Off–Diagonal Jacobi Matrices as a Model for a Spectral Transition of Anderson Type

Domingos H. U. Marchetti* and **Walter F. Wreszinski†**
Instituto de Física
Universidade de São Paulo
Caixa Postal 66318
05315 São Paulo, SP, Brasil

Leonardo F. Guidi‡
Instituto de Matemática
Universidade Federal do Rio Grande do Sul
Av. Bento Gonçalves, 9500 - Prédio 43-111
91509 Porto Alegre, RS, Brasil

Renato M. Angelo§
Departamento de Física
Universidade Federal do Paraná
Caixa Postal 19044
81531 Curitiba, PR, Brasil

Abstract

We introduce a class of Jacobi matrices which model a deterministic (sparse) disorder in the sense that the perturbation of the Laplacean consists of a (direct) sum of fixed off–diagonal two by two matrices placed at sites whose distances from one another grow exponentially. We prove that the spectrum is the set $[-2, 2]$. For "small coupling" there is (dense) pure point spectrum and for "large coupling" the support of the singular continuous spectral measure contains a set of positive Lebesgue measure, if the Prüfer angles are uniformly distributed. There is compelling numerical evidence for the latter property, which is briefly discussed.

*Partially supported by CNPq and FAPESP. E-mail: *marchett@if.usp.br*
†Partially supported by CNPq. E-mail: *wreszins@fma.if.usp.br*
‡Supported by FAPESP under grant 03/13772-0. E-mail: *guidi@mat.ufrgs.br*
§E-mail: *renato@fisica.ufpr.br*

1 Introduction and Summary

A basic open problem in the theory of disordered systems is the Anderson transition from localized to extended states, demonstrated for the Cayley tree in a remarkable paper [K]. Up to now, there is no proof of a spectral transition from absolutely continuous (a.c.) to dense pure point (p.p.) in lattice models with random or deterministic disorder, when a "coupling constant" and/or disorder vary. Following a suggestion of Howland [H], the existence of a spectral transition is addressed in this paper for a class of off–diagonal Jacobi matrices using the powerful methods of Last and Simon [LS] and Kiselev, Last and Simon [KLS], in which the nature of the spectrum is related to properties of transfer matrices.

We consider off–diagonal Jacobi matrices

$$J = \begin{pmatrix} 0 & p_0 & 0 & 0 & \cdots \\ p_0 & 0 & p_1 & 0 & \cdots \\ 0 & p_1 & 0 & p_2 & \cdots \\ 0 & 0 & p_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{1.1}$$

with $p_n = p_n(\kappa)$, $n = 0, 1, 2, \ldots$, real–valued functions of a parameter κ which plays the role of a "coupling constant". Without loss of generality, $0 \leq \kappa \leq 1$ and we assume that $p_n : [0,1] \longrightarrow \mathbb{R}_+$ is a continuous function for each j. Furthermore, as far as "sparseness" condition is concerned, $(p_n)_{n\geq 0}$ is required to be of the form

$$p_n = \begin{cases} 1 - r_j & \text{if} \quad n = a_j \in A \\ 1 & \text{if} \quad n \notin A \end{cases} , \tag{1.2}$$

with

$$r_j(0) = 0 \qquad \text{and} \qquad r_j(1) = 1 , \tag{1.3}$$

where $A = \{a_j\}_{j\geq 1}$ is a set of strictly positive integers such that

$$a_j - a_{j-1} \geq 2 , \qquad j = 2, 3, \ldots , \tag{1.4}$$

and

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = \beta > 1 , \tag{1.5}$$

where β is the "sparseness parameter".

A concrete example satisfying (1.3) is given by

$$r_j = \frac{\kappa}{1 + (1-\kappa)\,\zeta_j} \tag{1.6}$$

with $\zeta_j \geq 0$. If $\zeta_j \to \infty$ as $j \to \infty$, then

$$r_j(\kappa) \to 0 , \tag{1.7}$$

for all $\kappa \in [0,1)$ and $r_j(1) = \kappa$. If $\zeta_j = 0$ for all j, then

$$r_j(\kappa) = \kappa\,. \tag{1.8}$$

In both cases (either $\zeta_j \to \infty$ or $\zeta_j = 0$), as κ varies in the interval $(0,1)$, J interpolates continuously two opposite situations: J has pure a.c. spectrum at $\kappa = 0$ and purely point spectrum at $\kappa = 1$. Note that, in both cases, $p_n(1) = 0$ for all $n \in A$ and J becomes a direct sum of finite matrices with dense p.p. spectrum if (1.5) holds. This is Howland's remark [H], which shows off–diagonal Jacobi matrices as natural examples in which a spectral transition might be expected.

Jacobi matrices (1.1) can be written as

$$(Ju)_n = p_n\ u_{n+1} + p_{n-1}\ u_{n-1}\ , \tag{1.9}$$

$n \in \mathbb{N}$, with $p_{-1} = 1$ fixed and boundary condition $u_{-1} = 0$. A Jacobi matrix J_α is said to satisfy a α–boundary condition at -1 if u_{-1} is given by

$$u_{-1}\cos\alpha - u_0 \sin\alpha = 0 \tag{1.10}$$

($J = J_0$ satisfies a Dirichlet 0–boundary condition at -1. See also (2.12)). We shall call the class of Jacobi matrices (1.9) satisfying (1.10) for some $\alpha \in [0,\pi)$ and (1.2)-(1.5) Howland's class $\mathcal{H}$, and the subset of those satisfying (1.7), $\mathcal{H}_0$. Note that $\mathcal{H}\backslash\mathcal{H}_0$ includes those in which (1.8) holds.

Let J^0 be the free Jacobi matrix

$$J^0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

The spectrum $\sigma\left(J^0\right) = [-2,2]$ is purely absolutely continuous. If $J \in \mathcal{H}_0$, $Q = J - J^0$ is a compact perturbation of J^0 and thus, by Weyl's invariance theorem, the essential spectrum of J remains the same as that of J^0, $\sigma_{\mathrm{ess}}(J) = \sigma(J^0) = [-2,2]$. The spectrum of J in this subclass is described as follows.

Theorem 1.1 *Let $J \in \mathcal{H}_0$ with $0 < \kappa < 1$. Then $\sigma(J)$ is purely a.c. if $\sum_{k=1}^{\infty} |r_k|^2 < \infty$ and purely s.c. if $\sum_{k=1}^{\infty} |r_k|^2 = \infty$ and the "sparseness parameter" is $\beta = \infty$.*

Theorem 1.1 may be proven along the lines of Theorem 1.7 of Kiselev-Last-Simon [KLS]. It shows that there is no spectral transition in the subclass $\mathcal{H}_0$ as κ varies in $(0,1)$, at least under the sparseness condition (1.5). In order to occur a transition at a nontrivial value of κ, J has to be so that (1.7) does not hold (i.e. $r_n \nrightarrow 0$) and it is likely to be from s.c. to (dense) p.p., with a.c. spectrum just for $\kappa = 0$.

When $r_n \nrightarrow 0$, i.e. (1.8), the perturbation is not compact, which is a common feature with the theory of (random) disordered systems. This is the case we consider in the rest of the paper. Weyl's invariance principle is thus not applicable, and it is necessary to determine the spectrum. This is done in Section 2, where we also introduce the necessary notation. In Section 3 we show that a special choice of Prüfer (modulus and angle) variables is particularly convenient in the off–diagonal case. In Section 4 we recall the basic results of Last and Simon [LS] that we need, and state and prove our main Lemma, which roughly states that the spectral transition, from s.c. at small coupling to (dense) p.p. at large coupling, occurs whenever the corresponding Prüfer angles are uniformly distributed (u.d.). In the present case, the Prüfer angles are expected to have a continuous asymptotic distribution function (a.d.f.), which is near uniform for sufficiently small p. However, even for some values of p for which the a.d.f. is not uniform, the results are expected to hold because the strong numerical evidence presented in Section 5. Section 6 is reserved to a conclusion with open problems.

2 Some Notation and a Theorem on the Spectrum

A bounded Jacobi matrix J is a matrix of a self–adjoint operator H on a separable Hilbert space $\mathscr{H}$: $J_{ij} = (\varphi_i, H\varphi_j)$, where $\{\varphi_j\}_{j\geq 0}$ is an orthonormal base of $\mathscr{H}$. The spectral measure $\mu = d\rho$ associated to J is given by the 00–element of the resolvent matrix [A, S]

$$(J - \lambda I)^{-1}_{00} = \left(\varphi_0, (H-\lambda)^{-1}\varphi_0\right) = \int_{-\infty}^{\infty} \frac{1}{x-\lambda} d\rho(x)\,. \tag{2.1}$$

We introduce, for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, a 2×2 transfer matrix

$$T(n, n-1; \lambda) = \begin{pmatrix} \dfrac{\lambda}{p_n} & \dfrac{-p_{n-1}}{p_n} \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

associated to the $l_2(\mathbb{N})$–solutions u of $(J - \lambda I)\, u = 0$, and set

$$T(n;\lambda) := T(n, n-1; \lambda)\, T(n-1, n-2; \lambda) \cdots T(0, -1; \lambda)\,, \tag{2.3}$$

with $p_{-1} = 1$. If $J \in \mathcal{H}$, in view of (1.2) and (1.4), there are three situations to be considered: for $n \notin \{a_j, a_j+1\}_{j\geq 1}$, $T(n, n-1; \lambda)$ is equal to the free transfer matrix

$$T_0(\lambda) = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}; \tag{2.4}$$

for $n = a_j + 1$ for some $j \geq 1$,

$$T(n, n-1; \lambda) = T_+(a_j; \lambda) := \begin{pmatrix} \lambda & -1 + r_j \\ 1 & 0 \end{pmatrix}; \tag{2.5}$$

and for $n = a_j$ for some $j \geq 1$,

$$T(n, n-1; \lambda) = T_{-}(a_j; \lambda) := \left(\begin{array}{cc} \dfrac{\lambda}{1-r_j} & \dfrac{-\lambda}{1-r_j} \\ 1 & 0 \end{array} \right) . \tag{2.6}$$

Let $\lambda = 2\cos\varphi$ where $\varphi \in (0, \pi)$ (see Theorem 2.1 for a proof that this parameterization is exhaustive), and

$$U = \left(\begin{array}{cc} 0 & \sin\varphi \\ 1 & -\cos\varphi \end{array} \right) \tag{2.7}$$

be the nonsingular matrix with

$$U^{-1} = \left(\begin{array}{cc} \cot\varphi & 1 \\ \csc\varphi & 0 \end{array} \right) .$$

The matrix U has a remarkable property of transforming $T_0(\lambda)$ into a pure rotation

$$R(\varphi) = \left(\begin{array}{cc} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{array} \right) = UT_0(\lambda)U^{-1} . \tag{2.8}$$

It will be most convenient to introduce, for each $a_j \in A$, the matrices $P_{+-}(a_j)$ given by

$$T_{+-}^{U}(a_j) = R(\varphi) P_{+-}(a_j) R(\varphi) \tag{2.9}$$

where $T_{+-}^{U}(a_j) = UT_{+}(a_j) T_{-}(a_j) U^{-1}$. From (2.3) - (2.9) and, since a product of rotations is a rotation, we have

$$UT(n; \lambda)U^{-1} = R((n-a_k)\varphi) P_{+-}(a_k) R((a_k - a_{k-1})\varphi) \cdots P_{+-}(a_1) R((a_1)\varphi) \tag{2.10}$$

which will be useful in Section 3.

