The Molchanov–Vainberg Laplacian

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Abstract

It is well known that the Green function of the standard discrete Laplacian $\Delta_{st}\psi(n) = (2d)^{-1}\sum_{|n-m|=1}\psi(m)$ on $l^2(\mathbb{Z}^d)$ exhibits a pathological behavior in dimension $d \geq 3$. In particular, the estimate

$$< \delta_0|(\Delta_{st} - E - i0)^{-1}\delta_n >= O(|n|^{-\frac{d-1}{2}})$$

fails for $0 < |E| < 1 - 2/d$. This fact complicates the study of the scattering theory of discrete Schrödinger operators. Molchanov and Vainberg suggested the following alternative to the standard discrete Laplacian,

$$\Delta\psi(n) = 2^{-d}d! \sum_{|n-m|=\sqrt{d}} \psi(m),$$

and conjectured that the estimate

$$< \delta_0|(\Delta - E - i0)^{-1}\delta_n >= O(|n|^{-\frac{d-1}{2}})$$

holds for all $0 < |E| < 1$. In this paper we prove this conjecture.

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1 Introduction

In the recent years there has been considerable interest in the scattering theory of discrete random Schrödinger operators on the lattice $\mathbb{Z}^d$ [B, BBP, CK, CS, HK, JL1, JL2, JM, JMP, K, MV1, MV2, RS, SV]. Typical models considered in the literature have the general form

$$H_\omega = -\Delta_{st} + \sum_{n\in\Gamma} V_\omega(n) < \delta_n|\cdot > \delta_n, \tag{1}$$

where δ is the Kronecker delta on $\mathbb{Z}^d$ and $V_\omega(n)$ are independent random variables indexed by $n \in \Gamma \subseteq \mathbb{Z}^d$. Δ_{st} is the *standard* discrete Laplacian,

$$\Delta_{st}\psi(n) = \frac{1}{2d}\sum_{|n-m|=1}\psi(m),$$

where $\psi \in l^2(\mathbb{Z}^d)$ and $|n| = \sqrt{\sum_j n_j^2}$. The symbol of Δ_{st} is $\frac{1}{d}\sum_{j=1}^d \cos x_j$ and its spectrum is absolutely continuous and equal to $[-1,1]$. Concerning the random potential V_ω in (1), the following two cases are distinguished in the literature: sparse random potentials, for which

$$\#\{n \in \Gamma\ ;\ |n| \leq R\}/R^d \to 0$$

as $R \to \infty$, and decaying random potentials, for which $\mathbb{E}(|V_\omega(n)|^2) \to 0$ as $|n| \to \infty$.

Practically all recent works on the scattering theory of Hamiltonians (1) make use of the stationary scattering theory. An important ingredient of the stationary methods is an *a priori* estimate on the Green function

$$G_E(n) := < \delta_0|(\Delta_{st} - E - i0)^{-1}\delta_n >$$

for energies E in the spectrum of Δ_{st}. In contrast to the standard continuous Laplacian on $L^2(\mathbb{R}^d)$, the Green function $G_E(n)$ is not necessarily $O(|n|^{-\frac{d-1}{2}})$ for $d \geq 3$. The reason for this is that the constant energy surfaces

$$\{x \in \mathbb{T}^d\ ;\ \frac{1}{d}\sum_{j=1}^d \cos x_j = E\}$$

are not convex for $|E| < 1 - \frac{2}{d}$. Hence, while the expected decay holds for $1 - \frac{2}{d} < |E| < 1$, it fails otherwise [SV]. This property of the standard Laplacian is an unfortunate obstacle in the development of the scattering theory of (1).

On physical grounds the choice of the standard discrete Laplacian is primarily dictated by custom and convenience, due to its relation to the standard random walk on $\mathbb{Z}^d$. Indeed, equivalent choices are possible.[1] Since Δ_{st} is ill

[1] By *equivalent*, we mean that by taking the scaling limit, $l^2(\mathbb{Z}^d) \to L^2(\mathbb{R}^d)$, one recovers the continuous Laplacian.

