Supersymmetric Dirichlet operators, spectral gaps, and correlations

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Abstract

In this article we construct a supersymmetric extension of the Dirichlet operator associated with a tempered Gibbs measure on $\mathbb{R}^{\mathbb{Z}^d}$. Under fairly general assumptions on the interaction potentials we show that the Dirichlet operator (resp. its supersymmetric extension) is essentially selfadjoint on the set of smooth, bounded cylinder functions (resp. differential forms), for all inverse temperatures. Assuming that the single-site potentials have a non-degenerated minimum and no other critical point we prove that, for sufficiently large inverse tempertures, one observes a number of subsequent gaps in the spectrum of the Dirichlet operator. For translation invariant systems with a sufficiently weak (but in general infinite range) ferromagnetic interaction, we prove the validity of a formula for the leading asymptotics of the correlation of two spin variables, as their distance and the inverse temperature tend to infinity, which has originally been derived in [33] for finite-dimensional Gibbs measures. In the appendix we add a remark on the uniqueness of tempered Gibbs measures at low temperatures.

Keywords: Dirichlet operator, spectral gap, supersymmetry, correlation.

1 Introduction

In many recent articles the spectral theory of Witten-Laplacians [35] in large dimension has been studied and applied to the analysis of lattice systems with real-valued spins in classical statistical mechanics [10, 11, 12, 18, 22, 24, 29, 33, 34]. These Witten-Laplacians are Hodge-Laplacians corresponding to a deformed exterior derivative and act on the space of differential forms. Their restriction to zero-forms are unitarily equivalent to Dirichlet operators associated with finite dimensional Gibbs measures of the spin system. An identity due to Helffer and Sjöstrand represents the truncated two-point correlation functions of the spin systems as matrix elements of the inverse of a Witten-Laplacian on one-forms. The references cited above deal with finite dimensional spin systems and thier purpose is to obtain results which are uniform as the number of spins increases. On the other hand there are many recent results on the theory of (infinite-dimensional) Gibbs states and related Dirichlet forms and operators [4, 5, 6, 7] that suggest to apply the techniques of [10, 29] in a suitably adapted form directly to infinite-dimensional spin systems. This is carried out in the present paper. Namely, we pick some tempered Gibbs state, μ_β, on $\mathbb{R}^{\mathbb{Z}^d}$ and introduce an exterior derivative, $\bar{d}_\beta$, acting in $\mathscr{H} = L^2(\mu_\beta)\widehat{\otimes}\mathfrak{F}_{\mathrm{f}}$, where $\mathfrak{F}_{\mathrm{f}}$ is the fermionic Fock space over $\ell^2_{\mathbb{C}}(\mathbb{Z}^d)$. With this we define an infinite-dimensional analogue of the Witten-Laplacian setting $L_\beta := \bar{d}_\beta \bar{d}_\beta^* + \bar{d}_\beta^* \bar{d}_\beta$. It splits into a direct sum $L_\beta = \bigoplus_{k=0}^\infty L_\beta^{(k)}$, where $L_\beta^{(k)}$ acts in the k-particle sector of $\mathscr{H}$, or, in the space of k-forms. We show that L_β is essentially selfadjoint on the space of smooth, bounded, finitely based forms, for all inverse temperatures $\beta \geqslant 1$, under rather general assumptions on the interaction potentials: The main assumption on the on-site potentials is that, for large spin values, their second derivatives can be controlled by their first derivatives. The interaction between different spins is allowed to have infinite range as long as it is sufficiently weak and decays exponentially. Using the essential selfadjointness of L_β we show that it is supersymmetric in the sense that it can be written as the square of a selfadjoint supercharge. Additionally assuming that the on-site potentials possess a unique, non-degenerated minimum and no other critical point, and supposing that the inverse temperature β is sufficiently large, we derive Hodge decompositions of the spaces of k-forms and locate a number of gaps in the spectrum of $L_\beta^{(k)}$, provided k is small enough depending on β. We apply our results to the asymptotic analysis of the truncated correlation of two spin variables with respect to a pure Gibbs measure that is invariant under translations and space inversions on the lattice. Since the assumptions required for the analysis of the correlations are fulfilled by a class of lattice models where the set of tempered Gibbs measures consists of only one point, we also show how one can prove the uniqueness of the tempered Gibbs measure at low temperatures without assuming the on-site potentials to be strictly convex or the interaction to have finite range.

We note that supersymmetric Dirichlet operators are also constructed in [2] in the case of tempered Gibbs measures, and in [1] on a countable product of compact Riemannian manifolds. In an infinite-dimensional setting the Helffer-Sjöstrand formula for the correlation is also used in [16, 21, 30]. We also remark that a first version of the present work is contained in [28].

The plan of the paper is the following: In Section 0.2 we formulate our hypotheses on the lattice spin model precisely and recall the notion of a tempered Gibbs measure as well as some basic properties of it derived in [6]. In Section 0.3 we construct the operators $\bar{d}_\beta$ and L_β and state and prove our results on essential selfadjointness, Hodge decompositions, and spectral gaps, refering to some relative form bounds involving L_β from Section 0.6. In Section 0.4 we prove that L_β can be represented as the square of a selfadjoint supercharge. In Section 0.5 we show that L_β is essentially selfadjoint on the space of smooth, bounded, finitely based forms. Here we first apply a criterion from [26] in order to prove that $L_\beta^{(0)}$ is essentially selfadjoint on smooth, bounded, finitely based functions. Then we obtain the result for L_β by applying the Kato-Rellich theorem together with a relative operator bound which is derived in Section 0.6. The relative bounds are obtained by combinig ideas from [10] with DLR-equations. The proof of the main theorem on the spectral gaps is also contained in Section 0.6. In Section 0.7 we show that the Helffer-Sjöstrand formula for the correlation functions of pure, tempered Gibbs measures is valid, for sufficiently large β. We also introduce some further hypotheses on the interaction potentials, which allow us to derive the main theorem on the correlation asymptotics which is proved in Section 0.8. In this last section we follow closely the lines of [10] with some small modifications. As a starting point we use an asymptotic formula for certain families of convolution operators due to [33, 10]. Apart from this we present everything in detail in order to be more selfcontained. In Appendix A we collect some basic and elementary results refered to in the main text. In Appendix .2 we sketch how one can prove uniqueness of tempered Gibbs measures at low temperatures. Here, the on-site potentials are allowed to have several local minima besides a unique global minimum and the interaction may still have infinite range. This situation does not seem to be covered by the existing literature on lattice models with non-compact spin spaces (e.g. [15, 19]). Of course, we only consider temperatures below some possible critical value where a phase transition might occur [20].

2 Tempered Gibbs states for the lattice model

For some $d \in \mathbb{N}$, we consider a classical continuous lattice spin system on the lattice $\mathbb{Z}^d$, which is endowed with the Euclidean distance. The configuration space is $X := \mathbb{R}^{\mathbb{Z}^d}$, the set of all real sequences $(x_k)_{k\in\mathbb{Z}^d}$. For subsets $\Lambda \subseteq \mathbb{Z}^d$, and $x \in X$, or $x \in \mathbb{R}^\Lambda$, we write $x_\Lambda := (x_k)_{k\in\Lambda}$. If, moreover, $\varnothing \neq \Sigma, \Upsilon \subsetneq \mathbb{Z}^d$, and $\Sigma \cup \Upsilon = \Lambda$, we also write $x_\Lambda = (x_\Sigma|x_\Upsilon) = (x_\Upsilon|x_\Sigma)$. Constant sequences in X are denoted by $\mathbf{0} := (0)_{k\in\mathbb{Z}^d}$, $\mathbf{1} := (1)_{k\in\mathbb{Z}^d}$ etc. We equip X with the product topology so that X becomes a Polish space. Furthermore, we consider X as a measurable space equipped with the Borel σ-algebra $\mathfrak{B}(X)$ induced by the product topoplogy.

Let $\mathfrak{P}_\mathrm{f}$ denote the set of all finite subsets of $\mathbb{Z}^d$. The lattice spin system investigated in this article is given by a family of interaction potentials, $\Phi := \{\Phi_\Lambda\}_{\Lambda\in\mathfrak{P}_\mathrm{f}}$, where $\Phi_\Lambda : \mathbb{R}^\Lambda \to \mathbb{R}$, for all $\Lambda \in \mathfrak{P}_\mathrm{f}$. In the following we always assume that Φ fulfills

Hypothesis 1 *(i) For all $\Lambda \in \mathfrak{P}_\mathrm{f}$, we have $\Phi_\Lambda \in C^3(\mathbb{R}^\Lambda, \mathbb{R})$, and Φ_Λ is invariant under permutations of the coordinates.*

(ii) There exist constants $b_1, b_2, c, e > 0$, and some $L > 2$, such that, for all $t \in \mathbb{R}$, and $k \in \mathbb{Z}^d$, we have

$$|\Phi^{(\ell)}_{\{k\}}(t)| \leqslant c\,(1+|t|)^{L-\ell}, \qquad \ell = 0,1,2, \tag{1}$$

$$\Phi''_{\{k\}}(t) \geqslant -e, \tag{2}$$

$$\big|\, \Phi''_{\{k\}}(t) - \Phi''_{\{k\}}(0) \,\big| \leqslant b_1\,|\Phi'_{\{k\}}(t)| + b_2 . \tag{3}$$

Furthermore, there are constants $a, \sigma > 0$ and $\rho \geqslant 0$ such that

$$\Phi'_{\{k\}}(t)\,\mathrm{sgn}(t) \geqslant a\,\max\{|t|^{1+\sigma}, |t|\}, \qquad \text{for} \qquad |t| \geqslant \rho . \tag{4}$$

(iii) *For all $\Lambda \in \mathfrak{P}_{\mathrm{f}}$, $\#\Lambda \geqslant 2$, we have $\Phi_\Lambda(0) = 0$ and $\nabla\Phi_\Lambda(0) = 0$, and there exists a constant $J_\Lambda > 0$ such that, for all $x_\Lambda \in \mathbb{R}^\Lambda$, and $j, k, \ell \in \Lambda$, we have*

$$\big|\,\partial_k\partial_\ell\Phi_\Lambda(x_\Lambda)\,\big|\,,\ \big|\,\partial_j\partial_k\partial_\ell\Phi_\Lambda(x_\Lambda)\,\big| \ \leqslant\ J_\Lambda\ . \tag{5}$$

The constants J_Λ are invariant under translations on the lattice, i.e. $J_\Lambda = J_{\Lambda+k}$, for all $\Lambda \in \mathfrak{P}_{\mathrm{f}}$, and $k \in \mathbb{Z}^d$. Writing

$$\mathscr{J}_p(\varkappa) \quad := \quad \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni 0,\\ \#\Lambda\geqslant 2}} J_\Lambda(\#\Lambda)^p\, \exp\Big(\varkappa \max_{k\in\Lambda}|k|\Big)\,, \tag{6}$$

where $\varkappa \geqslant 0$, $p \in \mathbb{N}$, we moreover assume that there is some $\gamma > 0$ such that $\mathscr{J}_2(\gamma) < \infty$, and that $\mathscr{J}_1(0) < a$.

Remark 2.1 If Φ fulfills Hypothesis 1 with $b_2 = \rho = 0$, then (3) and (4) imply that the on-site potential $\Phi_{\{k\}}$ has a unique, non-degenerate local minimum at zero and no other critical point. Moreover, it holds $\Phi''_{\{k\}}(0) \geqslant a > \mathscr{J}_1(0)$, for all $k \in \mathbb{Z}^d$,

Hypothesis 2 Φ *fulfills Hypothesis 1 with $b_2 = \rho = 0$.*

Next, we recall the notions of local Hamilton functions, partition functions, local specifications, and tempered Gibbs measure for the interaction Φ; see, e.g. [31]. To this end we first introduce a scale of Hilbert subspaces, $(S_p, \langle\,\cdot\,|\,\cdot\,\rangle_p)$, of X by

$$S_p \quad := \quad \Big\{x \in X : |x|_p = \Big(\sum_{k\in\mathbb{Z}^d}(1+|k|)^{2p}|x_k|^2\Big)^{1/2} \ < \ \infty\Big\}\,,$$

where $p \in \mathbb{Z}$. We set $S' := \bigcup_{p\in\mathbb{N}} S_{-p}$, and denote the complexification of $(S_p, \langle\,\cdot\,|\,\cdot\,\rangle_p)$ by $(S_{p,c}, \langle\,\cdot\,|\,\cdot\,\rangle_{p,c})$. Clearly, for $p \in \mathbb{Z}$, the dual space of S_p is S_{-p}, and the dual pairing of S_p and S_{-p} is given by the scalar product on $S_0 = \ell^2_{\mathbb{R}}(\mathbb{Z}^d)$, i.e. we write $\langle\, x\,|\,y\,\rangle_0$, for the dual pairing of $x \in S_{-p}$, and $y \in S_p$. It holds $S_p \in \mathfrak{B}(X)$, for $p \in \mathbb{Z}$, and the Borel-σ-algebra on S_p defined by the Hilbert space topology on S_p satisfies $\mathfrak{B}(S_p) = \big\{\,A \cap S_p \ :\ A \in \mathfrak{B}(X)\,\big\}$; see [14, Chapter 1].

The Hamilton function in the volume $\Upsilon \in \mathfrak{P}_{\mathrm{f}}$ with boundary condition $y \in S'$ is given by

$$H_\Upsilon(x_\Upsilon|y_{\Upsilon^c}) \ := \sum_{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\Lambda\cap\Upsilon\neq\varnothing} \Phi_\Lambda(x_{\Lambda\cap\Upsilon}|y_{\Lambda\cap\Upsilon^c})\,, \qquad x_\Upsilon \in \mathbb{R}^\Upsilon\ . \tag{7}$$

Due to Lemma .1.1 the sum in (7) converges uniformly on closed balls in $S_{-p} \times S_{-p}$, where $p > d/2$, and it holds $\inf_{x\in X} H_\Upsilon(x_\Upsilon|y_{\Upsilon^c}) > -\infty$, for all $\Upsilon \in \mathfrak{P}_{\mathrm{f}}$, and $y \in S'$. In particular, $H_\Upsilon(\,\cdot\,|\,\cdot\,)$ becomes a $\mathfrak{B}(X) \times \mathfrak{B}(X)$-measurable function, if we set $H_\Upsilon(\,\cdot\,|\,\cdot\,) \equiv \infty$ on $(X \times X) \setminus (S' \times S')$.

The partition function in the volume $\Upsilon \in \mathfrak{P}_{\mathrm{f}}$ with boundary condition $y \in S'$ at inverse temperature $\beta \in [1,\infty)$ is defined by

$$\mathscr{Z}_\Upsilon(\beta, y) \;:=\; \int\limits_{\mathbb{R}^\Upsilon} \exp\Big(-\beta\, H_\Upsilon(x_\Upsilon | y_{\Upsilon^c})\Big)\, dx_\Upsilon \;, \tag{8}$$

where dx_Υ denotes the Lebesgue measure on $(\mathbb{R}^\Upsilon, \mathfrak{B}(\mathbb{R}^\Upsilon))$. Again due to Lemma .1.1 $\mathscr{Z}_\Upsilon(\beta, y)$ is finite, for $\Upsilon \in \mathfrak{P}_{\mathrm{f}}$, $\beta \in [1,\infty)$, and $y \in S'$.

The local specification associated with the interaction Φ at inverse temperature $\beta \in [1,\infty)$ is by definition given by the family of stochastic kernels

$$\mu_{\beta,\Upsilon} : \mathfrak{B}(X) \times X \;\longrightarrow\; [0,1]\;, \qquad \beta \in [1,\infty)\;,\; \Upsilon \in \mathfrak{P}_{\mathrm{f}}\;,$$

where, for all $A \in \mathfrak{B}(X)$,

$$\mu_{\beta,\Upsilon}(A, y) \;:=\; \frac{1}{\mathscr{Z}_\Upsilon(\beta,y)} \int\limits_{\mathbb{R}^\Upsilon} \mathbf{1}_A(x_\Upsilon | y_{\Upsilon^c}) \exp\Big(-\beta\, H_\Upsilon(x_\Upsilon|y_{\Upsilon^c})\Big)\, dx_\Upsilon\;, \tag{9}$$

if $y \in S'$, and $\mu_{\beta,\Upsilon}(A,y) := 0$, if $y \in X \setminus S'$. By Fubini's theorem the map $X \ni y \mapsto \mu_{\beta,\Upsilon}(A,y)$ is $\mathfrak{B}(X)$-measurable, for all $\beta \in [1,\infty)$, $\Upsilon \in \mathfrak{P}_{\mathrm{f}}$, and $A \in \mathfrak{B}(X)$.

Definition 2.2 *Assume that Φ fulfills Hypothesis 1. A Gibbs measure for the interaction Φ at inverse temperature $\beta \in [1,\infty)$ is a probability measure μ_β on the measurable space $(X, \mathfrak{B}(X))$ that satisfies the Dobrushin-Lanford-Ruelle (DLR) equilibrium equations,*

$$\int\limits_X \mu_{\beta,\Upsilon}(A,y)\, d\mu_\beta(y) \;=\; \mu_\beta(A)\;, \qquad \Upsilon \in \mathfrak{P}_{\mathrm{f}}\;,\;\; A \in \mathfrak{B}(X)\;. \tag{10}$$

A Gibbs measure μ_β is called tempered iff there exists some $p > d/2$ such that

$$\mu_\beta(S_{-p}) \;=\; 1\;. \tag{11}$$

The set of tempered Gibbs measures for the interaction Φ at inverse temperature $\beta \in [1,\infty)$ is denoted by $\mathscr{G}_t(\beta,\Phi)$.

In the following we denote by $C^k(S_{-p})$, $k \in \mathbb{N}$, the set of all complex-valued, k-times continuously Fréchet differentiable functions on S_{-p}. Furthermore, we set, for $k = 0,1,2,\ldots$,

$$\begin{aligned} C_0^k(S_{-p}) &= \big\{ f \in C^k(S_{-p}) :\; \exists\, R > 0 : f(x) = 0\;,\; |x|_{-p} \geqslant R \big\}\;, \\ C_{\mathrm{pol}}^k(S_{-p}) &= \big\{ f \in C^k(S_{-p}) :\; \exists\, C > 0\, \exists\, m \in \mathbb{N}\, \forall\, x \in S_{-p} \;: \\ &\qquad\qquad \sum_{\ell=0}^{k} \|f^{(\ell)}(x)\| \leqslant C(1 + |x|_{-p})^m \;\big\}\;. \end{aligned}$$

As usual, we say that the partial derivative in direction $e_k := (\delta_{k\ell})_{\ell\in\mathbb{Z}^d}$, $k \in \mathbb{Z}^d$, of a function $f : S_{-p} \to \mathbb{C}$ at $x \in S_{-p}$ exists if and only if the limit $\lim\limits_{\tau\to 0} \frac{1}{\tau}(f(x+\tau\, e_k) - f(x))$ exists, which is denoted by $\partial_k f(x)$ in this case. In Lemma .1.3 we verify that, for $x \in S_{-p}$, $p > d/2$, the derivative of the real function $H_{\{k\}}(\,\cdot\,|x_{\{k\}^c})$ at x_k is given by

$$\partial_k H(x) \;:=\; \frac{d}{dt}\Big|_{x_k} H_{\{k\}}(\,t\,|x_{\{k\}^c}) \;=\; \sum_{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni k} \partial_k\Phi_\Lambda(x_\Lambda)\;. \tag{12}$$

Due to Lemma .1.2 the sum in (12) converges uniformly on closed balls in S_{-p} and it holds $\partial_k H \in C_{\mathrm{pol}}(S_{-p})$, for $k \in \mathbb{Z}^d$, and $p > d/2$.

Theorem 2.3 ([6]) *Assume that Φ fulfills Hypothesis 1. Then the following assertions hold true.*

(i) For $\beta \in [1,\infty)$, the set $\mathscr{G}_t(\beta,\Phi)$ is not empty.

(ii) For $\beta \in [1,\infty)$, $\mathscr{G}_t(\beta,\Phi)$ equals the set of all probability measures μ_β on $(X,\mathfrak{B}(X))$ that satisfy, for some $p > d/2$, *the temperedness condition (11) and the following integration by parts formula, for all $k \in \mathbb{Z}^d$, and $f \in C_0^1(S_{-p})$,*

$$\int_X \partial_k f(x)\, d\mu_\beta(x) \;=\; \int_X f(x)\,\beta\,\partial_k H(x)\, d\mu_\beta(x)\;. \tag{13}$$

(iii) For $\beta \in [1,\infty)$, $\mu_\beta \in \mathscr{G}_t(\beta,\Phi)$, and all $p > d/2$, *we have $\mu_\beta(S_{-p}) = 1$. Furthermore, we have $\mu_\beta(V) > 0$, for each non-empty open subset $V \subset X$.*

(iv) For all $\beta \in [1,\infty)$, $\lambda > 0$, and $p > Ld/2$, the following à-priori estimates are valid:

$$\sup_{\mu_\beta\in\mathscr{G}_t(\beta,\Phi)}\ \sup_{k\in\mathbb{Z}^d} \int_X e^{\lambda\,|x_k|^2}\, d\mu_\beta(x) \quad<\quad \infty\,, \tag{14}$$

$$\sup_{\mu_\beta\in\mathscr{G}_t(\beta,\Phi)} \int_X e^{\lambda\,|x|^2_{-p}}\, d\mu_\beta(x) \quad<\quad \infty\,. \tag{15}$$

PROOF: Hypothesis 1 and its immediate implications (36), (72), and (74), show that the results of [6] can be applied in our situation and yield all assertions of the theorem. We note that all proofs in [6] are written down for pair-interactions. It is, however, easy to see that they work also under our hypotheses. ∎

Remark 2.4 ([4, 27]) (i) In the following we often use the space $\mathscr{F}C_b^\infty(X)$ of smooth, bounded, finitely based functions on X. By definition f belongs to $\mathscr{F}C_b^\infty(X)$, if there exist $N \in \mathbb{N}$, $k_1,\dots,k_N \in \mathbb{Z}^d$, and $f_N \in C_b^\infty(\mathbb{R}^N)$ such that $f(x) = f_N(x_{k_1},\dots,x_{k_N})$, for all $x \in X$. Since $\mu_\beta(V) > 0$, for all $\beta \in [1,\infty)$, $\mu_\beta \in \mathscr{G}_t(\beta,\Phi)$, and all non-empty open subsets $V \subset X$, we know that the equivalence class of $f \in \mathscr{F}C_b^\infty(X)$ in $L^2(\mu_\beta) \equiv L^2(X,\mathfrak{B}(X),\mu_\beta)$ contains precisely one representative belonging to $\mathscr{F}C_b^\infty(X)$. We shall therefore not distinguish between $\mathscr{F}C_b^\infty(X)$ and the set of equivalence classes of its elements. A monotone class argument then shows that $\mathscr{F}C_b^\infty(X)$ is dense in $L^2(\mu_\beta)$.

(ii) Using Lebesgue's dominated convergence theorem it is easy to see that (13) and (15) together imply that (13) holds also for $f \in C^1_{\rm pol}(S_{-p})$ if $p > L\,d/2$. As a consequence we get, for $f, g \in C^1_{\rm pol}(S_{-p})$, $p > L\,d/2$, and $k \in \mathbb{Z}^d$,

$$f \;=\; g \quad \mu\text{-a.e.} \quad\Longrightarrow\quad \partial_k f \;=\; \partial_k g \quad \mu\text{-a.e.}$$

Therefore, each partial derivative considered as an operator acting on $L^2(\mu_\beta)$ is well-defined on the domain $C^1_{\rm pol}(S_{-p})$, $p > L\,d/2$, (and, in particular, on $\mathscr{F}C_b^\infty(X)$). Furthermore, we readily infer from (13) and the Leibnitz rule that $(\partial_k, C^1_{\rm pol}(S_{-p}))$, $p > L\,d/2$, (respectively $(\partial_k, \mathscr{F}C_b^\infty(X))$) is closable since its adjoint is densely defined.

