On the essential spectrum of the translation invariant Nelson model

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Summary. Let $\mathbb{R}^\nu \ni \xi \to \Sigma_{\mathrm{ess}}(\xi)$ denote the bottom of the essential spectrum for the fiber Hamiltonians of the translation invariant massive Nelson model, which describes a ν-dimensional electron linearly coupled to a scalar massive radiation field. We prove that, away from a locally finite set, Σ_{ess} is an analytic function of total momentum.

1 The model and the result

Let $\mathfrak{h}_{\mathrm{ph}} := L^2(\mathbb{R}^\nu_k)$ and $\mathcal{F} = \Gamma(\mathfrak{h}_{\mathrm{ph}})$ denote the bosonic Fock space constructed from $\mathfrak{h}_{\mathrm{ph}}$. We write $p = -\mathrm{i}\nabla_{\mathrm{x}}$ for the momentum operator in $\mathcal{K} := L^2(\mathbb{R}^\nu_{\mathrm{x}})$. The translation invariant Nelson Hamiltonian describing a ν-dimensional electron (or positron) linearly coupled to a massive scalar radiation field has the form

$$H \ := \ \Omega(p) \otimes 1\!\!1_{\mathcal{F}} \ + \ 1\!\!1_{\mathcal{K}} \otimes d\Gamma(\omega) \ + \ V \ , \ \ \text{on}\ \mathcal{K} \otimes \mathcal{F}\ ,$$

where

$$V \ := \ \int_{\mathbb{R}^\nu} \Big\{ e^{-\mathrm{i}k\cdot \mathrm{x}} v(k)\, 1\!\!1_{\mathcal{K}} \otimes \mathbf{a}^*(k) \ + \ e^{\mathrm{i}k\cdot\mathrm{x}}\, v(k)\, 1\!\!1_{\mathcal{K}} \otimes \mathbf{a}(k) \Big\}\, dk\ .$$

We assume that the form factor v satisfies

$$\begin{aligned} &v \in L^2(\mathbb{R}^\nu_k)\ , \ \ v \text{ real valued}\ , \ v \neq 0 \text{ a.e.} \\ &\text{and } \forall O \in \mathcal{O}(\nu) \ : \ v(Ok) \ = \ v(k) \text{ a.e.}\ , \end{aligned} \tag{1}$$

which implies a UV-cutoff. Here $\mathcal{O}(\nu)$ denotes the orthogonal group. The physically interesting choices for the dispersion relations Ω and ω are $\Omega(\eta) = \eta^2/2M$, $\Omega(\eta) = \sqrt{\eta^2 + M^2}$ and $\omega(k) = \sqrt{k^2 + m^2}$, where $M, m > 0$ are the electron and boson masses. We will however work with general forms of both Ω and ω. As for ω, this is partly motivated by the similarity with the Polaron model, cf. [5, 11], where ω is not explicitly known. We make no attempt here to say anything about the Polaron model.

The operator H commutes with the total momentum $p \otimes 1\!\!1_{\mathcal{F}} + 1\!\!1_{\mathcal{K}} \otimes d\Gamma(k)$ and hence fibers as $H \sim \oint_{\mathbb{R}^\nu} H(\xi) d\xi$, where the fiber Hamiltonians $H(\xi)$, $\xi \in \mathbb{R}^\nu$, are operators on $\mathcal{F}$ given by

$$H(\xi) \ = \ H_0(\xi) \ + \ \Phi(v) \ \ \text{where} \ \ H_0(\xi) \ = \ d\Gamma(\omega) \ + \ \Omega(\xi - d\Gamma(k)) \tag{2}$$

and the interaction is

$$\Phi(v) \;=\; \int_{\mathbb{R}^\nu} \big\{ v(k)\,\mathbf{a}^*(k) \;+\; v(k)\,\mathbf{a}(k) \big\}\, dk\,. \tag{3}$$

We formulate precise assumptions on Ω and ω, which are satisfied by the examples mentioned above. We use the standard notation $\langle t\rangle := (1+t^2)^{1/2}$.

Condition 1 (The particle dispersion relation) *Let $\Omega \in C^\infty(\mathbb{R}^\nu)$. There exists $s_\Omega \in \{0,1,2\}$ such that*

i) There exists $C>0$ such that $\Omega(\eta) \geq C^{-1}\langle\eta\rangle^{s_\Omega} - C$.
ii) For any multi-index α there exists C_α such that $|\partial^\alpha \Omega(\eta)| \leq C_\alpha \langle\eta\rangle^{s_\Omega - |\alpha|}$.
iii) Ω is rotation invariant, i.e., $\Omega(O\eta) = \Omega(\eta)$, for all $O \in O(\nu)$.
iv) The function $\eta \to \Omega(\eta)$ is real analytic.

Condition 2 (The photon dispersion relation) *Let $\omega \in C^\infty(\mathbb{R}^\nu)$ satisfy*

i) There exists $m>0$, the photon mass, such that $\inf_{k\in\mathbb{R}^\nu}\omega(k) = \omega(0) = m$.
ii) $\omega(k) \to \infty$, in the limit $|k| \to \infty$.
iii) There exist $s_\omega \geq 0$, $C_\omega > 0$, and for any multi-index α with $|\alpha| \geq 1$, a C_α such that $\omega(k) \geq C_\omega^{-1}\langle k\rangle^{s_\omega} - C_\omega$ and $|\partial_k^\alpha \omega(k)| \leq C_\alpha \langle k\rangle^{s_\omega - |\alpha|}$.
iv) ω is rotation invariant, i.e., $\omega(O\eta) = \omega(\eta)$, for all $O \in O(\nu)$.
v) ω is real analytic.
vi) ω is strictly subadditive, i.e. $\omega(k_1) + \omega(k_2) > \omega(k_1 + k_2)$ for all $k_1, k_2 \in \mathbb{R}^\nu$.

Remarks: 1) The assumption that the photons are massive is essential.
2) One could weaken the assumption $v \in L^2(\mathbb{R}^\nu)$ by taking instead $v/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$. This is a weaker ultraviolet condition, which still allows for the construction of the Hamiltonian. See [2].
3) The subadditivity assumption is discussed at the end of this section.
4) Condition 2 vi) follows from subadditivity $\omega(k_1)+\omega(k_2) \geq \omega(k_1+k_2)$ together with Condition 2 i), iv), and v).
5) The assumptions (1), Condition 1 ii), Condition 2 i), ii), and iii) can be relaxed, cf. [10].

We introduce the bottom of the spectrum and essential spectrum as functions of total momentum

$$\Sigma_0(\xi) \;:=\; \inf\sigma\big(H(\xi)\big) \;\text{ and }\; \Sigma_{\mathrm{ess}}(\xi) \;:=\; \inf\sigma_{\mathrm{ess}}\big(H(\xi)\big)\,.$$

The energy of a system of n non-interacting bosons, with momenta $\underline{k} \in \mathbb{R}^{n\nu}$, $\underline{k} = (k_1,\dots,k_n)$, and one interacting electron with momentum $\xi - k^{(n)}$, where $k^{(n)} = k_1 + \cdots + k_n$, is

$$\Sigma_0^{(n)}(\xi;\underline{k}) \;:=\; \Sigma_0\big(\xi - k^{(n)}\big) \;+\; \sum_{j=1}^n \omega(k_j) \tag{4}$$

and the smallest of such energies

$$\Sigma_0^{(n)}(\xi) \;:=\; \inf_{\underline{k}\in\mathbb{R}^{n\nu}} \Sigma_0^{(n)}(\xi;\underline{k})\,, \tag{5}$$

which is a threshold energy for the model. Due to the assumption of strict subadditivity of ω, Condition 2 vi), we have

$$\Sigma_0^{(n)}(\xi) \;<\; \Sigma_0^{(n')}(\xi)\,, \quad \text{for } \; n \;<\; n'\,. \tag{6}$$

The function Σ_{ess} can be expressed in terms of Σ_0

$$\Sigma_{\mathrm{ess}}(\xi) \;=\; \Sigma_0^{(1)}(\xi)\,. \tag{7}$$

This is the content of the HVZ theorem, see [9, Theorem 1.2 and Corollary 1.4] and [11, Section 4]. Write $\mathcal{I}_0 := \{\xi \in \mathbb{R}^\nu | \Sigma_0(\xi) < \Sigma_{\mathrm{ess}}(\xi)\}$ (ξ's with an isolated groundstate) and for $\xi \in \mathbb{R}^\nu$: $\mathcal{I}_0^{(1)}(\xi) := \{k \in \mathbb{R}^\nu : \xi - k \in \mathcal{I}_0\}$.

We recall that $H(\xi)$ is self-adjoint on $\mathcal{D} = \mathcal{D}(H_0(\xi))$, which is independent of ξ. The functions $\xi \to \Sigma_0(\xi), \Sigma_{\mathrm{ess}}(\xi), \Sigma_0^{(n)}(\xi)$ are Lipschitz continuous, rotation invariant, and go to infinity at infinity. For a treatment of the second quantization formalism used in the formulation of the model see [1] or the brief overviews given in [4] and [9]. See also the recent monograph [12] by Spohn, for up to date material on models of non-relativistic QED.

