Break-up of resonant invariant curves in billiards and dual billiards associated to perturbed circular tables

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Abstract

Two area-preserving twist maps are associated to a smooth closed convex table: the (classical) billiard map and the dual billiard map. For circular tables, they are integrable and their phase spaces are foliated by invariant curves. The invariant curves with a rational rotation number are resonant and do not persist under generic perturbations. We present a criterion, obtained by a standard Melnikov approach from classical variational techniques, to distinguish when a given resonant invariant curve of some of these billiard maps does not persist under a concrete perturbation of a circular table.

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1 Introduction

One of the fundamental questions in dynamical systems is the effect that small perturbations of a dynamical system cause on its unperturbed invariant sets. For example, the hyperbolic fixed points of a smooth map persist under small enough smooth perturbations, whereas the maximal Diophantine tori of a completely integrable analytic non-degenerate Hamiltonian system persist under small enough analytic Hamiltonian perturbations. The first claim follows directly from the Implicit Function Theorem and the second one is the central topic of the more elaborate KAM Theory. On the other hand, it is

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well-known that the invariant sets formed by a continuum of smaller invariant subsets hardly persist. The simplest example of these fragile invariant objects are the resonant invariant curves of maps. An invariant curve of a map is called *resonant* when all its points remain fixed under some power of the map.

In this paper, we establish a criterion to decide when a given resonant invariant curve of an integrable area-preserving twist map does not persist under a concrete perturbation of the map. The twist character of the map is crucial, because our results are obtained from the variational principle that these maps verify. The criterion is applied to the classical and dual billiard maps associated to small perturbations of circular tables.

G. D. Birkhoff [2] introduced the problem of *convex billiard tables* more than 75 years ago as a way to describe the motion of a free particle inside a closed convex curve such that it reflects at the boundary according to the law "angle of incidence equals to angle of reflection". He realized that this billiard motion can be modeled by an area-preserving twist diffeomorphism defined on the open cylinder $\mathbb{T} \times (-1, 1)$. Since then, billiards have become a paradigm of conservative dynamical systems, because *in the billiard problem the formal side, usually so formidable in dynamics, almost completely disappears, and only the interesting questions need to be considered* [2, page 170]. We refer to the monographes [7,12] for the last developments in billiard dynamics.

On the other hand, following J.K. Moser, the dual billiard problem *may serve as a crude model for planetary motion* [9, page 66]. The dual billiard map associated to a smooth (strictly) convex closed plane curve $\Gamma \subset \mathbb{R}^2$ is defined on the unbounded component of $\mathbb{R}^2 \setminus \Gamma$ as follows: the image of a point equals to its reflection in the tangency point of the oriented tangent line to Γ through the initial point. The dual billiard motion can also be modeled by an area-preserving twist diffeomorphism defined on the half-cylinder $\mathbb{T} \times (0, +\infty)$. Smooth dual billiards have been studied, for instance, in [10,11,5,3].

The periodic (classical) billiard trajectories are inscribed polygons whose consecutive sides make equal angles with the curve Γ, whereas the periodic dual billiard trajectories are circumscribed polygons whose sides are bisected by their tangency points about Γ. Given a rational number p/q, where p and q are relatively prime, we say that the trajectory is p/q-periodic when the polygon has q sides and makes p turns inside (resp., about) Γ in the classical (resp., dual) billiard. We shall call these polygons *p/q-gons*. In classical billiards, $q \geq 2$. In dual billiards, $q \geq 3$. Besides, it is well-known that the classical (resp., dual) billiard dynamics verifies a variational principle in which it is established that its p/q-periodic trajectories are critical points of the length functional defined on the set of p/q-gons inscribed in Γ (resp., of the extrema of the area functional defined on the set of p/q-gons circumscribed about Γ).

The following existence results are obtained using the above characterization of periodic billiard trajectories as critical points of the functionals length and area. They hold when the billiard curve is smooth and strictly convex, and essentially go back to G.D. Birkhoff:

- *(Classical version)* For any integers $q \geq 2$ and $1 \leq p \leq q/2$ relatively prime, there exist at least two geometrically distinct p/q-periodic classical billiard trajectories [12, §2.5].
- *(Dual version)* For any integers $q \geq 3$ and $1 \leq p \leq q/2$ relatively prime, there exist at least a p/q-periodic dual billiard trajectory [12, §4.1].

We ask p and q to be relatively prime because, otherwise, the p/q-periodic trajectories could be a polygon with fewer vertices traversed several times.

Let us consider now the billiard dynamics associated to a circumference Γ_0 of radius r_0. It is an exercise in elementary geometry to check that:

- *(Classical version)* For any integers $q \geq 2$ and $1 \leq p \leq q/2$ relatively prime, there exists a continuous family $\mathcal{I}_0^{p/q}$ of regular p/q-gons inscribed in Γ_0 whose consecutive sides make equal angles with Γ_0.
- *(Dual version)* For any integers $q \geq 3$ and $1 \leq p \leq q/2$ relatively prime, there exists a continuous family $\mathcal{C}_0^{p/q}$ of regular p/q-gons circumscribed about Γ_0 whose sides are bisected by their tangency points about Γ_0.

