k h^{2k}(\widetilde{R}_q(\lambda)\mathcal{Q})^k\widetilde{R}_q(\lambda)\omega_+(z, hD_z) + \mathcal{R}_M(h)\omega_+(z, hD_z)$$

where $\mathcal{Q} = Q^w(y, z, h^2D_y, hD_z)$ and

$$\mathcal{R}_M(h) = h^{2M+2}\widetilde{R}_q(\lambda)\int_0^1 \frac{(1-t)^{M+1}}{M!}(Id + th^2\mathcal{Q}\widetilde{R}_q(\lambda))^{-M-2}dt(\mathcal{Q}\widetilde{R}_q(\lambda))^{M+1}.$$

In particular, estimate (4.1) and $i')$ yield

$$\|\mathcal{R}_M(h)\omega_+(z, hD_z)\|_{-\alpha+\delta,-\alpha} = \mathcal{O}(h^M).$$

Combining again estimate $i')$ with (4.6), we obtain $i)$. To prove $ii)$, it suffices to combine $ii')$, $iii')$ and (4.6). Indeed, for $k = 1, \ldots, M$, choosing $R > 0$ such that $\mathcal{Q} = \mathcal{Q}\,1_{B(0,R)}(z)$, we have

$$\|\omega_-(z, hD_z)(\widetilde{R}_q(\lambda)\mathcal{Q})^k\widetilde{R}_q(\lambda)\omega_+(z, hD_z)\|_{\alpha,-\alpha}$$
$$\leq \|(\widetilde{R}_q(\lambda)\mathcal{Q})^k\|)\|_{-\alpha,-\alpha}\|1_{B(0,R)}(z)\widetilde{R}_q(\lambda)\omega_+(z, hD_z)\|_{\alpha,-\alpha} = \mathcal{O}(h^\infty)$$

thanks to $iii')$. This achieves the proof of $ii)$. The proof of $iii)$ is similar and left to the reader. □

4.2. **Egorov Lemma.** The starting point of this section is the following formula

$$\mathcal{P}_{N_0}(\lambda)^{-1} = ih^{-1}\int_0^T e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)}dt + \mathcal{P}_{N_0}(\lambda)^{-1}e^{-ih^{-1}T\mathcal{P}_{N_0}(\lambda)}, \tag{4.7}$$

where we would like to neglect the second term of the right hand side. For this purpose, we need a sort of Egorov Lemma for our problem, but we can not apply standard results as our operators are h-pseudo in the variable z and h^2-pseudo in the variable y. Nevertheless, working with operator valued symbol we can establish the results we need. For $t > 0$, let $\phi_t :\ T^*\mathbb{R}^{n-1} \to T^*\mathbb{R}^{n-1}$ be defined by

$$\phi_t(y, z, \eta, \xi) = \exp(tH_{p_{\eta,y}})(z, \xi).$$

Lemma 4.2. *Let $\omega_1, \omega_2 \in S^0(T^*\mathbb{R}^{n-1})$ such that $\operatorname{supp}\omega_2 \cap \phi_t(\operatorname{supp}(\omega_1)) = \emptyset$ and ω_1 is compactly supported, then*

$$\|\omega_2^w(y, z, h^2D_y, hD_z)e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)}\omega_1^w(y, z, h^2D_y, hD_z)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty),$$

for all $\alpha > 1/2$.

Proof. As $e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)} = \mathrm{diag}(e^{-ih^{-1}t(P_q(h)-\lambda)}, q = 1, \dots, q_0)$, it suffices to prove that

$$\omega_2^w(y,z,h^2D_y,hD_z)e^{-ih^{-1}tP_q(h)}\omega_1^w(y,z,h^2D_y,hD_z) = \mathcal{O}(h^\infty).$$

For this purpose we build a parametrix for $e^{-ih^{-1}tP_q(h)}\omega_1^w(y,z,h^2D_y,hD_z)$ and we first search $F(t) = f^w(t,y,z,h^2D_y,hD_z)$ such that

$$ih\partial_t F(t) = [P_q(h), F(t)] \text{ and } f_{|t=0} = \omega_1. \tag{4.8}$$

We look for f under the form $f = \sum_{j\geq 0} h^j f_j$. From Proposition 3.4, we have $P_q(h) = p_q^w(y,z,h^2D_y,hD_z)$ with $p_q = \sum_{j\geq 0} h^j p_{q,j}(y,z,\eta,\xi)$. Hence, denoting $\widetilde{f}(z,\xi) = f^w(y,z,h^2D_y,\xi)$ and $\widetilde{p}_q(z,\xi) = p_q^w(y,z,h^2D_y,\xi)$, equation (4.8) becomes

$$ih\partial_t\widetilde{f}(t) = \widetilde{p}_q\sharp\widetilde{f} - \widetilde{f}\sharp\widetilde{p}_q \quad \text{and } f_{|t=0} = \omega_1, \tag{4.9}$$

where $\sharp$ denotes the composition law of symbols in the class $S^0(T^*\mathbb{R}_z^{n-2}, \mathcal{L}(L^2(\mathbb{R}_y))$. In particular, we have $\widetilde{p}_q\sharp\widetilde{f} = \sum_{j\geq 0} h^j \widetilde{p}_q\sharp_j\widetilde{f}$ with

$$\widetilde{p}_q\sharp_j\widetilde{f} = \sum_{\alpha+\beta=j} \frac{(-1)^\alpha}{(2i)^j\alpha!\beta!}(\partial_z^\alpha\partial_\xi^\beta\widetilde{p}_q)(\partial_\xi^\alpha\partial_z^\beta\widetilde{f}).$$

Consequently, (4.9) yields

$$ih\partial_t\widetilde{f}(t) = \sum_{j\geq 0} h^j(\widetilde{p}_q\sharp_j\widetilde{f} - \widetilde{f}\sharp_j\widetilde{p}_q)$$

and using symbolic calculus in $S^0(T^*\mathbb{R}_y)$, we obtain

$$ih\partial_t f(t) = \sum_{j,l\geq 0} h^{j+2l}(p_q\sharp_j\Box_l f - f\sharp_j\Box_l p_q),$$

where

$$p_q\sharp_j\Box_l f = \sum_{\alpha+\beta=j}\sum_{\gamma+\delta=l} \frac{(-1)^{\alpha+\gamma}}{(2i)^{j+l}\alpha!\beta!\gamma!\delta!}(\partial_y^\gamma\partial_\eta^\delta\partial_z^\alpha\partial_\xi^\beta p_q)(\partial_\eta^\gamma\partial_y^\delta\partial_\xi^\alpha\partial_z^\beta f).$$

Using the asymptotic expansions of f and p, we have to solve

$$i\partial_t f_n = \sum_{j+m+k+2l=n+1} (p_{q,k}\sharp_j\Box_l f_m - f_m\sharp_j\Box_l p_{q,k}),\ \forall n \in \mathbb{N}. \tag{4.10}$$

We solve these transport equations by induction and we show that $\mathrm{supp}\, f_n \subset \phi_{-t}(\mathrm{supp}\,\omega_1)$.

Case $n = 0$**.** We have to solve

$$i\partial_t f_0 = p_{q,0}\sharp_1\Box_0 f_0 - f_0\sharp_1\Box_0 p_{q,0} = iH_{p_{n,y}} f_0.$$

Combined with the initial condition $f_{0|t=0} = \omega_1$, this yields $f_0 = \omega_1 \circ \phi_t$.

Case $n \geq 1$**.** Equation (4.10) becomes

$$i\partial_t f_n = iH_{p_{n,y}} f_n + a_n(t),$$

with

$$a_n(t) = \sum_{\substack{j+m+k+2l=n+1 \\ m<n}} (p_{q,k}\sharp_j\Box_l f_m - f_m\sharp_j\Box_l p_{q,k}).$$

Together with the initial condition $f_{n|t=0} = 0$, this equation is solved by $f_n(t) = \frac{1}{i}\int_0^t a_n(s) \circ \phi_{t-s} ds$. Moreover, f_m, $m < n$ being supported in $\phi_{-t}(\operatorname{supp}\omega_1)$, $a_n(s)$ is supported in $\phi_{-s}(\operatorname{supp}\omega_1)$. Consequently, $f_n(t)$ is supported in $\phi_{-t}(\operatorname{supp}\omega_1)$.