The authors talk at the QMATH9 meeting, was devoted to an overview of results for the spectral functions introduced above. Drawing mostly on work of Fröhlich [6, 7], Spohn [11], and the author [9]. One of these results, [9, Theorem 1.9] (Theorem 1.11 in the mp_arc version), states that $\mathbb{R} \ni t \to \Sigma_{\mathrm{ess}}(t\boldsymbol{u})$ is a real analytic function away from a closed countable set, under the additional assumption that ω is also convex. Here $\boldsymbol{u}$ is an arbitrary unit vector. This prompted the following question from Heinz Siedentop: "Is this optimal?". Here is the answer:

Theorem 1. *Fix a unit vector $\boldsymbol{u} \in \mathbb{R}^\nu$. Suppose (1) and Conditions 1 and 2. Then there exists a locally finite set $\mathcal{S} \subset \mathbb{R}$ such that $\mathbb{R}\backslash\mathcal{S} \ni t \to \Sigma_{\mathrm{ess}}(t\boldsymbol{u})$ is analytic. For any connected component $I = (a,b)$ of $\mathbb{R}\backslash\mathcal{S}$ we have either: Σ_{ess} is constant on I, or there exists an analytic function $I \ni t \to \theta(t) \in \mathcal{I}_0^{(1)}(t\boldsymbol{u})$ such that for $t \in I$*

$$\Sigma_{\mathrm{ess}}\big(t\boldsymbol{u}\big) \;=\; \Sigma_0^{(1)}\big(t\boldsymbol{u};\theta(t)\boldsymbol{u}\big) \;\; \textit{and} \;\; \nabla\Sigma_{\mathrm{ess}}\big(t\boldsymbol{u}\big) \;=\; \nabla\omega\big(\theta(t)\boldsymbol{u}\big)\,.$$

In the latter case, there furthermore exist integers $1 \le p,q < \infty$ such that the functions $(a,a{+}\delta) \ni t \to \theta(a{+}(t{-}a)^p))$ and $(b{-}\delta,b) \ni t \to \theta(b{-}(b{-}t)^q)$ extend analytically through a respectively b. (Here δ is chosen such that $a{+}\delta^p, b{-}\delta^q \in I$.)

Remarks: 1) The positivity part of (1) is an input to a Perron-Frobenius argument, see [7, Section 2.4] and [9, Section 3.3], which ensures that the groundstate of each $H(\xi)$ is non-degenerate. This, together with analytic perturbation theory, implies that $\xi \to \Sigma_0(\xi)$ is analytic in $\mathcal{I}_0$, see [7, Theorem 3.6]. This is in fact the information we need to make the proof of Theorem 1 work. Hence the conclusion of the theorem

remains true also for the uncoupled system ($v \equiv 0$) although (1) is not satisfied in this case.
2) If subadditivity of ω is not assumed we are faced with two problems: I) We would need to understand the breakup of degenerate critical points of $\underline{k} \to \Sigma_0^{(n)}(\xi; \underline{k})$ for any n, not just $n = 1$. This is a much more difficult problem (but probably doable). II) The crossing of thresholds $\Sigma_0^{(n)}(\xi)$ may be associated with the disappearance of the groundstate $\Sigma_0(\xi)$ into the essential spectrum. The strategy of the proof below would require that $\Sigma_0(\xi)$ (suitably reparameterized as in Theorem 1) continues analytically into the essential spectrum. This is beyond current technology.

In Section 2 below, we study analytic functions of two complex variables which are of the form of $f(x-y)+g(y)$, cf. the definition (4) of $\Sigma_0^{(1)}(\cdot;\cdot)$. In Section 3 we apply the results of Section 2 to prove Theorem 1. In Appendix A we recall basic properties of Riemannian covering spaces.

2 A complex function of two variables

We write $(\mathcal{R}_p, \pi_p)$ for the p'th Riemannian cover over $\mathbb{C}\backslash\{0\}$. See Appendix A.

For $z \in \mathbb{C}$ and $r > 0$ we introduce the notation

$$D(z,r) := \{z' \in \mathbb{C} \,:\, |z - z'| < r\}\,, \;\; D'(z,r) \;:=\; D(z,r)\backslash\{z\}\,,$$
$$D'_p(r) := \pi_p^{-1}(D'(0,r)) \;\subset\; \mathcal{R}_p\,.$$

We furthermore use the abbreviations $D(r) \equiv D(0,r)$ and $D'(r) = D'(0,r)$.

Fix $x_0, y_0 \in \mathbb{C}$ and $r_0 > 0$. Let f, g be analytic in $D(x_0 - y_0, 2r_0)$ and $D(y_0, r_0)$ respectively. We define:

$$H(x,y) \;:=\; f(x-y) \;+\; g(y) \;\;\text{for}\;\; (x,y) \;\in \mathcal{D} \;:=\; D(x_0,r_0) \times D(y_0,r_0)\,.$$

The function H is an analytic function of two variables in the polydisc $\mathcal{D}$.

We suppose that $y \to H(x_0, y)$ has a critical point at $y = y_0$, that is:

$$(\partial_y H)(x_0, y_0) \;=\; 0\,. \tag{8}$$

The aim of this section is to catalogue the breakup of the critical point, counting multiplicity, when x_0 is replaced by an x near x_0. We wish to determine the sets

$$\Theta(x) \;:=\; \{y \in D(y_0, r_y) \;:\; (\partial_y H)(x,y) \;=\; 0\}\,, \tag{9}$$

for $x \in D'(x_0, r_x)$ and r_x, r_y small enough.

We write f and g as convergent power series in the discs $D(x_0 - y_0, 2r_0)$ and $D(y_0, r_0)$ respectively

$$f(z) \;=\; \sum_{k=0}^{\infty} f_k(z - (x_0 - y_0))^k \;\;\text{and}\;\; g(z) \;=\; \sum_{k=0}^{\infty} g_k(y - y_0)^k\,. \tag{10}$$

We can without loss of generality assume $f_0 = g_0 = 0$, and (8) implies $f_1 = g_1$, and hence we can in addition assume $f_1 = g_1 = 0$. (The critical points are independent of f_0, f_1, g_0 and g_1). To summarize

$$f_0 \;=\; f_1 \;=\; g_0 \;=\; g_1 \;=\; 0\,. \tag{11}$$

For a function h, analytic in an open set $\mathcal{U} \subset \mathbb{C}$, we recall that h has a zero of order k at z_0 if $(z-z_0)^{-k}h$ is analytic near z_0, and $(z-z_0)^{-k-1}h$ is singular at z_0. Equivalently h has a zero of order k at z_0 if $\frac{\partial^\ell h}{\partial z^\ell}(z_0) = 0$, for $0 \le \ell < k$, and $\frac{\partial^k h}{\partial z^k}(z_0) \neq 0$. We will use the following notation for roots of unity. For $k \ge 1$, we write the k solutions of $\alpha^k = 1$ as

$$\alpha_\ell^k \;:=\; e^{\mathrm{i}2\pi\ell/k}, \;\text{ for }\; \ell \in \{1,\dots,k\}\,. \tag{12}$$

By the p'th root of a complex non-zero constant $C \sim (|C|, \arg(C))$, where $0 \le \arg(C) < 2\pi$, we understand

$$C^{1/p} \;:=\; |C|^{1/p}\, e^{\mathrm{i}\arg C/p}\,. \tag{13}$$

In the following we write κ_H for the order of the zero y_0 for the analytic function $(\partial_y H)(x_0,\cdot)$, κ_g for the order of the zero y_0 of $y \to g(y)$, and κ_f for the order of the zero $x_0 - y_0$ of $z \to f(z)$ (recall (11)). We furthermore abbreviate

$$\begin{aligned} F &:= \frac{\frac{\partial^{\kappa_f} f}{\partial z^{\kappa_f}}(x_0-y_0)}{(\kappa_f-1)!} \;=\; \kappa_f f_{\kappa_f}\,, \quad G \;:=\; \frac{\frac{\partial^{\kappa_g} g}{\partial z^{\kappa_g}}(y_0)}{(\kappa_g-1)!} \;=\; \kappa_g g_{\kappa_g}\,, \\ M &:= \frac{(\partial_y^{\kappa_H+1}H)(x_0,y_0)}{\kappa_H!} \;=\; (\kappa_H+1)\big((-1)^{\kappa_H+1} f_{\kappa_H+1} \;+\; g_{\kappa_H+1}\big) \;\neq\; 0\,. \end{aligned} \tag{14}$$