(iii) In view of its characterization by integration by parts formulas the set $\mathscr{G}_t(\beta,\Phi)$ is obviously convex. We denote the set of its extreme points by $\mathscr{G}_t(\beta,\Phi)_{\rm extr}$. Elements of $\mathscr{G}_t(\beta,\Phi)_{\rm extr}$ are called pure, or, extreme Gibbs measures. A well-known theorem due to Dynkin and Föllmer says that, since $\mathscr{G}_t(\beta,\Phi)$ is not empty and X is a Polish space, each $\mu_\beta \in \mathscr{G}_t(\beta,\Phi)$ can be represented as an integral over extreme points of $\mathscr{G}_t(\beta,\Phi)$; see, e.g. [31, Theorem 2.2]. In particular, this shows that

$$\mathscr{G}_t(\beta,\Phi)_{\rm extr} \;\neq\; \varnothing\;.$$

3 Supersymmetric Dirichlet operators

In the following we fix some $p \in \mathbb{N}$ with $p > L\,d/2$. We recall that S_{-p} has full measure with respect to all $\mu_\beta \in \mathscr{G}_t(\beta,\Phi)$ and that the integration by parts formula (13) is applicable to $f \in C^1_{\rm pol}(S_{-p})$.

The fermionic Fock space over $\ell^2_{\mathbb{C}}(\mathbb{Z}^d)$ is defined as the topological direct sum $\mathfrak{F}_{\rm f} := \bigoplus_{k=0}^\infty \mathfrak{F}_{\rm f}^k$, where $\mathfrak{F}_{\rm f}^0 := \mathbb{C}$, and $\mathfrak{F}_{\rm f}^k$ is the canonical completion of the k-fold algebraic anti-symmetric tensor product of $\ell^2_{\mathbb{C}}(\mathbb{Z}^d)$. We let $\mathfrak{I}^k$ denote the set of all lexicographically ordered k-tupels of elements in $\mathbb{Z}^d$, and set $e_I := e_{i_1}\wedge\cdots\wedge e_{i_k}$, for $I = (i_1,\dots,i_k) \in \mathfrak{I}^k$, $k \in \mathbb{N}$, and $e_\varnothing = 1 \in \mathfrak{F}_{\rm f}^0$, $\mathfrak{I}^0 := \{\varnothing\}$. Then an orthonormal basis of $\mathfrak{F}_{\rm f}^k$ is given by $\mathscr{B}_k = \{e_I \,:\, I \in \mathfrak{I}^k\}$. If we further identify $e_I \in \mathfrak{F}_{\rm f}^k$ with the corresponding element $(\dots,0,e_I,0,\dots) \in \mathfrak{F}_{\rm f}$, an orthonormal basis of $\mathfrak{F}_{\rm f}$ is given by $\mathscr{B}_k = \{e_I \,:\, I \in \bigcup_{k=0}^\infty \mathfrak{I}^k\}$. We write $F^k := \langle\,\mathscr{B}_k\,\rangle$ and $F := \langle\,\mathscr{B}\,\rangle$ for the respective sets of finite linear combinations of basis vectors. Furthermore, we write $a^\dagger(v)$ for the exterior multiplication on F by a vector $v \in \ell^2_{\mathbb{C}}(\mathbb{Z}^d)$. By continuity $a^\dagger(v)$ extends to a bounded operator on $\mathfrak{F}_{\rm f}$. We preserve the same notation for its extension. $a^\dagger(v)$ and its adjoint, $a(v) := (a^\dagger(v))^*$, fulfill the following canonical anti-commutation relations (CAR),

$$\begin{aligned}\{\,a(v)\,,\,a^\dagger(w)\,\} &= \langle\,v\,|\,w\,\rangle_{\ell^2_{\mathbb{C}}(\mathbb{Z}^d)}\,\mathbf{1}\;,\\ \{\,a(v)\,,\,a(w)\,\} &= \{\,a^\dagger(v)\,,\,a^\dagger(w)\,\} \;=\; 0\;,\end{aligned}$$

for $v, w \in \ell^2_{\mathbb{C}}(\mathbb{Z}^d)$. We abbreviate $a_k^\dagger := a^\dagger(e_k)$ and $a_k := a(e_k)$, for $k \in \mathbb{Z}^d$. In the sequel we construct differential operators acting between the following complete tensor products of Hilbert spaces

$$\mathscr{H}^k \;\equiv\; \mathscr{H}^k(\beta) \;:=\; L^2(\mu_\beta) \,\widehat{\otimes}\, \mathfrak{F}_{\mathrm{f}}^k \;, \qquad k = 0, 1, 2, \ldots \;,$$

or on their topological direct sum $\mathscr{H} \;\equiv\; \mathscr{H}(\beta) := L^2(\mu_\beta) \,\widehat{\otimes}\, \mathfrak{F}_{\mathrm{f}}$. The algebraic tensor products

$$\Omega^k \;:=\; \mathscr{F}C_b^\infty(X) \otimes F^k \quad \text{and} \quad \Omega \;:=\; \mathscr{F}C_b^\infty(X) \otimes F$$

are dense subspaces of $\mathscr{H}^k$ and $\mathscr{H}$ respectively.

First, we introduce an exterior derivative, d_β, on the domain

$$\mathcal{D}(d_\beta) \;:=\; C^1_{\mathrm{pol}}(S_{-p}) \otimes F$$

setting

$$d_\beta \;:=\; \beta^{-1/2} \sum_{k \in \mathbb{Z}^d} \partial_k \otimes a_k^\dagger \;.$$

In other words the restriction, $d_\beta^{(k)}$, of d_β to $\mathcal{D}(d_\beta^{(k)}) = \mathcal{D}(d_\beta) \cap \mathscr{H}^k$, $k = 0, 1, 2, \ldots$, is given by

$$d_\beta^{(k)} f \, e_I \;:=\; \beta^{-1/2} \, \nabla f \wedge e_I \;,$$

for $f \in C^1_{\mathrm{pol}}(S_{-p})$, $I \in \mathfrak{I}^k$, plus linear extension. Here $\nabla f \in S_{p,c}$ is the unique vector such that $\langle \nabla f(x) \,|\, v \rangle_{0,c} = f'(x)v$, for all $v \in S_{-p}$. We note that d_β is well-defined because of Remark 0.2.4(ii) and the à-priori estimates from Theorem 0.2.3, and since, for all $f = \sum_{\nu=1}^n f_\nu \otimes e_{I_\nu} \in \mathcal{D}(d_\beta)$, and $x \in S_{-p}$, it holds $\nabla f_\nu(x) \in S_{p,c} \subset \ell^2_{\mathbb{C}}(\mathbb{Z}^d)$, and $|\nabla f_\nu(x)|_{p,c} = \|f_\nu'\|_{\mathscr{L}(S_{-p,c},\mathbb{C})} \leqslant C_\nu (1 + |x|_{-p})^{m_\nu}$, for some $C_\nu \in (0, \infty)$, and $m_\nu \in \mathbb{N}$. Clearly, $\Omega \subset \mathcal{D}(d_\beta)$, and d_β maps Ω into itself since an element of $\mathscr{F}C_b^\infty(X)$ has only finitely many non-vanishing partial derivatives. Since the partial derivatives of $f \in \mathscr{F}C_b^\infty(X)$ commute while the exterior product is anti-commutative we further have

$$d_\beta^{(k+1)} \circ d_\beta^{(k)} \;=\; 0 \qquad \text{on } \Omega^k \;, \text{ for } k = 0, 1, 2, \ldots \;. \tag{16}$$

Next, we define a coderivative, δ_β, on the domain $\mathcal{D}(\delta_\beta) := \mathcal{D}(d_\beta)$ by

$$\delta_\beta \;=\; \beta^{-1/2} \sum_{k \in \mathbb{Z}^d} \big(-\partial_k \,+\, \beta\, \partial_k H \big) \otimes a_k \;.$$

In other words, for $k = 0, 1, 2, \ldots$, the restriction of δ_β to $\mathcal{D}(d_\beta^{(k+1)})$ is given by

$$\begin{aligned} &\delta_\beta^{(k+1)} f \,\otimes\, e_{i_1} \wedge \cdots \wedge e_{i_{k+1}} \\ &\quad=\; \beta^{-1/2} \sum_{\nu=1}^{k+1} (-1)^{\nu+1} \big(\partial_{i_\nu} f \,+\, \partial_{i_\nu} H \, f \big) \otimes\, e_{i_1} \wedge \cdots \wedge \widehat{e_{i_\nu}} \wedge \cdots \wedge e_{i_{k+1}} \end{aligned}$$

for $f \in C^1_{\rm pol}(S_{-p})$, $i_1, \ldots, i_{k+1} \in \mathbb{Z}^d$, and linear extension. Here the symbol $\widehat{\ }$ indicates the omission of the vector under it. To check that δ_β is well-defined we just recall that $\partial_k H \in C_{\rm pol}(S_{-p})$, for $k \in \mathbb{Z}^d$. Using integration by parts it is easy to see that δ_β (respectively $\delta_\beta^{(k+1)}$) is a formal adjoint of d_β (respectively $d_\beta^{(k)}$).

Next, we define a positive symmetric operator $\mathscr{L}_\beta$ acting in $\mathscr{H}$ on the domain $\mathcal{D}(\mathscr{L}_\beta) := \Omega$ by

$$\mathscr{L}_\beta \ := \ d_\beta \delta_\beta \ + \ \delta_\beta d_\beta \ .$$

In order to show that $\mathscr{L}_\beta$ is well-defined we only have to check that $\delta_\beta \, \Omega \subseteq \mathcal{D}(d_\beta)$, since $d_\beta \, \Omega \subseteq \Omega \subset \mathcal{D}(\delta_\beta)$ is evident. The former inclusion, however, follows from Lemma .1.4 where we show that, for all $j \in \mathbb{Z}^d$, the function $\partial_j H$ is actually contained in $C^1_{\rm pol}(S_{-p})$. Its Gâteaux derivative in direction e_i is given by

$$\partial_i \partial_j H(x) \ = \sum_{\substack{\Lambda \in \mathfrak{P}_{\rm f}: \\ \Lambda \ni i,j}} \partial_i \partial_j \Phi_\Lambda(x_\Lambda) \ = \ \partial_j \partial_i H(x) \ , \qquad x \in S_{-p} \ . \tag{17}$$

We claim that

$$\mathscr{L}_\beta \ = \ \mathscr{L}_\beta^{(0)} \otimes \mathbf{1} \ + \ d\Gamma(H'') \ , \tag{18}$$

where

$$\mathscr{L}_\beta^{(0)} f \ = \sum_{k \in \mathbb{Z}^d} \Big(- \beta^{-1} \, \partial_k^2 f \ + \ \partial_k H \partial_k f \Big) \ ,$$

for $f \in \mathcal{D}\big(\mathscr{L}_\beta^{(0)}\big) := \Omega^0$, and

$$d\Gamma(H'') \ := \sum_{i,j \in \mathbb{Z}^d} \partial_i \partial_j H \ \otimes \ a_i^\dagger a_j \ , \qquad \mathcal{D}(d\Gamma(H'')) \ := \ \Omega \ . \tag{19}$$

Here $\mathscr{L}_\beta^{(0)}$ is well-defined because of (77) and the à-priori estimate (15). We show in Lemma .1.5 that $d\Gamma(H'')$ is well-defined, too. To prove (18) we observe that due to the CAR the following identities hold true on Ω,

$$\begin{aligned}
& d_\beta \delta_\beta \ + \ \delta_\beta d_\beta \\
& \quad = \ \beta^{-1} \sum_{i,j \in \mathbb{Z}^d} \Big\{ \partial_i \, (-\partial_j + \beta \, \partial_j H \,) \ \otimes \ a_i^\dagger a_j \ + \ (-\partial_i + \beta \, \partial_i H \,) \, \partial_j \ \otimes \ a_i a_j^\dagger \Big\} \\
& \quad = \ \beta^{-1} \sum_{i,j \in \mathbb{Z}^d} \Big\{ (-\partial_i + \beta \, \partial_i H \,) \, \partial_j \ \otimes \ \delta_{ij} \, \mathbf{1} \\
& \qquad \quad + \Big(\partial_i \, (-\partial_j + \beta \, \partial_j H \,) \ - \ (-\partial_j + \beta \, \partial_j H \,) \, \partial_i \Big) \ \otimes \ a_i^\dagger a_j \Big\} \\
& \quad = \ \mathscr{L}_\beta^{(0)} \otimes \mathbf{1} \ + \sum_{i,j \in \mathbb{Z}^d} \partial_i \partial_j H \ \otimes \ a_i^\dagger a_j \ .
\end{aligned}$$

In Section 0.5 we prove the following

Proposition 3.1 *Assume that Φ fulfills Hypothesis 1. Then $\mathscr{L}_\beta$ is essentially selfadjoint on Ω.*

In particular, $\mathscr{L}_\beta^{(k)}$ is essentially selfadjoint on Ω^k, for $k = 0, 1, 2, \ldots$. We denote the unique selfadjoint extensions of $\mathscr{L}_\beta$ (resp. $\mathscr{L}_\beta^{(k)}$) by L_β (resp. $L_\beta^{(k)}$).

We know that d_β is closable since its adjoint is an extension of δ_β and thus densely defined. We denote the closure of d_β by $\bar{d}_\beta$. It is easy to see that $\bar{d}_\beta = \bigoplus_{k=0}^\infty \bar{d}_\beta^{(k)}$, where $\bar{d}_\beta^{(k)}$ denotes the closure of $d_\beta^{(k)}$, for $k = 0, 1, 2, \ldots$.

Lemma 3.2 *Assume that Φ fulfills Hypothesis 1. Then it holds*

$$\mathrm{Ran}\ \bar{d}_\beta \subset \mathcal{D}(\bar{d}_\beta) \qquad \text{and} \qquad \bar{d}_\beta \circ \bar{d}_\beta = 0 .$$

Proof: To begin with we assume that the closure of d_β starting from $\mathcal{D}(d_\beta) = C^1_{\mathrm{pol}}(S_{-p}) \otimes F$ coincides with the closure of $d_\beta{\restriction}_\Omega$. We note that $d_\beta{\restriction}_\Omega$ is closable since its adjoint extends $\delta_\beta{\restriction}_\Omega$ and is thus densely defined. Under our assumption we find, for each $g \in \mathcal{D}(\bar{d}_\beta)$, a sequence $(g_n)_{n\in\mathbb{N}} \subset \Omega$ such that $g_n \to g$ and $d_\beta g_n \to \bar{d}_\beta g$. We have $d_\beta g_n \in \Omega$ and $\bar{d}_\beta \circ d_\beta g_n = d_\beta \circ d_\beta g_n = 0$, for all $n \in \mathbb{N}$. Since $\bar{d}_\beta$ is closed it follows that $\bar{d}_\beta g \in \mathcal{D}(\bar{d}_\beta)$ and $\bar{d}_\beta \circ \bar{d}_\beta g = 0$.

It therefore remains to show that $\bar{d}_\beta = \overline{d_\beta{\restriction}_\Omega}$. So we suppose that $(f_n)_{n\in\mathbb{N}} \subset C^1_{\mathrm{pol}}(S_{-p}) \otimes F$ is a sequence such that $f_n \to f \in \mathscr{H}$, and $d_\beta f_n \to g \in \mathscr{H}$, as $n \to \infty$. Each f_n has the form $f_n = \sum_{\nu=1}^N f_n^{(\nu)}\, e_{I_\nu^{(n)}}$, where $N \equiv N(n) \in \mathbb{N}$, $f_n^{(\nu)} \in C^1_{\mathrm{pol}}(S_{-p})$, $I_\nu^{(n)} \in \bigcup_{k=0}^\infty \mathfrak{I}^k$, for $\nu \in \{1, \ldots, N\}$. We pick some increasing sequence $(\Upsilon_m)_{m\in\mathbb{N}} \subset \mathfrak{P}_{\mathrm{f}}$, $\Upsilon_m \nearrow \mathbb{Z}^d$, as $m \to \infty$, and define the finite-dimensional orthogonal projector $P_m :=$ by

$$S_{-p} \ni x \longmapsto P_m(x) := \sum_{k\in\Upsilon_m} x_k\, e_k = \sum_{k\in\Upsilon_m} \langle\, e_k^{-p} \,|\, x \,\rangle_{-p}\, e_k^{-p} , \qquad (20)$$

where $e_k^{-p} := (1 + |k|)^p e_k$, $k \in \mathbb{Z}^d$, are orthonormal basis vectors in S_{-p}. Since $P_m \to \mathbf{1}$ strongly, and since $f_n^{(\nu)}$ is continuous, we have $f_n^{(\nu)}(P_m x) \to f_n^{(\nu)}(x)$ pointwise, for all $x \in S_{-p}$. Using Lebesgue's dominated convergence theorem and the à-priori estimates from Theorem 0.2.3 we infer that $f_{n,m}^{(\nu)} := f_n^{(\nu)} \circ P_m \to f_n^{(\nu)}$, as $m \to \infty$, in $\mathscr{H}^0$, for all $n, \nu \in \mathbb{N}$. Furthermore, we have $\nabla(f_{n,m}^{(\nu)})(x) = P_m \nabla f_n(P_m x) \to \nabla f_n^{(\nu)}(x)$, since f_n' is continous. Therefore, we get $\nabla(f_{n,m}^{(\nu)}) \to \nabla f_n^{(\nu)}$ in $\mathscr{H}^1$ again from Lebesgue's dominated convergence theorem and, hence,

$$d_\beta \sum_{\nu=1}^N f_{n,m}^{(\nu)}\, e_{I_\nu^{(n)}} = \beta^{-1/2} \sum_{\nu=1}^N P_m((\nabla f_n^{(\nu)}) \circ P_m) \wedge e_{I_\nu^{(n)}} \longrightarrow d_\beta f_n .$$

This shows that the closure of d_β starting from $C^1_{\mathrm{pol}}(S_{-p}) \otimes F$ coincides with its closure starting from $\mathscr{F}C^1_{\mathrm{pol}}(X) \otimes F$, where $\mathscr{F}C^1_{\mathrm{pol}}(X)$ is defined analogously to $\mathscr{F}C_b^\infty(X)$. Again in view of the à-priori estimates from Theorem 0.2.3 we can construct, for all $n, m \in \mathbb{N}$, and $\varepsilon > 0$, some $f_{n,m,\varepsilon}^{(\nu)} \in \mathscr{F}C_b^\infty(X)$, such that $\|f_{n,m}^{(\nu)} - f_{n,m,\varepsilon}^{(\nu)}\|_{\mathscr{H}^0} < \frac{\varepsilon}{N\#\Upsilon_m}$, and $\|\partial_k f_{n,m}^{(\nu)} - \partial_k f_{n,m,\varepsilon}^{(\nu)}\|_{\mathscr{H}^0} < \frac{\varepsilon}{N\#\Upsilon_m}$, for all $\nu \in \{1, \ldots, N(n)\}$, and $k \in \Upsilon_m$, multiplying $f_{n,m}^{(\nu)}$ with a suitable cut-off function and using Friedrichs mollifiers. Consequently, the closure of d_β starting from $\mathscr{F}C^1_{\rm pol}(X)\otimes F$ coincides with its closure starting from Ω. ■

Therefore, the sequence

$$0 \xrightarrow{\bar d_\beta^{(-1)}\equiv 0} \mathcal{D}(\bar d_\beta^{(0)}) \xrightarrow{\bar d_\beta^{(0)}} \mathcal{D}(\bar d_\beta^{(1)}) \xrightarrow{\bar d_\beta^{(1)}} \cdots$$

defines a complex of closed operators between the Hilbert spaces $\mathscr{H}^k$, $k = 0,1,2,\dots$. In the next proposition we derive a weak Hodge decomposition associated to this complex. To this end we introduce the following abbreviations:

$$R_k \;:=\; \mathrm{Ran}\; \bar d_\beta^{(k-1)}\;, \qquad R_k^* \;:=\; \mathrm{Ran}\; d_\beta^{(k)*}\;, \qquad k=0,1,2,\dots\;.$$

Proposition 3.3 *Assume that Φ fulfills Hypothesis 1. Then the following assertions hold true.*

(i) We have $\mathcal{D}(L_\beta)=\mathcal{D}(\bar d_\beta d_\beta^*)\cap\mathcal{D}(d_\beta^*\bar d_\beta)$ and

$$L_\beta \;=\; \bar d_\beta d_\beta^* + d_\beta^*\bar d_\beta\;.$$

(ii) The following weak Hodge decompositions are valid:

$$\mathscr{H}^k \;=\; \ker L_\beta^{(k)} \oplus \overline{R_k}\oplus\overline{R_k^*}\;, \qquad k\in\mathbb{N}\;, \tag{21}$$

$$\mathscr{H}^0 \;=\; \ker L_\beta^{(0)} \oplus \overline{R_0^*}\;. \tag{22}$$

Proof: In order to prove (i) we first recall that a theorem of von Neumann implies that both $\bar d_\beta d_\beta^*$ and $d_\beta^*\bar d_\beta$ are selfadjoint and, in particular, closed. Since Lemma 0.3.2 implies $\langle\, \bar d_\beta d_\beta^* f\,|\, d_\beta^*\bar d_\beta g\,\rangle = 0$, for all $f,g\in\mathcal{D}(\bar d_\beta d_\beta^*)\cap\mathcal{D}(d_\beta^*\bar d_\beta)$, an easy argument shows that the sum $\bar d_\beta d_\beta^* + d_\beta^*\bar d_\beta$ is closed, too. Since it is also symmetric and extends $\mathscr{L}_\beta = d_\beta\delta_\beta + \delta_\beta d_\beta$, it is equal to L_β because of Proposition 0.3.1. To prove (ii) we note that, by Proposition 0.3.1 and [25, Chapter VI, Corollary 2.4], $L_\beta^{(k)}$ is identical to the unique selfadjoint operator associated to the closed form

$$\mathcal{D}(\bar d_\beta^{(k)})\cap\mathcal{D}(d_\beta^{(k-1)*})\ni f,g \;\longmapsto\; \langle\, \bar d_\beta^{(k)} f\,|\,\bar d_\beta^{(k)} g\,\rangle + \langle\, d_\beta^{(k-1)*} f\,|\, d_\beta^{(k-1)*} g\,\rangle\;.$$

Here $d_\beta^{(-1)*}\equiv 0$. Now the claim follows from [8, Proposition A.1]. ■

Remark 3.4 We briefly describe the connection between $L_\beta^{(0)}$ and certain Dirichlet forms associated with μ_β. (In this remark all operators and forms are considered on the real Hilbert space $L^2_{\mathbb{R}}(\mu_\beta)$. We indicate this by a subscript $\mathbb{R}$.) A form $\mathcal{Q}$ on $L^2_{\mathbb{R}}(\mu_\beta)$ is called a Dirichlet form iff it is densely defined, nonnegative, symmetric, and closed, and, for each $T:\mathbb{R}\to\mathbb{R}$ satisfying $T(0)=0$, and $|T(s)-T(t)| \leqslant |s-t|$, for all $s,t \in \mathbb{R}$, and $u \in \mathcal{D}(\mathcal{Q})$, we have $T \circ u \in \mathcal{D}(\mathcal{Q})$ and $\mathcal{Q}(T \circ u, T \circ u) \leqslant \mathcal{Q}(u,u)$ [7]. The form $\widetilde{\mathcal{E}}$ given by

$$\widetilde{\mathcal{E}}(u,v) \;:=\; \int\limits_X \sum_{k \in \mathbb{Z}^d} \partial_k u \, \partial_k v \, d\mu_\beta \;=\; \langle \, u \, | \, \mathscr{L}^{(0)}_{\beta,\mathbb{R}} \, v \, \rangle_{L^2_{\mathbb{R}}(\mu_\beta)} \,,$$

for $u,v \in \mathcal{D}(\widetilde{\mathcal{E}}) := \mathscr{F}C_b^\infty(X,\mathbb{R})$, is closable, since $\mathscr{L}^{(0)}_{\beta,\mathbb{R}}$ is a densely defined, symmetric, non-negative operator. We denote the closure of $\widetilde{\mathcal{E}}$ by $\mathcal{E}$. It is called the minimal form associated to μ_β. Another possibility to obtain a closed form associated to μ_β is the following: First we define $\mathscr{E}_k$ as the closure of

$$\widetilde{\mathscr{E}}_k(u,v) \;:=\; \int\limits_X \partial_k u \, \partial_k v \, d\mu_\beta \,, \quad u,v \in \mathcal{D}(\widetilde{\mathscr{E}}_k) := \mathscr{F}C_b^\infty(X,\mathbb{R}) \,,$$

for $k \in \mathbb{Z}^d$. Again $\widetilde{\mathscr{E}}_k$ is closable since $-\beta^{-1}\partial_k^2 + \partial_k H \partial_k$ is a symmetric, non-negative operator on the dense domain $\mathscr{F}C_b^\infty(X,\mathbb{R})$. Then

$$\begin{aligned} \mathcal{D}(\mathscr{E}) \;&:=\; \Big\{ u \in \bigcap_{k \in \mathbb{Z}^d} \mathcal{D}(\mathscr{E}_k) \,:\, \sum_{k \in \mathbb{Z}^d} \mathscr{E}_k(u,u) \,<\, \infty \Big\} \,, \\ \mathscr{E}(u,v) \;&:=\; \sum_{k \in \mathbb{Z}^d} \mathscr{E}_k(u,v) \,, \qquad u,v \in \mathcal{D}(\mathscr{E}) \,, \end{aligned}$$

defines a Dirichlet form in the above mentioned sense [7, Theorem 3.8]. $\mathscr{E}$ is called the maximal Dirichlet form. However, since the generators of $\mathcal{E}$ and $\mathscr{E}$ both are selfadjoint extensions of $\mathscr{L}^{(0)}_{\beta,\mathbb{R}}$ they must be equal, as $\mathscr{L}^{(0)}_{\beta,\mathbb{R}}$ is essentially selfadjoint. We thus have $\mathcal{E} = \mathscr{E}$, too. Due to [4, 5] the latter equality implies that the following statements are equivalent:

(i) μ_β is a pure Gibbs state, that is, $\mu_\beta \in \mathscr{G}_t(\beta,\Phi)_{\mathrm{extr}}$.