suited for the study of the scattering theory, Molchanov and Vainberg [MV1] suggested the following alternative, which we will call the *Molchanov-Vainberg Laplacian*:

$$\Delta\psi(n) = 2^{-d}d! \sum_{|m-n|=\sqrt{d}} \psi(m).$$

In other words, $\Delta\psi(n)$ is equal to a constant times the sum of the $\psi(m)$'s for m varying over the vertices of the hypercube of side 2 centered at n. These vertices are the neighbours of n along the full diagonals. The symbol of Δ is

$$\Phi(x) := \prod_{j=1}^{d} \cos x_j,$$

as the reader may easily verify.[2] Consequently, the spectrum of Δ is absolutely continuous and equal to $[-1,1]$. Since the Fourier transform is unitary, the Green function of Δ for $z \notin \mathbb{R}$ is

$$G_z(n) := <\ \delta_0|(\Delta - z)^{-1}\delta_n\ >= (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{e^{in\cdot x}}{\Phi(x) - z}\, \mathrm{d}x.$$

Molchanov and Vainberg [MV1] conjectured that the constant energy surfaces of Δ,

$$\Gamma(E) := \{x \in \mathbb{T}^d\ ;\ \prod_{j=1}^{d} \cos x_j = E\},$$

are strictly convex and that the bound $G_{E+i0}(n) = O(|n|^{-\frac{d-1}{2}})$ holds. In this paper we prove this conjecture.

Let $D_\varepsilon = \mathbb{C}_+ \cup [-1+\varepsilon, -\varepsilon] \cup [\varepsilon, 1-\varepsilon]$, where $0 < \varepsilon < \frac{1}{2}$. Our main result is:

Theorem 1 (1) *For* $0 < |E| < 1$ *the surface* $\Gamma(E)$ *is strictly convex.*
(2) *The function* $\mathbb{C}_+ \ni z \mapsto G_z(n)$ *extends continuously to* D_ε *and satisfies*

$$\sup_{E \in D_\varepsilon} |G_E(n)| \le C_\varepsilon |n|^{-\frac{d-1}{2}}.$$

Remark: *One may derive the complete asymptotic expansion of* $G_E(n)$ *on* D_ε*, see Remark at the end of Section 3.*

Applications of Theorem 1 to the scattering theory of random Schrödinger operators with sparse random potentials will be discussed in a forthcoming article [JP].

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[2]In other words $\mathcal{F}\Delta\mathcal{F}^{-1} = \Phi$, where $(\mathcal{F}\psi)(x) = (2\pi)^{-\frac{d}{2}} \sum_{n\in\mathbb{Z}^d} e^{in\cdot x}\psi(n)$ is the usual Fourier transform acting on $\psi \in l^2(\mathbb{Z}^d)$ and Φ denotes the operator of multiplication by the function $\Phi(x)$ on $L^2(\mathbb{T}^d)$. This result follows from Euler's formula for the cosine.

2 Level Surfaces of Φ

2.1 Connected Components

Consider first the covering $\widehat{\Gamma(t)}:=\{x \in \mathbb{R}^d \ ; \ \Phi(x) = t\}$ of $\Gamma(t)$.

If $t = 0$, $\widehat{\Gamma(t)}$ consists of the hyperplanes $x_j = (2n+1)\frac{\pi}{2}$ for $n \in \mathbb{Z}$ and $j = 1, \cdots, d$. These hyperplanes divide the space into open hypercubes which are called *cells*. The cells admit a good bicoloration in the following sense: starting from a set of two colours, say red and blue, it is possible to paint each cell in such a way that the $2d$ neighbours of any red cell are blue and *vice versa*. Let us accomplish this, the cell containing O being painted red.

When x varies continuously, $\Phi(x)$ changes sign each time one of the previous hyperplanes is crossed. It follows that the connected components of $\widehat{\Gamma(t)}$ are enclosed in the red cells when $t > 0$, each red cell containing one component. Moreover, these components are all congruent. The situation is similar when $t < 0$, but with the blue cells.

Finally, $\Gamma(t)$ is obtained from $\widehat{\Gamma(t)}$ by restricting it to the torus, where $0 < |t| < 1$ is fixed. One easily concludes that $\Gamma(t)$ consists of 2^{d-1} identical connected components.

