Classical Mechanics

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1. General principles

Classical Mechanics is a theory of point particles motions. If $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ are the particles positions in a Cartesian inertial system of coordinates, the equations of motion are determined by their masses $(m_1, \ldots, m_n)$, $m_j > 0$, and by the *potential energy* of interaction $V(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ as

$$m_i \ddot{\mathbf{x}}_i = -\partial_{\mathbf{x}_i} V(\mathbf{x}_1, \ldots, \mathbf{x}_n), \qquad i = 1, \ldots, n \tag{1.1}$$

here $\mathbf{x}_i = (x_{i1}, \ldots, x_{id})$ are coordinates of the i-th particle and $\partial_{\mathbf{x}_i}$ is the gradient $(\partial_{x_{i1}}, \ldots, \partial_{x_{id}})$; d is the space dimension (*i.e.* $d = 3$, usually). The potential energy function will be supposed "smooth", *i.e. analytic* except, possibly, when two positions coincide. The latter exception is necessary to include the important cases of gravitational attraction or, when dealing with electrically charged particles, of Coulomb interaction. A basic result is that if V is bounded below the equation (1.1) admits, given initial data $\mathbf{X}_0 = \mathbf{X}(0), \dot{\mathbf{X}}_0 = \dot{\mathbf{X}}(0)$, a unique global solution $t \to \mathbf{X}(t)$, $t \in (-\infty, \infty)$; otherwise a solution can fail to be global if and only if, in a finite time, it reaches infinity or a singularity point (*i.e.* a configuration in which two or more particles occupy the same point: an event called a *collision*).

In Eq. (1.1) $-\partial_{\mathbf{x}_i} V(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is the *force* acting on the points. More general forces are often admitted. For instance velocity dependent *friction forces*: they are not considered here because of their phenomenological nature as models for microscopic phenomena which should also, in principle, be explained in terms of conservative forces (furthermore, even from a macroscopic viewpoint, they are rather incomplete models as they should be considered together with the important heat generation phenomena that accompany them). Another interesting example of forces not corresponding to a potential are certain velocity dependent forces like the *Coriolis force* (which however appears only in non inertial frames of reference) and the closely related *Lorentz force* (in electromagnetism): they could be easily accomodated in the upcoming Hamiltonian formulation of mechanics, see Appendix A2.

The *action principle* states that an equivalent formulation of the equations (1.1) is that a motion $t \to \mathbf{X}_0(t)$ satisfying (1.1) during a time interval $[t_1, t_2]$ and leading from $\mathbf{X}^1 = \mathbf{X}_0(t_1)$ to $\mathbf{X}^2 = \mathbf{X}_0(t_2)$, renders stationary the *action*

$$\mathcal{A}(\{\mathbf{X}\})=\int_{t_1}^{t_2}\Big(\sum_{i=1}^n\frac12\,m_i\,\dot{\mathbf{X}}_i(t)^2-V(\mathbf{X}(t))\Big)\,dt \tag{1.2}$$

within the class $\mathcal{M}_{t_1,t_2}(\mathbf{X}^1,\mathbf{X}^2)$ of smooth (*i.e.* analytic) "motions" $t\to\mathbf{X}(t)$ defined for $t\in[t_1,t_2]$ and leading from $\mathbf{X}^1$ to $\mathbf{X}^2$.

The function $\mathcal{L}(\mathbf{Y},\mathbf{X})=\frac12\sum_{i=1}^n m_i\mathbf{y}_i^2-V(\mathbf{X})\stackrel{def}{=}K(\mathbf{Y})-V(\mathbf{X})$, $\mathbf{Y}=(\mathbf{y}_1,\ldots,\mathbf{y}_n)$, is called the *Lagrangian* function and the action can be written as $\int_{t_1}^{t_2}\mathcal{L}(\dot{\mathbf{X}}(t),\mathbf{X}(t))\,dt$. The quantity $K(\dot{\mathbf{X}}(t))$ is called *kinetic energy* and motions satisfying (1.1) conserve *energy* as time t varies, *i.e.*

$$K(\dot{\mathbf{X}}(t))+V(\mathbf{X}(t))=E=const \tag{1.3}$$

Hence the action principle can be intuitively thought of as saying that motions proceed by keeping constant the energy, sum of the kinetic and potential energies, while trying to share as evenly as possible their (average over time) contribution to the energy.

In the special case in which V is translation invariant motions conserve *linear momentum* $\mathbf{Q}\stackrel{def}{=}\sum_i m_i\dot{\mathbf{x}}_i$; if V is rotation invariant around the origin O motions conserve *angular momentum* $\mathbf{M}\stackrel{def}{=}\sum_i m_i\,\mathbf{x}_i\wedge\dot{\mathbf{x}}_i$, where $\wedge$ denotes the vector product in $\mathbb{R}^d$, *i.e.* it is the tensor $(\mathbf{a}\wedge\mathbf{b})_{ij}=a_ib_j-b_ia_j$, $i,j=1,\ldots,d$: if the dimension $d=3$ the $\mathbf{a}\wedge\mathbf{b}$ will be naturally regarded as a vector. More generally to any continuous symmetry group of the Lagrangian correspond conserved quantities: this is formalized in the *Noether theorem*.

It is convenient to think that the scalar product in $\mathbb{R}^{dn}$ is defined in terms of the ordinary scalar product in $\mathbb{R}^d$, $\mathbf{a}\cdot\mathbf{b}=\sum_{j=1}^d a_jb_j$, by $(\mathbf{v},\mathbf{w})=\sum_{i=1}^n m_i\mathbf{v}_i\cdot\mathbf{w}_i$: so that kinetic energy and line element ds can be written as $K(\dot{\mathbf{X}})=\frac12(\dot{\mathbf{X}},\dot{\mathbf{X}})$ and, respectively, $ds^2=\sum_{i=1}^n m_id\mathbf{x}_i^2$. Therefore the metric generated by the latter scalar product can be called *kinetic energy metric.*

The interest of the kinetic metric appears from the *Maupertuis' principle* (*equivalent* to (1.1)): the principle allows us to identify the trajectory traced in $\mathbb{R}^d$ by a motion that leads from $\mathbf{X}^1$ to $\mathbf{X}^2$ moving with energy E. Parameterizing such trajectories as $\tau\to\mathbf{X}(\tau)$ by a parameter τ varying in $[0,1]$ so that the line element is $ds^2=(\partial_\tau\mathbf{X},\partial_\tau\mathbf{X})d\tau^2$, the principle states that the trajectory of a motion with energy E which leads from $\mathbf{X}^1$ to $\mathbf{X}^2$ makes stationary, among the analytic curves $\boldsymbol{\xi}\in\mathcal{M}_{0,1}(\mathbf{X}^1,\mathbf{X}^2)$, the function

$$L(\boldsymbol{\xi})=\int_{\boldsymbol{\xi}}\sqrt{E-V(\boldsymbol{\xi}(s))}\,ds \tag{1.4}$$

so that the possible trajectories traced by the solutions of (1.1) in $\mathbb{R}^{nd}$ and with energy E can be identified with the *geodesics* of the metric $dm^2\stackrel{def}{=}(E-V(\mathbf{X}))\cdot ds^2$.

References: [LL68], [Ga83]

2. Constraints

Often particles are subject to *constraints* which force the motion to take place on a surface $M \subset \mathbb{R}^{nd}$: *i.e.* $\mathbf{X}(t)$ is forced to be a point on the manifold M. A typical example is provided by *rigid systems* in which motions are subject to forces which keep the mutual distances of the particles constant: $|\mathbf{x}_i - \mathbf{x}_j| = \rho_{ij}$ with ρ_{ij} time independent positive quantities. In essentially all cases the forces that imply constraints, called *constraints reactions*, are velocity dependent and, therefore, *are not* in the class of *conservative* forces considered here, cf. (1.1). Hence, from a fundamental viewpoint admitting only conservative forces, constrained systems should be regarded as idealizations of systems subject to conservative forces which *approximately* imply the constraints.

In general the ℓ-dimensional manifold M will not admit a global system of coordinates: however it will be possible to describe points in the vicinity of any $\mathbf{X}^0 \in M$ by using $N = nd$ coordinates $\mathbf{q} = (q_1, \ldots, q_\ell, q_{\ell+1}, \ldots, q_N)$ varying in an open ball $B_{\mathbf{X}^0}$: $\mathbf{X} = \mathbf{X}(q_1, \ldots, q_\ell, q_{\ell+1}, \ldots, q_N)$.

The q-coordinates can be chosen *well adapted* to the surface M and to the kinetic metric, *i.e.* so that the points of M are identified by $q_{\ell+1} = \ldots = q_N = 0$ (which is the meaning of "adapted") and furthermore infinitesimal displacements $(0, \ldots, 0, d\varepsilon_{\ell+1}, \ldots, d\varepsilon_N)$ out of a point $\mathbf{X}^0 \in M$ are orthogonal to M (in the kinetic metric) and have a length independent of the position of $\mathbf{X}^0$ on M (which is the meaning of 'well adapted" to the kinetic metric).

Motions constrained on M arise when the potential V has the form

$$V(\mathbf{X}) = V_a(\mathbf{X}) + \lambda\, W(\mathbf{X}) \tag{2.1}$$

where W is a smooth function which reaches its minimum value, say equal to 0, precisely on the manifold M while V_a is another smooth potential. The factor $\lambda > 0$ is a parameter called the *rigidity* of the constraint.

A particularly interesting case arises when, furthermore, the level surfaces of W have the geometric property of being "parallel" to the surface M: in the precise sense that the matrix $\partial^2_{q_i q_j} W(\mathbf{X})$, $i, j > \ell$ is positive definite and $\mathbf{X}$-independent for all $\mathbf{X} \in M$ in a system of coordinates well adapted to the kinetic metric.

A potential W with the latter properties can be called an *approximately ideal constraint reaction.* In fact it can be proved that, given an initial datum $\mathbf{X}^0 \in M$ with velocity $\dot{\mathbf{X}}^0$ tangent to M, *i.e.* given an initial datum whose coordinates in a local system of coordinates are $(\mathbf{q}_0, \mathbf{0})$ and $(\dot{\mathbf{q}}_0, \mathbf{0})$ with $\mathbf{q}_0 = (q_{01}, \ldots, q_{0\ell})$ and $\dot{\mathbf{q}}_0 = (\dot{q}_{01}, \ldots, \dot{q}_{0\ell})$, the motion generated by (1.1) with V given by (2.1) is a motion $t \to \mathbf{X}_\lambda(t)$ which

(1) as $\lambda \to \infty$ tends to a motion $t \to \mathbf{X}_\infty(t)$,
(2) as long as $\mathbf{X}_\infty(t)$ stays in the vicinity of the initial data, say for $0 \le t \le t_1$, so that it can be described in the above local adapted coordinates, its coordinates have the form $t \to (\mathbf{q}(t), \mathbf{0}) = (q_1(t), \ldots, q_\ell(t), 0, \ldots, 0)$: *i.e.* it is a motion developing on the constraint surface M,
(3) the curve $t \to \mathbf{X}_\infty(t)$, $t \in [0, t_1]$, as an element of the space $\mathcal{M}_{0,t_1}(\mathbf{X}^0, \mathbf{X}_\infty(t_1))$ of analytic curves on M connecting $\mathbf{X}^0$ to $\mathbf{X}_\infty(t_1)$, renders the action

$$A(\mathbf{X})=\int_0^{t_1}\big(K(\dot{\mathbf{X}}(t))-V_a(\mathbf{X}(t))\big)\,dt \tag{2.2}$$

stationary.

The latter property can be formulated "intrinsically", *i.e.* referring only to M as a surface, via the restriction of the metric ds^2 to line elements $d\mathbf{s}=(dq_1,\ldots,dq_\ell,0,\ldots,0)$ tangent to M at the point $\mathbf{X}=(\mathbf{q}_0,0,\ldots,0)\in M$; we write $ds^2=\sum_{i,j}^{1,\ell}g_{ij}(\mathbf{q})dq_i\,dq_j$. The $\ell\times\ell$ symmetric positive definite matrix g is called the *metric* on M induced by the kinetic metric. Then the action in (2.2) can be written as

$$\mathcal{A}(\mathbf{q})=\int_0^{t_1}\Big(\frac12\sum_{i,j}^{1,\ell}g_{ij}(\mathbf{q}(t))\dot q_i(t)\dot q_j(t)-\overline{V}_a(\mathbf{q}(t))\Big)\,dt \tag{2.3}$$

where $\overline{V}_a(\mathbf{q})\stackrel{def}{=}V_a(\mathbf{X}(q_1,\ldots,q_\ell,0,\ldots,0))$: the function

$$\mathcal{L}(\boldsymbol{\eta},\mathbf{q})\stackrel{def}{=}\frac12\sum_{i,j}^{1,\ell}g_{ij}(\mathbf{q})\eta_i\eta_j-\overline{V}_a(\mathbf{q})\equiv\frac12 g(\mathbf{q})\boldsymbol{\eta}\cdot\boldsymbol{\eta}-\overline{V}_a(\mathbf{q}) \tag{2.4}$$

is called the *constrained Lagrangian* of the system.

An important property is that the constrained motions conserve the energy defined as $E=\frac12(g(\mathbf{q})\dot{\mathbf{q}},\dot{\mathbf{q}})+\overline{V}_a(\mathbf{q})$, see Sec. 3.

The constrained motion $\mathbf{X}_\infty(t)$ of energy E satisfies the Maupertuis' principle in the sense that the curve on M on which the motion develops renders

$$L(\boldsymbol{\xi})=\int_{\boldsymbol{\xi}}\sqrt{E-V_a(\boldsymbol{\xi}(s))}\,ds \tag{2.5}$$

stationary among the (smooth) curves that develop on M connecting two fixed values $\mathbf{X}_1$ and $\mathbf{X}_2$. In the particular case in which $\ell=n$ this is again Maupertuis' principle for unconstrained motions under the potential $V(\mathbf{X})$. In general ℓ is called the *number of degrees of freedom* because a complete description of the initial data requires 2ℓ coordinates $\mathbf{q}(0),\dot{\mathbf{q}}(0)$.

If W is minimal on M but the condition on W of having level surfaces parallel to M is not satisfied, *i.e.* if W is not an approximate ideal constraint reaction, it remains still true that the limit motion $\mathbf{X}_\infty(t)$ takes place on M. However, *in general*, it will not satisfy the above variational principles. For this reason motions arising as limits as $\lambda\to\infty$ of motions developing under the potential (2.1) with W having minimum on M and level curves parallel (in the above sense) to M are called *ideally constrained motions* or motions subject by ideal constraints to the surface M.

As an example suppose that W has the form $W(\mathbf{X})=\sum_{i,j\in\mathcal{P}}w_{ij}(|\mathbf{x}_i-\mathbf{x}_j|)$ with $w_{ij}(|\boldsymbol{\xi}|)\ge0$ an analytic function vanishing only when $|\boldsymbol{\xi}|=\rho_{ij}$ for i,j in some set of pairs $\mathcal{P}$ and for some given distances ρ_{ij} (for instance $w_{ij}(\boldsymbol{\xi})=(\boldsymbol{\xi}^2-\rho_{ij}^2)^2\,\gamma$, $\gamma>0$). Therefore *the so constrained motions $\mathbf{X}_\infty(t)$ of the body satisfy the variational principles* mentioned in connection with (2.3) and (2.5): in other words the above natural way of realizing a rather general rigidity constraint is ideal.

This is the modern viewpoint on the physical meaning of the constraints reactions: looking at motions in an inertial cartesian system it will appear that the system is subject to the *applied forces* with potential $V_a(\mathbf{X})$ and to *constraint forces* which are *defined* as the differences $\mathbf{R}_i = m_i\ddot{\mathbf{x}}_i + \boldsymbol{\partial}_{\mathbf{x}_i}V_a(\mathbf{X})$. The latter reflect the action of the forces with potential $\lambda W(\mathbf{X})$ in the limit of infinite rigidity ($\lambda \to \infty$).

In applications *sometimes* the action of a constraint can be regarded as ideal: so that the motion will verify the mentioned variational principles and the $\mathbf{R}$ can be computed as the differences between the $m_i\ddot{\mathbf{x}}_i$ and the active forces $-\boldsymbol{\partial}_{\mathbf{x}_i}V_a(\mathbf{X})$. In dynamics problems it is, *however*, a very difficult and important matter, particularly in engineering, to judge whether a system of particles can be considered as subject to ideal constraints: this leads to important decisions in the construction of machines. It simplifies the calculations of the efforts of the materials but a misjudgement can have serious consequences about stability and safety. For statics problems the difficulty is of lower order: usually assuming that the constraints reaction is ideal leads to an overestimate of the requirements for stability of equilibria. Hence employing the action principle to statics problems, where it constitutes the *virtual works principle*, generally leads to economic problems rather than to safety issues. Its discovery even predates Newtonian mechanics.

References: [Ar68], [Ga83].

3. Lagrange and Hamilton form of the equations of motion

The stationarity condition for the action $\mathcal{A}(\mathbf{q})$, cf. (2.3),(2.4), is formulated in terms of the Lagrangian $\mathcal{L}(\boldsymbol{\eta},\boldsymbol{\xi})$, see (2.4), by

$$\frac{d}{dt}\partial_{\eta_i}\mathcal{L}(\dot{\mathbf{q}}(t),\mathbf{q}(t)) = \partial_{q_i}\mathcal{L}(\dot{\mathbf{q}}(t),\mathbf{q}(t)), \qquad i=1,\ldots,\ell \tag{3.1}$$

which is a second order differential equation called *Lagrangian equation of motion.* It can be cast in "normal form": for this purpose, adopting the convention of "summation over repeated indices", introduce the "generalized momenta"

$$p_i \stackrel{def}{=} g(\mathbf{q})_{ij}\dot{q}_j, \qquad i=1,\ldots,\ell \tag{3.2}$$

Since $g(\mathbf{q}) > 0$ the motions $t \to \mathbf{q}(t)$ and the corresponding velocities $t \to \dot{\mathbf{q}}(t)$ can be described equivalently by $t \to (\mathbf{q}(t),\mathbf{p}(t))$: and the equations of motion (3.1) become the first order equations

$$\dot{q}_i = \partial_{p_i}\mathcal{H}(\mathbf{p},\mathbf{q}), \qquad \dot{p}_i = -\partial_{q_i}H(\mathbf{p},\mathbf{q}) \tag{3.3}$$

where the function $\mathcal{H}$, called *Hamiltonian* of the system, is defined by

$$\mathcal{H}(\mathbf{p},\mathbf{q}) \stackrel{def}{=} \frac{1}{2}(g(\mathbf{q})^{-1}\mathbf{p},\mathbf{p}) + \overline{V}_a(\mathbf{q}) \tag{3.4}$$

Eq. (3.3), regarded as equations of motion for phase space points $(\mathbf{p},\mathbf{q})$, are called *Hamilton equations*. In general $\mathbf{q}$ are local coordinates on M and motions are specified by giving $\mathbf{q},\dot{\mathbf{q}}$ or $\mathbf{p},\mathbf{q}$.

Looking for a coordinate free representation of motions consider the pairs $\mathbf{X},\mathbf{Y}$ with $\mathbf{X}\in M$ and $\mathbf{Y}$ a vector $\mathbf{Y}\in T_{\mathbf{X}}$ tangent to M at the point $\mathbf{X}$. The collection of pairs $(\mathbf{Y},\mathbf{X})$ is denoted $T(M)=\cup_{\mathbf{X}\in M}\big(T_{\mathbf{X}}\times\{\mathbf{X}\}\big)$ and a motion $t\to(\dot{\mathbf{X}}(t),\mathbf{X}(t))\in T(M)$ in local coordinates is represented by $(\dot{\mathbf{q}}(t),\mathbf{q}(t))$. The space $T(M)$ can be called the space of initial data for Lagrange's equations of motion: it has 2ℓ dimensions (more fancily known as the *tangent bundle* of M).

Likewise the space of initial data for the Hamilton equations will be denoted $T^*(M)$ and it consists in pairs $\mathbf{X},\mathbf{P}$ with $\mathbf{X}\in M$ and $\mathbf{P}=g(\mathbf{X})\mathbf{Y}$ with $\mathbf{Y}$ a vector tangent to M at $\mathbf{X}$. The space $T^*(M)$ is called the *phase space* of the system: it has 2ℓ dimensions (and it is occasionally called the *cotangent bundle* of M).

Immediate consequence of (3.3) is $\frac{d}{dt}\mathcal{H}(\mathbf{p}(t),\mathbf{q}(t))\equiv 0$ and it means that $\mathcal{H}(\mathbf{p}(t),\mathbf{q}(t))$ is constant along the solutions of the (3.3). Remarking that $\mathcal{H}(\mathbf{p},\mathbf{q})=\frac{1}{2}(g(\mathbf{q})\dot{\mathbf{q}},\dot{\mathbf{q}})+\overline{V}_a(\mathbf{q})$ is the sum of the kinetic and potential energies it follows that the conservation of $\mathcal{H}$ along solutions has the meaning of *energy conservation* in presence of ideal constraints.

Let S_t be the *flow* generated on the phase space variables $(\mathbf{p},\mathbf{q})$ by the solutions of the equations of motion (3.3), *i.e.* let $t\to S_t(\mathbf{p},\mathbf{q})\equiv(\mathbf{p}(t),\mathbf{q}(t))$ denote a solution of (3.3) with initial data $(\mathbf{p},\mathbf{q})$. Then a (measurable) set Δ in phase space evolves in time t into a new set $S_t\Delta$ *with the same volume*: this is obvious because the Hamilton equations (3.3) have manifestly 0 divergence ("*Liouville's theorem*").

The Hamilton equations also satisfy a variational principle, called *Hamilton action principle*: *i.e.* if $\mathcal{M}_{t_1,t_2}((\mathbf{p}_1,\mathbf{q}_1),(\mathbf{p}_2,\mathbf{q}_2);M)$ denotes the space of the analytic functions $\boldsymbol{\varphi}:\,t\to(\boldsymbol{\pi}(t),\boldsymbol{\kappa}(t))$ which in the time interval $[t_1,t_2]$ lead from $(\mathbf{p}_1,\mathbf{q}_1)$ to $(\mathbf{p}_2,\mathbf{q}_2)$, then the condition that $\boldsymbol{\varphi}_0(t)=(\mathbf{p}(t),\mathbf{q}(t))$ satisfies (3.3) can be equivalently formulated by requiring that the function

$$\mathcal{A}_{\mathcal{H}}(\boldsymbol{\varphi}) \stackrel{def}{=} \int_{t_1}^{t_2}\Big(\boldsymbol{\pi}(t)\cdot\dot{\boldsymbol{\kappa}}(t)-\mathcal{H}(\boldsymbol{\pi}(t),\boldsymbol{\kappa}(t))\Big)\,dt \tag{3.5}$$

be stationary for $\boldsymbol{\varphi}=\boldsymbol{\varphi}_0$: in fact the (3.3) are the stationarity conditions for the Hamilton action (3.5) on $\mathcal{M}_{t_0,t_1}((\mathbf{p}_1,\mathbf{q}_1),(\mathbf{p}_2,\mathbf{q}_2);M)$. And since the derivatives of $\boldsymbol{\pi}(t)$ do not appear in (3.5) stationarity is even achieved in the larger space $\mathcal{M}_{t_1,t_2}(\mathbf{q}_1,\mathbf{q}_2;M)$ of the motions $\boldsymbol{\varphi}:\,t\to(\boldsymbol{\pi}(t),\boldsymbol{\kappa}(t))$ leading from $\mathbf{q}_1$ to $\mathbf{q}_2$ *without* any restriction on the initial and final momenta $\mathbf{p}_1,\mathbf{p}_2$ (which therefore cannot be prescribed *a priori* independently of $\mathbf{q}_1,\mathbf{q}_2$). If the prescribed data $\mathbf{p}_1,\mathbf{q}_1,\mathbf{p}_2,\mathbf{q}_2$ are not compatible with the equations of motion, *e.g.* $H(\mathbf{p}_1,\mathbf{q}_2)\neq H(\mathbf{p}_2,\mathbf{q}_2)$, then the action functional has no stationary trajectory in $\mathcal{M}_{t_1,t_2}((\mathbf{p}_1,\mathbf{q}_1),(\mathbf{p}_2\mathbf{q}_2);M)$.

References: [LL68], [Ar68], [Ga83]

4. Canonical transformations of phase space coordinates

The Hamiltonian form, (3.4), of the equations of motion turns out to be quite useful in several problems. It is therefore important to remark that it is *invariant* under a special class of transformations of coordinates, called *canonical transformations.*

Consider a local change of coordinates on phase space, *i.e.* a smooth smoothly invertible map $\mathcal{C}(\boldsymbol{\pi},\boldsymbol{\kappa})=(\boldsymbol{\pi}',\boldsymbol{\kappa}')$ between an open set U in the phase space of a ℓ degrees of freedom Hamiltonian system, into an open set U' in a 2ℓ dimensional space. The change of coordinates is said *canonical* if for any solution $t\to(\boldsymbol{\pi}(t),\boldsymbol{\kappa}(t))$ of equations like (3.3), for any Hamiltonian $\mathcal{H}(\boldsymbol{\pi},\boldsymbol{\kappa})$ defined on U, the $\mathcal{C}$–image $t\to(\boldsymbol{\pi}'(t),\boldsymbol{\kappa}'(t))=\mathcal{C}(\boldsymbol{\pi}(t),\boldsymbol{\kappa}(t))$ are solutions of the (3.3) with the "same" Hamiltonian, *i.e.* with Hamiltonian $\mathcal{H}'(\boldsymbol{\pi}',\boldsymbol{\kappa}')\stackrel{def}{=}\mathcal{H}(\mathcal{C}^{-1}(\boldsymbol{\pi}',\boldsymbol{\kappa}'))$.