(ii) $\mathcal{E}$ is irreducible, that is, any $u \in \mathcal{D}(\mathcal{E})$ satisfying $\mathcal{E}(u,u) = 0$ is constant μ_β-almost everywhere.

(iii) $\ker L^{(0)}_{\beta,\mathbb{R}} \,=\, \mathbb{R}\,1$.

Proposition 3.5 *Assume that Φ fulfills Hypothesis 2 and that $\mu_\beta \in \mathscr{G}_t(\beta,\Phi)_{\mathrm{extr}}$. Then, for each $\ell \in \mathbb{N}$, there is some $\beta_0 \in [1,\infty)$ such that the following assertions hold true, for $\beta \in [\beta_0,\infty)$:*

(i) $L^{(k)}_\beta$ is strictly positive, for $1 \leqslant k \leqslant \ell+1$.

(ii) For $1 \leqslant k \leqslant \ell$, the subspaces R_k and R_k^* are closed and the following Hodge decompositions are valid:

$$\mathscr{H}^0 \;=\; \mathbb{C}\,1 \oplus R_0^* \,, \tag{23}$$

$$\mathscr{H}^k \;=\; R_k \oplus R_k^* \,, \qquad 1 \leqslant k \leqslant \ell \,. \tag{24}$$

(iii) For $0 \leqslant k \leqslant \ell$, there exist unitary operators $U_k : R_k^* \to R_{k+1}$ having the property

$$L_\beta^{(k)}\restriction_{R_k^*} \;=\; U_k^* \, L_\beta^{(k+1)}\restriction_{R_{k+1}} U_k \; .$$

Consequently,

$$\sigma(L_\beta^{(k)}\restriction_{R_k^*}) \;=\; \sigma(L_\beta^{(k+1)}\restriction_{R_{k+1}}) \; , \qquad 0 \leqslant k \leqslant \ell \; .$$

(iv) For $0 \leqslant k \leqslant \ell$, the orthogonal projection, $P_{R_k^*}$, onto R_k^* in $\mathscr{H}^k$ is given by

$$P_{R_k^*} \;=\; d_\beta^{(k)*} \big(\bar{d}_\beta^{(k)} d_\beta^{(k)*}\restriction_{R_{k+1}} \big)^{-1} \bar{d}_\beta^{(k)} \;=\; d_\beta^{(k)*} \big(L_\beta^{(k+1)} \big)^{-1} \bar{d}_\beta^{(k)} \; . \qquad (25)$$

In particular, we have

$$P^\perp_{\ker L_\beta^{(0)}} \;=\; P_{R_0^*} \;=\; d_\beta^{(0)*} \big(L_\beta^{(1)} \big)^{-1} \bar{d}_\beta^{(0)} \; . \qquad (26)$$

PROOF: Part (i) is a direct consequence of Lemma 0.6.6. The remaining assertions follow directly from [24, Theorem 1.3] together with the weak Hodge decompositions (21) and (22) and Remark 0.3.4. ■

Next, we define an operator $H''(0) \in \mathscr{L}(\ell^2_{\mathbb{C}}(\mathbb{Z}^d))$ with matrix elements

$$H''(0)_{k\ell} \;:=\; \sum_{\substack{\Lambda \in \mathfrak{P}_{\mathfrak{f}}: \\ \Lambda \ni k, \ell}} \partial_k \partial_\ell \Phi_\Lambda(0) \; ,$$

for $k, \ell \in \mathbb{Z}^d$. Using Hypothsesis 1 it is easily seen that $H''(0)$ is bounded with $\|H''(0)\| \leqslant \sup_{k\in\mathbb{Z}^d} \sum_{\ell\in\mathbb{Z}^d} |H''(0)_{k\ell}| \leqslant c + \mathscr{J}_1(0)$. Here we use the symmetry of $H''(0)$ and interpolation. Under Hypothesis 2 we moreover have $H''(0) \geqslant (a - \mathscr{J}_1(0))\,\mathbf{1} > 0$ in the sense of quadratic forms on $\ell^2_{\mathbb{C}}(\mathbb{Z}^d)$.

Theorem 3.6 *Assume that Φ fulfills Hypothesis 2. Let $n \in \mathbb{N}$ and suppose that $\eta_0 := \frac{2}{5}((n+1)\, m_\star - n\, m^\star) > 0$, where*

$$m_\star \;:=\; \inf \sigma(H''(0)) \qquad m^\star \;:=\; \sup \sigma(H''(0)) \; .$$

Then, for all $\eta \in (0, \eta_0)$, we find some $\beta_0 \geqslant 1$ such that, for all $\beta \geqslant \beta_0$, we have

$$\sigma(L_\beta^{(0)}) \,\cap\, (\,0\,,\,(n+1)\,m_\star - \eta\,) \;\subset\; \{0\} \cup \bigcup_{k=1}^{n} [\,k\,m_\star - \eta\,,\,k\,m^\star + \eta\,] \; ,$$

and the spectral projection of $L_\beta^{(0)}$ corresponding to the interval $[m_\star - \eta\,,\,m^\star + \eta]$ has an infinite-dimensional range.

The proof of Theorem 0.3.6 is given at the end of Section 0.6.

4 Verification of the supersymmetry

A selfadjoint operator H in some Hilbert space $\mathscr{K}$ is called supersymmetric, iff there exists another selfadjoint operator Q acting in $\mathscr{K}$ and an orthogonal decomposition of $\mathscr{K}$ into an even and odd part, $\mathscr{K} = \mathscr{K}_+ \oplus \mathscr{K}_-$, where $\mathscr{K}_\pm$ is the eigenspace corresponding to the eigenvalue ± 1 of some selfadjoint operator $\gamma \in \mathscr{L}(\mathscr{K})$, such that

$$H \,=\, Q^2\,, \qquad Q\,\gamma \,+\, \gamma\, Q \,=\, 0\,, \qquad \gamma^2 = \mathbf{1}\,.$$

See, e.g., [17, Chapter 6.3]. In this case the socalled supercharge Q maps $\mathcal{D}(Q) \cap \mathscr{K}_\pm$ into $\mathscr{K}_\mp$. We observe that L_β is supersymmetric in this sense provided that Φ fulfills Hypothesis 1. Indeed, we just have to set $\mathscr{K} = \mathscr{H}$,

$$\mathscr{K}_+ \,=\, \bigoplus_{i=0}^{\infty} \mathscr{H}^{2i}\,, \qquad \mathscr{K}_- \,=\, \bigoplus_{i=0}^{\infty} \mathscr{H}^{2i+1}\,, \qquad \gamma\!\restriction_{\mathscr{K}_\pm} = \pm\mathbf{1}_{\mathscr{K}_\pm}\,,$$

and $Q_\beta \,=\, d_\beta^* \,+\, \bar{d}_\beta$. Then we have

$$Q_\beta^2 \,=\, \big(d_\beta^* \,+\, \bar{d}_\beta\big)^2 \,=\, \bar{d}_\beta d_\beta^* + d_\beta^* \bar{d}_\beta \,=\, L_\beta$$

by the complex property for $\bar{d}_\beta$ and d_β^* and Proposition 0.3.3. It only remains to show that Q_β is selfadjoint on $\mathcal{D}(\bar{d}_\beta) \cap \mathcal{D}(d_\beta^*)$. To this end we first observe that, as $d_\beta + \delta_\beta$ is symmetric on $\mathcal{D}(d_\beta + \delta_\beta) = C^1_{\mathrm{pol}}(S_{-p}) \otimes F$, its closure exists and is also symmetric, and it holds

$$d_\beta + \delta_\beta \,\subset\, \overline{d_\beta + \delta_\beta} \,\subset\, \big(\overline{d_\beta + \delta_\beta}\big)^* \,\subset\, (d_\beta + \delta_\beta)^*\,.$$

Moreover, the sesquilinear forms E_1 and E_2 defined by

$$\mathsf{E}_1(f,g) \,:=\, \big\langle\, \overline{d_\beta + \delta_\beta}\, f \,\big|\, \overline{d_\beta + \delta_\beta}\, g \,\big\rangle\,, \qquad f,g \in \mathcal{D}\big(\overline{d_\beta + \delta_\beta}\big)\,,$$

respectively

$$\mathsf{E}_2(f,g) \,:=\, \big\langle\, (d_\beta + \delta_\beta)^*\, f \,\big|\, (d_\beta + \delta_\beta)^*\, g \,\big\rangle\,, \qquad f,g \in \mathcal{D}\big((d_\beta + \delta_\beta)^*\big)\,,$$

are both densely defined, closed and non-negative, since $\overline{d_\beta + \delta_\beta}$ and $(d_\beta + \delta_\beta)^*$ are closed operators. The unique non-negative selfadjoint operator L_1 associated to E_1 extends $\mathscr{L}_\beta$ by [25, Ch. VI, §2, Cor. 2.4], because we have

$$\mathsf{E}_1(f,g) \,=\, \big\langle\, f \,\big|\, \mathscr{L}_\beta\, g \,\big\rangle\,, \qquad f \in C^1_{\mathrm{pol}}(S_{-p}) \otimes F\,,\; g \in \Omega\,,$$

by the integration by parts formula, and $C^1_{\mathrm{pol}}(S_{-p}) \otimes F$ is a core for E_1. We recall that $(d_\beta + \delta_\beta)\,\Omega \,\subset\, C^1_{\mathrm{pol}}(S_{-p}) \otimes F$ and $\mathscr{L}_\beta = (d_\beta + \delta_\beta)^2$. Similarly, we obtain

$$\mathsf{E}_2(f,g) \,=\, \big\langle\, f \,\big|\, \mathscr{L}_\beta\, g \,\big\rangle\,, \qquad f \in \mathcal{D}\big((d_\beta + \delta_\beta)^*\big)\,,\; g \in \Omega\,,$$

by definition of the adjoint. Consequently, the unique non-negative selfadjoint operator L_2 associated to E_2 extends $\mathscr{L}_\beta$, too. Hence, $\mathsf{L}_1 = \mathsf{L}_2$, as $\mathscr{L}_\beta$ is essentially selfadjoint. Since E_i is uniquely determined by L_i, for $i = 1, 2$ [25, Ch. VI, §2, Thm. 2.7], it follows that

$$\mathcal{D}\big(\overline{d_\beta + \delta_\beta}\big) \;=\; \mathcal{D}\big((d_\beta + \delta_\beta)^*\big) \;.$$

Therefore, $d_\beta + \delta_\beta$ is essentially selfadjoint on $C^1_{\mathrm{pol}}(S_{-p}) \otimes F$. As $Q_\beta = \bar{d}_\beta + d^*_\beta$ is a symmetric closed extension of $d_\beta + \delta_\beta$ by the complex property for $\bar{d}_\beta$ and d^*_β, we conclude that Q_β is selfadjoint.

5 Essential selfadjointness

In order to prove that $\mathscr{L}_\beta$ is essentially selfadjoint on Ω we first prove that this holds true for $\mathscr{L}^{(0)}_\beta$ using a result from [26]. Then the essential selfadjointness of $\mathscr{L}_\beta$ is implied by the Kato-Rellich theorem and a relative operator bound proved in Section 0.6 involving $\mathscr{L}_\beta$ and a comparision operator which is obviously essentially selfadjoint.

In the following proof we set

$$\mathfrak{b}(x) \;:=\; -\sum_{k\in\mathbb{Z}^d} \partial_k H(x)\, e_k \;, \qquad x \in S_{-p} \;. \tag{27}$$

Due to Lemma .1.2 this defines a vector field $\mathfrak{b} : S_{-p} \to S_{-q}$, for $p > d/2$, $q > p\,(L-1) + d/2$.

Proposition 5.1 *Assume that Φ fulfills Hypothesis 1. Then $\mathscr{L}^{(0)}_\beta$ is essentially selfadjoint on Ω^0.*

PROOF: According to [26, Theorem 1] it suffices to verify the following assertion:

There exist $q > L\,d/2$, a sequence of vector fields $(\mathfrak{b}_n)_{n\in\mathbb{N}}$, $\mathfrak{b}_n : S_{-q} \to S_{-q}$, and an increasing sequence of finite sets $(\Upsilon_n)_{n\in\mathbb{N}} \subset \mathfrak{P}_{\mathrm{f}}$, $\Upsilon_n \nearrow \mathbb{Z}^d$, as $n \to \infty$, such that

(i) $\mathfrak{b}_n \circ P_n \;=\; P_n \circ \mathfrak{b}_n \;=\; \mathfrak{b}_n$, for all $n \in \mathbb{N}$, where P_n is given by (20).

(ii) $\mathfrak{b}_n\big(\iota_n(\,\cdot\,)\big)_k \in C^1_b(\mathbb{R}^{\Upsilon_n}, \mathbb{R})$ with globally Hölder continuous first order partial derivatives, for all $n \in \mathbb{N}$, and $k \in \Upsilon_n$, where, for $n \in \mathbb{N}$,

$$\begin{aligned} \iota_n : \mathbb{R}^{\Upsilon_n} \;&\longrightarrow\; X \;, \\ x_{\Upsilon_n} \;&\longmapsto\; (x_{\Upsilon_n} | \mathbf{0}_{\Upsilon_n^c}) \;. \end{aligned}$$

(iii) There exists some $c_+ \in (0, \infty)$ such that, for all $n \in \mathbb{N}$, and all $x, y \in \mathbb{R}^{\Upsilon_n}$,

$$\langle\, \iota_n\, y \,|\, \mathfrak{b}_n'(\iota_n\, x)\, \iota_n\, y \,\rangle_q \;\leqslant\; c_+\, |\iota_n\, y|_q^2 \;.$$

(iv) It holds
$$\int\limits_X \left|\mathfrak{b} - \mathfrak{b}_n\right|_{-q}^2 d\mu_\beta \;\longrightarrow\; 0\,, \qquad n\to\infty\,.$$

In order to verify this we fix some increasing sequence of finite sets, $(\Upsilon_n)_{n\in\mathbb{N}} \subset \mathfrak{P}_{\mathrm{f}}$ as above and some boundary condition $y \in \ell^\infty_{\mathbb{R}}(\mathbb{Z}^d)$. Moreover, we pick some $\chi \in C^\infty(\mathbb{R},\mathbb{R})$, such that $\chi(t) = -\chi(-t)$, for $t\in\mathbb{R}$, $\chi(t) = t$, for $|t| \leqslant 1$, $\chi(t) = \pm 2$, for $\pm t \geqslant 3$, and $0 \leqslant \chi'(t) \leqslant 1$, for $t\in\mathbb{R}$. We fix $p, q \in \mathbb{N}$ such that $q > p(L-1) + d/2$, and $p > (L+1)d/2$. Furthermore, we set $R(n,k) := n\,(1+|k|)^{p\,(L-1)}$ and $\chi_{n,k} := R(n,k)\,\chi(\,\cdot\,/R(n,k))$, for all $n\in\mathbb{N}$, and $k\in\mathbb{Z}^d$. Then we have $\chi_{n,k}(t) = t$, for $|t| \leqslant R(n,k)$, $\chi_{n,k}(t) = \pm 2\,R(n,k)$, for $\pm t \geqslant 3\,R(n,k)$, and $0 \leqslant \chi'_{n,k}(t) \leqslant 1$, for all $t\in\mathbb{R}$. With this we construct a sequence of vector fields $(\mathfrak{b}_n)_{n\in\mathbb{N}}$, $\mathfrak{b}_n : S_{-q} \to S_{-q}$, $n\in\mathbb{N}$, which approximates $\mathfrak{b}$, setting

$$\begin{aligned}\mathfrak{b}_n(x) \;\; := \;\; & -\sum_{k\in\Upsilon_n} \Big\{ \chi_{n,k}\big(\Phi'_{\{k\}}(x_k)\big) + \chi_{n,k}\Big(\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\subseteq\Upsilon_n,\\ \#\Lambda\geqslant 2}} \partial_k\Phi_\Lambda(x_\Lambda)\Big) \\ & + \chi_{n,k}\Big(\sum_{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\cap\Upsilon_n^c\neq\varnothing} \partial_k\Phi_\Lambda(x_{\Lambda\cap\Upsilon_n}|y_{\Lambda\cap\Upsilon_n^c})\Big)\Big\}\,e_k\,,\end{aligned} \tag{28}$$

for $x\in S_{-q}$, and $n\in\mathbb{N}$. Then (i) is obviously fulfilled. We note that $\mathfrak{b}_n$ is well-defined on S_{-q} by (77).

To check (ii) we first note that by construction each vector component $\mathfrak{b}_n(\,\cdot\,)_k$, $k\in\Upsilon_n$, $n\in\mathbb{N}$, considered as a function on $\mathbb{R}^{\Upsilon_n}$ is uniformly bounded on $\mathbb{R}^{\Upsilon_n}$. All partial derivatives of $\mathfrak{b}_n(\,\cdot\,)_k$ exist and are continuous due to Lemma .1.4. Moreover, (4) implies that $\partial_k\chi_{n,k}\big(\Phi'_{\{k\}}(x_k)\big)$ has compact support in $\mathbb{R}$, for all $n\in\mathbb{N}$, and $k\in\mathbb{Z}^d$. Using (5) we thus see that $\mathfrak{b}_n(\,\cdot\,)_k$ is contained in $C_b^1(\mathbb{R}^{\Upsilon_n},\mathbb{R})$, for all $k\in\mathbb{Z}^d$, and $n\in\mathbb{N}$. Moreover,

$$\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni j,k,\ell}} \big|\,\partial_j\partial_k\partial_\ell\Phi_\Lambda(x_\Lambda)\,\big| \;\leqslant\; \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni j,k,\ell}} J_\Lambda \;<\; \infty\,,$$

for all $x\in S_{-q}$, and $j,k,\ell\in\mathbb{Z}^d$. This implies that $\mathfrak{b}_n(\,\cdot\,)_k \in C_b^2(\mathbb{R}^{\Upsilon_n},\mathbb{R})$, for all $k\in\mathbb{Z}^d$, and $n\in\mathbb{N}$. Therefore, $\mathfrak{b}_n(\,\cdot\,)_k$ has globally Hölder continuous first order partial derivatives, for all $k\in\mathbb{Z}^d$, and $n\in\mathbb{N}$. Altogether, we see that (ii) is fulfilled.

Next, we consider (iv). For $x \in S_{-p} \subset S_{-q}$, we have

$$
\begin{aligned}
|\mathfrak{b}(x) - \mathfrak{b}_n(x)|_{-q}^2 \;&\leqslant\; \sum_{k \in \Upsilon_n^c} (1+|k|)^{-2q}\, |\partial_k H(x)|^2 \\
&+\; 3 \sum_{k \in \Upsilon_n} (1+|k|)^{-2q} \\
&\quad \cdot \Big| \sum_{\substack{\Lambda \in \mathfrak{P}_{\mathrm{f}} :\, \#\Lambda \geqslant 2, \\ k \in \Lambda \subseteq \Upsilon_n}} \partial_k \Phi_\Lambda(x_\Lambda) \;-\; \chi_{n,k}\Big(\sum_{\substack{\Lambda \in \mathfrak{P}_{\mathrm{f}} :\, \#\Lambda \geqslant 2, \\ k \in \Lambda \subseteq \Upsilon_n}} \partial_k \Phi_\Lambda(x_\Lambda) \Big) \Big|^2 \\
&+\; 3 \sum_{k \in \Upsilon_n} (1+|k|)^{-2q} \\
&\quad \cdot \Big\{ \sum_{\substack{\Lambda \in \mathfrak{P}_{\mathrm{f}} \,:\, \Lambda \ni k\,, \\ \#\Lambda \geqslant 2\,,\ \Lambda \cap \Upsilon_n^c \neq \varnothing}} \Big(\big|\, \partial_k \Phi_\Lambda(x_\Lambda) \,\big| \;+\; \big|\, \partial_k \Phi_\Lambda(x_{\Lambda \cap \Upsilon_n} | y_{\Lambda \cap \Upsilon_n^c}) \,\big| \Big) \Big\}^2 \\
&+\; 3 \sum_{k \in \Upsilon_n} (1+|k|)^{-2q} \Big| \, \Phi'_{\{k\}}(x_k) \;-\; \chi_{n,k}\big(\Phi'_{\{k\}}(x_k) \big) \Big|^2 \\
&=: \; \clubsuit + \spadesuit + \heartsuit + \diamondsuit\,. \qquad (29)
\end{aligned}
$$

Using (1) we see that $\big|\Phi'_{\{k\}}(x_k)\big| \geqslant R(n,k)$ implies $(1+|x_k|)^{L-1} \geqslant R(n,k)/c$ and, hence, $(1+|x|_{-p})^{L-1} \geqslant n/c$, for all $k \in \mathbb{Z}^d$, $n \in \mathbb{N}$, and $x \in S_{-p}$. Therefore, we obtain for the last term on the right side of (29)

$$
\begin{aligned}
\frac{\diamondsuit}{12} \;&\leqslant\; \sum_{k \in \Upsilon_n} (1+|k|)^{-2q} c^2\, \mathbf{1}_{\{|\Phi'_{\{k\}}(x_k)| \geqslant R(n,k)\}}\, (1+|x_k|)^{2(L-1)} \\
&\leqslant\; \sum_{k \in \Upsilon_n} (1+|k|)^{-2q+2p(L-1)} c^2\, \mathbf{1}_{\{(1+|x|_{-p})^{L-1} \geqslant n/c\}}\, (1+|x|_{-p})^{2(L-1)}.
\end{aligned}
$$