According to (1.2) and (1.8), in the equations above

$$r_j = 1 - p_{a_j} \tag{2.11}$$

where $\left(p_{a_j}\right)_{j\geq 1} \equiv 0$ refers to p. p. case, and $\left(p_{a_j}\right)_{j\geq 1} \equiv 1$ to the unperturbed (a.c.) case. We have

Theorem 2.1 *If $J \in \mathcal{H}$ and $0 \leq r_j \leq 1$ for all $j \in \mathbb{N}$, i. e. $p_{a_j} \in [0, 1]$, then*

$$\sigma(J) = [-2, 2] \ ,$$

is the essential spectrum of J.

Proof. We restrict ourselves to the case $J = J_0$ given by (1.9) with 0–boundary condition at -1. We have

$$J_\alpha = J_0 + \tan\alpha \ \delta_0 \ . \tag{2.12}$$

Clearly, $(\delta_0 u)_n = \delta_{n0}\ u_0$, is a rank one perturbation of J_0 and $\sigma_{\text{ess}}(J_\alpha) = \sigma_{\text{ess}}(J_0)$, by Weyl's invariance principle. Thus the choice $J = J_0$ entails no loss of generality.

Let $J \in \mathcal{H}$ with $(p_n)_{n\geq 0}$ so that $0 \leq p_{a_j} \leq 1$, $\forall j$. To prove $\sigma(J) \subseteq [-2,2]$, let $u = (u_n)_{n\geq 0} \in l_2(\mathbb{N})$. We have

$$(u, Ju) = \sum_{n=0}^{\infty} (u_n\, u_{n+1})\, h_n \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \tag{2.13}$$

where

$$h_n = \begin{pmatrix} 0 & p_n \\ p_n & 0 \end{pmatrix} .$$

Inserting

$$h_n = \lambda_n^+\, P_n^+ + \lambda_n^-\, P_n^-$$

into (2.13), where $P_n^{\pm}$ projects onto the eigendirection associated to the eingenvalue $\lambda_n^{\pm} = \pm p_n$, we have

$$-2\,\bar{\lambda} \leq \frac{(u, Ju)}{(u,u)} \leq 2\,\bar{\lambda}$$

where $\bar{\lambda} = \sup_n \lambda_n^+ = \sup_n p_n = 1$. Hence

$$\sigma(J) \subseteq \left[-2\,\bar{\lambda}, 2\,\bar{\lambda}\right] = [-2,2] . \tag{2.14}$$

Note that, because J is a symmetric off–diagonal matrix, the spectrum is symmetric and $\inf_n \lambda_n^- = -\sup_n \lambda_n^+ = -\bar{\lambda}$. In order to prove the opposite inclusion

$$\sigma(J) \supseteq [-2,2] , \tag{2.15}$$

we use Weyl's criterion (see Theorem VII.12 of [S1]): if A is a bounded self–adjoint operator in separable Hilbert space $\mathscr{H}$, λ is a point of the spectrum $\sigma(A)$ of A if and only if there exist a sequence $(\psi_n)_{n\geq 0}$ in $\mathscr{H}$, with $\|\psi_n\| = 1$, such that

$$\lim_{n\to\infty} \|(A-\lambda)\,\psi_n\| = 0 . \tag{2.16}$$

Let $\lambda = 2\cos\varphi$, $\varphi \in [0,\pi]$, and define $\psi_n = (1/\sqrt{n})\,(e^{i\varphi}, e^{i2\varphi}, \ldots, e^{in\varphi}, 0, \ldots)$. Clearly $\psi_n \in l_2(\mathbb{N})$ and we claim that

$$\|(J-\lambda)\,\psi_n\| \leq c\frac{\ln n}{\sqrt{n}} \tag{2.17}$$

holds with $c = c(\beta)$ independent of n for any $J \in \mathcal{H}$. In order to prove (2.17), notice that $(J-\lambda)\,\psi_n$ consists of the action on ψ_n of a sum of local matrices, each bounded in norm by one. Two of them involve the extremes $e^{i\varphi}$ and $e^{in\varphi}$, and there are $O(\ln n)$ non–diagonal matrices, each acting just on two successive components of ψ_n. The $O(\ln n)$

comes from the sequence $(a_j)_{j\geq 1}$ satisfying the sparseness condition (1.5), $a_j \sim \beta^j$, with at most r points a_j within $[1,n]$, where r is such that $\beta^r \leq n$ or, equivalently,

$$r \leq \frac{\ln n}{\ln \beta} \, .$$

This proves (2.17) and therefore (2.15).

A value λ in the spectrum $\sigma(J)$ is non–essential if it is an isolated eigenvalue with finite multiplicity. The essential spectrum is the complementary set in $\sigma(J)$ of the non–essential values. According to the quoted theorem of [S1], $\lambda \in \sigma(J)$ belongs to the essential spectrum if $\{\psi_n\}$ can be chosen to be orthogonal. Since $\{\psi_n\}$ are linearly independent vectors of $l_2(\mathbb{N})$, the proof of Theorem 2.1 is concluded by the Gram–Schmidt process.

$\square$

3 Prüfer Variables

We introduce the so called (modified) Prüfer variables for the off–diagonal Jacobi matrices in consideration. For $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, let $T(n, n-1;\lambda)$ be the 2×2 transfer matrix given by (2.2) and note that, if $\mathbf{u}_n = \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$ is a vector in $\mathbb{C}^2$ composed by the components of a $l_2(\mathbb{N})$–solution $u = (u_0, u_1, \ldots)$ of $(J_\alpha - \lambda)\, u = 0$, then

$$\mathbf{u}_{n+1} = T(n, n-1;\lambda)\, \mathbf{u}_n$$

holds for $n = 0, 1, \ldots$, with initial condition $\mathbf{u}_0 = \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ satisfying (1.10) for some $\alpha \in [0,\pi]$.

Writing

$$T_0^k = \underbrace{T_0\, T_0 \cdots T_0}_{k-\text{times}}$$

and

$$T_{+,-}(a;\lambda) = T_+(a;\lambda)\, T_-(a;\lambda) \, ,$$

where T_0 and $T_\pm$ is given by (2.4)-(2.6), equation (2.3) can be written as

$$T\,(n;\lambda) = T_0^{n-a_k-1}\, T_{+,-}(a_k)\, T_0^{a_k-a_{k-1}-2}\, T_{+,-}(a_{k-1}) \cdots T_{+,-}(a_1)\, T_0^{a_1-1} \, , \tag{3.1}$$

from which equation (2.10) can be obtained following the indications of Section 2. Inserting

$$R(-\varphi)UT_+(a)U^{-1} = \begin{pmatrix} p_a & 0 \\ (1-p_a)\cot\varphi & 1 \end{pmatrix}$$

and

$$UT_-(a)U^{-1}R(-\varphi) = \begin{pmatrix} 1 & 0 \\ (1/p_a - 1)\cot\varphi & 1/p_a \end{pmatrix}$$

into (2.9) gives

$$P_{+-}(a) = \begin{pmatrix} p_a & 0 \\ q_a \cot\varphi & 1/p_a \end{pmatrix} \tag{3.2}$$

where

$$q_a = \frac{1}{p_a} - p_a.$$

We now use equation (2.10), where R and $P_{+,-}$ are given by (2.8) and (3.2), respectively, to define the Prüfer variables $(R_k, \theta_k)_{k\geq 0}$ as follows. Let

$$\mathbf{v}_k = (R_{k-1}\cos\theta_k, R_{k-1}\sin\theta_k) \tag{3.3}$$

and

$$\widetilde{\mathbf{v}}_k = \left(R_k \cos\widetilde{\theta}_k, R_k \sin\widetilde{\theta}_k\right) \tag{3.4}$$

be given by

$$\mathbf{v}_k = R\left((a_k - a_{k-1})\varphi\right)\widetilde{\mathbf{v}}_{k-1} \tag{3.5}$$

and

$$\widetilde{\mathbf{v}}_k = P_{+-}\left(a_k\right)\mathbf{v}_k\,, \tag{3.6}$$

with $a_0 = 0$ and initial condition

$$\widetilde{\mathbf{v}}_0 = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix} = U\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \sin\alpha\ \sin\varphi \\ \cos\alpha - \sin\alpha\ \cos\varphi \end{pmatrix}. \tag{3.7}$$

Since $\partial\theta_0/\partial\alpha = -\sin\varphi$, for each $\varphi \in (0,\pi)$ there is a one–to–one correspondence between the initial Prüfer angle θ_0 and the α–boundary condition at -1 of (1.9). With these notations equation (2.10) reads

$$\widetilde{\mathbf{v}}_k = UT\left(a_k;\lambda\right)U^{-1}\mathbf{v}_0$$

and we have

$$R_k^2 = \|\mathbf{v}_{k+1}\|^2 = \|\widetilde{\mathbf{v}}_k\|^2 = R_{k-1}^2\left\|\begin{pmatrix} p_{a_k} & 0 \\ q_{a_k}\cot\varphi & 1/p_{a_k} \end{pmatrix}\begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix}\right\|^2$$

and

$$\theta_{k+1} = \arg\left(P_{+-}(a_k)\mathbf{v}_k\right) + \left(a_{k+1} - a_k\right)\varphi$$

which yield

$$\frac{R_k^2}{R_{k-1}^2} = p_{a_k}^2\cos^2\theta_k + \left(q_{a_k}\cot\varphi\cos\theta_k + \frac{1}{p_{a_k}}\sin\theta_k\right)^2 \tag{3.8}$$

and

$$\theta_{k+1} = \tan^{-1}\left(\frac{1}{p_{a_k}^2}\tan\theta_k + \frac{q_{a_k}}{p_{a_k}}\cot\varphi\right) + \left(a_{k+1} - a_k\right)\varphi \tag{3.9}$$

for $k \geq 1$ and

$$\theta_1 = \theta_0 + a_1\varphi \ . \tag{3.10}$$

We observe that θ_k and $\widetilde{\theta}_k$ are, respectively, Prüfer angles immediately before and after the perturbation at a_k takes place. The Euclidean norm of $UT\left(n;\lambda\right)U^{-1}$, given by

$$\left\|UT\left(n;\lambda\right)U^{-1}\mathbf{v}_0\right\|^2 = \left\|UT\left(a_N;\lambda\right)U^{-1}\mathbf{v}_0\right\|^2 = R_N^2, \tag{3.11}$$

for any n such that $a_N \leq n < a_{N+1}$, can thus be written as

$$\begin{aligned} R_N^2 &= \frac{R_N^2}{R_{N-1}^2}\frac{R_{N-1}^2}{R_{N-2}^2}\cdots\frac{R_1^2}{R_0^2} \\ &= \left(\exp\left\{\frac{1}{N}\sum_{k=1}^{N}\left(f\left(p_{a_k},\theta_k\right) - \ln p_{a_k}^2\right)\right\}\right)^N , \end{aligned} \tag{3.12}$$

where

$$f(p,\theta) = \ln\left(p^4\cos^2\theta + \left(\sin\theta + \left(1-p^2\right)\cot\varphi\cos\theta\right)^2\right) \tag{3.13}$$

with the Prüfer angles $\left(\theta_k\right)_{k\geq 0}$ satisfying the recursion relation

$$\theta_{k+1} = g(p_{a_k},\theta_k) + \left(a_{k+1} - a_k\right)\varphi \tag{3.14}$$

where

$$g(p,\theta) = \tan^{-1}\left(\frac{1}{p^2}\left(\tan\theta + \cot\varphi\right) - \cot\varphi\right) \tag{3.15}$$

for $k \geq 1$, with θ_1 given by (3.10).