2.2 Regularity and Convexity

$\Gamma(t)$ is a regular smooth surface for $0 < |t| < 1$, since for all $x \in \Gamma(t)$

$$\|\nabla\Phi(x)\|^2 = t^2 \sum_{j=1}^{d} \tan^2 x_j \neq 0.$$

Assuming first that $0 < t < 1$, one proves that any component, $\mathcal{R}(t)$, of the region $\{x \in \mathbb{R}^d \ ; \ \Phi(x) \geq t\}$ is convex. Given an arbitrarily fixed $\xi = (\xi_1, \cdots, \xi_d) \in \mathcal{R}(t)$, one defines for $j = 1, \cdots, d$,

$$f_j(u) = \begin{cases} \cos(u) & \text{if } \cos(\xi_j) > 0, \text{ with dom } f_j = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[+ 2k\pi, \\ -\cos(u) & \text{if } \cos(\xi_j) < 0, \text{ with dom } f_j = \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[+ 2l\pi, \end{cases}$$

where the choice of k or l is determined by $\xi_j \in \text{dom } f_j$. This last definition does not depend on $\xi \in \mathcal{R}(t)$.

Since $\Phi(\xi) = \cos\xi_1 \ldots \cos\xi_d \geq t > 0$, $\cos\xi_j$ is negative an even number of times, which implies that $\Phi(x) = f_1(x_1) \ldots f_d(x_d)$ on $\mathcal{R}(t)$. Note that $f_j(u)$ is strictly positive, concave, and hence logarithmically concave on its domain. Consequently,

Lemma 2 *$\mathcal{R}(t)$ is convex.*

Proof: Let $r, s \in [0,1]$ such that $r+s = 1$ and $x, y \in \mathcal{R}(t)$. Then, $rx+sy \in \mathcal{R}(t)$ since

$$\begin{aligned} \Phi(rx+sy) &= f_1(rx_1+sy_1) \ldots f_d(rx_d+sy_d) \\ &\geq f_1(x_1)^r f_1(y_1)^s \ldots f_d(x_d)^r f_d(y_d)^s \geq t. \end{aligned}$$

□

The case $t < 0$ is similar, starting from $(-\cos x_1)\cos x_2 \ldots \cos x_d \geq -t$.

Let $\mathcal{D}(t) = \partial\mathcal{R}(t)$ be a component of $\Gamma(t)$ and $N(x) = \frac{\nabla\Phi(x)}{\|\nabla\Phi(x)\|}$.

Theorem 3 *For any* $\omega \in \mathrm{S}^{d-1}$ *there exists exactly one point* $x = \mathrm{x}(\omega, t) \in \mathcal{D}(t)$ *such that* $N(x) = \omega$. *Moreover,* $\mathrm{x}(\omega, t)$ *extends to a holomorphic function in a neighbourhood of* $\mathrm{S}^{d-1} \times \{t\}$.

Proof: $N(x) = \omega$ iff x satisfies under the constraint $\Phi(x) = t$ the relations

$$-t \tan x_j = C\omega_j, \;\; j = 1, \cdots, d$$

for a scalar $C > 0$. Then, $x_j = \arctan \frac{-C\omega_j}{t}$, where the branch of arctan is fixed by the choice of $\mathcal{D}(t)$. Moreover, $\sec^2 x_j = 1 + (\frac{C\omega_j}{t})^2$, so the constraint implies

$$t^2 \left(1 + \frac{C^2\omega_1^2}{t^2}\right) \cdots \left(1 + \frac{C^2\omega_d^2}{t^2}\right) = 1. \tag{2}$$

This equation has a unique positive root $C = \mathrm{C}(\omega, t)$. The desired x is then given by $x_j = \arctan \frac{-\mathrm{C}(\omega,t)\omega_j}{t}$. Finally, $\mathrm{x}(\omega, t)$ is holomorphic since the left side of (2) has a non vanishing derivative in C. □

Note that the inverse, (N, Φ), of x also extends to a holomorphic function in a neighbourhood of $\mathcal{D}(t)$.

2.3 Gaussian Curvature

In this section we prove Part (1) of Theorem 1.

A regular smooth surface $\mathcal{S}$ may be expressed locally as the graph of a smooth function $h : U \subseteq \mathbb{R}^{d-1} \to \mathbb{R}$. The Gaussian curvature of $\mathcal{S}$ in a neighbourhood of a point $(u, h(u))$ is then given by

$$\frac{\det \mathrm{D}^2 h(u)}{(1 + \|\nabla h(u)\|^2)^2},$$

where $u \in U$. For $\mathcal{S} = \Gamma(t)$, we will prove that this curvature never vanishes. This computation together with Lemma 2 gives the following:

Theorem 4 *For* $0 < |t| < 1$ *any component of* $\Gamma(t)$ *is strictly convex.*