The condition that a transformation of coordinates is canonical is obtained by using the arbitrariness of the function $\mathcal{H}$ and is simply expressed as a necessary and sufficient property of the Jacobian L

$$L=\begin{pmatrix}A & B\\ C & D\end{pmatrix},\qquad A_{ij}=\partial_{\pi_j}\pi'_i,\ B_{ij}=\partial_{\kappa_j}\pi'_i,\ C_{ij}=\partial_{\pi_j}\kappa'_i,\ D_{ij}=\partial_{\kappa_j}\kappa'_i, \tag{4.1}$$

where $i,j=1,\ldots,\ell$. Let $E=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ denote the $2\ell\times2\ell$ matrix formed by four $\ell\times\ell$ blocks, equal to the 0 matrix or, as indicated, to the $\pm$ identity matrix; then, if a superscript T denotes matrix transposition, the condition that the map be canonical is that

$$L^{-1}=EL^TE^T\qquad\text{or}\quad L^{-1}=\begin{pmatrix}D^T & -B^T\\ -C^T & A^T\end{pmatrix} \tag{4.2}$$

which immediately implies that $\det L=\pm1$. In fact it is possible to show that (4.2) implies $\det L=1$. The (4.2) is equivalent to the four relations $AD^T-BC^T=1$, $-AB^T+BA^T=0$, $CD^T-DC^T=0$, $-CB^T+DA^T=1$. More explicitly, since the first and the fourth coincide, the relations are

$$\{\pi'_i,\kappa'_j\}=\delta_{ij},\quad \{\pi'_i,\pi'_j\}=0,\quad \{\kappa'_i,\kappa'_j\}=0 \tag{4.3}$$

where, for any two functions $F(\boldsymbol{\pi},\boldsymbol{\kappa}),G(\boldsymbol{\pi},\boldsymbol{\kappa})$, the *Poisson bracket* is

$$\{F,G\}(\boldsymbol{\pi},\boldsymbol{\kappa})\stackrel{def}{=}\sum_{k=1}^{\ell}\Big(\partial_{\pi_k}F(\boldsymbol{\pi},\boldsymbol{\kappa})\,\partial_{\kappa_k}G(\boldsymbol{\pi},\boldsymbol{\kappa})-\partial_{\kappa_k}F(\boldsymbol{\pi},\boldsymbol{\kappa})\,\partial_{\pi_k}G(\boldsymbol{\pi},\boldsymbol{\kappa})\Big) \tag{4.4}$$

The latter satisfies *Jacobi's identity*: $\{\{F,G\},Q\}+\{\{G,Q\},F\}+\{\{Q,F\},G\}=0$, for any three functions F,G,Q on phase space. It is quite useful to remark that if $t\to(\mathbf{p}(t),\mathbf{q}(t))=S_t(\mathbf{p},\mathbf{q})$ is a solution to Hamilton equations with Hamiltonian $\mathcal{H}$ then, given any *observable* $F(\mathbf{p},\mathbf{q})$, it "evolves" as $F(t)\stackrel{def}{=}F(\mathbf{p}(t),\mathbf{q}(t))$ satisfying $\partial_t F(\mathbf{p}(t),\mathbf{q}(t))=\{\mathcal{H},F\}(\mathbf{p}(t),\mathbf{q}(t))$. Requiring the latter identity to hold for all observables F is *equivalent* to requiring that the $t\to(\mathbf{p}(t),\mathbf{q}(t))$ be a solution of Hamilton's equations for $\mathcal{H}$.

Let $\mathcal{C}:U\longleftrightarrow U'$ be a smooth, smoothly invertible, transformation between two 2ℓ dimensional sets: $\mathcal{C}(\boldsymbol{\pi},\boldsymbol{\kappa})=(\boldsymbol{\pi}',\boldsymbol{\kappa}')$. Suppose that there is a function $\Phi(\boldsymbol{\pi}',\boldsymbol{\kappa})$ defined on a suitable domain W and such that

$$\mathcal{C}(\boldsymbol{\pi},\boldsymbol{\kappa})=(\boldsymbol{\pi}',\boldsymbol{\kappa}')\ \Rightarrow\ \begin{cases}\boldsymbol{\pi}=\partial_{\boldsymbol{\kappa}}\Phi(\boldsymbol{\pi}',\boldsymbol{\kappa})\\ \boldsymbol{\kappa}'=\partial_{\boldsymbol{\pi}'}\Phi(\boldsymbol{\pi}',\boldsymbol{\kappa})\end{cases}\tag{4.5}$$

then $\mathcal{C}$ is canonical. This is because (4.5) implies that if $\boldsymbol{\kappa},\boldsymbol{\pi}'$ are varied and if $\boldsymbol{\pi},\boldsymbol{\kappa}',\boldsymbol{\pi}',\boldsymbol{\kappa}$ are related by $\mathcal{C}(\boldsymbol{\pi},\boldsymbol{\kappa})=(\boldsymbol{\pi}',\boldsymbol{\kappa}')$, then $\boldsymbol{\pi}\cdot d\boldsymbol{\kappa}+\boldsymbol{\kappa}'\cdot d\boldsymbol{\pi}'=d\Phi(\boldsymbol{\pi}',\boldsymbol{\kappa})$: which implies

$$\boldsymbol{\pi}\cdot d\boldsymbol{\kappa}-\mathcal{H}(\boldsymbol{\pi},\boldsymbol{\kappa})dt\equiv\boldsymbol{\pi}'\cdot d\boldsymbol{\kappa}'-\mathcal{H}(\mathcal{C}^{-1}(\boldsymbol{\pi}',\boldsymbol{\kappa}'))dt+d\Phi(\boldsymbol{\pi}',\boldsymbol{\kappa})-d(\boldsymbol{\pi}'\cdot\boldsymbol{\kappa}')\tag{4.6}$$

and means that the Hamiltonians $\mathcal{H}(\mathbf{p},\mathbf{q}))$ and $\mathcal{H}'(\mathbf{p}',\mathbf{q}'))\stackrel{def}{=}\mathcal{H}(\mathcal{C}^{-1}(\mathbf{p}',\mathbf{q}'))$ have Hamilton actions $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{A}_{\mathcal{H}'}$ differing by a constant, if evaluated on corresponding motions $(\mathbf{p}(t),\mathbf{q}(t))$ and $(\mathbf{p}'(t),\mathbf{q}'(t))=\mathcal{C}(\mathbf{p}(t),\mathbf{q}(t))$.

The constant depends only on the initial and final values $(\mathbf{p}(t_1),\mathbf{q}(t_1))$ and $(\mathbf{p}(t_2),\mathbf{q}(t_2))$ and, respectively, $(\mathbf{p}'(t_1),\mathbf{q}'(t_1))$ and $(\mathbf{p}'(t_2),\mathbf{q}'(t_2))$ so that if $(\mathbf{p}(t),\ \mathbf{q}(t))$ makes $\mathcal{A}_{\mathcal{H}}$ extreme also $(\mathbf{p}'(t),\mathbf{q}'(t))=\mathcal{C}(\mathbf{p}(t),\mathbf{q}(t))$ makes $\mathcal{A}_{\mathcal{H}'}$ extreme.

Hence if $t\to(\mathbf{p}(t),\mathbf{q}(t))$ solves the Hamilton equations with Hamiltonian $\mathcal{H}(\mathbf{p},\mathbf{q})$ then the motion $t\to(\mathbf{p}'(t),\mathbf{q}'(t))=\mathcal{C}(\mathbf{p}(t),\mathbf{q}(t))$ solves the Hamilton equations with Hamiltonian $\mathcal{H}'(\mathbf{p}',\mathbf{q}')=\mathcal{H}(\mathcal{C}^{-1}(\mathbf{p}',\mathbf{q}'))$ *no matter which is the function* $\mathcal{H}$: therefore the transformation is canonical. The function Φ is called its *generating function*.

Eq. (4.5) provides a way to construct canonical maps. Suppose that a function $\Phi(\boldsymbol{\pi}',\boldsymbol{\kappa})$ is given and defined on some domain W; then setting $\begin{cases}\boldsymbol{\pi}=\partial_{\boldsymbol{\kappa}}\Phi(\boldsymbol{\pi}',\boldsymbol{\kappa})\\ \boldsymbol{\kappa}'=\partial_{\boldsymbol{\pi}'}\Phi(\boldsymbol{\pi}',\boldsymbol{\kappa})\end{cases}$ and inverting the first equation in the form $\boldsymbol{\pi}'=\boldsymbol{\Xi}(\boldsymbol{\pi},\boldsymbol{\kappa})$ and substituting the value for $\boldsymbol{\pi}'$, thus obtained, in the second equation, a map $\mathcal{C}(\boldsymbol{\pi},\boldsymbol{\kappa})=(\boldsymbol{\pi}',\boldsymbol{\kappa}')$ is defined on some domain (where the mentioned operations can be performed) and if such domain is open and not empty then $\mathcal{C}$ is a canonical map.

For similar reasons if $\Gamma(\boldsymbol{\kappa},\boldsymbol{\kappa}')$ is a function defined on some domain then setting $\boldsymbol{\pi}=\partial_{\boldsymbol{\kappa}}\Gamma(\boldsymbol{\kappa},\boldsymbol{\kappa}')$, $\boldsymbol{\pi}'=-\partial_{\boldsymbol{\kappa}'}\Gamma(\boldsymbol{\kappa},\boldsymbol{\kappa}')$ and solving the first relation to express $\boldsymbol{\kappa}'=\boldsymbol{\Delta}(\boldsymbol{\pi},\boldsymbol{\kappa})$ and substituting in the second relation a map $(\boldsymbol{\pi}',\boldsymbol{\kappa}')=\mathcal{C}(\boldsymbol{\pi},\boldsymbol{\kappa})$ is defined on some domain (where the mentioned operations can be performed) and if such domain is open and not empty then $\mathcal{C}$ is a canonical map.

Likewise canonical transformations can be constructed starting from *a priori* given functions $F(\boldsymbol{\pi},\boldsymbol{\kappa}')$ or $G(\boldsymbol{\pi},\boldsymbol{\pi}')$. And the most general canonical map can be generated locally (*i.e.* near a given point in phase space) by a single one of the above four ways, possibly composed with a few "trivial" canonical maps in which one pair of coordinates

(π_i,κ_i) is transformed into $(-\kappa_i,\pi_i)$. The necessity of including also the trivial maps can be traced to the existence of *homogeneous* canonical maps, *i.e.* maps such that $\boldsymbol{\pi}\cdot d\boldsymbol{\kappa} = \boldsymbol{\pi}'\cdot d\boldsymbol{\kappa}'$ (*e.g.* the identity map, see below or (9.7) for nontrivial examples) which are action preserving hence canonical, but which evidently cannot be generated by a function $\Phi(\boldsymbol{\kappa},\boldsymbol{\kappa}')$ although they can be generated by a function depending on $\boldsymbol{\pi}',\boldsymbol{\kappa}$.

Simple examples of homogeneous canonical maps are maps in which the coordinates $\mathbf{q}$ are changed into $\mathbf{q}'=\mathbf{R}(\mathbf{q})$ and, correspondingly, the $\mathbf{p}$'s are transformed as $\mathbf{p}' = \big(\partial_{\mathbf{q}}\mathbf{R}(\mathbf{q})\big)^{-1\,T}\mathbf{p}$, linearly: indeed this map is generated by the function $F(\mathbf{p}',\mathbf{q}) \stackrel{def}{=} \mathbf{p}'\cdot\mathbf{R}(\mathbf{q})$.

For instance consider the map "Cartesian–polar" coordinates $(q_1,q_2)\longleftrightarrow(\rho,\theta)$ with (ρ,θ) the polar coordinates of $\mathbf{q}$ (namely $\rho=\sqrt{q_1^2+q_2^2}, \theta=$ atan q_2/q_1) and let $\mathbf{n} \stackrel{def}{=} \mathbf{q}/\,|\mathbf{q}| = (n_1,n_2)$ and $\mathbf{t}=(-n_2,n_1)$. Setting $p_\rho \stackrel{def}{=} \mathbf{p}\cdot\mathbf{n}$, $p_\theta \stackrel{def}{=} \rho\,\mathbf{p}\cdot\mathbf{t}$, the map $(p_1,p_2,q_1,q_2)\longleftrightarrow(p_\rho,p_\theta,\rho,\theta)$ is homogeneous canonical (because $\mathbf{p}\cdot d\mathbf{q} = \mathbf{p}\cdot\mathbf{n}\,d\rho+\mathbf{p}\cdot\mathbf{t}\,\rho\,d\theta = p_\rho\,d\rho+p_\theta\,d\theta$).

As a further example any area preserving map $(p,q)\longleftrightarrow(p',q')$ defined on an open region of the plane $\mathbb{R}^2$ is canonical: because in this case the matrices A,B,C,D are just numbers which satisfy $AD-BC=1$ and, therefore, the (4.2) holds.

References: [LL68], [Ga83]

5. Quadratures

The simplest mechanical systems are *integrable by quadratures.* For instance the Hamiltonian on $\mathbb{R}^2$

$$\mathcal{H}(p,q)=\frac{1}{2m}p^2+V(q) \tag{5.1}$$

generates a motion $t\to q(t)$ with initial data $q_0,\dot q_0$ such that $\mathcal{H}(p_0,q_0)=E$, *i.e.* $\frac12 m\dot q_0^2+V(q_0)=E$, satisfying $\dot q(t)=\pm\sqrt{\frac{2}{m}(E-V(q(t)))}$. If the equation $E=V(q)$ has only two solutions $q_-(E)<q_+(E)$ and $|\partial_q V(q_\pm(E))|>0$ the motion is periodic with period

$$T(E)=2\int_{q_-(E)}^{q_+(E)}\frac{dx}{\sqrt{\frac{2}{m}(E-V(x))}}. \tag{5.2}$$

The special solution with initial data $q_0=q_-(E),\dot q_0=0$ will be denoted $Q(t)$ and it is an analytic function (by the general regularity theorem on ordinary differential equations). For $0\le t\le\frac{T}{2}$ or for $\frac{T}{2}\le t\le T$ it is given, respectively, by

$$t=\int_{q_-(E)}^{Q(t)}\frac{dx}{\sqrt{\frac{2}{m}(E-V(x))}},\qquad\text{or}\qquad t=\frac{T}{2}-\int_{q_+(E)}^{Q(t)}\frac{dx}{\sqrt{\frac{2}{m}(E-V(x))}}. \tag{5.3}$$

The most general solution with energy E has the form $q(t)=Q(t_0+t)$ where t_0 is defined by $q_0=Q(t_0),\dot q_0=\dot Q(t_0)$, *i.e.* it is the time needed to the "standard solution" $Q(t)$ to reach the initial data for the new motion.

If the derivative of V vanishes in one of the extremes or if one at least of the two solutions $q_\pm(E)$ does not exist the motion is not periodic and it may be unbounded:

nevertheless it is still expressible via integrals of the type (5.2). If the potential V is periodic in q and the variable q is considered varying on a circle then essentially *all solutions are periodic*: exceptions can occurr if the energy E has a value such that $V(q) = E$ admits a solution where V has zero derivative,

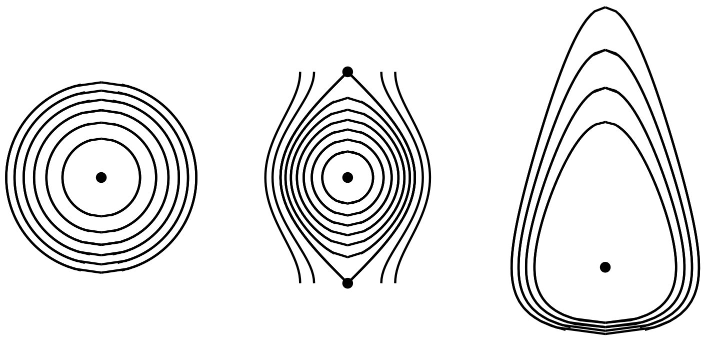
Typical examples are the *harmonic oscillator*, the *pendulum*, the *Kepler oscillator*: whose Hamiltonians, if m, ω, g, h, G, k are positive constants, are respectively

$$\frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2, \qquad \frac{p^2}{2m} + mg(1 - \cos\frac{q}{h}), \qquad \frac{p^2}{2m} - mk\frac{1}{|q|} + m\frac{G^2}{2q^2} \tag{5.4}$$

the latter has a potential which is singular at $q = 0$ but if $G \neq 0$ the energy conservation forbids too close an approach to $q = 0$ and the singularity becomes irrelevant.

The integral in (5.3) is called a *quadrature* and the systems in (5.1) are therefore *integrable by quadratures*. Such systems, at least in the cases in which the motion is periodic, are best described in new coordinates in which periodicity is more manifest. Namely when $V(q) = E$ has only two roots $q_\pm(E)$ and $\mp V'(q_\pm(E)) > 0$ the *energy–time* coordinates can be used by replacing $q, \dot q$ or p, q by E, τ where τ is the time needed to the standard solution $t \to Q(t)$ to reach the given data, *i.e.* $Q(\tau) = q, \dot Q(\tau) = \dot q$. In such coordinates the motion is simply $(E, \tau) \to (E, \tau + t)$ and, of course, the variable τ has to be regarded as varying on a circle of radius $T/2\pi$. The E, τ variables are a kind of polar coordinates as can be checked by drawing the curves of constant E, "energy levels", in the plane p, q in the cases in (5.4), see Fig. 1.

In the harmonic oscillator case all trajectories are periodic. In the pendulum case all motions are periodic except the ones which separate the oscillatory motions (the closed curves in the second drawing) from the rotatory motions (the apparently open curves) which, in fact, are on closed curves as well if the q coordinate, *i.e.* the vertical coordinate in Fig. 1, is regarded as "periodic" with period $2\pi h$. In the Kepler case only the negative energy trajectories are periodic and a few of them are drawn in Fig. 1. The single dots represent the equilibrium points in phase space.



(1)

Fig. 1: The energy levels of the harmonic oscillator, the pendulum, and the Kepler motion

The region of phase space where motions are periodic is a set of points (p, q) with the topological structure of $\cup_{u\in U}(\{u\} \times C_u)$ where u is a coordinate varying in an open interval U, for instance the set of values of the energy, and C_u is a closed curve whose points (p, q) are identified by a coordinate, for instance by the time necessary to an arbitrarily fixed datum with the same energy to evolve into (p, q).

In the above cases, (5.4), if the "radial" coordinate is chosen to be the energy the set U is the interval $(0,+\infty)$ for the harmonic oscillator, $(0,2mg)$ or $(2mg,+\infty)$ for the pendulum, and $(-\frac{1}{2}\frac{mk^2}{G^2},0)$ in the Kepler case. The fixed datum for the reference motion can be taken, in all cases, of the form $(0,q_0)$ with the time coordinate t_0 given by (5.3).

It is remarkable that the energy-time coordinates are canonical coordinates: for instance in the vicinity of (p_0,q_0) and if $p_0>0$ this can be seen by setting

$$S(q,E)=\int_{q_0}^{q}\sqrt{2m(E-V(x))}\,dx \tag{5.5}$$

and checking that $p=\partial_q S(q,E)$, $t=\partial_E S(q,E)$ are identities if (p,q) and (E,t) are coordinates for the same point so that the criterion expressed by (4.6) applies.

It is convenient to standardize the coordinates by replacing the time variable by an angle$\alpha=\frac{2\pi}{T(E)}t$; and instead of the energy any invertible function of it can be used.

It is natural to look for a coordinate $A=A(E)$ such that the map $(p,q)\longleftrightarrow(A,\alpha)$ is a canonical map: this is easily done as the function

$$\hat S(q,A)=\int_{q_0}^{q}\sqrt{2m(E(A)-V(x))}\,dx \tag{5.6}$$

generates (locally) the correspondence between $p=\sqrt{2m(E(A)-V(q))}$ and $\alpha=E'(A)\int_0^q \frac{dx}{\sqrt{2m^{-1}(E(A)-V(x))}}$. Therefore, by the criterion (4.6), if $E'(A)=\frac{2\pi}{T(E(A))}$, *i.e.* if $A'(E)=\frac{T(E)}{2\pi}$, the coordinates (A,α) will be canonical coordinates. Hence, by (5.2), $A(E)$ can be taken equal to

$$A=\frac{1}{2\pi}2\int_{q_-(E)}^{q_+(E)}\sqrt{2m(E-V(q))}\,dq\equiv\frac{1}{2\pi}\oint p\,dq \tag{5.7}$$

where the last integral is extended to the closed curve of energy E, see Fig. 1. The *action–angle coordinates* (A,α) are defined in open regions of phase space covered by periodic motions: in action–angle coordinates such regions have the form $W=J\times\mathbb{T}$ of a product of an open interval J and a one dimensional "torus" $\mathbb{T}=[0,2\pi]$ (*i.e.* a unit circle).

References: [LL68], [Ar68], [Ga83].

6. Quasi periodicity and integrability

A Hamiltonian is called integrable in an open region $W\subset T^*(M)$ of phase space if
(1) there is an analytic and non singular (i.e. with non zero Jacobian) change of coordinates $(\mathbf{p},\mathbf{q})\longleftrightarrow(\mathbf{I},\boldsymbol{\varphi})$ mapping W into a set of the form $\mathcal{I}\times\mathbb{T}^\ell$ with $\mathcal{I}\subset\mathbb{R}^\ell$ (open) and furthermore
(2) the flow $t\to S_t(\mathbf{p},\mathbf{q})$ on phase space is transformed into $(\mathbf{I},\boldsymbol{\varphi})\to(\mathbf{I},\boldsymbol{\varphi}+\boldsymbol{\omega}(\mathbf{I})\,t)$ where $\boldsymbol{\omega}(\mathbf{I})$ is a smooth function on $\mathcal{I}$.

This means that in suitable coordinates, that can be called "integrating coordinates", the system appears as a set of ℓ points with coordinates $\boldsymbol{\varphi}=(\varphi_1,\ldots,\varphi_\ell)$ moving on a unit circle at angular velocities $\boldsymbol{\omega}(\mathbf{I})=(\omega_1(\mathbf{I}),\ldots,\omega_\ell(\mathbf{I}))$ depending on the actions of the initial data.

A system integrable in a region W which in integrating coordinates $\mathbf{I},\boldsymbol{\varphi}$ has the form $\mathcal{I}\times\mathbb{T}^\ell$ is said *anisochronous* if $\det\partial_{\mathbf{I}}\boldsymbol{\omega}(\mathbf{I})\neq 0$. It is said *isochronous* if $\boldsymbol{\omega}(\mathbf{I})\equiv\boldsymbol{\omega}$ is *independent on* $\mathbf{I}$. The motions of integrable systems are called *quasi periodic* with frequency *spectrum* $\boldsymbol{\omega}(\mathbf{I})$, or with *frequencies* $\boldsymbol{\omega}(\mathbf{I})/2\pi$, in the coordinates $(\mathbf{I},\boldsymbol{\varphi})$.

Clearly an integrable system admits ℓ independent constants of motion, the $\mathbf{I}=(I_1,\ldots,I_\ell)$ and, for each choice of $\mathbf{I}$, the other coordinates vary on a "standard" ℓ-dimensional torus $\mathbb{T}^\ell$: hence it is possible to say that a phase space region of integrability is *foliated into ℓ-dimensional invariant tori $\mathcal{T}(\mathbf{I})$* parameterized by the values of the constants of motion $\mathbf{I}\in\mathcal{I}$.

If an integrable system is anisochronous then it is canonically integrable: *i.e.* it is possible to define on W a canonical change of coordinates $(\mathbf{p},\mathbf{q})=\mathcal{C}(\mathbf{A},\boldsymbol{\alpha})$ mapping W onto $J\times\mathbb{T}^\ell$ and such that $\mathcal{H}(\mathcal{C}(\mathbf{A},\boldsymbol{\alpha}))=h(\mathbf{A})$ for a suitable h: so that, if $\boldsymbol{\omega}(\mathbf{A})\stackrel{def}{=}\partial_{\mathbf{A}}h(\mathbf{A})$, the equations of motion become

$$\dot{\mathbf{A}}=\mathbf{0},\qquad\dot{\boldsymbol{\alpha}}=\boldsymbol{\omega}(\mathbf{A})\tag{6.1}$$

Given a system $(\mathbf{I},\boldsymbol{\varphi})$ of coordinates integrating an anisochronous system the construction of action–angle coordinates can be performed, *in principle*, via a classical procedure (under a few extra assumptions).

Let $\gamma_1,\ldots,\gamma_\ell$ be ℓ *topologically independent* circles on $\mathbb{T}^\ell$, for definiteness let $\gamma_i(\mathbf{I})=\{\boldsymbol{\varphi}\,|\,\varphi_1=\varphi_2=\ldots=\varphi_{i-1}=\varphi_{i+1}=\ldots=0,\ \varphi_i\in[0,2\pi]\}$, and set

$$A_i(\mathbf{I})=\frac{1}{2\pi}\oint_{\gamma_i(\mathbf{I})}\mathbf{p}\cdot d\mathbf{q}\tag{6.2}$$

If the map $\mathbf{I}\longleftrightarrow\mathbf{A}(\mathbf{I})$ is analytically invertible as $\mathbf{I}=\mathbf{I}(\mathbf{A})$ the function

$$S(\mathbf{A},\boldsymbol{\varphi})=(\lambda)\int_{\mathbf{0}}^{\boldsymbol{\varphi}}\mathbf{p}\cdot d\mathbf{q}\tag{6.3}$$

is well defined if the integral is over any path λ joining the points $(p(\mathbf{I}(\mathbf{A}),\mathbf{0}),\,q(\mathbf{I}(\mathbf{A}),\mathbf{0}))$ and $(p(\mathbf{I}(\mathbf{A}),\,\boldsymbol{\varphi})),\,q(\mathbf{I}(\mathbf{A}),\boldsymbol{\varphi})$ lying on the torus parameterized by $\mathbf{I}(\mathbf{A})$.

The key remark in the proof that (6.3) really defines a function of the only variables $\mathbf{A},\boldsymbol{\varphi}$ is that *anisochrony implies the vanishing of the Poisson brackets* (cf. (4.4)): $\{I_i,I_j\}=0$ (hence also $\{A_i,A_j\}\equiv\sum_{h,k}\partial_{I_k}\partial_{I_h}\{I_k,I_h\}=0$). And the property $\{I_i,I_j\}=0$ can be checked to be precisely the integrability condition for the differential form $\mathbf{p}\cdot d\mathbf{q}$ restricted to the surface obtained by varying $\mathbf{q}$ while $\mathbf{p}$ is constrained so that $(\mathbf{p},\mathbf{q})$ stays on the surface $\mathbf{I}$=constant, *i.e.* on the invariant torus of the points with fixed $\mathbf{I}$.

The latter property is necessary and sufficient in order that the function $S(\mathbf{A},\boldsymbol{\varphi})$ be well defined (*i.e.* be independent on the integration path λ) up to an additive quantity of the form $\sum_i 2\pi n_i A_i$ with $\mathbf{n}=(n_1,\ldots,n_\ell)$ integers..

