Geometric Scattering on Compact Riemannian Manifolds and Spectral Theory of Automorphic Functions

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Abstract. We show that the spectral properties of the Laplace–Beltrami operator on a compact Riemannian manifold with n semi-lines attached to it are similar to those for a finite-volume hyperbolic manifold with n cusps. Our results are further justification of the Gromov–Novikov thesis concerning relations between Hyperbolic Geometry on infinity and One-Dimensional Geometry. As an application of the corresponding results we obtain a relation between the scattering matrix on a compact Riemann surface of constant negative curvature and the Selberg zeta function for this surface.

1 Introduction

The starting point of our work is the following remark given by S. P. Novikov [13]: "As Misha Gromov often explains in his lectures, Hyperbolic Geometry is visible from the infinity as one-dimensional one. Therefore, we may conclude, that for the discrete groups in 2D Lobachevski Plane with Noncompact Fundamental Domain of finite volume, Spectral Theory of the Laplace–Beltrami Operator should look in a sense 'similar' to the one on the graphs with k tails. We discussed this analogy with D. Kazhdan, who pointed out to me that for arithmetic subgroups there are many discrete eigenvalues drown in the continuous spectrum. They disappear after nonarithmetic perturbation as Peter Sarnak pointed out. In the case of graphs with k tails, we have simplified version of this picture for operators with symmetry: exceptional eigenvalues disappear after generic nonsymmetric perturbation."

We announce here some results concerning a more realistic model of the mentioned Spectral Theory; for this purpose we consider a compact Riemannian manifold X_0 of dimension d, $d \leq 3$, with n one-dimensional tails attached to it [3], [4], [5]. The resulting space X, the so-called "horned manifold", is a particular case of a hybrid manifold or a graph decorated by a manifold [16]; on the other hand, this space can be considered as a limit of manifolds with cusps with respect to the Hausdorff–Gromov metric [10]. To define the Laplace–Beltrami operator on X (or generally a Schrödinger operator on X) we use the operator extension theory; in our approach, the main role is played by the Krein resolvent formula [1], [2], [14]. Such an approach to hybrid manifolds was proposed in [8].

2 Basic notations and definitions

X_0 denotes a compact connected Riemannian manifold of dimension d, $0 < d < 4$.

$g_{\mu\nu}$ is the metric on X_0, $g := \det(g_{\mu\nu})$.

$d\lambda(x)$ and $r(x,y)$ denote the Riemannian measure and geodesic distance on X_0, respectively.

$\{q_1, \ldots, q_n\}$ is a subset of X_0 (the points q_j are mutually distinct).

$\mathbb{R}_j^+$ ($j = 1, \ldots, n$) are copies of the half-line $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

X is the topological space obtained from the disjoint union $X_0 \sqcup \mathbb{R}_1^+ \sqcup \ldots \sqcup \mathbb{R}_n^+$ by gluing the point $0 \in \mathbb{R}_j^+$ to the point q_j.

$d\mu(x)$ denotes the sum of the measure $d\lambda(x)$ on X_0 and the Lebesgue measures dx on $\mathbb{R}_j^+$ ($j = 1, \ldots, n$) (therefore we identify the space $L^2(X, d\mu)$ with the direct sum $L^2(X_0, d\lambda) \oplus L^2(\mathbb{R}_1^+, dx) \oplus \ldots \oplus L^2(\mathbb{R}_n^+, dx)$).

$\mathcal{A}_\mu$ ($\mu = 1, \ldots, d$) and P are real-valued smooth functions on X_0 ($\mathcal{A}_\mu$ we will consider as the components of a vector potential $\mathcal{A}$, giving rise to a magnetic field on X_0, and P is viewed as the scalar potential of an electric field on X_0).