Proof: We treat the case $0 < t < 1$, the other case being similar. W.l.o.g. the component under consideration is given by $\{x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[^d \; ; \; \prod_{j=1}^d \cos x_j = t\}$. Let $m = d - 1$, $u = (x_1, \cdots, x_m)$, and $h = x_d$. The equation defining the previous component becomes $\cos x_1 \ldots \cos x_m \cos h = t$ or, each factor being positive,

$$\ln\cos x_1 + \cdots + \ln\cos x_m + \ln\cos h - \ln t = 0. \tag{3}$$

Since the component is symmetric with respect to the hyperplanes $x_j = 0$ and $x_j = x_l$, where $j,\, l \in \{1, \dots, d\}$ are distinct, it suffices to show the result for the fundamental domain $h \leq x_1 \leq \cdots \leq x_m \leq 0$. There, $h \neq 0$ since $t \neq 1$. The derivative with respect to h of the left side of (3) is $-\tan h$, which does not vanish. Consequently, an implicit function $h = \mathrm{h}(u,t)$ satisfying (3) in a neighbourhood of an arbitrary point of the fundamental domain exists, is analytic, and induces a local parametrization of the previous component.

We thus have to show that the Hessian $\mathrm{D}_u^2\mathrm{h}(u,t)$ is not singular. Differentiating (3) with respect to x_j gives

$$-\tan x_j - (\tan h)\partial_{x_j} h = 0,$$

where $h = \mathrm{h}(u,t)$. Differentiating this last equation with respect to x_l gives

$$\begin{array}{ll} -\sec^2 x_j - (\sec^2 h)(\partial_{x_j} h)^2 - (\tan h)\partial_{x_j}^2 h = 0 & \text{if } l = j, \\ -(\sec^2 h)\partial_{x_j} h\, \partial_{x_l} h - (\tan h)\partial_{x_j}\partial_{x_l} h = 0 & \text{otherwise.} \end{array}$$

Let $a_{jl} = -(\tan h)\partial_{x_j}\partial_{x_l} h$. Since $\tan h \neq 0$, it suffices to show that $\det[a_{jl}] \neq 0$. By the above,

$$a_{jl} = \begin{cases} \sec^2 x_j + (\sec^2 h)(\partial_{x_j} h)^2 & \text{if } l = j, \\ (\sec^2 h)\partial_{x_j} h\, \partial_{x_l} h & \text{otherwise.} \end{cases}$$

Let C_l be the l^{th} column of $[a_{jl}]$. Explicitly,

$$\begin{array}{rcl} C_1 & = & \sec^2 x_1 [1\ 0 \dots 0]^{\mathrm{t}} + (\sec^2 h)\partial_{x_1} h [\partial_{x_1} h \dots \partial_{x_m} h]^{\mathrm{t}}, \\ C_2 & = & \sec^2 x_2 [0\ 1 \dots 0]^{\mathrm{t}} + (\sec^2 h)\partial_{x_2} h [\partial_{x_1} h \dots \partial_{x_m} h]^{\mathrm{t}}, \\ \dots & & \\ C_m & = & \sec^2 x_m [0\ 0 \dots 1]^{\mathrm{t}} + (\sec^2 h)\partial_{x_m} h [\partial_{x_1} h \dots \partial_{x_m} h]^{\mathrm{t}}. \end{array}$$

Since $\wedge$ is alternate, the only non vanishing terms in $\det[a_{jl}] = C_1 \wedge \cdots \wedge C_m$ are $\sec^2 x_1 \dots \sec^2 x_m$ and m terms similar to $(\partial_{x_1} h)^2 \sec^2 h \sec^2 x_2 \dots \sec^2 x_m$, where the index varies in the obvious way. All these terms are non negative and the first one is strictly positive. Thus, the Gaussian curvature is strictly positive on the chosen fundamental domain and the result follows. $\square$

2.4 Analytic Parametrization

In this section we limit our considerations to a specific cell in $\mathbb{R}^d$, say $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[^d$. Exactly one component, $\mathcal{D}(t)$, of $\Gamma(t)$ is enclosed in this cell for $0 < t < 1$. We show that $\mathcal{D}(t)$ inherits the C^ω-structure of the sphere. This last is specified by the following system of local parametrizations: each hyperplane of coordinates divides S^{d-1} into two hemispheres, whose projections on D^{d-1} are denoted γ_j^{-1}; one then constitutes $\{\gamma_j : \mathrm{D}^{d-1} \to \mathrm{S}^{d-1}\}_{j=1}^{2d}$.