4 The Main Theorem

In this section we prove the main Theorem connecting continuous asymptotic distribution of the Prüfer angles with the spectral transition. We combine Proposition 4.2 stated below with the powerful theorems of Y. Last and B. Simon [LS] relating the nature of the spectrum of J to the properties of the transfer matrix $T\left(n;\lambda\right)$: the essential support Σ_{ac} of the a.c. part μ_{ac} of spectral measure $\mu = d\rho$ (2.1) is given by

$$\Sigma_{\rm ac} = \left\{\lambda : \liminf_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\left\|T\left(n;\lambda\right)\right\|^2 < \infty\right\} \tag{4.1}$$

By Theorems 1.6 and 1.7 of [LS] the eigenvalue equation $\left(J - \lambda I\right)u = 0$ has no $l_2\left(\mathbb{N}\right)$–solutions if

$$\sum_{n=0}^{\infty}\left\|T\left(n;\lambda\right)\right\|^{-2} = \infty \tag{4.2}$$

(see also Theorem 2.1 in [SS]) and an $l_2(\mathbb{N})$–solution if

$$N_{\mathrm{pp}} \equiv \sum_{n=0}^{\infty} \|T(n;\lambda)\|^2 \left(\sum_{k=n}^{\infty} \|T(k;\lambda)\|^{-2} \right)^2 < \infty \,. \tag{4.3}$$

We say that the Prüfer angles $(\theta_k)_{k\geq 1}$ defined by (3.3) have an asymptotic distribution function modulo π (abbreviated a.d.f. mod π) $\nu(x)$ if, for any $\theta \in [0,\pi]$

$$\lim_{N\to\infty} \frac{\mathrm{card}\left(\{k : \theta_k \in [0,\theta),\, 1 \leq k \leq N\}\right)}{N} = \nu(\theta) \tag{4.4}$$

where card(S), the cardinality of the set S, is the number of elements in S. Note that the function $\nu : [0,\pi] \to [0,1]$ is nondecreasing right continuous with $\nu(0) = 0$ and $\nu(\pi) = 1$. The sequence $(\theta_k)_{k\geq 1}$ need not have an a.d.f mod π but upper

$$\limsup_{N\to\infty} \frac{\mathrm{card}\left(\{k : \theta_k \in [0,\theta),\, 1 \leq k \leq N\}\right)}{N} = \Phi_+(\theta)$$

and lower

$$\liminf_{N\to\infty} \frac{\mathrm{card}\left(\{k : \theta_k \in [0,\theta),\, 1 \leq k \leq N\}\right)}{N} = \Phi_-(\theta)$$

distribution functions, with $0 \leq \Phi_-(\theta) \leq \Phi_+(\theta) \leq 1$ for $\theta \in [0,\pi]$, always exist. If $\Phi_- \equiv \Phi_+$, the sequence $(\theta_k)_{k\geq 1}$ has an a.d.f. mod π. If $\Phi_-(\theta) = \Phi_+(\theta) = \theta$ for $\theta \in [0,\pi]$, the sequence $(\theta_k)_{k\geq 1}$ is uniformly distributed modulo π (abbreviated u.d. mod π).

For simplicity, and without loss of generality, we fix here and henceforth $p_{a_k} = p$ for all k, $0 < p < 1$, and take $A = (a_k)_{k\geq 1}$ so that

$$a_k - a_{k-1} = \beta^k \tag{4.5}$$

with $a_0 = 0$ i. e., $a_k = a_k(\beta) = \left(\beta^{k+1} - \beta\right)/(\beta - 1)$, $\beta > 1$. In this case (3.12) and (3.14) read

$$R_N^2 = \left(\frac{1}{p^2} \exp\left\{ \frac{1}{N} \sum_{k=1}^{N} f(\theta_k) \right\} \right)^N \tag{4.6}$$

with

$$\theta_k = g(\theta_{k-1}) + \beta^k \varphi \tag{4.7}$$

where $f(\theta) = f(p,\theta)$ is given by (3.13), $g(\theta) = g(p,\theta)$ by (3.15) and θ_1 by (3.10).

Let us assume that the sequence $(\theta_k)_{k\geq 1}$ given by (4.7) has the continuous a.d.f. mod π, $\nu(\theta)$. We have (see Theorem 7.2, Chap. 1 of [KN]):

Theorem 4.1 *A sequence $\omega = (\theta_k)_{k\geq 1}$ of real numbers has the continuous a.d.f. mod π, $\nu(\theta)$, if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\theta_k) = \frac{1}{\pi} \int_0^{\pi} f(\theta)\, d\nu(\theta) \,, \tag{4.8}$$

holds for every continuous periodic function f of period π.

The other ingredient is the following result

Proposition 4.2 *Suppose*

$$[a,b] \subset \sigma\left(J_\alpha\right) \tag{4.9}$$

holds for all $\alpha \in [0,\pi]$ *and there exists* $A \subset [a,b]$ *of Lebesgue measure zero independent of* α *such that, if* $\lambda \in [a,b] \setminus A$,

$$\left(J - \lambda I\right) u = 0 \tag{4.10}$$

has a solution $u \in l_2\left(\mathbb{N}\right)$, *where* J *is given by (1.9) for* $n = 1, 2, \ldots$, *and* u_0 *may take any value. Then* $\sigma(J_\alpha)$ *has only dense p.p. spectrum in* $[a,b]$ *for a.e. (w.r.t. Lebesgue measure)* α*–boundary condition (1.10).*

Proof. Take $\lambda \in [a,b] \setminus A$. By (4.10), (1.9) has, for this λ, a l_2–solutions on $(0,\infty)$. Thus, by Theorem $II.3$ of [S2], either **1**. λ is an eigenvalue of J with Neumann b. c. (i. e., corresponding to $u_{-1} = 0$ in (1.10), or $\alpha = 0$), or **2**. $G(\lambda) = \displaystyle\int d\rho(y)/\left(\lambda - y\right)^2 < \infty$, where ρ is the spectral measure (2.1). The first alternative **1**. can hold at most for λ belonging to a countable set; but, by hypothesis, (4.10) holds in a set of positive Lebesgue measure, i. e., the complement of a set of zero Lebesgue measure. Thus, even excluding the countable set satisfying the first alternative, we are led to the statement that $G(\lambda) < \infty$ for a. e. λ in $[a,b]$, and therefore, by the Simon–Wolff criterion [SW] (see also Theorem $II.5$ of [S2]), J_α has only p. p. spectrum in $[a,b]$ for a. e. boundary condition α, which is a nontrivial statement by assumption (4.9). Finally, the inclusion of those λ satisfying the first alternative does not alter the statement of the proposition. □

The assumption of the following theorem – our main result – has not yet been proven, but compelling numerical evidence for it exists: see Section 5.

Theorem 4.3 *Let* $J \in \mathcal{H}\backslash\mathcal{H}_0$ *with* $r_j = 1 - p$ *for all* $j \in \mathbb{N}$, $p \in (0,1)$ *and* $\beta \in \mathbb{N}$, $\beta \geq 2$. *Let*

$$I_1 \equiv \left\{ \lambda \in [-2,2] : \left(\frac{p}{1-p^2}\right)^2 (\beta - 1)\left(4 - \lambda^2\right) > 1 \right\} \tag{4.11}$$

and

$$I_2 \equiv \left\{ \lambda \in [-2,2] : \left(\frac{p}{1-p^2}\right)^2 (\beta^3 - 1)(4 - \lambda^2) < 1 \right\} \quad . \tag{4.12}$$

If the Prüfer angles $\left(\theta_k\right)_{k\geq 1}$ *are uniformly distributed modulo* π *for* $\varphi \in [0,\pi]\backslash A_{\theta_0}$, *where* A_{θ_0} *is a set of Lebesgue measure zero, possibly* θ_0*–dependent, then*

a. *there exist a set* A_1 *of Lebesgue measure zero such that the spectrum restricted to the set* $I_1\backslash A_1$ *is purely singular continuous;*

b. *the spectrum* $\sigma\left(J_\alpha\right)$ *of* J_α *is dense pure point when restricted to* I_2 *for almost every* $\alpha \in [0,\pi)$, *where* α *characterizes the boundary condition (1.10).*

Remark 4.4 *1. Take e.g. $\beta = 2$: (4.11) holds for p sufficiently close to 1, i.e. for small "coupling" $\kappa = 1 - p$, and λ never in a neighborhood U near the edges of the band ± 2, while (4.12) holds for p sufficiently close to zero ("large" κ) and λ always close to the band edges (see Figure 1). These features are also expected in the Anderson model, with the continuous part of the spectrum being, however, absolutely continuous.*

2. *In Section 5 we present numerical evidence that the sequence $(\theta_k)_{k\geq 1}$ has a continuous a.d.f. mod π for any parameter values $p \in (0,1)$ and $\beta = 2$ and becomes u.d. mod π in the limit as p goes to both 0 and 1.*

3. *The proof of Theorem 4.3 holds even when the Prüfer angles are not nearly uniformly distributed under the assumption that the corresponding l.r.s. of the integral (4.17), with $d\nu(\theta)$ replacing $d\theta$, is close to the r.h.s. of (4.17). If $(\theta_k)_{k\geq 1}$has a continuous a.d.f. mod π, the r.h.s. of (4.8) with f given by (4.14) can be evaluated numerically in order to determine intervals I_1 and I_2 similar to (4.11) and (4.12). Figures 2 and 3 show the profile of the integral as a function of φ for $p = 0.1$ and $p = 0.3$. The dots represent the numerical integration of (4.17) with the Lebesgue measure $d\theta$ replaced by the Stieltjes-Lebesgue measure $d\nu$ where ν is the distribution given in Figures 5 and 6.*