Of course, each family is obtained just by rotating any of its elements with respect to the center of the circumference Γ_0. In particular, all the polygons of $\mathcal{I}_0^{p/q}$ have the same perimeter: $l_0^{p/q} = 2q\sin(\pi p/q) r_0$, and all the polygons of $\mathcal{C}_0^{p/q}$ have the same area: $A_0^{p/q} = q\tan(\pi p/q) r_0^2$. This follows also from the variational principles, since any functional is constant on each continuum set formed by critical points. Thus, if a family $\mathcal{I}_0^{p/q}$ (resp., $\mathcal{C}_0^{p/q}$) persists under some perturbation $\Gamma_\epsilon = \Gamma_0 + \mathrm{O}(\epsilon)$ of the circumference, then the polygons of the perturbed family $\mathcal{I}_\epsilon^{p/q}$ (resp., $\mathcal{C}_\epsilon^{p/q}$) not need to be regular, but they shall still have the same perimeter (resp., area).

A criterion to decide if these continuous families of polygons persist under small perturbations of Γ_0 is given below. Let $\mathbf{n}_\theta = (\cos\theta, \sin\theta)$. We write the perturbation in polar coordinates as follows:

$$\Gamma_\epsilon = \{ r_\epsilon(\theta)\mathbf{n}_\theta : \theta \in \mathbb{T} \}, \qquad r_\epsilon(\theta) = r_0 + \epsilon r_1(\theta) + \mathrm{O}(\epsilon^2), \tag{1}$$

for some smooth function $r_\epsilon : \mathbb{T} \to \mathbb{R}$.

Theorem 1 *Let $\sum_{j\in\mathbb{Z}} \hat{r}_1^j \, \mathrm{e}^{\mathrm{i}j\theta}$ be the Fourier expansion of $r_1(\theta)$. Let $q \geq 2$ be any integer. If there exists some $j \in q\mathbb{Z}\setminus\{0\}$ such that $\hat{r}_1^j \neq 0$, then the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ do not persist under the perturbation (1).*

As a first example of this criterion, we realize that the perturbations given by

$$r_\epsilon(\theta) = r_0 + \epsilon\alpha \frac{\cos\theta - \alpha}{1 - 2\alpha\cos\theta + \alpha^2} = r_0 + \epsilon \sum_{0\neq j \in \mathbb{Z}} \frac{\alpha^{|j|}}{2} \mathrm{e}^{\mathrm{i}j\theta}, \qquad \alpha \in (0,1)$$

destroy the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ for *any* rational p/q. On the contrary, it turns out that the elliptic perturbation given by

$$r_\epsilon(\theta) = r_0 \sqrt{\frac{1-\epsilon}{1-\epsilon\cos^2\theta}} = r_0 + \epsilon r_0(\cos 2\theta - 1)/4 + \mathrm{O}(\epsilon^2)$$

preserves the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ for all rationals p/q such that $q \geq 3$, whereas it destroys the family $\mathcal{I}_0^{1/2}$. The break-up of the family $\mathcal{I}_0^{1/2}$ follows directly from our criterion. The persistence of the other families has to do with the fact that, under this second perturbation, the perturbed circumference Γ_ϵ becomes an ellipse. We recall that elliptic (classical and dual) billiards are integrable, because ellipses verify the celebrated Poncelet's Porism [12, §4.3].

In Section 2 we introduce the maps and the variational principles we will be working with. It also contains the general Melnikov theory for resonant curves of twist maps. The proofs of our criterion for classical billiards and dual billiards are contained in Section 3 and Section 4, respectively.

2 Break-up of resonant invariant curves in twist maps

For the sake of simplicity, we will assume that the objects here considered are smooth. For a general background on twist maps we refer, for instance, to the book [6, §9.3] or to the review [8].

2.1 Twist maps

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\mathbb{A} = \mathbb{T} \times \mathbb{R}$, and $\pi_1 : \mathbb{A} \to \mathbb{T}$ be the natural projection. Sometimes it is convenient to work in the universal cover $\mathbb{R}$ of $\mathbb{T}$. We will use the coordinates (s, y) for $\mathbb{A}$ and (x, y) for $\mathbb{R}^2$. The horizontal and vertical directions in these coordinates will be called the angular and radial directions, respectively. A tilde will always denote the lift of a point, function or set to the universal cover. If g is a real-valued function, $\partial_i g$ denotes the derivative with respect to the ith variable. We will consider certain diffeomorphisms defined on an open cylinder of the form $C = \mathbb{T} \times Y$, $Y = (y_-, y_+)$, for some $-\infty \leq y_- < y_+ \leq +\infty$. Then $\tilde{C} = \mathbb{R} \times Y$ is an open strip of the plane.

A diffeomorphism $f : C \to C$ is called an *area-preserving twist map* when it preserves area, orientation, and verifies the classical *twist condition*: If $\tilde{f}(x, y) = (x', y')$ is a lift of f then

$$\partial_2 x'(x, y) > 0.$$

In what follows the lift $\tilde{f}$ remains fixed. We also assume, although it is not essential, that f verifies some *rigid boundary conditions.* To be more precise, if y_- and y_+ are finite, we suppose that the twist map f can be extended continuously to the closed cylinder $\bar{C} = \mathbb{T} \times \bar{Y}$, $\bar{Y} = [y_-, y_+]$, as a rigid rotation on the boundaries. That is, there exist some *boundary frequencies* $\omega_\pm \in \mathbb{R}$, $\omega_- < \omega_+$, such that $\tilde{f}(x, y_\pm) = (x + \omega_\pm, y_\pm)$. When $y_- = -\infty$ (resp., $y_+ = +\infty$), the lower (resp., upper) boundary condition is stated in terms of the corresponding limit. These rigid boundary conditions are verified by the classical and dual billiard maps considered in this paper.