By Theorem 3 the following system then yields a C^ω-structure on $\mathcal{D}(t)$:

$$\{\sigma_j(\cdot, t) : \mathrm{D}^{d-1} \to \mathcal{D}(t),\, \sigma_j(u,t) = \mathrm{x}(\gamma_j(u), t)\}_{j=1}^{2d}.$$

Thus,

Corollary 5 *Each component $\mathcal{D}(t)$ of $\Gamma(t)$ is C^ω-diffeomorphic to* S^{d-1}.

Let us compute the element of surface. To this end we consider the determinants $M_j^{(l)}(u,t)$ of the submatrices of format $d-1 \times d-1$ obtained from $[\partial_{u_k}\sigma_j^{(l)}(u,t)]_{k,l}$ by removing its l^{th} column, where $\sigma_j = (\sigma_j^{(1)}, \cdots, \sigma_j^{(d)})$. Let

$$D_j(u,t) = (M_j^{(1)}(u,t), -M_j^{(2)}(u,t), \cdots, (-1)^{d-1}M_j^{(d)}(u,t)).$$

By definition the element of surface is $\|D_j(u,t)\|\,\mathrm{d}u$.

Lemma 6 $\|D_j(u,t)\| = \|\nabla\Phi(\sigma_j(u,t))\|\,|\det \mathrm{D}\sigma_j(u,t)|$, *where the derivative of σ_j is taken with respect to (u,t).*

Proof: The chain rule applied to $\Phi(\sigma_j(u,t)) = t$ gives

$$\nabla\Phi(\sigma_j(u,t))\,\mathrm{D}\sigma_j(u,t) = [0\ \cdots\ 0\ 1],$$

which we abbreviate $\nabla\Phi\,\mathrm{D}\sigma_j = e_d{}^{\mathrm{t}}$. Since $(u,t) \mapsto \sigma_j(u,t)$ is bijective, analytic, and has an analytic inverse, $\mathrm{D}\sigma_j$ is invertible. Thus,

$$\nabla\Phi = e_d{}^{\mathrm{t}}(\mathrm{D}\sigma_j)^{-1} = \frac{1}{\det \mathrm{D}\sigma_j} e_d{}^{\mathrm{t}}(\mathrm{adj}\mathrm{D}\sigma_j),$$

where adj stands for the classical adjoint. The result follows from $e_d{}^{\mathrm{t}}(\mathrm{adj}\mathrm{D}\sigma_j) = (-1)^{d-1}D_j(u,t)$. □

The whole $\mathcal{D}(t)$ except a set of measure zero may be covered using two parametrizations, σ_j and σ_k, associated with opposite hemispheres, $\gamma_j(\mathrm{D}^{d-1})$, $\gamma_k(\mathrm{D}^{d-1}) \subset \mathrm{S}^{d-1}$, and admitting holomorphic extensions. Using such parametrizations for each component of $\Gamma(t)$, one obtains:

Corollary 7 *For a fixed $0 < |t| < 1$, let $f(x)$ be holomorphic in a neighbourhood of $\Gamma(t)$. Then, the Fourier transform*

$$(\mathbf{F}f)(n,t) := \int_{\Gamma(t)} e^{in\cdot x} f(x)\,\mathrm{d}x$$

is holomorphic at t, where $n \in \mathbb{Z}^d$ and $\mathrm{d}x$ denotes the element of surface.

We close this section by presenting a local parametrization of $\mathcal{D}(t)$ used in the sequel.

Let $\mathcal{R}(t)$ be the interior of $\mathcal{D}(t)$. For an arbitrarily fixed $\omega \in \mathrm{S}^{d-1}$, let Ω be a point in $\mathcal{R}(t)$ incident to the line through $\mathrm{x}(\omega,t)$ and parallel to ω, and let $\omega^\perp$ be the hyperplane perpendicular to ω through Ω. Let R be the rigid motion sending e_d to ω and O to Ω. Finally, let $V = \omega^\perp \cap \mathcal{R}(t)$.

By convexity of $\mathcal{D}(t)$ the ray parallel to ω emanating from a given $v \in V$ intersects $\mathcal{D}(t)$ at exactly one point, $Q(v)$, whose normal is not parallel to $\omega^{\perp}$. Using the natural embedding $\mathbb{R}^{d-1} \subset \mathbb{R}^d$, the composition $Q \circ R$ then gives a parametrization $\tau : \Upsilon \subset \mathbb{R}^{d-1} \to \mathcal{D}(t)$ compatible with the C^{ω}-structure on $\mathcal{D}(t)$.