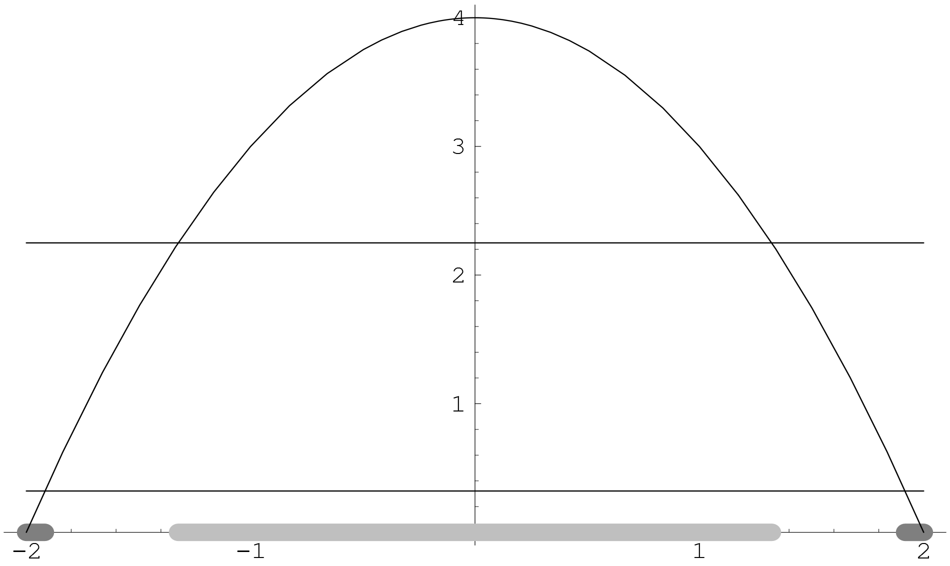

Figure 1: Intervals supporting s.c. (light gray) and p.p. (medium gray) spectrum of J for $\beta = 2$ and $p = 1/2$.

Proof. We must verify that the conditions on p, β and $\lambda = 2\cos\varphi$ which define (4.11) and (4.12) imply (4.2) and (4.3), respectively, for some $J = J_\alpha$ satisfying α–boundary condition at -1.

By assumption (4.8) the large N–behavior of R_N is governed by (4.6) with the sum in the exponent substituted by the integral (4.8). More precisely, Koksma's inequality gives (see Theorem 5.1 and Corollary 1.1 in Chap. 2 of [KN])[1]

$$\left| \frac{1}{N} \sum_{k=1}^{N} f\left(\theta_k\right) - \frac{1}{\pi} \int_0^{\pi} f(\theta)\, d\theta \right| \leq \frac{D_N^*}{\pi} \int_0^{\pi} \left| f'(\theta) \right| \, d\theta \tag{4.13}$$

where the discrepancy

$$D_N^* := \sup_{0<\alpha\leq\pi} \left| \frac{\operatorname{card}\left(\{k : \theta_k \in [0,\alpha),\ 1 \leq k \leq N\}\right)}{N} - \alpha \right|$$

goes to 0 as N goes to ∞.

In order to employ Theorem 4.1 it is necessary to ascertain that f is a Riemann–integrable periodic function of period π, and compute its integral. By simple trigonometric identities (3.13) may be written in the form

$$f(\theta) = \ln\left(a + b\cos 2\theta + c\sin 2\theta\right) \tag{4.14}$$

where

$$\begin{aligned} 2a &= \left(1-p^2\right)^2 \cot^2\varphi + 1 + p^4 \\ 2b &= \left(1-p^2\right)^2 \cot^2\varphi - 1 + p^4 \end{aligned}$$

and

$$c = \left(1-p^2\right)\cot\varphi\,,$$

which shows that f is periodic with period π. An explicitly computation gives

$$\begin{aligned} a + b\cos 2\theta + c\sin 2\theta &\geq \min_{\theta}\left(a + b\cos 2\theta + c\sin 2\theta\right) \\ &= a - \sqrt{b^2+c^2} = a - \sqrt{a^2-p^4} > 0 \end{aligned} \tag{4.15}$$

since

$$0 < b^2 + c^2 = a^2 - p^4 \tag{4.16}$$

if $0 < p < 1$, and this implies that $f(\theta)$ is Riemann–integrable. From [PBM], **29** p. 546 and (4.16), we have

$$\begin{aligned} \frac{1}{\pi}\int_0^{\pi} f(\theta)\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \ln\left(a + b\cos x + c\sin x\right)\, dx \\ &= \ln\left(\frac{a + \sqrt{a^2-b^2-c^2}}{2}\right) \\ &= \ln\left(\frac{(1-p^2)^2\csc^2\varphi}{4} + p^2\right)\ . \end{aligned} \tag{4.17}$$

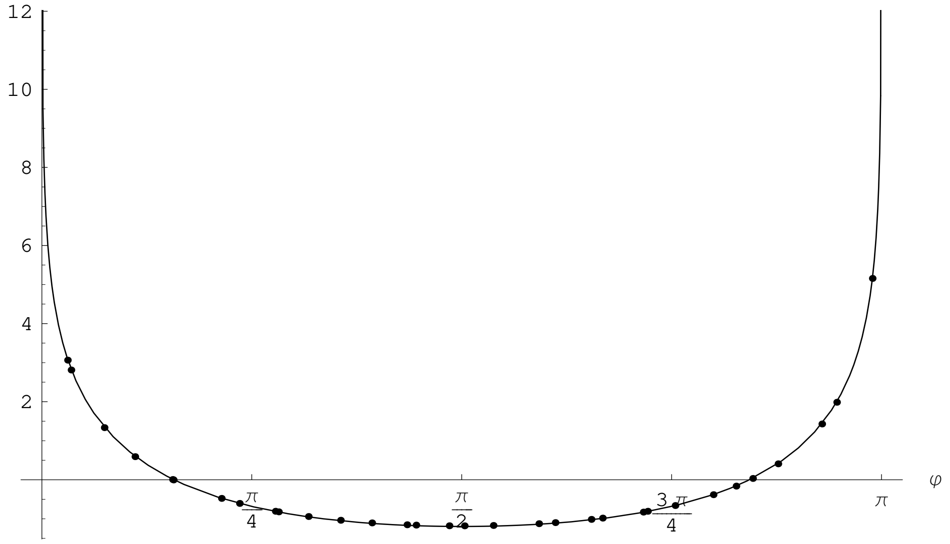

Figure 2: Profile of the integral (4.17) as a function of φ for $p = 0.1$

Let, now

$$r\,(p, \varphi) \equiv 1 + \frac{(1-p^2)^2 \csc^2 \varphi}{4p^2} \ . \tag{4.18}$$

By (4.6), (4.13), (4.14) and (4.17),

$$C_N^{-1}\, r^N \leq R_N^2 \leq C_N\ r^N \ , \qquad \text{for } \varphi \in [0, \pi] \backslash A_{\theta_0} \tag{4.19}$$

where A_{θ_0} is a set of Lebesgue measure 0, possibly depending on θ_0, mentioned in the hypothesis, and some finite constant $C_N > 1$ such that $\lim_{N\to\infty} C_N^{1/N} = 1$. Note that $\sup_{\theta\in[0,\pi]} |f'(\theta)| < \infty$ in view of (4.15), so that the r.h.s. of (4.13) indeed yields a C_N with the stated properties.

Let n so that $a_N \leq n < a_{N+1}$ holds for some $N \in \mathbb{N}$. We have

$$\begin{aligned} \sum_{m\geq n} \|T\,(m;\lambda)\|_U^{-2} &= \sum_{k\geq N} \|T\,(a_k;\lambda)\|_U^{-2} \sum_{a_k \leq m < a_{k+1}} 1 - \|T\,(a_N;\lambda)\|_U^{-2} \sum_{a_N \leq m < n} 1 \\ &= \sum_{k\geq N+1} \|T\,(a_k;\lambda)\|_U^{-2}\ \beta^{k+1} + \|T\,(a_N;\lambda)\|_U^{-2} \sum_{n\leq m < a_{N+1}} 1 \end{aligned} \tag{4.20}$$

By Proposition A.1, (A.4) and (A.5) of Appendix A,

$$\|T\,(m;\lambda)\|^{-2} \leq C^2\, \|T\,(a_k;\lambda)\|_U^{-2} \leq C^2 \left(\max_{i\in\{1,2\}} R_k(\theta_0^i) \right)^{-2} \tag{4.21a}$$

[1]The Lebesgue measure $d\theta$ may be substituted by the Riemann–Stieltjes measure $d\nu(\theta)$ in this Theorem.

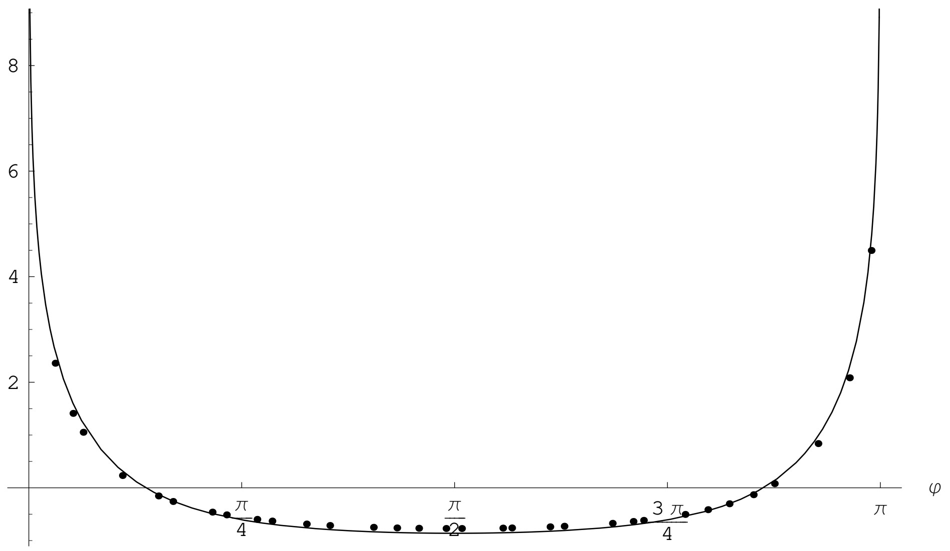

Figure 3: Profile of the integral (4.17) as a function of φ for $p = 0.3$

and

$$\|T\,(m;\lambda)\|^{-2} \geq C^{-2}\,\|T\,(a_k;\lambda)\|_U^{-2} \geq \widetilde{C}^{-2}\left(\max_{i\in\{1,2\}} R_k(\theta_0^i)\right)^{-2} . \tag{4.22a}$$

Let, now

$$S_{N,M}^{\pm} \equiv \beta \sum_{k=N}^{M} C_k^{\pm 1}\left(\frac{\beta}{r}\right)^k . \tag{4.23}$$

By (4.19) - (4.23),

$$\widetilde{C}^{-2}\,S_{N+1,M}^{-} \leq \sum_{m=n}^{a_{M+1}} \|T\,(m;\lambda)\|^{-2} \leq C^2\,S_{N,M}^{+} \tag{4.24}$$

holds for all integers $M \geq N+1$ and $\lambda \in B$ where[2]

$$B \equiv 2\cos\left([0,\pi]\setminus A'\right) , \tag{4.25a}$$

and

$$A' = A_{\theta_0^1} \cup A_{\theta_0^2} \tag{4.26a}$$

is a fixed set of Lebesgue measure zero, once we have chosen an arbitrary **but from now on fixed** set $\{\theta_0^1, \theta_0^2\}$, with $\theta_0^1 \neq \theta_0^2$ (see (A.4) of Appendix A). The further assertions of lemma depend solely on the bounds (4.24).