Finally, we have

$$e^{-ih^{-1}tP_q(h)}\omega_1(y, z, h^2D_y, hD_z) = F(t)e^{-ih^{-1}tP_q(h)} + \mathcal{O}(h^\infty)$$

uniformly with respect to t in any compact set. This yields the result in L^2 norm. To prove the result in L^2_α, it suffices to work as in Appendix B of [15]. $\square$

From this proposition, we can deduce a convenient formula for the scattering amplitude. First we introduce the analogous operators to $R_\pm$ on the sphere. Let us denote $R^s_- : L^2(\mathbb{R}_y \times S^{n-3}_\omega)^{q_0} \to L^2(\mathbb{R}^2_{x,y} \times S^{n-3}_\omega)$ the operator defined by $R^s_-(\varphi_1, \dots, \varphi_{q_0}) = \sum_{q=1}^{q_0} \varphi_q \Phi_q$ and let $R^s_+ = (R^s_-)^*$.

Working as in [22], it follows from Lemmas 4.1 and 4.2 and formulas (3.8) and (4.7) that for $T_0 > 0$ large enough

$$\begin{aligned}\mathcal{T}(\lambda, h) = 2\pi h^{-3}\int_0^{T_0} \mathrm{U}^* R^s_+ \mathcal{G}_h(\lambda)\mathcal{A}_{N_0} e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)} B_{N_0}\mathcal{G}_h(\lambda)^* R^s_- \,\mathrm{U}\, dt \\ + \mathcal{O}(h^{N_0-2}),\end{aligned} \tag{4.11}$$

with $\mathcal{G}_h(\lambda) = R^s_-\mathcal{F}_h(\lambda)$. A simple calculation shows that

$$\mathcal{G}_h(\lambda)(\varphi_1, \dots, \varphi_{q_0}) = (\widetilde{\mathcal{F}}_0(h^{-2}\lambda_1)\varphi_1, \dots, \widetilde{\mathcal{F}}_0(h^{-2}\lambda_{q_0})\varphi_{q_0}).$$

From this formula and by definition of $\widetilde{\mathcal{F}}_0(E)$, we obtain the following formula for the scattering amplitude

$$\begin{aligned}T(\omega, \omega', \lambda b, b) = c_0(h)\,\mathrm{U}^* R^s_+ \int_0^{T_0}\int_{\mathbb{R}^{n-2}} e_-(z, \lambda, \omega', h)\mathcal{A}_{N_0} \\ e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)}\mathcal{B}_{N_0} e_+(z, \lambda, \omega, h) dz dt R^s_- \,\mathrm{U} \\ + \mathcal{O}(h^{N_0-2}),\end{aligned} \tag{4.12}$$

where we set $e_\pm(z, \lambda, \omega, h) = \operatorname{diag}(e_{q,\pm}(z, \lambda, \omega, h),\ q = 1, \dots, q_0)$, with

$$e_{q,\pm}(z, \lambda, \omega, h) = (\lambda_q)^{\frac{n-4}{4}} e^{\pm ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle}$$

and

$$c_0(h) = (1/2)h^{-(n-1)}(2\pi)^{-(n-3)}. \tag{4.13}$$

Applying Proposition 3.5 with N large enough, (4.12) becomes

$$\begin{aligned}T(\omega, \omega', \lambda b, b) =& c_0(h)\,\mathrm{U}^* R^s_+ \int_0^{T_0}\int_{\mathbb{R}^{n-2}} e_-(z, \lambda, \omega', h) A^w_{N_0}(y, z, h^2D_y, hD_z, h) \\ & e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)} B^w_{N_0}(y, z, h^2D_y, hD_z, h) e_+(z, \lambda, \omega, h) dz dt R^s_- \,\mathrm{U} \\ & + \mathcal{O}(h^{N_0-2}).\end{aligned}$$

Moreover, denoting $A_{N_0} = (a_{pq})_{p,q=1,\dots,q_0}$ and $B_{N_0} = (b_{pq})_{p,q=1,\dots,q_0}$, a simple matrix calculus shows that

$$T(\omega, \omega', \lambda b, b) = c_0(h)\,\mathrm{U}^* \int_0^{T_0}\int_{\mathbb{R}^{n-2}} R^s_+ G(z, t, \omega, \omega', \lambda, h) R^s_- dz dt \,\mathrm{U} + \mathcal{O}(h^{N_0-2}),$$

with $G(z,t,\omega,\omega',\lambda,h)=(g_{pq}(z,t,\omega,\omega',\lambda,h))_{p,q=1,\dots,q_0}$ and

$$
\begin{aligned}
(4.14)\qquad g_{pq}(z,t,\omega,\omega',\lambda,h)=\sum_{k=1}^{q_0} & e_{p-}(z,\lambda,\omega',h)a_{pk}^w(\dots)\\
& e^{-ih^{-1}t(p_k^w(\dots)-\lambda_k)}b_{kq}^w(\dots)e_{q+}(z,\lambda,\omega,h),
\end{aligned}
$$

where in the preceding formula, $(\dots)$ stands for (y,z,h^2D_y,hD_z). Moreover, as $z\mapsto e_{p\pm}(z,\lambda,\omega,h)$ is microlocalized near $\xi^2=\lambda_q$, Lemma 4.2 shows that for $p\neq q$, $\|g_{pq}(\omega,\omega',\lambda,h)\|_{L^2,L^2}=\mathcal{O}(h^\infty)$ uniformly with respect to $t\in[0,T]$ and $z\in\mathbb{R}^{n-2}$, and it follows that

$$
(4.15)\qquad T(\omega,\omega',\lambda b,b)=c_0(h)\sum_{q=1}^{q_0}\mathrm{U}^*\, f_q(\omega,\omega',\lambda,h)\otimes\Pi_q\,\mathrm{U}+\mathcal{O}(h^{N_0-2}),
$$

with $f_q(\omega,\omega',\lambda,h)=\int_0^{T_0}\int_{\mathbb{R}^{n-2}}g_{qq}(z,t,\omega,\omega',\lambda,h)dzdt$.

4.3. **Microlocalization near the classical flow.** Here we follow the construction of section 3.2 of [22] and adapt it to our case. Let us denote

$$
\begin{aligned}
f_{q,k}(\omega,\omega',\lambda,h)=\int_0^{T_0}\int_{\mathbb{R}^{n-2}} & e_{q-}(z,\lambda,\omega',h)a_{qk}^w(\dots)e^{ih^{-1}t(\lambda_k-p_k^w(\dots))}\\
& b_{kq}^w(\dots)e_{q+}(z,\lambda,\omega,h)dzdt,
\end{aligned}
$$

so that $f_q=\sum_{k=1}^{q_0}f_{q,k}$. We fix $\epsilon>0$ sufficiently small and for $q\in\{1,\dots,q_0\}$ and $l\in\{1,\dots,l_q\}$ we introduce

$$
\mathcal{Z}_{q,l}=\{(y,\eta,z)\in\mathbb{R}^2\times\Upsilon_\omega\,|\,|z-\tilde z_{q,l}(\eta,y)|<\epsilon\}.
$$

and

$$
\widetilde{\mathcal{Z}}_{q,l}=\{(y,\eta,z)\in\operatorname{supp}(b_{kq})\,|\,z=Z_{q,\infty}(s,\eta,y,\tilde z_j(\eta,y),\omega),\ s<0\}.
$$

Taking ϵ sufficiently small, $\mathcal{Z}_{q,l}\cap\mathcal{Z}_{q,l'}=\emptyset$ for all $l\neq l'$. As in [22], there exists $S_1>>S_0>>1$ such that

$$
\widetilde{\mathcal{Z}}_{q,l}\subset\Sigma_{q,l,-}=\{(y,\eta,Z_{q,\infty}(s,\eta,y,\tilde z,\omega)),\ (y,\eta,\tilde z)\in\mathcal{Z}_{q,l},\ -S_1<s<-S_0\}.
$$

Let $\sigma_{q,l,-}\in S^0(T^*\mathbb{R}^n)$ with support contained in $\Sigma_{q,l}$ and such that $\sigma_{q,l,-}=1$ on $\widetilde{\mathcal{Z}}_{q,l}$. Using the fact that for (y,η) fixed, $z\mapsto b_{kq}(y,\eta,z,hD_z)e_{q+}(z)$ is microlocalized near $\operatorname{supp}(b_{kq}(y,\eta,.))\times\{\xi=\sqrt{\lambda_q}\omega\}$ it follows from Lemma 4.2 that

$$
\begin{aligned}
f_{q,k}(\omega,\omega',\lambda,h)=\sum_{l=1}^{l_q}\int_0^{T_0}\int_{\mathbb{R}^{n-2}} & e_{q-}(z,\lambda,\omega',h)a_{qk}^w(\dots)e^{ih^{-1}t(\lambda_k-p_k^w(\dots))}\\
& \sigma_{q,l,-}^w(y,h^2D_y,z)b_{kq}^w(\dots)e_{q+}(z,\lambda,\omega,h)dzdt+\mathcal{O}(h^\infty).
\end{aligned}
$$