Let $D = \{(x, x') \in \mathbb{R}^2 : \omega_- < x' - x < \omega_+\}$. Then there exists a function $h : D \to \mathbb{R}$ such that $\tilde{f}(x, y) = (x', y')$ if and only if

$$y = -\partial_1 h(x, x'), \qquad y' = \partial_2 h(x, x'). \tag{2}$$

The function h is called the *Lagrangian* or *generating function* of f.

Remark 2 *In order to give a geometric interpretation of h, we note that if y_- is finite (as for classical and dual billiard maps), then the quantity $h(x, x')$ equals to the area of the region in the strip $\tilde{C}$ enclosed by the lower boundary $\tilde{C}_- = \mathbb{R} \times \{y_-\}$, the vertical line $\tilde{Y}(x') = \{x'\} \times Y$, and the $\tilde{f}$-image of the vertical line $\tilde{Y}(x) = \{x\} \times Y$, see [6, §9.3].*

The term Lagrangian is due to the fact that twist maps satisfy a variational principle, since their orbits are in correspondence with the critical points of some real-valued functionals, called *actions.* To clarify this, we describe the periodic version of this principle. A point $(x, y) \in \tilde{C}$ is said to be a *p/q-periodic* point of $\tilde{f}$, for some integers p and q relatively prime, whenever $\tilde{f}^q(x, y) = (x + 2\pi p, y)$. Obviously, $\omega_- < 2\pi p/q < \omega_+$. Then the corresponding point $(s, y) \in C$, $x = \tilde{s}$, is a periodic point of period q by f that is translated by $2\pi p$ in the base by the lift. Analogously, a $\tilde{f}$-orbit $\{(x_k, y_k)\}_{k\in\mathbb{Z}}$ is *p/q-periodic* when $(x_{k+q}, y_{k+q}) = (x_k + 2\pi p, y_k)$ for all $k \in \mathbb{Z}$. In the periodic variational principle [8, §V], it is stated that the p/q-periodic points of the lift are in one-to-one correspondence with the critical points of the *p/q-periodic action* $W^{p/q} : \mathbb{R}^q \to \mathbb{R}$ defined by

$$W^{p/q}(x_0, \ldots, x_{q-1}) = h(x_0, x_1) + h(x_1, x_2) + \cdots + h(x_{q-1}, x_0 + 2\pi p).$$

2.2 Integrable twist maps

Let $f_0 : C \to C$ be an integrable twist map, that is, an area-preserving twist map with a lift $\tilde{f}_0 : \tilde{C} \to \tilde{C}$ of the form

$$(x', y') = \tilde{f}_0(x, y) = (x + \omega(y), y)$$

such that $\omega'(y) > 0$ for all $y \in Y$ and the limits $\omega_\pm = \lim_{y \to y_\pm} \omega(y)$ are finite. Its Lagrangian $h_0(x, x')$ depends only on the difference $x' - x$, since $(\partial_1 + \partial_2) h_0(x, x') = y' - y \equiv 0$. In fact, $h_0(x, x') = l_0(x' - x)$, for some function $l_0(\omega)$ such that $l_0'(\omega(r)) = r$.

Integrable twist maps leave all the horizontal lines $\mathbb{R}(y) := \mathbb{R} \times \{y\}$ invariant. We restrict our study to the resonant ones, that is, those whose rotation number is rational. If $p/q \in \mathbb{Q}$ and $\omega_- < 2\pi p/q < \omega_+$, let $y_0^{p/q} = \omega^{-1}(2\pi p/q)$ and $T_0^{p/q} = \mathbb{R}(y_0^{p/q})$. Any orbit

$$(x_k^{p/q}, y_k^{p/q}) = \tilde{f}_0^k(x_0^{p/q}, y_0^{p/q}), \qquad x_k^{p/q} = x + 2\pi k p/q, \qquad y_k^{p/q} = y_0^{p/q},$$

contained in $T_0^{p/q}$ is p/q-periodic. Finally, we stress that all these p/q-periodic orbits have the same p/q-periodic action, namely

$$\sum_{k=1}^{q} h_0(x_{k-1}^{p/q}, x_k^{p/q}) = \sum_{k=1}^{q} l_0(x_k^{p/q} - x_{k-1}^{p/q}) \equiv q l_0(2\pi p/q). \tag{3}$$

2.3 Radial curves and radial potentials

Let f_ϵ be a perturbation of an integrable twist map and let $\tilde{f}_\epsilon$ be a fixed lift. Moreover, let $h_\epsilon = h_0 + \epsilon h_1 + \mathrm{O}(\epsilon^2)$ be the perturbed Lagrangian.

As it is well-known, the resonant invariant curves $T_0^{p/q}$ do not persist under generic perturbations, but following G. D. Birkhoff [2, §VI] and V. I. Arnold [1, §20], there exist a couple of *radial curves*

$$T_\epsilon^{p/q} = \left\{ (x, y_\epsilon^{p/q}(x)) : x \in \mathbb{R} \right\}, \qquad \widehat{T}_\epsilon^{p/q} \left\{ (x, \hat{y}_\epsilon^{p/q}(x)) : x \in \mathbb{R} \right\}$$

close to $T_0^{p/q}$ such that $\tilde{f}_\epsilon^q$ projects $T_\epsilon^{p/q}$ onto $\widehat{T}_\epsilon^{p/q}$ along the radial direction.

This follows directly from the next lemma.