Proposition 1. *Let f, g, x_0, y_0 and r_0 be as above, with $\kappa_g, \kappa_f, \kappa_H < \infty$. Then there exist $0 < r_x, r_y \le r_0$, such that for $x \in D'(x_0, r_x)$ the set of solutions $\Theta(x) \subset D'(y_0, r_y)$ consists of precisely κ_H distinct points, all of which are zeroes of order 1 for $(\partial_y H)(x,\cdot)$. We have the following description of $\Theta(x)$:*

I The case $\kappa_g \le \kappa_H$: *There are analytic functions $\theta_\ell : D(r_x) \to D(r_y)$, $\ell \in \{1,\dots,\kappa_g-2\}$, and $\theta : D'_{\kappa_H-\kappa_g+2}(r_x) \to D'(r_y)$, such that $\Theta(x) = y_0 + (\cup_{\ell=1}^{\kappa_g-2}\{\theta_\ell(x-x_0)\}) \cup \theta(\pi^{-1}_{\kappa_H-\kappa_g+2}(\{x-x_0\}))$. We have the asymptotics*

$$\begin{aligned} \theta_\ell(x) &= \frac{\alpha_\ell^{\kappa_g-1}}{\alpha_\ell^{\kappa_g-1}-1}\, x \;+\; O\big(|x|^2\big)\,, \;\; \ell \;\in\; \{1,\dots,\kappa_g-2\}\,, \\ \theta(\mathbf{x}) &= C_I \mathbf{x}^{1/(\kappa_H-\kappa_g+2)} \;+\; O\big(|\mathbf{x}|^{-2/(\kappa_H-\kappa_g+2)}\big) \end{aligned}$$

where $C_I = [-(\kappa_g-1)G/M]^{1/(\kappa_H-\kappa_g+2)}$.

II The case $\kappa_g = \kappa_H + 1$: *Here $\kappa_f \ge \kappa_g$ and we write $\kappa_f - 1 = p\kappa_H + q$ and $d = (q, \kappa_H)$ (the greatest common divisor), where $p \ge 1$, $0 \le q < \kappa_H$. There exists d analytic functions $\theta_\ell : D'_{\kappa_H/d}(r_x) \to D'(r_y)$ such that $\Theta(x) =$*

$y_0 + \cup_{\ell=1}^d \theta_\ell(\pi_{\kappa_H/d}^{-1}(\{x-x_0\}))$. We have two possible asymptotics: If $\kappa_f = \kappa_g$ (and hence $q=0$, $d=\kappa_H$, and $D_1'(r_x) \equiv D'(r_x)$) then

$$\theta_\ell(x) \;=\; \frac{C_{II}\alpha_\ell^{\kappa_H}}{C_{II}\alpha_\ell^{\kappa_H}\,-\,1}\,x \;+\; O\big(|x|^2\big)\;,\;\; \ell \,\in\, \{1,\ldots,\kappa_H\}\;,$$

where $C_{II} = ((-1)^{\kappa_H}F/G)^{1/\kappa_H} \neq 1$. If $\kappa_f > \kappa_g$ then

$$\theta_\ell(\mathbf{x}) \;=\; C_{II}'\,\alpha_\ell^d\,\pi_{\kappa_H/d}(\mathbf{x})^p\,\big(\mathbf{x}^{\frac{1}{\kappa_H/d}}\big)^{\frac{q}{d}} \;+\; O\big(|\mathbf{x}|^{\kappa_f/\kappa_H}\big)\;,\;\; \ell \,\in\, \{1,\ldots,d\}\;,$$

where $C_{II}' = (F/G)^{1/\kappa_H}$.

III The case $\kappa_g > \kappa_H + 1$: *Write $\kappa_g - 1 = p\kappa_H + q$ and $d = (\kappa_H, q)$, where $p \geq 1$ and $0 \leq q < \kappa_H$. There are d analytic functions $\theta_\ell : D_{\kappa_H/d}'(r_x) \to D'(r_y)$ such that $\Theta(x) = y_0 + \cup_{\ell=1}^d \theta_\ell(\pi_{\kappa_H/d,x_0}^{-1}(\{x-x_0\}))$. We have the asymptotics, with $C_{III} = (-G/M)^{1/\kappa_H}$ and $\ell \in \{1,\ldots,d\}$,*

$$\theta_\ell(\mathbf{x}) \;=\; \pi_{\kappa_H/d}(\mathbf{x}) \;+\; C_{III}\,\alpha_\ell^d\,\pi_{\kappa_H/d}(\mathbf{x})^p\,\big(\mathbf{x}^{\frac{1}{\kappa_H/d}}\big)^{\frac{q}{d}} \;+\; O\big(|\mathbf{x}|^{\kappa_g/\kappa_H}\big)\;.$$

If $q = 0$ and hence $d = \kappa_H$ in II and III, then the maps θ_ℓ, a priori defined on $D_1'(r_x) \equiv D'(0,r_x)$, extend to analytic maps from $D(0,r_x)$ by the prescription $\theta_\ell(0) := y_0$. (Note the convention $(0,p) = p$ for $p \neq 0$.)

Remarks: 1) If $\kappa_f = \infty$, then $H(x,y) = g(y)$, and $\theta(x) \equiv y_0$ is the solution to (8). If $\kappa_g = \infty$ then $g = 0$ and $\theta(x) = x$ is the solution to (8). If $\kappa_H = \infty$ then $H(x_0,y) \equiv H(x_0,y_0)$ and hence $g(y) = H(x_0,y_0) - f(x-y)$.
2) If $\kappa_H = 1$ We get an analytic solution $x \to \theta(x)$ of (8) from the implicit function theorem, cf. [8, Theorem I.B.4]. Proposition 1 II and III then states the possible asymptotics.
3) A particular consequence is that degenerate critical points are isolated.
4) In the proof we handle the error term by a fixed point argument. This implies the following important observation. If $x_0, y_0 \in \mathbb{R}$ and f and g are real analytic. Then a branch $\mathbb{R}\backslash\{x_0\} \ni x \to \theta(x) \in \Theta(x)$ is real valued if and only if $\theta(x)$ is real to leading order (the order needed to uniquely determine θ.)

Proof. The plan of the proof is as follows. First we identify enough terms in an asymptotic expansion $\theta(x) = \tilde{\theta}(x) + z(x)$ of the critical points so that we can separate them. Secondly we use a fixed point argument to show that the remainder, $x \to z(x)$, vanishes at a faster rate than the leading order term $\tilde{\theta}$. Note that it is a general result that for x close to x_0, $\partial_y H(x,\cdot)$ has precisely κ_H zeroes counting their orders. See [8, Lemma 1.B.3]. Our task is to account for those κ_H zeroes. We remark that we could have simply postulated the form of the leading order terms $\tilde{\theta}$, but at the cost of transparency.

We can assume without loss of generality that $x_0 = y_0 = 0$. We wish to solve, for a fixed x in a neighbourhood of 0,

$$(\partial_y H)(x,y) \;=\; 0\;. \tag{15}$$

We begin by collecting some facts. Compute

$$\forall \ell \;:\; (\partial_y^\ell H)(0,0) \;=\; (-1)^\ell \frac{\partial^\ell f}{\partial y^\ell}(0) \;+\; \frac{\partial^\ell g}{\partial y^\ell}(0)\,. \tag{16}$$

We thus find from (8) and (10), recall (11), that

$$\forall\, \ell \,\le\, \kappa_H \;:\; (-1)^{\ell+1} f_\ell \;=\; g_\ell\,, \tag{17}$$

and from the definition of κ_g that

$$\forall\, \ell \,<\, \min\{\kappa_H+1, \kappa_g\} \;:\; f_\ell \;=\; g_\ell \;=\; 0\,. \tag{18}$$

Below we will use the following notation for remainders in expansions. Let h be an analytic function in a disc $D(z_0, r_h)$ with expansion $h(z) = \sum_{k=0}^\infty h_k(z-z_0)^k$. We write for $z \in D(z_0, r_h)$

$$R_h^\ell(z) \;:=\; \sum_{k=\ell+1}^\infty k\, h_k\, (z-z_0)^{k-\ell-1}\,. \tag{19}$$

Note that R_h^ℓ are bounded analytic functions in $D(z_0, r_h/2)$.

We separate into the three cases I, II, and III.