Then the action–angle variables are defined by the canonical change of coordinates with $S(\mathbf{A},\boldsymbol{\varphi})$ as generating function, *i.e.* by setting

$$\alpha_i = \partial_{A_i} S(\mathbf{A},\boldsymbol{\varphi}), \qquad I_i = \partial_{\varphi_i} S(\mathbf{A},\boldsymbol{\varphi}). \tag{6.4}$$

and, since the computation of $S(\mathbf{A},\boldsymbol{\varphi})$ is "reduced to integrations" which can be regarded as a natural extension of the quadratures discussed in the one dimensional cases, also such systems are called *integrable by quadratures*. The just described construction is a version of the more general *Arnold-Liouville theorem*.

In practice, however, the actual evaluation of the integrals in (6.2),(6.3) can be difficult: its analysis in various cases (even as "elementary" as the pendulum) has in fact led to key progress in various domains, *e.g.* in the theory of special functions and in group theory.

In general any surface on phase space on which the restriction of the differential form $\mathbf{p}\cdot d\mathbf{q}$ is locally integrable is called a *Lagrangian manifold*: hence the invariant tori of an anisochronous integrable system are Lagrangian manifolds.

If an integrable system is anisochronous it cannot admit more than ℓ independent constants of motion; furthermore it does not admit invariant tori of dimension $>\ell$. Hence ℓ-dimensional invariant tori are called *maximal*.

Of course invariant tori of dimension $<\ell$ can also exist: this happens when the variables $\mathbf{I}$ are such that the frequencies $\boldsymbol{\omega}(\mathbf{I})$ admit nontrivial rational relations; *i.e.* there is an integer components vector $\boldsymbol{\nu}\in\mathbb{Z}^\ell$, $\boldsymbol{\nu}=(\nu_1,\ldots,\nu_\ell)\ne 0$ such that

$$\boldsymbol{\omega}(\mathbf{I})\cdot\boldsymbol{\nu} = \sum_i \omega_i(\mathbf{I})\,\nu_i = \mathbf{0} \tag{6.5}$$

in this case the invariant torus $\mathcal{T}(\mathbf{I})$ is called *resonant*. If the system is anisochronous then $\det \partial_{\mathbf{I}}\boldsymbol{\omega}(\mathbf{I})\ne 0$ and, therefore, the resonant tori are associated with values of the constants of motion $\mathbf{I}$ which *form a set of* 0*-measure* in the space $\mathcal{I}$ but *which is not empty and dense*.

Examples of isochronous systems are the systems of harmonic oscillators, *i.e.* systems with Hamiltonian $\sum_{i=1}^\ell \frac{1}{2m_i}p_i^2+\frac12\sum_{i,j}^{1,\ell} c_{ij}q_iq_j$ where the matrix v is a positive definite matrix. This is an isochronous system with frequencies $\boldsymbol{\omega}=(\omega_1,\ldots,\omega_\ell)$ whose squares are the eigenvalues of the matrix $m_i^{-\frac12}c_{ij}m_j^{-\frac12}$. It is integrable in the region W of the data $\mathbf{x}=(\mathbf{p},\mathbf{q})\in\mathbb{R}^{2\ell}$ such that, setting $A_\beta=\frac{1}{2\,\omega_\beta}\big((\sum_{i=1}^\ell \frac{v_{\beta,i}p_i}{\sqrt{m_i}})^2+\omega_\beta^2(\sum_{i=1}^\ell\frac{v_{\beta,i}q_i}{\sqrt{m_i}^{-1}})^2\big)$ for all eigenvectors $\mathbf{v}_\beta$, $\beta=1,\ldots,\ell$, of the above matrix, the vectors $\mathbf{A}$ have all components >0 .

Even though this system is isochronous it, nevertheless, admits a system of canonical action–angle coordinates in which the Hamiltonian takes the simplest form

$$h(\mathbf{A})=\sum_{\beta=1}^\ell \omega_\beta A_\beta \equiv \boldsymbol{\omega}\cdot\mathbf{A} \tag{6.6}$$

with $\alpha_\beta = -\operatorname{atan}\frac{\sum_{i=1}^\ell \frac{v_{\beta,i}p_i}{\sqrt{m_i}}}{\sum_{i=1}^\ell \sqrt{m_i}\omega_\beta v_{\beta,i}q_i}$ as conjugate angles.

An example of anisochronous system is the *free rotators* or *free wheels*: *i.e.* ℓ noninteracting points on a circle of radius R or ℓ noninteracting coaxial wheels of radius R. If $J_i=m_iR^2$ or, respectively, $J_i=\frac12 m_iR^2$ are the inertia moments and if the positions are determined by ℓ angles $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)$ the angular velocities are constants related to the angular momenta $\mathbf{A}=(A_1,\ldots,A_\ell)$ by $\omega_i=A_i/J_i$. The Hamiltonian and the specrum are

$$h(\mathbf{A})=\sum_{i=1}^{\ell}\frac{1}{2J_i}A_i^2,\qquad \boldsymbol{\omega}(\mathbf{A})=\Big(\frac{1}{J_i}A_i\Big)_{i=1,\ldots,\ell} \tag{6.7}$$

References: [LL68], [Ar68], [Ga83], [Fa98].

7. Multidimensional quadratures: central motion

Important mechanical systems with more than one degree of freedom are integrable by canonical quadratures in vast regions of phase space. This is checked by showing that there is a foliation into invariant tori $\mathcal{T}(\mathbf{I})$ of dimension equal to the number ℓ of degrees of freedom parameterized by ℓ constants of motion $\mathbf{I}$ *in involution*, *i.e.* such that $\{I_i,I_j\}=0$. One then performs, if possible, the construction of the action–angle variables by the quadratures discussed in the previous section.

The above procedure is well exemplified by the theory of the planar motion of a unit mass attracted by a coplanar center of force: the Lagrangian is, in polar coordinates (ρ,θ), $\mathcal{L}=\frac{m}{2}(\dot\rho^2+\rho^2\dot\theta^2)-V(\rho)$. The planarity of the motion is not a strong resttriction as central motion takes always place on a plane

Hence the equations of motion are $\frac{d}{dt}m\rho^2\dot\theta=0$, *i.e.* $m\rho^2\dot\theta=G$ is a constant of motion (it is the angular momentum), and $\ddot\rho=-\partial_\rho V(\rho)+\partial_\rho\frac{m}{2}\rho^2\dot\theta^2=-\partial_\rho V(\rho)+\frac{G^2}{m\,\rho^3}\stackrel{def}{=}-\partial_\rho V_G(\rho)$. So that energy conservation yields a second constant of motion E

$$\frac{m}{2}\dot\rho^2+\frac{1}{2}\frac{G^2}{m\rho^2}+V(\rho)=E=\frac{1}{2m}p_\rho^2+\frac{1}{2m}\frac{p_\theta^2}{\rho^2}+V(\rho) \tag{7.1}$$

The *r.h.s.* is the Hamiltonian for the system, derived from $\mathcal{L}$, if p_ρ,p_θ denote conjugate momenta of ρ,θ: $p_\rho=m\dot\rho$ and $p_\theta=m\rho^2\dot\theta$ (note that $p_\theta=G$).

Suppose $\rho^2V(\rho)\xrightarrow[\rho\to0]{}0$: then the singularity at the origin cannot be reached by any motion starting with $\rho>0$ if $G>0$. Assume also that the function $V_G(\rho)\stackrel{def}{=}\frac12\frac{G^2}{m\rho^2}+V(\rho)$ has only one minimum $E_0(G)$, no maximum and no horizontal inflection and tends to a limit $E_\infty(G)\le\infty$ when $\rho\to\infty$. Then the system is integrable in the domain $W=\{(\mathbf{p},\mathbf{q})\,|\,E_0(G)<E<E_\infty(G),\,G\ne0\}$.

This is checked by introducing a "standard" periodic solution $t\to R(t)$ of $m\ddot\rho=-\partial_\rho V_G(\rho)$ with energy $E_0(G)<E<E_\infty(G)$ and initial data $\rho=\rho_{E,-}(G)$, $\dot\rho=0$ at time $t=0$, where $\rho_{E,\pm}(G)$ are the two solutions of $V_G(\rho)=E$, see Section 5: this is a periodic analytic function of t with period $T(E,G)=2\int_{\rho_{E,-}(G)}^{\rho_{E,+}(G)}\frac{dx}{\sqrt{\frac{2}{m}(E-V_G(x))}}$.

The function $R(t)$ is given, for $0\le t\le\frac12T(E,G)$ or for $\frac12T(E,G)\le t\le T(E,G)$, by the quadratures

$$t=\int_{\rho_{E,-}(G)}^{R(t)}\frac{dx}{\sqrt{\frac{2}{m}(E-V_G(x))}},\quad\text{or}\quad t=\frac{T(E,G)}{2}-\int_{\rho_{E,+}(G)}^{R(t)}\frac{dx}{\sqrt{\frac{2}{m}(E-V_G(x))}} \tag{7.2}$$

respectively. The analytic regularity of $R(t)$ follows from the general existence, uniqueness and regularity theorems applied to the differential equation for $\ddot\rho$.

Given an initial datum $\dot\rho_0,\rho_0,\dot\theta_0,\theta_0$ with energy E and angular momentum G define t_0 to be the time such that $R(t_0)=\rho_0,\dot R(t_0)=\dot\rho_0$: then $\rho(t)\equiv R(t+t_0)$ and $\theta(t)$ can be computed as $\theta(t)=\theta_0+\int_0^t\frac{G}{mR(t'+t_0)^2}dt'$: a second quadrature. Therefore we can use as coordinates for the motion E,G,t_0, which determine $\dot\rho_0,\rho_0,\dot\theta_0$ and a fourth coordinate that determines θ_0 which could be θ_0 itself but which is conveniently determined, via the second quadrature, as follows.

The function $G\,m^{-1}\,R(t)^{-2}$ is periodic with period $T(E,G)$, hence it can be expressed in a Fourier series $\chi_0(E,G)+\sum_{k\ne0}\chi_k(E,G)e^{\frac{2\pi}{T(E,G)}\,i\,t\,k}$; the quadrature for $\theta(t)$ can be performed by integrating the series terms. Setting $\overline\theta(t_0)\stackrel{def}{=}\frac{T(E,G)}{2\pi}\sum_{k\ne0}\frac{\chi_k(E,G)}{k}e^{\frac{2\pi}{T(E,G)}\,i\,t_0\,k}$ and $\varphi_1(0)=\theta_0-\overline\theta(t_0)$ the $\theta(t)=\theta_0+\int_0^t\frac{G}{mR(t'+t_0)^2}dt'$ becomes

$$\varphi_1(t)=\varphi_1(0)+\chi_0(E,G)\,t \tag{7.3}$$

Hence the system is integrable and the spectrum is $\boldsymbol\omega(E,G)=(\omega_0(E,G),\,\omega_1(E,G))\equiv(\omega_0,\omega_1)$ with $\omega_0\stackrel{def}{=}\frac{2\pi}{T(E,G)}$ and $\omega_1\stackrel{def}{=}\chi_0(E,G)$, while $\mathbf I=(E,G)$ are constants of motion and the angles $\boldsymbol\varphi=(\varphi_0,\varphi_1)$ can be taken $\varphi_0\stackrel{def}{=}\omega_0t_0,\varphi_1\stackrel{def}{=}\theta_0-\overline\theta(t_0)$. At E,G fixed the motion takes place on a 2-dimensional torus $\mathcal T(E,G)$ with φ_0,φ_1 as angles.

In the anisochronous cases, *i.e.* when $\det\partial_{E,G}\boldsymbol\omega(E,G)\ne0$, canonical action–angle variables conjugated to $(p_\rho,\rho,p_\theta,\theta)$ can be constructed via (6.2),(6.3) by using two cycles γ_1,γ_2 on the torus $\mathcal T(E,G)$. It is convenient to choose
(1) γ_1 as the cycle consisting of the points $\rho=x,\theta=0$ whose first half (where $p_\rho\ge0$) consists in the set $\rho_{E,-}(G)\le x\le\rho_{E,+}(G)$, $p_\rho=\sqrt{2m(E-V_G(x))}$ and $d\theta=0$;
(2) γ_2 as the cycle $\rho=const,\theta\in[0,2\pi]$ on which $d\rho=0$ and $p_\theta=G$ obtaining

$$A_1=\frac{2}{2\pi}\int_{\rho_{E,-}(G)}^{\rho_{E,+}(G)}\sqrt{2m(E-V_G(x))}dx,\qquad A_2=G \tag{7.4}$$

According to the general theory (cf. Sect. 6) a generating function for the canonical change of coordinates from $(p_\rho,\rho,p_\theta,\theta)$ to action–angle variables is (if, to fix ideas, $p_\rho>0$)

$$S(A_1,A_2,\rho,\theta)=G\theta+\int_{\rho_{E,-}}^{\rho}\sqrt{2m(E-V_G(x))}dx \tag{7.5}$$

In terms of the above ω_0,χ_0 the Jacobian matrix $\frac{\partial(E,G)}{\partial(A_1,A_2)}$ is computed from (7.4),(7.5) to be $\begin{pmatrix}\omega_0&\chi_0\\0&1\end{pmatrix}$. It follows $\partial_ES=t$, $\partial_GS=\theta-\overline\theta(t)-\chi_0t$ so that, see (6.4),

$$\alpha_1\stackrel{def}{=}\partial_{A_1}S=\omega_0t,\qquad\alpha_2=\stackrel{def}{=}\partial_{A_2}S=\theta-\overline\theta(t) \tag{7.6}$$

and $(A_1,\alpha_1),(A_2,\alpha_2)$ are the action–angle pairs.

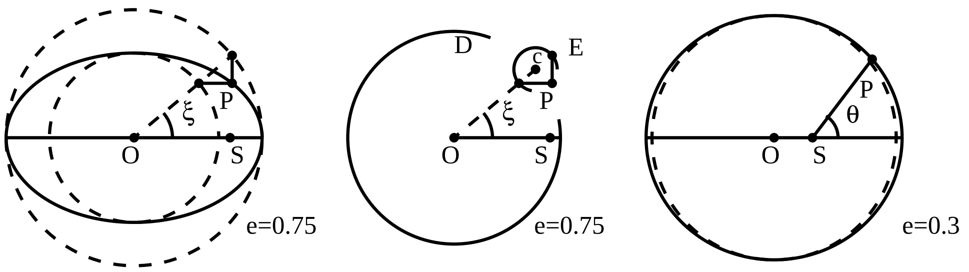
References: [LL68], [Ga83].

8. Newtonian potential and Kepler's laws

The anisochrony property, *i.e.* $\det \frac{\partial(\omega_0,\chi_0)}{\partial(A_1,A_2)} \neq 0$ or, equivalently, $\det \frac{\partial(\omega_0,\chi_0)}{\partial(E,G)} \neq 0$, *is not satisfied* in the important cases of the harmonic potential and of the Newtonian potential. Anisochrony being only a sufficient condition for canonical integrability it is still possible (and true) that, nevertheless, in both cases the canonical transformation generated by (7.5) integrates the system. This is expected since the two potentials are limiting cases of anisochronous ones (*e.g.* $|\mathbf{q}|^{2+\varepsilon}$ and $|q|^{-1-\varepsilon}$ with $\varepsilon \to 0$).

The Newtonian potential $\mathcal{H}(\mathbf{p},\mathbf{q}) = \frac{1}{2m}\mathbf{p}^2 - \frac{k\,m}{|\mathbf{q}|}$ is integrable in the region $G \neq 0$, $E_0(G) = -\frac{k^2m^3}{2G^2} < E < 0, |G| < \sqrt{\frac{k^2m^3}{-2E}}$. Proceeding as in Section 7 one finds integrating coordinates and that the integrable motions develop on ellipses with one focus on the center of attraction S so that motions are periodic hence not anisochronous: nevertheless the construction of the canonical coordinates via (6.2),(6.3),(6.4) (hence (7.5)), works and leads to canonical coordinates (L',λ',G',γ'). To obtain action–angle variables with a simple interpretation it is convenient to perform on the variables (L',λ',G',γ') (constructed by following the procedure just indicated) a further trivial canonical tranformation by setting $L = L' + G'$, $G = G'$, $\lambda = \lambda'$, $\gamma = \gamma' - \lambda'$; then

- λ *average anomaly*, is the time necessary to the point P to move from the pericenter to its actual position in units of the period and times 2π
- L *action*, is essentially the energy $E = -\frac{k^2m^3}{2L^2}$,
- G *angular momentum*,
- γ *axis longitude*, is the angle between a fixed axis and the major axis of the ellipse oriented from the center of the ellipse O to the center of attraction S.



(2)

Fig. 2: Eccentric and true anomalies of P which moves on a small circle E centered at a point c moving on the circle D located half way between the two concentric circles containing the Keplerian ellipse: the anomaly of c with respect to the axis OS is ξ. The circle D is *eccentric* with respect to S and therefore ξ is, still today, called eccentric anomaly while the circle D is, in ancient terminology, the *deferent circle* (eccentric circles were introduced in Astronomy by Ptolemy). The small circle E on which the point P moves is, in ancient terminology, an *epicycle*. The deferent and the epicyclical motions are synchronous (*i.e.* have the same period); Kepler discovered that his key *a priori* hypothesis of inverse proportionality between angular velocity on the deferent and distance between P and S (*i.e.* $\rho\dot\xi = constant$) implied both synchrony and elliptical shape of the orbit, with focus in S. The latter law is equivalent to $\rho^2\dot\theta = constant$ (because of the identity $a\dot\xi = \rho\dot\theta$). Small eccentricity ellipses can be hardly distinguished from circles.

The *eccentricity* of the ellipse is e such that $G = \pm L\sqrt{1-e^2}$. The ellipse equation is $\rho = a\,(1 - e\cos\xi)$ where ξ is the *eccentric anomaly* (see Fig. 2), $a = \frac{L^2}{km^2}$ is the major semiaxis and ρ is the distance to the center of attraction S.

Finally the relations between eccentric anomaly ξ, average anomaly λ, *true anomaly* θ (the latter is the polar angle), and SP–distance ρ are given by Kepler's equations

$$\begin{aligned} \lambda =&\xi - e\sin\xi, \qquad (1-e\cos\xi)(1+e\cos\theta)=1-e^2,\\ \lambda =&(1-e^2)^{\frac32}\int_0^\theta \frac{d\theta'}{(1+e\cos\theta')^2}, \qquad \frac{\rho}{a}=\frac{1-e^2}{1+e\cos\theta} \end{aligned} \tag{8.1}$$

and this relation between true anomaly and average anomaly can be inverted in the form

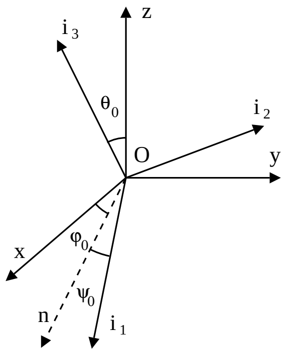
$$\xi=\lambda+g_\lambda, \qquad \theta=\lambda+f_\lambda \ \Rightarrow\ \frac{\rho}{a}=\frac{1-e^2}{1+e\cos(\lambda+f_\lambda)} \tag{8.2}$$

where $g_\lambda=g(e\sin\lambda,e\cos\lambda)$, $f_\lambda=f(e\sin\lambda,e\cos\lambda)$ and $g(x,y), f(x,y)$ are suitable functions analytic for $|x|,|y|<1$. Furthermore $g(x,y)=x\,(1+y+\ldots)$, $f(x,y)=2x\,(1+\frac54 y+\ldots)$ and $\ldots$ denote terms of degree 2 or higher in x,y, containing only even powers of x.

References: [LL68], [Ga83].

9. Rigid body

Another fundamental integrable system is the rigid body in absence of gravity and with a fixed point O. It can be naturally described in terms of the *Euler angles* $\theta_0,\varphi_0,\psi_0$, see Fig. 3, and their derivatives $\dot\theta_0,\dot\varphi_0,\dot\psi_0$.



(3)

Fig. 3: The Euler angles of the comoving frame $\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3$ with respect to a fixed frame $\mathbf{x},\mathbf{y},\mathbf{z}$. The direction $\mathbf{n}$ is the "node line, intersection between the planes $\mathbf{x},\mathbf{y}$ and $\mathbf{i}_1,\mathbf{i}_2$.

Let I_1,I_2,I_3 be the three *principal inertia moments* of the body along the three *principal axes* with unit vectors $\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3$. The inertia moments and the principal axes are the eigenvalues and the associated unit eigenvectors of the 3×3 *inertia matrix* $\mathcal{I}$ which is defined by $\mathcal{I}_{hk}=\sum_{i=1}^n m_i(\mathbf{x}_i)_h(\mathbf{x}_i)_k$, where $h,k=1,2,3$ and $\mathbf{x}_i$ is the position of the i-th particle in a reference frame with origin at O and in which all particles are at rest: this *comoving frame* exists as a consequence of the rigidity constraint. The principal axes form a coordinate system which is comoving as well: *i.e.* also in the frame $(O;\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3)$ the particles are at rest.

The Lagrangian is simply the kinetic energy: *i.e.* we imagine the rigidity constraint to be ideal, *e.g.* realized by internal central forces in the limit of infinite rigidity, as mentioned in Sec. 3. The *angular velocity* of the rigid motion is defined by

$$\boldsymbol{\omega} = \dot{\theta}_0 \mathbf{n} + \dot{\varphi}_0 \mathbf{z} + \dot{\psi}_0 \mathbf{i}_3 \tag{9.1}$$

expressing that a generic infinitesimal motion must consist of a variation of the three Euler angles and therefore it has to be a rotation of speeds $\dot{\theta}_0, \dot{\varphi}_0, \dot{\psi}_0$ around the axes $\mathbf{n}, \mathbf{z}, \mathbf{i}_3$ as shown in Fig. 3.

Let $(\omega_1, \omega_2, \omega_3)$ be the components of $\boldsymbol{\omega}$ along the pricipal axes $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$: for brevity the latter axes will often be called $\mathbf{1}, \mathbf{2}, \mathbf{3}$. Then the angular momentum $\mathbf{M}$, with respect to the pivot point O, and the kinetic energy K can be checked to be

$$\mathbf{M} = I_1\omega_1\mathbf{i}_1 + I_2\omega_2\mathbf{i}_2 + I_3\omega_3\mathbf{i}_3, \qquad K = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_2\omega_3^2) \tag{9.2}$$

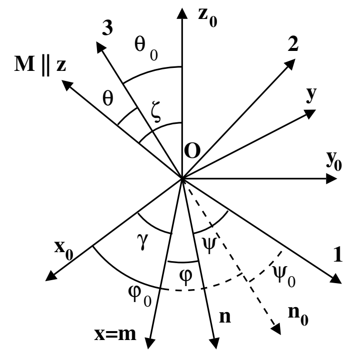
and are constants of motion. From Fig.3 it follows that $\omega_1 = \dot{\theta}_0 \cos\psi_0 + \dot{\varphi}_0 \sin\theta_0 \sin\psi_0$, $\omega_2 = -\dot{\theta}_0 \sin\psi_0 + \dot{\varphi}_0 \sin\theta_0 \cos\psi_0$ and $\omega_3 = \dot{\varphi}_0 \cos\theta_0 + \dot{\psi}_0$ so that the Lagrangian, uninspiring at first, is

$$\begin{aligned}\mathcal{L} \stackrel{def}{=} & \frac{1}{2} I_1(\dot{\theta}_0 \cos\psi_0 + \dot{\varphi}_0 \sin\theta_0 \sin\psi_0)^2 + \\ & + \frac{1}{2} I_2(-\dot{\theta}_0 \sin\psi_0 + \dot{\varphi}_0 \sin\theta_0 \cos\psi_0)^2 + \frac{1}{2} I_3(\dot{\varphi}_0 \cos\theta_0 + \dot{\psi}_0)^2\end{aligned} \tag{9.3}$$

Angular momentum conservation does not imply that the components ω_j are constants because also $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ change with time according to $\frac{d}{dt}\mathbf{i}_j = \boldsymbol{\omega} \wedge \mathbf{i}_j$, $j = 1, 2, 3$. Hence $\dot{\mathbf{M}} = \mathbf{0}$ becomes, by the first of (9.2) and denoting $I\boldsymbol{\omega} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$, the *Euler equations* $I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \wedge I\boldsymbol{\omega} = \mathbf{0}$, or

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1. \quad I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 \tag{9.4}$$

which can be considered together with the conserved quantities, (9.2).



(4)

Fig. 4: The laboratory frame, the angular momentum frame, the comoving frame (and the Deprit angles).

Since angular momentum is conserved it is convenient to introduce the *laboratory frame* $(O; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ with fixed axes $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$ and (see Fig. 4):
(1) $(O; \mathbf{x}, \mathbf{y}, \mathbf{z})$: the *momentum frame* with fixed axes, but with $\mathbf{z}$-axis oriented as $\mathbf{M}$, and $\mathbf{x}$-axis coinciding with the node (*i.e.* the intersection) of the plane $\mathbf{x_0 y_0}$ and the plane

$\mathbf{xy}$ (orthogonal to $\mathbf{M}$). Therefore $\mathbf{x},\mathbf{y},\mathbf{z}$ is determined by the *two* Euler angles ζ,γ of $(O;\mathbf{x},\mathbf{y},\mathbf{z})$ in $(O;\mathbf{x_0},\mathbf{y_0},\mathbf{z_0})$,
(2) $(O;\mathbf{1},\mathbf{2},\mathbf{3})$: the *comoving frame*, *i.e.* the frame fixed with the body, and with unit vectors $\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3$ parallel to the principal axes of the body. The frame is determined by three Euler angles $\theta_0,\varphi_0,\psi_0$
(3) The Euler angles of $(O;\mathbf{1},\mathbf{2},\mathbf{3})$ with respect to $(O;\mathbf{x},\mathbf{y},\mathbf{z})$ which are denoted θ,φ,ψ
(4) G: the total angular momentum: $G^2=\sum_j I_j^2\omega_j^2$
(5) M_3: the angular momentum along the $\mathbf{z}_0$ axis; $M_3=G\cos\zeta$
(6) L: the projection of $\mathbf{M}$ on the axis $\mathbf{3}$, $L=G\cos\theta$

The quantities $G,M_3,L,\varphi,\gamma,\psi$ determine $\theta_0,\varphi_0,\psi_0$ and $\dot\theta_0,\dot\varphi_0,\dot\psi_0$, or the $p_{\theta_0},p_{\varphi_0},p_{\psi_0}$ variables conjugated to $\theta_0,\varphi_0,\psi_0$ as shown by the following comment.