Since it holds

$$
\sum_{k \in \Upsilon_n} (1+|k|)^{-2q+2p(L-1)} \;<\; |\mathbf{1}|^2_{-q+p(L-1)} \;<\; \infty\,, \qquad (30)
$$

for all $n \in \mathbb{N}$, because of our choice of p and q, we see that $\diamondsuit$ tends to zero, as $n \to \infty$, pointwise, for $x \in S_{-p}$. Using (77) we get for the first term on the right side of (29)

$$
\clubsuit \;\leqslant\; \mathrm{const}\, (1+|x|_{-p})^{2(L-1)} \sum_{k \in \Upsilon_n^c} (1+|k|)^{-2q+2p(L-1)} \;\longrightarrow\; 0\,,
$$

as $n \to \infty$, for all $x \in S_{-p}$. Using (36) and (75) we see that the third term on the right side of (29) can be estimated as follows,

$$
\begin{aligned}
\heartsuit \;\leqslant\; & 3\,(\,2\,|x|_{-p} \,+\, |y|_{-p}\,)^2 \sum_{k\in\Upsilon_n} (1+|k|)^{-2q+2p} \\
& \cdot\Big(\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\Lambda\ni k\,,\\ \#\Lambda\geqslant 2\,,\ \Lambda\cap\Upsilon_n^c\neq\varnothing}} J_\Lambda (\#\Lambda)^{\frac12} \Big(1+\max_{\ell\in\Lambda}|k-\ell|\Big)^p \Big)^2 \\
\leqslant\; & 3\,C(p)\,(\,2\,|x|_{-p} \,+\, |y|_{-p}\,)^2 \sum_{k\in\Upsilon_m^c} (1+|k|)^{-2q+2p} \\
& + 3\,|\mathbf{1}|^2_{-q+p}(\,2\,|x|_{-p} \,+\, |y|_{-p}\,)^2 \\
& \cdot \max_{k\in\Upsilon_m}\Big(\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\Lambda\ni k\,,\\ \#\Lambda\geqslant 2\,,\ \Lambda\cap\Upsilon_n^c\neq\varnothing}} J_\Lambda (\#\Lambda)^{\frac12} \Big(1+\max_{\ell\in\Lambda}|k-\ell|\Big)^p \Big)^2,
\end{aligned}
\tag{31}
$$

where $m<n$, and $C(p)>0$. Here we observe that, for $m\in\mathbb{N}$,

$$
\max_{k\in\Upsilon_m} \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\Lambda\ni k\,,\\ \#\Lambda\geqslant 2\,,\ \Lambda\cap\Upsilon_n^c\neq\varnothing}} J_\Lambda(\#\Lambda)\Big(1+\max_{\ell\in\Lambda}|k-\ell|\Big)^p \;\leqslant\; \frac{\mathrm{const}(p)}{\mathrm{dist}(\Upsilon_m,\Upsilon_n^c)}\,. \tag{32}
$$

Consequently, for all $x\in S_{-p}$, $\heartsuit$ can be bounded by an arbitrarily small number by choosing m and $n>m$ large enough. Now we estimate the second term on the right side of (29). To begin with we observe that an estimate analogous to (77) yields

$$
\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\Lambda\ni k,\\ \#\Lambda\geqslant 2}} |\partial_k\Phi_\Lambda(x_\Lambda)| \;\leqslant\; C'(p)\,(1+|k|)^p\,|x|_{-p}\,, \tag{33}
$$

for all $x\in S_{-p}$, where $C'(p)>0$. As a consequence we have

$$
\begin{aligned}
\mathcal{M}_n^{(k)} \;&:=\; \Big\{ x\in S_{-p}\,:\, \Big| \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\#\Lambda\geqslant 2,\\ k\in\Lambda\subseteq\Upsilon_n}} \partial_k\Phi_\Lambda(x_\Lambda)\Big| \;\geqslant\; R(n,k)\Big\} \\
&\subset\; \Big\{ x\in S_{-p}\,:\, |x|_{-p} \;\geqslant\; \frac{n}{C'(p)}\Big\} \;=:\; \mathcal{N}_n\,,
\end{aligned}
\tag{34}
$$

for $n\in\mathbb{N}$, and $k\in\Upsilon_n$. In view of (34) and (33), we get, for all $n\in\mathbb{N}$,

$$
\begin{aligned}
\spadesuit \;&\leqslant\; 3\sum_{k\in\Upsilon_n}(1+|k|)^{-2q}\Big(\mathbf{1}_{\mathcal{M}_n^{(k)}}(x)\cdot 2\,\Big| \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\#\Lambda\geqslant 2,\\ k\in\Lambda\subseteq\Upsilon_n}} \partial_k\Phi_\Lambda(x_\Lambda)\Big|\Big)^2 \\
&\leqslant\; \mathrm{const}\;|\mathbf{1}|^2_{-q+p}\,\mathbf{1}_{\mathcal{N}_n}(x)\,|x|^2_{-p}\,.
\end{aligned}
\tag{35}
$$

Hence, also $\spadesuit$ converges to zero, as n tends to infinity, pointwise, for each $x\in S_{-p}$. Now, since $\mu_\beta(X\setminus S_{-p})=0$ by Theorem 0.2.3, we infer from (29)-(35) together with the à-priori estimate (15) and Lebesgue's dominated convergence theorem that (iv) is fulfilled.

Finally, we verify (iii). For $n \in \mathbb{N}$, $x \in S_{-q}$, $h \in S_q$, and $z_\Lambda = (x_{\Lambda\cap\Upsilon_n}|y_{\Lambda\cap\Upsilon_n^c})$, we get

$$
\begin{aligned}
\langle\, \mathfrak{b}_n'(x)\, h\, ,\, h\,\rangle_q \;&\leqslant\; e\,|h|_q^2 \\
&\quad + \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\\ \#\Lambda\geqslant 2}} \sum_{k,\ell\in\Lambda} (1+|k-\ell|)^q \big|\,\partial_k\partial_\ell\Phi_\Lambda(z_\Lambda)\,\big|\,(1+|k|)^q|h_k|\,(1+|\ell|)^q|h_\ell| \\
&\leqslant\; e\,|h|_q^2 + \sum_{k\in\mathbb{Z}^d} \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2\\ \Lambda\ni k}} \sum_{\ell\in\Lambda} J_\Lambda\,(1+|k-\ell|)^q(1+|k|)^{2q}|h_k|^2 \\
&\leqslant\; \mathrm{const}(q)\,|h|_q^2\,.
\end{aligned}
$$

Here we use that $0 \leqslant \chi_{n,k}' \leqslant 1$, for all $k \in \mathbb{Z}^d$, and $n \in \mathbb{N}$, in the first step. In the second step we apply the Cauchy-Schwarz inequality. ■

Proof of Proposition 0.3.1 : Proposition 0.5.1, Lemma 0.6.5 below, and the Kato-Rellich theorem imply that $\mathscr{L}_\beta^{(k)}$ is essentially selfadjoint, for $k \in \mathbb{N}$. This shows that $\mathscr{L}_\beta$ is essentially selfadjoint, too, since the orthogonal sum $L_\beta := \bigoplus_{k\in\mathbb{N}_0} \mathscr{L}_\beta^{(k)**}$ defines a selfadjoint operator that extends $\mathscr{L}_\beta$ and $\mathcal{D}(\mathscr{L}_\beta) = \mathrm{span}\,\{\mathcal{D}(\mathscr{L}_\beta^{(k)}) : k = 0,1,2,\dots\}$ is a core of L_β. ■

6 Relative bounds at small temperature

In this section we derive relative operator and form bounds which show that, for large β, L_β is a small perturbation of some simpler comparison operator. Crucial from a technical point point of view is the use of the DLR equations in Lemma 0.6.1, which allows us to derive estimates inspired by [10]. We point out that in order to obtain relative operator bounds it suffices to assume that Φ fulfills Hypothesis 1. The relative form bounds are derived using Hypothesis 2.

Lemma 6.1 *Assume that Φ fulfills Hypothesis 1. Then, for all $k \in \mathbb{Z}^d$, it holds*

$$
\tfrac{\beta^2}{4}\,|\partial_k H|^2 - \tfrac{\beta}{2}\,\partial_k^2 H \;\leqslant\; \big(-\partial_k + \beta\,\partial_k H\,\big)\,\partial_k
$$

in the sense of quadratic forms on $\Omega^0 \subset \mathscr{H}^0$.

Proof: Using (73), (75), and Lemmas .1.3 and .1.4, we first observe that, for all $k \in \mathbb{Z}^d$, $v \in \Omega^0$, and μ_β-almost everywhere (dropping the arguments

$(x_k|x_{\{k\}^c})$ of the integrants) we have

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(\tfrac{\beta^2}{4}\, |\partial_k H|^2 - \tfrac{\beta}{2}\, \partial_k^2 H \Big)\, |v|^2 e^{-\beta\, H_{\{k\}}}\, dx_k \\
&\quad\leqslant \int_{\mathbb{R}} e^{-\frac{\beta}{2}\, H_{\{k\}}}\, \bar{v} \Big(-\, \partial_k^2 + \tfrac{\beta^2}{4}\, |\partial_k H|^2 - \tfrac{\beta}{2}\, \partial_k^2 H \Big)\, e^{-\frac{\beta}{2}\, H_{\{k\}}}\, v\, dx_k \\
&\quad= \int_{\mathbb{R}} e^{-\frac{\beta}{2}\, H_{\{k\}}}\, \bar{v} \big(-\, \partial_k + \tfrac{\beta}{2}\, \partial_k H \big) \big(\partial_k + \tfrac{\beta}{2}\, \partial_k H \big)\, e^{-\frac{\beta}{2}\, H_{\{k\}}}\, v\, dx_k \\
&\quad= \int_{\mathbb{R}} \Big| \big(\partial_k + \tfrac{\beta}{2}\, \partial_k H \big) e^{-\frac{\beta}{2}\, H_{\{k\}}}\, v \Big|^2 dx_k \\
&\quad= \int_{\mathbb{R}} e^{-\beta\, H_{\{k\}}}\, |\partial_k v|^2\, dx_k \ .
\end{aligned}$$

Due to the previous estimate, the DLR equations (10), and the integration by parts formula (13) we further get, for all $k \in \mathbb{Z}^d$, and $v \in \Omega^0$,

$$\begin{aligned}
&\int_X \big(\tfrac{\beta^2}{4}\, |\partial_k H(y)|^2 - \tfrac{\beta}{2}\, \partial_k^2 H(y) \big)\, |v(y)|^2\, d\mu_\beta(y) \\
&\quad= \int_X \frac{1}{\mathscr{Z}_{\{k\}}(\beta, y)} \int_{\mathbb{R}} \Big(\tfrac{\beta^2}{4}\, |\partial_k H(x_k|y_{\{k\}^c})|^2 - \tfrac{\beta}{2}\, \partial_k^2 H(x_k|y_{\{k\}^c}) \Big) \\
&\qquad \cdot |v(x_k|y_{\{k\}^c})|^2 e^{-\beta\, H_{\{k\}}(x_k|y_{\{k\}^c})}\, dx_k\, d\mu_\beta(y) \\
&\quad\leqslant \int_X \frac{1}{\mathscr{Z}_{\{k\}}(\beta, y)} \int_{\mathbb{R}} e^{-\beta\, H_{\{k\}}(x_k|y_{\{k\}^c})}\, |\partial_k v(x_k|y_{\{k\}^c})|^2\, dx_k\, d\mu_\beta(y) \\
&\quad= \int_X |\partial_k v(y)|^2\, d\mu_\beta(y) \\
&\quad= \int_X \overline{v(y)}\, \big(-\, \partial_k + \beta\, \partial_k H(y) \big)\, \partial_k\, v(y)\, d\mu_\beta(y) \ .
\end{aligned}$$

■

Lemma 6.2 *Assume that Φ fulfills Hypothesis 1. Then, for all $\Lambda \in \mathfrak{P}_{\mathrm{f}}$, $\#\Lambda \geqslant 2$, $k \in \Lambda$, and $x_\Lambda \in \mathbb{R}^\Lambda$, it holds*

$$|\partial_k \Phi_\Lambda(x_\Lambda)| \ \leqslant \ J_\Lambda (\#\Lambda)\, \rho + \frac{J_\Lambda}{a} \sum_{\ell \in \Lambda} |\Phi'_{\{\ell\}}(x_\ell)| \ .$$

Proof: Using the fundamental theorem of calculus and Hypothesis 1(iii) we obtain

$$|\partial_k \Phi_\Lambda(x_\Lambda)| \ \leqslant \ J_\Lambda \sum_{\ell \in \Lambda} |x_\ell| \ . \tag{36}$$

On the other hand (4) implies

$$|x_\ell| \;\leqslant\; \frac{1}{a}\,|\Phi'_{\{\ell\}}(x_\ell)| \,+\, \rho\,. \tag{37}$$

■

Lemma 6.3 *Assume that* Φ *fulfills Hypothesis 1. Then there is an operator* $\Xi \in \mathscr{L}(\ell^\infty_{\mathbb{R}}(\mathbb{Z}^d))$ *with positive matrix elements* $\Xi_{k\ell}$*,* $k,\ell \in \mathbb{Z}^d$*, such that*

$$\frac{1}{2}\,|\Phi'_{\{k\}}(x_k)|^2 \;\leqslant\; \|\Xi\| \sum_{\ell\in\mathbb{Z}^d} \Xi_{k\ell}\,|\partial_\ell H(x)|^2 \,+\, \rho^2\,\mathscr{J}_1(0)^2\,\|\Xi\|^2$$

holds, for all $k \in \mathbb{Z}^d$*, and* $x \in S_{-p}$*,* $p > d/2$*.*

Proof: Lemma 0.6.2 implies

$$\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2\\ \Lambda\ni k}} |\,\partial_k\Phi_\Lambda(x_\Lambda)\,| \;\leqslant\; \rho\,\mathscr{J}_1(0) \,+\, \frac{1}{a}\sum_{\ell\in\mathbb{Z}^d} J_{k\ell}\,|\Phi'_{\{\ell\}}(x_\ell)|\,, \tag{38}$$

where we set

$$J_{k\ell} \;:=\; \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni k,\ell}} J_\Lambda\,, \tag{39}$$

for $k,\ell \in \mathbb{Z}^d$. Since we have

$$|\Phi'_{\{k\}}(x_k)| \;\leqslant\; |\partial_k H(x)| \,+\, \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni k}} |\,\partial_k\Phi_\Lambda(x_\Lambda)\,|\,,$$

Equation (38) yields

$$\sum_{\ell} \big(\,\delta_{k\ell} - \frac{1}{a}\,J_{k\ell}\,\big)\,|\Phi'_{\{\ell\}}(x_\ell)| \;\leqslant\; |\partial_k H(x)| \,+\, \rho\,\mathscr{J}_1(0)\,. \tag{40}$$

Due to Hypothesis 1(iii) we have $J_{k\ell} = J_{0\ell-k}$ and $\sum_{m\in\mathbb{Z}^d} J_{0m} = \mathscr{J}_1(0) < a$. This shows that the inverse, Ξ, of the operator Ξ^{-1} with matrix elements $(\Xi^{-1})_{k\ell} := \delta_{k\ell} - \frac{1}{a}\,J_{k\ell}$ is given by a $\mathscr{L}(\ell^\infty_{\mathbb{R}}(\mathbb{Z}^d))$-norm convergent Neumann series. We recall that the norm on $\mathscr{L}(\ell^\infty_{\mathbb{R}}(\mathbb{Z}^d))$ is given by the row sum norm. Furthermore, we observe that all matrix elements of Ξ are positive. An application of Ξ to (40) thus gives

$$|\Phi'_{\{k\}}(x_k)| \;\leqslant\; \sum_{\ell\in\mathbb{Z}^d} \Xi_{k\ell}\,|\partial_\ell H(x)| \,+\, \rho\,\mathscr{J}_1(0)\,\|\Xi\|\,,$$

which together with the Cauchy-Schwarz inequality implies the asserted estimate. ■

Lemma 6.4 *Assume that Φ fulfills Hypothesis 1. Then there exists a constant $C \in (0, \infty)$, such that, for all $k \in \mathbb{Z}^d$, and $\beta \geqslant 1$, we have*

$$| \, \Phi'_{\{k\}} \, |^2 \;\leqslant\; \frac{C}{\beta} \, \big(\mathscr{L}_\beta^{(0)} + 1 \big) \, + \, \rho^2 \, C$$

in the sense of quadratic forms on Ω^0.

Proof: Lemma 0.6.1 and Hypothesis 1 imply

$$\begin{aligned}
\frac{1}{4} \, | \, \partial_k H \, |^2 \;\; & \leqslant \;\; \frac{1}{4} \, | \, \partial_k H \, |^2 \, - \, \frac{1}{2\beta} \, \partial_k^2 H \, + \, \frac{1}{2\beta} \, \Phi''_{\{k\}}(0) \\
& \quad + \frac{1}{2\beta} \, \big| \Phi''_{\{k\}}(x_k) - \Phi''_{\{k\}}(0) \big| \, + \, \frac{\mathscr{J}_1(0)}{2\beta} \\
& \leqslant \;\; \frac{1}{\beta^2} \, (-\partial_k + \beta \, \partial_k H) \partial_k \, + \, \frac{\epsilon}{2\beta} \, |\Phi'_{\{k\}}(x_k)|^2 \\
& \quad + \frac{c + \mathscr{J}_1(0) + b_2 + b_1^2/(4\epsilon)}{2\beta} \, ,
\end{aligned} \tag{41}$$

for all $k \in \mathbb{Z}^d$, $\beta \in [1, \infty)$, and $\epsilon > 0$. Combining (41) with Lemma 0.6.3 we see that, for all $\epsilon > 0$, there is a constant C_ϵ such that

$$\begin{aligned}
|\Phi'_{\{k\}}(x_k)|^2 \;\; & \leqslant \;\; \frac{2 \, \|\Xi\|^2}{\beta^2} \sum_{\ell \in \mathbb{Z}^d} \big(- \partial_\ell + \beta \partial_\ell H \big) \, + \, \frac{C_\epsilon}{\beta} \, + \, 2\rho^2 \, \mathscr{J}_1(0)^2 \|\Xi\|^2 \\
& \quad + \frac{\epsilon \, \|\Xi\|}{\beta} \sum_{\ell \in \mathbb{Z}^d} \Xi_{k\ell} |\Phi'_{\{\ell\}}(x_\ell)|^2 \, ,
\end{aligned} \tag{42}$$

for all $k \in \mathbb{Z}^d$, and $\beta \in [1, \infty)$. Choosing $\epsilon > 0$ sufficiently small we can ensure that the norm of the inverse of the operator $\Theta \in \mathscr{L}(\ell^\infty_{\mathbb{R}}(\mathbb{Z}^d))$ given by the matrix elements $\Theta_{k\ell} := \delta_{k\ell} - \frac{\epsilon \, \|\Xi\|}{\beta} \Xi_{k\ell}$ is bounded uniormly in $\beta \in [1, \infty)$. An application of Θ^{-1} to (42) shows that the assertion of the lemma holds true. ■

In the next two lemmas we derive relative bounds on $\mathscr{L}_\beta$ with respect to some comparison operator $\mathscr{A}_\beta$ acting in $\mathscr{H}$. $\mathscr{A}_\beta$ is defined on the dense domain Ω by

$$\mathscr{A}_\beta \; := \; \mathscr{L}_\beta^{(0)} \otimes \mathbf{1} \, + \, \mathbf{1} \otimes d\Gamma(H''(0)) \; = \; \mathscr{L}_\beta^{(0)} \otimes \mathbf{1} \, + \, \mathbf{1} \otimes \sum_{k, \ell \in \mathbb{Z}^d} H''(0)_{k\ell} \, a_k^\dagger a_\ell \, .$$

Since $\mathscr{L}_\beta^{(0)}$ is essentially selfadjoint on Ω^0, and $H''(0)$ is bounded on $\ell^2_{\mathbb{C}}(\mathbb{Z}^d)$, we know that $\mathscr{A}_\beta$ is again essentially selfadjoint. The same holds true for its restrictions, $\mathscr{A}_\beta^{(k)}$, to Ω^k, for $k \in \mathbb{N}$. We denote the unique selfadjoint extensions of $\mathscr{A}_\beta^{(k)}$, $k \in \mathbb{N}$, and $\mathscr{A}_\beta$ by $A_\beta^{(k)}$, $k \in \mathbb{N}$, and A_β.

For $x \in S_{-p}$, $p > d/2$, we further define an operator $W(x)$ with matrix elements

$$W_{ij}(x) \; := \sum_{\Lambda \in \mathfrak{P}_{\mathrm{f}} : \, \Lambda \ni i,j} \big(\partial_i \partial_j \Phi_\Lambda(x_\Lambda) \, - \, \partial_i \partial_j \Phi_\Lambda(0) \big) \, ,$$

for $i,j\in\mathbb{Z}^d$. Then we obtain a well-defined operator $d\Gamma(W):\Omega\to\mathscr{H}$ and it obviously holds $d\Gamma(W)=d\Gamma(H'')-d\Gamma(H''(0))=\mathscr{L}_\beta-\mathscr{A}_\beta$. We denote the restriction of $d\Gamma(W)$ to Ω^k by $d\Gamma^{(k)}(W)$, $k\in\mathbb{N}$.