Note that $\Upsilon \to R^{-1}\mathcal{D}(t)$, $v \mapsto (v, \omega \cdot \tau(v))$ is a local parametrization of $R^{-1}\mathcal{D}(t)$. Since the curvature is invariant under rotation, it follows from Section 2.3 that

$$\det \mathrm{D}^2_v(\omega \cdot \tau(v)) > 0. \tag{4}$$

3 Decay of the Resolvent

In this section we prove Part (2) of Theorem 1. Our proof, based on the stationary phase method, uses of the following classical result (*cf.* [P, S]):

Proposition 8 *Let $I(r,t) = \int_{\mathbb{R}^d} e^{ir\varphi(x,t)} f(x,t)\,\mathrm{d}x$, where $r > 0$, $t \in T$, T is a compact set in $\mathbb{R}^m$, m is arbitrary, φ and f are smooth on $\mathbb{R}^d \times T$, φ is real valued, and $f(x,t) = 0$ when x is outside a compact subset $K \subseteq \mathbb{R}^d$.*

1. *If $\nabla_x \varphi(x,t) \neq 0$ for all $t \in T$ and $x \in K$, then $I(r,t) = O(r^{-\infty})$.*
2. *If, for all $t \in T$, $\varphi(x,t)$ admits exactly one critical point $x_0 = x_0(t)$ in K (i.e. $\nabla_x \varphi(x_0,t) = 0$), which is not degenerate (i.e. $\det \mathrm{D}^2_x \varphi(x_0,t) \neq 0$), then $I(r,t) = O(r^{-\frac{d}{2}})$.*

These estimates hold when $r \to \infty$ and are uniform in $t \in T$. Moreover, 1 holds for $I(r,t)$ over $\mathbb{T}^d$ under the same assumptions except that the compactness of $\operatorname{supp} f$ *is replaced with the periodicity of f.*

In the sequel $r\omega$ stands for the polar form of $n \in \mathbb{Z}^d$. For a fixed non zero $-1 < E < 1$, let χ be a smooth compactly supported function on $\mathbb{T}^d$ satisfying

$$\chi(x) = \begin{cases} 1 & \text{if } |\Phi(x) - E| < \frac{\delta}{2} \\ 0 & \text{if } |\Phi(x) - E| > \delta, \end{cases}$$

where $0 < \delta < \operatorname{dist}(E, \{-1, 0, 1\})$. Finally, let $\phi = \frac{1}{\|\nabla\Phi\|}$. Then,

Theorem 9 *$G_{E+i0}(n)$ exists and is equal to $(2\pi)^{-d}$ times*

$$\pi i(\mathbf{F}\phi)(n,E) + \text{p.v.}\int_{|t-E|\le\delta} \frac{(\mathbf{F}\chi\phi)(n,t)}{t-E}\,\mathrm{d}t + O(r^{-\infty})$$

when $r \to \infty$, uniformly in ω and uniformly in E on each compact, where p.v.$\int$ *stands for the Cauchy principal value.*

Proof: Consider the second term in the following decomposition, where $z \in \mathbb{C}_+$:

$$(2\pi)^d G_z(n) = \int_{\mathbb{T}^d} \frac{\chi(x)e^{in\cdot x}}{\Phi(x)-z}\,\mathrm{d}x + \int_{\mathbb{T}^d} \frac{(1-\chi(x))e^{in\cdot x}}{\Phi(x)-z}\,\mathrm{d}x.$$

By the dominated convergence theorem its limit exists when $z \to E$. Moreover, integration by parts shows that this limit is $O(|n|^{-\infty})$ (*cf.* Proposition 8).