Moreover, if we define

$$
\overline{\mathcal{Z}}_{q,l}=\{(y,\eta,z)\in\mathbb{R}^2\times\operatorname{supp}(a_{qk})\,|\,z=Z(t,\eta,y,\tilde z,\sqrt{\lambda_q}\omega),\tilde z\in\tilde{\mathcal{Z}}_{q,l}\},
$$

then there exists $T_1>>1$ such that

$$
\overline{\mathcal{Z}}_{q,l}\subset\Sigma_{q,l,+}=\{(y,\eta,z)\,|\,z=Z(t,\eta,y,\tilde z,\sqrt{\lambda_q}\omega),\tilde z\in\Sigma_{q,l,-},\,T_1<t<T_0\}.
$$

Let $\sigma_{q,l,+} \in S^0(\mathbb{R}^n)$ with support contained in $\Sigma_{q,l,+}$ and such that $\sigma_{q,l,+} = 1$ on $\overline{\mathcal{Z}}_{q,l}$. As precedently, Lemma 4.2 yields

$$f_{q,k}(\omega,\omega',\lambda,h) = \sum_{l=1}^{l_q} \int_{T_1}^{T_0} \int_{\mathbb{R}^{n-2}} F_{q,k,l}(z,t,\omega,\omega',\lambda,h)dzdt + \mathcal{O}(h^\infty). \tag{4.16}$$

with

$$\begin{aligned} F_{q,k,l} =& e_{q-}(z,\lambda,\omega',h) a^w_{qk}(y,z,h^2D_y,hD_z)\sigma^w_{q,l,+}(y,h^2D_y,z) \\ & e^{ih^{-1}t(\lambda_k - p^w_k(y,z,h^2D_y,hD_z))}\sigma^w_{q,l,-}(y,h^2D_y,z) b^w_{kq}(y,z,h^2D_y,hD_z)e_{q+}(z,\lambda,\omega,h) \end{aligned}$$

Therefore, for (z,t) fixed, we have to analyze the operator $F_{q,k,l} : L^2(\mathbb{R}_y) \to L^2(\mathbb{R}_y)$. We will approximate the operator $e^{ih^{-1}tp^w_k(\dots)}\sigma^w_{q,l,-}(\dots)b^w_{kq}(\dots)e_{q+}$ by a pseudo-differential operator in the variable y.

5. Approximation of the evolution

Let us denote $P_k(h) = p^w_k(y,z,h^2D_y,hD_z)$, which is self-adjoint on $L^2(\mathbb{R}^{n-1}_{y,z})$ with domain $L^2(\mathbb{R}_y, H^2(\mathbb{R}^{n-2}_z))$. Let $\varphi \in C_0^\infty(\mathbb{R}_y)$ and $\psi(t) \in L^2(\mathbb{R}^{n-1}_{y,z})$ be defined by

$$\psi(t) = e^{-ih^{-1}tP_k(h)}(\sigma^w_{q,l,-}(y,z,h^2D_y,hD_z)b^w_{kq}(y,z,h^2D_y,hD_z)e_{q+}(z,\lambda,\omega,h)\varphi(y)).$$

We look for $\psi(t)$ under the form $\tau^w_{q,k,l}(t,y,z,h^2D_y,h)\varphi$ with

$$\tau_{q,k,l}(t,y,z,\eta,h) = \sum_{j\geq 0} h^j \tau_{q,k,l,j}(t,y,z,\eta)e^{ih^{-1}S_{q,k,l}(t,y,z,\eta)}$$

and $\tau_{q,k,l,j}(t,.), S_{q,k,l}(t,.) \in S^0(\mathbb{R}^n_{y,z,\eta}, h^2)$. Remark that for such $S_{q,k,l}$ and for $z \in \mathbb{R}^{n-1}$ fixed, the symbol $(y,\eta) \mapsto \tau_{q,k,l}(t,y,\eta,z,h)$ belongs to the class $S^{1/2}(\mathbb{R}^n_{y,\eta}, h^2)$. By definition, we have

$$\begin{cases} (ih\partial_t - P_k(h))\psi = 0 \\ \psi_{|t=0} = \sigma^w_{q,l,-}(y,z,h^2D_y,hD_z)b^w_{kq}(y,z,h^2D_y,hD_z)e_{q+}(z,\lambda,\omega,h)\varphi(y). \end{cases}$$

Replacing $\psi(t)$ by $\tau^w(t,y,z,h^2D_y,h)\varphi$ (and forgetting the index q,k,l), it comes

$$\begin{cases} ih\partial_t\tau^w(t,y,z,h^2D_y,h) - P_k(h)\tau^w(t,y,z,h^2D_y,h) = 0 \\ \tau(0,y,z,\eta,h) = \sigma^w_{q,l,-}(y,z,h^2D_y,hD_z)b^w_{kq}(y,z,\eta,hD_z))e_{q+}(z,\lambda,\omega,h). \end{cases} \tag{5.1}$$

The first step is to compute the operator $p^w_k(y,z,h^2D_y,hD_z)\tau^w(t,y,z,h^2D_y,h) : L^2(\mathbb{R}_y) \to L^2(\mathbb{R}^{n-1}_{y,z})$. For (y,η) fixed, the function $z \mapsto \tau(t,y,z,\eta,h)$ can be considered as a multiplication operator from $\mathbb{R}$ into $L^2(\mathbb{R}^{n-2}_z)$. Hence, it follows from the symbolic calculus in $S^0(\mathbb{R}^2_{y,\eta}, \mathcal{L}(L^2(\mathbb{R}_z)), h^2) \times S^{1/2}(\mathbb{R}^2_{y,\eta}, \mathcal{L}(\mathbb{R}, L^2(\mathbb{R}_z)), h^2)$ that

$$p^w_k(y,z,h^2D_y,hD_z)\tau^w(t,y,z,h^2D_y,h) = \eta^w_k(t,y,z,h^2D_y,h)$$

with $\eta_k(t,.) \in S^1(\mathbb{R}^n_{y,z,\eta})$ and

$$\eta_k(t,y,z,\eta,h) = \sum_{\alpha,\beta\in\mathbb{N}} \frac{h^{2(\alpha+\beta)}(-1)^\alpha}{(2i)^{\alpha+\beta}\alpha!\beta!}\partial^\alpha_y\partial^\beta_\eta p_k(y,z,\eta,hD_z,h)\partial^\alpha_\eta\partial^\beta_y\tau(t,y,z,\eta,h).$$

Formally, we obtain $\eta_k(t,y,z,\eta,h) = L(y,z,\eta,hD_y,hD_z,hD_\eta,h)\tau(t,y,z,\eta,h)$ with

$$L(y,z,\eta,y^*,z^*,\eta^*) = \sum_{\alpha,\beta,m\in\mathbb{N}} \frac{h^{\alpha+\beta+m}(-1)^\alpha}{2^{\alpha+\beta}\alpha!\beta!}\partial^\alpha_y\partial^\beta_\eta p_{k,m}(y,z,\eta,z^*)(\eta^*)^\alpha(y^*)^\beta.$$

On the other hand, we have also the formal expansion

$$\sigma_{q,l,-}^w(y,\eta,z)b_{kq}^w(y,z,\eta,hD_z))e_{q+}(z,\lambda,\omega,h) = c_{q,k,l}(y,z,\eta,h)e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle},$$

with $c_{q,k,l}(y,z,\eta,h) = \sum_{\gamma\in\mathbb{N}} h^\gamma c_{q,k,l,\gamma}(y,z,\eta)$,

$$c_{q,k,l,0}(y,z,\eta) = \lambda_q^{\frac{n-4}{4}}\sigma_{q,l,-}(y,\eta,z)b_{q,k}(y,z,\eta,\sqrt{\lambda_q}\omega) \tag{5.2}$$

and $\operatorname{supp} c_{q,k,l,\gamma} \subset \Sigma_{q,l,-}$. Therefore, (5.1) gives

$$\begin{cases} (ih\partial_t - L^w(y,z,\eta,hD_y,hD_z,hD_\eta,h))\tau(t,y,z,\eta,h) = 0 \\ \tau(0,y,z,\eta,h) = \sum_{\gamma\in\mathbb{N}} h^\gamma c_{q,k,l,\gamma}(y,z,\eta)e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle} \end{cases} \tag{5.3}$$