Lemma 6.5 *Assume that Φ fulfills Hypothesis 1. Then, for all $k\in\mathbb{N}$, and $\beta\in[1,\infty)$, $d\Gamma^{(k)}(W)$ is infinitesimally $\mathscr{A}_\beta^{(k)}$-bounded. More precisely, for each $\epsilon>0$, we find some constant $c(\epsilon)\in(0,\infty)$ such that, for all $\beta\in[1,\infty)$, $k\in\mathbb{N}$, and $v\in\Omega^k$,*

$$\|\,d\Gamma^{(k)}(W)\,v\,\|^2\;\leqslant\;\epsilon\beta^{-1}k^2\,\|\,\mathscr{A}_\beta^{(k)}\,v\,\|^2\,+\,c(\epsilon)(\beta^{-1}+b_2+\rho^2)\,k^2\,\|\,v\,\|^2\,.$$

Proof: For $i,j\in\mathbb{Z}^d$, we set

$$\widetilde{J}_{ij}\;:=\;\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni i,j}}J_\Lambda(\#\Lambda)\,,\qquad \widehat{J}_{ij}\;:=\;\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni i,j}}J_\Lambda(\#\Lambda)^2\,.$$

Then Hypothesis 1(iii) implies $\widetilde{J}_{ij}=\widetilde{J}_{0j-i}$ and $\sum_{i\in\mathbb{Z}^d}e^{2\varkappa|i|}\widetilde{J}_{0i}\leqslant\mathscr{J}_2(2\varkappa)<\infty$, for $\varkappa\in[0,\gamma/2]$. Similarly, we have $\widehat{J}_{ij}=\widehat{J}_{0j-i}$ and $\sum_{i\in\mathbb{Z}^d}\widehat{J}_{0i}<\infty$, where we use $(\#\Lambda)\leqslant(1+2\max_{\ell\in\Lambda}|\ell|)^d$. Using this we observe that due to (3), (5), and (37), we find some $C_\gamma>0$ such that, for all $\varkappa\in[0,\gamma/2]$, $i,j\in\mathbb{Z}^d$, and $x\in S_{-p}$, $p>d/2$,

$$\begin{aligned}
&e^{2\varkappa|i-j|}|W_{ij}(x)|^2\\
&\leqslant\; e^{2\varkappa|i-j|}\Big(\delta_{ij}\,b_1\,|\Phi'_{\{i\}}(x_i)|\,+\,\delta_{ij}\,b_2\,+\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni i,j}}J_\Lambda\sum_{\ell\in\Lambda}|x_\ell|\Big)^2\\
&\leqslant\;\Big(\delta_{ij}\,b_1\,|\Phi'_{\{i\}}(x_i)|\,+\,\delta_{ij}\,b_2\\
&\qquad+\frac{e^{\varkappa|i-j|}}{a}\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni i,j}}J_\Lambda\sum_{\ell\in\Lambda}|\Phi'_{\{\ell\}}(x_\ell)|\,+\,\rho\,e^{\varkappa|i-j|}\widetilde{J}_{ij}\Big)^2\\
&\leqslant\;\Big(\delta_{ij}\,b_1^2\,+\,\delta_{ij}\,b_2\,+\,e^{2\varkappa|i-j|}\widetilde{J}_{ij}\,+\,\frac{e^{2\varkappa|i-j|}}{a^2}\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni i,j}}J_\Lambda\Big)\\
&\qquad\cdot\Big(\delta_{ij}\,|\Phi'_{\{i\}}(x_i)|^2\,+\,\delta_{ij}\,b_2\,+\,\rho^2\,\widetilde{J}_{ij}\,+\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni i,j}}J_\Lambda\Big(\sum_{\ell\in\Lambda}|\Phi'_{\{\ell\}}(x_\ell)|\Big)^2\,\Big)\\
&\leqslant\;C_\gamma\Big(\delta_{ij}|\Phi'_{\{i\}}(x_i)|^2+\delta_{ij}b_2+\rho^2\widetilde{J}_{ij}\,+\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\#\Lambda\geqslant 2,\\ \Lambda\ni i,j}}J_\Lambda(\#\Lambda)\sum_{\ell\in\Lambda}|\Phi'_{\{\ell\}}(x_\ell)|^2\Big).
\end{aligned}$$

Using (43) and Lemma 0.6.4 we get the following bound in the sense of quadratic forms on Ω^0,

$$\begin{aligned}
e^{2\varkappa|i-j|}|W_{ij}|^2\;&\leqslant\;(\delta_{ij}\,+\,\widehat{J}_{ij})\,\frac{C_\gamma C}{\beta}\,(\mathscr{L}_\beta^{(0)}+1)\\
&\quad+C_\gamma\Big(\delta_{ij}\,b_2\,+\,\rho^2\,\widetilde{J}_{ij}\,+\,\rho^2\,C\,(\delta_{ij}\,+\,\widehat{J}_{ij})\Big)\\
&=\;\Gamma_{ij}\,(\,\beta^{-1}\mathscr{L}_\beta^{(0)}+\beta^{-1}+b_2+\rho^2\,)\,,
\end{aligned}\tag{43}$$

where $\Gamma_{ij} = \Gamma_{ji} \geqslant 0$, for $i,j \in \mathbb{Z}^d$, such that $\sup_{i\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \Gamma_{ij} < \infty$. Now, for $v \in \Omega$, we have

$$\begin{aligned} \Big\| \sum_{i,j\in\mathbb{Z}^d} W_{ij} \otimes a_i^\dagger a_j \, v \Big\|^2 &= \Big\langle v \Big| \sum_{i,j,k,\ell\in\mathbb{Z}^d} W_{ij}\, W_{k\ell} \otimes a_i^\dagger a_j a_k^\dagger a_\ell \, v \Big\rangle \\ &\leqslant \Big| \Big\langle v \Big| \sum_{i,j,k,\ell\in\mathbb{Z}^d} W_{ij}\, W_{k\ell} \otimes a_i^\dagger a_k^\dagger a_j a_\ell \, v \Big\rangle \Big| \\ &\quad + \Big| \Big\langle v \Big| \sum_{i,k,\ell\in\mathbb{Z}^d} W_{ik}\, W_{k\ell} \otimes a_i^\dagger a_\ell \, v \Big\rangle \Big| . \end{aligned} \tag{44}$$

The first summand on the right side of (44) is less or equal to

$$\begin{aligned} &\int_X \sum_{i,j,k,\ell\in\mathbb{Z}^d} |W_{ij}|\, |W_{k\ell}|\, \| \, a_k a_i \, v \, \|_{\mathfrak{F}_\mathrm{f}} \, \| \, a_j a_\ell \, v \, \|_{\mathfrak{F}_\mathrm{f}} \, d\mu_\beta \\ &\quad \leqslant \int_X \Big(\sum_{i,j,k,\ell\in\mathbb{Z}^d} |W_{ij}|\, |W_{k\ell}|\, \| \, a_k a_i \, v \, \|_{\mathfrak{F}_\mathrm{f}}^2 \Big)^{1/2} \\ &\qquad \cdot \Big(\sum_{i,j,k,\ell\in\mathbb{Z}^d} |W_{ij}|\, |W_{k\ell}|\, \| \, a_j a_\ell \, v \, \|_{\mathfrak{F}_\mathrm{f}}^2 \Big)^{1/2} d\mu_\beta \\ &\quad = \int_X \sum_{i,j,k,\ell\in\mathbb{Z}^d} |W_{ij}|\, |W_{k\ell}|\, \| \, a_k a_i \, v \, \|_{\mathfrak{F}_\mathrm{f}}^2 \, d\mu_\beta \, . \end{aligned} \tag{45}$$

Here we use the symmetry of W in the last step. Now, for $\varkappa \in [0, \gamma/2]$, $i,j,k,\ell \in \mathbb{Z}^d$, and all $x \in S_{-p}$, $p > d/2$, we have

$$\begin{aligned} &|W_{ij}(x)|\, |W_{k\ell}(x)| \\ &\quad \leqslant \frac{e^{-2\varkappa|k-\ell|+2\varkappa|i-j|}}{2} |W_{ij}(x)|^2 + \frac{e^{-2\varkappa|i-j|+2\varkappa|k-\ell|}}{2} |W_{k\ell}(x)|^2 \, , \end{aligned} \tag{46}$$

and using (43) we get

$$\begin{aligned} &\int_X \sum_{i,j,k,\ell\in\mathbb{Z}^d} e^{-2\varkappa|k-\ell|+2\varkappa|i-j|} |W_{ij}|^2 \, \| \, a_k a_i \, v \, \|_{\mathfrak{F}_\mathrm{f}}^2 \, d\mu_\beta \\ &\quad = \sum_{i,j,k,\ell\in\mathbb{Z}^d} e^{-2\varkappa|k-\ell|} \Big\langle a_k a_i \, v \Big| e^{2\varkappa|i-j|} |W_{ij}|^2 \otimes a_k a_i \, v \Big\rangle \\ &\quad \leqslant C' \Big(\sum_{\ell\in\mathbb{Z}^d} e^{-2\varkappa|\ell|} \Big) \Big(\sup_{i\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \Gamma_{ij} \Big) \\ &\qquad \cdot \sum_{i,k\in\mathbb{Z}^d} \Big\langle a_k a_i \, v \Big| \big(\frac{1}{\beta} (\mathscr{L}_\beta^{(0)} + 1) + b_2 + \rho^2 \big) \otimes a_k a_i \, v \Big\rangle \\ &\quad = C'' \Big\langle v \Big| \big(\frac{1}{\beta} (\mathscr{L}_\beta^{(0)} + 1) + b_2 + \rho^2 \big) \otimes \sum_{i,k\in\mathbb{Z}^d} a_i^\dagger a_k^\dagger a_k a_i \, v \Big\rangle \\ &\quad = C'' \Big\langle v \Big| \big(\frac{1}{\beta} (\mathscr{L}_\beta^{(0)} + 1) + b_2 + \rho^2 \big) \otimes (N^2 - N) \, v \Big\rangle \, , \end{aligned} \tag{47}$$

where $C', C'' > 0$, and where $N = d\Gamma(\mathbf{1}) = \sum_{i\in\mathbb{Z}^d} a_i^\dagger a_i$ is the number operator on $\mathfrak{F}_{\mathrm{f}}$. Combining the previous estimate and an obvious analogue with (46) we see that the term in the last line of (45) and, hence, the term in the second line of (44) is bounded by the last term in (47). Similarly, we see that the term in the last line of (44) can be estimated as follows,

$$\begin{aligned} &\Big|\Big\langle v \,\Big|\, \sum_{i,k,\ell\in\mathbb{Z}^d} W_{ik}\, W_{k\ell} \otimes\, a_i^\dagger a_\ell\, v \Big\rangle\Big| \\ &\leqslant\;\; C'''\Big\langle v \,\Big|\, (\,\frac{1}{\beta}(\mathscr{L}_\beta^{(0)} + 1) + b_2 + \rho^2\,) \otimes N\, v \Big\rangle\,. \end{aligned} \tag{48}$$

Combining (44)-(48) we obtain the following estimate, for all $\varepsilon > 0$, $k\in\mathbb{N}$, and $v\in\Omega^k$,

$$\begin{aligned} \|\, d\Gamma^{(k)}(W)\, v\,\|^2 \;&=\; \Big\|\sum_{i,j\in\mathbb{Z}^d} W_{ij}\, a_i^\dagger a_j\, v\,\Big\|^2 \\ &\leqslant\;\; \frac{c_1\, k^2}{\beta}\,\langle\, v\,|\,\mathscr{L}_\beta^{(0)}\otimes\mathbf{1}\, v\,\rangle \;+\; c_1\,\big(\beta^{-1} \;+\; b_2+\rho^2)\big)\, k^2\,\|\, v\,\|^2 \\ &\leqslant\;\; \frac{c_1\, k^2}{\beta}\,\langle\, v\,|\,\mathscr{A}_\beta^{(k)}\, v\,\rangle \;+\; c_1\,\big((1+e+\mathscr{J}_1(0))\beta^{-1} + b_2+\rho^2\big)\, k^2\,\|\, v\,\|^2 \\ &\leqslant\;\; \frac{\epsilon\, k^2}{\beta}\,\|\,\mathscr{A}_\beta^{(k)}\, v\,\|^2 \;+\; \big(c_2(\epsilon)\beta^{-1} \;+\; c_1 k^2(b_2+\rho^2)\big)\, k^2\,\|\, v\,\|^2\;, \end{aligned}$$

where $0 < c_1, c_2(\epsilon) < \infty$. Here we observe in the second step that $H''(0) + e + \mathscr{J}_1(0) \geqslant 0$ by Hypothesis 1. ∎

Next, we derive the relative form bound. Actually, we prove a stronger result stating that the relative form bound still holds true after the conjugation with appropriate exponential weights. Such weighted estimates are needed in Section 0.8 in order to obtain the correlation asymptotics. So, given any globally Lipschitz continuous function $\theta : \mathbb{R}^d \to [0,\infty)$, we introduce an operator Θ acting in $\ell^2_{\mathbb{C}}(\mathbb{Z}^d)$ with domain

$$\mathcal{D}(\Theta) \;:=\; \big\{\, x\in\ell^2_{\mathbb{C}}(\mathbb{Z}^d)\,:\,\sum_{i\in\mathbb{Z}^d} e^{2\,\theta(i)}\,|x_i|^2 \;<\;\infty\,\big\}\;,$$

setting

$$(\Theta\, x)_i \;:=\; e^{\theta(i)}\, x_i\;, \qquad i\in\mathbb{Z}^d\;,\;\; x\in\mathcal{D}(\Theta)\;.$$

Lemma 6.6 *Asssume that Φ fulfills Hypothesis 2. Then there exists a constant $C > 0$ such that, for all $k\in\mathbb{N}$, and $\beta\in[1,\infty)$, it holds*

$$\pm\big(\,\mathscr{L}_\beta^{(k)} \;-\; \mathscr{A}_\beta^{(k)}\,\big) \;\leqslant\; \frac{k\, C}{\beta^{1/2}}\,\mathscr{A}_\beta^{(k)}\;,$$

in the sense of quadratic forms on Ω^k. Furthermore, if $\theta\; : \mathbb{R}^d \to [0,\infty)$ is any globally Lipschitz continous function with Lipschitz constant $c\in[0,\gamma]$, then *there exists a constant $C > 0$ such that, for all $k \in \mathbb{N}$, $\beta \in [1, \infty)$, and $u, v \in \Omega^k$, we have*

$$\big| \langle\, u \,|\, d\Gamma^{(k)}\big(\, \Theta\, W\, \Theta^{-1} \,\big)\, v \,\rangle \big| \;\leqslant\; \frac{k\, C}{\beta^{1/2}} \, \big\| \, \big(\mathscr{A}_\beta^{(k)}\big)^{1/2} \, u \, \big\| \, \big\| \, \big(\mathscr{A}_\beta^{(k)}\big)^{1/2} \, v \, \big\| \, . \tag{49}$$

Proof: For $u, v \in \Omega$, we have

$$\begin{aligned}
&|\langle\, u \,|\, d\Gamma(\Theta\, W\, \Theta^{-1})\, v \,\rangle| \\
&\quad = \quad \Big| \Big\langle u \Big| \sum_{k,\ell \in \mathbb{Z}^d} e^{\theta(k) - \theta(\ell)} W_{k\ell} \otimes \, a_k^\dagger a_\ell \, v \Big\rangle \Big| \\
&\quad \leqslant \quad \int\limits_X \sum_{k,\ell \in \mathbb{Z}^d} e^{c|k-\ell|} |W_{k\ell}| \, \| \, a_k u \, \| \, \| \, a_\ell v \, \| \, d\mu_\beta \\
&\quad \leqslant \quad \frac{\epsilon}{2} \int\limits_X \sum_{k,\ell \in \mathbb{Z}^d} e^{c|k-\ell|} |W_{k\ell}| \, \| \, a_k u \, \|^2 \, d\mu_\beta \\
&\qquad\qquad + \frac{1}{2\epsilon} \int\limits_X \sum_{k,\ell \in \mathbb{Z}^d} e^{c|k-\ell|} |W_{k\ell}| \, \| \, a_\ell v \, \|^2 \, d\mu_\beta
\end{aligned} \tag{50}$$

Using (3) and (5) we further get, for $\beta \geqslant 1$, $k, \ell \in \mathbb{Z}^d$, and $x \in S_{-p}$, $p > d/2$,

$$\begin{aligned}
&e^{c|k-\ell|} |W_{k\ell}(x)| \\
&\quad \leqslant \quad \delta_{k\ell} \, b_1 \, |\Phi'_{\{k\}}(x_k)| \; + \; \frac{e^{c|k-\ell|}}{a} \sum_{\substack{\Lambda \in \mathfrak{P}_{\mathrm{f}} :\, \#\Lambda \geqslant 2, \\ \Lambda \ni k, \ell}} J_\Lambda \sum_{m \in \Lambda} |\Phi'_{\{m\}}(x_m)| \\
&\quad \leqslant \quad \beta^{1/2} \Big(\delta_{k\ell} \, |\Phi'_{\{k\}}(x_k)|^2 \; + \; \frac{e^{c|k-\ell|}}{a} \sum_{\substack{\Lambda \in \mathfrak{P}_{\mathrm{f}} :\, \#\Lambda \geqslant 2, \\ \Lambda \ni k, \ell}} J_\Lambda \sum_{m \in \Lambda} |\Phi'_{\{m\}}(x_m)|^2 \Big) \\
&\qquad + \frac{1}{4\beta^{1/2}} \Big(\, \delta_{k\ell} \, b_1^2 + \frac{e^{c|k-\ell|}}{a} \, \widetilde{J}_{k\ell} \Big) \, .
\end{aligned}$$

Applying Lemma 0.6.4 (with $\rho = 0$) we obtain

$$\begin{aligned}
&\int\limits_X \sum_{k,\ell \in \mathbb{Z}^d} e^{c|k-\ell|} |W_{k\ell}| \, \| \, a_k u \, \|^2 \, d\mu_\beta \\
&\quad \leqslant \quad \frac{C'}{\beta^{1/2}} \, \langle\, u \,|\, (\mathscr{L}_\beta^{(0)} + 1) \otimes N \, u \,\rangle \\
&\quad \leqslant \quad \frac{C''}{\beta^{1/2}} \, \langle\, u \,|\, (\mathscr{L}_\beta^{(0)} \otimes N \; + \; \mathbf{1} \otimes d\Gamma(H''(0)))\, u \,\rangle \; ,
\end{aligned}$$

where $C', C'' > 0$. Here we use that $H''(0) \geqslant (a - \mathscr{J}_1(0))\mathbf{1} > 0$. A similar estimate holds for the term in the last line of (50). Using that, for $s, t \geqslant 0$, it holds $\inf_{\epsilon\in(0,\infty)}\{\epsilon\, s^2 + t^2/\epsilon\} = 2st$, we thus obtain

$$
\begin{aligned}
&|\langle\, u \,|\, d\Gamma(\Theta\, W\, \Theta^{-1})\, v\,\rangle| \\
&\quad\leqslant\; \frac{C''}{\beta^{1/2}}\Big(\langle\, u \,|\, (\mathscr{L}_\beta^{(0)} \otimes N \,+\, \mathbf{1} \otimes d\Gamma(H''(0)))\, u\,\rangle\Big)^{1/2} \\
&\qquad\cdot\Big(\langle\, v \,|\, (\mathscr{L}_\beta^{(0)} \otimes N \,+\, \mathbf{1} \otimes d\Gamma(H''(0)))\, v\,\rangle\Big)^{1/2} ,
\end{aligned}
$$

which implies the assertion of the lemma. ■

In the following we denote the spectral projection corresponding to the interval J of some selfadjoint operator T in some Hilbert space by $E_J(T)$, and set

$$D_J(T) \;:=\; \dim \mathrm{Ran}\;\; E_J(T) \,.$$

Corollary 6.7 *Assume that Φ fulfills Hypothesis 2. Let $\ell \in \mathbb{N}$, and let $a, b \geqslant 0$ and $0 < \varepsilon < 1$ be such that $a + 2\,\varepsilon < b$,*

$$(a-\varepsilon, a+\varepsilon) \subset \varrho(A_\beta^{(\ell)}), \quad \textit{and} \quad (b-\varepsilon, b+\varepsilon) \subset \varrho(A_\beta^{(\ell)}) \,. \tag{51}$$

Furthermore let J denote one of the intervals (a,b), $(a,b]$, $[a,b)$, or $[a,b]$. Then there exists some $\beta_0 \in [1,\infty)$ such that

$$D_J(L_\beta^{(\ell)}) \;=\; D_J(A_\beta^{(\ell)}) \,, \tag{52}$$

for all $\beta \in [\beta_0,\infty)$. Moreover, if $k \in \mathbb{N}$ and if (51) holds, for all $1 \leqslant \ell \leqslant k$, then (52) holds simultaneously, for all $1 \leqslant \ell \leqslant k$, and $\beta \in [\beta_0,\infty)$.

PROOF: The proof is identical to that of [29, Proposition 2.3]. ■

PROOF OF THEOREM 0.3.6: We prove the theorem by induction on $n \in \mathbb{N}$. So let $n = 1$ and $\eta > 0$ be such that $2\eta < 2m_\star - m^\star$. Since $A_\beta^{(1)} \geqslant m_\star$, Corollary 0.6.7 ensures that $L_\beta^{(1)} \geqslant m_\star - \eta/2$, for all sufficiently large β. By Proposition 0.3.5(ii) and (iii) we see that $\sigma(L_\beta^{(0)}) \cap (0\,,\, m_\star - \eta/2) = \varnothing$, for β large enough. Since

$$\sigma(A_\beta^{(1)}) \;=\; \sigma(L_\beta^{(0)}) \;+\; \sigma(H''(0)) \,,$$

it follows that

$$
\begin{aligned}
\sigma(A_\beta^{(1)}) \cap (m^\star\,,\, 2m_\star - \eta/2) \;&=\; \varnothing \,, \\
\sigma(A_\beta^{(1)}) \cap (-\infty\,,\, 2m_\star - \eta/2) \;&=\; \sigma(H''(0)) \,,
\end{aligned}
$$

for large β. Clearly, it holds

$$\mathrm{Ran}\; E_{(-\infty\,,\,2m_\star-\eta/2)}(A_\beta^{(1)}) \;=\; \mathrm{Ran}\; E_{[m_\star\,,\,m^\star]}(A_\beta^{(1)}) \;=\; \mathbb{C}\,1 \otimes \ell^2_{\mathbb{C}}(\mathbb{Z}^d) \,,$$

for sufficiently large values of β. Moreover, we know that $A_\beta^{(2)} \geqslant 2m_\star$ Now Corollary 0.6.7 again implies that there is some $\beta_0 \geqslant 1$, such that, for all $\beta \geqslant \beta_0$, we have $L_\beta^{(2)} \geqslant 2m_\star - \eta$ and $\sigma(L_\beta^{(1)}) \cap (m^\star + \eta\,,\, 2m_\star - \eta) = \varnothing$, and, hence, $\sigma(L_\beta^{(0)}) \cap (m^\star + \eta\,,\, 2m_\star - \eta) = \varnothing$. Since we may suppose that β_0 is so large that, for all $\beta \geqslant \beta_0$, we also have the unitary equivalence $L_\beta^{(1)} \restriction_{R_1^*} = U_1^* \, L_\beta^{(2)} \restriction_{R_2} U_1$, we see that all vectors contained in Ran $E_{[m_\star - \eta\,,\, m^\star + \eta]}(L_\beta^{(1)})$ belong to R_1. Then, using the unitary equivalence $L_\beta^{(0)} \restriction_{R_0^*} = U_0^* \, L_\beta^{(1)} \restriction_{R_1} U_0$ we obtain again by Corollary 0.6.7,

$$
\begin{aligned}
D_{[m_\star - \eta\,,\, m^\star + \eta]}(L_\beta^{(0)}) &= D_{[m_\star - \eta\,,\, m^\star + \eta]}(L_\beta^{(1)}) \\
&= D_{[m_\star - \eta\,,\, m^\star + \eta]}(A_\beta^{(1)}) \;=\; \infty \,.
\end{aligned}
$$

Now suppose that the theorem holds for some $n \in \mathbb{N}$, assume that $\tilde\eta_0 = ((n+2)m_\star - (n+1)m^\star) > 0$, and let $\eta \in (0, \tilde\eta_0)$. Then there is a $\beta_0 > 0$ such that, for $\beta \geqslant \beta_0$, we have

$$
\sigma(A_\beta^{(1)}) \cap (-\infty\,,\, (n+2)m_\star - \eta/2\,) \;\subset\; \bigcup_{\ell=1}^{n+1} [\,\ell m_\star - \eta/2\,,\, \ell m^\star + \eta/2]\,.
$$

Using Corollary 0.6.7 and the unitary equivalence of $L_\beta^{(0)} \restriction_{R_0^*}$ and $L_\beta^{(1)} \restriction_{R_1}$ we see that the assertion of the theorem holds true also for $n+1$. ■

7 The Helffer-Sjöstrand formula for the correlation

Using Proposition 0.3.5 it is easy to derive the Helffer-Sjöstrand formula for the truncated two-point correlation function of $\mu_\beta \in \mathscr{G}_t(\beta, \Phi)_{\text{extr}}$, provided Φ fulfills Hypothesis 2 and the inverse temperature is large enough. In the following we denote the orthogonal projection on $\mathscr{H}^1$ onto the closed subspace $\mathbb{C}\,1 \otimes \ell^2_{\mathbb{C}}(\mathbb{Z}^d)$ by P_1 and write $P_1^\perp = \mathbf{1} - P_1$. Furthermore, we introduce the Feshbach operator $\mathcal{F}_\beta$ associated with P_1 and $L_\beta^{(1)}$ setting

$$
\begin{aligned}
\mathcal{F}_\beta \;&:=\; P_1 L_\beta^{(1)} P_1 \,-\, P_1 L_\beta^{(1)} P_1^\perp \big(P_1^\perp L_\beta^{(1)} P_1^\perp\big)^{-1} P_1^\perp L_\beta^{(1)} P_1 \\
&=\; H''(0) P_1 \,+\, P_1 W P_1 \,-\, P_1 W P_1^\perp \big(P_1^\perp L_\beta^{(1)} P_1^\perp\big)^{-1} P_1^\perp W P_1 \,,
\end{aligned}
\tag{53}
$$

where $W = d\Gamma^{(1)}(W)$. It follows from [9, Theorem II.1] and Lemma 0.6.5 that, for sufficiently large values of β, the Feshbach operator $\mathcal{F}_\beta : P_1\mathscr{H}^1 \to P_1\mathscr{H}^1$ is well-defined, continuously invertible, and fulfills

$$
\mathcal{F}_\beta^{-1} \;=\; P_1 \big(L_\beta^{(1)}\big)^{-1} P_1 \,. \tag{54}
$$

Proposition 7.1 *Assume that Φ fulfills Hypothesis 2 and that $\mu_\beta \in \mathscr{G}_t(\beta)_{\rm extr}$. Then there is some $\beta_0 \geqslant 1$ such that, for all $\beta \geqslant \beta_0$, the following formula holds true, for all $f, g \in \mathcal{D}(L_\beta^{(0)})$:*

$$\begin{aligned} \mu_\beta(\bar f\,; g) \quad &:= \quad \int\limits_X \overline{f(x)} \left(g(x) \,-\, \int\limits_X g(y)\, d\mu_\beta(y) \right) d\mu_\beta(x) \\ &= \quad \big\langle\, \bar d_\beta^{(0)} f \,\big|\, \big(L_\beta^{(1)}\big)^{-1}\, \bar d_\beta^{(0)} g \,\big\rangle_{\mathscr{H}^1} \end{aligned} \tag{55}$$

In particular, we have, for all $\beta \geqslant \beta_0$, and $k, \ell \in \mathbb{Z}^d$,

$$\begin{aligned} \mu_\beta(k\,;\ell) \quad &:= \quad \mu_\beta(x_k\,; x_\ell) \;=\; \beta^{-1} \big\langle\, e_k \,\big|\, \big(L_\beta^{(1)}\big)^{-1}\, e_\ell \,\big\rangle_{\mathscr{H}^1} \\ &= \quad \beta^{-1} \big\langle\, e_k \,\big|\, \mathcal{F}_\beta^{-1}\, e_\ell \,\big\rangle_{\mathscr{H}^1}\,, \end{aligned} \tag{56}$$

where $e_k \equiv 1 \otimes e_k$, for $k \in \mathbb{Z}^d$.