It then suffices to analyze the first term. One may express it as a sum of integrals over the open cells of $\mathbb{T}^d$ and then break each cell into two halves covered by surfaces of the form $\mathcal{D}(t)^+ = \sigma_j(\mathrm{D}^{d-1}, t)$ and $\mathcal{D}(t)^- = \sigma_k(\mathrm{D}^{d-1}, t)$ respectively, where γ_j and γ_k parametrize opposite hemispheres (see Section 2.4). For one of these halves, say $\mathcal{H}^+ = \bigcup_{0<t<1} \mathcal{D}(t)^+$, the change of variables $x = \sigma_j(u,t)$ gives

$$\begin{aligned}\int_{\mathcal{H}^+} \frac{\chi(x)e^{in\cdot x}}{\Phi(x)-z}\,\mathrm{d}x &= \int_0^1 \int_{\mathrm{D}^{d-1}} \frac{\chi(\sigma_j(u,t))e^{in\cdot\sigma_j(u,t)}}{\Phi(\sigma_j(u,t))-z} |\det D\sigma_j(u,t)|\,\mathrm{d}u\,\mathrm{d}t \\ &= \int_{E-\delta}^{E+\delta} \frac{1}{t-z} \int_{\mathcal{D}(t)^+} \phi(x)\chi(x)e^{in\cdot x}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Reuniting all pieces together,

$$\begin{aligned}\int_{\mathbb{T}^d} \frac{\chi(x)e^{in\cdot x}}{\Phi(x)-z}\,\mathrm{d}x &= \int_{E-\delta}^{E+\delta} \frac{1}{t-z} \int_{\Gamma(t)} \phi(x)\chi(x)e^{in\cdot x}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{E-\delta}^{E+\delta} \frac{1}{t-z} (\mathbf{F}\phi\chi)(n,t)\,\mathrm{d}t.\end{aligned}$$

Since $(\mathbf{F}\phi\chi)(n,t)$ is holomorphic at $t = E$, the result follows. $\square$

It remains to establish the decay of both non negligible terms in the last theorem. In what follows t varies in a neighbourhood of a compact interval $T \subset]-1,1[$ not containing 0. Again, our considerations are limited to the component $\mathcal{D}(t)$ of $\Gamma(t)$ enclosed in a fixed cell. We define $\mathcal{S} := \bigcup_{t\in T} \mathcal{D}(t)$.

Theorem 10 *If $f(x)$ is smooth in a neighbourhood of $\mathcal{S}$, then $(\mathbf{F}f)(n,t) = O(r^{-\frac{d-1}{2}})$ uniformly in $\omega \in \mathrm{S}^{d-1}$ and $t \in T$ when $r \to \infty$.*

Proof: By compactness of S^{d-1} it suffices to prove the result for ω in a closed neighbourhood of an arbitrarily fixed ω_0, say, for $|\omega - \omega_0| \le \varepsilon$, where ε is arbitrarily small.

Let $\{\sigma_j(\cdot,t)\}_{j=1}^{2d}$ be the system of local parametrizations described in Section 2.4, where $\sigma_j(u,t) = \mathrm{x}(\gamma_j(u),t)$. One assumes w.l.o.g. that γ_1 parametrizes a hemisphere containing ω_0, and γ_2, the opposite hemisphere. For an open interval $0 \notin I \subset]-1,1[$ containing T, let $U_j = \sigma_j(\mathrm{D}^{d-1}, I)$, so $\{U_j\}_{j=1}^{2d}$ is an open covering of $\mathcal{S}$. Note that $\mathrm{x}(\omega_0,t) \in U_1$ and $\mathrm{x}(-\omega_0,t) \in U_2$. By continuity of x, $\mathrm{x}(\omega,t)$ and $\mathrm{x}(-\omega,t)$ vary in compact sets $K_1 \subset U_1$ and $K_2 \subset U_2$ respectively when t varies in T and ω, in $|\omega-\omega_0| \le \varepsilon$ for a sufficiently small ε.

Consequently, there exists a partition of unity $\{\chi_j\}_{j=1}^{2d}$ subordinate to $\{U_j\}_{j=1}^{2d}$ satisfying $\chi_j|K_j = 1$ for $j = 1, 2$. Restricting the Fourier transform to $\mathcal{D}(t)$, one obtains

$$(\mathbf{F}f)(n,t) = \sum_j \int_{\mathrm{D}^{d-1}} e^{ir\omega\cdot\sigma_j(u,t)} \chi_j(\sigma_j(u,t)) f(\sigma_j(u,t)) \|D_j(u,t)\| \, \mathrm{d}u. \tag{5}$$

Let $\varphi_j(u) = \omega \cdot \sigma_j(u,t)$ be the phase. φ_j is stationary if and only if its gradient is 0, in which case $\omega \cdot \partial_{u_k}\sigma_j(u,t) = 0$ for each $k = 1, \cdots, d-1$, i.e. $\omega \perp \mathcal{D}(t)$. Thus, $\mathrm{x}(\pm\omega, t)$ are the only stationary phase points. Since they lie in K_1 and K_2 respectively, they affect only the first two integrals, separately.