Hence, our problem is reduced to apply Maslov theory (see [13]) to the preceding system. For $N \in \mathbb{N}$ to be chosen large enough, let $L_N \in S^0(T^*\mathbb{R}^n, \langle y^*\rangle^N\langle\eta^*\rangle^N)$ be defined by

$$L_N(y,z,\eta,y^*,z^*,\eta^*) = \sum_{|\alpha+\beta+m|\le N} \frac{h^{\alpha+\beta+m}(-1)^\alpha}{2^{\alpha+\beta}\alpha!\beta!}\partial_y^\alpha\partial_\eta^\beta p_{k,m}(y,z,\eta,z^*)(\eta^*)^\alpha(y^*)^\beta.$$

Its principal symbol l_0 is given by

$$l_0(y,z,\eta,y^*,z^*,\eta^*) = |z^*|^2 + V(\eta,y,z)$$

so that the corresponding Hamiltonian system is

$$\begin{cases} \dot{Z} = 2Z^*, \ \ \dot{Z}^* = -\nabla_z V(\Theta,Y,Z) \\ \dot{Y} = 0, \ \ \dot{Y}^* = -\nabla_y V(\Theta,Y,Z) \\ \dot{\Theta} = 0, \ \ \dot{\Theta}^* = -\nabla_x V(\Theta,Y,Z) \end{cases} \tag{5.4}$$

Remark that (Y,Z,Θ,Z^*) is independent of the initial value of Y^* and Θ^*. To stay coherent with the notations of the previous parts, we denote by $(Z,Z^*)(t,\tilde\eta,\tilde y,\tilde z,\tilde z^*)$ the solution of the two first equations of (5.4) such that $(Z,Z^*,Y,\Theta)_{|t=0} = (\tilde z,\tilde z^*,\tilde y,\tilde\eta)$. Remark that $(Z,Z^*)(t,\tilde\eta,\tilde y,\tilde z,\tilde z^*) = \exp(tH_{p_{\tilde\eta,\tilde y}})(\tilde z,\tilde z^*)$.

Lemma 5.1. *For $(\tilde y,\tilde z,\tilde\eta) \in \Sigma_{q,j}$ and $T_1 < t < T_0$, the point $(Y,Z,\Theta)(t,\tilde y,\tilde z,\tilde\eta,\sqrt{\lambda_q}\omega)$ is non-focal in the Maslov sense;*

$$D_q(t,\tilde y,\tilde z,\tilde\eta) := \det\frac{\partial(Y,Z,\Theta)}{\partial(\tilde y,\tilde z,\tilde\eta)}(\tilde y,\tilde z,\tilde\eta,\sqrt{\lambda_q}\omega) \neq 0.$$

Proof. As $Y(t,\tilde\eta,\tilde y,\tilde z) = \tilde y$ and $\Theta(t,\tilde\eta,\tilde\eta,\tilde z) = \tilde\eta$,

$$D_q(t,\tilde y,\tilde z,\tilde\eta) = \det\frac{\partial Z}{\partial\tilde z}(t,\tilde\eta,\tilde y,\tilde z,\sqrt{\lambda_q}\omega).$$

Denoting $D_{q,\infty}(t,\tilde y,\tilde z,\tilde\eta) = \det\frac{\partial Z_{q,\infty}}{\partial\tilde z}(t,\tilde\eta,\tilde y,\tilde z)$, we have as in [22]

$$D_q(t,\tilde y,\tilde z,\tilde\eta) = D_{q,\infty}(t+s,\tilde y,\tilde z,\tilde\eta)D_{q,\infty}(s,\tilde y,\tilde z,\tilde\eta)^{-1}.$$

The numerator of the right hand side of this equation is different from zero thanks to (1.6) whereas Assumption 2 insures that the denominator is never null. $\square$

Let us denote

$$L_N(h) = L_N^w(y,z,\eta,hD_y,hD_z,hD_\eta,h).$$

Proposition 5.2. *Let $c_l \in C^\infty(\mathbb{R}^n_{y,z,\eta})$ be supported in $\Sigma_{q,l,-}$ for some $l \in \{1,\dots,l_q\}$. Suppose additionally that $\mathrm{supp}_z\, c_l$ is compact and $(y,\eta) \mapsto c_l(y,z,\eta)$ is constant at infinity. Then there exists some functions $\tau_{q,k,l,j} \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_{y,z,\eta})$, $j \in \mathbb{N}$ such that $\mathrm{supp}_z\, \tau_{q,k,l,j}$ is compact, $(y,\eta) \mapsto \tau_{q,k,l,j}(y,z,\eta)$ is constant at infinity and*

$$\begin{aligned}&(e^{-ih^{-1}tL_N(h)}c_l e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle})(t,y,z,\eta) = \\ &\qquad e^{ih^{-1}S_{q,l}(t,y,\tilde z,\eta)-i\mu_{q,l}\pi/2}|D_q(t,y,\tilde z,\eta)|^{-1/2}\sum_{j=1}^N h^j\tau_{q,k,l,j}(t,y,\tilde z,\eta) \\ &\qquad + h^{N+1}r_N(t,y,z,\eta,h)\end{aligned}\tag{5.5}$$

with $r_N \in S^{1/2}(\mathbb{R}^n, h^2)$, $z = Z(t,\eta,y,\tilde z,\sqrt{\lambda_q}\omega)$, $S_{q,l}$ is the action along the trajectory joining $\tilde z$ and z

$$\begin{aligned}S_{q,l}(t,y,\tilde z,\eta) = \int_0^t (|Z^*(s,\eta,y,\tilde z,\sqrt{\lambda_q}\omega)|^2 - V(\eta,y,Z(s,\eta,y,\tilde z,\sqrt{\lambda_q}\omega))ds \\ + \sqrt{\lambda_q}\langle\tilde z,\omega\rangle\end{aligned}\tag{5.6}$$

and $\mu_{q,l}$ is the path index of this trajectory. Moreover, $\tau_{q,k,l,0}(t,y,\tilde z,\eta) = c_l(y,\tilde z,\eta)$ and $\mu_{q,l}$ is independent on (y,η).

Proof. We apply Maslov strategy (see [13]) to the preceding system. In particular, we need to work with compactly supported initial data. For this purpose, we split our problem in two; one concerning large (y,η), the other concerning small one. More precisely, let $\chi \in C_0^\infty(\mathbb{R}^2)$ be such that $\chi(y,\eta)W(\eta,y,z) = W(\eta,y,z)$ and $\tilde c_l = (1-\chi)c_l$ is independent of (y,η). Let us set $\tau_{q,k,l}(t,y,z,\eta,h) = e^{-ih^{-1}tL_N(h)}c_l(y,\eta,z)e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle}$, then

$$L_N(h)(1-\chi) = (1-\chi)(h^2D_z^2 + V^\infty(z) + \Lambda_k)$$

so that

$$\begin{aligned}\tau_{q,k,l}(t,y,z,\eta,h) =& e^{-ih^{-1}tL_N(h)}\chi(y,\eta)c_l(y,\eta,z)e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle} \\ &+ e^{-ith^{-1}(h^2D_z^2+V^\infty(z))}\tilde c_l(z)e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle}.\end{aligned}$$

Thanks to Lemma 5.1, it follows from the Maslov theory for the Schrödinger equation (c.f. Theorem 12.5 in [13]) that the second term of the right hand side of the preceding equation has the required form. Hence, it remains to find an approximate solution of $(ih\partial_t - L_N(h))\tau = 0$ with initial condition $\tau_{|t=0} = \chi(y,\eta)c_l(y,\eta,z)e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle}$. Using again Lemma 5.1, it is clear that $(y,z) = (y, Z(t,\eta,y,\tilde z,\sqrt{\lambda_q}\omega),\eta))$ is non-focal in the Maslov sense for $T_1 < t < T_0$ and $(y,\tilde z,\eta) \in \Sigma_{q,l,-}$ so that we can build an approximate solution τ_{ap} under the form

$$\tau_{ap}(t,y,z,\eta,h) = e^{ih^{-1}S_{q,l}(t,y,\tilde z,\eta)-i\mu_{q,l}\pi/2}|D_l(t,y,\tilde z,\eta)|^{-1/2}\sum_{j=0}^N h^j\tau_{q,k,l,j}(t,y,\tilde z,\eta)$$

where $S_{q,l}$ defined by (5.6), $\mu_{q,l}$ is the path index of the above trajectory, $\tau_{q,k,l,j}$ are smooth with respect to (t,y,z,η) and compactly supported with respect to