H_0 denotes the Schrödinger operator which is a unique self-adjoint extension of the operator τ defined on $C_0^\infty(X)$ by the expression

$$\tau := -g^{-1/2}(x)\left(\partial_\mu + i\mathcal{A}_\mu(x)\right) g^{1/2}(x) g^{\mu\nu}(x)\left(\partial_\nu + i\mathcal{A}_\nu(x)\right) + P(x)\,.$$

$G_0(x,y;\zeta)$ is the Green function of H_0, i.e. the integral kernel of the resolvent $(H_0-\zeta)^{-1}$.

S_0 denotes the restriction of H_0 to the set $\{f \in \mathcal{D}(H_0) :\, f(q_j) = 0 \,\forall\, j\}$.

S_j $(j = 1,\ldots,n)$ is the closure in $L^2(\mathbb{R}_j^+)$ of $-d^2/dx^2$ defined on $C_0^\infty(0,\infty)$.

H_j $(j = 1,\ldots,n)$ is the self-adjoint extension of S_j determined by the Neumann boundary condition at the point 0.

$S := S_0 \oplus S_1 \oplus \ldots \oplus S_n$.

$H^0 = H_0 \oplus H_1 \oplus \ldots \oplus H_n$.

$Q_0(\zeta)$ is the Krein Q-matrix defined by the relations $[Q_0(\zeta)]_{lm} = G_0(q_l,q_m;\zeta)$ if $l \neq m$, and $[Q_0(\zeta)]_{ll} = F_1(q_l,q_l;\zeta)$ with F_1 determined from the following expansion of $G_0(x,q;\zeta)$ near a fixed point $q \in X_0$:

$$G_0(x,q;\zeta) = F_0(x,q) + F_1(x,q;\zeta) + R(x,q;\zeta)\,.$$

Here F_1 is a continuous functions, F_0 is independent of the spectral parameter ζ and has the form:

$$F_0(x,q) = \begin{cases} -\dfrac{c_1(x,q)}{2} r(x,q)\,, & \text{if} \quad d=1; \\ -\dfrac{c_2(x,q)}{2\pi} \ln r(x,q)\,, & \text{if} \quad d=2; \\ \dfrac{c_3(x,q)}{4\pi} [r(x,q)]^{-1}\,, & \text{if} \quad d=3; \end{cases}$$

where $c_j(x,q)$ is a continuous functions of x and the remainder term R has the following behavior as $x \to q$: $R(x,q;\zeta) = o(r(x,q))$ if $d=1$, and $R(x,q;\zeta) = o(1)$ otherwise.

Let $f \in \mathcal{D}(S_0^*)$, then $a_j(f)$ and $b_j(f)$ denote the coefficients of the expansion $f(x) = a_j(f)F_0(x,q_j) + b_j(f) + R_j(x)$ where the remainder term $R_j(x)$ has the same behavior as $R(x,q;\zeta)$ above.

$Q(\zeta)$ is the $2n \times 2n$- matrix of the block form

$$Q(\zeta) = \begin{bmatrix} Q_0(\zeta) & 0 \\ 0 & D(\zeta) \end{bmatrix},$$

where $D(\zeta) = (-\zeta)^{-1/2} I$ with the unit $n \times n$-matrix I.

By definition, the *Schrödinger operator H* on the hybrid manifold X is an arbitrary self-adjoint extension of S. We restrict ourselves to a generic case of self-adjoint extensions defined by an Hermitian matrix L of dimension $2n \times 2n$ having the following block form

$$L = \begin{bmatrix} B & A \\ A^* & C \end{bmatrix},$$

where $B = (\beta_{jk})$ and $C = (\gamma_{jk})$ are Hermitian $n \times n$-matrices whereas $A = (\alpha_{jk})$ is an arbitrary complex $n \times n$-matrix. Then L defines H through the following boundary conditions

$$\begin{aligned} b_j(f_0) &= \sum_{k=1}^{n} \beta_{jk} a_k(f_0) - \sum_{k=1}^{n} \alpha_{jk} f_k{}'(0), \\ f_j(0) &= \sum_{k=1}^{n} \bar{\alpha}_{kj} a_k(f_0) - \sum_{k=1}^{n} \gamma_{jk} f_k{}'(0), \quad j = 1, \ldots, n. \end{aligned}$$

Here $f = (f_0, f_1, \ldots, f_n) \in L^2(X_0) \oplus L^2(\mathbb{R}_1^+) \oplus \ldots \oplus L^2(\mathbb{R}_n^+)$ is an element of $\mathcal{D}(S^*)$.