Case I ($\kappa_g \le \kappa_H$): Expand the left-hand side of (15), using (17), (18), and the notation (19):

$$\begin{aligned}
(\partial_y H)(x,y) &= \sum_{\ell=\kappa_g}^{\kappa_H+1} \ell\,\big[-\, f_\ell\,(x-y)^{\ell-1} \;+\; g_\ell\, y^{\ell-1}\big] \\
&\quad -\, R_f^{\kappa_H+1}(x-y)\,(x-y)^{\kappa_H+1} \;+\; R_g^{\kappa_H+1}(y)\, y^{\kappa_H+1} \\
&= \sum_{\ell=\kappa_g}^{\kappa_H} \ell\, g_\ell\, \big[y^{\ell-1} \;-\; (y-x)^{\ell-1}\big] \\
&\quad +\, M\, y^{\kappa_H} + (\kappa_H+1)\, f_{\kappa_H+1}((x-y)^{\kappa_H} - y^{\kappa_H}) \\
&\quad -\, R_f^{\kappa_H+1}(x-y)\,(x-y)^{\kappa_H+1} \;+\; R_g^{\kappa_H+1}(y)\, y^{\kappa_H+1}\,.
\end{aligned} \tag{20}$$

First we look for solutions to (15) with asymptotics $\theta(x) \sim |x|$. That is, the leading order term should solve $y^{\kappa_g-1} - (y-x)^{\kappa_g-1} = 0$, i.e. if we put $y = \beta x$ then β should solve $(\beta/(\beta-1))^{\kappa_g-1} = 1$. This gives the following $\kappa_g - 2$ solutions for β

$$\beta_\ell \;=\; \frac{\alpha_\ell^{\kappa_g-1}}{\alpha_\ell^{\kappa_g-1} - 1}, \;\text{ for }\; \ell \in \{1, \kappa_g-2\}\,. \tag{21}$$

Note that $\alpha_{\kappa_g-1}^{\kappa_g-1} = 1$ does not give rise to a solution. We thus find in this case $\tilde\theta_\ell(x) = \beta_\ell x$. Recall the notation α_ℓ^k from (12).

Secondly we look for solutions to (15) with asymptotics $\theta(x) \sim |x|^\rho$ for some $0 < \rho < 1$. Expanding the terms $(y-x)^{\ell-1}$ in binomial series we identify the highest order terms and are led to require $(\kappa_g - 1)Gxy^{\kappa_g-2} + My^{\kappa_H} = 0$. This gives the equation $y^{\kappa_H-\kappa_g+2} = -[(\kappa_g - 1)G/M]x$. (We note that the $\kappa_g - 2$ zero solutions are the ones we identified in the first step above.) We use the map $\mathcal{R}_{p_I} \ni \mathbf{x} \to \mathbf{x}^{1/p_I}$ introduced in (32), $p_I = \kappa_H - \kappa_g + 2$, to express the solution

$$\theta(\mathbf{x}) = C_I \mathbf{x}^{1/p_I} , \text{ where } C_I = [-(\kappa_g - 1)G/M]^{1/p_I} . \tag{22}$$

Case II ($\kappa_g = \kappa_H + 1$): We expand again

$$(\partial_y H)(x, y) = -F(x-y)^{\kappa_f-1} + G y^{\kappa_g-1} - R_f^{\kappa_f+1}(x-y)(x-y)^{\kappa_f} + R_g^{\kappa_g+1}(y) y^{\kappa_g} . \tag{23}$$

First we consider the case $\kappa_f = \kappa_g = \kappa_H + 1$. This is similar to the first step above. We look for solutions with the asymptotics $\theta(x) \sim |x|$, and put $\tilde{\theta}(x) = \beta x$. We get the equation $(-1)^{\kappa_H+1}F(\beta - 1)^{\kappa_H} + G\beta^{\kappa_H} = 0$. This leads us to consider the equation $(\beta/(\beta - 1))^{\kappa_H} = (-1)^{\kappa_H} F/G$, which has κ_H solutions

$$\beta_\ell = \frac{C_{II}\alpha_\ell^{\kappa_H}}{C_{II}\alpha_\ell^{\kappa_H} - 1} , \text{ for } \ell \in \{1, \dots, \kappa_H\} . \tag{24}$$

Here $C_{II} := ((-1)^{\kappa_H} F/G)^{1/\kappa_H}$. We note that since $(-1)^{\kappa_H+1}F + G = M \neq 0$, cf. (14), we must have $0 < \arg(C_{II}) < 2\pi/\kappa_H$. This observation ensures that we avoid any singularity in the case $|F| = |G|$.

Secondly we assume $\kappa_f > \kappa_g$ and look for solutions to (15) with asymptotics $\theta(x) \sim |x|^\rho$, for some $\rho > 1$. This leads to the equation

$$-F x^{\kappa_f-1} + G y^{\kappa_g-1} = 0 .$$

(Here $M = G$.) Let $\kappa_f - 1 = p\kappa_H + q$ and $d = (q, \kappa_H)$ as in the statement of the proposition. We express the solutions as d analytic maps from $\mathcal{R}_{\kappa_H/d}$ to $\mathbb{C}\backslash\{0\}$

$$\tilde{\theta}_\ell(\mathbf{x}) = C'_{II}\, \alpha_\ell^d\, \pi_{\kappa_H/d}(\mathbf{x})^p\, (\mathbf{x}^{1/(\kappa_H/d)})^{q/d} , \; C'_{II} = (F/G)^{1/\kappa_H} . \tag{25}$$

Case III ($\kappa_g > \kappa_H + 1$): Here we must have $\kappa_f = \kappa_H + 1$. We write down the expansion

$$(\partial_y H)(x, y) = M(y-x)^{\kappa_H} + G y^{\kappa_g-1} - R_f^{\kappa_H+2}(x-y)(x-y)^{\kappa_H+1} + R_g^{\kappa_g+1}(y) y^{\kappa_g} .$$

Here the asymptotics is the same to leading order, namely $y \sim x$. Write $\tilde{\theta}(x) = x + \hat{\theta}$, and look for $\hat{\theta}$ with the asymptotics $\hat{\theta} \sim |x|^\rho$, $\rho > 1$. This gives the equation for $\hat{\theta}$

$$M\, \hat{\theta}^{\kappa_H} + G\, x^{\kappa_g-1} = 0 ,$$

As above let $\kappa_g - 1 = p\kappa_H + q$ and $d = (q, \kappa_H)$. We express the solutions as d analytic maps from $\mathcal{R}_{\kappa_H/d}$ to $\mathbb{C}\backslash\{0\}$, with $C_{III} = (-G/M)^{1/\kappa_H}$,

$$\tilde{\theta}_\ell(\mathbf{x}) \;=\; \pi_{\kappa_H/d}(\mathbf{x}) \;+\; C_{III}\,\alpha_\ell^d\,\pi_{\kappa_H/d}(\mathbf{x})^p\,(\mathbf{x}^{1/(\kappa_H/d)})^{q/d}\,. \tag{26}$$

We have now determined the leading order term in all cases. We proceed to show by a fixed point argument that indeed there is a zero of order 1 near each of the terms identified above. We introduce function spaces

$$\mathcal{Z}_p(\rho, C) \;:=\; \left\{\, z \,\in\, C(D_p'(r_x)\,;\,\mathbb{C}) \mid |z(\mathbf{x})| \;\leq\; C\,|\mathbf{x}|^\rho \,\right\},$$

equipped with sup-norm. If $p = 1$ we identify $D_1'(r_x) \equiv D'(r_x)$. We now describe the procedure which we follow below, so as to cut short the individual arguments. First we write the actual branch of critical points as a sum $\theta(x) = \tilde{\theta}(x) + z(x)$, where $\tilde{\theta}$ is the leading order term as derived above and z is an element of a suitable $\mathcal{Z}_p$. We plug this into an expansion of $(\partial_y H)(x, y)$ and identify leading order terms. These are of two types. One is linear in z and the others are independent of z. The term linear in z is used to construct maps T on $\mathcal{Z}_p$, by $(Tz)(x) = z(x) - (\partial_y H)(x, \tilde{\theta}(x) + z(x))/h(x)$, if hz is the term linear in z. The remaining leading order terms now become leading order terms for Tz and their decay determine the decay of the remainder and hence ρ. The constant C is chosen such that T maps $\mathcal{Z}_p$ into itself. Finally, since terms in Tz depending on z are of higher order, we can choose r_x small enough such that T becomes a contraction. Its unique fixed point z_0 is the desired correction to the leading order contribution found above. Note that a fixed point satisfies $(\partial_y H)(x, \tilde{\theta}(x) + z_0(x)) = 0$.

Case I: Write $\theta_\ell(x) = \beta_\ell x + z(x)$, cf. (21), and look for z vanishing faster than $|x|$ at 0. The leading order term linear in z in (20) is

$$(\kappa_g - 1)\,G\left[(\beta_\ell x)^{\kappa_g - 2} \;-\; ((\beta_\ell - 1)x)^{\kappa_g - 2}\right] z \;=\; (\kappa_g - 1)\,G\,\gamma_\ell\,x^{\kappa_g - 2}\,z$$

A computation yields $\gamma_\ell := \beta_\ell^{\kappa_g - 2} - (\beta_\ell - 1)^{\kappa_g - 2} \neq 0$. Define maps on $\mathcal{Z}_1(2, C)$

$$(T_\ell z)(x) := z(x) \;-\; \frac{(\partial_y H)(x, \beta_\ell x + z(x))}{(\kappa_g - 1)G\gamma_\ell x^{\kappa_g - 2}}\;.$$

The contributions to $T_\ell z$ which scale as $|x|^2$ (the slowest appearing rate) are

$$-\,\frac{(\kappa_g + 1)g_{\kappa_g + 1}\left[\beta_\ell^{\kappa_g} \;-\; (\beta_\ell - 1)^{\kappa_g}\right]}{(\kappa_g - 1)G\gamma_\ell}\,x^2\,,\quad \text{for}\;\; \kappa_g \;<\; \kappa_H\,,$$
$$-\,\frac{M\beta_\ell^{\kappa_H} \;+\; (\kappa_H + 1)f_{\kappa_H + 1}[(1 - \beta_\ell)^{\kappa_H} \;-\; \beta_\ell^{\kappa_H}]}{(\kappa_g - 1)G\gamma_\ell}\,x^2\,,\quad \text{for}\;\; \kappa_g \;=\; \kappa_H\,.$$

We now choose C large enough such that the norm of the coefficients above are less than C. Choosing r_x sufficiently small turns T_ℓ into contractions on $\mathcal{Z}_1(2, C)$.