[2] Here, $f(A)$ denotes the set $\{f(x) : x \in A\}$.

By (4.18), (4.19), (A.4) and (4.25)

$$B \subseteq \left\{ \lambda : \lim_{n\to\infty} \|T(n;\lambda)\| = \infty \right\} \tag{4.27}$$

and lies in the complement of Σ_{ac} by (4.1). Thus, except for a set

$$A_1 = A' \cup A'' \tag{4.28}$$

of Lebesgue measure zero - where A'' is a set of zero Lebesgue measure connected with the definition of the essential support Σ_{ac} of μ_{ac} (see, e.g., [CFKS], pg. 179, for this definition) - the set

$$B_1 \equiv 2\cos\left([0,\pi] \setminus A_1\right) \tag{4.29}$$

lies in the singular spectrum. By (4.23) and the l.h.s. of (4.24), (4.2) holds (taking $M \to \infty$) if

$$4p^2(\beta-1) > (1-p^2)^2 \csc^2\varphi\,, \tag{4.30}$$

which, using $4\csc^{-2}\varphi = 4-\lambda^2$, may be written in the form (4.11) and defines I_1. Thus, by (4.29) the spectrum is purely singular continuous when restricted to the set

$$I_1 \setminus (A_1 \cap I_1)$$

which is assertion a. of Theorem 4.3.

We now prove assertion b., assuming that

$$4p^2(\beta-1) < (1-p^2)^2 \csc^2\varphi \tag{4.31}$$

which, by (4.18) and (4.23), implies that $S^+_{N,M}$ converges for each N as $M\to\infty$. Using again (4.24) and denoting by

$$S^+_N = \lim_{M\to\infty} S^+_{N,M} \tag{4.32}$$

we find

$$\sum_{m=n}^{\infty} \|T(m;\lambda)\|^{-2} \le C^2\, S^+_N\ . \tag{4.33}$$

By (4.19), (4.21a), (4.32) and (4.33), and recalling the notation in (4.3), there exist finite constants C' and C'' such that

$$N_{\text{pp}} \le C' \sum_{N=1}^{\infty} \beta^N R_N^2 \left(S_N^+\right)^2 \le C'' \sum_{N=1}^{\infty} C_N^3 \left(\frac{4p^2\beta^3}{(1-p^2)^2\csc^2\varphi + 4p^2} \right)^N \ .$$

The above series converges if

$$\frac{p^2}{(1-p^2)^2}(\beta^3-1)(4-\lambda^2) < 1 \tag{4.34}$$

which defines I_2 in (4.12). Thus, by (4.3) and (4.29) if λ belongs to the set

$$B_2 = I_2 \backslash (A_1 \cap I_2)$$

the eigenvalue equation $(J - \lambda I) u = 0$ has an $l_2(\mathbb{N})$–solution. Since

$$I_2 \subset \sigma(J_\alpha) \ , \qquad \forall \alpha \in [0, \pi),$$

by Theorem 2.1, the assumptions of Proposition 4.2 hold and J_α has only (dense) p.p. spectrum in I_2 for almost every $\alpha \in [0,\pi)$, concluding the proof of assertion b. of Theorem 4.3.

□

Remark 4.5 *It follows from a general result of rank one perturbations (see e.g. Theorem $II.11$ in [S2]) that, for a dense G_δ set of α–boundary conditions at -1, J_α has purely s.c. spectrum on I_2. The G_δ set in $[0,\pi]$ has, by Proposition 4.2, Lebesgue measure zero.*

5 Uniform Distribution of Prüfer Angles

In this section we present the compelling numerical evidence that the Prüfer angles have an a.d.f. mod π for almost every $\varphi \in [0,\pi)$. We have computed the asymptotic density function $\rho = d\nu/d\theta$ for $\beta = 2$, φ a typically dyadic irrational and p^2 varying over the values 0.01, 0.1, 0.3, 0.5, 0.9 and 0.99. The corresponding hystograms presented in Figures 4-9, each containing 10^7 iteration points of the map (4.7), indicate that the sequence $(\theta_k)_{k\geq 1}$ has continuous a.d.f. mod π, $\nu(\theta)$ which tends to the identity θ when $p \to 0$ and 1. These claims will now be examined numerically using two criteria whose background is given in following.

Given a sequence of numbers $(u_k)_{k\geq 1}$ let $(S_n)_{n\geq 1}$ be given by

$$S_n = \frac{1}{n} \sum_{k=1}^{n} e^{2ihu_k} .$$

According to the Weyl criterion ([KN], Theorem 2.1 of Chap. 1), the sequence $(u_k)_{k\geq 1}$ is u.d. mod π if and only if $\lim_{n\to\infty} S_n = 0$ for every integer $h \neq 0$. If $u_k = u_k(x)$ is defined for each x lying in some interval $[a,b] \subset \mathbb{R}$, then $S_n = S_n(x)$ is defined for each $x \in [a,b]$. Denoting by $\|f\|_{L_2([a,b])} = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}$ the L_2 norm, the general metric result is as follows (see Theorem 4.2, Chap. 1, of [KN] for a proof)

Theorem 5.1 *If the numerical series*

$$\sum_{n=1}^{\infty} \frac{1}{n} \|S_n\|^2_{L_2([a,b])}$$

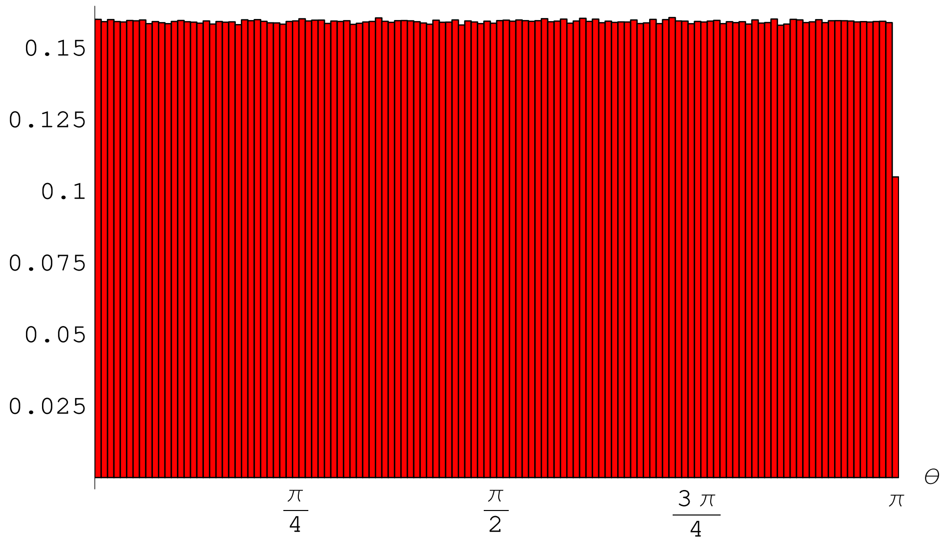

Figure 4: Hystograms with 10^7 iteration points of the map (4.7) for $\beta = 2$, φ a dyadic irracional and $p^2 = 0.01$

converges for each integer $h \neq 0$ then the sequence $\{u_k\}$ is u.d. mod π for almost all $x \in [a, b]$, i. e. $\{u_k\}$ is u.d. mod π for every x lying in $[a, b]$ apart from a set with Lebesgue measure 0.

The powerful metric result is ([KN], Theorem 4.3 of Chap. 1):

Theorem 5.2 (Koksma's Metric Theorem) *Let $(u_k(x))_{k\geq 1}$ be a sequence of real valued continuously differentiable functions defined in an interval $[a, b]$. If for any two positive integers $k \neq m$, $u'_k - u'_m$ is a monotone function of x such that $|u'_k - u'_m| \geq K > 0$ holds uniformly in k, m and $x \in [a, b]$, then $(u_k)_{k\geq 1}$ is u.d. mod π for almost all $x \in [a, b]$.*

Theorem 5.2 cannot be applied for the sequence $(\theta_k)_{k\geq 1}$ with $p \in (0, 1)$ in the present form. However, for convenience, the proof of Koksma's theorem will be included in order to identify the problems and to make understandable our numerical results.
Proof. By definition,

$$\begin{aligned} \|S_n\|^2_{L_2([a,b])} &= \frac{1}{n^2} \sum_{k,m=1}^{n} \int_a^b e^{2i(u_k(x)-u_m(x))}\,dx \\ &\leq \frac{|b-a|}{n} + \frac{2}{n^2} \sum_{k=2}^{n} \sum_{m=1}^{k-1} \left| \int_a^b e^{2i(u_k(x)-u_m(x))}\,dx \right| . \end{aligned} \tag{5.1}$$

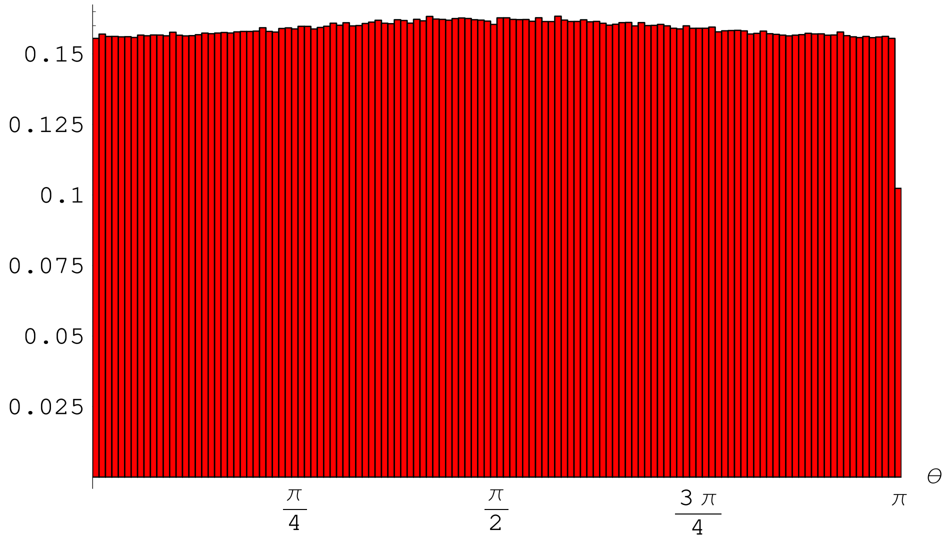

Figure 5: Hystograms with 10^7 iteration points of the map (4.7) for $\beta = 2$, φ a dyadic irracional and $p^2 = 0.1$

Under the hypothesis, the second mean value theorem holds and there exists c , with $a < c < b$, such that

$$\left| \int_a^b e^{2i(u_k(x)-u_m(x))}\, dx \right| = \left| \frac{1}{2i} \int_a^b \frac{de^{2i(u_k(x)-u_m(x))}}{u'_k(x) - u'_m(x)} \right| \leq \frac{1}{|u'_k(a) - u'_m(a)|} + \frac{1}{|u'_k(b) - u'_m(b)|} . \tag{5.2}$$

Ordering $\{u'_k(x)\}_{k=1}^n$ such that $u'_1(x) < u'_2(x) < \cdots < u'_n(x)$ holds for all x, the difference of any two consecutive numbers satisfies $u'_k(x) - u'_{k-1}(x) > K$ and we have

$$\sum_{m=1}^{k-1} \frac{1}{|u'_k(x) - u'_m(x)|} < 2 \sum_{m=1}^{n} \frac{1}{mK} < \frac{2}{K} \ln 3n .$$

Putting all together, yields

$$||S_n||^2_{L_2([a,b])} < \frac{|b-a|}{n} + \frac{4}{nK} \ln 3n \tag{5.3}$$

which combined with Theorem 5.1 concludes the proof.