(y,η). Moreover, denoting $l_1(y,z,\eta,y^*,z^*,\eta^*)$ the sub-principal symbol of $L_N(h)$, it follows from Theorem 10.5 in [13] that

$$\tau_{q,k,l,0}(t,y,z,\eta)=\exp\left(\int_0^t M(s)ds\right)\chi(y,\eta)c_l(y,\tilde z,\eta),$$

with

$$M(s)=\frac12\left(\frac{\partial^2 l_0}{\partial z\partial z^*}+\frac{\partial^2 l_0}{\partial y\partial y^*}+\frac{\partial^2 l_0}{\partial\eta\partial\eta^*}-2l_1\right)((Y,Z,\Theta,Y^*,Z^*,\Theta^*)(s,y,\tilde z,\eta,\sqrt{\lambda_q}\omega))$$

An easy calculation shows that l_1 vanishes along the Hamiltonian flow so that $M(s)$ is identically zero, and then

$$\tau_{q,k,l,0}(t,y,z,\eta)=\chi(y,\eta)c_l(y,\tilde z,\eta).$$

Thanks to Assumption 2 and using Definition 7.4 in [13] of the path index, it is clear that $\mu_{q,l}$ is locally constant with respect to (y,η). Indeed, for (y_0,η_0) fixed, if $(z^*_{(\alpha)},z_{(\beta)})$ is a local system of focal coordinates associate to the parameter (y_0,η_0) then for (y,η) close to (y_0,η_0), it is still a system of focal coordinates. Moreover, the number of negative eigenvalues of $\partial z_{(\alpha)}/\partial z^*_{(\alpha)}$ at a non-singular point is locally constant so that the index of a path contained in a chart is independent of (y,η). Using formula (7.4) of [13], it follows that $\mu_{q,l}$ is locally constant. As (y,η) takes its values in a connected set, $\mu_{q,l}$ is independent of (y,η).

Moreover, $\tau_{q,k,l,j}$ being compactly supported with respect to (y,η), it follows from Theorem 3.15 in [13] that $(ih\partial_t-L_N(h))\tau_{ap}=h^{N+1}r_{N+1}(t,y,z,\eta)$ with r_{N+1} in $S^{1/2}(\mathbb{R}_t\times\mathbb{R}^n_{y,z,\eta},h^2)$ and the proof is complete. □

Let us fix $N\in\mathbb{N}$ to be chosen large enough at the end of the paper. From the preceding proposition it follows that one can find some functions $\tau_{q,k,l,j}\in C^\infty(\mathbb{R}_t\times\mathbb{R}^n_{y,z,\eta})$, $l=1,\dots,l_q$, $j=1,\dots,N$ such that

$$\tau_{q,k,l}(t,y,z,\eta,h)=e^{ih^{-1}S_{q,l}(t,y,\tilde z,\eta)-i\mu_{q,l}\pi/2}|D_l(t,y,\tilde z,\eta)|^{-1/2}\sum_{j=1}^N h^j\tau_{q,k,l,j}(t,y,\tilde z,\eta)$$

with $z=Z(t,\eta,y,\tilde z,\sqrt{\lambda_q}\omega)$, satisfies

$$(5.7)\qquad\begin{cases}(ih\partial_t-L_N^w(y,z,\eta,hD_y,hD_z,hD_\eta,h))\tau_{q,k,l}=h^{N+1}r_N(t,y,z,\eta,h)\\ \tau_{q,k,l}(0,y,z,\eta,h)=\displaystyle\sum_{\gamma=0}^N h^\gamma c_{k,q,\gamma}(y,z,\eta)e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle}\end{cases}$$

with $r_N(t,.)\in S^{1/2}(\mathbb{R}^n_{y,z,\eta},h^2)$ and $\tau_{q,k,l,0}(t,y,\tilde z,\eta)=c_{q,k,l,0}(y,\tilde z,\eta)$. For large (y,η), $S_{q,l}$ and $\tau_{q,k,l,j}$ depend only on z. Therefore, for all $\alpha,\beta\in\mathbb{N}$, $\partial_y^\alpha\partial_\eta^\beta S_{q,l}$ and $\partial_y^\alpha\partial_\eta^\beta\tau_{q,k,l,j}$ are bounded and it follows that $\tau_{q,k,l}$ belongs to $S^{1/2}(\mathbb{R}^n_{y,z,\eta},h^2)$. Hence, the symbolic calculus in $S^0(\mathbb{R}^2_{y,\eta},h^2)\times S^{1/2}(\mathbb{R}^2_{y,\eta},h^2)$ shows that

$$(ih\partial_t-P_k(h))\tau_{q,k,l}(t,y,z,h^2D_y,h)=h^{N+1}r_N(t,y,z,h^2D_y,h)$$

with $r_N\in S^{1/2}(\mathbb{R}^n_{y,z,\eta},h^2)$. Using this approximate solution, we obtain

$$F_{q,k,l}=e_{q-}(z,\lambda,\omega',h)a^w_{qk}(y,z,h^2D_y,hD_z)\sigma^w_{q,l,+}(y,h^2D_y,z)\tau_{q,k,l}(t,y,z,h^2D_y)+\mathcal{O}(h^N)$$

so that the scattering amplitude is given by

(5.8)
$$f_q(\omega,\omega',\lambda,h)=\sum_{j=0}^{N}h^j\sum_{k=1}^{q_0}\sum_{l=1}^{l_q}\int_{T_1}^{T_0}e^{ih^{-1}t\lambda_k}\int_{\mathbb{R}_z^{n-2}}\mathcal{G}^w_{q,k,l,j}(t,z,\lambda,y,h^2D_y,h)dtdz+\mathcal{O}(h^N)$$

with

$$\begin{aligned}\mathcal{G}_{q,k,l,j}(t,z,y,\eta,h)=&e_{q-}(z,\lambda,\omega',h)a^w_{qk}(y,z,\eta,hD_z)\sigma^w_{q,l,+}(y,\eta,z)\\&e^{ih^{-1}S_{q,l}(t,y,\tilde z,\eta)-i\mu_{q,l}\pi/2}|D_l(t,y,\tilde z,\eta)|^{-1/2}\tau_{q,k,l,j}(t,y,\tilde z,\eta).\end{aligned}$$

and $z=Z(t,\eta,y,\tilde z,\sqrt{\lambda_q}\omega)$. Using th fact that V is compactly supported with respect to z, we can intertwine the integrals in (5.8) so that

$$f_q(\omega,\omega',\lambda,h)=\widetilde{F}^w_q(\omega,\omega',y,h^2D_y,\lambda,h)+\mathcal{O}(h^N)$$

with

$$\widetilde{F}_q(\omega,\omega',y,\eta,\lambda,h)=$$
$$\sum_{j=0}^{N}h^j\sum_{k=1}^{q_0}\sum_{l=1}^{l_q}\int_{T_1}^{T_0}e^{ih^{-1}t\lambda_k}\int_{\mathbb{R}_z^{n-2}}\mathcal{G}_{q,k,l,j}(t,z,y,\eta,h)dzdt. \tag{5.9}$$

We will apply the stationnary phase method to this integral. For brevity, we will consider only the leading term. Moreover, denoting δ_{qk} the Kronecker symbol, it follows from the proof of Proposition 3.5 that

$$a_{q,k}(y,z,\eta,\xi)=2ih\delta_{qk}\nu_0(\xi)\nabla\chi_1(z).\xi+\mathcal{O}(h^2) \tag{5.10}$$

and

$$b_{q,k}(y,z,\eta,\xi)=2ih\delta_{qk}\nu_0(\xi)\nabla\chi_2(z).\xi+\mathcal{O}(h^2), \tag{5.11}$$

in $S^0(\mathbb{R}^{n+1})$. Combining these equations with the explicit form of $e_{q,-}$, we obtain $f_q(\omega,\omega',\lambda,h)=F^w_q(\omega,\omega',y,h^2D_y,\lambda,h)+\mathcal{O}(h)$ in $S^{1/2}(\mathbb{R}^2,h^2)$ with