Considering Fig. 4, the angles ζ,γ determine location, in the fixed frame $(O;\mathbf{x_0},\mathbf{y_0},\mathbf{z_0})$ of the direction of $\mathbf{M}$ and the node line $\mathbf{m}$, which are respectively the $\mathbf{z}$-axis and the $\mathbf{x}$ axis of the fixed frame associated with the angular momentum; the angles θ,φ,ψ *then* determine the position of the comoving frame with respect to the fixed frame $(O;\mathbf{x},\mathbf{y},\mathbf{z})$, hence its position with respect to $(O;\mathbf{x_0},\mathbf{y_0},\mathbf{z_0})$, *i.e.* $(\theta_0,\varphi_0,\psi_0)$. From this and G it is possible to determine $\boldsymbol{\omega}$ because

$$\cos\theta=\frac{I_3\omega_3}{G},\ \tan\psi=\frac{I_2\omega_2}{I_1\omega_1},\quad \omega_2^2=I_2^{-2}(G^2-I_1^2\omega_1^2-I_3^2\omega_3^2) \tag{9.5}$$

and, from (9.1), $\dot\theta_0,\dot\varphi_0,\dot\psi_0$ are determined.

The Lagrangian (9.3) gives immediately (after expressing $\boldsymbol{\omega}$, *i.e.* $\mathbf{n},\mathbf{z},\mathbf{i_3}$, in terms of the Euler angles $\theta_0,\varphi_0,\psi_0$) an expression for the variables $p_{\theta_0},p_{\varphi_0},\ p_{\psi_0}$ conjugated to $\theta_0,\varphi_0,\psi_0$

$$p_{\theta_0}=\mathbf{M}\cdot\mathbf{n_0}\qquad p_{\varphi_0}=\mathbf{M}\cdot\mathbf{z_0}\qquad p_{\psi_0}=\mathbf{M}\cdot\mathbf{i_3} \tag{9.6}$$

and in principle we could proceed to compute the Hamiltonian.

However the computation *can be avoided* because of the very remarkable property (Deprit), which can be checked with some patience, making use of (9.6) and of elementary spherical trigonometry identities,

$$M_3d\gamma+Gd\varphi+Ld\psi=p_{\varphi_0}d\varphi_0+p_{\psi_0}d\psi_0+p_{\theta_0}d\theta_0 \tag{9.7}$$

which means that *the map* $\big((M_3,\gamma),(L,\psi),(G,\varphi)\big)\longleftrightarrow\big((p_{\theta_0},\theta_0),(p_{\varphi_0},\varphi_0),(p_{\psi_0},\psi_0)\big)$ *is a canonical map.* And in the new coordinates the kinetic energy, hence the Hamiltonian, takes the form

$$K=\frac12\Big[\frac{L^2}{I_3}+(G^2-L^2)\Big(\frac{\sin^2\psi}{I_1}+\frac{\cos^2\psi}{I_2}\Big)\Big] \tag{9.8}$$

This again shows that G,M_3 are constants of motion, and the L,ψ variables are determined by a quadrature, because the Hamilton equation for ψ combined with the energy conservation yields

$$\dot\psi=\pm\big(\frac{1}{I_3}-\frac{\sin^2\psi}{I_1}-\frac{\cos^2\psi}{I_2}\big)\sqrt{\frac{2E-G^2\big(\frac{\sin^2\psi}{I_1}+\frac{\cos^2\psi}{I_2}\big)}{\frac{1}{I_3}-\frac{\sin^2\psi}{I_1}-\frac{\cos^2\psi}{I_2}}} \tag{9.9}$$

In the integrability region this motion is periodic with some period $T_L(E,G)$. Once $\psi(t)$ is determined the Hamilton equation for φ leads to the further quadrature

$$\dot\varphi=\big(\frac{\sin^2\psi(t)}{I_1}+\frac{\cos^2\psi(t)}{I_2}\big)\,G \tag{9.10}$$

which determines a second periodic motion with period $T_G(E,G)$. The γ, M_3 are constants and, therefore, the motion takes place on three dimensional invariant tori $\mathcal{T}_{E,G,M_3}$ in phase space each of which is *always* foliated into two dimensional invariant tori parameterized by the angle γ which is constant by (9.8) (because K is M_3-independent): the latter are in turn foliated by one dimensional invariant tori, *i.e.* by periodic orbits, with E,G such that the value of $T_L(E,G)/T_G(E,G)$ is rational.

Note that if $I_1=I_2=I$ the above analysis is extremely simplified. Furthermore if gravity g acts on the system the Hamiltonian will simply change by the addition of a potential $-mgz$ if z is the height of the center of mass. Then, see Fig. 4, if the center of mass of the body is on the axis $\mathbf{i}_3$ and $z=h\cos\theta_0$ and h is the distance of the center of mass from O, since $\cos\theta_0=\cos\theta\cos\zeta-\sin\theta\sin\zeta\cos\varphi$, the Hamiltonian will become $\mathcal{H}=K-mgh\cos\theta_0$ or

$$\mathcal{H}=\frac{G^2}{2I_3}+\frac{G^2-L^2}{2I}-m\,g\,h\,\Big(\frac{M_3L}{G^2}-\big(1-\frac{M_3^2}{G^2}\big)^{1/2}\big(1-\frac{L^2}{G^2}\big)^{1/2}\cos\varphi\Big) \tag{9.11}$$

so that, again, the system is integrable by quadratures (with the roles of ψ and φ "interchanged" with respect to the previous case) in suitable regions of phase space. This is called the *Lagrange's gyroscope.*

A less elementary integrable case is when the inertia moments are related as $I_1=I_2=2I_3$ and the center of mass is in the plane $\mathbf{i}_1,\mathbf{i}_2$ (rather than on the $\mathbf{i}_3$ axis) and only gravity acts, besides the constraint force on the pivot point O; this is called the *Kowalevskaia's gyroscope.*

References: [Ga83].

10. Other quadratures

An interesting classical integrable motion is that of a point mass attracted by two equal mass centers of gravitational attraction. Or a point ideally constrained to move on the surface of a general ellipsoid.

New integrable systems have been discovered quite recently and have generated a wealth of new developments ranging from group theory (as integrable systems are closely related to symmetries) to partial differential equations.

It is convenient to extend the notion of integrability by saying that a system is integrable in a region W of phase space if

(1) there is a change of coordinates $(\mathbf{p},\mathbf{q})\in W \longleftrightarrow \{\mathbf{A},\boldsymbol{\alpha},\mathbf{Y},\mathbf{y}\}\in (U\times\mathbb{T}^\ell)\times(V\times\mathbb{R}^m)$ *where* $U\subset\mathbb{R}^\ell$, $V\subset\mathbb{R}^m$, *with* $\ell+m\ge 1$, *are open sets and*
(2) the $\mathbf{A},\mathbf{Y}$ *are constants of motion while the other coordinates vary "linearly":*

$$(\boldsymbol{\alpha},\mathbf{y})\to(\boldsymbol{\alpha}+\boldsymbol{\omega}(\mathbf{A},\mathbf{Y})t,\,\mathbf{y}+\mathbf{v}(\mathbf{A},\mathbf{Y})t) \tag{10.1}$$

where $\boldsymbol{\omega}(\mathbf{A},\mathbf{Y}),\mathbf{v}(\mathbf{A},\mathbf{Y})$ *are smooth functions.*

In the new sense the systems studied in the previous sections are integrable in much wider regions (essentially on the entire phase space with the exception of a set of data which lie on lower dimensional surfaces forming sets of zero volume). The notion is convenient also because it allows us to say that even the systems of free particles are integrable.

Two very remarkable systems integrable in the new sense are the Hamiltonian systems, respectively called *Toda lattice* (Kruskal, Zabusky), and *Calogero lattice*, (Calogero, Moser); if $(p_i,q_i)\in\mathbb{R}^2$ they are

$$\begin{aligned}\mathcal{H}_T(\mathbf{p},\mathbf{q})=&\frac{1}{2m}\sum_{i=1}^n p_i^2+\sum_{i=1}^{n-1} g\,e^{-\kappa(q_{i+1}-q_i)}\\ \mathcal{H}_C(\mathbf{p},\mathbf{q})=&\frac{1}{2m}\sum_{i=1}^n p_i^2+\sum_{i<j}^{n}\frac{g}{(q_i-q_j)^2}+\frac12\sum_{i=1}^n m\omega^2 q_i^2\end{aligned} \tag{10.2}$$

where $m>0$ and $\kappa,\omega,g\ge 0$. They describe the motion of n interacting particles on a line.

The integration method for the above systems is again to find first the constants of motion and later to look for quadratures, when appropriate. The constants of motion can be found with the method of the *Lax pairs*. One shows that there is a pair of self adjoint $n\times n$ matrices $M(\mathbf{p},\mathbf{q}),N(\mathbf{p},\mathbf{q})$ such that the equations of motion become

$$\tfrac{d}{dt}M(\mathbf{p},\mathbf{q})\;=\;i\,\big[M(\mathbf{p},\mathbf{q}),N(\mathbf{p},\mathbf{q})\big],\qquad i=\sqrt{-1} \tag{10.3}$$

which imply that $M(t)=U(t)M(0)U(t)^{-1}$, with $U(t)$ a unitary matrix. When the equations can be written in the above form it is clear that the n eigenvalues of the matrix $M(0)=M(\mathbf{p}_0,\mathbf{q}_0)$ are constants of motion. When appropriate, *e.g.* in the Calogero lattice case with $\omega>0$, it is possible to proceed to find canonical action–angle coordinates: a task that is quite difficult due to the arbitrariness of n, but which is possible.

The Lax pairs for the Calogero lattice (with $\omega=0,g=m=1$) are

$$M_{hh}=p_h,\quad N_{hh}=0,\ \text{and}\ M_{hk}=\frac{i}{(q_h-q_k)},\ \ N_{hk}=\frac{1}{(q_h-q_k)^2}\ \ h\ne k \tag{10.4}$$

while for the Toda lattice (with $m=g=\frac12\kappa=1$) the non zero matrix elements of M,N are

$$
\begin{aligned}
M_{hh} = p_h, \qquad & M_{h,h+1} = M_{h+1,h} = e^{-(q_h - q_{h+1})}, \\
N_{h,h+1} = -N_{h+1,h} = & \; i\, e^{-(q_h - q_{h+1})}
\end{aligned}
\tag{10.5}
$$

which are checked by first trying the case $n = 2$.

Another integrable system (SUTHERLAND) is

$$
\mathcal{H}_S(\mathbf{p},\mathbf{q}) = \frac{1}{2m}\sum_{i=k}^{n} p_k^2 + \sum_{h<k}^{n} \frac{g}{\sinh^2(q_h - q_k)}
\tag{10.6}
$$

whose Lax pair is related to that of the Calogero lattice.

By taking suitable limits as $n \to \infty$ and as the other parameters tend to 0 or ∞ at suitable rates integrability of a few differential equations, among which the *Korteweg-deVries equation* or the *nonlinear Schödinger equation*, can be derived.

As mentioned in Section 1 symmetry properties under continuous groups imply existence of constants of motion. Hence it is natural to think that integrability of a mechanical system reflects enough symmetry to imply the existence of as many constants of motion, independent and in involution, as the number n of degrees of freedom.

This is in fact *always true*, and in some respects it is a tautological statement in the anisochronous cases. Integrability in a region W implies existence of canonical action–angle coordinates $(\mathbf{A}, \boldsymbol{\alpha})$, see Section 6, and the Hamiltonian depends solely on the $\mathbf{A}$'s: therefore its restriction to W is invariant with respect to the action of the continuous commutative group $\mathcal{T}^n$ of the *translations of the angle variables.* The actions can be seen as constants of motion whose existence follows from Noether's theorem, at least in the anisochronous cases in which the Hamiltonian formulation is equivalent to a Lagrangian formulation.

What is nontrivial is to recognize, prior to realizing integrability, that a system admits this kind of symmetry: in most of the interesting cases the systems either do not exhibit obvious symmetries or they exhibit symmetries apparently unrelated to the group $\mathcal{T}^n$, which *nevertheless* imply existence of constants of motion in number sufficient for integrability. Hence nontrivial integrable systems possess a "hidden" symmetry under $\mathcal{T}^n$: the rigid body is an example.

However very often the symmetries of a Hamiltonian H which imply integrability also imply partial isochrony, *i.e.* imply that the number of independent frequencies is smaller than n, see Section 6. Even in such cases often a map exists from the original coordinates $(\mathbf{p}, \mathbf{q})$ to the integrating variables $(\mathbf{A}, \boldsymbol{\alpha})$ in which $\mathbf{A}$ are constants of motion and the $\boldsymbol{\alpha}$ are uniformly rotating angles (some of which are *also* constant) with spectrum $\boldsymbol{\omega}(\mathbf{A})$ which is the gradient $\boldsymbol{\partial}_{\mathbf{A}} h(\mathbf{A})$ for some function $h(\mathbf{A})$ depending only on a few of the $\mathbf{A}$ coordinates. However the map might fail to be *canonical.* The system is then said to be *bihamiltonian*: in the sense that one can represent motions in two systems of canonical coordinates, not related by a canonical transformation, and by two Hamiltonian functions H and $H' \equiv h$ which generate the *same* motions in the respective coordinates (the latter changes of variables are sometimes called "canonical with respect to the pair H, H'" while the transformations considered in Section 4 are called completely canonical).

References: [CD82].

11. Generic nonintegrability

It is natural to try to prove that a system "close" to an integrable one has motions with properties very close to quasi–periodic. This is indeed the case but in a rather subtle way. That there is a problem is easily seen in the case of a perturbation of an anisochronous integrable system.

Assume that a system is integrable in a region W of phase space which, in the integrating action–angle variables $(\mathbf{A},\boldsymbol{\alpha})$, has the standard form $U\times\mathbb{T}^\ell$ with a Hamiltonian $h(\mathbf{A})$ with gradient $\boldsymbol{\omega}(\mathbf{A})=\partial_{\mathbf{A}}h(\mathbf{A})$. If the forces are perturbed by a potential which is smooth then the new system will be described, in the same coordinates, by a Hamiltonian like

$$\mathcal{H}_\varepsilon(\mathbf{A},\boldsymbol{\alpha})=h(\mathbf{A})+\varepsilon f(\mathbf{A},\boldsymbol{\alpha}) \tag{11.1}$$

with h,f analytic in the variables $\mathbf{A},\boldsymbol{\alpha}$.

If the system really behaved like the unperturbed one it ought to have ℓ constants of motion of the form $F_\varepsilon(\mathbf{A},\boldsymbol{\alpha})$ analytic in ε near $\varepsilon=0$ and *uniform*, *i.e.* single valued (which is the same as periodic) in the variables $\boldsymbol{\alpha}$. However the following theorem (POINCARÉ) shows that this is somewhat unlikely a possibility.

If the matrix $\boldsymbol{\partial}^2_{\mathbf{A}\mathbf{A}}h(\mathbf{A})$ has rank ≥ 2 the Hamiltonian (11.1) "generically" (an intuitive notion precised below) cannot be integrated by a canonical transformation $\mathcal{C}_\varepsilon(\mathbf{A},\boldsymbol{\alpha})$ which
(1) reduces to the identity as $\varepsilon\to 0$,
(2) is analytic in ε near $\varepsilon=0$ and in $(\mathbf{A},\boldsymbol{\alpha})\in U'\times\mathbb{T}^\ell$, with $U'\subset U$ open.
Furthermore no uniform constants of motion $F_\varepsilon(\mathbf{A},\boldsymbol{\alpha})$, defined for ε near 0 and $(\mathbf{A},\boldsymbol{\alpha})$ in an open domain $U'\times\mathbb{T}^\ell$, exist other than the functions of $\mathcal{H}_\varepsilon$ itself.

Integrability in the sense (1),(2) can be called *analytic integrability* and it is the strongest (and most naive) sense that can be given to the attribute.

The first part of the theorem, *i.e.* (1),(2), holds simply because, if integrability was assumed, a generating function of the integrating map would have the form $\mathbf{A}'\cdot\boldsymbol{\alpha}+\Phi_\varepsilon(\mathbf{A}',\boldsymbol{\alpha})$ with Φ admitting a power series expansion in ε as $\Phi_\varepsilon=\varepsilon\Phi^1+\varepsilon^2\Phi^2+\ldots$. Hence Φ^1 would have to satisfy

$$\boldsymbol{\omega}(\mathbf{A}')\cdot\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi^1(\mathbf{A}',\boldsymbol{\alpha})+f(\mathbf{A}',\boldsymbol{\alpha})=\overline{f}(\mathbf{A}') \tag{11.2}$$

where $\overline{f}(\mathbf{A}')$ depends only on $\mathbf{A}'$ (hence integrating both sides with $d\boldsymbol{\alpha}$ it appears that $\overline{f}(\mathbf{A}')$ must coincide with the average of $f(\mathbf{A}',\boldsymbol{\alpha})$ over $\boldsymbol{\alpha}$).

This implies that the Fourier transform $f_{\boldsymbol\nu}(\mathbf{A})$, $\boldsymbol\nu\in\mathbb{Z}^\ell$, should satisfy

$$f_{\boldsymbol\nu}(\mathbf{A}')=0\qquad\text{if } \boldsymbol\omega(\mathbf{A}')\cdot\boldsymbol\nu=0,\ \boldsymbol\nu\ne\mathbf{0}\tag{11.3}$$

which is equivalent to the existence of $\widetilde f_{\boldsymbol\nu}(\mathbf{A}')$ such that $f_{\boldsymbol\nu}(\mathbf{A})=\boldsymbol\omega(\mathbf{A}')\cdot\boldsymbol\nu\widetilde f_{\boldsymbol\nu}(\mathbf{A})$ for $\boldsymbol\nu\ne\mathbf{0}$. But since there is no relation between $\boldsymbol\omega(\mathbf{A})$ and $f(\mathbf{A},\boldsymbol\alpha)$ this property "generically" will not hold in the sense that as close as wished to a f which satisfies the property (11.3) there will be another f which does not satisfy it esentially no matter how "closeness" is defined, *e.g.* with respect to the metric $||f-g||=\sum_{\boldsymbol\nu}|f_{\boldsymbol\nu}(\mathbf{A})-g_{\boldsymbol\nu}(\mathbf{A})||$. This is so because the rank of $\boldsymbol\partial^{\,2}_{\mathbf{AA}}h(\mathbf{A})$ is higher than 1 and $\boldsymbol\omega(\mathbf{A})$ varies at least on a two dimensional surface, so that $\boldsymbol\omega\cdot\boldsymbol\nu=0$ becomes certainly possible for some $\boldsymbol\nu\ne\mathbf{0}$ while $f_{\boldsymbol\nu}(\mathbf{A})$ in general will not vanish, so that Φ^1, hence Φ_ε, does not exist.

This means that close to a function f there is a function f' which violates (11.3) for some $\boldsymbol\nu$. Of course this depends on what is meant by "close": however here essentially any topology introduced on the space of the functions f will make the statement correct. For instance if the distance between two functions is defined by $\sum_{\boldsymbol\nu}\sup_{\mathbf{A}\in U}|f_{\boldsymbol\nu}(\mathbf{A})-g_{\boldsymbol\nu}(\mathbf{A})|$ or by $\sup_{\mathbf{A},\boldsymbol\alpha}|f(\mathbf{A},\boldsymbol\alpha)-g(\mathbf{A},\boldsymbol\alpha)|$.

The idea behind the last of the theorem statement is in essence the same: consider, for simplicity, the anisochronous case in which the matrix $\boldsymbol\partial^{\,2}_{\mathbf{AA}}h(\mathbf{A})$ has maximal rank ℓ, *i.e.* the determinant $\det\boldsymbol\partial^{\,2}_{\mathbf{AA}}h(\mathbf{A})$ does not vanish. Anisochrony implies that $\boldsymbol\omega(\mathbf{A})\cdot\boldsymbol\nu\ne0$ for all $\boldsymbol\nu\ne\mathbf{0}$ and for $\mathbf{A}$ on a dense set and this property will be used repeatedly in the following analysis.

Let $B(\varepsilon,\mathbf{A},\boldsymbol\alpha)$ be a "uniform" constant of motion, meaning that it is single–valued and analytic in the non simply connected region $U\times\mathbb{T}^\ell$ and, for ε small,

$$B(\varepsilon,\mathbf{A},\boldsymbol\alpha)=B_0(\mathbf{A},\boldsymbol\alpha)+\varepsilon B_1(\mathbf{A},\boldsymbol\alpha)+\varepsilon^2B_2(\mathbf{A},\boldsymbol\alpha)+\ldots\tag{11.4}$$

The condition that B is a constant of motion can be written order by order in its expansion in ε: the first two orders are

$$\begin{aligned}&\boldsymbol\omega(\mathbf{A})\cdot\partial_{\boldsymbol\alpha}B_0(\mathbf{A},\boldsymbol\alpha)=0,\\&\partial_{\mathbf{A}}f(\mathbf{A},\boldsymbol\alpha)\cdot\partial_{\boldsymbol\alpha}B_0(\mathbf{A},\boldsymbol\alpha)-\partial_{\boldsymbol\alpha}f(\mathbf{A},\boldsymbol\alpha)\cdot\partial_{\mathbf{A}}B_0(\mathbf{A},\boldsymbol\alpha)+\\&\qquad\qquad+\boldsymbol\omega(\mathbf{A})\cdot\partial_{\boldsymbol\alpha}B_1(\mathbf{A},\boldsymbol\alpha)=0\end{aligned}\tag{11.5}$$

Then the above two relations *and anisochrony* imply that B_0 must be a function of $\mathbf{A}$ only and the second that $\boldsymbol\omega(\mathbf{A})\cdot\boldsymbol\nu$ and $\partial_{\mathbf{A}}B_0(\mathbf{A})\cdot\boldsymbol\nu$ vanish simoultaneously for all $\boldsymbol\nu$. Hence the gradient of B_0 *must* be proportional to $\boldsymbol\omega(\mathbf{A})$, *i.e.* to the gradient of $h(\mathbf{A})$: $\boldsymbol\partial_{\mathbf{A}}B_0(\mathbf{A})=\lambda(\mathbf{A})\boldsymbol\partial_{\mathbf{A}}h(\mathbf{A})$. Therefore generically (because of the anisochrony) it must be that B_0 depends on $\mathbf{A}$ through $h(\mathbf{A})$: $B_0(\mathbf{A})=F(h(\mathbf{A}))$ for some F.

Looking again, with the new information, at the second of (11.5) it follows that at fixed $\mathbf{A}$ the $\boldsymbol\alpha$-derivative in the direction $\boldsymbol\omega(\mathbf{A})$ of B_1 equals $F'(h(\mathbf{A}))$ times the $\boldsymbol\alpha$-derivative of f, *i.e.* $B_1(\mathbf{A},\boldsymbol\alpha)=f(\mathbf{A},\boldsymbol\alpha)\,F'(h(\mathbf{A}))+C_1(\mathbf{A})$.

Summarizing: the constant of motion B has been written as $B(\mathbf{A},\boldsymbol\alpha)=F(h(\mathbf{A}))+\varepsilon F'(h(\mathbf{A}))\,f(\mathbf{A},\boldsymbol\alpha)+\varepsilon C_1(\mathbf{A})+\varepsilon^2B_2+\ldots$ which is equivalent to $B(\mathbf{A},\boldsymbol\alpha)=F(\mathcal{H}_\varepsilon)+\varepsilon(B_0'+\varepsilon B_1'+\ldots)$ and therefore $B_0'+\varepsilon B_1'+\ldots$ is another analytic constant of motion. Repeating the argument also $B_0'+\varepsilon B_1'+\ldots$ must have the form $F_1(\mathcal{H}_\varepsilon)+\varepsilon\,(B_0''+\varepsilon B_1''+\ldots)$; conclusion

$$B=F(\mathcal{H}_\varepsilon)+\varepsilon\,F_1(\mathcal{H}_\varepsilon)+\varepsilon^2F_2(\mathcal{H}_\varepsilon)+\ldots+\varepsilon^nF_n(\mathcal{H}_\varepsilon)+O(\varepsilon^{n+1}) \tag{11.6}$$

By analyticity $B=F_\varepsilon(\mathcal{H}_\varepsilon(\mathbf{A},\boldsymbol{\alpha}))$ for some F_ε: hence generically all constants of motion are trivial.

Therefore a system close to integrable cannot behave as it would naively be expected. The problem, however, was not manifest until POINCARÉ's proof of the above results: because in most applications the function f has only finitely many Fourier components, or at least is replaced by an approximation with this property, so that at least (11.3) and even a few of the higher order constraints like (11.5) become possible in open regions of action space. In fact it may happen that the values of $\mathbf{A}$ of interest are restricted so that $\boldsymbol{\omega}(\mathbf{A})\cdot\boldsymbol{\nu}=0$ only for "large" values of $\boldsymbol{\nu}$ and there $f_{\boldsymbol{\nu}}=0$. *Nevertheless* the property that $f_{\boldsymbol{\nu}}(\mathbf{A})=(\boldsymbol{\omega}(\mathbf{A})\cdot\boldsymbol{\nu})\,\widetilde{f}_{\boldsymbol{\nu}}(\mathbf{A})$ (or the analogous higher order conditions, *e.g.* (11.5)), which we have seen to be necessary for analytic integrability of the perturbed system, can be checked to fail in important problems, if no approximation is made on f. Hence a conceptual problem arises.

References: [Po].

12. Perturbing functions

To check, in a given problem, the nonexistence of nontrivial constants of motion along the lines indicated in Sect. 11 it is necessary to express the potential, usually given in Cartesian coordinates as $\varepsilon V(\mathbf{x})$, in terms of the action–angle variables of the unperturbed, integrable, system.

In particular the problem arises when trying to check nonexistence of nontrivial constants of motion when the anisochrony assumption (cf. Sect.11) *is not satisfied.* Usually it becomes satisfied "to second order" (or higher): but to show this a more detailed information on the structure of the perturbing function expressed in action–angle variables is needed. For instance this is often necessary even when the perturbation is approximated by a trigonometric polynomial, as it is essentially always the case in Celestial Mechanics.

Finding explicit expressions for the action–angle variables is in itself a rather nontrivial task which leads to many problems of intrinsic interest even in seemingly simple cases. For instance, in the case of the planar gravitational central motion, the Kepler equation $\lambda=\xi-\varepsilon\sin\xi$, see the first of (8.1), must be solved expressing ξ in terms of λ, see the first of (8.2). It is obvious that for small ε the variable ξ can be expressed as an analytic function of ε: *nevertheless* the actual construction of this expression leads to several problems. For small ε an interesting algorithm is the following.