□

In order to apply Theorem 5.2, $\theta'_k(\varphi) - \theta'_m(\varphi)$ has to be a monotone function for each interval $[a, b]$ independent of k and m. But this is not true for the sequence (4.7) for $0 < p < 1$. As we shall see, the number of monotone intervals of $\theta'_k - \theta'_m$ in $I \subset (0, \pi)$,

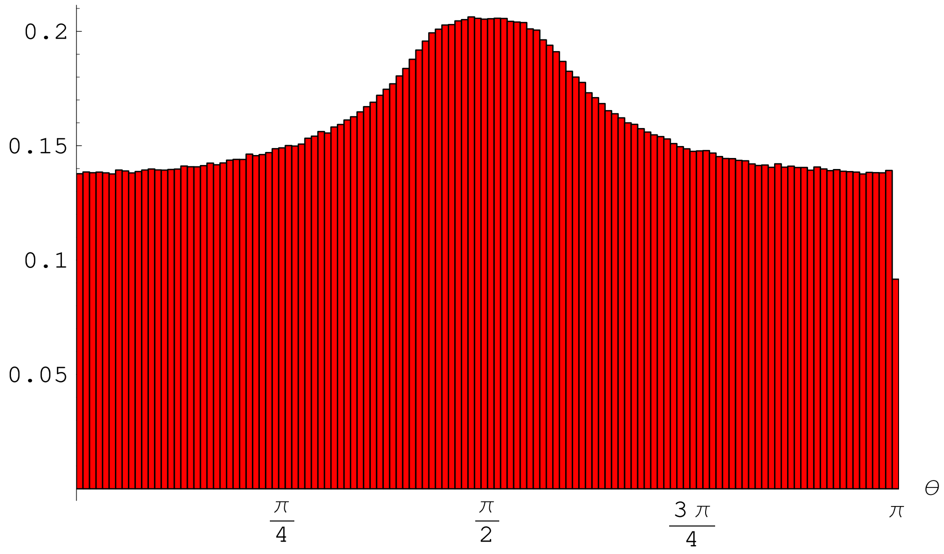

Figure 6: $\beta = 2$, φ a diadic irracional and $p^2 = 0.3$

with $k < m$ fixed, for which we may apply Koksma's Theorem, is proportional to β^k and this number is compensated by the constant K of Theorem 5.2 which, in our case, has the same k dependence.

We now need some properties of (3.15).

Proposition 5.3 *The function g given by (3.15) maps $[-\pi/2, \pi/2]$ into itself and has three fixed point solutions: $\theta^s_\pm = \pm\dfrac{\pi}{2}$ are two stable solutions and $\theta^u = \varphi - \dfrac{\pi}{2}$, $\varphi \in (0, \pi)$, is unstable. g is monotone increasing with $g'(\theta^s_\pm) = p^2$, $g'(\theta^u) = 1/p^2$ and there are two inflection points $\theta^i_\pm$ such that*

$$g'(\theta^i_-) = c_1 p^2 \le g'(\theta) \le \frac{c_2}{p^2} = g'(\theta^i_+) \tag{5.4}$$

holds for $c_{1(2)} = 1 + O(p^2)$.

Proof. Writing $g(\theta) = \tan^{-1}(\zeta/p^2)$ where

$$\zeta(\theta) = \tan\theta + (1 - p^2)\cot\varphi, \tag{5.5}$$

it follows from

$$g'(\theta) = \frac{p^2}{p^4 + \zeta^2}\sec^2\theta \tag{5.6}$$

that g is a monotone increasing function with $g(-\pi/2) = -\pi/2$ and $g(\pi/2) = \pi/2$. Hence $g : [-\pi/2, \pi/2] \to [-\pi/2, \pi/2]$. As θ goes to $\pm\pi/2$, $\sec^2\theta/\zeta^2 \to 1$ and $g'(\theta) \to p^2$.

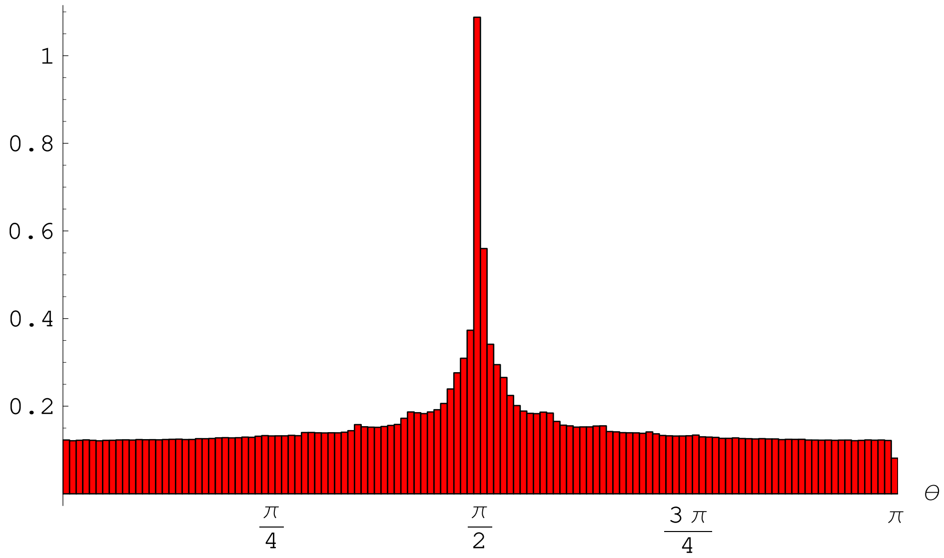

Figure 7: $\beta = 2$, φ a diadic irracional and $p^2 = 0.5$

Since $p < 1$, $\theta = \pm\pi/2$ are stable fixed points of g. To find the unstable fixed point we observe that

$$\tan g(\theta) + \cot\varphi = \frac{1}{p^2}(\tan\theta + \cot\varphi)\ ,$$

together with the fixed point equation $g(\theta) = \theta$ and $p < 1$, implies

$$\tan\theta^u + \cot\varphi = 0 \tag{5.7}$$

which is equivalent to $\theta^u = \varphi - \pi/2$. At $\theta = \theta^u$ equation (5.5) reads $\zeta(\theta^u) = p^2 \cot\varphi$ which together with $\sec^2\theta^u = 1 + \tan^2\theta^u = 1 + \cot^2\varphi$ and (5.6) gives

$$g'(\theta^u) = \frac{1}{p^2}\,.$$

A direct computation yields

$$g''(\theta) = \frac{2p^2\sec^2\theta}{(p^4 + \zeta^2)^2}\left[(p^4 + \zeta^2)\tan\theta - \zeta\sec^2\theta\right] \tag{5.8}$$

from where we get the two inflection points by solving $(p^4 + \zeta^2)\tan\theta - \zeta\sec^2\theta = 0$ for $\tan\theta$:

$$\theta^i_\pm = \tan^{-1}\left(\mp(\cot\varphi)^{\pm1}\left(1 \pm \frac{p^2}{\sec^2\varphi} + O\left(p^4\right)\right)\right) \tag{5.9}$$

gives $\theta^i_+ = \theta^u + O(p^2)$ and $\theta^i_- = \varphi + O(p^2)$, if $\varphi \in [0, \pi/2)$ and $\theta^i_- = \varphi - \pi + O(p^2)$ if $\varphi \in [\pi/2, \pi)$. Equation (5.4) follows by perturbation around $p = 0$ and this concludes the proof of Proposition 5.3.

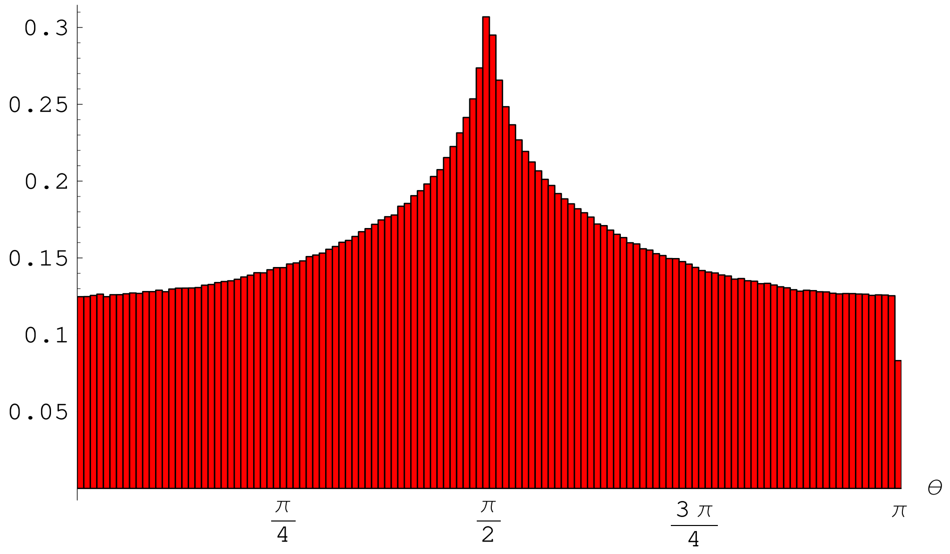

Figure 8: $\beta = 2$, φ a diadic irracional and $p^2 = 0.9$

$\square$

If $p^2\beta > c_2 = 1 + O(p^2)$, then from (4.7) and (5.4)

$$\begin{aligned} \theta_k'(\varphi) &= g'(\theta_{k-1})\theta_{k-1}'(\varphi) + \beta^k \\ &\le \frac{c_2}{p^2}\theta_{k-1}' + \beta^k \le \frac{1}{1 - c_2/(p^2\beta)}\beta^k\,, \end{aligned} \tag{5.10}$$

by induction. Equations (5.10) and (5.4) imply

$$\begin{aligned} \theta_k'(\varphi) - \theta_{k-1}'(\varphi) &= \left(g'(\theta_{k-1}) - 1\right)\ \theta_{k-1}'(\varphi) + \beta^k \\ &\ge -\left(1 - p^2\right)\theta_{k-1}' + \beta^k \\ &\ge \frac{\beta p^2 - c_2 - p^2 + p^4}{p^2\beta - c_2}\beta^k \end{aligned}$$

and hence

$$\theta_k'(\varphi) - \theta_m'(\varphi) \ge K\beta^k \tag{5.11}$$

where

$$K = K(p,\beta) = \frac{\beta\left(\beta p^2 - c_2 - p^2 + p^4\right)}{\left(p^2\beta - c_2\right)(\beta - 1)}$$

is a strictly positive constant by hypothesis.