Now write $\theta(\mathbf{x}) = C_I \mathbf{x}^{1/p_I} + z(\mathbf{x})$ where $z \in \mathcal{Z}_{p_I}(\kappa_g/p_I, C)$. The term in (20) which is linear in z and of leading order is

$$\begin{aligned}&\left[(\kappa_g - 2)\,(\kappa_g - 1)\,G\,\pi_{p_I}(\mathbf{x})\,(C_I\mathbf{x}^{1/p_I})^{\kappa_g - 3} \;+\; \kappa_H\,M\,(C_I\mathbf{x}^{1/p_I})^{\kappa_H - 1}\right] z\\ &\quad = p_I\,M\,C_I^{\kappa_H - 1}\,(\mathbf{x}^{1/p_I})^{\kappa_H - 1}\,z\,,\end{aligned}$$

where we used (33). Note that the coefficient is non-zero. Define a map

$$(Tz)(\mathbf{x}) \ := \ z(\mathbf{x}) \ - \ \frac{(\partial_y H)(\pi_{p_I}(\mathbf{x}), C_I\mathbf{x}^{1/p_I} + z(\mathbf{x}))}{p_I M(C_I\mathbf{x}^{1/p_I})^{\kappa_H - 1}} \ .$$

We wish to show that $T : \mathcal{Z}_{p_I}(\kappa_g/p_I, C) \to \mathcal{Z}_{p_I}(\kappa_g/p_I, C)$ if C is large enough. Let $C(\kappa_g) = f_{\kappa_g+1}$ if $\kappa_g < \kappa_H$ and $C(\kappa_g) = g_{\kappa_H+1}$ if $\kappa_g = \kappa_H$. The term in Tz which vanish to lowest order is

$$\frac{(\kappa_g + 1)\, C(\kappa_g)\, \pi_{p_I}(\mathbf{x})\, (C_I\mathbf{x}^{1/p_I})^{\kappa_g - 1}}{p_I M(C_I\mathbf{x}^{1/p_I})^{\kappa_H - 1}} \ = \ O\big(|\mathbf{x}|^{\frac{2}{p_I}}\big) \ ,$$

Choosing C and r_x as above finishes case I.

Case II: Consider the case $\kappa_f = \kappa_g = \kappa_H + 1$. Let $\tilde\theta_\ell(x) = \beta_\ell x$, where β_ℓ is as in (24). We define maps on $\mathcal{Z}_1(2, C)$

$$(T_\ell z)(x) \ := \ z \ - \ \frac{(\partial_y H)(x, \beta_\ell x + z(x))}{(\kappa_g - 1)\, A_\ell\, x^{\kappa_g - 2}} \ ,$$

which is well defined since

$$\begin{aligned} A_\ell &:= (-1)^{\kappa_H+1}\, F\, (\beta_\ell - 1)^{\kappa_H - 1} \ + \ G\, \beta_\ell^{\kappa_H - 1} \\ &= -\, G\, (\beta_\ell - 1)^{\kappa_H}\, \beta_\ell^{-1}\, C_{II}^{\kappa_H - 1} \big(C_{II} \ - \ (\alpha_\ell^{\kappa_H})^{\kappa_H - 1}\big) \ \neq \ 0 \, . \end{aligned}$$

Here we used (24), the definition of C_{II}, and that $0 < \arg(C_{II}) < 2\pi/\kappa_H$. The leading order contributions to $T_\ell z$ are

$$[(\kappa_g - 1)\, A_\ell]^{-1} \big(R_f^{\kappa_f+1}\, (1 - \beta_\ell)^{\kappa_g} \ + \ R_g^{\kappa_g+1}\, \beta_\ell^{\kappa_g}\big)\, x^2 \ = \ O(|x|^2) \, .$$

As above this estimate suffice.

Next we turn to the case $\kappa_f > \kappa_g$. Let $\tilde\theta_\ell$ be as in (25) and define maps on $\mathcal{Z}_{\kappa_H/d}(\kappa_f/\kappa_H, C)$ by

$$(T_\ell z)(\mathbf{x}) \ := \ z \ - \ \frac{(\partial_y H)(\pi_{\kappa_H/d}(\mathbf{x}), \tilde\theta_\ell(\mathbf{x}) + z(\mathbf{x}))}{(\kappa_g - 1) G \tilde\theta_\ell(\mathbf{x})^{\kappa_g - 2}} \ .$$

Let $\rho_{\mathrm{II}} := \min\{2\kappa_f - \kappa_g - 1, \kappa_f + \kappa_g - 2\}$. The leading order terms in $T_\ell z$ are

$$-\, \frac{(\kappa_f - 1) F \pi_{\kappa_H/d}(\mathbf{x})^{\kappa_f - 2} \tilde\theta_\ell(\mathbf{x}) - R_f^{\kappa_f+1} \pi_{\kappa_H/d}(\mathbf{x})^{\kappa_f}}{(\kappa_g - 1) G \tilde\theta_\ell(\mathbf{x})^{\kappa_g - 2}} \ = \ O\big(|\mathbf{x}|^{\rho_{\mathrm{II}}}\big) \, .$$

Since $2\kappa_f - \kappa_g - 1 \geq \kappa_f$ and $\kappa_f + \kappa_g - 2 \geq \kappa_f$, we have $\rho_{\mathrm{II}} \geq \kappa_f$ and conclude, as above, case II.

Case III: Let $\tilde\theta_\ell$ be as in (26) and write $\theta(\mathbf{x}) = \tilde\theta_\ell(\mathbf{x}) + z(\mathbf{x})$, where we take z from $\mathcal{Z}_{\kappa_H/d}(\kappa_g/\kappa_H, C)$. We define maps

$$(T_\ell z)(\mathbf{x}) \ := \ z \ - \ \frac{(\partial_y H)(\pi_{\kappa_H/d}(\mathbf{x}), \tilde\theta_\ell(\mathbf{x}) + z(\mathbf{x}))}{\kappa_H M[\tilde\theta_\ell(\mathbf{x}) - \pi_{\kappa_H/d}(\mathbf{x})]}$$

Let $\rho_{\mathrm{III}} := \min\{2\kappa_g - \kappa_H - 2, \kappa_H + \kappa_g - 1\}$. The leading order terms in $T_\ell z$ are

$$\frac{(\kappa_g - 1)G\pi_{\kappa_H/d}(\mathbf{x})^{\kappa_g - 2}\tilde{\theta}_\ell(\mathbf{x}) + R_g^{\kappa_g+1}\pi_{\kappa_H/d}(\mathbf{x})^{\kappa_g}}{\kappa_H M[\tilde{\theta}_\ell(\mathbf{x}) - \pi_{\kappa_H/d}(\mathbf{x})]} \;=\; O\big(|\mathbf{x}|^{\rho_{\mathrm{III}}}\big)\,.$$

Since $2\kappa_g - \kappa_H - 2 \geq \kappa_g$ and $\kappa_g + \kappa_H - 1 \geq \kappa_g$, we conclude, as above, case III.

Finally we address analyticity of the a priori continuous solutions found above. Maps from $D'(r_x)$ are analytic by the analytic implicit function theorem [8, Theorem I.B.4], and maps from Riemanian covers are locally analytic by the same argument, and hence analytic. That the maps above defined on $D'(r_x)$ extend to analytic functions on the whole disc $D(r_x)$ follows from [3, Theorem V.1.2]. □

3 The essential spectrum

In this section we use Proposition 1 to prove Theorem 1.

Let $\boldsymbol{u} \in \mathbb{R}^\nu$ be a unit vector. We introduce

$$\sigma(t) \;:=\; \Sigma_0(t\boldsymbol{u}) \;\text{ and }\; \sigma^{(1)}(t;s) \;:=\; \sigma(t\,-\,s)\;+\;\omega(s)\,,$$

where we abuse notation and identify $\omega(s) \equiv \omega(s\boldsymbol{u})$. We furthermore write

$$\sigma^{(1)}(t) \;:=\; \inf_{s\in\mathbb{R}} \sigma^{(1)}(t;s)$$

and $\mathcal{I}_0^{(1)}(t) := \{s \in \mathbb{R} | s\boldsymbol{u} \in \mathcal{I}_0^{(1)}(t\boldsymbol{u})\}$.