(5.12)
$$\begin{aligned}F_q(\omega,\omega',y,\eta,\lambda,h)=\lambda_q^{(n-4)/4}\sum_{l=1}^{l_q}\int_{T_1}^{T_0}\int_{\mathbb{R}_z^{n-2}}e^{ih^{-1}(t\lambda_q-\sqrt{\lambda_q}\langle z,\omega'\rangle)}a_{qq}(y,z,\eta,\sqrt{\lambda_q}\omega')\\ \sigma^w_{q,l,+}(y,\eta,z)e^{ih^{-1}S_{q,l}(t,y,\tilde z,\eta)-i\mu_{q,l}\pi/2}|D_l(t,y,\tilde z,\eta)|^{-1/2}\tau_{q,q,l,0}(t,y,\tilde z,\eta)dzdt.\end{aligned}$$

with $z=Z(t,y,\tilde z,\eta,\sqrt{\lambda_q}\omega)$. From Lemma 5.1 it is possible to make the change of variable $z\mapsto Z(t,\eta,y,z,\sqrt{\lambda_q}\omega)$ so that

$$F_q(\omega,\omega',y,\eta,\lambda,h)=\lambda_q^{(n-4)/4}\sum_{l=1}^{l_q}N_{q,l}(\omega,\omega',\lambda,y,\eta,h) \tag{5.13}$$

where

$$\begin{aligned}N_{q,l}(\omega,\omega',\lambda,y,\eta,h)=\int_{T_1}^{T_0}\int_{\mathbb{R}_z^{n-2}}&e^{ih^{-1}\Psi_{q,l}(t,y,z,\eta)-i\mu_{q,l}\pi/2}\\&\tilde g_{q,l}(t,y,z,\eta)|D(t,y,z,\eta)|^{1/2}dzdt,\end{aligned}$$

with

$$\Psi_{q,l}(t,y,z,\eta)=t\lambda_q+S_{q,l}(t,z)-\langle Z(t,\eta,y,z,\sqrt{\lambda_q}\omega),\sqrt{\lambda_q}\omega'\rangle \tag{5.14}$$

and

$$\begin{aligned}\tilde{g}_{q,l}(t,y,z,\eta) =& a_{qq}(y, Z(t,\eta,y,z,\sqrt{\lambda_q}\omega),\eta,\sqrt{\lambda_q}\omega')\\ &\sigma_{q,l,+}(y, Z(t,\eta,y,z,\sqrt{\lambda_q}\omega),\eta) b_{qq}(y,z,\eta,\sqrt{\lambda_q}\omega)\sigma_{q,-}(y,z,\eta).\end{aligned}$$

Hence the proof is reduced to show that F_q converges to the announced limit. This is done in the following section.

6. Conclusion by stationary phase method

Here again we follow closely the proof of [22]. The only significant difference is that we have to show that $F_q(.,h)$ as well as its derivatives have an asymptotic behavior when h goes to zero. On the other hand, the computations are considerably simplified, thanks to the compactness of the support of V in the variable z. The main tool is the stationary phase method which allows to control the derivatives. We treat only the leading term and prove the following

Proposition 6.1. *There exists a sequence $(N_{q,l,j})$, $j=1,\dots,N$ of symbols in $S^0(\mathbb{R}^2_{y,\eta})$ such that*

$$N_{q,l}(y,\eta,h) = c_2(h) e^{ih^{-1}\mathbf{S}_{\mathbf{q},\mathbf{l}}(y,\eta) - i\mu_{q,l}\pi/2} \sum_{j=0}^{N} h^j N_{q,l,j}(y,\eta) + h^{N+1}\mathcal{S}_N(y,\eta,h) \tag{6.1}$$

with $\mathcal{S}_N \in S^{1/2}(\mathbb{R}^2_{y,\eta}, h^2)$,

$$N_{q,l,0} = \hat{\sigma}(\eta, y, \tilde{z}_l(\eta,y))^{-1/2} \tag{6.2}$$

and $c_2(h) = -\frac{(2\pi h)^{(n+1)/2}}{4\pi^2}\exp(i(n-3)\pi/4)$.

Before proving this proposition, we make some reductions and prove some preparation lemmas. We start by a change of variable $(s,z) \in]-S_1, -S_0[\cap \mathcal{Z}_{q,l} \mapsto Z_{q,\infty}(s,\eta,y,z,\omega) \in \Sigma_{q,l,-}$ (note that S_0 being large, Assumption 2 shows that it is a change of variable) so that

$$N_{q,l} = \int_{T_1}^{T_0}\int_{-S_1}^{-S_0} e^{ih^{-1}t\lambda_q - i\mu_{q,l}\pi/2} I_{q,l}(t,s,\omega,\omega',\lambda,y,\eta,h)\,ds\,dt$$

with

$$I_{q,l} = \int_{\mathbb{R}^{n-2}} e^{ih^{-1}\Phi_{q,l}(t,s,z,y,\eta)} g_{q,l}(t,s,z,y,\eta)|D_{q,\infty}(t+s,z)|^{1/2}|D_{q,\infty}(s,z)|^{1/2}dz$$

with

$$\Phi_{q,l} = S_{q,l}(t,y,Z_{q,\infty}(s,\eta,y,z,\omega),\eta) - \langle Z_{q,\infty}(t+s,\eta,y,z,\omega), \sqrt{\lambda_q}\omega'\rangle$$

and

$$\begin{aligned}g_{q,l} = & a_{qq}(y, Z_{q,\infty}(t+s,\eta,y,z,\omega),\eta,\sqrt{\lambda_q}\omega')\sigma_{q,l,+}(y,\eta,Z_{q,\infty}(t+s,\eta,y,z,\omega))\\ & b_{qq}(y, Z_{q,\infty}(s,\eta,y,z,\omega),\eta,\sqrt{\lambda_q}\omega)\sigma_{q,l,-}(y,\eta,Z_{q,\infty}(s,\eta,y,z,\omega))\end{aligned}$$

Lemma 6.2. *The only stationary point of the phase $\Phi_{q,l}$ is $z=\tilde{z}_{q,l}(\eta,y)$ and at this point we have*

i) $\det(\partial^2\Phi_{q,l}/\partial z^2) = \lambda_q^{(n-4)/2}\widehat{\sigma}_q(\eta,y,\tilde{z}_{q,l})D_{q,\infty}(t+s,\tilde{z}_{q,l})$.

ii) $sgn(\partial^2\Phi_{q,l}/\partial z^2) = n-3$

Proof. By definition of $\Phi_{q,l}$ and from the Hamilton-Jacobi theory, we know that

$$\partial\Phi_{q,l}/\partial z_\alpha = \langle Z^*_{q,\infty}(t+s,\eta,y,z,\omega) - \sqrt{\lambda_q}\omega', \partial Z_{q,\infty}(t+s,\eta,y,z,\omega)/\partial z_\alpha\rangle. \tag{6.3}$$

The potential V being compactly supported with respect to z, for large time, the classical trajectories are straight line. In particular, for $(t,s) \in]T_1, T_0[\times]-S_1, -S_0[$, we have

$$Z^*_{q,\infty}(t+s,\eta,y,z) = \sqrt{\lambda_q}\theta_{q,\infty}(\eta,y,z) \tag{6.4}$$

$$\partial Z_{q,\infty}(t+s,\eta,y,z,\omega)/\partial z_\alpha = 2(t+s)\partial Z^*_{q,\infty}(t+s,\eta,y,z,\omega)/\partial z_\alpha + c(z) \tag{6.5}$$

where $c(z) \in R^{n-2}$ does not depend on time. Moreover, from Assumption 2 we know that for $(y,z,\eta) \in \operatorname{supp}\sigma_{q,l,-}$, $\partial Z^*_{q,\infty}(t+s,\eta,y,z,\omega)/\partial z_\alpha \neq 0$. Therefore the main term in (6.3) when $t+s$ becomes large is $(t+s)\langle -\sqrt{\lambda_q}\omega', \partial Z^*_{q,\infty}(t+s,\eta,y,z,\omega)/\partial z_\alpha\rangle$. As $\|Z^*_{q,\infty}\| = \sqrt{\lambda_q}$, it follows from (6.3) that the only stationary points is $z = \tilde z_{q,l}$. Combining (6.3) and (6.4), we obtain easily

$$\begin{aligned}&\frac{\partial^2\Phi_{q,l}}{\partial z_\alpha\partial z_\beta}(t,s,\tilde z_{q,l}(\eta,y),y,\eta) = \\ &\qquad\sqrt{\lambda_q}\langle\frac{\partial\theta_{q,\infty}}{\partial z_\beta}(\eta,y,\tilde z_{q,l}(\eta,y)), \frac{\partial Z_{q,\infty}}{\partial z_\alpha}(t+s,\eta,y,\tilde z_{q,l}(\eta,y),\omega)\rangle.\end{aligned} \tag{6.6}$$