Let $h(\lambda)=\xi-\lambda$ so that the equation to solve (*i.e.* the first of (8.1)) is

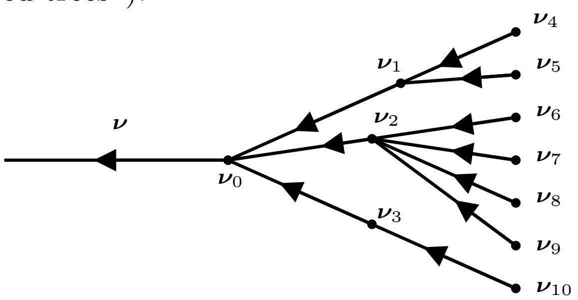
$$h(\lambda)=\varepsilon\sin(\lambda+h(\lambda))\equiv-\varepsilon\,\frac{\partial c}{\partial\lambda}(\lambda+h(\lambda)) \tag{12.1}$$

where $c(\lambda)=\cos\lambda$; the function $\lambda\to h(\lambda)$ should be periodic in λ, with period 2π, and analytic in ε,λ for ε small and λ real. If $h(\lambda)=\varepsilon h^{(1)}+\varepsilon^2h^{(2)}+\ldots$ the Fourier transform of $h^{(k)}(\lambda)$ satisfies the recursion relation

$$h_\nu^{(k)} = -\sum_{p=1}^\infty \frac{1}{p!} \sum_{\substack{k_1+\ldots+k_p=k-1\\ \nu_0+\nu_1+\ldots+\nu_p=\nu}} (i\nu_0)c_{\nu_0}(i\nu_0)^p \prod h_{\nu_j}^{(k_j)}, \qquad k>1 \tag{12.2}$$

with c_ν the Fourier transform of the cosine ($c_{\pm1}=\frac{1}{2}$, $c_\nu=0$ if $\nu\ne\pm1$), and (of course) $h_\nu^{(1)}=-i\nu c_\nu$. Eq. (12.2) is obtained by expanding the *r.h.s.* of (12.1) in powers of h and then taking the Fourier transform of both sides retaining only terms of order k in ε.

Iterating the above relation imagine to draw all trees θ with k "branches", or "lines", *distinguished by a label taking k values*, and k nodes and attach to each node v a *harmonic* label $\nu_v=\pm1$ as in Fig. 5. The trees will assumed to start with a *root line vr* linking a point r and the "first node" v, see Fig. 5, and then bifurcate arbitrarily (such trees are sometimes called "rooted trees").



(5)

Fig. 5: An example of a tree graph and of its labels. It contains only one simple node (1). Harmonics are indicated next to their nodes. Labels distinguishing lines are not marked.

Imagine the tree oriented from the endpoints towards the root r (not to be considered a node) and given a node v call v' the node immediately following it. If v is the first node before the root r let $v'=r$ and $\nu_{v'}=1$. For each such decorated tree define its *numerical value*

$$\mathrm{Val}(\theta) = \frac{-i}{k!} \prod_{lines\, l=v'v} (\nu_{v'}\nu_v) \prod_{nodes} c_{\nu_v} \tag{12.3}$$

and define a *current* $\nu(l)$ on a line $l=v'v$ to be the sum of the harmonics of the nodes preceding v': $\nu(l)=\sum_{w\le v}\nu_v$. Call $\nu(\theta)$ the current flowing in the root branch and call order of θ the number of nodes (or branches). Then

$$h_\nu^{(k)} = \sum_{\substack{\theta,\,\nu(\theta)=\nu\\ order(\theta)=k}} \mathrm{Val}(\theta) \tag{12.4}$$

provided trees are considered identical if they can be overlapped (labels included) after suitably scaling the lengths of their branches and pivoting them around the nodes out of which they emerge (the root is always imagined fixed at the origin).

If the trees are stripped of the harmonic labels their number is finite and it can be estimated $\le k!4^k$ (because the labels which distinguish the lines can be attached to an unlabeled tree in many ways). The harmonic labels (*i.e.* $\nu_v=\pm1$) can be laid down in 2^k ways, and the value of each tree can be bounded by $\frac{1}{k!}2^{-k}$ (because $c_{\pm1}=\frac{1}{2}$).

Hence $\sum_\nu |h^{(k)}_\nu| \le 4^k$, which gives a (rough) estimate of the radius of convergence of the expansion of h in powers of ε: namely .25 (easily improvable to 0.3678 if $4^k k!$ is replaced by k^{k-1} using Cayley's formula for the enumeration of rooted trees). A simple expression for $h^{(k)}(\psi)$ (LAGRANGE) is $h^{(k)}(\psi) = \frac{1}{k!}\partial_\psi^{k-1} \sin^k \psi$ (also readable from the tree representation): the actual radius of convergence, first determined by LAPLACE, of the series for h can be also determined from the latter expression for h (ROUCHÉ) or directly from the tree representation: it is $\sim .6627$.

One can find better estimates or at least more efficient methods for evaluating the sums in (12.4): in fact in performing the sum in (12.4) important *cancellations* occurr. For instance the harmonic labels can be subject to the further strong constraint that *no line carries zero current* because the sum of the values of the trees of fixed order and with at least one line carrying 0 current vanishes.

The above expansion can also be simplified by "partial resummations". For the purpose of an example, call *simple* the nodes with one entering and one exiting line (see Fig. 5). Then all tree graphs which on any line between two non simple nodes contain any number of simple nodes can be eliminated. This is done by replacing, in evaluating the (remaining) trees value, the factors $\nu_{v'}\nu_v$ in (12.3) by $\nu_{v'}\nu_v/(1-\varepsilon\cos\psi)$: then the value of a tree θ becomes a function of ψ and ε to be denoted $\mathrm{Val}(\theta)_\psi$ *and* (12.4) is replaced by

$$h(\psi) = \sum_{k=1}^\infty \sum^{\quad *}_{\substack{\theta,\,\nu(\theta)=\nu\\ order(\theta)=k}} \varepsilon^k\, e^{i\,\nu\,\psi}\, \mathrm{Val}(\theta)_\psi \tag{12.5}$$

where the $*$ means that the trees are *subject to the further restriction of not containing any simple node.* It should be noted that the above graphical representation of the solution of the Kepler equation is strongly reminiscent of the representations of quantities in terms of graphs that occurr often in quantum field theory. Here the trees correspond to "Feynman graphs", the factors associated with the nodes are the "couplings", the factors associated with the lines are the "propagators" and the resummations are analogous to the "self-energy resummations", while the mentioned cancellations can be related to the class of identities called "Ward identities". Not only the analogy can be shown to be not superficial, but it turns out to be even very helpful in key mechanical problems: see Appendix A1.

The existence of a vast number of identities relating the tree values is shown already by the "simple" form of Lagrange's series and by the even more remarkable resummation (LEVI-CIVITA) leading to

$$h(\psi) = \sum_{k=1}^\infty \frac{(\varepsilon\sin\psi)^k}{k!}\Big(\frac{1}{1-\varepsilon\cos\psi}\partial_\psi\Big)^k\,\psi \tag{12.6}$$

It is even possible further collection of the series terms to express it as series with much better convergence properties, for instance its terms can be reorganized and collected (*resummed*) so that h is expressed as a power series in the parameter

$$\eta = \frac{\varepsilon\, e^{\sqrt{1-\varepsilon^2}}}{1+\sqrt{1-\varepsilon^2}} \tag{12.7}$$

with radius of convergence 1, which corresponds to $\varepsilon = 1$ (via a simple argument by LEVI-CIVITA). The analyticity domain for the Lagrange series is $|\eta| < 1$. This also determines the Laplace radius value: it is the point closest to the origin of the complex curve $|\eta(\varepsilon)| = 1$: it is imaginary so that it is the root of the equation $\varepsilon e^{\sqrt{1+\varepsilon^2}}/(1 + \sqrt{1+\varepsilon^2}) = 1$.

The analysis provides an example, in a simple case of great interest in applications, of the kind of computations actually necessary to represent the perturbing function in terms of action–angle variables. The property that the function $c(\lambda)$ in (12.1) is the cosine has been used only to limit the range of the label ν to be ± 1; hence the same method, with similar results, can be applied to study the inversion of the relation between the average anomaly λ and the true anomaly θ and to get, for instance, quickly the properties of f, g in (8.2).

References: [LC].

13. Lindstedt and Birkhoff series. Divergences.

Nonexistence of constants of motion, rather than being the end of the attempts to study by perturbation methods motions close to integrable ones, marks the beginning of renewed efforts to understand their nature.

Let $(\mathbf{A}, \boldsymbol{\alpha}) \in U \times \mathbb{T}^\ell$ be action–angle variables defined in the integrability region for an analytic Hamiltonian and let $h(\mathbf{A})$ be its value in the action–angle coordinates. Suppose that $h(\mathbf{A})$ is *anisochronous* and let $f(\mathbf{A}, \boldsymbol{\alpha})$ be an analytic perturbing function. Consider, for ε small, the Hamiltonian $\mathcal{H}_\varepsilon(\mathbf{A}, \boldsymbol{\alpha}) = \mathcal{H}_0(\mathbf{A}) + \varepsilon f(\mathbf{A}, \boldsymbol{\alpha})$.

Let $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(\mathbf{A}_0) \equiv \boldsymbol{\partial}_{\mathbf{A}}\mathcal{H}_0(\mathbf{A})$ be the frequency spectrum, see Section 6, of one of the invariant tori of the unperturbed system corresponding to an action $\mathbf{A}_0$. Short of integrability the question to ask at this point is whether the perturbed system admits an analytic invariant torus on which motion is quasi periodic with
(1) the same spectrum $\boldsymbol{\omega}_0$ and
(2) depends analytically on ε at least for ε small and
(3) reduces to the "unperturbed torus" $\{\mathbf{A}_0\} \times \mathbb{T}^\ell$ as $\varepsilon \to 0$.

More concretely the question is

Are there functions $\mathbf{H}_\varepsilon(\boldsymbol{\psi}), \mathbf{h}_\varepsilon(\boldsymbol{\psi})$ analytic in $\boldsymbol{\psi} \in \mathbb{T}^\ell$ and in ε near 0, vanishing as $\varepsilon \to 0$ and such that the torus with parametric equations

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{H}_\varepsilon(\boldsymbol{\psi}), \qquad \boldsymbol{\alpha} = \boldsymbol{\psi} + \mathbf{h}_\varepsilon(\boldsymbol{\psi}) \qquad \boldsymbol{\psi} \in \mathbb{T}^\ell \tag{13.1}$$

is invariant and, if $\boldsymbol{\omega}_0 \stackrel{def}{=} \boldsymbol{\omega}(\mathbf{A}_0)$, the motion on it is simply $\boldsymbol{\psi} \to \boldsymbol{\psi} + \boldsymbol{\omega}_0 t$, i.e. it is quasi periodic with spectrum $\boldsymbol{\omega}_0$?

In this context POINCARÉ's theorem of Sect.11 had followed another key result, earlier developed in particular cases and completed by him, which provides a *partial* answer to the question.

Suppose that $\boldsymbol{\omega}_0=\boldsymbol{\omega}(\mathbf{A}_0)\in\mathbb{R}^\ell$ satisfies a *Diophantine property*, namely suppose that there exist constants $C,\tau>0$ such that

$$|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|\ge\frac{1}{C|\boldsymbol{\nu}|^\tau},\qquad\text{for all } \mathbf{0}\ne\boldsymbol{\nu}\in\mathbb{Z}^\ell \tag{13.2}$$

which, for each $\tau>\ell-1$ *fixed*, is a property enjoyed by all $\boldsymbol{\omega}\in\mathbb{R}^\ell$ *but for a set of zero measure*. Then the motions on the unperturbed torus run over trajectories that *fill the torus densely* because of the "irrationality" of $\boldsymbol{\omega}_0$ implied by (13.2). Writing Hamilton's equations,, $\dot{\boldsymbol{\alpha}}=\partial_{\mathbf{A}}\mathcal{H}_0(\mathbf{A})+\varepsilon\boldsymbol{\partial}_{\mathbf{A}}f(\mathbf{A},\boldsymbol{\alpha})$, $\dot{\mathbf{A}}=-\varepsilon\boldsymbol{\partial}_{\boldsymbol{\alpha}}f(\mathbf{A},\boldsymbol{\alpha})$ with $\mathbf{A},\boldsymbol{\alpha}$ given by (13.1) with $\boldsymbol{\psi}$ replaced by $\boldsymbol{\psi}+\boldsymbol{\omega}t$, and using the density of the unperturbed trajectories implied by (13.2), the condition that (13.1) are equations for an invariant torus on which the motion is $\boldsymbol{\psi}\to\boldsymbol{\psi}+\boldsymbol{\omega}_0t$ are

$$\begin{aligned}&\boldsymbol{\omega}_0+(\boldsymbol{\omega}_0\cdot\boldsymbol{\partial}_{\boldsymbol{\psi}})\mathbf{h}_\varepsilon(\boldsymbol{\psi})=\boldsymbol{\partial}_{\mathbf{A}}\mathcal{H}_0(\mathbf{A}_0+\mathbf{H}_\varepsilon(\boldsymbol{\psi}))+\varepsilon\boldsymbol{\partial}_{\mathbf{A}}f(\mathbf{A}_0+\mathbf{H}_\varepsilon(\boldsymbol{\psi}),\boldsymbol{\psi}+\mathbf{h}_\varepsilon(\boldsymbol{\psi}))\\&(\boldsymbol{\omega}_0\cdot\boldsymbol{\partial}_{\boldsymbol{\psi}})\mathbf{H}_\varepsilon(\boldsymbol{\psi})=-\varepsilon\boldsymbol{\partial}_{\boldsymbol{\alpha}}f(\mathbf{A}_0+\mathbf{H}_\varepsilon(\boldsymbol{\psi}),\boldsymbol{\psi}+\mathbf{h}_\varepsilon(\boldsymbol{\psi}))\end{aligned}\tag{13.3}$$

The theorem referred above (Poincaré) is that

If the unperturbed system is anisochronous and $\boldsymbol{\omega}_0=\boldsymbol{\omega}(\mathbf{A}_0)$ satisfies (13.2) for some $C,\tau>0$ there exist two well defined power series $\mathbf{h}_\varepsilon(\boldsymbol{\psi})=\sum_{k=1}^\infty\varepsilon^k\,\mathbf{h}^{(k)}(\boldsymbol{\psi})$ and $\mathbf{H}_\varepsilon(\boldsymbol{\psi})=\sum_{k=1}^\infty\varepsilon^k\,\mathbf{H}^{(k)}(\boldsymbol{\psi})$ which solve (13.3) to all orders in ε. The series for $\mathbf{H}_\varepsilon$ is uniquely determined, and such is also the series for $\mathbf{h}_\varepsilon$ up to the addition of an arbitrary constant at each order, so that it is unique if $\mathbf{h}_\varepsilon$ is required, as henceforth done with no loss of generality, to have zero average over $\boldsymbol{\psi}$.

The algorithm for the construction is illustrated in a simple case in Section 14, (14.4),(14.5). Convergence of the above series, called *Lindstedt series*, even for ε small has been a problem for a rather long time. Poincaré proved existence of the formal solution; but his other result, discussed in Sect.11, casts doubts on convergence although *it does not exclude it* as was immediately stressed by several authors (including Poincaré himself). The result of Sect. 11 shows impossibility of solving (13.3) *for all* $\boldsymbol{\omega}_0$'s near a given spectrum, analytically and uniformly, but it does not exclude the possibility of solving it *for a single* $\boldsymbol{\omega}_0$.

The theorem admits several extensions or analogues: an interesting one is to the case of isochronous unperturbed systems:

Given the Hamiltonian $\mathcal{H}_\varepsilon(\mathbf{A},\boldsymbol{\alpha})=\boldsymbol{\omega}_0\cdot\mathbf{A}+\varepsilon f(\mathbf{A},\boldsymbol{\alpha})$, with $\boldsymbol{\omega}_0$ satisfying (13.2) and f analytic, there exist power series $\mathcal{C}_\varepsilon(\mathbf{A}',\boldsymbol{\alpha}')$, $u_\varepsilon(\mathbf{A}')$ such that $\mathcal{H}_\varepsilon(\mathcal{C}_\varepsilon(\mathbf{A}',\boldsymbol{\alpha}'))=\boldsymbol{\omega}_0\cdot\mathbf{A}'+u_\varepsilon(\mathbf{A}')$ holds as an equality between formal power series (i.e. order by order in ε) and at the same time the $\mathcal{C}_\varepsilon$ regarded as a map satisfies order by order the condition (i.e. (4.3)) that it is a canonical map.

This means that there is a generating function $\mathbf{A}'\cdot\boldsymbol{\alpha}+\Phi_\varepsilon(\mathbf{A}',\boldsymbol{\alpha})$ also defined by a formal power series $\Phi_\varepsilon(\mathbf{A}',\boldsymbol{\alpha})=\sum_{k=1}^\infty\varepsilon^k\,\Phi^{(k)}(\mathbf{A}',\boldsymbol{\alpha})$, *i.e.* such that if $\mathcal{C}_\varepsilon(\mathbf{A}',\boldsymbol{\alpha}')=(\mathbf{A},\boldsymbol{\alpha})$ then it is true, order by order in powers of ε, that $\mathbf{A}=\mathbf{A}'+\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi_\varepsilon(\mathbf{A}',\boldsymbol{\alpha})$ and $\boldsymbol{\alpha}'=\boldsymbol{\alpha}+\boldsymbol{\partial}_{\mathbf{A}'}\Phi_\varepsilon(\mathbf{A}',\boldsymbol{\alpha})$. The series for $\Phi_\varepsilon,u_\varepsilon$ are called *Birkhoff series.*

In this isochronous case *if Birkhoff series were convergent* for small ε and $(\mathbf{A}',\boldsymbol{\alpha})$ in a region of the form $U\times\mathbb{T}^\ell$, with $U\subset\mathbb{R}^\ell$ open and bounded, it would follow that, for small ε, $\mathcal{H}_\varepsilon$ would be integrable in a large region of phase space (*i.e.* where the generating function can be used to build a canonical map: this would essentially be $U\times\mathbb{T}^\ell$ deprived of a small layer of points near the boundary of U). However convergence for ε small is false (in general) as shown by the simple two dimensional example

$$\mathcal{H}_\varepsilon(\mathbf{A},\boldsymbol{\alpha})=\boldsymbol{\omega}_0\cdot\mathbf{A}+\varepsilon\,(A_2+f(\boldsymbol{\alpha})),\qquad(\mathbf{A},\boldsymbol{\alpha})\in\mathbb{R}^2\times\mathbb{T}^2\tag{13.4}$$

with $f(\boldsymbol{\alpha})$ an arbitrary analytic function with *all* Fourier coefficients $f_{\boldsymbol{\nu}}$ positive for $\boldsymbol{\nu}\neq\mathbf{0}$ and $f_{\mathbf{0}}=0$. In the latter case the solution is

$$u_\varepsilon(\mathbf{A}')=\varepsilon A_2,\quad \Phi_\varepsilon(\mathbf{A}',\boldsymbol{\alpha})=\sum_{k=1}^\infty\varepsilon^k\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathbb{Z}^2}f_{\boldsymbol{\nu}}\,e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}}\frac{(i\,\nu_2)^k}{(i(\omega_{01}\nu_1+\omega_{02}\nu_2))^{k+1}}\tag{13.5}$$

The series does not converge: in fact its convergence would imply integrability and, consequently, bounded trajectories in phase space: however the equations of motion for (13.4) can be easily solved *explicitly* and in any open region near a given initial data there are other data which have unbounded trajectories if $\omega_{01}/(\omega_{02}+\varepsilon)$ is rational.

Nevertheless even in this elementary case a formal sum of the series yields

$$u(\mathbf{A}')=\varepsilon A_2',\quad \Phi_\varepsilon(\mathbf{A}',\boldsymbol{\alpha})=\varepsilon\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathbb{Z}^2}\frac{f_{\boldsymbol{\nu}}\,e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}}}{i(\omega_{01}\nu_1+(\omega_{20}+\varepsilon)\nu_2)}\tag{13.6}$$

and the series in (13.6) (no longer a power series in ε) is really convergent if $\boldsymbol{\omega}=(\omega_{01},\omega_{02}+\varepsilon)$ is a Diophantine vector (by (13.2), because analyticity implies exponential decay of $|f_{\boldsymbol{\nu}}|$). Remarkably for such values of ε the Hamiltonian $\mathcal{H}_\varepsilon$ *is integrable and it is integrated* by the canonical map generated by (13.6), in spite of the fact that (13.6) is obtained, from (13.5), via the *non rigorous sum rule*

$$\sum_{k=0}^\infty z^k=\frac{1}{1-z}\qquad\text{for } z\neq1\tag{13.7}$$

(applied to cases with $|z|\geq1$, which are certainly realized for a dense set of ε's even if $\boldsymbol{\omega}$ is Diophantine because the z's have values $z=\frac{\nu_2}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}$). In other words the integration of the equations is elementary and once performed it becomes apparent that, if $\boldsymbol{\omega}$ is diophantine, the solutions can be rigorously found from (13.6). Note that, for instance, this means that relations like $\sum_{k=0}^\infty 2^k=-1$ are really used to obtain (13.6) from (13.5).

Another extension of Lindstedt series arises in a perturbation of an anisochronous system when asking the question of what happens to the unperturbed invariant tori $\mathcal{T}_{\boldsymbol{\omega}_0}$ on which the spectrum *is resonant*, *i.e.* $\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}=0$ for some $\boldsymbol{\nu}\neq\mathbf{0}$, $\boldsymbol{\nu}\in\mathbb{Z}^\ell$. The result is that even in such case there is a formal power series solutions showing that *at least a few* of the (infinitely many) invariant tori into which $\mathcal{T}_{\boldsymbol{\omega}_0}$ is in turn foliated in the unperturbed case can be formally continued at $\varepsilon \neq 0$, see Section 15.

References: [Po].

14. Quasi periodicity and KAM stability

To discuss more advanced results it is convenient to restrict attention to a special (non trivial) paradigmatic case

$$\mathcal{H}_\varepsilon(\mathbf{A},\boldsymbol{\alpha}) = \frac{1}{2}\mathbf{A}^2 + \varepsilon\, f(\boldsymbol{\alpha}) \tag{14.1}$$

In this simple case (called *Thirring model*: representing ℓ particles on a circle interacting via a potential $\varepsilon f(\boldsymbol{\alpha})$) the equations for the maximal tori (13.3) reduce to equations for the only functions $\mathbf{h}_\varepsilon$:

$$(\boldsymbol{\omega}\cdot\boldsymbol{\partial}_{\boldsymbol{\psi}})^2\mathbf{h}_\varepsilon(\boldsymbol{\psi}) = -\varepsilon\,\boldsymbol{\partial}_{\boldsymbol{\alpha}} f(\boldsymbol{\psi}+\mathbf{h}_\varepsilon(\boldsymbol{\psi})), \qquad \boldsymbol{\psi}\in\mathbb{T}^\ell \tag{14.2}$$

as the second of (13.3) simply becomes the definition of $\mathbf{H}_\varepsilon$ because the *r.h.s.* does not involve $\mathbf{H}_\varepsilon$.

The real problem is therefore whether the formal series considered in Section 13 converge at least for small ε: and the example (13.4) on the Birkhoff series shows that sometimes *sum rules* might be needed in order to give a meaning to the series. In fact whenever a problem (of physical interest) admits a formal power series solution which is not convergent, or which it is not known whether it is convergent, then one should look for sum rules for it.

The modern theory of perturbations starts with the proof of the convergence for ε small enough of the Lindstedt series (Kolmogorov). The general "KAM" result is

Consider the Hamiltonian $\mathcal{H}_\varepsilon(\mathbf{A},\boldsymbol{\alpha}) = h(\mathbf{A}) + \varepsilon f(\mathbf{A},\boldsymbol{\alpha})$, defined in $U = V\times\mathbb{T}^\ell$ with $V\subset\mathbb{R}^\ell$ open and bounded and with $f(\mathbf{A},\boldsymbol{\alpha}), h(\mathbf{A})$ analytic in the closure $\overline{V}\times\mathbb{T}^\ell$ where $h(\mathbf{A})$ is also anisochronous; let $\boldsymbol{\omega}_0 \stackrel{def}{=} \boldsymbol{\omega}(\mathbf{A}_0) = \partial_{\mathbf{A}} h(\mathbf{A}_0)$ and assume that $\boldsymbol{\omega}_0$ satisfies (13.2). Then
(1) there is $\varepsilon_{C,\tau}>0$ such that the Lindstedt series converges for $|\varepsilon|<\varepsilon_{C,\tau}$,
(2) its sum yields two function $\mathbf{H}_\varepsilon(\boldsymbol{\psi}), \mathbf{h}_\varepsilon(\boldsymbol{\psi})$ on $\mathbb{T}^\ell$ which parameterize an invariant torus $\mathcal{T}_{C,\tau}(\mathbf{A}_0,\varepsilon)$,
(3) on $\mathcal{T}_{C,\tau}(\mathbf{A}_0,\varepsilon)$ the motion is $\boldsymbol{\psi}\to\boldsymbol{\psi}+\boldsymbol{\omega}_0 t$, see (13.1).
(4) the set of data in U which belong to invariant tori $\mathcal{T}_{C,\tau}(\mathbf{A}_0,\varepsilon)$ with $\boldsymbol{\omega}(\mathbf{A}_0)$ satisfying (13.2) with prefixed C,τ has complement with volume $<$ const C^{-a} for a suitable $a>0$ and with area also $<$ const C^{-a} on each nontrivial surface of constant energy $\mathcal{H}_\varepsilon = E$.

In other words for ε small the spectra of most unperturbed quasi periodic motions can still be found as spectra of perturbed quasi periodic motions developing on tori which are close to the corresponding unperturbed ones (*i.e.* with the same spectrum).

This is a *stability result*: for instance in systems with two degrees of freedom the invariant tori of dimension 2 which lie on a given energy surface, which has 3 dimensions, will separate the points on the energy surface into the set which is "inside" the torus and the set which is "outside": hence an initial datum starting (say) inside cannot reach the outside. Likewise a point starting between two tori has to stay in between forever: and if the two tori are close this means that motion will stay very localized in action space, with a trajectory accessing only points close to the tori and coming close to all such points, within a distance of the order of the distance between the confining tori. The case of three or more degrees of freedom is quite different, see Sect.17,19.