Lemma 3 *There exist two 2π-periodic functions $y_\epsilon^{p/q}, \hat{y}_\epsilon^{p/q} : \mathbb{R} \to Y$ such that:*

i) $y_\epsilon^{p/q}(x), \hat{y}_\epsilon^{p/q}(x) = y_0^{p/q} + \mathrm{O}(\epsilon)$, *uniformly in* $x \in \mathbb{R}$.
ii) $f_\epsilon^q(x, y_\epsilon^{p/q}(x)) = (x, \hat{y}_\epsilon^{p/q}(x))$, *for all* $x \in \mathbb{R}$.

PROOF. To obtain the 2π-periodic function $y_\epsilon^{p/q}(x)$, it suffices to realize that

$$G^{p/q}(y, \epsilon; x) := \pi_1(\tilde{f}_\epsilon^q(x, y)) - x - 2\pi p$$

is 2π-periodic in x and verifies the hypotheses

$$G(y_0^{p/q}, 0; x) = 0, \qquad \partial_1 G(y_0^{p/q}, 0; x) = q\omega'(y_0^{p/q}) \neq 0$$

of the Implicit Function Theorem at the point $(y, \epsilon) = (y_0^{p/q}, 0)$. Here, we have used the twist condition $\omega'(y) > 0$. Besides, y and ϵ are the variables, whereas x is considered just a parameter. Once proved the existence and uniqueness of $y_\epsilon^{p/q} : \mathbb{R} \to Y$, we determine $\hat{y}_\epsilon^{p/q} : \mathbb{R} \to Y$ by means of property ii). Finally, the uniformity in $x \in \mathbb{R}$ follows from the periodicity of the functions. $\square$

Corollary 4 *The radial curves $T_\epsilon^{p/q}$ and $\widehat{T}_\epsilon^{p/q}$ have the following properties:*

i) They have a non-empty intersection.
ii) Their intersection points are p/q-periodic points of $\tilde{f}_\epsilon$.
iii) They coincide identically if and only if $T_0^{p/q}$ persists.

Therefore, it is rather useful to quantify the separation between the radial curves $T_\epsilon^{p/q}$ and $\widehat{T}_\epsilon^{p/q}$, which is done in the next lemma. A dot means derivative.

Lemma 5 $\hat{y}_\epsilon^{p/q}(x) - y_\epsilon^{p/q}(x) = \dot{L}_\epsilon^{p/q}(x)$, *where $L_\epsilon^{p/q} : \mathbb{R} \to \mathbb{R}$ is the function*

$$L_\epsilon^{p/q}(x) = \sum_{k=1}^{q} h_\epsilon(\bar{x}_{k-1}^{p/q}(x;\epsilon), \bar{x}_k^{p/q}(x;\epsilon)), \quad \bar{x}_k^{p/q}(x;\epsilon) = \pi_1\Big(\tilde{f}_\epsilon^k(x, y_\epsilon^{p/q}(x))\Big).$$

PROOF. We shall not write the dependence on ϵ and p/q. Given any $x \in \mathbb{R}$, we introduce the notations

$$(\bar{x}_k, y_k) = \tilde{f}^k(x, y(x)), \qquad d_k = d_k(x) := \partial_1 \bar{x}_k(x;\epsilon), \qquad k = 0, \ldots, q.$$

Then $\bar{x}_0 = x$ and $\bar{x}_q = x + 2\pi p$, so $d_0 = d_q = 1$. Besides, $y_0 = y(x)$ and $y_q = \hat{y}(x)$. From the implicit equations (2), we get that $\partial_1 h(\bar{x}_0, \bar{x}_1) = -y_0$,

$\partial_2 h(\bar{x}_{q-1}, \bar{x}_q) = y_q$, and $\partial_2 h(\bar{x}_{k-1}, \bar{x}_k) + \partial_1 h(\bar{x}_k, \bar{x}_{k+1}) = 0$ for $k = 1, \ldots, q-1$. Therefore, $\dot{L}(x) = \partial_1 h(\bar{x}_0, \bar{x}_1) d_0 + \sum_{k=1}^{q-1} (\partial_2 h(\bar{x}_{k-1}, \bar{x}_k) + \partial_1 h(\bar{x}_k, \bar{x}_{k+1})) d_k + \partial_2 h(\bar{x}_{q-1}, \bar{x}_q) d_q = \hat{y}(x) - y(x)$. $\square$

Corollary 6 *The resonant curve $T_0^{p/q}$ persists if and only if $\dot{L}_\epsilon^{p/q}(x) \equiv 0$.*

Since the function $L_\epsilon^{p/q}(x)$ is 2π-periodic, it can be considered as a function defined on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We shall say that $L_\epsilon^{p/q} : \mathbb{T} \to \mathbb{R}$ is the *radial potential* of the resonant radial invariant curve $T_0^{p/q}$ under the perturbation f_ϵ.

2.4 The Melnikov radial potential

Once introduced the radial potential $L_\epsilon^{p/q}(x) = L_0^{p/q}(x) + \epsilon L_1^{p/q}(x) + \mathrm{O}(\epsilon^2)$, it is rather natural to extract information from its low-order terms. This is the main idea behind any Melnikov approach to a perturbative problem, see [4]. The zero-order term $L_0^{p/q}(x)$ is constant (and so useless), since

$$L_0^{p/q}(x) = \sum_{k=1}^{q} h_0(x_{k-1}^{p/q}, x_k^{p/q}) = \sum_{k=1}^{q} l_0(x_k^{p/q} - x_{k-1}^{p/q}) \equiv q l_0(2\pi p/q),$$

where $x_k^{p/q} = x_k^{p/q}(x) := \bar{x}_k^{p/q}(x; 0) = \pi_1(\tilde{f}_0^k(x, y_0^{p/q})) = x + 2\pi k p/q$, compare with (3). We shall say that the first-order term $L_1^{p/q} : \mathbb{R} \to \mathbb{R}$ is the *Melnikov radial potential* of the resonant invariant curve $T_0^{p/q}$ under the perturbation f_ϵ. In the following corollary, it is displayed its more important property in relation with the goals of this paper, whereas in the proposition below it is given a closed formula for its computation.