Taking the determinant of both part of the above equation, we obtain $i)$. Moreover, we may assume that $t+s$ is large so that

$$\partial Z_{q,\infty}/\partial z_\alpha \sim \sqrt{\lambda_q}(t+s)\partial\theta_{q,\infty}/\partial z_\alpha.$$

Therefore, $\partial^2\Phi_{q,l}/\partial z^2$ is positive and $ii)$ is a consequence of $i)$. □

Lemma 6.3. *Let* $\mathbf{S_{q,l}}$, $l = 1,\dots,l_q$ *be as in Theorem 2. Then*

$$\Phi_{q,l}(t,s,\tilde z_{q,l}(\eta,y),y,\eta) = \mathbf{S_{q,l}}(y,\eta) - t\lambda_q.$$

Proof. From formula (5.6), we know that

$$\begin{aligned}&S_{q,l}(t, Z_{q,\infty}(s,\eta,y,\tilde z_{q,l}(\eta,y),\omega)) = \sqrt{\lambda_q}\langle Z_{q,\infty}(s,\eta,y,\tilde z_{q,l}(\eta,y)),\omega\rangle \\ &+\int_0^t |Z^*_{q,\infty}(\sigma+s,\eta,y,\tilde z_{q,l}(\eta,y))|^2 - V(\eta,y,Z_{q,\infty}(\sigma+s,\eta,y,\tilde z_{q,l}(\eta,y),\omega)d\sigma\end{aligned}$$

Making the change of variables $\sigma \mapsto \sigma - s$ and using the fact that for

$$\sigma \le s,\ Z_{q,\infty}(\sigma) = 2\sqrt{\lambda_q}\sigma\omega + z \tag{6.7}$$

it is clear that

$$S_{q,l}(t, Z_{q,\infty}(s,\eta,y,\tilde z_{q,l}(\eta,y),\omega)) = \lambda_q(2s+t) + \mathbf{S_{q,l}}(y,\eta).$$

Plugging this expression into (5.14) and using (1.5), we obtain the announced result. □

Proof of Proposition 6.1. We apply the stationary phase method to the integral $I_{q,l}$. From (6.7) one deduces easily that for $\sigma \in]-S_1, -S_0[$, $D_{q,\infty}(s,\tilde z_{q,l}) = 1$. Therefore, combining Lemmas 6.2 and 6.3 with the fact that $\widehat\sigma(\eta,y,\tilde z_{q,l})$ is bounded from below by a strictly positive constant uniformly with respect to $(y,\eta)\in\mathbb{R}^2$, we have

$$I_{q,l}=c_2(h)\widehat{\sigma}_q(\eta,y,\tilde{z}_{q,l}(y,\eta))^{-1/2}e^{ih^{-1}(\mathbf{S_{q,1}}-t\lambda_q)}\sum_{j=0}^{N}h^j\varrho_{q,l,j}(t,s,y,\eta)\\+h^{N+1}R_N(t,s,y,\eta,h)$$

with $\varrho_{q,l,j}(t,s,.)\in S^0(\mathbb{R}^2_{y,\eta})$, $\varrho_{q,l,0}(t,s,y,\eta)=-g_{q,l}(t,s,\tilde{z}_{q,l}(y,\eta),y,\eta)$, and

$$R_N(t,s,y,\eta,h)=\int_0^1(1-\sigma)^N\int_{\mathbb{R}_z^{n-2}}e^{i\sigma h^{-1}\Phi_{q,l}(t,s,z,y,\eta)}\rho_{q,l}(t,s,z,y,\eta)dzd\sigma \tag{6.8}$$

with $\rho_{q,l}(t,s,z,.)\in S^0(\mathbb{R}_{y,\eta})$. Integrating with respect to t and s, if comes that (6.1) is satisfied with

$$N_{q,l,j}=\widehat{\sigma}_q(\eta,y,\tilde{z}_{q,l}(y,\eta))\int_{T_1}^{T_0}\int_{-S_1}^{-S_0}\varrho_{q,l,j}(t,s,y,\eta)dsdt$$

and

$$\mathcal{S}_N(y,\eta,h)=\int_{T_1}^{T_0}\int_{-S_1}^{-S_0}R_N(t,s,y,\eta,h)dsdt.$$

As $\tilde{z}_{q,l}$ belong to the class $S^0(\mathbb{R}^2)$, it is clear that the $N_{q,l,j}$ belong also to $S^0(\mathbb{R}^2)$. Moreover,

$$N_{q,l,0}=-\widehat{\sigma}_q(\eta,y,\tilde{z}_{q,l})^{-1/2}J_{q,l}$$

with

$$J_{q,l}=\int_{-S_1}^{-S_0}\int_{T_1}^{T_0}g_{q,l}(t,s,\tilde{z}_{q,l}(y,\eta),y,\eta)dtds$$

and we have to calculate the above integral. For $(t,s)\in]T_0,T_1[\times]-S,-S_0[$, we have

$$\sigma_{q,l,+}(y,\eta,Z_{q,\infty}(t+s,\eta,y,z,\omega))=\sigma_{q,l,-}(y,\eta,Z_{q,\infty}(s,\eta,y,z,\omega))=1.$$

Combining this result with equations (5.10) and (5.11) we obtain

$$J_{q,l}=2\lambda_q\int_{-S_1}^{-S_0}\int_{T_1}^{T_0}\langle\nabla\chi_1(Z_{q,\infty}(t+s)),\omega'\rangle\langle\nabla\chi_2(Z_{q,\infty}(s)),\omega\rangle dsdt+\mathcal{O}(h)$$

where we denote $Z_{q,\infty}(\sigma)=Z_{q,\infty}(\sigma,\eta,y,\tilde{z}_{q,l},\omega)$. From (1.4) and (6.7) we deduce that

$$J_{q,l}=\int_{-S_1}^{-S_0}\int_{T_1}^{T_0}\frac{d}{dt}(\chi_1(Z_{q,\infty}(t+s)))\frac{d}{ds}(\chi_2(Z_{q,\infty}(s)))dsdt+\mathcal{O}(h)=-1+\mathcal{O}(h)$$

This achieves to prove (6.2) and it remains to show that $\mathcal{S}_N$ is in $S^{1/2}(\mathbb{R}^2,h^2)$. As we integrate on a compact set, it suffices to do it for R_N. For this purpose, it suffices to notice that when we derive equation (6.8) with respect to (y,η), we lost exactly h^{-1} when the derivatives work on the term $e^{i\sigma h^{-1}\Phi_{q,l}}$. This achieves the proof of Proposition 6.1 □

Finally, combining equations (2.2), (4.13), (4.15), (5.12), (5.13) and (6.2), we obtain the announced result.

7. Scattering matrix in dimension 3

In this section, we prove Theorem 1. The proof is very close to that of Theorem 2 and we just sketch it. From formula (4.15), we know that

$$S_{11}(\lambda b, b) = Id + c_0(h)\sum_{q=1}^{q_0} \mathrm{U}^* f_q(\lambda, h) \otimes \Pi_q \mathrm{U} + \mathcal{O}(h^{N_0-2}),$$

with

$$f_q(\lambda, h) = \lambda_q^{-1/2} \sum_{k=1}^{q_0} \int_{T_1}^{T_0} \int_{\mathbb{R}} e^{-ih^{-1}\sqrt{\lambda_q}z} a^w_{qk}(\ldots) \\ e^{ih^{-1}t(\lambda_k - p^w_k(\ldots))} b^w_{kq}(\ldots) e^{ih^{-1}\sqrt{\lambda_q}z} dzdt,$$

where $(\ldots) = (y, z, h^2 D_y, hD_z)$ and $h = b^{-1/2}$ and $T_1 >> 1$ is such that for all $t \in [0, T_1]$ and all $\tilde{z} \in \operatorname{supp}\chi_1'$, $Z(t, \tilde{\eta}, \tilde{y}, \tilde{z}, \sqrt{\lambda_q}) \notin \operatorname{supp}\chi_2'$. Now, we have to approximate the evolution in the same way as in section 5. We treat separately the case $1 \leq q \leq q_1 - 1$ and $q_1 \leq q \leq q_0$.

