A NOTE ON A THEOREM OF MASSERA

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Abstract. In the present paper we consider a non autonomous inhomogeneous ω-periodic linear differential equation on a reflexive Banach space and show that if it has a bounded solution then it has an ω-periodic solution.

1. Introduction

In this paper we consider a non autonomous inhomogeneous ω-periodic linear differential equation on a reflexive Banach space and show that if it has a bounded solution then it has an ω-periodic solution.

Theorems that deduce existence of periodic solutions from assumption of existence for bounded solutions have been studied by many authors.

First Massera [6] proved such a theorem for a linear inhomogeneous system of finite number of ODE and for two dimensional nonlinear ODE.

A problem of existence of $m\omega$-periodic ($m > 0$, integer) solution for ordinary differential system that possesses bounded solution with some stability property was considered in [7].

Chow [2] obtained Massera's type result for linear functional-differential equations with finite delay.

In [4] are shown such a sort results for functional-differential equations with infinite delay and for some class of integral equations.

In [5] different aspects of the Massera type results for non linear functional-differential equations are considered and examples of existence and non existence are given.

Other results for periodic solutions to functional-differential equations with delay are presented in [1],[3].

2. Setup of the Problem

Consider an inhomogeneous linear system of ordinary differential equations:

$$\dot{x} = A(t)x + b(t), \quad x = (x_1, \dots, x_n). \tag{2.1}$$

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The matrix A and vector b are continuous and ω-periodic in $t \geq 0$.

The celebrated Massera's theorem states that if system (2.1) has a bounded solution then it has an ω-periodic solution.

Our aim is to generalize this theorem to the case of any reflexive Banach space E.

Nevertheless, it is convenient to reformulate our problem in terms of Poincare's mappings. In such a terms our theorem can be applied to linear PDE.

Give an informal description of transfer from the differential problem to mapping of the space E. Exact result is presented in the next sections, it is concerned to a stable point of some Poinrare's mapping.

So assume that $b(t), x \in E$ and $A(t)$ is an ω-periodic family of linear operators of E, $b(t)$ is also ω-periodic.

Introduce a linear homogeneous equation

$$\dot{x} = A(t)x, \quad x(t_0) = \hat{x}.$$

Suppose that there exists a family of the Cauchy operators $\{K(t, t_0)\}_{t, t_0 \in \mathbb{R}}$ such that the solution to this equation presents as follows

$$x(t) = K(t, t_0)\hat{x}.$$

According to Du Hammel's principle, present system (2.1) with given $\hat{x} = x(t_0)$ as an integral equation:

$$x(t) = K(t, t_0)\hat{x} + \int_{t_0}^{t} K(t, \tau)b(\tau)\, d\tau. \tag{2.2}$$

The initial value $\hat{x} = x(0) = x(\omega)$ for an ω-periodic solution to problem (2.2) is found from the equation:

$$x(\omega) = K(\omega, 0)x(\omega) + \int_{0}^{\omega} K(\omega, \tau)b(\tau)\, d\tau.$$

Thus, writing $g = \int_0^\omega K(\omega, \tau)b(\tau)\, d\tau$ and $Q = K(\omega, 0)$ we look for a fixed point of a mapping

$$Px = Qx + g. \tag{2.3}$$

If equation (2.2) has a bounded solution with initial value $x(0) = x_0$ then sequence $\{P^n x_0\}_{n \in \mathbb{N}}$ is bounded in E.

3. Main theorem

Let $Q : E \to E$ be a linear bounded transformation of a reflexive Banach space E, let g be an element of E and the mapping P be given by (2.3).

Theorem 1. *Assume that there exists an element $x_0 \in E$ such that the sequence $\{P^n x_0\}_{n \in \mathbb{N}}$ is bounded. Then the mapping P has a fixed point $\hat{x} \in E$ i.e. $P(\hat{x}) = \hat{x}$.*

4. Proof of Theorem 1

Let V be another reflexive Banach space. Consider a bounded linear operator $T : V \to V$ and construct operators $T_n, \quad n \in \mathbb{N}$ as follows:

$$T_n = \frac{1}{n} \sum_{k=1}^{n} T^k.$$

Lemma 1 (á la Yosida's ergodic theorem [8]). *If for some $z_0 \in V$ the sequence $\{T^n z_0\}$ is bounded:*

$$\sup_{n \in \mathbb{N}} \{\|T^n z_0\|\} = c < \infty$$

then the sequence $\{T_n z_0\}$ contains a subsequence $\{T_{n'} z_0\}$ such that $\{T_{n'} z_0\} \to \hat{z}$ weakly as $n' \to \infty$ and

$$T\hat{z} = \hat{z}. \tag{4.1}$$

Proof. Note that the sequence $\{T_n z_0\}$ is also bounded. Indeed,

$$\|T_n z_0\| \le \frac{1}{n} \sum_{k=1}^{n} \|T^k z_0\| \le c.$$

Thus, we can pick the announced subsequence.

We shall prove that the element $\hat{z}$ is desired fixed point of T.

Since the sequence $\{T^n z_0\}$ is bounded we obtain

$$TT_{n'} z_0 - T_{n'} z_0 = \frac{1}{n'} \Big(T^{n'+1} z_0 - T z_0 \Big) \to 0 \quad \text{as} \quad n' \to \infty.$$

For any $f \in V^*$ this implies:

$$(TT_{n'} z_0, f) - (T_{n'} z_0, f) \to 0. \tag{4.2}$$

On the other hand the following formulas hold true:

$$(T_{n'} z_0, f) \to (\hat{z}, f), \tag{4.3}$$

$$(TT_{n'} z_0, f) = (T_{n'} z_0, T^* f) \to (\hat{z}, T^* f) = (T\hat{z}, f). \tag{4.4}$$

Gathering formulas (4.2), (4.3), (4.4) we see (4.1). □

Now we are ready to prove Theorem 1.

As the space V take a space $E \times \mathbb{R}$. As the mapping $T : V \to V$ take a mapping $(x, y) \mapsto (Qx + gy, y)$. Applying Lemma 1 to the mapping T and a point $z_0 = (x_0, 1)$ we obtain a fixed point $\hat{z} = (\hat{x}, \hat{y})$ for the mapping T.

Obviously, if we show that $\hat{y} = 1$ then $\hat{x}$ is a fixed point of P and the Proof is concluded.

Let $z = (x, y) \in V$ then define $p \in V^*$ as follows: $(z, p) = y$. Note that $(T^n z, p) = (z, p)$ and $(T_n z, p) = (z, p)$. Therefore we have:

$$1 = (T_{n'} z_0, p) \to (\hat{z}, p)$$

and this implies that $(\hat{z}, p) = 1$.

Theorem 1 is proved.

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