Now, let $\mathcal{P}_{k,m} = \{I_i\}$ be the partition of $I \subset (0,\pi)$ into pairwise disjoint open intervals $I_i = (a_i, a_{i+1})$ such that $\theta_k'(\varphi) - \theta_m'(\varphi)$ is a monotone function in each I_i and

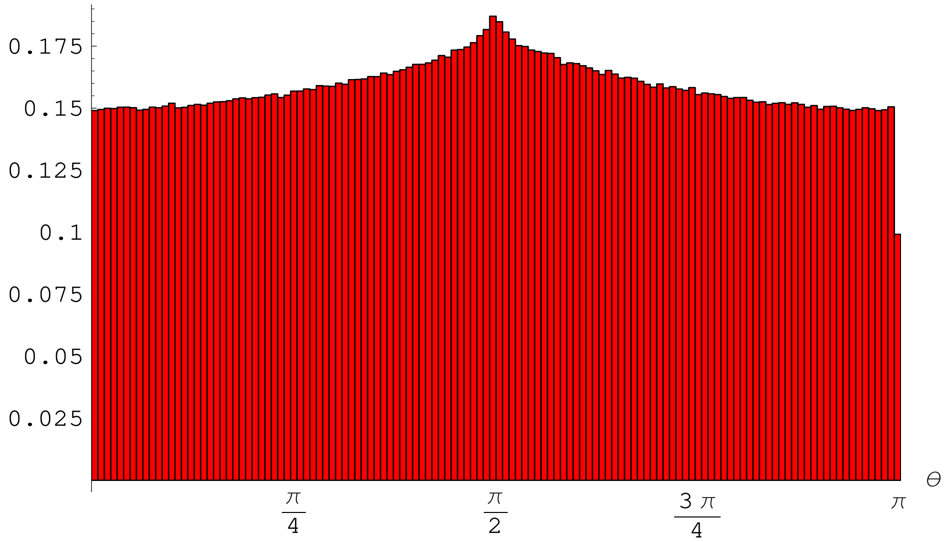

Figure 9: $\beta = 2$, φ a diadic irracional and $p^2 = 0.99$

$\bigcup_i \overline{I}_i = I$ ($\overline{A}$ means the closure of A). Since

$$\begin{aligned} \theta_k'(\varphi) &\geq c_1 p^2 \theta_{k-1}'(\varphi) + \beta^k \\ &\geq \frac{1}{1 - c_1 p^2/\beta} \beta^k \end{aligned} \tag{5.12}$$

and g has two inflection points at $\theta_{\pm}^i \in [-\pi/2, \pi/2]$, we have

Lemma 5.4

$$card\,(\mathcal{P}_{k,m}) \leq 20 \frac{\beta+1}{\beta(\beta-1)} \beta^k \equiv C\beta^k$$

holds for any $m < k$.

Proof. Define

$$\zeta_k = \tan\theta_k \tag{5.13}$$

and

$$\omega_k = \tan\beta^k\varphi \tag{5.14}$$

for any $k \in \mathbb{N}$, with $\omega = \omega_0 = \tan\varphi$, and note that $(\zeta_k)_{k\geq 0}$ satisfies a rational linear (Möbius) transformation

$$\zeta_k = \frac{1 - p^2 + p^2\omega\omega_{k-1} + \omega\zeta_{k-1}}{p^2\omega - (1-p^2)\,\omega_{k-1} - \omega\omega_{k-1}\zeta_{k-1}} \tag{5.15}$$

with $\zeta_0 = \tan\theta_0$, in view of (4.7), (3.15) and

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\ \tan\beta}\ . \tag{5.16}$$

In the following, we are going to show that $\zeta_k = C_k/D_k$ is a rational function with polynomials C_k and D_k in ω of degree not exceeding $C\beta^k$ for some constant C depending on β.

Taking the derivative of (5.13) with respect to φ, we have $\zeta_k' = \sec^2\theta_k\ \theta_k'$, which leads to the conclusion that

$$\theta_k' = \frac{\zeta_k'}{1+\zeta_k^2} \tag{5.17}$$

and

$$\theta_k'' = \frac{(1+\zeta_k^2)\,\zeta_k'' - 2\zeta_k\,(\zeta_k')^2}{(1+\zeta_k^2)^2} \tag{5.18}$$

are rational function of ω. Since the number of zeros of $\theta_k''(\varphi)$ counts the monotone intervals of $\theta_{j+1}'(\varphi)$ in $I \subset (0,\pi)$, Lemma 5.4 will be established by the fundamental theorem of Algebra and the fact that, as ω varies over $\mathbb{R}$,

$$\omega = \tan\varphi \tag{5.19}$$

has exactly two solutions in $(0,\pi)$, $\varphi_1(\omega)$ and $\varphi_2(\omega)$.

We begin by showing that $\omega_k = A_{\beta^k}/B_{\beta^k}$ is a rational function with A_n and B_n polynomials in ω of order n and $n-1$, if n is odd, or $n-1$ and n, if n is even, respectively. Using (5.16),

$$\frac{A_n}{B_n} = \tan\left((n-1)\varphi + \varphi\right) = \frac{\tan(n-1)\varphi + \omega}{1 - \omega\tan(n-1)\varphi}$$

implies the recursion

$$\begin{aligned} A_n &= A_{n-1} + \omega B_{n-1} \\ B_n &= B_{n-1} - \omega A_{n-1} \end{aligned}$$

with initial data $A_0 = 0$ and $B_0 = 1$, which by iteration yields

$$\begin{aligned} A_{2n}(\omega) &= (-1)^{n-1}\,2n\omega^{2n-1} + \text{ lower order terms} \\ B_{2n}(\omega) &= (-1)^n\,\omega^{2n} + \text{ lower order terms} \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} A_{2n+1}(\omega) &= (-1)^n\,\omega^{2n+1} + \text{ lower order terms} \\ B_{2n+1}(\omega) &= (-1)^n\,(2n+1)\omega^{2n} + \text{ lower order terms}\ . \end{aligned} \tag{5.21}$$

Now, let us assume that $C_k(\omega)$ and $D_k(\omega)$ are polynomials of degree

$$\deg(C_k), \ \deg(D_k) \le C\beta^k . \tag{5.22}$$

Clearly, since $\zeta_0 = \tan\theta_0$ is a constant in ω, C_0 and D_0 satisfy (5.22) for any $C \ge 0$. From (5.15),

$$\begin{aligned} C_{k+1} &= \left(1-p^2\right) B_{\beta^k} D_k + p^2 \omega A_{\beta^k} D_k + \omega B_{\beta^k} C_k \\ D_{k+1} &= p^2 \omega B_{\beta^k} D_k - \left(1-p^2\right) A_{\beta^k} D_k - \omega A_{\beta^k} C_k \end{aligned}$$

are polynomials in ω whose degree, in view of (5.20), (5.21) and (5.22), is bounded by

$$C\beta^k + \beta^k + 1 \le C\beta^{k+1}$$

provided

$$C \ge \frac{\beta+1}{\beta(\beta-1)} .$$

This concludes the induction. (5.18) is a rational function, $\theta_k'' = \dfrac{P_k}{Q_k}$, of ω with polynomials P_k and Q_k of degree not exceeding $10C\beta^k$ and, from the Fundamental Theorem of Algebra and observation (5.19), the number of zeros of $\theta_k''(\varphi) - \theta_m''(\varphi)$, $m < k$, in $[a,b] \subset (0,\pi)$ does not exceed $20C\beta^k$. This concludes the proof of Lemma 5.4.

$\square$

The estimate (5.2) together with (5.4) and (5.11), gives

$$\begin{aligned} \left| \int_I e^{2i(\theta_k(\varphi)-\theta_m(\varphi))} d\varphi \right| &\le \sum_{I_i \in \mathcal{P}_{k,m}} \left| \int_{I_i} e^{2i(\theta_k(\varphi)-\theta_m(\varphi))} d\varphi \right| \\ &\le \sum_i \frac{2}{|\theta_k'(a_i) - \theta_m'(a_i)|} \le \frac{2C}{K} \end{aligned} \tag{5.23}$$

and this implies, in view of (5.1), that $(1/n)\left\|S_n\right\|^2_{L_2(I)}$ is bounded by C'/n which does not fulfill the necessary requirement to apply Theorem 5.1.

As a matter of fact, this bound is not expected to be improved. According to the results shown in Figures 4 - 9, $S_n(\varphi)$ does not converge to 0 for any $\varphi \in [0,\pi]$ if $p \in (0,1)$. Note that, by the Weyl criterion together with (4.8),

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2ih\theta_k(\varphi)} \\ &= \int e^{2ih\theta} \, d\nu(\theta) \equiv \omega_h \end{aligned} \tag{5.24}$$

converges to 0 for every $h \neq 0$ if and only if $(\theta_k)_{k\ge1}$ is uniformly distributed, i.e. $\nu(\theta) = \theta$.

Despite of this, we have tested numerically Koksma's criterion for the sequence $(\theta_k)_{k\geq 1}$. Figure 10 plots $\|S_n\|^2_{L_2(I)}$ for $\beta = 2$ and $p = 0.3$ as n varies from 10^3 to 10^4. We have used Monte Carlo method of integration for an interval I of length 2^{-990} around $\varphi = 0.89375$ with the standard deviation from predicted value kept below 2%. According to the best interpolation fit $\|S_n\|^2_{L_2(I)}$ approaches to a constant as fast as $c_1 - c_2 n^{-0.9}$. For sequences satisfying the assumptions of Theorem 5.2 (see equation (5.3)), $\|S_n\|^2_{L_2(I)}$ converges to 0 as fast as $cn^{-1}\ln n$. For the aditive part of the sequence (4.7), $x_k = \beta^k\varphi$, the situation is even better with $\|S_n\|^2_{L_2(I)}$ decaying as $1/n$ without the log correction.