For $q \in \{1, \ldots, q_1 - 1\}$, $(\tilde{y}, \tilde{z}, \tilde{\eta}) \in \mathbb{R}^3$ and $-t >> 1$, one has

$$Z(t, \tilde{\eta}, \tilde{y}, \tilde{z} + \delta_{\tilde{z}}, \sqrt{\lambda_q}) = \sqrt{\lambda_q}t + (z + \delta_{\tilde{z}}) = \sqrt{\lambda_q}(t + \frac{\delta_{\tilde{z}}}{\sqrt{\lambda_q}}) + z.$$

Hence, for all $t \in \mathbb{R}$,

$$Z(t, \tilde{\eta}, \tilde{y}, \tilde{z} + \delta_{\tilde{z}}, \sqrt{\lambda_q}) = Z(t + \frac{\delta_{\tilde{z}}}{\sqrt{\lambda_q}}, \tilde{\eta}, \tilde{y}, \tilde{z} + \delta\tilde{z}, \sqrt{\lambda_q})$$

and for $t \in [0, T_0]$, we get

$$D_q(t, \tilde{y}, \tilde{z}, \tilde{\eta}) = \det \frac{\partial(Y, Z, \Theta)}{\partial(\tilde{y}, \tilde{z}, \tilde{\eta})} = \det \frac{\partial Z}{\partial \tilde{z}}(t, \tilde{\eta}, \tilde{y}, \tilde{z}, \sqrt{\lambda_q}) \\ = 2\lambda_q^{-1/2} Z^*(t, \tilde{\eta}, \tilde{y}, \tilde{z}, \sqrt{\lambda_q}).$$

As $V < \lambda_q$, $Z^* \neq 0$ so that D_q is not null and there is no focal points on this trajectory. Moreover, for $t \in [T_0, T_1]$ we get $D_q(t, \tilde{y}, \tilde{z}, \tilde{\eta}) = 2$. In particular the conclusion of Lemma 5.1 remains valid and we can apply Maslov theory. Working as in section 5, we get $f_q(\lambda, h) = F^w_q(y, h^2 D_y, \lambda, h) + \mathcal{O}(h)$ with

$$F_q(y, \eta, \lambda, h) = \lambda_q^{-1/2} \int_{T_1}^{T_0} \int_{\mathbb{R}} e^{ih^{-1}(t\lambda_q - \sqrt{\lambda_q}z)} a_{qq}(y, z, \eta, \sqrt{\lambda_q}) \\ e^{ih^{-1}S_q(t, y, \tilde{z}, \eta)} |D_q(t, y, \tilde{z}, \eta)|^{-1/2} b_{qq}(y, \tilde{z}, \eta, \sqrt{\lambda_q}) dzdt,$$

with

$$S_q(t, y, \tilde{z}, \eta) = \int_0^t (|Z^*(s, \eta, y, \tilde{z}, \sqrt{\lambda_q})|^2 - V(\eta, y, Z(s, \eta, y, \tilde{z}, \sqrt{\lambda_q}))ds \\ + \sqrt{\lambda_q}\tilde{z} \tag{7.1}$$

and $z = Z(t, \tilde{y}, \tilde{z}, \tilde{\eta}, \sqrt{\lambda_q})$. Making the change of coordinates $z \mapsto Z(t, y, z, \eta, \sqrt{\lambda_q})$ and using equations (5.10), (5.11), we obtain

$$F_q(y, \eta, \lambda, h) = -4h^2 \lambda_q^{1/2} \int_{T_1}^{T_0} \int_{\mathbb{R}} e^{ih^{-1}\psi(t,z)} \chi_1'(Z(t, \eta, y, z, \sqrt{\lambda_q})) \\ |D_q(t, y, z, \eta)|^{1/2} \chi_2'(z) dzdt + \mathcal{O}(h^3),$$

with

$$\psi(t,z) = S_q(t,y,z,\eta) + t\lambda_q - Z(t,\eta,y,z,\sqrt{\lambda_q}).$$

Thanks to the fact that the variable z takes its values in $\mathbb{R}$, a simple calculus shows that

$$\psi(t,z) = \sqrt{2}\int_z^{Z(t,y,z,\eta,\sqrt{\lambda_q})} \sqrt{\lambda_q - V(\eta,y,z')} - \sqrt{\lambda_q}dz'.$$

Let $\chi_{2\pm} \in C^\infty(\mathbb{R})$ be such that $\chi_{2\pm}(z) = 1$ for $\mp z \geq 0$ and $\chi_{2\pm}(z) = \chi_2(z)$ for $\pm z \geq 0$. Then $\chi_2' = \chi_{2+}' + \chi_{2-}'$ and $F_q = F_{q+} + F_{q-}$ where $F_{q\pm}$ is obtained from F_q by replacing χ_2 by $\chi_{2\pm}$. For $z \in \operatorname{supp}\chi_{2+}'$ and $t \geq 0$, we have $[z, Z(t,y,z,\eta,\sqrt{\lambda_q})] \cap \operatorname{supp} V = \emptyset$ so that $\psi(t,z) = 0$. Hence

$$F_{q+}(y,\eta,\lambda,h) = -4h^2\lambda_q^{1/2}\int_{T_1}^{T_0}\int_{\mathbb{R}} \chi_1'(Z(t,\eta,y,z,\sqrt{\lambda_q}))\chi_{2+}'(z)dzdt + \mathcal{O}(h^3).$$

Moreover, for $t \in [T_1, T_0]$, $\frac{dZ}{dt}(t,\eta,y,z,\sqrt{\lambda_q}) = 2\sqrt{\lambda_q}$ and we get

$$F_{q+}(y,\eta,\lambda,h) = -2h^2\int_{T_1}^{T_0}\int_{\mathbb{R}} \frac{d}{dt}\chi_1(Z(t,\eta,y,z,\sqrt{\lambda_q}))\chi_{2+}'(z)dzdt + \mathcal{O}(h^3).$$

By definition of T_0 and T_1, it follows that

$$F_{q+}(y,\eta,\lambda,h) = -2h^2 + \mathcal{O}(h^3). \tag{7.2}$$

Similarly, for $z \in \operatorname{supp}\chi_{2-}'$ and $t \in [T_1, T_0]$, we have

$$V(\eta,y,z') = 0,\ \forall z' \in \mathbb{R} \setminus [z, Z(t,\eta,y,z,\sqrt{\lambda_q}))]$$

so that

$$\psi(t,z) = \sqrt{2}\int_{-\infty}^{+\infty} \sqrt{\lambda_q - V(\eta,y,z')} - \sqrt{\lambda_q}dz'. \tag{7.3}$$

In particular it doesn't depend on t and z. As in the case of F_{q+} it follows that

$$F_{q-}(y,\eta,\lambda,h) = 2h^2 e^{ih^{-1}\psi(t,z)} + \mathcal{O}(h^3), \tag{7.4}$$

with $\psi(t,z)$ given by (7.3). Combining equations (4.13), (7.2) and (7.4) we obtain

$$\forall q \in \{1,\dots,q_1-1\},\ c_0(h)f_q(\lambda,h) = -Id + s_{d,q}(\lambda, b^{-1}D_y, y) \tag{7.5}$$

with $s_{d,q}$ as in the theorem.

The case $q \in \{q_1,\dots,q_0\}$ can be treated by the same method. As classical particles starting from $x << -1$ with initial speed equal to $\sqrt{\lambda_q}$ escape with speed $-\sqrt{\lambda_q}$, the contribution of the term F_{q-} is $\mathcal{O}(b^{-\infty})$. Similarly, particles starting from $x >> 1$ with initial speed equal to $\sqrt{\lambda_q}$ escape with speed $\sqrt{\lambda_q}$ and the contribution of the term F_{q+} is $-Id$. Therefore, we obtain

$$\forall q \in \{q_1,\dots,q_0\},\ c_0(h)f_q(\lambda,h) = -Id + \mathcal{O}(b^{-\infty}). \tag{7.6}$$

From equations (7.5) and (7.6), we deduce the asymptotics announced for S_{11}. The case sof S_{21} and S_{12} are similar and left to the reader. □

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