We begin with three lemmata and a proposition. The first lemma is a special case of [9, Section 3.2].

Lemma 1. *Assume $v \in L^2(\mathbb{R}^\nu)$ and Conditions 1 i), ii) and 2 i), ii), vi). Let $\xi \in \mathbb{R}^\nu$ and $k \in \mathbb{R}^\nu$. If $\Sigma_0^{(1)}(\xi; k) < \Sigma_0^{(2)}(\xi)$, then $k \in \mathcal{I}_0^{(1)}(\xi)$.*

Using this lemma, cf. (6), we find that

$$\Sigma_{\mathrm{ess}}(\xi) \;=\; \min\{E \mid (\xi, E) \;\in\; \mathcal{T}_0^{(1)}\}\,, \tag{27}$$

where $\mathcal{T}_0^{(1)}$ is the set of thresholds coming from one-photon excitations of the ground state. It is defined by

$$\mathcal{T}_0^{(1)} := \big\{(\xi, E) \in \mathbb{R}^{\nu+1} \,|\, \exists k \in \mathbb{R}^\nu : E = \Sigma_0^{(1)}(\xi; k) \;\text{ and }\; (\xi; k) \in \mathrm{Crit}_0^{(1)}\big\}\,,$$
$$\mathrm{Crit}_0^{(1)} := \big\{(\xi, k) \in \mathbb{R}^{2\nu} \mid k \in \mathcal{I}_0^{(1)}(\xi) \;\text{ and }\; \nabla_k \Sigma_0^{(1)}(\xi; k) \;=\; 0\big\}\,.$$

There are obvious extensions to higher photon number, which must be included in (27), if ω is not subadditive.

The next lemma is the key to the applicability of Proposition 1

Lemma 2. *Let $(\xi,k)\in \mathrm{Crit}_0^{(1)}$, such that $\xi\neq 0$ and $\nabla\omega(k)\neq 0$. Then there exists $\theta\in\mathbb{R}$ such that $k=\theta\xi$.*

Proof. Let $\xi\neq 0$ and $k\in\mathbb{R}^\nu$ be a critical point $\nabla_k\Sigma_0^{(1)}(\xi;k)=0$. Then

$$\nabla\Sigma_0(\xi\,-\,k)\;=\;\nabla\omega(k) \tag{28}$$

Write $\nabla\omega(k)=c_1k$ and $\nabla\Sigma_0(\xi-k)=c_2(\xi-k)$, using rotation invariance. Here $c_1\neq 0$. From (28) we find $c\xi=(1+c)k$, where $c=c_2/c_1$. Since $\xi\neq 0$, we conclude the result. □

We write in the following, for $r\geq 0$, $B(r)=\{k\in\mathbb{R}^\nu||k|=r\}$. For a unit vector $\boldsymbol{u}$ and radii $r_1,r_2\geq 0$ we write for $t\in\mathbb{R}$

$$\mathcal{C}_t\;\equiv\;\mathcal{C}_t(r_1,r_2;\boldsymbol{u})\;:=\;\{k\,\in\,B(r_1)\mid t\,\boldsymbol{u}\,-\,k\,\in\,B(r_2)\}\,.$$

We leave the proof of the following lemma to the reader. (Draw a picture.) It deals with the stability of critical points which are not covered by Lemma 2.

Lemma 3. *Let $r_1,r_2\geq 0$, $\boldsymbol{u}\in\mathbb{R}^\nu$ be a unit vector, and assume $\nu\geq 2$. Suppose $t_0\in\mathbb{R}$ and $\mathcal{C}_{t_0}\neq\emptyset$. There exists a neighbourhood $\mathcal{U}$ of t_0, such that:*

i) If $\mathcal{C}_{t_0}\not\subset\{-r_1\boldsymbol{u},+r_1\boldsymbol{u}\}$ and $t\in\mathcal{U}$, then $\mathcal{C}_t\neq\emptyset$.
ii) If $\mathcal{C}_{t_0}\subset\{-r_1\boldsymbol{u},+r_1\boldsymbol{u}\}$, then $\mathcal{C}_{t_0}=\{k_0\}$. Let $\sigma=\boldsymbol{u}\cdot k_0(t_0-\boldsymbol{u}\cdot k_0)\in\{-r_1r_2,0,+r_1r_2\}$ ($\sigma=0$ iff either r_1 or r_2 equals 0) and $t\in\mathcal{U}\backslash\{t_0\}$. If $\sigma(t-t_0)>0$ then $\mathcal{C}_t\neq\emptyset$, and if $\sigma(t-t_0)\leq 0$, then $\mathcal{C}_t=\emptyset$.

Proposition 2. *Let $t_0\in\mathbb{R}$ be such that $\sigma^{(1)}(t_0)<\Sigma_0^{(2)}(t_0\boldsymbol{u})$. There exist $0<\delta\leq 1$ and an analytic function $\mathcal{U}_\delta\backslash\{t_0\}\ni t\to\theta(t)$, where $\mathcal{U}_\delta=(t_0-\delta,t_0+\delta)$, such that $\sigma^{(1)}(t)=\sigma^{(1)}(t;\theta(t))$, for $t\in\mathcal{U}_\delta\backslash\{t_0\}$. Furthermore, there exist integers $1\leq p_\ell,p_r<\infty$, such that the functions $(t_0-\delta,t_0)\ni t\to\theta(t_0-(t_0-t)^{p_\ell})$ and $(t_0,t_0+\delta)\ni t\to\theta(t_0+(t-t_0)^{p_r})$ extend analytically through t_0.*

Remark: The reason for only studying $\sigma^{(1)}$ where it is smaller than $\Sigma_0^{(2)}$, is the need for having global minima of $s\to\sigma^{(1)}(t;s)$ in $\mathcal{I}_0^{(1)}(t)$, cf. Lemma 1. This may not be true in general. If $\nu=1,2$ or ω is convex, this consideration is unnecessary. See [9, Theorem 1.5 i)] and Lemma 2

Proof. Pick $\tilde\delta>0$ such that $\sigma^{(1)}(t)<\Sigma_0^{(2)}(t\boldsymbol{u})$, for $|t-t_0|\leq\tilde\delta$. For $t\in\mathcal{U}_{\tilde\delta}$, let $\mathcal{G}_t=\{s\in\mathbb{R}|\sigma^{(1)}(t;s)=\sigma^{(1)}(t)\}$ be the set of global minima for $s\to\sigma^{(1)}(t;s)$. We recall from [9, Proof of Theorem 1.9 (Theorem 1.11 in the mp_arc version)] (an application of Lemma 1) that the sets $\mathcal{G}_t$ are finite, and all $s\in\mathcal{G}_t$ are zeros of finite order for the analytic function $\mathcal{I}_0^{(1)}(t)\ni s\to\partial_s\sigma^{(1)}(t;s)$.

Secondly we remark that for any $\tilde t>0$ and $\bar\sigma\in\mathbb{R}$, the set $\{(t,s)\in\mathbb{R}^2||t|\leq\tilde t$ and $\sigma^{(1)}(t;s)\leq\bar\sigma\}$ is compact. From this remark and the finiteness of the $\mathcal{G}_t$'s we conclude from Proposition 1 and a compactness argument that the set

$$\widetilde{\mathcal{S}}:=\{t\in\mathcal{U}_{\tilde\delta}|\exists s\in\mathcal{G}_t,n\in\mathbb{N}\text{ s.t. }\partial_s^2\sigma^{(1)}(t;s)=0\;\text{ and }\;\partial^n\sigma(t-s)\neq 0\}$$

is locally finite. In particular, for $t \in \mathcal{U}_{\tilde\delta}\backslash\widetilde{\mathcal{S}}$ the global minima $s \in \mathcal{G}_t$ are all either simple zeroes of $s \to \partial_s\sigma^{(1)}(t;s)$, or zeroes of infinite order for $s \to \partial\sigma(t-s)$.

Suppose first that $t_0 \notin \widetilde{\mathcal{S}}$. For $s_0 \in \mathcal{G}_{t_0}$ which are simple zeroes of $s \to \partial_s\sigma^{(1)}(t;s)$ we obtain from the analytic implicit function theorem analytic solutions θ to $\partial_s\sigma^{(1)}(t;\theta(t)) = 0$, defined in a neighbourhood of t_0. For $s_0 \in \mathcal{G}_{t_0}$ which are zeroes of infinite order of $s \to \partial\sigma(t_0-s)$ (and not already included in the first case), we take $\theta(t) \equiv s_0$ which solves $\partial_s\sigma^{(1)}(t;\theta(t)) = 0$ near t_0. See Remark 1), with $\kappa_f = \infty$, after Proposition 1. We have thus for some $0 < \delta' < \tilde\delta$ constructed $|\mathcal{G}_{t_0}|$ analytic functions θ_ℓ defined in $\mathcal{U}_{\delta'}$, such that $\mathcal{G}_{t_0} = \{\theta_\ell(t_0)\}$ and $\mathcal{G}_t \subset \{\theta_\ell(t)\}$ (by continuity) for $t \in \mathcal{U}_{\delta'}$. Hence $\sigma^{(1)}(t) = \min_{1\le\ell\le|\mathcal{G}_{t_0}|}\sigma^{(1)}(t;\theta_\ell(t))$, for $t \in \mathcal{U}_{\delta'}$.