Proof: First we observe that, due to Theorem 0.3.4(iii),

$$\mu_\beta(\bar f; g) \quad = \quad \big\langle\, f \,\big|\, P_1^\perp\, g \,\big\rangle_{\mathscr{H}^1} \;=\; \big\langle\, f \,\big|\, P^\perp_{\ker L_\beta^{(0)}}\, g \,\big\rangle_{\mathscr{H}^1}\,,$$

for all $f, g \in \mathcal{D}(L_\beta^{(0)})$. Formula (55) is then a consequence of Proposition 0.3.5(iv), and (56) follows from (55) and (54). ■

The formula for the correlation derived in the preceeding proposition together with Lemma 0.6.6 allows to apply the detailed asymptotic analysis of the correlation of two spin variables developed in [10, 33, 34] also to pure, tempered Gibbs measures $\mu_\beta \in \mathscr{G}_t(\beta, \Phi)_{\rm extr}$, provided the interaction Φ fulfills Hypothesis 3 below and μ_β is invariant under translations and space inversion on the lattice. The latter statement means that it holds $\mu_\beta = \mu_\beta \circ s_k$, and $\mu_\beta = \mu_\beta \circ \imath$, where $(s_k(x))_\ell = x_{\ell+k}$, and $(\imath(x))_\ell = x_{-\ell}$, for $k, \ell \in \mathbb{Z}^d$, $x \in X$.

Hypothesis 3 *We assume that the lattice dimension is at least two, $d \geqslant 2$, that Φ fulfills Hypothesis 2, and, additionally, that the following holds true:*

(i) The interaction is translation invariant and invariant under space inversions on the lattice, i.e. $\Phi_\Lambda = \Phi_{\Lambda+k}$ and $\Phi_\Lambda = \Phi_{-\Lambda}$, for all $\Lambda \in \mathfrak{P}_{\rm f}$, and $k \in \mathbb{Z}^d$.

(ii) For all $c_1 > 0$ there is some $c_2 > 0$ such that

$$J_{0k} \quad = \sum_{\substack{\Lambda \in \mathfrak{P}_{\rm f}:\, \Lambda \geqslant 2, \\ \Lambda \ni 0, k}} J_\Lambda \;\leqslant\; c_2\, e^{-c_1 |k|}\,, \qquad k \in \mathbb{Z}^d\,.$$

(iii) For all $k \in \mathbb{Z}^d$, $k \neq 0$, it holds $H''(0)_{k0} \leqslant 0$, and, for all $k \in \mathbb{Z}^d$ with $|k| = 1$, it holds $H''(0)_{k0} < 0$.

(iv) It holds

$$-\sum_{k\in\mathbb{Z}^d\setminus\{0\}} H''(0)_{k0} \ < \ H''(0)_{00} \ .$$

In order to write down the formula for the leading asymptotics of (56) we introduce some further notation. First, we define a vector $v = (v(k))_{k\in\mathbb{Z}^d} \in \ell^1_{\mathbb{R}}(\mathbb{Z}^d)$ setting $v(0) = 0$ and, for $k \in \mathbb{Z}^d \setminus \{0\}$,

$$v(k) \ := \ -\frac{H''(0)_{k0}}{H''(0)_{00}} \ .$$

Next, we set, for $y \in \mathbb{R}^d$,

$$F(y) \ := \ \sum_{k\in\mathbb{Z}^d} e^{k\cdot y}\, v(k) \ .$$

Then F is a well-defined, smooth, strictly positive function on $\mathbb{R}^d$, such that $F(0) < 1$, and $\ln F$ is strictly convex on $\mathbb{R}^d$ [33]. For σ in a neighbourhood of 1 the sets

$$D_\sigma \ := \ \{y \in \mathbb{R}^d \ : \ F(y) \ < \ \sigma\}$$

are strictly convex and pointsymmetric with respect to the origin. Their boundaries

$$\Sigma_\sigma \ := \ \partial D_\sigma$$

are smooth submanifolds of $\mathbb{R}^d$ whose exterior normal fields

$$N_\sigma \ : \ \Sigma_\sigma \ \longrightarrow \ S^{d-1}$$

are diffeomorphisms. For $\mathring{k} \in S^{d-1}$, we let $y_\sigma(\mathring{k}) := N_\sigma^{-1}(\mathring{k})$ denote the unique point on Σ_σ at which its exterior normal field points in direction $\mathring{k}$. Then the support function associated to F and D_σ is given by

$$\mathcal{S}_\sigma(k) \ := \ \sup_{y\in\overline{D}_\sigma} y\cdot k \ = \ |k|\, y_\sigma(\mathring{k})\cdot\mathring{k} \ ,$$

for $k \in \mathbb{R}^d$, where $\mathring{k} = k/|k|$. The support function $\mathcal{S}_\sigma$ is continuous on $\mathbb{R}^d$, smooth on $\mathbb{R}^d \setminus \{0\}$, and defines a norm on $\mathbb{R}^d$. Finally, we set

$$\partial_\perp^2 F(y_\sigma(\mathring{k})) \quad := \quad F''\restriction_{T_{y_\sigma(\mathring{k})}\Sigma_\sigma} \ ,$$

where $T_{y_\sigma(\mathring{k})}\Sigma_\sigma$ denotes the tangent space of Σ_σ at $y_\sigma(\mathring{k})$. Clearly the $(d-1)\times(d-1)$-matrix $\partial_\perp^2 F(y_\sigma(\mathring{k}))$ is positive definite since F is strictly convex.

Theorem 7.2 *Assume that Φ fulfills Hypothesis 3 and assume that μ_β is a pure, tempered Gibbs state for the interaction Φ that is translation invariant and invariant under space inversions on the lattice. Then we have $\mu_\beta(k\,;\ell) =$*

$\mu_\beta(k-\ell\,;0)$, for $k,\ell\in\mathbb{Z}^d$, and the following asymptotic formula holds true, as β tends to infinity, for all $k\in\mathbb{Z}^d\setminus\{0\}$, where $\mathring{k}:=k/|k|$:

$$\mu_\beta(k\,;0) \;=\; \frac{1}{\beta}\,\frac{1}{(H''(0))_{00}}\,\frac{|\nabla F(y_1(\mathring{k}))|^{\frac{d-3}{2}}}{\sqrt{\det\partial_\perp^2 F(y_1(\mathring{k}))}}\,\frac{1+\mathcal{O}(\frac{1}{|k|})+\mathcal{O}(\beta^{-1/2})}{(2\pi|k|)^{\frac{d-1}{2}}}$$
$$\cdot\,\exp\Big(-\big(1+\mathcal{O}(\beta^{-1/2})\big)\,\mathcal{S}_1(k)\Big)\,. \qquad (57)$$

Here the $\mathcal{O}$-symbols are uniform in β and in k.

Proceeding as in [10] we prove Theorem 0.7.2 in Section 0.8.

Remark 7.3 We remark that because of the one-well shape of the on-site potential $\Phi_{\{0\}}=\Phi_{\{k\}}$ assumed in Hypothesis 3 one expects that there exists only one tempered Gibbs measure for Φ at each $\beta\geqslant 1$. This unique tempered Gibbs measure μ_β would consequently be invarint under translations and space inversion, since by general arguments the set of all tempered Gibbs measures $\mathscr{G}_t(\beta,\Phi)$ is always invariant under all symmetries of the interaction [13, Proposition V.3]. In Appendix .2 we sketch how one can prove the uniqueness of the tempered Gibbs measure at low temperatures under additional assumptions on the interaction potentals but under weaker assumptions on the on-site potentials as compared to Hypothesis 3.

8 Derivation of the correlation asymptotics

In this section we prove Theorem 0.7.2 following [10] with a few small modifications. We always suppose that the assumptions of Theorem 0.7.2 are fulfilled.

To begin with we define a family of convolution kernels $\{v_\beta\}_{\beta\in[1,\infty]}$, where $v_\infty\equiv v$, and

$$v_\beta(k) \;=\; \delta_0(k)\;-\;\frac{\langle\, e_k\,|\mathcal{F}_\beta\, e_0\,\rangle_{\mathscr{H}^1}}{\langle\, e_0\,|\mathcal{F}_\beta\, e_0\,\rangle_{\mathscr{H}^1}}\,, \qquad \beta\in[1,\infty)\,. \qquad (58)$$

We infer from Hypothesis 3(iii) and (iv) that $v\in\ell^1_{\mathbb{R}}(\mathbb{Z}^d)$ and $\|\,v\,\|_1<1$. Together with Lemma 0.8.6 below this shows that also $\|\,v_\beta\,\|_1<1$, for sufficiently large $\beta\in[1,\infty)$. Therefore, for sufficiently large $\beta\in[1,\infty]$, the convolution operator $(\delta_0\;-\;v_\beta)*$ is contained in $\mathscr{L}(\ell^\infty_{\mathbb{R}}(\mathbb{Z}^d),\ell^1_{\mathbb{R}}(\mathbb{Z}^d))$, and it is easily verified that its inverse convolution kernel is given by

$$E_\beta(k)\;:=\;\frac{1}{(2\pi)^d}\int\limits_{\mathbb{T}^d}\frac{e^{ik\cdot x}}{1-\check{v}_\beta(x)}\,dx\;=\;((1-\check{v}_\beta)^{-1})^\wedge(k)\,, \qquad k\in\mathbb{Z}^d\,. \qquad (59)$$

Here $\mathbb{T}^d=(\mathbb{R}/2\pi\,\mathbb{Z})^d$ is the d-dimensional torus, $^\wedge$ denotes Fourier transformation, and $\check{v}_\beta$ is the Fourier series associated with v_β,

$$\check{v}_\beta(x)\;=\;\sum_{k\in\mathbb{Z}^d}u(k)\,e^{-ik\cdot x}\,, \qquad x\in\mathbb{T}^d\,.$$

Since $|\check{v}_\beta(x)| \leqslant \| v_\beta \|_1 < 1$, for $x \in \mathbb{T}^d$, the integrals in (59) exist, and $(E_\beta(k))_{k\in\mathbb{Z}^d} \in \ell^\infty_{\mathbb{R}}(\mathbb{Z}^d)$.

Lemma 8.1 *For $k \in \mathbb{Z}^d$, and $\beta \in [1,\infty)$, it holds*

$$\mu_\beta(k\,;0) \;=\; \frac{E_\beta(k)}{\beta \,\langle\, e_0 \,|\mathcal{F}_\beta\, e_0 \,\rangle_{\mathscr{H}^1}} \,.$$

Proof: For all $\ell \in \mathbb{Z}^d$, we define an operator $\mathfrak{S}_\ell \in \mathscr{L}(\mathscr{H}^1)$ by $\mathfrak{S}_\ell f \otimes e_k = (f \circ s_\ell) \otimes e_{k+\ell}$, for $f \in \mathscr{H}^0$, and linear and continuous extension. Here we set $(s_\ell(x))_i = x_{i+\ell}$, for $x \in X$, and $i \in \mathbb{Z}^d$. It is clear that $[\mathscr{L}^{(1)}_\beta\,, \mathfrak{S}_\ell] = 0$ on Ω^1, because the interaction Φ is translation invariant. Since $\mathscr{L}^{(1)}_\beta$ is essentially self-adjoint on Ω^1 and $\mathfrak{S}_\ell$ is bounded, and since μ_β is translation invariant, it follows that $L^{(1)}_\beta$ and, hence, $(L^{(1)}_\beta)^{-1}$ commute with $\mathfrak{S}_\ell$. Since also $[P_1\,, \mathfrak{S}_\ell] = 0$, we get $[\mathcal{F}_\beta\,, \mathfrak{S}_\ell] = [\mathcal{F}^{-1}_\beta\,, \mathfrak{S}_\ell] = 0$ on $\mathrm{Ran}\ P_1 = \mathbb{C}\,1 \otimes \ell^2_{\mathbb{C}}(\mathbb{Z}^d)$. Since μ_β is translation invariant we further know that $\mathfrak{S}^*_\ell = \mathfrak{S}_{-\ell}$. Therefore, it follows that, for $\beta \in [1,\infty)$, and $k \in \mathbb{Z}^d$, we have

$$\begin{aligned}
&\sum_{\ell\in\mathbb{Z}^d} \langle\, e_{k-\ell} \,|\mathcal{F}_\beta\, e_0 \,\rangle_{\mathscr{H}^1} \langle\, e_\ell \,|\mathcal{F}^{-1}_\beta\, e_0 \,\rangle_{\mathscr{H}^1} \\
&\quad= \sum_{\ell\in\mathbb{Z}^d} \langle\, \mathcal{F}_\beta\, e_k \,|\, e_\ell \,\rangle_{\mathscr{H}^1} \langle\, e_\ell \,|\mathcal{F}^{-1}_\beta\, e_0 \,\rangle_{\mathscr{H}^1} \;=\; \delta_0(k)\,,
\end{aligned}$$

since $\sum_{\ell\in\mathbb{Z}^d} \langle\, e_\ell \,|\, \cdot \,\rangle_{\mathscr{H}^1}\, e_\ell$ converges strongly to the identity in Ran P_1. Similarly, we see that $\langle\, e_{(\,\cdot\,)} \,|\mathcal{F}^{-1}_\beta\, e_0 \,\rangle_{\mathscr{H}^1}$ is also a left inverse kernel for $\langle\, e_{(\,\cdot\,)} \,|\mathcal{F}_\beta\, e_0 \,\rangle_{\mathscr{H}^1}$. From the Helffer-Sjöstrand formula (56) we thus infer the assertion of the lemma. ■

The next theorem is due to [10, 33]. A detailed workout is also contained in [28]. In the following $\widetilde{\Gamma}_\beta \subset \mathbb{R}^d$ denotes an open set with the property that there is some $\delta > 0$ such that

$$\overline{\widetilde{\Gamma}_\beta + \{\, |x| < \delta \,\}} \;\subset\; \Big\{ y \in \mathbb{R}^d : \sum_{k\in\mathbb{Z}^d} e^{y\cdot k} |v_\beta(k)| < \infty \Big\} \;=:\; \Gamma_\beta\,.$$

Theorem 8.2 ([10, 33]) *Let $(v_\beta)_{\beta\in[1,\infty]} \subset \ell^1_{\mathbb{R}}(\mathbb{Z}^d)$, $v \equiv v_\infty$, be a family of convolution kernels having the following properties:*

(i) $\| v \|_1 < 1$, $v(k) \geqslant 0$, *for* $k \in \mathbb{Z}^d$, $\{k \in \mathbb{Z}^d : |k| = 1\} \subset \mathrm{supp}\, v$, *and, for all* $c_1 > 0$, *there is some* $c_2 > 0$ *such that* $v(k) \leqslant c_2\, e^{-c_1|k|}$, *for* $k \in \mathbb{Z}^d$.

(ii) For all $\beta \in [1,\infty]$, and $k \in \mathbb{Z}^d$, it holds $v_\beta(k) = v_\beta(-k)$.

(iii) There exist constants $\sigma_1 \in (1, 2 - F(0))$ and $\beta_0 \geqslant 1$, such that, for all multiindices $\alpha, \alpha' \in \mathbb{N}^d_0$, there is some $C(\alpha,\alpha') \in (0,\infty)$, such that, for all $\beta \in [\beta_0,\infty)$,

$$D_{\sigma_1}(v) \;\;\subseteq\;\; \widetilde{\Gamma}_\beta\,,$$

and

$$\sup_{\substack{x+iy\in\\ \mathbb{T}^d+iD_{\sigma_1}}} \left|\partial_x^\alpha \partial_y^{\alpha'} \check{v}(x+iy) \;-\; \partial_x^\alpha \partial_y^{\alpha'} \check{v}_\beta(x+iy)\right| \;\leqslant\; C(\alpha,\alpha')\,\beta^{-1/2}\,.$$

Then the leadings asymptotics of $E_\beta(k)$, as β and $|k|$ tend to infinity, are given by the right side of Formula (57), where the $\mathcal{O}$-symbols are uniform in β and $k \in \mathbb{Z}^d \setminus \{0\}$.

In order to prove Theorem 0.7.2 it therefore suffices to verify conditions (i)-(iii) of Theorem 0.8.2 for the family of convolution kernels given by (58). Conditions (i) and (ii) follow immediately from Hypothesis 3 and the invariance of μ_β under space inversions on the lattice. It remains to check Condition (iii). This is done in Lemma 0.8.7 below which requires some preparations. The strategy of the remaining part of this section is taken over from [10] with a few modifications.

We start by introducing a family of weight functions $\theta_{\sigma,R}$, $\sigma \in [1,\sigma_0]$, $R \geqslant 1$, on $\mathbb{R}^d$, where $\sigma_0 \in (1, 2 - F(0))$ is arbitrary but fixed. Let $\chi \in C_0^\infty(\mathbb{R}^d, [0,\infty))$ satisfy $\operatorname{supp}\chi \subset \{|x| < 1\}$ and $\int_{\mathbb{R}^d} \chi = 1$. Setting $\chi_R := R^{-d}\chi(\cdot/R)$ we define

$$\theta_{\sigma,R}(k) \;:=\; (1+1/R)\,\chi_R * \mathcal{S}_\sigma(k)\;, \tag{60}$$

for $k \in \mathbb{R}^d$, $R \geqslant 1$, and $\sigma \in [1,\sigma_0]$. We remark that the prefactor $(1+1/R)$ in the definition (60) does not appear in [10]. Its advantage is that [10, Lemma I.7(iii)] is not needed anymore.

Lemma 8.3 *The weight function $\theta_{\sigma,R} : \mathbb{R}^d \to (0,\infty)$ given by (60) has the following properties:*

(i) For all $R \geqslant 1$, $\sigma \in [1,\sigma_0]$, and $k, \ell \in \mathbb{R}^d$, we have

$$\left|\,\theta_{\sigma,R}(k) \;-\; \theta_{\sigma,R}(\ell)\,\right| \;\leqslant\; (1+1/R)\,\mathcal{S}_\sigma(k-\ell)\,.$$

(ii) Let $\sigma_1 \in (1,\sigma_0)$. Then there is some $R_0 \geqslant 1$, such that, for all $R \geqslant R_0$, and $\sigma \in [1,\sigma_1]$, we have $\nabla\theta_{\sigma,R}(k) \in \overline{D}_{\sigma_0}$, where D_{σ_0}.

(iii) If $c > 0$ denotes a uniform Lipschitz constant for $\{\mathcal{S}_\sigma\}_{\sigma\in[1,\sigma_0]}$, then we have, for $R \geqslant 1$, and $\sigma \in [1,\sigma_0]$,

$$\left\|\,\theta_{\sigma,R} \;-\; (1+1/R)\,\mathcal{S}_\sigma\,\right\|_\infty \;\leqslant\; 4\,c\,R\,.$$

(iv) There exists a constant $c' > 0$, neither depending on σ nor R, such that

$$\left|\,\theta_{\sigma,R}(k) \;-\; \theta_{\sigma,R}(\ell) \;-\; \nabla\theta_{\sigma,R}(\ell)\cdot(k-\ell)\,\right| \;\leqslant\; \frac{c'\,|k-\ell|^2}{R}\;,$$

for all $k, \ell \in \mathbb{R}^d$, $R \geqslant 1$, and $\sigma \in [1,\sigma_0]$.

PROOF: (i) Since $\mathcal{S}_\sigma$ is a norm on $\mathbb{R}^d$ we obtain, for all $R \geqslant 1$, $\sigma \in [1, \sigma_0]$, and $k, \ell \in \mathbb{R}^d$,

$$\begin{aligned} \theta_{\sigma,R}(k) &\leqslant (1+1/R) \int\limits_{\mathbb{R}^d} \big(\mathcal{S}_\sigma(\ell - m) + \mathcal{S}_\sigma(k-\ell) \big) \, \chi_R(m) \, dm \\ &= \theta_{\sigma,R}(\ell) + (1+1/R) \, \mathcal{S}_\sigma(k-\ell) \, . \end{aligned}$$

The same estimates hold with k and ℓ exchanged. Hence, we obtain the asserted estimate because $\mathcal{S}_\sigma$ is even.

(ii) Since $\mathcal{S}_\sigma(k) = (N_\sigma)^{-1}(\mathring{k}) \cdot k$, for $k \in \mathbb{R}^d \setminus \{0\}$, we find for the derivative of the support function

$$\mathcal{S}_\sigma'(k) = \langle \, (N_\sigma^{-1})'(\mathring{k}) \, |k|^{-1} P_k^\perp \, \cdot \, | \, k \, \rangle_{\mathbb{R}^d} + \langle \, (N_\sigma)^{-1}(\mathring{k}) \, | \, \mathbf{1}_{\mathbb{R}^d} \cdot \rangle_{\mathbb{R}^d} \, ,$$

where P_k is the projection onto the subspace spanned by k in $\mathbb{R}^d$. Since

$$(N_\sigma^{-1})'(\mathring{k}) : T_{\mathring{k}} \, S^{d-1} \longrightarrow T_{y_\sigma(k)} \Sigma_\sigma \perp k \, ,$$

the map from $\mathbb{R}^d \setminus \{0\}$ into $\mathscr{L}(\mathbb{R}^d, \mathbb{R})$ given by

$$k \longmapsto \langle \, (N_\sigma^{-1})'(\mathring{k}) \, |k|^{-1} P_k^\perp \, \cdot \, | \, k \, \rangle_{\mathbb{R}^d}$$

is identically zero. This implies that, for all $k \in \mathbb{R} \setminus \{0\}$, and $\sigma \in [1, \sigma_1]$, it holds

$$\nabla \mathcal{S}_\sigma(k) = N_\sigma^{-1}(\mathring{k}) \in \Sigma_\sigma = \partial \overline{D}_\sigma \, . \tag{61}$$

Now let $\sigma_1 \in (1, \sigma_0)$. The strict convexity of F implies

$$\overline{D}_\sigma \subsetneq \overline{D}_{\sigma'} \, , \qquad 1 \leqslant \sigma < \sigma' \, , \tag{62}$$

and since $\overline{D}_{\sigma_0}$ is convex and point symmetric with respect to the origin, we find some $R_0 \geqslant 1$ such that $(1+1/R) \, y \in \overline{D}_{\sigma_0}$, for all $R \geqslant R_0$, and $y \in \bigcup_{\sigma \in [1, \sigma_1]} \overline{D}_\sigma$. Consequently, we get

$$(1+1/R) \, \nabla \mathcal{S}_\sigma(k) \in \overline{D}_{\sigma_0} \, , \qquad k \in \mathbb{R}^d \, , \ \ \sigma \in [1, \sigma_1] \, , \ \ R \geqslant R_0 \, . \tag{63}$$

On the other hand we have, for $k \in \mathbb{R}^d$, $R \geqslant R_0$, and $\sigma \in [1, \sigma_0]$,

$$\nabla_k \theta_{\sigma,R}(k) = \int\limits_{\mathbb{R}^d} (1+1/R) \, \nabla_\ell \mathcal{S}_\sigma(\ell) \, \chi_R(k-\ell) \, d\ell \, .$$

Now we obtain the claim because χ_R is a probability density.