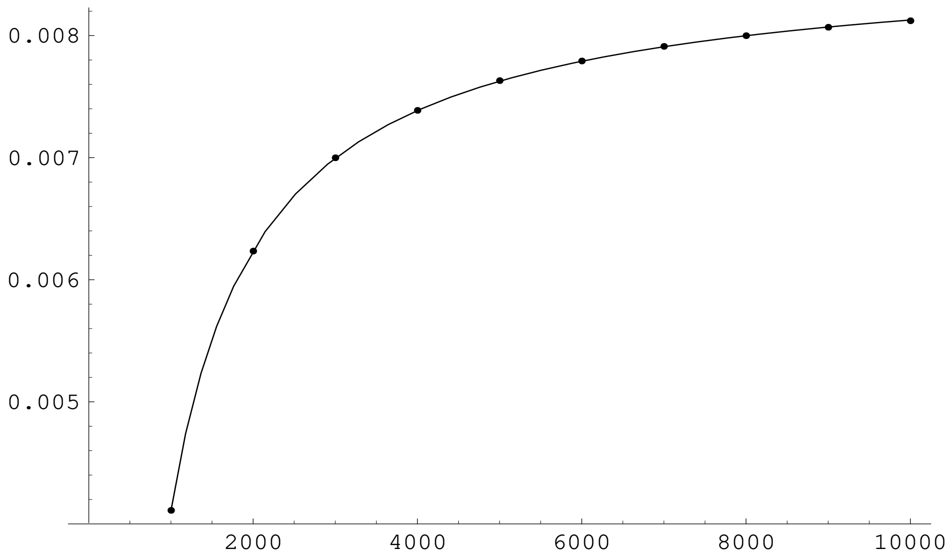

Figure 10: Plots of $\|S_n\|^2_{L_2(I)}$ for $\beta = 2$ and $p = 0.3$

An extension of Koksma's metric theorem to a nonuniform distribution ν is an open problem. However, if somehow one is able to subtract from S_n its limit point ω_h, the integral on the second line of (5.1) would, instead of behave like (5.23), decay as fast as $c_2 n^{-0.9}$ acording to the behavior of Figure 10 for $c_1 - \|S_n\|^2_{L_2(I)}$. Hence, the decay may be interpreted as an indication that the double sum in (5.1) would grow like $n^{1+\delta}$, $\delta \simeq 0.1$, and the sequence $(\theta_k)_{k\geq 1}$ would have continuous a.d.f. mod π for almost every $\varphi \in [0,\pi]$.

Finally, we verify the following criterion due to Wiener–Schoenberg (see Theorem 7.5, Chap 1 of [KN]): $(\theta_k)_{k\geq 1}$ has a continuous a.d.f. mod π if and only if $\omega_h = \lim_{n\to\infty} S_n$ exists for every positive integer h and

$$\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} |\omega_h|^2 = 0 \ .$$

Figure 11 is a log–log plot of $\sum_{h=1}^{H} |\omega_h|^2 / H$, with each point ω_h obtained by 10^7 iteration of (4.7) and φ a fixed dyadic irracional, as H varies from 2 to 6000.

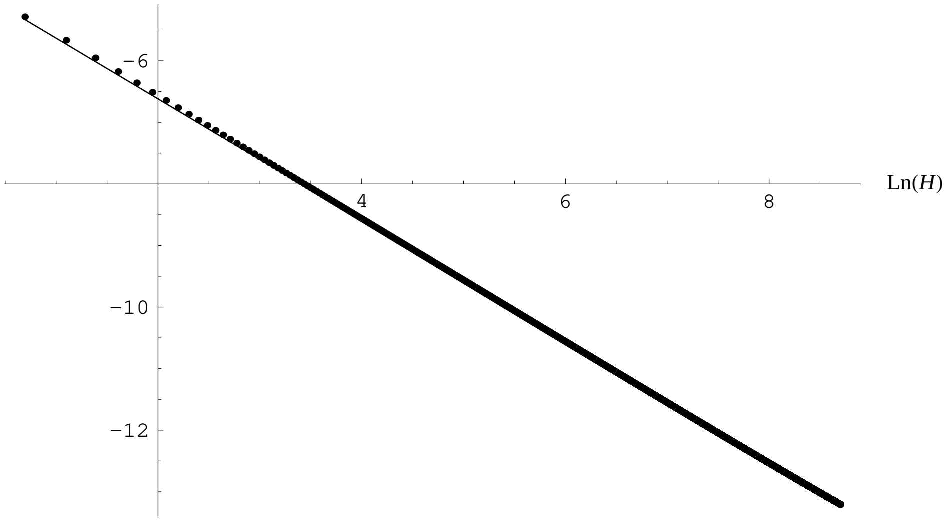

Figure 11: Wiener-Schoenberg criterion

6 Conclusion: Open Problems

The skeptical reader might say that we have transformed a difficult problem in functional analysis and mathematical physics into an equally difficult problem in the theory of (classical) dynamical systems. That may be so, but, although a numerical analysis of exotic spectral types is notoriously unreliable, that is no so for classical maps, and, indeed, the numerical evidence in Section 5 is compelling! Thus we are confident of having "made a case" for the Anderson transition in the present model, although a mathematical proof remains open. Given the map (4.7), the well–known [AA] ergodicity of the shift $\varphi \longrightarrow 2\varphi$ (mod 1) suggests that the property of uniform distribution of the Prüfer angles is very natural. We have seen that the assumption of **u. d.** leads to sharp bounds on the transfer matrix norm. In contrast, previous analysis by [KLS] and [R] is based on the entirely different idea of **independence**. Their applicability in the above mentioned references was, however, restricted to the case (1.7).

The singular character of the contiuous spectrum of the present model is due to its underlying one–dimensionality (see e.g. [KLS]). Already the separable Hamiltonian model described by the Kronecker sum

$$H = J \otimes I + I \otimes J \ , \tag{6.1}$$

with I the identity matrix, could present a transition from a.c. to p.p. spectrum. The reason is that the spectral measure of H, given by (6.1), is the convolution of the spectral measure of J with itself, and the convolution of two s.c. measures may be absolutely continuous (see [C], Exercise 9, pg. 165).

Finally, the above analysis introduced a class of dynamical systems which may be termed "perturbations of the dyadic shift" and are of independent interest. We hope to come back to this problem elsewhere.

A Equivalence of Matrix Norms

Let $\|A\|_U := \|UAU^{-1}\|$ be a matrix norm defined by the invertible matrix U given by (2.7) with $\varphi \in (0,\pi)$, where

$$\|B\| = \sup_{\mathbf{v}\in\mathbb{C}^2} \frac{\|B\ \mathbf{v}\|}{\|\mathbf{v}\|}$$

is the spectral matrix norm induced by the Euclidean vector norm on $\mathbb{C}^2$. We have

Proposition A.1 (Norm Equivalence)

$$C^{-1}\,\|A\| \le \|A\|_U \le C\,\|A\|$$

holds with

$$C = \sqrt{\frac{1+|\cos\varphi|}{1-|\cos\varphi|}}\,.$$

Proof. Given $\mathbf{v} = (v_1, v_2) \in \mathbb{C}^2$, we have

$$\|U\mathbf{v}\|^2 = |v_1|^2 + |v_2|^2 - 2\Re e\,(v_1\overline{v_2})\,\cos\varphi$$

which together with

$$0 \le |v_1 \pm v_2|^2 = |v_1|^2 + |v_2|^2 \pm 2\Re e\,(v_1\overline{v_2})\ ,$$

yields

$$(1-|\cos\varphi|)\,\|\mathbf{v}\|^2 \le \|U\mathbf{v}\|^2 \le (1+|\cos\varphi|)\,\|\mathbf{v}\|^2 \tag{A.1}$$

and the vector norm $\|\mathbf{v}\|_U := \|U\mathbf{v}\|$ is equivalent to the norm $\|\mathbf{v}\|$. Since U define an isomorphism in $\mathbb{C}^2$,

$$\|A\|_U = \sup_{\mathbf{v}\in\mathbb{C}^2} \frac{\|UAU^{-1}\mathbf{v}\|}{\|\mathbf{v}\|} = \sup_{\mathbf{v}\in\mathbb{C}^2} \frac{\|UA\mathbf{v}\|}{\|U\mathbf{v}\|} = \sup_{\mathbf{v}\in\mathbb{C}^2} \frac{\|A\mathbf{v}\|_U}{\|\mathbf{v}\|_U}$$

and the result follows from (A.1).

□

Proposition A.1 and (3.11) imply

$$\|T(n;\lambda)\| \le C \sup_{\theta_0} \left\|UT(n;\lambda)U^{-1}\mathbf{v}_0\right\| = C\sup_{\theta_0} R_N(\theta_0) \tag{A.2}$$

and

$$\|T(n;\lambda)\| \ge C^{-1}\left\|UT(n;\lambda)U^{-1}\mathbf{v}_0\right\| = C^{-1}R_N(\theta_0) \tag{A.3}$$

for n so that $a_N \le n < a_{N+1}$ holds for some $N \in \mathbb{N}$. We use the notation $R_n(\theta_0)$ to indicate the dependence of R_n on the initial Prüfer angle θ_0. As in Theorem 2.3 of Kiselev, Last and Simon [KLS] (see also Lemma 2.2) we can get rid of the sup in (A.2) using the fact that T, and consequently $T_U = UTU^{-1}$, is a unimodal matrix. We have

$$\|T_U\mathbf{v}_0\|^2 = \sin^2(\theta_0-\theta^*)\,\|T\|_U^2 + \cos^2(\theta_0-\theta^*)\,\|T\|_U^{-2} ,$$

where θ^* is the angle in which the infimum is attained, and

$$\|T\|_U \le \frac{1}{|\sin(\theta_0-\theta^*)|}\,\|T_U\mathbf{v}_0\| \ .$$

Now, for any pair $\theta_0^1,\ \theta_0^2 \in [0,\pi]$ with $0 < |\theta_0^1-\theta_0^2| < \pi/2$,

$$\left|\sin\left(\theta_0^i-\theta^*\right)\right|^{-1} \le \left|\sin\left(\theta_0^1-\theta_0^2\right)/2\right|^{-1}$$

holds for at least one of $i \in \{1,2\}$. Hence (A.2) can be replaced by

$$\|T(n;\lambda)\| \le \frac{C}{|\sin(\theta_0^1-\theta_0^2)/2|}\max_{i\in\{1,2\}} R_N(\theta_0^i) \equiv \widetilde{C}\max_{i\in\{1,2\}} R_N(\theta_0^i) \ . \tag{A.4}$$

Since θ_0 in (A.3) is arbitrary, we write

$$\|T(n;\lambda)\| \ge C^{-1}\ \max_{i\in\{1,2\}} R_N(\theta_0^i) \ . \tag{A.5}$$

These estimates play an important role in the proof of Lemma 4.3.

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