If $|\mathcal{G}_{t_0}| = 1$ take $\delta = \delta'$ and $\theta = \theta_1$. If $|\mathcal{G}_{t_0}| > 1$ choose $0 < \delta < \delta'$, ℓ_1, and ℓ_2 such that the following choice works: $\theta(t) = \theta_{\ell_1}(t)$, for $t_0 - \delta < t < t_0$, and $\theta(t) = \theta_{\ell_2}(t)$, for $t_0 < t < t_0 + \delta$. This proves the result if $t_0 \notin \widetilde{\mathcal{S}}$.

It remains to treat $t_0 \in \widetilde{\mathcal{S}}$. Here we get from Proposition 1 a $0 < \delta' < \tilde\delta$ and two families of analytic functions $\{\theta_\ell^{\text{left}}\}$ and $\{\theta_\ell^{\text{right}}\}$ defined in $(t_0-\delta', t_0)$ and $(t_0, t_0+\delta')$ respectively, which parameterize the critical points for t near t_0, which comes from $\mathcal{G}_{t_0}$. Furthermore $\mathcal{G}_t \subset \{\theta_\ell^{\text{left}}(t)\}$, $t_0 - \delta' < t < t_0$ and $\mathcal{G}_t \subset \{\theta_\ell^{\text{right}}(t)\}$, $t_0 < t < t_0 + \delta'$. Note that the number of branches to the left and to the right need not be the same, but both are finite.

We are finished if we can prove that, for $\ell \neq \ell'$, the function $\sigma^{(1)}(t;\theta_\ell^{\text{left}}(t)) - \sigma^{(1)}(t;\theta_{\ell'}^{\text{left}}(t))$ is either identically zero, or it does not vanish on a sequence of t's converging to t_0 from the left. Similarly for the right of t_0. (If there is only one branch, then it continues analytically through t_0 and we are done).

In the following we work to the left of t_0 and drop the superscript 'left'. The region to the right of t_0 can be treated similarly. There exists $1 \le p, p' < \infty$ and analytic functions $\theta : \mathcal{R}_p \to \mathbb{C}$ and $\theta' : \mathcal{R}_{p'} \to \mathbb{C}$ such that θ_ℓ and $\theta_{\ell'}$ are branches of θ and θ' respectively. See Proposition 1. That is, there exist $0 \le q < p$ and $0 \le q' < p'$ such that

$$\theta_\ell(t) \;=\; \theta(R_p^\rho(t_0-t)) \;\text{ and }\; \theta_{\ell'}(t) \;=\; \theta'(R_{p'}^{\rho'}(t_0-t))\,,$$

where $\rho = \pi + 2\pi q$ and $\rho' = \pi + 2\pi q'$. Recall notation from (30) and (31). Here we used the canonical embedding $\mathbb{R}\backslash\{0\} \ni r \to (t,0) \in \mathcal{R}_p$, for any p. Since $z \to \theta(R_p^\rho(P_p(z)))$ and $z \to \theta'(R_{p'}^{\rho'}(P_{p'}(z)))$ are analytic and bounded functions from $D'(r)$ (for some $r > 0$), they extend to analytic functions on $D(r)$. We can hence define an analytic function

$$h(z) = \sigma^{(1)}(z;\theta(R_p^\rho(P_p((t_0-z)^{p'})))) - \sigma^{(1)}(z;\theta'(R_{p'}^{\rho'}(P_{p'}((t_0-z)^{p}))))$$

in $D(r^{1/pp'})$. The function h is either identically zero or has only isolated zeroes (one is at t_0). This now implies that $\sigma^{(1)}(t;\theta_\ell(t)) - \sigma^{(1)}(t;\theta_{\ell'}(t)) = h(t_0 - (t_0 - t)^{1/pp'})$ is either identically zero or has finitely many zeroes near t_0. We can now choose $0 < \delta < \delta'$ and θ as above. This concludes the proof. $\square$

Proof of Theorem 1: Proposition 2 covers the case $\nu = 1$. In the following we assume $\nu \geq 2$. It suffices to prove the theorem locally near any $t_0 \in \mathbb{R}$. For the global minima at $t \in \mathbb{R}$ we write

$$\mathcal{M}_t := \left\{ k \in \mathbb{R}^\nu \mid \Sigma_0^{(1)}(t\boldsymbol{u}; k) = \Sigma_0^{(1)}(t\boldsymbol{u}) \right\}$$

and we introduce two subsets

$$\mathcal{M}_t^{\|} := \left\{ k \in \mathcal{M}_t \mid k \parallel \boldsymbol{u} \right\} \text{ and } \mathcal{M}_t^0 := \left\{ k \in \mathcal{M}_t \mid \nabla\omega(k) = 0 \right\}.$$

We begin with the following note. Let $t_0 \in \mathbb{R}$. If $\mathcal{M}_{t_0}^{\|} = \emptyset$ ($\mathcal{M}_{t_0}^0 = \emptyset$) then there exists a neighbourhood $\mathcal{U} \ni t_0$ such that for $t \in \mathcal{U}$ we have $\mathcal{M}_t^{\|} = \emptyset$ ($\mathcal{M}_t^0 = \emptyset$).

Let $t_0 \in \mathbb{R}$. First consider the case $\mathcal{M}_{t_0}^0 = \emptyset$. For $t \in \mathcal{U}$, chosen as above, we have $\Sigma_{\mathrm{ess}}(t\boldsymbol{u}) = \sigma^{(1)}(t)$, which by Proposition 2 concludes the proof.

We can now assume that $\mathcal{M}_{t_0}^0 \neq \emptyset$. By analyticity and rotation invariance, the set of k's such that $\nabla\omega(k) = 0$ is a set of concentric balls, with a locally finite set of radii. If $k \in \mathcal{M}_{t_0}^0$ then $Ok \in \mathcal{M}_{t_0}^0$ for any $O \in \mathcal{O}(\nu; \boldsymbol{u})$, where $\mathcal{O}(\nu; \boldsymbol{u}) := \{O \in \mathcal{O}(\nu) | O\boldsymbol{u} = \boldsymbol{u}\}$.

Let

$$\widetilde{\mathcal{M}}_t^0 = \left\{ k \in \mathcal{I}_0^{(1)}(t) \mid (t\boldsymbol{u}, k) \in \mathrm{Crit}_0^{(1)} \text{ and } \nabla\omega(k) = 0 \right\},$$
$$\tilde{\sigma}^{(1)}(t) := \min_{k \in \widetilde{\mathcal{M}}_t^0} \Sigma_0^{(1)}(t\boldsymbol{u}; k),$$

with the convention that $\tilde{\sigma}^{(1)}(t) = +\infty$ if $\widetilde{\mathcal{M}}_t^0 = \emptyset$. Then by Lemma 2

$$\Sigma_0^{(1)}(t\boldsymbol{u}) = \min\{\sigma^{(1)}(t), \tilde{\sigma}^{(1)}(t)\}.$$

We work only to the left of t_0, i.e. we take $t \leq t_0$. The case $t \geq t_0$ can be treated similarly.

We proceed to find a $\delta > 0$ such that $\sigma^{(1)}$ is either constant equal to $\Sigma_0^{(1)}(t_0\boldsymbol{u})$, for $t_0 - \delta < t \leq t_0$ or it satisfies $\tilde{\sigma}^{(1)}(t) \geq \sigma^{(1)}(t) = \Sigma_0^{(1)}(t\boldsymbol{u})$, for $t_0 - \delta < t \leq t_0$. This concludes the result, since both $\sigma^{(1)}$ and a constant function, suitably reparameterized, continues analytically though t_0. See Proposition 2.