(iii) is clear and (iv) follows readily from Taylor's formula, integration by parts, and (61) and (62). ■

In the following we set $\lambda := H''(0)_{00}$ and, for $R \geqslant 1$, and $\sigma \in [1, \sigma_0]$,

$$B_{\sigma,R} := \Theta_{\sigma,R} \, H''(0) \, \Theta_{\sigma,R}^{-1} - H''(0) \, , \tag{64}$$

$$\Re \, B_{\sigma,R} := \frac{1}{2} \big(\Theta_{\sigma,R} \, H''(0) \, \Theta_{\sigma,R}^{-1} + \Theta_{\sigma,R}^{-1} \, H''(0) \, \Theta_{\sigma,R} \big) - H''(0), \tag{65}$$

where

$$(\Theta_{\sigma,R})_{k\ell} \;:=\; \exp\big(\,\theta_{\sigma,R}(k)\,\big)\,\delta_{k\ell}\,, \qquad k,\ell\in\mathbb{Z}^d\,.$$

The next lemma shows that $B_{\sigma,R}$ and $\Re B_{\sigma,R}$ are bounded operators on $\ell^2_{\mathbb{C}}(\mathbb{Z}^d)$.

Lemma 8.4 *Let $\sigma_1\in(1,\sigma_0)$. Then, for all $\varepsilon>0$, we find some $R_0\geqslant 1$, such that, for all $R>R_0$, and $\sigma\in[1,\sigma_1]$, the operator defined in (65) is contained in $\mathscr{L}(\ell^2_{\mathbb{C}}(\mathbb{Z}^d))$ and satifies*

$$\big\|\,\Re B_{\sigma,R}\,\big\|_{\mathscr{L}(\ell^2_{\mathbb{C}}(\mathbb{Z}^d))} \;\leqslant\; \lambda\,\big(\,\varepsilon\,+\,e^{\varepsilon}\,\sigma_0\,-\,F(0)\,\big)\,.$$

PROOF: By the symmetry of $\Re B_{\sigma,R}$ we get

$$\begin{aligned}\big\|\,\Re B_{\sigma,R}\,\big\|_{\mathscr{L}(\ell^2_{\mathbb{C}}(\mathbb{Z}^d))} \;&\leqslant\; \sup_{k\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}\big|\,(\Re B_{\sigma,R})_{k\ell}\,\big| \\ &\leqslant\; \lambda\,\sup_{k\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}\Big(\,\cosh\big(\,\theta_{\sigma,R}(k)-\theta_{\sigma,R}(\ell)\,\big)\,-\,1\Big)\,v(k-\ell) \\ &=\; \lambda\,\sup_{k\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}\cosh\big(\,\theta_{\sigma,R}(k)-\theta_{\sigma,R}(\ell)\,\big)\,v(k-\ell)\,-\,\lambda\,F(0)\,,\end{aligned}$$

for all $R\geqslant 1$, and $\sigma\in[1,\sigma_0]$. Now let $\varepsilon>0$. By Hypothesis 3 and Lemma 0.8.3(i) we find some $r>0$, such that, for all $k\in\mathbb{Z}^d$, $R\geqslant 1$, and $\sigma\in[1,\sigma_0]$, we have

$$\begin{aligned}&\sum_{\substack{\ell\in\mathbb{Z}^d:\\|k-\ell|>r}}\cosh\big(\,\theta_{\sigma,R}(k)-\theta_{\sigma,R}(\ell)\,\big)\,v(k-\ell) \\ &\qquad\leqslant\; \sum_{\substack{\ell\in\mathbb{Z}^d:\\|k-\ell|>r}}\exp\big(\,(1+1/R)\,\mathcal{S}_\sigma(k-\ell)\,\big)\,v(k-\ell) \\ &\qquad\leqslant\; \sum_{\substack{\ell\in\mathbb{Z}^d:\\|k-\ell|>r}}\exp\big(\,2\,c\,|k-\ell|\,\big)\,v(k-\ell)\;<\;\varepsilon\,,\end{aligned}$$

where $c>0$ is a uniform Lipschitz constant for $\{\mathcal{S}_\sigma\}_{\sigma\in[1,\sigma_0]}$. Furthermore, we have

$$\begin{aligned}&\sum_{\substack{\ell\in\mathbb{Z}^d:\\|k-\ell|\leqslant r}}\cosh\big(\,\theta_{\sigma,R}(k)-\theta_{\sigma,R}(\ell)\,\big)\,v(k-\ell) \\ &\qquad=\; \sum_{\substack{\ell\in\mathbb{Z}^d:\\|k-\ell|\leqslant r}}\cosh\big(\,\nabla\theta_{\sigma,R}(k)\cdot(k-\ell)\,+\,\delta(k,\ell)\,\big)\,v(k-\ell)\,,\end{aligned}$$

where $|\delta(k,\ell)|\leqslant c'|k-\ell|^2/R$, for all $k,\ell\in\mathbb{Z}^d$, because of Lemma 0.8.3(iii). Now we choose some $R_0\geqslant 1$, such that $c'r^2/R<\varepsilon$ and the assertion of Lemma 0.8.3(ii) hold, for all $R\geqslant R_0$. Then, observing that $\cosh(a+b)\leqslant e^{|b|}\cosh(a)$, for all $a,b\in\mathbb{R}$, we obtain, for all $R\geqslant R_0$, and $\sigma\in[1,\sigma_1]$,

$$\begin{aligned}\big\|\,\Re B_{\sigma,R}\,\big\|_{\mathscr{L}(\ell^2_{\mathbb{C}}(\mathbb{Z}^d))} \;&\leqslant\; \lambda\,\big(\,\varepsilon\,+\,e^{\varepsilon}\,\sup_{k\in\mathbb{Z}^d}F(\nabla\theta_{\sigma,R}(k))\big)\,-\,\lambda\,F(0) \\ &\leqslant\; \lambda\,\big(\,\varepsilon\,+\,e^{\varepsilon}\,\sigma_0\,-\,F(0)\,\big)\,.\end{aligned}$$

■

Next, we recall that $W = H'' - H''(0)$ and introduce the operators

$$W_{\sigma,R} \;:=\; \big(\mathbf{1}\otimes\Theta_{\sigma,R}\big)\, W\,\big(\mathbf{1}\otimes\Theta_{\sigma,R}^{-1}\big)\,, \tag{66}$$
$$L^{(1)}_{\beta,\sigma,R} \;:=\; \big(\mathbf{1}\otimes\Theta_{\sigma,R}\big)\, L^{(1)}_{\beta}\,\big(\mathbf{1}\otimes\Theta_{\sigma,R}^{-1}\big) \;=\; A^{(1)}_{\beta} + \mathbf{1}\otimes B_{\sigma,R} + W_{\sigma,R}\,,$$

for $\beta \in [1,\infty)$, $R \geqslant 1$, and $\sigma \in [1,\sigma_0]$. We note also that $[\,A^{(1)}_{\beta}\,,\,P_1\,] = 0$, where P_1 denotes the orthogonal projection in $\mathscr{H}^1$ onto the closed subspace $(\mathbb{C}1)\otimes\ell^2_{\mathbb{C}}(\mathbb{Z}^2)$.

Lemma 8.5 *Let $\sigma_1 \in (1,\sigma_0)$. Then there exist constants $\beta_0, R_0, K > 0$, such that, for all $\beta \geqslant \beta_0$, $R \geqslant R_0$, and $\sigma \in [1,\sigma_1]$, the following estimate holds true,*

$$\Big\|\, P_1^{\perp}\big(A^{(1)}_{\beta}\big)^{1/2}\big(\,L^{(1)}_{\beta,\sigma,R}\restriction_{\mathrm{Ran}\ P_1^{\perp}}\,\big)^{-1}\big(A^{(1)}_{\beta}\big)^{1/2}P_1^{\perp}\,\Big\| \;\leqslant\; K\,.$$

PROOF: Due to Theorem 0.3.6 we find, for each $\eta > 0$, some $\beta_1 \geqslant 1$, such that $A^{(1)}_{\beta}P_1^{\perp} \geqslant (2m_\star - \eta)\,P_1^{\perp}$ and, hence, $\big(A^{(1)}_{\beta}\restriction_{\mathrm{Ran}\ P_1^{\perp}}\big)^{-1} \leqslant (2m_\star-\eta)^{-1}\,P_1^{\perp}$ in the sense of quadratic forms on Ran $P_1^{\perp}$. In the same sense we thus get

$$\begin{aligned} &P_1^{\perp}\big(A^{(1)}_{\beta}\big)^{-1/2}\Big(A^{(1)}_{\beta} + \mathbf{1}\otimes\Re\, B_{\sigma,R} + W_{\sigma,R}\Big)\big(A^{(1)}_{\beta}\big)^{-1/2}P_1^{\perp} \\ &\geqslant\; \Big(1 - C\,\beta^{-1/2} - (2m_\star-\eta)^{-1}\big\|\,\Re\, B_{\sigma,R}\,\big\|_{\mathscr{L}(\ell^2_{\mathbb{C}}(\mathbb{Z}^d))}\Big)P_1^{\perp} \\ &\geqslant\; \Big(1 - C\,\beta^{-1/2} - \frac{1}{2}\,\frac{\varepsilon + e^{\varepsilon}\sigma_0 - F(0)}{1-F(0) - \eta/2}\Big)P_1^{\perp} \;\geqslant\; \mathrm{const} > \; 0\,. \end{aligned}$$

for all $\beta \geqslant \beta_0$, $\sigma \in [1,\sigma_1]$, and $R > R_0$, provided $\eta > 0$ and $\varepsilon > 0$ are chosen sufficiently small, and $\beta_0 \equiv \beta_0(\eta) \geqslant \beta_1$ and $R_0 \equiv R_0(\varepsilon) > 0$ are sufficiently large. (We recall that $\sigma_0 \in (1, 2 - F(0))$.) To obtain the previous estimate we apply Lemma 0.6.6 in the first step. In the second step we use Lemma 0.8.4 and $m_\star = \lambda\,(1 - F(0))$. The latter identity follows easily from the fact that $H''(0) \in \mathscr{L}(\ell^2_{\mathbb{C}}(\mathbb{Z}^d))$ is via Fourier transformation unitarily equivalent to the multiplication operator $M_\varphi \in \mathscr{L}(L^2(\mathbb{T}^d,(2\pi)^{-d}dx))$ induced by the function

$$\varphi(x) \;:=\; \lambda\Big(1 - \sum_{k\in\mathbb{Z}^d} v(k)\,\cos(k\cdot x)\Big)\,, \qquad x \in \mathbb{T}^d\,.$$

Clearly, we have $\inf\sigma(M_\varphi) = \inf_{x\in\mathbb{T}^d}\varphi(x) = \varphi(0) = \lambda\,(1 - F(0))$. The claim follows readily from the form bound above and (66). ■

Lemma 8.6 *Let $\sigma_1 \in (1,\sigma_0)$. Then there exist constants $\beta_0, R_0, K' > 0$, such that, for all $\beta \geqslant \beta_0$, $R \geqslant R_0$, and $\sigma \in [1,\sigma_1]$, the following estimate holds true,*

$$\sup_{k\in\mathbb{Z}^d} e^{\theta_{\sigma,R}(k)-\theta_{\sigma,R}(0)}\,\big|\,v(k) - v_\beta(k)\,\big| \;\leqslant\; K'\,\beta^{-1/2}\,. \tag{67}$$

PROOF: We set

$$\tilde{v}_\beta(k) \;:=\; \delta_0(k) \;-\; \frac{\langle\, e_k \,|\mathcal{F}_\beta\, e_0\,\rangle_{\mathscr{H}^1}}{\lambda}\,, \qquad k\in\mathbb{Z}^d\,.$$

Using (53) and $[P_1\,,\,\mathbf{1}\otimes\Theta_{\sigma,R}]=0$, we find

$$\begin{aligned}\lambda\, e^{\theta_{\sigma,R}(k)-\theta_{\sigma,R}(0)}\,\big(\,\tilde{v}_\beta(k)\;-\;v(k)\,\big) \;&=\; \big\langle\, 1\otimes e_k \,\big|\, W_{\sigma,R}\,1\otimes e_0\,\big\rangle_{\mathscr{H}^1}\\ &+\big\langle\, 1\otimes e_k \,\big|\, W_{\sigma,R}\,P_1^\perp\big(\,L^{(1)}_{\beta,\sigma,R}{\restriction}_{\mathrm{Ran}\;P_1^\perp}\,\big)^{-1}P_1^\perp\,W_{\sigma,R}\,1\otimes e_0\,\big\rangle_{\mathscr{H}^1}.\end{aligned} \tag{68}$$

Observing that, for all $k\in\mathbb{Z}^d$,

$$\big\|\,\big(A^{(1)}_\beta\big)^{1/2}\,1\otimes e_k\,\big\|^2_{\mathscr{H}^1} \;=\; \big\langle\, 1\otimes e_k\,\big|\,A^{(1)}_\beta\,1\otimes e_k\,\big\rangle_{\mathscr{H}^1} \;=\; \big(H''(0)\big)_{kk} \;=\; \lambda\,,$$

and using Lemma 0.6.6 we further get

$$\begin{aligned}&\Big|\big\langle\, 1\otimes e_k \,\big|\, W_{\sigma,R}\,1\otimes e_k\,\big\rangle_{\mathscr{H}^1}\Big| \\ &\leqslant\; \big\|\,\big(A^{(1)}_\beta\big)^{1/2}\,1\otimes e_k\,\big\|^2_{\mathscr{H}^1}\,\big\|\,\big(A^{(1)}_\beta\big)^{-1/2}\,W_{\sigma,R}\,\big(A^{(1)}_\beta\big)^{-1/2}\,\big\| \;\leqslant\; \lambda\, C\,\beta^{1/2}\,,\end{aligned} \tag{69}$$

for all $\beta\geqslant\beta_0$, $R\geqslant 1$, $\sigma\in[1,\sigma_0]$, and $k\in\mathbb{Z}^d$, provided $\beta_0\geqslant 1$ is chosen sufficiently large. Additonally using Lemma 0.8.5 we similarly obtain

$$\begin{aligned}&\Big|\big\langle\, 1\otimes e_k \,\big|\, W_{\sigma,R}\,P_1^\perp\big(\,L^{(1)}_{\beta,\sigma,R}{\restriction}_{\mathrm{Ran}\;P_1^\perp}\,\big)^{-1}P_1^\perp\,W_{\sigma,R}\,1\otimes e_0\,\big\rangle_{\mathscr{H}^1}\Big|\\ &\leqslant\; \big\|\,\big(A^{(1)}_\beta\big)^{1/2}\,1\otimes e_k\,\big\|^2\,\big\|\,\big(A^{(1)}_\beta\big)^{-1/2}\,W_{\sigma,R}\,\big(A^{(1)}_\beta\big)^{-1/2}\,\big\|^2\\ &\qquad\cdot\Big\|\,P_1^\perp\big(A^{(1)}_\beta\big)^{1/2}\big(\,L^{(1)}_{\beta,\sigma,R}{\restriction}_{\mathrm{Ran}\;P_1^\perp}\,\big)^{-1}\big(A^{(1)}_\beta\big)^{1/2}P_1^\perp\,\Big\|\\ &\leqslant\; \lambda\, C^2\,\beta^{-1}\,K\end{aligned} \tag{70}$$

for all $\beta\geqslant\beta_0$, $R\geqslant R_0$, $\sigma\in[1,\sigma_1]$, and $k\in\mathbb{Z}^d$, provided $\beta_0\geqslant 1$ and $R_0\geqslant 1$ are chosen sufficiently large. Combining the equations (68)-(70) we obtain (67) with v_β replaced by $\tilde{v}_\beta$. Using again the estimates above with $k=0$ we see that (67) holds, too. ■

Lemma 8.7 *Condition (iii) of Theorem 0.8.2 is fulfilled.*

PROOF: First we fix some $\sigma'\in(1,\sigma_0)$. Then, using Lemma 0.8.6, we choose $\beta'\geqslant 1$ and $R'\geqslant 1$ such that the estimate (67) holds true, for all $\beta\geqslant\beta'$, and $R\geqslant R'$. Fixing some $R''\geqslant R'$, and using Lemma 0.8.3(iii) and Hypothesis 3, we first obtain, for all $y\in\overline{D}_{\sigma'}$, and $\beta\geqslant\beta'$,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d} e^{y\cdot k}|v_\beta(k)|\\ &\leqslant\; \sum_{k\in\mathbb{Z}^d} e^{k\cdot y}\big(\,v(k)\;+\;\big|\,v(k)\;-\;v_\beta(k)\,\big|\,\big)\\ &\leqslant\; \sum_{k\in\mathbb{Z}^d}\exp\Big(\,|k|\,\max_{y\in\overline{D}_{\sigma'}}|y|\,\Big)\,v(k)\;+\;\sum_{k\in\mathbb{Z}^d} e^{\mathcal{S}_{\sigma'}(k)}\big|\,v(k)\;-\;v_\beta(k)\,\big|\\ &\leqslant\; C(\sigma')\;+\;\sup_{\ell\in\mathbb{Z}^d} e^{(1+1/R'')\,\mathcal{S}_{\sigma'}(\ell)}\,\big|\,v(\ell)\;-\;v_\beta(\ell)\,\big|\sum_{k\in\mathbb{Z}^d} e^{-\mathcal{S}_{\sigma'}(k)/R''}\,,\end{aligned} \tag{71}$$

where $C(\sigma') \in (0, \infty)$. The second summand in the last line of (71) is bounded by

$$
\begin{aligned}
& e^{2\,c\,R'' + \theta_{\sigma', R''}(0)} \sup_{\ell \in \mathbb{Z}^d} e^{\theta_{\sigma', R''}(\ell) - \theta_{\sigma', R''}(0)} \left| v(\ell) \,-\, v_\beta(\ell) \right| \sum_{k \in \mathbb{Z}^d} e^{-\frac{\mathcal{S}_{\sigma'}(k)}{R''}} \\
& \quad \leqslant \quad e^{2\,c\,R'' + \theta_{\sigma', R''}(0)}\, K'\, \beta^{-1/2} \sum_{k \in \mathbb{Z}^d} \exp\Big(-\, \tfrac{|k|}{R''} \min_{\check{n} \in S^{d-1}} \mathcal{S}_{\sigma'}(\check{n}) \Big) \\
& \quad \leqslant \quad C_1\, \beta^{-1/2} \,,
\end{aligned}
$$

where $C_1 \in (0, \infty)$. This shows that $\overline{D}_{\sigma'} \subseteq \Gamma_\beta$, for all $\beta \geqslant \beta'$. Next we pick some $\sigma_1 \in (1, \sigma')$. Then we find some open set $\widetilde{\Gamma}_\beta$ and some $\delta > 0$ such that, for all $\beta \geqslant \beta'$,

$$
D_{\sigma_1} \subsetneq \widetilde{\Gamma}_\beta \,, \quad \text{and} \qquad \widetilde{\Gamma}_\beta + \{|x| < \delta\} \subsetneq \overline{D}_{\sigma'} \subseteq \Gamma_\beta \,.
$$

Again using Lemma 0.8.6, we now choose $\beta_0 \geqslant \beta'$ and $R_0 \geqslant 1$ such that the estimate (67) holds true, for all $\beta \geqslant \beta_0$, and $R \geqslant R_0$. Fixing some $R \geqslant R_0$, and using Lemma 0.8.3(iii) we obtain, similarly to the previous estimate, for all $x + iy \in \mathbb{T}^d + iD_{\sigma_1}$, $\alpha, \alpha' \in \mathbb{N}_0^d$, and $\beta \geqslant \beta_0$,

$$
\begin{aligned}
& \left| \partial_x^{\alpha} \partial_y^{\alpha'} \check{v}(x + iy) \,-\, \partial_x^{\alpha} \partial_y^{\alpha'} \check{v}_\beta(x + iy) \right| \\
& \quad \leqslant \quad \sum_{k \in \mathbb{Z}^d} e^{k \cdot y} \left| v(k) \,-\, v_\beta(k) \right| |k|^{|\alpha| + |\alpha'|} \\
& \quad \leqslant \quad \sum_{k \in \mathbb{Z}^d} e^{H_{\sigma_1}(k)} \left| v(k) \,-\, v_\beta(k) \right| |k|^{|\alpha| + |\alpha'|} \\
& \quad \leqslant \quad e^{2\,c\,R + \theta_{\sigma_0, R}(0)}\, K'\, \beta^{-1/2} \sum_{k \in \mathbb{Z}^d} |k|^{|\alpha| + |\alpha'|} \exp\big(-\, \tfrac{|k|}{R} \min_{\check{n} \in S^{d-1}} \mathcal{S}_{\sigma_1}(\check{n}) \big) \\
& \quad \leqslant \quad C(\alpha, \alpha')\, \beta^{-1/2} \,,
\end{aligned}
$$

where $C(\alpha, \alpha') \in (0, \infty)$ neither depends on x, y, nor β. ■

A Some basic estimates

Here we derive some basic and elementary results we refered to in the main text. The first two lemmas originate from [6].

Lemma A.1 *The sum in (7) converges uniformly on closed balls in $S_{-p} \times S_{-p}$, for $p > d/2$, and we have $\inf_{x \in X} H_\Upsilon(x_\Upsilon | y_{\Upsilon^c}) \;>\; -\infty$, for all $y \in S'$. Moreover, the partition function defined in (8) is finite, for all $\Upsilon \in \mathfrak{P}_{\mathrm{f}}$, $\beta \in [1, \infty)$, and $y \in S'$.*

Proof: Due to (4) there is some $\tilde{a} > 0$ such that, for all $k \in \mathbb{Z}^d$, and $t \in \mathbb{R}$, we have

$$
\Phi_{\{k\}}(t)\, t \;\geqslant\; a\, |t|^{2+\sigma} \,-\, \tilde{a} \,, \tag{72}
$$

which permits to get

$$\Phi_{\{k\}}(t) \;\; \geqslant \;\; a\,(2+\sigma)^{-1}\,|t|^{2+\sigma} \;-\; \tilde{a} \;+\; \Phi_{\{k\}}(0)\;, \tag{73}$$

for $k \in \mathbb{Z}^d$. Furthermore, Equation (5) and the fundamental theorem of calculus imply that

$$|\Phi_\Lambda(x_\Lambda)| \;\leqslant\; J_\Lambda \Big(\sum_{k\in\Lambda} |x_k|\Big)^2 \tag{74}$$

holds, for all $\Lambda \in \mathfrak{P}_{\mathrm{f}}$, $\#\Lambda \geqslant 2$, and $x_\Lambda \in \mathbb{R}^\Lambda$. Moreover, we have

$$\begin{aligned}
\Big(\sum_{k\in\Lambda} |x_k|\Big)^2 \;\; &\leqslant \;\; (\#\Lambda) \sum_{k\in\Lambda} |x_k|^2 \\
&\leqslant \;\; (\#\Lambda)\Big(1+\max_{\ell\in\Lambda}|\ell|\Big)^{2p} \sum_{k\in\Lambda} (1+|k|)^{-2p}|x_k|^2 \\
&\leqslant \;\; (\#\Lambda)\,(1+|j|)^{2p}\Big(1+\max_{\ell\in\Lambda}|j-\ell|\Big)^{2p} |x|^2_{-p}\;,
\end{aligned} \tag{75}$$

for all $\Lambda \in \mathfrak{P}_{\mathrm{f}}$, $j \in \mathbb{Z}^d$. Using the previous estimates and Hypothesis 1(iii) the first assertion of the lemma is easily checked and we moreover see that, for all $p, q > d/2$, and $\Upsilon \in \mathfrak{P}_{\mathrm{f}}$, we find some constant $C(p,q,\Upsilon) > 0$ such that

$$\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\Lambda\cap\Upsilon\neq\varnothing,\\ \#\Lambda\geqslant 2}} |\Phi_\Lambda(x_{\Lambda\cap\Upsilon}|y_{\Lambda\cap\Upsilon^c})| \;\leqslant\; C(p,q,\Upsilon)\,\big(|x_\Upsilon|^2_{-p} + |y_{\Upsilon^c}|^2_{-q}\big)\;, \tag{76}$$

for $x \in X$, and $y \in S_{-q}$. Now (73) and (76) together show that the remaining assertions of the lemma hold true, too. ∎

Lemma A.2 *The sum in (12) converges uniformly on closed balls in S_{-p}, $p > d/2$, and it holds $\partial_k H \in C_{\mathrm{pol}}(S_{-p})$, for $k \in \mathbb{Z}^d$, and $p > d/2$. For $p > d/2$, $q > p\,(L-1) + d/2$, the sum in (27) defines a vector field $\mathfrak{b} : S_{-p} \to S_{-q}$.*

Proof: Using Hypothesis 1(iii), (36), and (75), we readily verify the first assertion of the lemma and obtain

$$\begin{aligned}
|\partial_k H(x)| \;\; &\leqslant \;\; |\partial_k \Phi_{\{k\}}(x_k)| \;+ \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\Lambda\ni k,\\ \#\Lambda\geqslant 2}} |\partial_k \Phi_\Lambda(x_\Lambda)| \\
&\leqslant \;\; c\,(1+|x_k|)^{L-1} \;+ \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\,:\,\Lambda\ni k,\\ \#\Lambda\geqslant 2}} J_\Lambda \sum_{\ell\in\Lambda} |x_\ell| \\
&\leqslant \;\; c\,((1+|k|)^p(1+|x|_{-p}))^{L-1} \;+\; C'(p)(1+|k|)^p\,|x|_{-p} \\
&\leqslant \;\; C''(p)\,(1+|k|)^{p(L-1)}\,(1+|x|_{-p})^{L-1}\;,
\end{aligned} \tag{77}$$

for all $k \in \mathbb{Z}^d$, $p > d/2$, and $x \in S_{-p}$, where $C'(p), C''(p) > 0$. In Lemma .1.4 we show that $\partial_k H : S_{-p} \to \mathbb{R}$ is Fréchet differentiable and, in particular, continious.