Let us prove that $\mathrm{x}(\omega,t)$ is not degenerate, the proof for $\mathrm{x}(-\omega,t)$ being similar. The parametrization (Υ, τ) defined at the end of Section 2.4 covers a neighbourhood of $\mathrm{x}(\omega,t)$. Thus, there exists a v_0 such that $\mathrm{x}(\omega,t) = \tau(v_0)$. On the other hand, there exists a u_0 such that $\mathrm{x}(\omega,t) = \sigma_1(u_0,t)$. Let $\mathcal{T} = \sigma_1(\cdot,t)^{-1} \circ \tau$, so the relation $\omega \cdot \tau(v) = (\varphi_1 \circ \mathcal{T})(v)$ holds locally. There, letting $u = \mathcal{T}(v)$,

$$\mathrm{D}_v^2(\omega \cdot \tau(v)) = \mathrm{D}_u^2\varphi_1(u)(\mathrm{D}_v\mathcal{T}(v))^2 + \mathrm{D}_u\varphi_1(u)\mathrm{D}_v^2\mathcal{T}(v).$$

Since φ_1 is stationary at u_0, (4) and the previous relation give $\det \mathrm{D}_u^2\varphi_1(u_0) > 0$, as desired.

Proposition 8 then completes the proof. $\square$

The previous theorem takes care of the first term in the decomposition provided by Theorem 9, taking $f = \phi$ and $t = E$. In order to study the second term, let E vary in an arbitrary interval $[a,b] \subset]-1,1[$ not containing zero. One may assume that δ is less than the distance between $[a,b]$ and $\{-1,0,1\}$, and apply the previous proof to $T = [a-\delta, b+\delta]$. The next result is an easy consequence of this proof.

Lemma 11 *Suppose $f(x)$ is smooth in the neighbourhood of $\mathcal{S}$. Then, $\partial_t(\mathbf{F}f)(n,t) = O(r^{-\frac{d-1}{2}})$ uniformly in $\omega \in \mathrm{S}^{d-1}$ and $t \in T$ when $r \to \infty$.*

Proof: From (5), setting $g_j(u,t) = \chi_j(\sigma_j(u,t))f(\sigma_j(u,t))\|D_j(u,t)\|$,

$$\begin{aligned} \partial_t(\mathbf{F}f)(n,t) &= \sum_j \int_{\mathrm{D}^{d-1}} \partial_t(e^{ir\omega\cdot\sigma_j(u,t)} g_j(u,t)) \, \mathrm{d}u \\ &= \sum_j \int_{\mathrm{D}^{d-1}} e^{ir\omega\cdot\sigma_j(u,t)}((ir\omega \cdot \partial_t\sigma_j(u,t))g_j(u,t) + \partial_t g_j(u,t)) \, \mathrm{d}u, \end{aligned}$$

where interchange of the order of differentiation and integration is justified by the dominated convergence theorem. Proposition 8 completes the proof. $\square$

Theorem 12 *Suppose $f(x)$ is smooth in the neighbourhood of $\mathcal{S}$. Then,*

$$\text{p.v.}\int_{|t-E|\leq\delta} \frac{(\mathbf{F}f)(n,t)}{t-E}\,\mathrm{d}t = O(r^{-\frac{d-1}{2}})$$

uniformly in $\omega \in \mathrm{S}^{d-1}$ and $E \in [a,b]$ when $r \to \infty$.

Proof: The result follows from the lemma and the estimate

$$\left|\text{p.v.}\int_{|t-E|\leq\delta} \frac{g(t)}{t-E}\,\mathrm{d}t\right| \leq 2\delta \max_{|t-E|\leq\delta} |g'(t)|,$$

which holds for any continuously differentiable g. □

Joining Theorems 4, 9, 10, and 12, we have proven Theorem 1.

Remark: *Using the same technique one may derive the complete asymptotic expansion of the Green function, $G_{E+i0}(r\omega) \sim r^{-\frac{d-1}{2}} \sum_{k=0}^{\infty} \mathrm{a}_k(E,\omega) r^{-\frac{k}{2}}$, where the a_k's are smooth and the estimates are uniform in ω and E, see [P].*

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