In the simple case of the rotators system (14.1) the equations for the parametric representation of the tori are the (14.2). The latter bear some analogy with the easier problem in (12.1): but the (14.2) are ℓ equations instead of one and they are differential equations rather than ordinary equations. Furthermore the function $f(\boldsymbol{\alpha})$ which plays here the role of $c(\lambda)$ in (12.1) has Fourier coefficients $f_{\boldsymbol{\nu}}$ with no restrictions on $\boldsymbol{\nu}$, while the Fourier coefficiens c_ν for c in (12.1) do not vanish only for $\nu=\pm1$.

The above differences are, to some extent, "minor" and the power series solution to (14.2) can be constructed *by the same algorithm* used in the case of (12.1): namely one forms trees as in Fig. 5 with the harmonic labels $\nu_v\in\mathbb{Z}$ replaced by $\boldsymbol{\nu}_v\in\mathbb{Z}^\ell$ (still to be thought of as possible harmonic indices in the Fourier expansion of the perturbing function f). All other labels affixed to the trees in Sec. 11 will be the same. In particular the *current flowing on a branch* $l=v'v$ will be defined as the sum of the harmonics of the nodes $w\le v$ preceding v:

$$\boldsymbol{\nu}(l)\stackrel{def}{=}\sum_{w\le v}\boldsymbol{\nu}_w \tag{14.3}$$

and we call $\boldsymbol{\nu}(\theta)$ the current flowing in the root branch.

This time the value $\mathrm{Val}(\theta)$ of a tree has to be defined differently because the equation (14.2) to be solved contains the differential operator $(\boldsymbol{\omega}_0\cdot\boldsymbol{\partial}_{\boldsymbol{\psi}})^2$ which in Fourier transform becomes multiplication of the Fourier component with harmonic $\boldsymbol{\nu}$ by $(i\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2$.

The variation due to the presence of the operator $(\boldsymbol{\omega}_0\cdot\boldsymbol{\partial}_{\boldsymbol{\psi}})^2$ and the necessity of its inversion in the evaluation of $\mathbf{u}\cdot\mathbf{h}^{(k)}_{\boldsymbol{\nu}}$, *i.e.* of the component of $\mathbf{h}^{(k)}_{\boldsymbol{\nu}}$ along an arbitrary unit vector $\mathbf{u}$, is nevertheless quite simple: the value of a tree graph θ of order k (*i.e.* with k nodes and k branches) has to be defined by (cf. (12.3))

$$\mathrm{Val}(\theta)\stackrel{def}{=}\frac{-i\,(-1)^k}{k!}\Big(\prod_{lines\,l=v'v}\frac{\boldsymbol{\nu}_{v'}\cdot\boldsymbol{\nu}_v}{\big(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(l)\big)^2}\Big)\Big(\prod_{nodes\,v}f_{\boldsymbol{\nu}_v}\Big) \tag{14.4}$$

where the $\boldsymbol{\nu}_{v'}$ appearing in the factor relative to the root line rv from the first node v to the root r, see Fig. 5, is interpreted as an unit vector $\mathbf{u}$ (it was interpreted as 1 in the "one dimensional" case (12.1)). The (14.4) makes sense only for trees in which no line carries $\mathbf{0}$ current. Then the component along $\mathbf{u}$ (the harmonic label attached to the root of a tree) of $\mathbf{h}^{(k)}$ is given, see also (12.4), by

$$\mathbf{u}\cdot\mathbf{h}_{\boldsymbol{\nu}}^{(k)}=\sum_{\substack{\theta,\,\boldsymbol{\nu}(\theta)=\boldsymbol{\nu}\\ order(\theta)=k}}^{*}\mathrm{Val}(\theta) \tag{14.5}$$

where the $*$ means that the sum is only over trees in which a non zero current $\boldsymbol{\nu}(l)$ flows on the lines $l\in\theta$. The quantity $\mathbf{u}\cdot\mathbf{h}_{\mathbf{0}}^{(k)}$ will be $\mathbf{0}$, see Section 13.

In the case of (12.1) zero current lines could appear: *but* the contributions from tree graphs containing at least one zero current line would cancel. In the present case the statement that the above algorithm actually gives $\mathbf{h}_{\boldsymbol{\nu}}^{(k)}$ *by simply ignoring* trees with lines with $\mathbf{0}$ current is non trivial. It has been POINCARÉ's contribution to the theory of Lindstedt series to show that *even in the general case*, cf.(13.3), the equations for the invariant tori can be solved by a formal power series. The (14.5) is proved by induction on k after checking it for the first few orders.

The algorithm just described leading to (14.4) can be extended to the case of the general Hamiltonian considered in the KAM theorem.

The convergence proof is more delicate than the (elementary) one for the equation (12.1). In fact the values of trees of order k can give large contributions to $\mathbf{h}_{\boldsymbol{\nu}}^{(k)}$: because the "new" factors $\big(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(l)\big)^2$, although not zero, can be quite small and their small size can overwhelm the smallness of the factors $f_{\boldsymbol{\nu}}$ and ε: in fact even if f is a trigonometric polynomial (so that $f_{\boldsymbol{\nu}}$ vanishes identically for $|\boldsymbol{\nu}|$ large enough) the currents flowing in the branches can be very large, of the order of the number k of nodes in the tree, see (14.3).

This is called the *small divisors* problem. The key to its solution goes back to a related work (SIEGEL) which shows that

Consider the contribution to the sum in (14.3) from graphs θ in which no pairs of lines which lie on the same path to the root carry the same current and, furthermore, the node harmonics are bounded by $|\boldsymbol{\nu}|\le N$ for some N. Then the number of lines ℓ in θ with divisor $\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\ell$ satisfying $2^{-n}<C\,|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\ell|\le 2^{-n+1}$ does not exceed $4\,N\,k2^{-n/\tau}$.

Hence, setting $F\overset{def}{=}C^2\max_{|\boldsymbol{\nu}|\le N}|f_{\boldsymbol{\nu}}|$, the corresponding $\mathrm{Val}(\theta)$ can be bounded by

$$\frac{1}{k!}F^kN^{2k}\prod_{n=0}^{\infty}2^{2n\,(4k2^{-n/\tau})}\overset{def}{=}\frac{1}{k!}B^k,\qquad B=FN^2 2^{\sum_n 8n2^{-n/\tau}} \tag{14.6}$$

since the product is convergent. In the case in which f is a trigonometric polynomial of degree N the above *restricted* contributions to $\mathbf{u}\cdot\mathbf{h}_{\boldsymbol{\nu}}^{(k)}$ would generate a convergent series for ε small enough. In fact the number of trees is bounded, as in Sec. 12, by $k!4^k(2N+1)^{\ell k}$ so that the series $\sum_{\boldsymbol{\nu}}|\varepsilon|^k\,|\mathbf{u}\cdot\mathbf{h}_{\boldsymbol{\nu}}^{(k)}|$ would converge for ε small (*i.e.* $|\varepsilon|<(B\cdot 4(2N+1)^\ell)^{-1}$).

Given this comment the analysis of the "remaining contributions" becomes the real problem and it requires new ideas because among the excluded trees there are some simple k-th order trees whose value alone, if considered separately from the other contributions, would generate a factorially divergent power series in ε.

However the contributions of all large valued trees of order k can be shown to cancel: although not exactly (unlike the case of the elementary problem of Sec. 12, where the cancellation is not necessary for the proof, in spite of its exact occurrence), but enough so that in spite of the existence of exceedingly large values of individual tree graphs their total sum can still be bounded by a constant to the power k so that the power series actually converges for ε small enough. The idea is discussed in Appendix A1.

References: [Po], [Ko55], [Mo62], [Ar68].

15. Resonances and their stability

A quasi periodic motion with r rationally independent frequencies is called *resonant* if r is strictly less than the number ℓ of degrees of freedom. The difference $s = \ell - r$ is the *degree* of the resonance.

Of particular interest are the cases of a perturbation of an integrable system in which resonant motions take place.

A typical example is the *n-body problem* which studies the mutual perturbations of the motions of $n-1$ particles gravitating around a more massive particle. If the particles masses can be considered negligible the system will consist of $n-1$ central Keplerian motions: it will therefore have $\ell = 3(n-1)$ degrees of freedom. In general, only one frequency per body occurs in absence of the perturbations (the period of the Keplerian orbit): hence $r \le n-1$ and $s \ge 2(n-1)$ (or in the planar case $s \ge (n-1)$) with equality holding when the periods are rationally independent.

Another example is the rigid body with a fixed point perturbed by a conservative force: in this case the unperturbed system has 3 degrees of freedom but, in general, only two frequencies, see Section 9 after (9.10).

Furthermore in the above examples there is the possibility that the independent frequencies assume, for special initial data, values which are rationally related, giving rise to resonances of even higher order (*i.e.* with smaller values of r).

In an integrable anisochronous system resonant motions will be dense in phase space because the frequencies $\boldsymbol{\omega}(\mathbf{A})$ will vary as much as the actions and therefore resonances of any order (*i.e.* any $r < \ell$) will be dense in phase space: in particular the periodic motions (*i.e.* the *highest order resonances*) will be dense.

Resonances, in integrable systems, can arise in *a priori stable* integrable systems and in *a priori unstable* systems: the first are systems whose Hamiltonian admits canonical action–angle coordinates $(\mathbf{A},\boldsymbol{\alpha}) \in U \times \mathbb{T}^\ell$ with $U \subset \mathbb{R}^\ell$ open, while the second are systems whose Hamiltonian has, in suitable *local* canonical coordinates, the form

$$\mathcal{H}_0(\mathbf{A}) + \sum_{i=1}^{s_1} \frac{1}{2}(p_i^2 - \lambda_i^2 q_i^2) + \sum_{j=1}^{s_2} \frac{1}{2}(\pi_j^2 + \mu_j^2 \kappa_j^2), \qquad \lambda_i, \mu_j > 0 \tag{15.1}$$

where $(\mathbf{A},\boldsymbol{\alpha}) \in U \times \mathbb{T}^r$, $U \in \mathbb{R}^r$, $(\mathbf{p},\mathbf{q}) \in V \subset \mathbb{R}^{2s_1}$, $(\boldsymbol{\pi},\boldsymbol{\kappa}) \in V' \subset \mathbb{R}^{2s_2}$ with V, V' neighborhoods of the origin and $\ell = r + s_1 + s_2$, $s_i \ge 0, s_1 + s_2 > 0$ and $\pm\sqrt{\lambda_j}, \pm\sqrt{\mu_j}$ are called *Lyapunov coefficients* of the resonance. The perturbations considered are supposed to have the form $\varepsilon f(\mathbf{A},\boldsymbol{\alpha},\mathbf{p},\mathbf{q},\boldsymbol{\pi},\boldsymbol{\kappa})$. The denomination of *a priori* stable or unstable refers to the properties of the "*a priori* given unperturbed Hamiltonian". The name of *a priori* unstable is certainly appropriate if $s_1 > 0$: here also $s_1 = 0$ is allowed for notational convenience implying that the Lyapunov coefficients in *a priori* unstable cases are all of order 1 (whether real, λ_j or imaginary $i\sqrt{\mu_j}$). In other words the *a priori* stable case, $s_1 = s_2 = 0$ in (15.1), is the only excluded case. Of course the stability properties of the motions when a perturbation acts will depend on the perturbation *in both cases*.

The *a priori stable* systems have usually a great variety of resonances (*e.g.* in the anisochronous case resonances of any dimension are dense). The *a priori unstable* systems have (among possible other resonances) some *very special* r-dimensional resonances occurring when the *unstable coordinates* $(\mathbf{p},\mathbf{q})$ and $(\boldsymbol{\pi},\boldsymbol{\kappa})$ are zero and the frequencies of the r action–angle coordinates are rationally independent.

In the first case, *a priori* stable, the general question is whether the resonant motions, which form invariant tori of dimension r arranged into families that fill ℓ dimensional invariant tori, continue to exists, in presence of small enough perturbations $\varepsilon f(\mathbf{A},\boldsymbol{\alpha})$, on slightly deformed invariant tori. Similar questions can be asked in the *a priori* unstable cases. To examine more closely the matter consider the formulation of the simplest problems.

A priori stable resonances: More precisely let $\{\mathbf{A}_0\}\times\mathbb{T}^\ell$ be the unperturbed invariant torus $\mathcal{T}_{\mathbf{A}_0}$ with spectrum $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(\mathbf{A}_0) = \partial_{\mathbf{A}}\mathcal{H}_0(\mathbf{A}_0)$ with only r rationally independent components. For simplicity suppose that $\boldsymbol{\omega}_0 = (\omega_1,\ldots,\omega_r,0,\ldots,0) \stackrel{def}{=} (\boldsymbol{\omega},\mathbf{0})$ with $\boldsymbol{\omega}\in\mathbb{R}^r$. The more general case in which $\boldsymbol{\omega}$ has only r rationally independent components can be reduced to special case above by a canonical linear change of coordinates at the price of changing the $\mathcal{H}_0$ to a new one, still quadratic in the actions but containing mixed products A_iB_j: the proofs of the results that are discussed here would not be really affected by such more general form of $\mathcal{H}$.

It is convenient to distinguish between the "fast" angles $\alpha_1,\ldots,\alpha_r$ and the "resonant" angles $\alpha_{r+1},\ldots,\alpha_\ell$ (also called "slow" or "secular") and call $\boldsymbol{\alpha} = (\boldsymbol{\alpha}',\boldsymbol{\beta})$ with $\boldsymbol{\alpha}'\in\mathbb{T}^r$ and $\boldsymbol{\beta}\in\mathbb{T}^s$. Likewise we distinguish the fast actions $\mathbf{A}' = (A_1,\ldots,A_r)$ and the resonant ones $A_{r+1},\ldots,A_\ell$ and set $\mathbf{A} = (\mathbf{A}',\mathbf{B})$ with $\mathbf{A}'\in\mathbb{R}^r$ and $\mathbf{B}\in\mathbb{R}^s$.

Therefore the torus $\mathcal{T}_{\mathbf{A}_0}$, $\mathbf{A}_0 = (\mathbf{A}'_0,\mathbf{B}_0)$, is in turn a *continuum of invariant tori* $\mathcal{T}_{\mathbf{A}_0,\boldsymbol{\beta}}$ with trivial parametric equations: $\boldsymbol{\beta}$ fixed, $\boldsymbol{\alpha}' = \boldsymbol{\psi}$, $\boldsymbol{\psi}\in\mathbb{T}^r$, and $\mathbf{A}' = \mathbf{A}'_0$, $\mathbf{B} = \mathbf{B}_0$. On each of them the motion is: $\mathbf{A}',\mathbf{B},\boldsymbol{\beta}$ constant and $\boldsymbol{\alpha}'\to\boldsymbol{\alpha}'+\boldsymbol{\omega}\,t$, with rationally independent $\boldsymbol{\omega}\in\mathbb{R}^r$.

Then the natural question is whether there exist functions $\mathbf{h}_\varepsilon,\mathbf{k}_\varepsilon,\mathbf{H}_\varepsilon,\mathbf{K}_\varepsilon$ smooth in ε near $\varepsilon = 0$ and in $\boldsymbol{\psi}\in\mathbb{T}^r$, vanishing for $\varepsilon = 0$, and such that the torus $\mathcal{T}_{\mathbf{A}_0,\boldsymbol{\beta}_0,\varepsilon}$ with parametric equations

$$\begin{aligned}\mathbf{A}' =&\mathbf{A}'_0 + \mathbf{H}_\varepsilon(\boldsymbol{\psi}), \qquad \boldsymbol{\alpha}' = \boldsymbol{\psi} + \mathbf{h}_\varepsilon(\boldsymbol{\psi}),\\ \mathbf{B} =&\mathbf{B}_0 + \mathbf{K}_\varepsilon(\boldsymbol{\psi}), \qquad \boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{k}_\varepsilon(\boldsymbol{\psi}),\end{aligned} \qquad \boldsymbol{\psi}\in\mathbb{T}^r \tag{15.2}$$

is invariant for the motions with Hamiltonian $\mathcal{H}_\varepsilon(\mathbf{A},\boldsymbol{\alpha}) = \frac{1}{2}{\mathbf{A}'}^2 + \frac{1}{2}\mathbf{B}^2 + \varepsilon f(\boldsymbol{\alpha}',\boldsymbol{\beta})$ and the motions on it are $\boldsymbol{\psi}\to\boldsymbol{\psi}+\boldsymbol{\omega}\,t$. The above property, when satisfied, is summarized by saying that the unperturbed resonant motions $\mathbf{A}=(\mathbf{A}'_0,\mathbf{B}_0)$, $\boldsymbol{\alpha}=(\boldsymbol{\alpha}'_0+\boldsymbol{\omega}'t,\boldsymbol{\beta}_0)$ can be *continued* in presence of perturbation εf, for small ε, to quasi periodic motions with the same spectrum and on a slightly deformed torus $\mathcal{T}_{\mathbf{A}'_0,\boldsymbol{\beta}_0,\varepsilon}$.

A priori unstable resonances: here the question is whether the *special* invariant tori continue to exist in presence of small enough perturbations, of course slightly deformed. This means asking whether, given $\mathbf{A}_0$ such that $\boldsymbol{\omega}(\mathbf{A}_0)=\partial_{\mathbf{A}}\mathcal{H}_0(\mathbf{A}_0)$ has rationally independent components, there are functions $(\mathbf{H}_\varepsilon(\boldsymbol{\psi}),\,\mathbf{h}_\varepsilon(\boldsymbol{\psi}))$, $(\mathbf{P}_\varepsilon(\boldsymbol{\psi}),\mathbf{Q}_\varepsilon(\boldsymbol{\psi}))$ and $(\boldsymbol{\Pi}_\varepsilon(\boldsymbol{\psi}),\mathbf{K}_\varepsilon(\boldsymbol{\psi}))$ smooth in ε near $\varepsilon=0$, vanishing for $\varepsilon=0$, analytic in $\boldsymbol{\psi}\in\mathbb{T}^r$ and such that the r-dimensional surface

$$\begin{aligned}\mathbf{A}=&\mathbf{A}_0+\mathbf{H}_\varepsilon(\boldsymbol{\psi}), &\qquad \boldsymbol{\alpha}=\boldsymbol{\psi}+\mathbf{h}_\varepsilon(\boldsymbol{\psi})&\\ \mathbf{p}=&\mathbf{P}_\varepsilon(\boldsymbol{\psi}), &\qquad \mathbf{q}=\mathbf{Q}_\varepsilon(\boldsymbol{\psi})&\qquad \boldsymbol{\psi}\in\mathbb{T}^r\\ \boldsymbol{\pi}=&\boldsymbol{\Pi}_\varepsilon(\boldsymbol{\psi}), &\qquad \boldsymbol{\kappa}=\mathbf{K}_\varepsilon(\boldsymbol{\psi})&\end{aligned}\tag{15.3}$$

is an invariant torus $\mathcal{T}_{\mathbf{A}_0,\varepsilon}$ on which the motion is $\boldsymbol{\psi}\to\boldsymbol{\psi}+\boldsymbol{\omega}(\mathbf{A}_0)\,t$. Again the above property is summarized by saying that the unperturbed special resonant motions can be *continued* in presence of perturbation εf for small ε to quasi periodic motions with the same spectrum and on a slightly deformed torus $\mathcal{T}_{\mathbf{A}_0,\varepsilon}$.

Some answers to the above questions are presented in the following section.

References: [GBG04].

16. Resonances and Lindstedt series

We discuss the equations (15.2) in the paradigmatic case in which the Hamiltonian $\mathcal{H}_0(\mathbf{A})$ is $\frac{1}{2}\mathbf{A}^2$ (cf. (14.1)). It will be $\boldsymbol{\omega}(\mathbf{A}')\equiv\mathbf{A}'$ so that $\mathbf{A}_0=\boldsymbol{\omega},\mathbf{B}_0=\mathbf{0}$ and the perturbation $f(\boldsymbol{\alpha})$ can be considered as a function of $\boldsymbol{\alpha}=(\boldsymbol{\alpha}',\boldsymbol{\beta})$: let $\overline{f}(\boldsymbol{\beta})$ be defined as its average over $\boldsymbol{\alpha}'$. The determination of the invariant torus of dimension r which can be continued in the sense discussed in Sect. 15 is easily understood in this case.

A resonant invariant torus which, among the tori $\mathcal{T}_{\mathbf{A}_0,\boldsymbol{\beta}}$, has parameric equations that can be continued as a *formal powers series* in ε is the torus $\mathcal{T}_{\mathbf{A}_0,\boldsymbol{\beta}_0}$ with $\boldsymbol{\beta}_0$ a stationarity point for $\overline{f}(\boldsymbol{\beta})$, *i.e.* an *equilibrium point for the average perturbation*: $\partial_{\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)=0$. In fact the following theorem holds

If $\boldsymbol{\omega}\in\mathbb{R}^r$ satisfies a Diophantine property and if $\boldsymbol{\beta}_0$ is a nondegenerate stationarity point for the "fast angle average" $\overline{f}(\boldsymbol{\beta})$ (i.e. such that $\det\partial^2_{\boldsymbol{\beta}\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)\neq0$), then the equations for the functions $\mathbf{h}_\varepsilon,\mathbf{k}_\varepsilon$:

$$\begin{aligned}(\boldsymbol{\omega}\cdot\partial_{\boldsymbol{\psi}})^2\mathbf{h}_\varepsilon(\boldsymbol{\psi})=&-\varepsilon\partial_{\boldsymbol{\alpha}'}f(\boldsymbol{\psi}+\mathbf{h}_\varepsilon(\boldsymbol{\psi}),\boldsymbol{\beta}_0+\mathbf{k}_\varepsilon(\boldsymbol{\psi}))\\ (\boldsymbol{\omega}\cdot\partial_{\boldsymbol{\psi}})^2\mathbf{k}_\varepsilon(\boldsymbol{\psi})=&-\varepsilon\partial_{\boldsymbol{\beta}}f(\boldsymbol{\psi}+\mathbf{h}_\varepsilon(\boldsymbol{\psi})+\mathbf{k}_\varepsilon(\boldsymbol{\psi}))\end{aligned}\tag{16.1}$$

can be formally solved in powers of ε.

Given the simplicity of the Hamiltonian that we are considering, *i.e.* (14.1), it is not necessary to discuss the functions $\mathbf{H}_\varepsilon, \mathbf{K}_\varepsilon$ because the equations that they should obey reduce to their definitions as in the case of Sec. 14, and for the same reason.

In other words also the resonant tori admit a Lindstedt series representation. *It is however very unlikely that the series are, in general, convergent.*

Physically this *new* aspect is due to the fact that the linearization of the motion near the torus $\mathcal{T}_{\mathbf{A}_0,\boldsymbol{\beta}_0}$ introduces *oscillatory motions* around $\mathcal{T}_{\mathbf{A}'_0,\boldsymbol{\beta}_0}$ with frequencies proportional to the square roots of the *positive* eigenvalues of the matrix $\varepsilon\partial^2_{\boldsymbol{\beta}\,\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)$: therefore it is naively expected that it has to be necessary that a Diophantine property be required on the vector $(\boldsymbol{\omega}, \sqrt{\varepsilon\mu_1}, \ldots)$ where $\varepsilon\mu_j$ are the positive eigenvalues. Hence some values of ε, those for which $(\boldsymbol{\omega}, \sqrt{\varepsilon\mu_1}, \ldots)$ is not a Diophantine vector or is too close to a non Diophantine vector, should be excluded or at least should be expected to generate difficulties. Note that the problem arises no matter what is supposed on the non degenerate matrix $\partial^2_{\boldsymbol{\beta}\,\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)$ since ε can have either sign; and no matter how $|\varepsilon|$ is supposed small. But we can expect that if the matrix $\partial^2_{\boldsymbol{\beta}\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)$ is (say) positive definite (*i.e.* $\boldsymbol{\beta}_0$ is a minimum point for $\overline{f}(\boldsymbol{\beta})$) then the problem should be easier for $\varepsilon < 0$ and viceversa if $\boldsymbol{\beta}_0$ is a maximum it should be easier for $\varepsilon > 0$ (*i.e.* in the cases in which the eigenvalues of $\varepsilon\partial^2_{\boldsymbol{\beta}\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)$ are negative and their roots do not have the interpretation of frequencies).

Technically the sums of the formal series can be given (so far) a meaning only via summation rules involving divergent series: typically one has to identify in the formal expressions (denumerably many) geometric series which although divergent can be given a meaning by applying the rule (13.7). Since the rule can only be applied if $z \neq 1$ this *leads to conditions on the parameter* ε, in order to exclude that the various z that have to be considered are too close to 1. Hence this stability result turns out to be *rather different* from the KAM result for the maximal tori. Namely the series can be given a meaning via summation rules provided f and $\boldsymbol{\beta}_0$ satisfy certain additional conditions and provided certain values of ε are excluded. An example of a theorem is the following:

Given the Hamiltonian (14.1) and a resonant torus $\mathcal{T}_{\mathbf{A}'_0,\boldsymbol{\beta}_0}$ with $\boldsymbol{\omega} = \mathbf{A}'_0 \in \mathbb{R}^r$ satisfying a Diophantine property let $\boldsymbol{\beta}_0$ be a non degenerate maximum point for the average potential $\overline{f}(\boldsymbol{\beta}) \stackrel{def}{=} (2\pi)^{-r}\int_{\mathbb{T}^r} f(\boldsymbol{\alpha}', \boldsymbol{\beta}) d^r\boldsymbol{\alpha}'$. Consider the Lindstedt series solution for the equations (16.1) of the perturbed resonant torus with spectrum $(\boldsymbol{\omega}, \mathbf{0})$. It is possible to express the single n-th order term of the series as a sum of many terms and then rearrange the series thus obtained so that the resummed series converges for ε in a domain $\mathcal{E}$ which contains a segment $[0, \varepsilon_0]$ and also contains a subset of $[-\varepsilon_0, 0]$ which, although with open dense complement, is so large that it has 0 as a Lebesgue density point. Furthermore the resummed series for $\mathbf{h}_\varepsilon, \mathbf{k}_\varepsilon$ define an invariant r dimensional analytic torus with spectrum $\boldsymbol{\omega}$.

More generally if $\boldsymbol{\beta}_0$ is only a nondegenerate stationarity point for $\overline{f}(\boldsymbol{\beta})$ the domain of definition of the resummed series is a set $\mathcal{E} \subset [-\varepsilon_0, \varepsilon_0]$ which on *both sides* of the origin has an open dense complement although it has 0 as a Lebesgue density point.