3 Spectral properties

If H_0 is the Laplace–Beltrami operator (i.e. $\mathcal{A} = 0$, $P = 0$), we call H the *Laplace–Beltrami operator* on X. The spectral properties of H are very similar to those for the automorphic Laplacian on the Lobachevski plane, i.e. for the Laplace–Beltrami operator on a Riemann surface with cusps. Recall very briefly the basic spectral properties of such an operator (see e.g., [9], [11], [12], [17], ...). Let $\mathbb{H}$ be the Lobachevski plane viewed as the Poincaré upper half-plane $\mathbb{C}^+ = \{z = x + iy : x \in \mathbb{R}, y > 0\}$, Γ be a Fuchsian group of the first kind which is not co-compact, n, $n \geq 1$, be the number of cusps for Γ, $-\Delta_\Gamma$

be the Laplace–Beltrami operator on the hyperbolic surface $X_\Gamma := \Gamma \backslash \mathbb{H}$. Then $\sigma(-\Delta_\Gamma)$, the spectrum of $-\Delta_\Gamma$, consists of three parts with mutually orthogonal invariant subspaces: (1) $\sigma_{\mathrm{ac}}(-\Delta_\Gamma)$, absolutely continuous spectrum, which fills the half-line $[1/4, \infty)$ with multiplicity n; (2) $\sigma_{\mathrm{cusp}}(-\Delta_\Gamma)$, the cuspidal spectrum of $-\Delta_\Gamma$, which consists of eigenvalues imbedded into the continuous spectrum; $\sigma_{\mathrm{cusp}}(-\Delta_\Gamma)$ can be empty; only $+\infty$ can be a limiting point of $\sigma_{\mathrm{cusp}}(-\Delta_\Gamma)$; points of $\sigma_{\mathrm{cusp}}(-\Delta_\Gamma)$ can have only finite degeneracy. Moreover, the resolvent $(-\Delta_\Gamma - \zeta)^{-1}$ is compact when restricted to the corresponding subspace $L^2_{\mathrm{cusp}}(X_\Gamma)$); (3) $\sigma_{\mathrm{res}}(-\Delta_\Gamma)$, the residual spectrum of $-\Delta_\Gamma$; it consists of points $\lambda = s(1-s)$, $s \in (1/2, 1]$, such that s is a pole of the determinant of the scattering matrix $\varphi_\Gamma(s)$; the corresponding subspace $L^2_{\mathrm{res}}(X_\Gamma)$ is finite-dimensional and $1 \in \sigma_{\mathrm{res}}(-\Delta_\Gamma)$. Hence, $L^2(X_\Gamma) = L^2_{\mathrm{ac}}(X_\Gamma) \oplus L^2_{\mathrm{cusp}}(X_\Gamma) \oplus L^2_{\mathrm{res}}(X_\Gamma)$. Whether $L^2_{\mathrm{cusp}}(X_\Gamma) \neq \{0\}$ is by no means obvious [15]. The complete set of generalized eigenfunctions of the continuous spectrum is given by analytic continuations of the Eisenstein series $E_j(z,s)$ $(j = 1, \ldots, n)$; for fixed s, $s = 1/2 + it$, $t \geq 0$, the functions $E_j(z,s)$ correspond to the spectral point $\lambda = s(1-s)$. The eigenfunctions from $L^2_{\mathrm{cusp}}(X_\Gamma)$ are the Maass cusp forms, they are automorphic functions which has no zero-th term in the Fourier expansions at any cusp. Finally, the eigenfunctions of the residual spectrum are residues of Eisenstein series at points $s = s_k$ corresponding to the points of this spectrum.