This shows that $\partial_k H \in C_{\mathrm{pol}}(S_{-p},\mathbb{R})$. For $p > d/2$, $q > p\,(L-1) + d/2$, and $x \in S_{-p}$, we moreover get

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2q}\,|\partial_k H(x)|^2 \\ &\quad\leqslant\; C''(p)^2\,(1+|x|_{-p})^{2(L-1)} \sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2q+2p(L-1)} \\ &\quad=\; C(p,q,L)\,(1+|x|_{-p})^{2(L-1)}\,, \end{aligned} \tag{78}$$

where $C(p,q,L) > 0$. This proves the last assertion of the lemma. ■

Lemma A.3 *For $k \in \mathbb{Z}^d$ and $x \in S_{-p}$, $p > d/2$, the the equality in (12) holds true.*

PROOF: Let $k \in \mathbb{Z}^d$ and $x \in S_{-p}$, $p > d/2$. Clearly, we have, for all $n \in \mathbb{N}$, and $h \in (0,1]$,

$$\begin{aligned} &\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni k,\\ \mathrm{diam}(\Lambda)\leqslant n}} \Phi_\Lambda(x_k + h|x_{\Lambda\setminus\{k\}}) \;-\; \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni k,\\ \mathrm{diam}(\Lambda)\leqslant n}} \Phi_\Lambda(x_\Lambda) \\ &\quad=\; \int_0^1 \sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni k,\\ \mathrm{diam}(\Lambda)\leqslant n}} \partial_k\Phi_\Lambda(x_k + t\,h|x_{\Lambda\setminus\{k\}})\,h\,dt\,, \end{aligned}$$

where $\mathrm{diam}(\Lambda) := \max_{k,\ell\in\Lambda}|k-\ell|$, for $\Lambda \in \mathfrak{P}_{\mathrm{f}}$. Due to (77) we may pass to the limit $n\to\infty$ on both sides of the previous equation by means of Lebesgue's dominated convergence theorem and obtain, for all $h \in (0,1]$,

$$\begin{aligned} &\big|\,H_{\{k\}}(x_k + h|x_{\{k\}^c}) \;-\; H_{\{k\}}(x_k|x_{\{k\}^c}) \;-\; \partial_k H(x)\,h\,\big| \\ &\quad=\; \Big|\int_0^1 \big(\partial_k H(x_k + t\,h|x_{\{k\}^c})\,h \;-\; \partial_k H(x)\,h\,\big)\,dt\Big| \\ &\quad\leqslant\; |h|\,\sup_{s\in[0,1]}\big|\,\partial_k H(x_k + s\,h\,|x_{\{k\}^c}) \;-\; \partial_k H(x)\,\big| \;=\; |h|\,o(1)\,. \end{aligned}$$

The supremum in the last line is $o(1)$ as h tends to zero, because $\partial_k H(\,\cdot\,|x_{\{k\}^c})$ is the uniform limit of a sequence of continuous functions in a compact neighborhood of x_k. ■

Lemma A.4 *Let $j \in \mathbb{Z}^d$ and $p > d/2$. Then $\partial_j H : S_{-p} \to \mathbb{R}$ is Fréchet differentiable. Its Gâteaux derivative in direction e_i, $i \in \mathbb{Z}^d$, is given by the function $\partial_i\partial_j H$ defined by (17). The sum in (17) converges uniformly on each closed ball in S_{-p}, and it holds $\partial_i\partial_j H \in C_{\mathrm{pol}}(S_{-p},\mathbb{R})$.*

PROOF: Approximating $\partial_i\partial_j H(x)$ by the sequence

$$\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni i,j,\\ \mathrm{diam}(\Lambda)\leqslant n}} \partial_i\partial_j\Phi_\Lambda(x_\Lambda)\,, \qquad n\in\mathbb{N}\,,$$

we easily see that $\partial_i\partial_j H$ converges uniformly on each closed ball in S_{-p}, $p > d/2$, and that it holds

$$\big|\partial_i\partial_j H(x)\big| \;\leqslant\; c\,\delta_{ij}\,(1+|x_j|)^{L-2} \,+\, J_{ij}\ , \tag{79}$$

where $J_{ij} = J_{0j-i}$ is defined by (39). This together with Hypothesis 1(iii) implies that there are constants $C(i), C(i,p) > 0$ such that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^d}(1+|j|)^{2p}|\partial_i\partial_j H(x)|^2 \\
&\quad\leqslant\; C(i)(1+|x_i|)^{2L-4} \,+\, 2\,(1+|i|)^{2p}\sum_{j\in\mathbb{Z}^d}(1+|i-j|)^{2p}J_{0j-i} \\
&\quad\leqslant\; C(i,p)\,(1+|x_i|)^{2L-4}\ .
\end{aligned}\tag{80}$$

This shows that, for $x \in S_{-p}$, $p > d/2$, the sequence $(\partial_i\partial_j H(x))_{j\in\mathbb{Z}^d}$ is contained in S_p. Now let $x, h \in S_{-p}$, $p > d/2$. Then we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni i,\\ \mathrm{diam}(\Lambda)\leqslant n}}\big(\partial_i\Phi_\Lambda(x_\Lambda+h_\Lambda)\,-\,\partial_i\Phi_\Lambda(x_\Lambda)\big) \\
&\quad=\;\int_0^1\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni i,\\ \mathrm{diam}(\Lambda)\leqslant n}}\sum_{j\in\Lambda}\partial_i\partial_j\Phi_\Lambda(x_\Lambda+th_\Lambda)\,h_j\,dt\ .
\end{aligned}\tag{81}$$

Here we observe that, for $n, m \in \mathbb{N}$, $n < m$,

$$\begin{aligned}
&\Big|\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni i,\\ n<\mathrm{diam}(\Lambda)\leqslant m}}\sum_{j\in\Lambda}\partial_i\partial_j\Phi_\Lambda(x_\Lambda+th_\Lambda)\,h_j\Big| \\
&\quad\leqslant\;\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni i,\\ n<\mathrm{diam}(\Lambda)\leqslant m}}\sum_{j\in\Lambda}J_\Lambda(1+|j|)^p(1+|j|)^{-p}\,|h_j| \\
&\quad\leqslant\;|h|_{-p}(1+|i|)^p\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}:\,\Lambda\ni i,\\ n<\mathrm{diam}(\Lambda)\leqslant m}}J_\Lambda\Big(\sum_{j\in\Lambda}(1+|i-j|)^{2p}\Big)^{1/2} \\
&\quad\leqslant\;C'(i,p,n,m)\,|h|_{-p}\ ,
\end{aligned}\tag{82}$$

where $C'(i,p,n,m) \to 0$, as $n, m \to \infty$. Similarly, we also find a majorant for the integrants in (82). It follows from Lebesgue's theorem and (80)-(82) that

$$\begin{aligned}
&\Big|\partial_i H(x+h)\,-\,\partial_i H(x)\,-\,\sum_{j\in\mathbb{Z}^d}\partial_i\partial_j H(x)\,h_j\Big| \\
&\quad=\;\Big|\int_0^1(\partial_i\partial_j H(x+th)\,-\,\partial_i\partial_j H(x))\,h_j\,dt\Big| \\
&\quad\leqslant\;|h|_{-p}\sup_{t\in[0,1]}\Big|(\partial_i\partial_j H(x+th)\,-\,\partial_i\partial_j H(x))_{j\in\mathbb{Z}^d}\Big|_p\ .
\end{aligned}\tag{83}$$

So it remains to show that the supremum in the last line of (83) is $o(1)$ as $|h|_{-p}$ tends to zero. This follows, however, from the following estimates:

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}^d}(1+|j|)^{2p}\Big|\,\partial_i\partial_j H(x+th)\;-\;\partial_i\partial_j H(x)\,\Big|^2\\
&\quad\leqslant\;\; 2\,(1+|i|)^{2p}\,|\Phi''_{\{i\}}(x_i+th_i)\;-\;\Phi''_{\{i\}}(x_i)|^2\\
&\qquad +2\sum_{j\in\mathbb{Z}^d}(1+|j|)^{2p}\Big|\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\colon\Lambda\ni i,j,\\ \#\Lambda\geqslant 2}}\;\sum_{\ell\in\Lambda}\int_0^1\partial_\ell\partial_i\partial_j\Phi_\Lambda(x_\Lambda+sth_\Lambda)\,t\,h_\ell\,ds\Big|^2 .
\end{aligned}
$$

Here we observe that

$$
\begin{aligned}
&\Big|\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\colon\Lambda\ni i,j,\\ \#\Lambda\geqslant 2}}\;\sum_{\ell\in\lambda}\int_0^1\partial_\ell\partial_i\partial_j\Phi_\Lambda(x_\Lambda+sth_\Lambda)\,t\,h_\ell\,ds\Big|^2\\
&\quad\leqslant\;\; t^2\,|h|^2_{-p}(1+|i|)^{2p}\Big(\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\colon\Lambda\ni i,j,\\ \#\Lambda\geqslant 2}}J_\Lambda\Big(\sum_{\ell\in\Lambda}(1+|i-\ell|)^{2p}\Big)^{1/2}\Big)^2.
\end{aligned}
$$

To complete the proof of the lemma we note that due to Hypothesis 1(iii)

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}^d}(1+|j-i|)^{2p}\Big(\sum_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}}\colon\Lambda\ni i,j,\\ \#\Lambda\geqslant 2}}J_\Lambda\Big(\sum_{\ell\in\Lambda}(1+|i-\ell|)^{2p}\Big)^{1/2}\Big)^2\\
&\qquad\qquad\leqslant\;\;\mathrm{const}(p,\gamma)\;\mathscr{J}_2(\gamma)\;<\;\infty\,.
\end{aligned}
$$

■

Lemma A.5 *The operator $d\Gamma(H'')$ introduced in (19) is well-defined on Ω.*

PROOF: Using a calculation analogous to (44) we get, for $v\in\Omega$,

$$
\begin{aligned}
&\|\,d\Gamma(H'')\,v\,\|^2\\
&\quad\leqslant\;\int_X\sum_{i,j,k,\ell\in\mathbb{Z}^d}|\partial_i\partial_j H|\,|\partial_k\partial_\ell H|\,\|\,a_k a_i\,v\,\|_{\mathfrak{F}_{\mathrm{f}}}\,\|\,a_j a_\ell\,v\,\|_{\mathfrak{F}_{\mathrm{f}}}\,d\mu_\beta\\
&\qquad+\int_X\sum_{i,k,\ell\in\mathbb{Z}^d}|\partial_i\partial_k H|\,|\partial_k\partial_\ell H|\,\|\,a_i\,v\,\|_{\mathfrak{F}_{\mathrm{f}}}\,\|\,a_\ell\,v\,\|_{\mathfrak{F}_{\mathrm{f}}}\,d\mu_\beta\,.
\end{aligned}
$$

Here the first integral on the right side is finite because the sum under the sign of integration is finite and because of (79) and the à-priori estimate (14). Note that $\sup_{x\in X}\|\,a_j a_\ell\,v(x)\,\|_{\mathfrak{F}_{\mathrm{f}}}<\infty$, for all $j,\ell\in\mathbb{Z}^d$. The second integral can be estimated as follows:

$$\begin{aligned}
&\int_X \sum_{i,k,\ell\in\mathbb{Z}^d} |\partial_i\partial_k H|\,|\partial_k\partial_\ell H|\,\|\,a_i\,v\,\|_{\mathfrak{F}_{\mathrm{f}}}\,\|\,a_\ell\,v\,\|_{\mathfrak{F}_{\mathrm{f}}}\,d\mu_\beta \\
&\quad\leqslant\; 4\sum_{i,k,\ell\in\mathbb{Z}^d}\int_X e^{-2\varkappa|k-\ell|+2\varkappa|i-k|}\Big(c^2\delta_{ik}(1+|x_i|)^{2L-4}+J_{ik}^2\Big)\,\|\,a_i\,v\,\|^2\,\mu_\beta \\
&\quad\leqslant\; C(\varkappa)\sum_{i\in\mathbb{Z}^d}\int_X c^2(1+|x_i|)^{2L-4}\,\|\,a_i\,v\,\|^2\,\mu_\beta \\
&\qquad\quad +C(\varkappa)\Big(\sum_{k\in\mathbb{Z}^d} e^{2\varkappa|k|}J_{0k}^2\Big)\,\langle\,v\,|\,N\,v\,\rangle \;<\;\infty\,.
\end{aligned}$$

Here $\varkappa\in(0,\gamma/2]$, $C(\varkappa):=4\sum_{\ell\in\mathbb{Z}^d}e^{-2\varkappa|\ell|}$, N is the number operator on $\mathfrak{F}_{\mathrm{f}}$, and we note that the sum in the third line is again finite. ■

B Uniqueness of tempered Gibbs measures at low temperature

In this appendix we always assume that the interaction $\Phi=\{\Phi_\Lambda\}_{\Lambda\in\mathfrak{P}_{\mathrm{f}}}$ fulfills the following

Hypothesis 4 *If $\#\Lambda>2$ then $\Phi_\Lambda\equiv 0$. For all $i\in\mathbb{Z}^d$, there is an even, strictly convex function $\phi_i\in C^3(\mathbb{R},[0,\infty))$ such that $\phi_i=\phi_{-i}$, $\phi_i(0)=0$, and, for all $k,\ell\in\mathbb{Z}^d$, we have $\Phi_{\{k,\ell\}}(x_k-x_\ell)=\phi_{k-\ell}(x_k-x_\ell)$. There exists $\kappa>0$ such that $\|\phi_i''\|_\infty,\|\phi_i'''\|_\infty\leqslant e^{-\kappa|i|}$. There is some $v\in C^3(\mathbb{R},\mathbb{R})$ such that, for all $k\in\mathbb{Z}^d$, we have $\Phi_{\{k\}}=v$. We assume that there exist $L>2$, $c_v,C_v,e,a,\sigma>0$, and $\rho\geqslant 0$, such that, for all $t\in\mathbb{R}$,*

$$|v^{(\ell)}(t)|\;\leqslant\;C_v(1+|t|)^{L-\ell}\,,\qquad \ell=0,1,2\,, \tag{84}$$

$$v''(t)\;\geqslant\;-e\,, \tag{85}$$

$$c_v\;\leqslant\;v''(0)\;\leqslant\;C_v\,, \tag{86}$$

$$|v''(t)-v''(0)|\;\leqslant\;C_c\left(|v'(t)|+\min\{1,|t|\}\right), \tag{87}$$

$$|v'(t)|\;\geqslant\;c_v\,\max\big\{\,|v'(s)|\,:\,|s|\leqslant|t|\,\big\}\quad\text{if}\quad |t|\geqslant\rho\,, \tag{88}$$

$$v'(t)\,\mathrm{sgn}(t)\;\geqslant\;a\,\max\{|t|^{1+\sigma},|t|\}\quad\text{if}\quad |t|\geqslant\rho\,. \tag{89}$$

The previous hypotheses on the on-site potential v imply that v has a unique global minimum at zero, which is non-degenerated, but may have further local minima [11]. We note that (73) is still satisfied. For sufficiently small $h\in\mathbb{R}$, we let t_h denote the point where the function $v(t)+ht$ attains its unique global minimum and set $v_h(t)=v(t+t_h)+ht-v(t_h)$. Then we observe that, for sufficiently small $\varepsilon>0$, Hypothesis 4 is fulfilled uniformly by the family

$\{v_h\}_{|h|\leqslant\varepsilon}$. That is, we may assume that (84)-(89) hold true for v_h where none of the constants depends on $h \in [-\varepsilon, \varepsilon]$.

Next, we define local Hamilton functions with free boundary conditions and exterior magnetic field strength $h \in [-\varepsilon, \varepsilon]$ setting, for $\Lambda \in \mathfrak{P}_{\mathrm{f}}$, $x_\Lambda \in \mathbb{R}^\Lambda$, and $g \geqslant 0$,

$$H^{\mathrm{f}}_{\Lambda,h}(x_\Lambda) \;:=\; \sum_{k\in\Lambda}(v(x_k) + h\,x_k) \;+\; \frac{g}{2}\sum_{k,\ell\in\Lambda}\phi_{k-\ell}(x_k - x_\ell)\;.$$

We also define corresponding local Gibbs measures with free boundary conditions at inverse temperature $\beta \geqslant 1$ by

$$d\mu^{\mathrm{f}}_{\Lambda,\beta,h}(x_\Lambda) \;:=\; \frac{1}{\mathscr{Z}^{\mathrm{f}}_\Lambda(\beta,h)}\;\exp\big(-\beta\,H^{\mathrm{f}}_{\Lambda,h}(x_\Lambda)\big)\,dx_\Lambda\;, \tag{90}$$

where $\mathscr{Z}^{\mathrm{f}}_\Lambda(\beta,h)$ is a normalization constant. Using (73) it is easy to see that there is some $C>0$ such that, for $k,\ell\in\mathbb{Z}^d$, $k\neq\ell$, $x_k,x_\ell\in\mathbb{R}$, and $h\in[-\varepsilon,\varepsilon]$, we have

$$\begin{aligned}&|\phi'_{k-\ell}(x_k - x_\ell)| \;+\; |\phi''_{k-\ell}(x_k - x_\ell) - \phi''_{k-\ell}(0)| \\ &\quad\leqslant\;\; C\,e^{-\kappa\,|k-\ell|}\sum_{\tau=k,\ell}\Big(\min\{|v_h(x_\tau)|, |v_h(x_\tau)|^{1/2}\} + \min\{1,|x_\tau|\}\Big)\;.\end{aligned} \tag{91}$$

Hypothesis 4 and (91) show that [11, Hypotheses 1&2] are fulfilled by the family of Hamilton functions $\{H^{\mathrm{f}}_{\Lambda,h}\}_{\substack{\Lambda\in\mathfrak{P}_{\mathrm{f}},\\ |h|\leqslant\varepsilon}}$ uniformly in $|h|\leqslant\varepsilon$. Therefore, [11, Theorem I.1] implies that there are constants $\alpha, g_0, K > 0$, and $\beta_0 \geqslant 1$, such that, for all $\Lambda\in\mathfrak{P}_{\mathrm{f}}$, $k,\ell\in\Lambda$, $\beta\geqslant\beta_0$, $g\in[0,g_0]$, and $h\in[-\varepsilon,\varepsilon]$, the truncated two-point correlation function of $\mu^{\mathrm{f}}_{\Lambda,\beta,h}$ satisfies

$$\big|\,\mu^{\mathrm{f}}_{\Lambda,\beta,h}(x_k; x_\ell)\,\big| \;\leqslant\; \frac{K}{\beta}\;e^{-\alpha\,|k-\ell|}\;. \tag{92}$$

Next, we note that the methods of [6] can be applied to obtain the following à-priori estimate,

$$\sup_{\Lambda\in\mathfrak{P}_{\mathrm{f}}}\;\sup_{k\in\Lambda}\int_{\mathbb{R}^\Lambda} e^{\lambda\,|x_k|^2}\,d\mu^{\mathrm{f}}_{\Lambda,\beta,0} \;\leqslant\; C(\lambda,\beta)\;, \tag{93}$$

for $\lambda\in(0,\infty)$, where $C(\lambda,\beta)>0$.

In order to prove that $\#\mathscr{G}_t(\beta,\Phi) = 1$ it suffices to show that the pressure,

$$p(h) \;:=\; -\lim_{\Lambda\nearrow\mathbb{Z}^d}\;\frac{1}{\#\Lambda}\;\ln\mathscr{Z}^{\mathrm{f}}_\Lambda(\beta,h)\;,\qquad h\in[-\varepsilon,\varepsilon]\,,$$

is differentiable at $h=0$ [13, Proposition VI.5]. Here the limit is taken in the van Hove sense [32] and exists due to [13, Proposition VI.1]. We fix some sequence $\{\Lambda_n\}_{n\in\mathbb{N}}\subset\mathfrak{P}_{\mathrm{f}}$ converging to $\mathbb{Z}^d$ in the van Hove sense. To prove that p

is differentiable at zero we show that the sequence of magnetizations, $(m_n)_{n\in\mathbb{N}}$, where

$$m_n(h) \;:=\; -\,\frac{1}{\#\Lambda_n}\,\frac{d}{dh}\;\ln \mathscr{Z}^{\mathrm{f}}_{\Lambda_n}(\beta,h) \;=\; \frac{1}{\#\Lambda_n}\sum_{k\in\Lambda_n}\int_{\mathbb{R}^{\Lambda_n}} x_k\, d\mu^{\mathrm{f}}_{\Lambda_n,\beta,h}\;,$$

contains a uniformly (in $h\in[-\varepsilon,\varepsilon]$) convergent subsequence provided $\varepsilon>0$ is small enough. First, we observe that the real sequence $(m_n(0))_{n\in\mathbb{N}}$ is bounded because of (93) and, hence, contains a convergent subsequence which we denote by $(m_{n_j}(0))_{j\in\mathbb{N}}$. Furthermore, it follows from (92) that there exist $\beta_0\geqslant 1$ and $g_0,K'>0$ such that, for all $j\in\mathbb{N}$, $\beta\geqslant\beta_0$, $g\in[0,g_0]$, and $|h|\leqslant\varepsilon$, it holds

$$\Big|\,\frac{d}{dh}\,m_{n_j}(h)\,\Big| \;=\; \Big|\,\frac{1}{\#\Lambda_{n_j}}\sum_{k,\ell\in\Lambda_{n_j}}\mu^{\mathrm{f}}_{\Lambda_{n_j},\beta,h}(x_k;x_\ell)\,\Big| \;\leqslant\; \frac{K'}{\beta}\;.$$

Consequently, the sequence of functions $(m_{n_j})_{j\in\mathbb{N}}$ is equicontinuous on $[-\varepsilon,\varepsilon]$ and convergent at zero and the theorem of Arzela and Ascoli implies that it contains a uniformly convergent subsequence.

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