Corollary 7 *If $L_1^{p/q}(x)$ is not constant, the curve $T_0^{p/q}$ does not persist.*

Proposition 8 *$L_1^{p/q}(x) = \sum_{k=1}^{q} h_1(x_{k-1}^{p/q}, x_k^{p/q})$, where $x_k^{p/q} = x + 2\pi k p/q$. In particular, $L_1^{p/q}(x)$ is a $2\pi/q$-periodic function.*

PROOF. Given any $x \in \mathbb{R}$, we introduce the notations

$$x_k^{p/q} = x_k^{p/q}(x) := \bar{x}_k^{p/q}(x; 0), \qquad z_k^{p/q} = z_k^{p/q}(x) := \partial_2 \bar{x}_k^{p/q}(x; 0),$$

for $k = 0, \ldots, q$. The perturbed Lagrangian is $h_\epsilon = h_0 + \epsilon h_1 + \mathrm{O}(\epsilon^2)$. Then the first order term of the radial potential is

$$
\begin{aligned}
L_1^{p/q}(x) = & \sum_{k=1}^{q} h_1(x_{k-1}^{p/q}, x_k^{p/q}) + \\
& \partial_1 h_0(x_0^{p/q}, x_1^{p/q}) z_0^{p/q} + \partial_2 h_0(x_{q-1}^{p/q}, x_q^{p/q}) z_q^{p/q} + \\
& \sum_{k=2}^{q-1} \Big(\partial_1 h_0(x_k^{p/q}, x_{k+1}^{p/q}) + \partial_2 h_0(x_{k-1}^{p/q}, x_k^{p/q}) \Big) z_k^{p/q}.
\end{aligned}
$$

Using the implicit equations (2) for the integrable twist map, the third line vanishes. The second line of the above equation also vanishes, due to the fact that $\bar{x}_0^{p/q}(x;\epsilon) = x$ and $\bar{x}_q^{p/q}(x;\epsilon) = x + 2\pi p$ for small values of ϵ. Besides, $x_k^{p/q} = x_k^{p/q}(x) = \bar{x}_k^{p/q}(x;0) = \pi_1\Big(\tilde{f}_0^k(x, y_0^{p/q})\Big) = x + 2\pi kp/q$.

Finally, it is clear that $L_1^{p/q}(x)$ is a periodic function with periods 2π and $2\pi p/q$. But p and q are relatively prime. Thus, $2\pi/q$ is also a period. □

3 Classical Billiards

Let Γ be a smooth closed strictly convex curve in the plane $\mathbb{R}^2$. Without loss of generality we can assume that Γ has length 2π; it is just a technical normalization condition. Let $\gamma : \mathbb{T} \to \Gamma$ be an arc-parameterization of this curve and $\tilde{\gamma} : \mathbb{R} \to \Gamma$ be a fixed lift. Finally, let us consider the open cylinder $C = \mathbb{T} \times (y_-, y_-)$, with $y_\pm = \pm 1$. Then we can model the classical billiard dynamics inside Γ by means of a map $f : C \to C$, $f(s,y) = (s', y')$, defined as follows. If the particle hits Γ at a point $\gamma(s)$ under an angle of incidence $\phi \in (0,\pi)$, then the next impact point is $\gamma(s')$ and the next angle of incidence is $\phi' \in (0,\pi)$. Here, $y = -\cos\phi$ and $y' = -\cos\phi'$. It is well-known [6, §9.2] that, in these coordinates, the billiard map f is an area-preserving twist map whose Lagrangian is

$$
h(x,x') = |\tilde{\gamma}(x) - \tilde{\gamma}(x')| .
$$

Besides, f verifies the rigid boundary conditions with $\omega_- = 0$ and $\omega_+ = 2\pi$.

Let us consider now the classical billiard dynamics inside a circumference Γ_0 of radius $r_0 = 1$, since Γ_0 must have length 2π. Then $\gamma_0(s) = \mathbf{n}_s = (\cos s, \sin s)$ is an arc-parameterization. Let $f_0 : C \to C$ be the area-preserving twist map that models the classical billiard dynamics inside Γ_0 in the coordinates (s,y). It is integrable. In fact, it is easy to check that

$$
\tilde{f}_0(x,y) = (x + \omega(y), y), \qquad \omega(y) = 2\cos^{-1}(-y),
$$

where $\cos^{-1} : (-1,1) \to (0,\pi)$ stands for the inverse of the cos function, and so $\omega : (-1,1) \to (0,2\pi)$ is a diffeomorphism.

Thus, given any integers $q \geq 2$ and $1 \leq p < q$ relatively prime, there exist only one radial coordinate $y_0^{p/q} \in (-1,1)$ verifying the equality

$$\omega(y_0^{p/q}) = 2\pi p/q,$$

and so there exists just one resonant invariant curve $T_0^{p/q} = \mathbb{R}(y_0^{p/q})$ with rotation number p/q. The curves $T_0^{p/q}$, $1 \leq p < q$, are in 2-to-1 correspondence with the continuous families $\mathcal{I}_0^{p/q}$, $1 \leq p \leq q/2$, of regular p/q-gons inscribed in Γ_0 mentioned in the Introduction (but in the case $q = 2$, of course). This has to do with the fact that a p/q-gon can be traversed clockwise or counterclockwise. So we can restrict our attention to the case $1 \leq p \leq q/2$.