The above theorem can be naturally extended to the general case in which the Hamiltonian is the most general perturbation of an anisochronous integrable system $\mathcal{H}_\varepsilon(\mathbf{A}, \boldsymbol{\alpha}) =$

$h(\mathbf{A})+\varepsilon f(\mathbf{A},\boldsymbol{\alpha})$ if $\partial^2_{\mathbf{A}\mathbf{A}}h$ is a non singular matrix and the resonance arises from a spectrum $\boldsymbol{\omega}(\mathbf{A}_0)$ which has r independent components (while the remaining are not necessarily $\mathbf{0}$).

We see that the convergence is a delicate problem for the Lindstedt series for nearly integrable resonant motions. They might even be divergent (mathematically a proof of divergence is an open problem but it is a very resonable conjecture in view of the above physical interpretation), *nevertheless* the above theorem shows that sum rules can be given that "sometimes", *i.e.* for ε in a large set near $\varepsilon = 0$, yield a true solution to the problem.

This is reminiscent of the phenomenon met in discussing perturbations of isochronous systems in (13.4), but it is a much more complex situation. And it leaves many open problems: *foremost of them is the question of uniqueness.* The sum rules of divergent series always contain some arbitrary choices: which lead to doubt about the uniqueness of the functions parameterizing the invariant tori constructed in this way. It might even be that the convergence set $\mathcal{E}$ may depend upon the arbitrary choices, and that considering several of them no ε with $|\varepsilon| < \varepsilon_0$ is left out.

The case of *a priori* unstable systems has also been widely studied: in this case too resonances with Diophantine r-dimensional spectrum $\boldsymbol{\omega}$ are considered. However in the case $s_2 = 0$ (called *a priori* unstable *hyperbolic resonance*) the Lindstedt series can be shown to be convergent while in the case $s_1 = 0$ (called *a priori* unstable *elliptic resonance*) or in the *mixed* cases $s_1, s_2 > 0$ extra conditions are needed. They involve $\boldsymbol{\omega}$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{s_2})$, cf. (15.1), and properties of the perturbations as well. It is also possible to study a slightly different problem: namely to look for conditions on $\boldsymbol{\omega}, \boldsymbol{\mu}, f$ which imply that for small ε invariant tori with spectrum ε-dependent but *close*, in a suitable sense, to $\boldsymbol{\omega}$ exist.

The literature is vast but it seems fair to say that, given the above comments, *particularly those concerning uniqueness and analyticity*, the situation is still quite unsatisfactory.

References: [GBG04].

17. Diffusion in phase space

The KAM theorem implies that a perturbation of an analytic anisochronous integrable system, *i.e.* with an analytic Hamiltonian $\mathcal{H}_\varepsilon(\mathbf{A},\boldsymbol{\alpha}) = \mathcal{H}_0(\mathbf{A}) + \varepsilon f(\mathbf{A},\boldsymbol{\alpha})$ and non degenerate Hessian matrix $\partial^2_{\mathbf{A}\mathbf{A}}h(\mathbf{A})$, generates large families of maximal invariant tori. Such tori lie on the energy surfaces but do not have codimension 1 on them, *i.e.* do not split the $(2\ell-1)$–dimensional energy surfaces into disconnected regions *except*, of course, in the case of 2–degrees of freedom systems, see Sect.14..

Therefore there might exist trajectories with initial data close in action space to $\mathbf{A}^i$ which reach phase space points close in action space to $\mathbf{A}^f \neq \mathbf{A}^i$ for $\varepsilon \neq 0$, *no matter how small.* Such *diffusion* phenomenon would occurr in spite of the fact that the corresponding trajectory has to move in a space in which very close to each $\{\mathbf{A}\}\times\mathbb{T}^\ell$ there is an invariant surface on which points move keeping $\mathbf{A}$ constant within $O(\varepsilon)$, which for ε small can be $\ll |\mathbf{A}^f - \mathbf{A}^i|$.

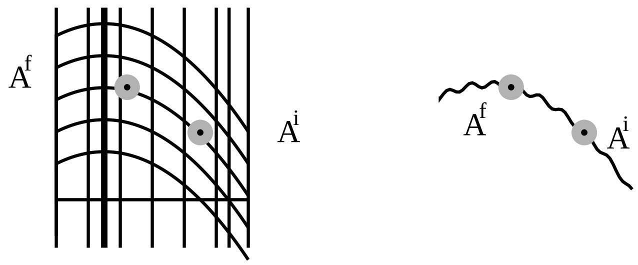
In *a priori* unstable systems (cf. Sect. 15) with $s_1 = 1, s_2 = 0$ it is not difficult to see that the corresponding phenomenon can actually happen: the paradigmatic example (ARNOLD) is the *a priori* unstable system

$$\mathcal{H}_\varepsilon = \frac{A_1^2}{2} + A_2 + \frac{p^2}{2} + g\,(\cos q - 1) + \varepsilon\,(\cos\alpha_1 + \sin\alpha_2)(\cos q - 1) \tag{17.1}$$

This is a system describing a motion of a "pendulum" ((p,q) coordinates) interacting with a "rotating wheel" ((A_1,α_1) coordinates) and a "clock" ((A_2,α_2) coordinates) *a priori* unstable near the points $p=0, q=0,2\pi$, ($s_1=1, s_2=0$, $\lambda_1=\sqrt{g}$, cf. (15.1)). And it can be proved that on the energy surface of energy E and for each $\varepsilon\ne0$ small enough (no matter how small) *there are initial data with action coordinates close to* $\mathbf{A}^i=(A_1^i,A_2^i)$ *with* $\frac12 {A_1^i}^2+A_2^i$ *close to* E *eventually evolving to a datum* $\mathbf{A}'=(A_1',A_2')$ with A_1' at distance from A_1^f smaller than an *arbitrarily* prefixed distance (of course with energy E). *Furthermore* during the whole process the pendulum energy stays close to 0 within $o(\varepsilon)$ (*i.e.* the pendulum swings following closely the unperturbed separatrices).

In other words (17.1) describes a machine (the pendulum) which, working approximately in a cycle, extracts energy from a reservoir (the clock) to transfer it to a mechanical device (the wheel). The statement that diffusion is possible means that the machine can work as soon as $\varepsilon\ne0$, if the initial actions and the initial phases (*i.e.* α_1,α_2,p,q) are suitably tuned (as functions of ε).

The peculiarity of the system (17.1) is that the unperturbed pendulum fixed points $P_\pm$ (*i.e.* the equilibria $p=0,q=0,2\pi$) *remain unstable equilibria even when* $\varepsilon\ne0$: and this is an important simplifying feature.



(6)

Fig. 6: The first drawing represents the $\varepsilon=0$ geometry: the "partial energy" lines are parabolae, $\frac12A_1^2+A_2=const.$ The vertical lines are the resonances $A_1=rational$ (*i.e.* $\nu_1A_1+\nu_2=0$). The disks are neighborhoods of the points $\mathbf{A}^i$ and $\mathbf{A}^f$ (the dots at their centers). The second drawing ($\varepsilon\ne0$) is an artist rendering of a trajectory in $\mathbf{A}$ space, driven by the pendulum swings to accelerate the wheel from A_1^i to A_1^f at the expenses of the clock energy, sneaking through invariant tori (not represented and approximately) located "away" from the intersections between resonances and partial energy lines (a dense set, however). The pendulum coordinates are not shown: its energy stays close to 0 within a power of ε. Hence the pendulum swings staying close to the separatrix. The oscillations symbolize the wiggly behavior of the partial energy $\frac12A_1^2+A_2$ in the process of sneaking between invariant tori which, because of their invariance, would be impossible without the pendulum. The energy $\frac12A_1^2$ of the wheel increases slightly at each pendulum swing: accurate estimates yield an increase of the wheel speed A_1 of the order of $\varepsilon/\log\varepsilon^{-1}$ at each swing of the pendulum implying a transition time of the order of $g^{-\frac12}\varepsilon^{-1}\log\varepsilon^{-1}$.

It permits bypassing the obstacle, arising in the analysis of more general cases, represented by the resonance surfaces consisting in the $\mathbf{A}$'s with $A_1\nu_1+\nu_2=0$: the latter correspond to harmonics (ν_1,ν_2) present in the perturbing function, *i.e.* the harmonics which would lead to division by zero in an attempt to construct (as necessary in the proof by Arnold's method) the parametric equations of the perturbed invariant tori with action close such $\mathbf{A}$'s. In the case of (17.1) the problem arises only on the resonance marked in Fig.6 by a heavy line *i.e.* $A_1=0$ corresponding to $\cos\alpha_1$ in (17.1).

If $\varepsilon=0$ the points P_- with $p=0, q=0$ and the point P_+ with $p=0, q=2\pi$ are both unstable equilibria (and they are of course the same point, if q is an angular variable). The unstable manifold (it is a curve) of P_+ coincides with the stable manifold of P_- and viceversa. So that the unperturbed system admits non trivial motions leading from P_+ to P_- and from P_- to P_+, both in a biinfinite time interval $(-\infty,\infty)$: the p,q variables describe a pendulum and $P_\pm$ are its unstable equilibria which are connected by the separatrices (which constitute the 0-energy surfaces for the pendulum).

The latter property remains true for more general *a priori* unstable Hamiltonians

$$\mathcal{H}_\varepsilon=\mathcal{H}_0(\mathbf{A})+\mathcal{H}_u(p,q)+\varepsilon\, f(\mathbf{A},\boldsymbol{\alpha},p,q),\qquad \text{in } (U\times\mathbb{T}^\ell)\times(\mathbb{R}^2) \tag{17.2}$$

where $\mathcal{H}_u$ is a one dimensional Hamiltonian which has two unstable equilibrium points P_+ and P_- linearly repulsive in one direction and linearly attractive in another which are connected by two *heteroclinic* trajectories which, as time tends to $\pm\infty$, approach P_- and P_+ and viceversa.

Actually the points need not be different but, if coinciding, the trajectories linking them must be nontrivial: in the case (17.1) the variable q can be considered an angle and then P_+ and P_- would coincide (but are connected by nontrivial trajectories, *i.e.* by trajectories that visit points different from $P_\pm$). Such trajectories are called *heteroclinic* if $P_+\ne P_-$ and *homoclinc* if $P_+=P_-$.

In the general case besides the homoclinicity (or heteroclinicity) condition certain weak genericity conditions, automatically satisfied in the example (17.1), have to be imposed in order to show that given $\mathbf{A}^i$ and $\mathbf{A}^f$ with the same unperturbed energy E one can find, for all ε small enough but not equal to 0, initial data (ε-dependent) with actions arbitrarily close to $\mathbf{A}^i$ which evolve to data with actions arbitrarily close to $\mathbf{A}^f$. This is a phenomenon called *Arnold diffusion.* Simple sufficient conditions for a transition from near $\mathbf{A}^i$ to near $\mathbf{A}^f$ are expressed by the following result

Given the Hamiltonian (17.2) with $\mathcal{H}_u$ admitting two hyperbolic fixed points $P_\pm$ with heteroclinic connections, $t\to(p_a(t),q_a(t))$, $a=1,2$, suppose that
(1) On the unperturbed energy surface of energy $E=\mathcal{H}(\mathbf{A}^i)+\mathcal{H}_u(P_\pm)$ there is a regular curve $\gamma\,:\, s\to\mathbf{A}(s)$ joining $\mathbf{A}^i$ to $\mathbf{A}^f$ such that the unperturbed tori $\{\mathbf{A}(s)\}\times\mathbb{T}^\ell$ can be continued at $\varepsilon\ne0$ into invariant tori $\mathcal{T}_{\mathbf{A}(s),\varepsilon}$ for a set of values of s which fills the curve γ leaving only gaps of size of order $o(\varepsilon)$.
(2) The $\ell\times\ell$ matrix D_{ij} of the second derivatives of the integral of f over the heteroclinic motions is not degenerate, i.e.

$$|\det D|=\Big|\det\,\Big(\int_{-\infty}^{\infty}dt\,\partial_{\alpha_i\alpha_j}f(\mathbf{A},\boldsymbol{\alpha}+\boldsymbol{\omega}(\mathbf{A})\,t,p_a(t),q_a(t))\Big)\Big|>c>0 \tag{17.3}$$

for all $\mathbf{A}$*'s on the curve* γ *and all* $\boldsymbol{\alpha} \in \mathbb{T}^2$.
Given arbitrarily $\rho > 0$, *for* $\varepsilon \neq 0$ *small enough there are initial data with action and energy closer than* ρ *to* $\mathbf{A}^i$ *and* E, *respectively, which after a long enough time acquire an action closer than* ρ *to* $\mathbf{A}^f$ *(keeping the initial energy).*

The above two conditions can be shown to hold generically for many pairs $\mathbf{A}^i \neq \mathbf{A}^f$ (and many choices of the curves γ connecting them) if the number of degree of freedom is ≥ 3. Thus the result, obtained by a simple extension of the argument originally outlined by Arnold to discuss the paradigmatic example (17.1), proves the existence of diffusion in *a priori unstable* system. The integral in (17.3) is called *Melnikov integral*.

The real difficulty is to estimate the time needed for the transition: it is a time that obviously has to diverge as $\varepsilon \to 0$. Assuming g fixed (*i.e.* ε–independent) a naive approach easily leads to estimates which can even be worse than $O(e^{a\varepsilon^{-b}})$ with some $a, b > 0$. It has finally been shown that in such cases the minimum time can be, for rather general perturbations $\varepsilon f(\boldsymbol{\alpha}, q)$, estimated above by $O(\varepsilon^{-1} \log \varepsilon^{-1})$, which is the best that can be hoped for under generic assumptions.

References: [Ar68], [CV00].

18. Long time stability of quasi periodic motions

A more difficult problem is whether the same phenomenon of migration in action space occurs in *a priori* stable systems. The root of the difficulty is a remarkable stability property of quasi periodic motions. Consider Hamiltonians $\mathcal{H}_\varepsilon(\mathbf{A}, \boldsymbol{\alpha}) = h(\mathbf{A}) + \varepsilon f(\mathbf{A}, \boldsymbol{\alpha})$ with $\mathcal{H}_0(\mathbf{A}) = h(\mathbf{A})$ *strictly convex, analytic and anisochronous* on the closure $\overline{U}$ of an open *bounded* region $U \subset \mathbb{R}^\ell$, and a perturbation $\varepsilon f(\mathbf{A}, \boldsymbol{\alpha})$ analytic in $\overline{U} \times \mathbb{T}^\ell$.

Then *a priori* bounds are available on how long it can possibly take to migrate from an action close to $\mathbf{A}_1$ to one close to $\mathbf{A}_2$: and the bound is of "exponential type" as $\varepsilon \to 0$ (*i.e.* it admits a lower bound which behaves as the exponential of an inverse power of ε). The simplest theorem is (Nekhorossev):

There are constants $0 < a, b, d, g, \tau$ *such that any initial datum* $(\mathbf{A}, \boldsymbol{\alpha})$ *evolves so that the* $\mathbf{A}$ *will not change by more than* $a\varepsilon^g$ *before a long time bounded below by* $\tau e^{b\varepsilon^{-d}}$.

Thus this puts an exponential bound, *i.e.* a bound exponential in an inverse power of ε, to the diffusion time: *before a time* $\tau e^{b\varepsilon^{-d}}$ *actions can only change by* $O(\varepsilon^g)$ so that their variation cannot be large no matter how small $\varepsilon \neq 0$ is chosen. This places a (long) lower bound to the time of diffusion in *a priori* stable systems.

The proof of the theorem provides, actually, an interesting and detailed picture of the actions variations showing that some actions ay vary slower than others.

The theorem is constructive, *i.e.* all constants $0 < a, b, d, \tau$ can be explicitly chosen and depend on $\ell, \mathcal{H}_0, f$ although some of them can be fixed to depend only on ℓ and on the minimum curvature of the convex graph of $\mathcal{H}_0$. Its proof can be adapted to cover many cases which do not fall in the class of systems with strictly convex unperturbed

Hamiltonian. And even to cases with a resonant unperturbed Hamiltonian.

However in important problems, *e.g.* in the 3 body problems met in Celestial Mechanics, there is empirical evidence that diffusion takes place at a fast pace (*i.e.* not exponentialy slow in the above sense) while the above results would forbid a rapid migration in phase space *if they applied*: however in such problems the assumptions of the theorem are not satisfied, because the unperturbed system is strongly resonant (as in the celestial mechanics problems where the number of independent frequencies is a fraction of the number of degrees of freedom and $h(\mathbf{A})$ is far from strictly convex), leaving wide open the possibility of observing rapid diffusion.

And changing the assumptions can dramatically change the results. For instance rapid diffusion can sometimes be proved even though it might be feared that it should require exponentially long times: an example that has been proposed is the case of a *three time scales system*, with Hamiltonian

$$\omega_1 A_1 + \omega_2 A_2 + \frac{p^2}{2} + g(1+\cos q) + \varepsilon f(\alpha_1,\alpha_2,p,q) \tag{18.1}$$

with $\boldsymbol{\omega}_\varepsilon \stackrel{def}{=} (\omega_1,\omega_2)$ with $\omega_1 = \varepsilon^{-\frac12}\overline{\omega}$, $\omega_2 = \varepsilon^{\frac12}\widetilde{\omega}$ and $\overline{\omega},\widetilde{\omega} > 0$ constants. The three scales are $\omega_1^{-1}, \sqrt{g^{-1}}, \omega_2^{-1}$. In this case there are many (although by no means all) pairs $\mathbf{A}_1, \mathbf{A}_2$ which can be connected within a time that can be estimated to be of order $O(\varepsilon^{-1}\log\varepsilon^{-1})$.

This is a rapid diffusion case in a *a priori* unstable system in which condition (17.3) is *not* satisfied: because the ε–dependence of $\boldsymbol{\omega}(\mathbf{A})$ implies that the lower bound c in (17.3) must depend on ε (and be exponentiallly small with an inverse power of ε as $\varepsilon \to 0$).

The unperturbed system in (18.1) is non resonant in the part $\mathcal{H}_0$ for $\varepsilon > 0$ outside a set of zero measure (*i.e.* where the vector $\boldsymbol{\omega}_\varepsilon$ satisfies a suitable Diophantine property) and, furthermore, it is *a priori* unstable: cases met in applications can be *a priori* stable and resonant (and often not anisochronous) in the part $\mathcal{H}_0$. And in such system not only the speed of diffusion is not understood but proposals to prove its existence, if present (as expected), have so far not given really satisfactory results.

References: [Ne77].

19. The three bodies problem

Mechanics and the three bodies problem can be almost identified in the sense that the motion of three gravitating masses has been since the beginning a key astronomical problem and at the same time the source of inspiration for many techniques: foremost among them the theory of perturbations.

As an introduction consider a special case. Let three masses $m_S = m_0, m_J = m_1, m_M = m_2$ interact via gravity, *i.e.* with interaction potential $-km_i m_j|\mathbf{x}_i - \mathbf{x}_j|^{-1}$: the simplest problem arises when the third body has a neglegible mass compared to the two others and the latter are supposed to be on a circular orbit; furthermore the mass m_J is εm_S with ε small and the mass m_M moves in the plane of the circular orbit. This will be called the *circular restricted three body problem*.

In a reference system with center S and rotating at the angular speed of J around S inertial forces (centrifugal and Coriolis') act. Supposing that the body J is located on the axis with unit vector $\mathbf{i}$ at distance R from the origin S, the acceleration of the point M is $\ddot{\boldsymbol{\varrho}} = \mathbf{F} + \omega_0^2(\boldsymbol{\varrho} - \frac{\varepsilon R}{1+\varepsilon}\mathbf{i}) - 2\boldsymbol{\omega}_0 \wedge \dot{\boldsymbol{\varrho}}$ if $\mathbf{F}$ is the force of attraction and $\boldsymbol{\omega}_0 \wedge \dot{\boldsymbol{\varrho}} \equiv \omega_0 \dot{\boldsymbol{\varrho}}^\perp$ where $\boldsymbol{\omega}_0$ is a vector with $|\boldsymbol{\omega}_0| = \omega_0$ and perpendicular to the orbital plane and $\boldsymbol{\varrho}^\perp \stackrel{def}{=} (-\rho_2, \rho_1)$ if $\boldsymbol{\varrho} = (\rho_1, \rho_2)$. Here, taking into account that the origin S rotates around the fixed center of mass, $\omega_0^2(\boldsymbol{\varrho} - \frac{\varepsilon R}{1+\varepsilon}\mathbf{i})$ is the centrifugal force while $-2\boldsymbol{\omega}_0 \wedge \dot{\boldsymbol{\varrho}}$ is the Coriolis force. The equations of motion can therefore be derived from a Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\boldsymbol{\varrho}}^2 - W + \omega_0 \boldsymbol{\varrho}^\perp \cdot \dot{\boldsymbol{\varrho}} + \frac{1}{2}\omega_0^2 \boldsymbol{\varrho}^2 - \omega_0^2 \frac{\varepsilon R}{1+\varepsilon} \boldsymbol{\varrho} \cdot \mathbf{i} \tag{19.1}$$

where $\omega_0^2 R^3 = k m_S (1+\varepsilon) \stackrel{def}{=} g_0$ and $W = -\frac{k m_S}{|\boldsymbol{\varrho}|} - \frac{k\, m_S\, \varepsilon}{|\boldsymbol{\varrho} - R\mathbf{i}|}$ if k is the gravitational constant, R the distance between S and J and finally the last three terms come from the Coriolis force (the first) and from the centripetal force (the other two, taking into account that the origin S rotates around the fixed center of mass).

Setting $g = g_0/(1+\varepsilon) \equiv k m_S$, the Hamiltonian of the system is

$$\mathcal{H} = \tfrac{1}{2}(\mathbf{p} - \omega_0 \boldsymbol{\varrho}^\perp)^2 - \tfrac{g}{|\boldsymbol{\varrho}|} - \tfrac{1}{2}\omega_0^2 \boldsymbol{\varrho}^2 - \varepsilon \tfrac{g}{R}\big(|\tfrac{\boldsymbol{\varrho}}{R} - \mathbf{i}|^{-1} - \tfrac{\boldsymbol{\varrho}}{R} \cdot \mathbf{i}\big) \tag{19.2}$$

The first part can be expressed immediately in the action–angle coordinates for the two body problem, cf. Sect. 8. Calling such coordinates $(L_0, \lambda_0, G_0, \gamma_0)$ and θ_0 the polar angle of M with respect to the ellipse major axis and λ_0 the mean anomaly of M on its ellipse, the Hamiltonian becomes, taking into account that for $\varepsilon = 0$ the ellipse axis rotates at speed $-\omega_0$,

$$\mathcal{H} = -\tfrac{g^2}{2L_0^2} - \omega_0 G_0 - \varepsilon \tfrac{g}{R}\big(|\tfrac{\boldsymbol{\varrho}}{R} - \mathbf{i}|^{-1} - \tfrac{\boldsymbol{\varrho}}{R} \cdot \mathbf{i}\big) \tag{19.3}$$

which is convenient if we study the *interior problem*, *i.e.* $|\boldsymbol{\varrho}| < R$. This can be expressed in the action–angle coordinates via (8.1), (8.2):

$$\begin{aligned} &\theta_0 = \lambda_0 + f_{\lambda_0}, && \theta_0 + \gamma_0 = \lambda_0 + \gamma_0 + f_{\lambda_0}, \\ &e = \big(1 - \tfrac{G_0^2}{L_0^2}\big)^{\frac{1}{2}}, && \tfrac{|\boldsymbol{\varrho}|}{R} = \tfrac{G_0^2}{gR} \tfrac{1}{1 + e\cos(\lambda_0 + f_{\lambda_0})}, \end{aligned} \tag{19.4}$$

where, see (8.2), $f_\lambda = (f(e\sin\lambda, e\cos\lambda)$ and $f(x,y) = 2x(1 + \frac{5}{4}y + \ldots)$ with the $\ldots$ denoting higher orders in x, y even in x. The Hamiltonian takes the form, if $\omega^2 = gR^{-3}$,

$$\mathcal{H}_\varepsilon = -\frac{g^2}{2L_0^2} - \omega G_0 + \varepsilon \frac{g}{R} F(G_0, L_0, \lambda_0, \lambda_0 + \gamma_0) \tag{19.5}$$

where the only important feature (for our purposes) is that $F(L, G, \alpha, \beta)$ is an *analytic function* of L, G, α, β near a datum with $|G| < L$ (*i.e.* $e > 0$) and $|\boldsymbol{\varrho}| < R$. *However* the domain of analyticity in G is rather small as it is constrained by $|G| < L$ excluding in particular the circular orbit case $G = \pm L$.

Note that apparently the KAM theorem fails to be applicable to (19.5) because the matrix of the second derivatives of $\mathcal{H}_0(L, G)$ has 0 determinant. Nevertheless the proof of the theorem goes through also in this case, with minor changes. This can be checked by studying the proof or, following a remark by POINCARÉ, by simply remarking that the "squared" Hamiltonian $\mathcal{H}'_\varepsilon \stackrel{def}{=} (\mathcal{H}_\varepsilon)^2$ has the form

$$\mathcal{H}'_\varepsilon = (-\frac{g^2}{2L_0^2} - \omega\, G_0)^2 + \varepsilon F'(G_0, L_0, \lambda_0, \lambda_0 + \gamma_0) \tag{19.6}$$

with F' still analytic. *But* this time det $\frac{\partial^2 \mathcal{H}'_0}{\partial(G_0, L_0)} = -6g^2 L_0^{-4}\omega_0^2 h \neq 0$ if $h = -g^2\, L_0^{-2} - 2\omega\, G_0 \neq 0$.

Therefore *the KAM theorem applies to* $\mathcal{H}'_\varepsilon$ and the key observation is that the orbits generated by the Hamiltonian $(\mathcal{H}_\varepsilon)^2$ are *geometrically the same* as those generated by the Hamiltonian $\mathcal{H}_\varepsilon$: they are only run at a different speed because of the need of a time rescaling by the *constant* factor $2\mathcal{H}_\varepsilon$.