To describe now the spectrum of the Laplace–Beltrami operator on the horned manifold X we introduce firstly the scattering matrix and the scattering states for H. Let us write the matrix $[Q(\zeta) - L]^{-1}$ in block form

$$[Q(\zeta) - L]^{-1} = \begin{bmatrix} N(\zeta) & W(\zeta) \\ M(\zeta) & V(\zeta) \end{bmatrix},$$

where $W(\zeta) = (w_{lm}(\zeta))$ and $V(\zeta) = (v_{lm}(\zeta))$ are $n \times n$-matrices.

Theorem 1. *There exists a discrete subset Z_H of $\mathbb{R}$ such that for every $j \in \{1, \ldots, n\}$ and every $k > 0$, $k^2 \notin Z_H$, the equation $Hf = k^2 f$ has a unique solution $f = (f_0, f_1, \ldots, f_n)$, $f \in L^2(X_0) \oplus L^2_{\mathrm{loc}}(\mathbb{R}^+_1) \oplus \ldots \oplus L^2_{\mathrm{loc}}(\mathbb{R}^+_n)$, satisfying the conditions:*

(i) $f_0(x) = -2 \sum\limits_{m=1}^{n} w_{mj}(k^2) G_0(x, q_m; k^2)$;

(ii) $f_l(x) = s_{lj}(k)\exp(ikx)$ for $l > 0,\, l \neq j$;

(iii) $f_j(x) = \exp(-ikx) + s_{jj}(k)\exp(ikx)$.

Here the complex numbers $s_{jj}(k)$ form a unitary matrix $\Sigma(k)$ of the form

$$\Sigma(k) = \left[C + A^*(Q_0(k^2) - B)^{-1}A + ik^{-1}I\right] \times$$
$$\left[C + A^*(Q_0(k^2) - B)^{-1}A - ik^{-1}I\right]^{-1}.$$

The matrix $\Sigma(k)$ is the *scattering matrix* for H. The solution f depends on $j \in \{1,\dots,n\}$, $x \in X$, and on $k \in (0,\infty)$: $f = f^j(x,k)$; as a function of k it has an analytic continuation to a meromorphic function on $\mathbb{C}$.

To avoid cumbersome statements in Theorem 2 below, we restrict ourselves to the case of positive operator H_0, in particular, we can choose $H_0 = -\Delta$, the Laplace–Beltrami operator on the compact manifold X_0.

Theorem 2. *There is a decomposition $L^2(X) = L^2_{\mathrm{ac}}(X) \oplus L^2_{\mathrm{cusp}}(X) \oplus L^2_{\mathrm{res}}(X)$ into subspaces which are invariant with respect to H such that the following statements are true.*

(1) *The spectrum of H in $L^2_{\mathrm{ac}}(X)$ is purely absolutely continuous and fills the semi-axis $[0,\infty)$ with multiplicity n. For $\varphi \in C_0^\infty(X)$, denote by $\widehat{\varphi}$ the function from $L^2((0,\infty);\mathbb{C}^n)$ with coordinates*

$$\widehat{\varphi}_j(k) = \int\limits_X \overline{f^j(x,k)}\,\varphi(x)\,d\mu(x).$$

Then the mapping $\varphi \mapsto \widehat{\varphi}$ is a Hilbert isomorphism of the spaces $L^2_{\mathrm{ac}}(X)$ and $L^2((0,\infty);\mathbb{C}^n)$.