This correspondence between resonant invariant curves of a (classical) billiard map and continuous families of inscribed polygons whose consecutive sides make equal angles with the billiard curve holds for any smooth strictly convex curve. Therefore, in order to prove the claim contained Theorem 1 about the families $\mathcal{I}_0^{p/q}$, it suffices to prove the same claim about the invariant curves $T_0^{p/q}$, which follows directly from Corollary 7 and the next Proposition.

Proposition 9 *The Melnikov radial potential of the resonant invariant curve $T_0^{p/q}$ of the classical billiard map inside the perturbed circumference (1) is*

$$L_1^{p/q}(x) = 2\sin(\pi p/q) \sum_{k=1}^{q} r_1(x + 2k\pi p/q) = 2q\sin(\pi p/q) \sum_{j \in q\mathbb{Z}} \hat{r}_1^j \, \mathrm{e}^{\mathrm{i}jx}.$$

PROOF. The second equality is trivial: the sum $\sum_{k=1}^{q} \mathrm{e}^{\mathrm{i}j(x+2k\pi p/q)}$ equals to $q\,\mathrm{e}^{\mathrm{i}jx}$ when $j \in q\mathbb{Z}$, and it vanishes otherwise. It remains to prove the first equality.

Let $\gamma_\epsilon(s)$ be an arc-parameterization of Γ_ϵ. Then $h_\epsilon(x,x') = |\tilde{\gamma}_\epsilon(x) - \tilde{\gamma}_\epsilon(x')|$ is the Lagrangian of the classical billiard map inside Γ_ϵ. We have shown in Proposition 8 that, if $h_\epsilon = h_0 + \epsilon h_1 + \mathrm{O}(\epsilon^2)$, the radial Melnikov potential is

$$L_1^{p/q}(x) = \sum_{k=1}^{q} h_1(x_{k-1}^{p/q}, x_k^{p/q}), \qquad x_k^{p/q} = \pi_1\left(\tilde{f}_0^k(x, y_0^{p/q})\right) = x + 2\pi kp/q.$$

Next, we compute the term $h_1(x_{k-1}^{p/q}, x_k^{p/q})$ that appears in this formula.

From equation (1), we get that the arc-parameterization has the form

$$\gamma_\epsilon(s) = r_\epsilon(\theta_\epsilon(s))\mathbf{n}_{\theta_\epsilon(s)}$$

for some 2π-periodic function $\theta_\epsilon(s)$. We know that $\gamma_0(s) = r_0\mathbf{n}_s$, so $\theta_0(s) = s$. Let $\theta_\epsilon(s) = s + \epsilon\theta_1(s) + \mathrm{O}(\epsilon^2)$ and $\gamma_\epsilon(s) = r_0\mathbf{n}_s + \epsilon\gamma_1(s) + \mathrm{O}(\epsilon^2)$. Then it turns out that $\gamma_1(s) = r_0\theta_1(s)\mathbf{t}_s + r_1(s)\mathbf{n}_s$, where $\mathbf{t}_s = (-\sin s, \cos s)$ and $\mathbf{n}_s = (\cos s, \sin s)$. In particular,

$$\tilde{\gamma}_1(x_k^{p/q}) = r_0\theta_1(x_k^{p/q})\mathbf{t}_k^{p/q} + r_1(x_k^{p/q})\mathbf{n}_k^{p/q}, \qquad \mathbf{n}_k^{p/q} = \mathbf{n}_{x_k^{p/q}}, \quad \mathbf{t}_k^{p/q} = \mathbf{t}_{x_k^{p/q}}.$$

On the other hand, $h_1(x_{k-1}^{p/q}, x_k^{p/q}) = \langle u_k^{p/q}, \tilde{\gamma}_1(x_{k-1}^{p/q}) - \tilde{\gamma}_1(x_k^{p/q})\rangle$, where

$$u_k^{p/q} = \frac{\mathbf{n}_{k-1}^{p/q} - \mathbf{n}_k^{p/q}}{|\mathbf{n}_{k-1}^{p/q} - \mathbf{n}_k^{p/q}|}.$$

Finally, we obtain the radial Melnikov potential in the desired form:

$$\begin{aligned}
L_1^{p/q}(x) &= \sum_{k=1}^{q} h_1(x_{k-1}^{p/q}, x_k^{p/q}) \\
&= \sum_{k=1}^{q} \langle u_k^{p/q}, \tilde{\gamma}_1(x_k^{p/q}) - \tilde{\gamma}_1(x_{k-1}^{p/q})\rangle \\
&= \sum_{k=1}^{q} \langle u_k^{p/q} - u_{k+1}^{p/q}, \tilde{\gamma}_1(x_k^{p/q})\rangle \\
&= 2\sin(\pi p/q) \sum_{k=1}^{q} r_1(x_k^{p/q}),
\end{aligned}$$

since $\langle u_k^{p/q} - u_{k+1}^{p/q}, \mathbf{t}_k^{p/q}\rangle = 0$, $\langle u_k^{p/q} - u_{k+1}^{p/q}, \mathbf{n}_k^{p/q}\rangle = 2\sin(\pi p/q)$, $x_{k+q}^{p/q} = x_k^{p/q} + 2\pi p$, and $u_{k+q}^{p/q} = u_k^{p/q}$. $\square$

4 Dual billiards

Let Γ be a smooth closed strictly convex curve in the plane $\mathbb{R}^2$. Let U_Γ be the unbounded component of $\mathbb{R}^2 \setminus \Gamma$, that is, the "exterior" of Γ. The dual billiard map $f : U_\Gamma \to U_\Gamma$ is defined as follows: $z' = f(z)$ is the reflection of z in the tangency point of the oriented tangent line to Γ through z. This map is area-preserving [10]. Next we shall describe some useful coordinates on U_Γ. We skip many details, which can be found in the nice paper of P. Boyland [3].