This shows that, given an unperturbed ellipse of parameters (L_0, G_0) such that $\boldsymbol{\omega} = (\frac{g^2}{L_0^3}, -\omega)$, $G_0 > 0$, with ω_1/ω_2 Diophantine, then the perturbed system admits a motion which is quasi periodic with spectrum proportional to $\boldsymbol{\omega}$ and takes place on an orbit which wraps around a torus remaining *forever close* to the unperturbed torus (which can be visualized as described by a point moving, according to the area law on an ellipse rotating at rate $-\omega_0$) with actions (L_0, G_0), provided ε is small enough. Hence

The KAM theorem answers, at least conceptually, the classical question: can a solution of the three body problem remain close to an unperturbed one forever? i.e. is it possible that a solar system is stable forever?

Assuming $e, |\boldsymbol{\varrho}|/R \ll 1$ and retaining only the lowest orders in e and $|\boldsymbol{\varrho}|/R \ll 1$ the Hamiltonian (19.5) simplifies into

$$\begin{aligned} \mathcal{H} = &- \tfrac{g^2}{2L_0^2} - \omega G_0 + \delta_\varepsilon(G_0) - \tfrac{\varepsilon g}{2R}\tfrac{G_0^4}{g^2R^2}\big(3\cos 2(\lambda_0 + \gamma_0) - \\ &- e\,\cos\lambda_0 - \tfrac{9}{2} e\cos(\lambda_0 + 2\gamma_0) + \tfrac{3}{2} e\cos(3\lambda_0 + 2\gamma_0)\big) \end{aligned} \tag{19.7}$$

where $\delta_\varepsilon(G_0) = -((1+\varepsilon)^{\frac{1}{2}} - 1)\,\omega\, G_0 - \frac{\varepsilon\, g}{2R}\frac{G_0^4}{g^2R^2}$ and $e = (1 - G_0^2/L_0^2)^{\frac{1}{2}}$.

It is an interesting exercise to estimate, assuming as model the (19.7) and following the proof of the KAM theorem, how small has ε to be if a planet with the data of Mercury can be stable forever on a (slowly precessing) orbit with actions close to the present day values under the influence of a mass ε times the solar mass orbiting on a circle, at a distance from the Sun equal to that of Jupiter. It is possible to follow either the above reduction to the ordinary KAM theorem or to apply directly to (19.7) the Lindstedt series expansion, proceeding along the lines of Sect. 14. The first approach is easy but the second is more efficient: unless the estimates are done in a particularly careful manner the value found for εm_S does not have astronomical interest.

References: [Ar68].

20. Rationalization and regularization of singularities

Often integrable systems have interesting data which lie on the boundary of the integrability domain. For instance the central motion when $L = G$ (circular orbits) or the rigid body in a rotation around one of the principal axes or the two body problem when $G = 0$ (collisional data). In such cases perturbation theory cannot be applied as discussed above. Typically the perturbation depends on quantities like $\sqrt{L-G}$ and is *not analytic* at $L = G$. Nevertheless it is sometimes possible to enlarge phase space and introduce new coordinates in the vicinity of the data which in the initial phase space are singular.

A notable example is the failure of the analysis of the circular restricted three body problem: it apparently fails when the orbit that we want to perturb is circular.

It is convenient to introduce the canonical coordinates L, λ and G, γ

$$L = L_0,\ G = L_0 - G_0,\ \lambda = \lambda_0 + \gamma_0,\ \gamma = -\gamma_0 \tag{20.1}$$

so that $e = \sqrt{2GL^{-1}}\sqrt{1 - G(2L)^{-1}}$ and $\lambda_0 = \lambda + \gamma$ and $\theta_0 = \lambda_0 + f_{\lambda_0}$ where f_{λ_0} is defined in (8.2) (see also (19.4)). Hence

$$\begin{aligned}
&\theta_0 = \lambda + \gamma + f_{\lambda+\gamma}, && \theta_0 + \gamma_0 = \lambda + f_{\lambda+\gamma},\\
&e = \sqrt{2G}\sqrt{\tfrac{1}{L}\big(1 - \tfrac{G}{2L}\big)}, && \tfrac{|\varrho|}{R} = \tfrac{L^2(1-e^2)}{gR}\tfrac{1}{1+e\cos(\lambda+\gamma+f_{\lambda+\gamma})},
\end{aligned} \tag{20.2}$$

and the Hamiltonian (19.7) takes the form

$$\mathcal{H}_\varepsilon = -\frac{g^2}{2L^2} - \omega L + \omega G + \varepsilon\frac{g}{R}F(L - G, L, \lambda + \gamma, \lambda) \tag{20.3}$$

In the coordinates L, G of (20.1) the unperturbed circular case corresponds to $G = 0$ and the (19.3) once expressed in the action–angle variables G, L, γ, λ is analytic in a domain whose size is controlled by $\sqrt{G}$. Nevertheless very often problems of perturbation theory can be "*regularized*".

This is done by "enlarging the integrability" domain by adding to it points (one or more) around the singularity (a boundary point of the domain of the coordinates) and introduce new coordinates to describe simultaneously the data close to the singularity and the newly added points: in many interesting cases the equations of motion are no longer singular, *i.e.* become analytic, in the new coordinates and therefore apt to describe the motions that reach the singularity in a finite time. *One can say that the singul;arity was only apparent.*

Perhaps this is best illustrated precisely in the above circular restricted three body problem. There the singularity is where $G = 0$, *i.e.* at a circular unperturbed orbit. If we describe the points with G small in a new system of coordinates obtained from the one in (20.1) by letting alone L, λ and setting

$$p = \sqrt{2G}\cos\gamma, \qquad q = \sqrt{2G}\sin\gamma \tag{20.4}$$

then p, q vary in a neighborhood of the origin with the origin itself *excluded*.

Adding the origin of the p, q plane then in a full neighborhood of the origin the Hamiltonian (19.3) is *analytic* in L, λ, p, q. This is because it is analytic, cf. (19.3),(19.4), as a function of L,λ and $e\cos\theta_0$ and of $\cos(\lambda_0{+}\theta_0)$. Since $\theta_0=\lambda{+}\gamma{+}f_{\lambda+\gamma}$ and $\theta_0{+}\lambda_0=\lambda{+}f_{\lambda+\gamma}$ by (19.4), the Hamiltonian (19.3) is analytic in $L,\lambda,e\cos(\lambda{+}\gamma{+}f_{\lambda+\gamma})$, $\cos(\lambda{+}f_{\lambda+\gamma})$ for e small (*i.e.* for G small) and, by (8.2), $f_{\lambda+\gamma}$ is analytic in $e\sin(\lambda+\gamma)$ and $e\cos(\lambda+\gamma)$. Hence the trigonometric identities

$$e\sin(\lambda+\gamma)\;=\;\tfrac{p\sin\lambda+q\cos\lambda}{\sqrt{L}}\sqrt{1-\tfrac{G}{2L}},\qquad e\cos(\lambda+\gamma)\;=\;\tfrac{p\cos\lambda-q\sin\lambda}{\sqrt{L}}\sqrt{1-\tfrac{G}{2L}}\tag{20.5}$$

together with $G=\frac12(p^2+q^2)$ imply that (20.3) is analytic near $p=q=0$ and $L>0,\lambda\in[0,2\pi]$. The Hamiltonian becomes analytic and the new coordinates are suitable to describe motions crossing the origin: *e.g.* setting $C\stackrel{def}{=}\frac12(1-\frac{p^2+q^2}{4L})\,L^{-\frac12}$ (19.7) becomes

$$\begin{aligned}\mathcal{H}=-\tfrac{g^2}{2L^2}-\omega L+\omega\tfrac12(p^2+q^2)+\delta_\varepsilon(\tfrac12(p^2+q^2))-\tfrac{\varepsilon g}{2R}\tfrac{(L-\frac12(p^2+q^2))^4}{g^2R^2}\cdot\\ \cdot\Big(3\cos2\lambda-\big((-11\cos\lambda+3\cos3\lambda)\,p-\,(7\sin\lambda+3\sin3\lambda)\,q\big)\,C\Big)\end{aligned}\tag{20.6}$$

The KAM theorem does not apply in the form discussed above to "cartesian coordinates" *i.e.* when, as in (20.6), the unperturbed system is not assigned in action–angle variables: however there are versions of the theorem (actually corollaries of it) which do apply and therefore it becomes possible to obtain some results even for the perturbations of circular motions by the techniques that have been illustrated here.

Likewise the Hamiltonian of the rigid body with a fixed point O and subject to analytic external forces becomes singular, if expressed in the action–angle coordinates of Deprit, when the body motion nears a rotation around a principal axis or more generally nears a configuration in which any two of the axes $\mathbf{i}_3$, $\mathbf{z}$, $\mathbf{z}_0$ coincide (*i.e.* any two among the principal axis, the angular momentum axis and the inertial z-axis coincide, see Section 9). Nevertheless by imitating the procedure just described in the simpler cases of the circular three body problem, it is possible to enlarge phase space so that in the new coordinates the Hamiltonian is analytic near the singular configurations.

A regularization also arises when considering collisional orbits in the unrestricted planar three body problem. In this respect a very remarkable result is the regularization of collisional orbits in the planar three body problem. After proving that if the total angular momentum does not vanish simoultaneous collisions of the three masses cannot happen within any finite time interval the question is reduced to the regularization of two bodies collisions, under the assumption that the total angular momentum does not vanish.

The *local* change of coordinates which changes the relative position coordinates (x,y) of two colliding bodies as $(x,y)\to(\xi,\eta)$ with $x+iy=(\xi+i\eta)^2$ is not one to one, hence it has to be regarded as an enlargement of the positions space, if points with different (ξ,η) are considered different. However the equations of motion written in the variables ξ,η have no singularity at $\xi,\eta=0$, (Levi-Civita).

Another celebrated regularization is the regularization of the *Schwartzschild metric*, *i.e.* of the general relativity version of the two body problem: it is however somewhat out of the scope of this review (Synge, Kruskal).

References: [LC].

Appendix A1. KAM resummation scheme

The idea to control the "remaining contributions" is to reduce the problem to the case in which there are no pairs of lines that follow each other in the tree order and which have the *same current*. Mark by a *scale label* "0" the lines of a tree whose divisors are > 1: these are lines which give no problems in the estimates. Then mark by a scale label "≥ 1" the lines with current $\boldsymbol{\nu}(l)$ such that $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(l)| \leq 2^{-n+1}$ for $n = 1$ (*i.e.* the remaining lines).

The lines labeled 0 are said to be *on scale* 0, while those labeled ≥ 1 are said to be *on scale* ≥ 1. A *cluster of scale* 0 will be a *maximal* collection of lines of scale 0 forming a connected subgraph of a tree θ.

Consider only trees $\theta_0 \in \Theta_0$ of the family Θ_0 of trees *containing no clusters of lines with scale label* 0 *which have only one line entering the cluster and one exiting it with equal current*.

It is useful to introduce the notion of a line ℓ_1 situated "*between*" two lines ℓ, ℓ' with $\ell' > \ell$: this will mean that ℓ_1 precedes ℓ' but not ℓ.

All trees θ in which there are some pairs $l' > l$ of consecutive lines of scale label ≥ 1 which have equal current and such that all lines between them bear scale label 0 are obtained by "inserting" on the lines of trees in Θ_0 with label ≥ 1 any number of clusters of lines and nodes, with lines of scale 0 and with the property that the sum of the harmonics of the nodes inserted *vanishes*.

Consider a line $l_0 \in \theta_0 \in \Theta_0$ linking nodes $v_1 < v_2$ and labeled ≥ 1 and imagine inserting on it a cluster γ of lines of scale 0 *with sum of the node harmonics vanishing and out of which emerges one line connecting a node v_{out} in γ to v_2 and into which enters one line linking v_1 to a node $v_{in} \in \gamma$*. The insertion of a k–lines, $|\gamma| = (k+1)$-nodes, cluster changes the tree value by replacing the line factor, that will be briefly called "value of the cluster γ,

$$\frac{\boldsymbol{\nu}_{v_1} \cdot \boldsymbol{\nu}_{v_2}}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(l_0)^2} \;\rightarrow\; \frac{(\boldsymbol{\nu}_{v_1} \cdot M(\gamma; \boldsymbol{\nu}(l_0))\, \boldsymbol{\nu}_{v_2})}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(l_0)^2} \frac{1}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(l_0)^2} \tag{A1.1}$$

where M is a $\ell \times \ell$ matrix $M_{rs}(\gamma, \boldsymbol{\nu}(l_0)) = \frac{\varepsilon^{|\gamma|}}{k!} \nu_{out,r} \nu_{in,s} \prod_{v \in \gamma} (-f_{\boldsymbol{\nu}_v}) \prod_{l \in \gamma} \frac{\boldsymbol{\nu}_v \cdot \boldsymbol{\nu}_{v'}}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(l)^2}$ if $\ell = v'v$ denotes a line linking v' and v. Therefore if all possible connected clusters are inserted and the resulting values are added up the result can be taken into account by attributing to the original line l_0 a factor like (A1.1) with $M^{(0)}(\boldsymbol{\nu}(l_0)) \stackrel{def}{=} \sum_\gamma M(\gamma; \boldsymbol{\nu}(l_0))$ replacing $M(\gamma; \boldsymbol{\nu}(l_0))$.

If several connected clusters γ are inserted on the same line and their values are summed the result is a modification of the factor associated with the line l_0 into

$$\sum_{k=0}^\infty \boldsymbol{\nu}_{v_1}\cdot\Big(\frac{M^{(0)}(\boldsymbol{\nu}(l_0))}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(l_0)^2}\Big)^k\boldsymbol{\nu}_{v_2}\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(l_0)^2}=(\boldsymbol{\nu}_{v_1}\cdot\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(l_0)^2-M^{(0)}(\boldsymbol{\nu}(l_0))}\boldsymbol{\nu}_{v_2}) \qquad (A1.2)$$

The series defining $M^{(0)}$ involves, *by construction*, only trees with lines of scale 0, hence with large divisors so that it converges to a matrix of small size of order ε (actually ε^2, looking more carefully) if ε is small enough.

Convergence *can* be established by simply remarking that the series defining $M^{(1)}$ is built with lines with values $>\frac{1}{2}$ of the propagator, so that it certainly converges for ε small enough (by the estimates in Section 12 where the propagators were identically 1) and the sum is of order ε (actually ε^2), hence <1. *However* such an argument cannot be repeated when dealing with lines with smaller propagators (which still have to be discussed). Therefore a method not relying on so trivial a remark on the size of the propagators has eventually to be used when considering lines of scale higher than 1, as it will soon become necessary.

The advantage of the collection of terms achieved with (A1.2) is that we can represent $\mathbf{h}$ as a sum of values of trees which are *simpler* because they contain no pair of lines of scale ≥ 1 with in between lines of scale 0 with total sum of the node harmonics vanishing. The price is that the divisors are now more involved and we even have a problem due to the fact that we have not proved that the series in (A1.2) converges. In fact it is a geometric series whose value is the *r.h.s.* of (A1.2) obtained by the sum rule (13.7) *unless we can prove that the ratio of the geometric series is* <1. This is trivial in this case by the previous remark: but it is better to remark that there is anothr reason for convergence, whose use is not really necesary here but it will become essential later.

The property that the ratio of the geometric series is <1 can be regarded as due to the consequence of the *cancellation* mentioned in Section 14 which can be shown to imply that the ratio is <1 because $M^{(0)}(\boldsymbol{\nu})=\varepsilon^2\,(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu})^2m^{(0)}(\boldsymbol{\nu})$ with $|m^{(0)}(\boldsymbol{\nu})|<D_0$ for some $D_0>0$ and for all $|\varepsilon|<\varepsilon_0$ for some ε_0: so that for small ε the divisor in (A1.2) is essentially still what it was before starting the resummation.

At this point an induction can be started. Consider trees evaluated with the new rule and place a scale lavel "≥ 2" on the lines with $|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(l)|\le 2^{-n+1}$ for $n=2$: leave the label "0" on the lines already marked so and label by "1" the other lines. The lines of scale "1" will satisfy $2^{-n}<|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(l)|\le 2^{-n+1}$ for $n=1$. And the graphs will now possibly contain lines *of scale* $0,1$ *or* ≥ 2 *while lines with label "*≥ 1*" no longer can appear*, by construction.

A *cluster of scale* 1 will be a maximal collection of lines of scales $0,1$ forming a connected subgraph of a tree θ and containing at least one line of scale 1.

The construction carried considering clusters of scale 0 can be repeated by considering trees $\theta_1\in\Theta_1$ with Θ_1 the collection of trees with lines marked $0,1$ or ≥ 2 and in which no pairs of lines with equal momentum appear to follow each other if between them there are only lines marked 0 or 1.

Insertion of connected clusters γ of such lines on a line l_0 of θ_1 leads to define a matrix $M^{(1)}$ formed by summing tree values of clusters γ with lines of scales 0 or 1 evaluated with the line factors defined in (A1.1) and with the restriction that *in* γ *there are no pairs of lines* $\ell<\ell'$ *with the same current and which follow each other while any line between them has lower scale (i.e.* 0), here between means preceding l' but not preceding l, as above.

Therefore a scale independent method has to devised to check convergence for $M^{(1)}$ and for the matrices to be introduced later to deal with even smaller propagators. This is achieved by the following extension of Siegel's theorem mentioned in Section 14:

Let $\boldsymbol{\omega}_0$ satisfy (13.2) and set $\boldsymbol{\omega} = C\boldsymbol{\omega}_0$. Consider the contribution to the sum in (14.3) from graphs θ in which
(1) no pairs $\ell' > \ell$ of lines which lie on the same path to the root carry the same current $\boldsymbol{\nu}$ if all lines ℓ_1 between them have current $\boldsymbol{\nu}(\ell_1)$ such that $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}(\ell_1)| > 2|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|$.
(2) the node harmonics are bounded by $|\boldsymbol{\nu}| \le N$ for some N.
Then the number of lines ℓ in θ with divisor $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$ satisfying $2^{-n} < |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| \le 2^{-n+1}$ does not exceed $4\,N\,k\,2^{-n/\tau}$, $n = 1, 2, \ldots$.

This implies, by the same estimates in (14.6), that the series defining $M^{(1)}$ converges. Again it must be checked that there are cancellations implying that $M^{(1)}(\boldsymbol{\nu}) = \varepsilon^2\,(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu})^2 m^{(1)}(\boldsymbol{\nu})$ with $|m^{(1)}(\boldsymbol{\nu})| < D_0$ for the *same* $D_0 > 0$ and the same ε_0.

At this point one deals with trees containing only lines carrying labels $0, 1, \ge 2$ and the line factors for the lines $\ell = v'v$ of scale 0 are $\boldsymbol{\nu}_{v'}\cdot\boldsymbol{\nu}_v/(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(\ell))^2$, those of the lines $\ell = v'v$ of scale 1 have line factors $\boldsymbol{\nu}_{v'}\cdot(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(\ell)^2 - M^{(0)}(\boldsymbol{\nu}(\ell)))^{-1}\boldsymbol{\nu}_v$ and those of the lines $\ell = v'v$ of scale ≥ 2 have line factors $\boldsymbol{\nu}_{v'}\cdot(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}(\ell)^2 - M^{(1)}(\boldsymbol{\nu}(\ell)))^{-1}\boldsymbol{\nu}_v$. *Furthermore no pair of lines of scale "1" or of scale "≥ 2" with the same momentum and with only lines of lower scale (i.e. of scale "0" in the first case or of scale "0","1" in the second) between then can follow each other.*

And so on until, after infinitely many steps, the problem is reduced to the evaluation of tree values in which each line carries a scale label n and there are no pairs of lines which follow each other and which have only lines of lower scale in between. Then the Siegel argument applies once more and the series so resummed is an absolutely convergent series of functions analytic in ε: hence the original series is convergent.

Although at each step there is a lower bound on the denominators it would not be possible to avoid using Siegel's theorem. In fact the lower bound would becomes worse and worse as the scale increases. In order to check the estimates of the constants D_0, ε_0 which control the scale independence of the convergence of the various series it is necessary to take advantage of the theorem, and of the absence at each step of the necessity of considering trees with pairs of consecutive lines with equal momentum and intermediate lines of higher scale.

One could also perform the analysis by bounding $h^{(k)}$ order by order with no resummations (*i.e.* without changing the line factors) and exhibiting the necessary cancellations. Or the paths that Kolmogorov, Arnold and Moser used to prove the first three (somewhat different) versions of the theorem, by successive approximations of th equations for the tori, can be followed.

The invariant tori are *Lagrangian manifolds* just as the unperturbed ones (cf. comments after (6.4)) and, in the case of the Hamiltonian (14.1) the generating function $\mathbf{A}\cdot\boldsymbol{\psi} + \Phi(\mathbf{A},\boldsymbol{\psi})$ can be expressed in terms of their parametric equations

$$
\begin{aligned}
&\Phi(\mathbf{A},\boldsymbol{\psi}) = G(\boldsymbol{\psi}) + \mathbf{a}\cdot\boldsymbol{\psi} + \mathbf{h}(\boldsymbol{\psi})\cdot(\mathbf{A}-\boldsymbol{\omega}-\Delta\mathbf{h}(\boldsymbol{\psi}))\\
&\boldsymbol{\partial}_{\boldsymbol{\psi}} G(\boldsymbol{\psi}) \stackrel{def}{=} -\Delta\mathbf{h}(\boldsymbol{\psi}) + \underset{\sim}{h}(\boldsymbol{\psi})\boldsymbol{\partial}_{\boldsymbol{\psi}}\Delta\underset{\sim}{h}(\boldsymbol{\psi}) - \mathbf{a}\\
&\mathbf{a} \stackrel{def}{=} \int(-\Delta\mathbf{h}(\boldsymbol{\psi}) + \underset{\sim}{h}(\boldsymbol{\psi})\boldsymbol{\partial}_{\boldsymbol{\psi}}\Delta\underset{\sim}{h}(\boldsymbol{\psi}))\frac{d\boldsymbol{\psi}}{(2\pi)^\ell} = \int \underset{\sim}{h}(\boldsymbol{\psi})\boldsymbol{\partial}_{\boldsymbol{\psi}}\Delta\underset{\sim}{h}(\boldsymbol{\psi})\frac{d\boldsymbol{\psi}}{(2\pi)^\ell}
\end{aligned}
\tag{A1.3}
$$

where $\Delta = (\boldsymbol{\omega}\cdot\boldsymbol{\partial}_{\boldsymbol{\psi}})$ and the invariant torus corresponds to $\mathbf{A}' = \boldsymbol{\omega}$ in the map $\boldsymbol{\alpha} = \boldsymbol{\psi} + \boldsymbol{\partial}_{\mathbf{A}}\Phi(\mathbf{A},\boldsymbol{\psi})$ and $\mathbf{A}' = \mathbf{A} + \boldsymbol{\partial}_{\boldsymbol{\psi}}\Phi(\mathbf{A},\boldsymbol{\psi})$. In fact by (A1.3) the latter becomes $\mathbf{A}' = \mathbf{A} - \Delta\mathbf{h}$ and, from the second of (13.3) written for f depending only on he angles $\boldsymbol{\alpha}$, it is $\mathbf{A} = \boldsymbol{\omega} + \Delta\mathbf{h}$ when $\mathbf{A},\boldsymbol{\alpha}$ are on the ivariant torus.

Note that if $\mathbf{a}$ exists it is necessarily determined by the third relation but the check that the second equation in (A1.3) is soluble (*i.e.* that the *r.h.s.* is an exact gradient up to a constant) is nontrivial. The canonical map generated by $\mathbf{A}\cdot\boldsymbol{\psi} + \Phi(\mathbf{A},\boldsymbol{\psi})$ is *also* defined for $\mathbf{A}'$ close to $\boldsymbol{\omega}$ and foliates the neighborhood of the invariant torus with other tori: of course for $\mathbf{A}' \neq \boldsymbol{\omega}$ the tori defined in this way are, in general, not invariant.

References: [GBG94].

Appendix A2. Coriolis and Lorentz forces. Larmor precession

Larmor precession is the part of the motion of an electrically charged particle caused by the action of a magnetic field $\mathbf{H}$ (in an inertial frame of reference). It is due to the *Lorentz force* which, on a unit mass with unit charge, produces an acceleration $\ddot{\boldsymbol{\varrho}} = \mathbf{v}\wedge\mathbf{H}$ if the speed of light is $c = 1$.

Therefore if $\mathbf{H} = H\mathbf{k}$ is directed along the $\mathbf{k}$ axis the acceleration it produces is the same that the Coriolis force would impress on a unit mass located in a reference frame which rotates with angular velocity $\omega_0\mathbf{k}$ around the $\mathbf{k}$ axis if $\mathbf{H} = 2\omega_0\mathbf{k}$.

The above remarks imply that a homogeneous sphere homogeneoulsy electrically charged with a unit charge and freely pivoting about its center in a constant magnetic field H directed along the $\mathbf{k}$ axis undergoes the same motion it would follow if not subject to the magnetic field but seen in a non inertial reference frame rotating at constant angular velocity ω_0 around the $\mathbf{k}$ axis if H and ω_0 are related by $H = 2\omega_0$: in this frame the Coriolis force is interpreted as a magnetic field.

This holds, however, only if the centrifugal force has zero moment with rerspect to the center: true in the spherical symmetry case only. In spherically non symmetric cases the centrifugal forces have in general non zero moment so the equivalence between Coriolis forces and magnetic fields is only approximate.

The *Larmor theorem* makes this more precise. It gives a quantitative estimate of the difference between the motion of a general system of particles of mass m in a magnetic field and the motion of the same particles in a rotating frame of reference but in absence of a magnetic field. The approximation is estimated in terms of the size of the *Larmor frequency* $eH/2mc$: which should be small compared to the other characteristic frequencies of the motion of the system: the physical meaning is that the centrifugal force should be small compared to the other forces.

The vector potential $\mathbf{A}$ for a constant magnetic field in the $\mathbf{k}$-direction $\mathbf{H} = 2\omega_0\mathbf{k}$ is $\mathbf{A} = 2\omega_0\mathbf{k} \wedge \boldsymbol{\varrho} \equiv 2\omega_0\boldsymbol{\varrho}^\perp$. Therefore, from the treatment of the Coriolis force in Section 19, see (19.2), the motion of a charge e with mass m in a magnetic field $\mathbf{H}$ with vector potential $\mathbf{A}$ and subject to other forces with potential W can be described, in an inertial frame and in generic units in which the speed of light is c, by a Hamiltonian

$$\mathcal{H} = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 + W(\boldsymbol{\varrho}) \tag{A2.1}$$

where $\mathbf{p} = m\dot{\boldsymbol{\varrho}} + \frac{e}{c}\mathbf{A}$ and $\boldsymbol{\varrho}$ are canonically conjugated.

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