(2) *If $L^2_{\mathrm{cusp}}(X) \neq \{0\}$, then the spectrum of H in $L^2_{\mathrm{cusp}}(X)$ (denoted by $\sigma_{\mathrm{cusp}}(H)$) is discrete and lies on the semi-axis $[0,\infty)$. For a generic choice of the set $\{q_1,\dots,q_n\} \subset X_0$ and of the matrix L, the set $\sigma_{\mathrm{cusp}}(H)$ consists of all the eigenvalues of H_0 which have in $\sigma(H_0)$ multiplicity m_0 greater than n. The multiplicity m of a number λ in the spectrum of H equals $m_0 - n$.*

Each eigenfunction g which corresponds to an eigenvalue $\lambda \in \sigma_{\mathrm{cusp}}(H)$ has the form $g = (g_0, g_1, \ldots, g_n)$ where $g_j = 0$ for $1 \le j \le n$ and g_0 is an eigenfunction of H_0 vanishing at all the points q_j $(j = 1, \ldots, n)$. Hence, if H_0 has only simple eigenvalues (this is a generic case), then $L^2_{\mathrm{cusp}}(X) = \{0\}$ and $\sigma_{\mathrm{cusp}}(H) = \emptyset$. In any case, $\sigma_{\mathrm{cusp}}(H)$ is contained in the spectrum of the point perturbation of H_0 determined by the matrix B, and the multiplicity m allows the estimate $m \le m_0 + 2n$.

(3) *The space $L^2_{\mathrm{res}}(X)$ is finite-dimensional with dimension $d \le 2n$; moreover, $d \ge 1$ for the Laplace–Beltrami operator H. For a generic choice of the matrix A, the number $\lambda = (ik)^2$, $k > 0$, belongs to σ_{res} if and only if ik is a pole of the scattering matrix Σ. In this case, the corresponding eigenfunctions are residues of the meromorphic continuation of the functions $f^j(x,k)$ from Theorem 1. In any case, if g is an eigenfunction for $\lambda \in \sigma_{\mathrm{res}}(H)$, then $g = (g_0, g_1, \ldots, g_n)$ where g_0 is a linear combination of the functions $x \mapsto G_0(x, q_j; \lambda)$ and $g_j(x) = c_j \exp(-\sqrt{-\lambda} x)$ for $j \ge 1$.*

Theorem 2 shows that there is a deep analogy

(1) between the functions $f^j(x,k)$ from Theorem 1 and the Eisenstein series;

(2) between eigenfunctions from the item (2) of Theorem 2 and the Maass cusp forms;

(3) between eigenfunctions from the item (3) of Theorem 2 and the Maass forms which are represented by incomplete theta-series.

4 A formula for the Selberg zeta function

We consider here for simplicity only a compact Riemann surface X_0 of constant curvature $K = -1$ with one attached semi-axis $\mathbb{R}^+$ at a point q, $q \in X_0$. The matrices A, B, and C are simply numbers α, β, γ, respectively; we will assume $\alpha \neq 0$. The scattering matrix $\Sigma(k)$ has in this case only one term, namely, the reflection amplitude $S(k,q)$, which depends on q. Using an expression for the

Selberg trace formula from [6], [7], we get the following relation between the Selberg zeta function $\mathsf{Z}(s)$ for X_0 and the reflection amplitude $S(k,q)$:

$$\frac{\mathsf{Z}'(s)}{\mathsf{Z}(s)} = (2s-1)\Big[2(g-1)\psi(s)+\beta-$$

$$\frac{|\alpha|^2\sqrt{s(s-1)}}{4\pi(g-1)}\int\limits_X \left(\mathrm{Ca}(S(\sqrt{s(s-1)};q))+\gamma\sqrt{s(s-1)}\,\right)^{-1}dq\Big]\,.$$

Here ψ is the logarithmic derivative of the Euler Γ-function, g is the genus of X, $\mathrm{Ca}(S)$ stands for the Cayley transform of S.

Conclusion. The statements of Theorems 1 and 2 reinforce the Gromov–Novikov thesis concerning relations between Hyperbolic Geometry on infinity and One-Dimensional Geometry. Namely, the spectral theory of the Laplace–Beltrami operator on a compact manifold with n attached semi-axes (infinitely thin horns) completely looks like the spectral theory of the automorphic Laplacian for a Fuchsian group of the first kind with n cusps.

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