To begin with, let $\gamma : \mathbb{T} \to \Gamma$ be an *envelope parameterization* of Γ, that is, a parameterization of the curve Γ such that

$$\dot{\gamma}(s) = \rho(s)\mathbf{t}_s, \qquad \mathbf{t}_s = (-\sin s, \cos s),$$

for some function $\rho(s) > 0$, which is the *radius of curvature* of Γ. Given any $s \in \mathbb{T}$, let $\mathcal{L}_s$ be the tangent line to Γ at the point $\gamma(s)$. We also consider the decomposition $\mathcal{L}_s = \mathcal{L}_s^- \cup \{\gamma(s)\} \cup \mathcal{L}_s^+$, where

$$\mathcal{L}_s^- = \{z \in \mathcal{L}_s : \langle z - \gamma(s), \mathbf{t}_s \rangle < 0\}, \quad \mathcal{L}_s^+ = \{z \in \mathcal{L}_s : \langle z - \gamma(s), \mathbf{t}_s \rangle > 0\}.$$

The *height function* $p(s)$ of Γ is the distance from the origin to the line $\mathcal{L}_s$. Then $\beta(s) = p(s)\mathbf{n}_s$ is the nearest point to the origin on the line $\mathcal{L}_s$. Boyland proves that the height function verifies the second order differential equation

$$p'' + p = \rho, \tag{4}$$

and that $p'(s)$ equals to the signed distance from $\gamma(s)$ to $\beta(s)$.

Finally, let $\ell(s,t)$ be the distance from $\gamma(s)$ to the intersection $\mathcal{L}_s \cap \mathcal{L}_t$. We note that $\ell(s,t) \neq \ell(t,s)$. This function can be expressed in terms of the radius of curvature or the height function. Concretely,

$$\ell(s,t) = \frac{\int_s^t \sin(v-s)\rho(v)\,\mathrm{d}v}{\sin(t-s)} = \frac{p(s)}{\sin(t-s)} - \frac{p(t)}{\tan(t-s)} + p'(t).$$

The first equality is contained in [3]. The second one follows from equation (4); it suffices to integrate by parts twice.

Given any point $z \in U_\Gamma$, let $s = s(z)$ be the unique angle such that $z \in \mathcal{L}_s^-$, and let $y = y(z) = \ell(z)^2/2$ with $\ell(z) = |z - \gamma(s)|$. The map $U_\Gamma \ni z \mapsto (s,y) \in C$ is an area-preserving diffeomorphism, where $C = \mathbb{T} \times (y_-, y_+)$ with $y_- = 0$ and $y_+ = +\infty$. The coordinates (s,y) will be called *envelope coordinates*. The dual billiard map $f : C \to C$, $f(s,y) = (s',y')$, is an area-preserving twist map in these envelope coordinates [5,3]. Besides, it verifies the rigid boundary conditions with $\omega_- = 0$ and $\omega_+ = \pi$.

From the the area-preserving character of the change $z \mapsto (s,y)$ and the geometric interpretation given in Remark 2, the Lagrangian of f is

$$h(x,x') = \int_x^{x'} y(x,u)\,\mathrm{d}u, \qquad y(x,u) = \tfrac{1}{2}\tilde{\ell}(x,u)^2.$$

Hence, the quantity $h(x,x')$ is the area bounded by the curve Γ and the tangent lines $\mathcal{L}_x^+$ and $\mathcal{L}_{x'}^-$.

Let us consider now the dual billiard dynamics outside a circumference Γ_0 of radius r_0. Its radius of curvature and its height function are $\rho_0(s) = p_0(s) = r_0$.

Then $\gamma_0(s) = r_0 \mathbf{n}_s = (r_0 \cos s, r_0 \sin s)$ is an envelope parameterization of Γ_0, and $\ell_0(s,t) = r_0(1-\cos(t-s))/\sin(t-s) = r_0 \tan \frac{t-s}{2}$.

Let $f_0 : C \to C$ be the area-preserving twist map that models the dual billiard dynamics outside Γ_0 in the coordinates (s,y). The map f_0 is integrable. In fact, it is easy to check that

$$\tilde{f}_0(x,y) = (x + \omega(y), y), \qquad \omega(y) = 2\tan^{-1}\left(\sqrt{2y}/r_0\right),$$

where $\tan^{-1} : (0,+\infty) \to (0,\pi/2)$ stands for the inverse of the tan function, and so $\omega : (0,+\infty) \to (0,\pi)$ is a diffeomorphism.

Thus, given any integers $q \geq 3$ and $1 \leq p < q/2$ relatively prime, there exist only one radial coordinate $y_0^{p/q} \in (0,+\infty)$ verifying the equality

$$\omega(y_0^{p/q}) = 2\pi p/q,$$

and so there exists just one resonant invariant curve $T_0^{p/q} = \mathbb{R}(y_0^{p/q})$ with rotation number p/q. It is clear that the curves $T_0^{p/q}$, $1 \leq p < q/2$, are in one-to-one correspondence with the continuous families $\mathcal{C}_0^{p/q}$, $1 \leq p < q/2$, of regular p/q-gons circumscribed about Γ_0 mentioned in the Introduction. Therefore, to prove the claim contained Theorem 1 about the families $\mathcal{C}_0^{p/q}$, it suffices to prove the same claim about the invariant curves $T_0^{p/q}$, which follows directly from Corollary 7 and the next Proposition.