First we consider the case where: **A)** Σ_0 is not constant on the connected component of $\mathcal{I}_0$ containing $t_0\boldsymbol{u} - k_0$, for any $k_0 \in \mathcal{M}_{t_0}^0$. **B)** For any $k_0 \in \mathcal{M}_{t_0}^0$ we have (with $r_1 = |k_0|$ and $r_2 = |t_0\boldsymbol{u} - k_0|$): $\mathcal{C}_{t_0}(r_1, r_2; \boldsymbol{u}) \subset \{-r_1\boldsymbol{u}, r_1\boldsymbol{u}\}$ and $\boldsymbol{u} \cdot k_0(t_0 - \boldsymbol{u} \cdot k_0) \leq 0$. Assuming **A)** and **B)** we have by Lemma 3 ii) a $\delta > 0$, such that

$$\mathcal{C}_t(r_1, r_2; \boldsymbol{u}) = \emptyset, \text{ for } t_0 - \delta < t < t_0. \tag{29}$$

We proceed to argue that **A)** and **B)** implies $\Sigma_0^{(1)}(t\boldsymbol{u}) = \sigma^{(1)}(t)$, for $t < t_0$. It suffices to find a $\delta > 0$ such that $\mathcal{M}_t^0 = \emptyset$, $t_0 - \delta < t < t_0$. Suppose to the contrary that there exists a sequence $t_n \to t_0$, with $t_n < t_0$ and $\mathcal{M}_{t_n}^0 \neq \emptyset$. Let $k_n \in \mathcal{M}_{t_n}^0$. We can assume, by possibly passing to a subsequence, that $k_n \to k_\infty$. Here we used that $\omega(k) \to \infty$ as $|k| \to \infty$. Clearly $k_\infty \in \mathcal{M}^0_{t_0}$, and hence $k_\infty \in \mathcal{C}_{t_0}(r_1, r_2; \boldsymbol{u})$ for some r_1, r_2. Since the possible r_1's and r_2's are isolated, we must have a $\bar{n}$ such that $\forall n > \bar{n}$: $k_n \in \mathcal{C}_{t_n}(r_1, r_2; \boldsymbol{u})$. This contradicts (29).

For the remaining case we assume one of the following: **C)** There exists $k_0 \in \mathcal{M}^0_{t_0}$ such that Σ_0 is constant on the connected component of $\mathcal{I}_0$ containing $t_0\boldsymbol{u} - k_0$. (The converse of **A)** above.) **D)** There exists $k_0 \in \mathcal{M}^0_{t_0}$ such that either $\mathcal{C}_{t_0}(r_1, r_2; \boldsymbol{u}) \not\subset \{-r_1\boldsymbol{u}, r_1\boldsymbol{u}\}$ or $\mathcal{C}_{t_0}(r_1, r_2; \boldsymbol{u}) \subset \{-r_1\boldsymbol{u}, r_1\boldsymbol{u}\}$ and $\boldsymbol{u} \cdot k_0(t_0 - \boldsymbol{u} \cdot k_0) > 0$. Again $r_1 = |k_0|$ and $r_2 = |t_0\boldsymbol{u} - k_0|$. (The converse of **B)** above.)

There exists $\delta > 0$ such that: In the case **C)**, for $t_0 - \delta < t < t_0$, there exists $k \in \widetilde{\mathcal{M}}^0_t$ with $\Sigma^{(1)}_0(t\boldsymbol{u}; k) = \Sigma^{(1)}_0(t_0\boldsymbol{u})$. In case **D)** we have by Lemma 3 i) and ii), for $t_0 - \delta < t < t_0$, likewise $k \in \widetilde{\mathcal{M}}^0_t$ with $\Sigma^{(1)}_0(t\boldsymbol{u}; k) = \Sigma^{(1)}_0(t_0\boldsymbol{u})$. Hence, if either **C)** or **D)** are satisfied we have $\tilde{\sigma}^{(1)}(t) \le \Sigma^{(1)}_0(t_0\boldsymbol{u})$, for $t_0 - \delta < t < t_0$.

In order to show the converse inequality $\tilde{\sigma}^{(1)}(t) \ge \Sigma^{(1)}_0(t_0\boldsymbol{u})$, we assume to the contrary that there exists a sequence $t_n \to t_0$ and $k_n \in \widetilde{\mathcal{M}}^0_{t_n}$ such that $\Sigma^{(1)}_0(t_n\boldsymbol{u}; k_n) < \Sigma^{(1)}_0(t_0\boldsymbol{u})$. As above we can assume $k_n \to k_\infty \in \mathcal{M}^0_{t_0}$. If Σ_0 is constant on the connected component of $t_0\boldsymbol{u} - k_\infty$, then $\Sigma^{(1)}_0(t_n\boldsymbol{u}; k_n) = \Sigma^{(1)}_0(t_0\boldsymbol{u})$ is a constant sequence for n large enough, which is a contradiction. If Σ_0 is not constant on the connected component of $t_0\boldsymbol{u} - k_\infty$, then $|k_n| = |k_\infty|$ and $|t_n\boldsymbol{u} - k_n| = |t_0\boldsymbol{u} - k_\infty|$, for n large enough, and again we conclude $\Sigma^{(1)}_0(t_n\boldsymbol{u}; k_n) = \Sigma^{(1)}_0(t_0\boldsymbol{u})$ is a constant sequence, which is a contradiction. □

A Riemannian covers

Let $\mathcal{R}_p = (0, \infty) \times \mathbb{R}/2\pi\mathbb{Z}$, equipped with the product topology. We write $\mathbf{z} = (|\mathbf{z}|, \arg(\mathbf{z}))$ for elements of $\mathcal{R}_p$ and introduce the p-cover $(\mathcal{R}_p, \pi_p)$ of $\mathbb{C}\backslash\{0\}$ by

$$\pi_p \ : \ \mathcal{R}_p \ \to \ \mathbb{C}\backslash\{0\} \ \text{ where } \ \pi_p(\mathbf{z}) \ := \ |\mathbf{z}|\, e^{\mathrm{i}\arg(\mathbf{z})} \,.$$

Note that π_p is locally a homeomorphism and thus provides a chart which turns $\mathcal{R}_p$ into an analytic surface. If $\mathcal{U} \subset \mathcal{R}_p$ is such that π_p is 1-1 on $\mathcal{U}$, write $\pi^{-1}_{\mathcal{U}}$ for the inverse homeomorphism from $\pi_p(\mathcal{U})$ to $\mathcal{U}$.

We will use the concept of analytic functions to, from, and between cover spaces. This is just special cases of what it means to be an analytic map between two analytic surfaces, cf. [3, Sections IX.6 and IX.7]

Let $\mathcal{V} \subset \mathbb{C}$, $\mathcal{V}_p \subset \mathcal{R}_p$, and $\mathcal{V}_q \subset \mathcal{R}_q$ be open sets, and $f_1 : \mathcal{V} \to \mathcal{R}_p$, $f_2 : \mathcal{V}_p \to \mathcal{R}_q$, and $f_3 : \mathcal{V}_q \to \mathbb{C}$ continuous maps.

We say f_1 is analytic if for any $z_0 \in \mathcal{V}$ there exists an open set $\mathcal{U} \subset \mathcal{V}$ with $z_0 \in \mathcal{U}$, such that the map $\mathcal{U} \ni z \to \pi_p(f_1(z))$ is analytic in the usual sense.

We say f_2 is analytic if for any $\mathbf{z}_0 \in \mathcal{V}_p$ there exists an open set $\mathcal{U}_p \subset \mathcal{V}_p$ with $\mathbf{z}_0 \in \mathcal{U}_p$ and $\pi_p : \mathcal{U}_p \to \mathbb{C}$ 1-1, such that $\pi_p(\mathcal{U}_p) \ni z \to \pi_q(f_2(\pi^{-1}_{\mathcal{U}_p}(z)))$ is analytic in the usual sense.

We say f_3 is analytic if for any $\mathbf{z}_0 \in \mathcal{V}_q$ there exists an open set $\mathcal{U}_q \subset \mathcal{V}_q$ with $\mathbf{z}_0 \in \mathcal{U}_q$ and $\pi_q : \mathcal{U}_q \to \mathbb{C}$ 1-1, such that the map $\pi_q(\mathcal{U}_q) \ni z \to f_3(\pi_{\mathcal{U}_q}^{-1}(z))$ is analytic in the usual sense.

With this definition it is easy to check that $f_2 \circ f_1$, $f_3 \circ f_2$ and $f_3 \circ f_2 \circ f_1$ are analytic maps. We give three examples which we use in Section 2. The first example is $P_p : \mathbb{C}\backslash\{0\} \to \mathcal{R}_p$, defined by

$$P_p(z) \ := \ (|z|^p, p\arg(z))\,. \tag{30}$$

Second example: Let $\rho \in \mathbb{R}$. We define a map $R_p^\rho : \mathcal{R}_p \to \mathcal{R}_p$ by

$$R_p^\rho(\mathbf{z}) \ := \ (|\mathbf{z}|, \arg(\mathbf{z}) \ + \ \rho \mod 2\pi p)\,. \tag{31}$$

The third example is the map $\mathcal{R}_p \ni \mathbf{z} \to \mathbf{z}^{1/p} \in \mathbb{C}\backslash\{0\}$ defined by

$$\mathbf{z}^{1/p} \ := \ |\mathbf{z}|^{1/p}\, e^{\mathrm{i}\arg(\mathbf{z})/p}\,. \tag{32}$$

The three examples above are all analytic and in addition bijections. We have

$$P_p(\mathbf{z}^{1/p}) \ = \ (P_p(\mathbf{z}))^{1/p} \ = \ \mathbf{z} \ \text{ and } \ (\mathbf{z}^{1/p})^p \ = \ \pi_p(\mathbf{z})\,. \tag{33}$$

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