Proposition 10 *The Melnikov radial potential of the resonant invariant curve $T_0^{p/q}$ of the dual billiard map outside the perturbed circumference (1) is*

$$\begin{aligned} L_1^{p/q}(x) &= 2r_0 \tan(\pi p/q) \sum_{k=1}^{q} r_1(x + 2k\pi p/q) - p r_0 \int_0^{2\pi} r_1(x)\, \mathrm{d}x \\ &= 2r_0 q \tan(\pi p/q) \sum_{j \in q\mathbb{Z}} \hat{r}_1^j \, \mathrm{e}^{\mathrm{i}jx} - p r_0 \hat{r}_1^0. \end{aligned}$$

PROOF. The second equality follows from a trivial computation with Fourier coefficients, as in the previous section. It remains to prove the first one.

Let $p_\epsilon(s) = r_0 + \epsilon p_1(s) + \mathrm{O}(\epsilon^2)$ and $\gamma_\epsilon(s) = r_0 \mathbf{n}_s + \mathrm{O}(\epsilon)$ be the height function and an envelope parameterization of the perturbed circumference (1). Then the perturbed Lagrangian is

$$h_\epsilon(x,x') = h_0(x,x') + \epsilon h_1(x,x') + \mathrm{O}(\epsilon^2) = \tfrac{1}{2} \int_x^{x'} \tilde{\ell}_\epsilon(x,u)^2 \, \mathrm{d}u,$$

where $\ell_\epsilon(s,t) = \ell_0(s,t) + \epsilon\ell_1(s,t) + \mathrm{O}(\epsilon^2) = \frac{p_\epsilon(s)}{\sin(t-s)} - \frac{p_\epsilon(t)}{\tan(t-s)} + p'_\epsilon(t)$.

First, we are going to show that $p_1(s) = r_1(s)$, and consequently

$$\ell_1(s,t) = \frac{r_1(s)}{\sin(t-s)} - \frac{r_1(t)}{\tan(t-s)} + r'_1(t).$$

To prove it, let us consider the triangle formed by $A = \alpha_\epsilon(s) = r_\epsilon(s)\mathbf{n}_s \in \Gamma_\epsilon$, $B = \beta_\epsilon(s) = p_\epsilon(s)\mathbf{n}_s \in \mathcal{L}_s$, and $C = \gamma_\epsilon(s) \in \Gamma_\epsilon \cap \mathcal{L}_s$. We recall that $\mathcal{L}_s$ is the tangent line to Γ_ϵ at the point C. We also know that $\mathrm{dist}(C,B) = |p'_\epsilon(s)| = \mathrm{O}(\epsilon)$. If $\hat{A}$, $\hat{B}$, and $\hat{C}$ are the angles of the triangle, then $\hat{B} = \pi/2$ and $\hat{C} = \mathrm{O}(\epsilon)$. The first claim follows from the orthogonality between $\mathcal{L}_s$ and $\mathbf{n}_s$. The second one follows from the fact that the line through the points $A, C \in \Gamma_\epsilon$ tends to the tangent line to the curve Γ_ϵ at the point C (that is, to $\mathcal{L}_s$) as A tends to C, because the curve is smooth. Hence

$$|r_\epsilon(s) - p_\epsilon(s)| = \mathrm{dist}(A,B) = \mathrm{dist}(C,B)\cdot\tan\hat{C} = |p'_\epsilon(s)|\tan\hat{C} = \mathrm{O}(\epsilon^2),$$

and so $r_1(s) = p_1(s)$.

Now we are ready to compute the first-order term $h_1(x,x')$ of the Lagrangian. To begin with, we recall that $\ell_0(s,t) = r_0\tan\frac{t-s}{2}$ and so

$$\ell_0(s,t)\ell_1(s,t) = r_0\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\tan\tfrac{t-s}{2}(r_1(s)+r_1(t))\Big\} - r_0 r_1(t).$$

Then, using that $h_1(x,x') = \int_x^{x'} \tilde{\ell}_0(x,u)\tilde{\ell}_1(x,u)\,\mathrm{d}u$, we get that

$$h_1(x,x') = r_0\tan\tfrac{x'-x}{2}(r_1(x)+r_1(x')) - r_0\int_x^{x'} r_1(u)\,\mathrm{d}u.$$

Finally, the radial Melnikov potential is

$$\begin{aligned} L_1^{p/q}(x) &= \sum_{k=1}^{q} h_1(x_{k-1}^{p/q}, x_k^{p/q}) \\ &= r_0\tan(\pi p/q)\sum_{k=1}^{q}(r_1(x_{k-1}^{p/q}) + r_1(x_k^{p/q})) - r_0\int_{x_0^{p/q}}^{x_q^{p/q}} r_1(u)\,\mathrm{d}u \\ &= 2r_0\tan(\pi p/q)\sum_{k=1}^{q} r_1(x_k^{p/q}) - pr_0\int_0^{2\pi} r_1(u)\,\mathrm{d}u, \end{aligned}$$

since $x_k^{p/q} = \pi_1\Big(\tilde{f}_0^k(x, y_0^{p/q})\Big) = x + 2\pi kp/q$ and $r_1(u)$ is 2π-periodic. $\square$

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