Ergodicty for a class of Markov processes and applications to randomly forced PDE's. I

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Dedicated to Leonid Romanovich Volevich on the occasion of his seventieth birthday

Abstract

The paper is devoted to studying the problem of ergodicity for dissipative PDE's perturbed by an external random force. We give a simple sufficient condition for uniqueness and stability of a stationary measure for a family of Markov processes. The result obtained is applied to the 2D NS system.

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Contents

0 Introduction

The problem of ergodicity for randomly forced PDE's was in the focus of attention during the last ten years. The first result in this direction was obtained by Flandoli and Maslowski [FM95], who studied the 2D Navier–Stokes (NS) system perturbed by a random force that is white in time and sufficiently irregular in the space variables x. They proved that the Markov family associated with the above problem has a unique stationary measure, and all other solutions converge to it in the total variation norm. The method used in [FM95] is based on the strong Feller property, which does not hold for smooth in x forcing. That case was first studied by Kuksin and the author [KS00]. We considered a large class of parabolic PDE's (including the 2D NS system and the complex Ginzburg–Landau equation) perturbed by a random force discrete in time and smooth in x. Under the condition that the perturbation is sufficiently non-degenerate, it was proved that there is a unique stationary measure, which satisfies a mixing property. E, Mattingly, Sinai [EMS01] and Bricmont, Kupiainen, Lefevere [BKL02] studied later the NS system on the 2D torus. Assuming that the right-hand side is white noise in time and trigonometric polynomial in x of sufficiently high degree, they established the uniqueness of stationary measure. Moreover, it was proved in [BKL02] that the model in question possesses the property of exponential mixing. Eckmann and Hairer [EH01] constructed an infinite-dimensional version of the Malliavin calculus and established the ergodicity for the real Ginzburg–Landau equation perturbed by a rough degenerate forcing. The above results were further developed in a number of works, and we refer the reader to the reviews [Bri02, Kuk02, Shi03] for more references and details.

The aim of this paper is to give a simple sufficient condition for ergodicity of Markov processes and to show that it applies to a class of randomly forced PDE's. Without going into details, we now outline the main idea, which is a variation of a classical coupling construction (for instance, see [Lin92, Ver87]). The scheme given below is not entirely accurate; however, it contains the essential points of the proof.

Let X be a separable Banach space with a norm $\|\cdot\|_X$ and let $(u_t, \mathbb{P}_u)$, $t \ge 0$, $u \in X$, be a Feller family of Markov processes in X. Here u_t is the trajectory of the process and $\mathbb{P}_u$ stands for the probability measure corresponding to the initial condition $u \in X$. Let $(u_t', \mathbb{P}_{u'})$ be another (independent) copy of the above family. Suppose that we can prove the following assertions.

- The stopping time $\tau_\varepsilon = \min\{t \ge 0 : \|u_t - u_t'\|_X \le \varepsilon\}$ is almost surely finite for any initial points $u, u' \in X$ and any $\varepsilon > 0$.
- There is a function $\delta(\varepsilon) \ge 0$ defined for $\varepsilon > 0$ and going to zero with ε such that, for any $u, u' \in X$ satisfying the inequality $\|u - u'\|_X \le \varepsilon$, we have
$$\big\|\mathbb{P}_u\{u_t \in \cdot\} - \mathbb{P}_{u'}\{u_t' \in \cdot\}\big\|_{\mathcal{L}}^* \le \delta(\varepsilon) \quad \text{for all } t \ge 0,$$
where $\|\cdot\|_{\mathcal{L}}^*$ stands for the dual Lipschitz norm on the space of probability measures on X (see Notation) and $\mathbb{P}\{\xi \in \cdot\}$ denotes the distribution of a random variable ξ under the law $\mathbb{P}$.

If these properties hold, then using the the strong Markov property one can show that

$$\big\|\mathbb{P}_u\{u_t \in \cdot\} - \mathbb{P}_{u'}\{u'_t \in \cdot\}\big\|_{\mathcal{L}}^* \to 0 \quad \text{as } t \to \infty \text{ for any } u, u' \in X.$$

This implies the ergodicity of the Markov family under study.

In the present paper, we show that the above idea applies to the 2D NS system (see Section 1.2 for the exact formulation of the abstract result). Let us note that the idea of coupling was used earlier in [KS01, KPS02, Mat02, MY02, Hai02, KS02] and enabled one to prove the property of exponential mixing for various randomly forced PDE's, including the 2D NS system perturbed by white noise force. Our result does not give an estimate for the rate of mixing. However, the proof presented here is much shorter, and we deal with a more general case in which the attractor of the unperturbed problem is non-trivial and the random force has a minimal regularity with respect to the space variables (see Remarks 2.4 and 2.5). In the second part of this work, we shall show that the same abstract result applies to other randomly forced PDE's.

The paper is organised as follows. Section 1 is devoted to an abstract theorem that gives a sufficient condition for ergodicity of Markov processes. In Section 2, we first recall a well-known result on the initial-boundary value problem for 2D NS equations and then formulate the main theorem on uniqueness and mixing property. The proofs are given in Section 3.

Notation

Let X be a separable Banach space with a norm $\|\cdot\|_X$. We shall use the following notation:
$B_X(R)$ is the ball in X of radius R centred at zero;
$\mathcal{B}(X)$ is the Borel σ-algebra of X;
$\mathcal{P}(X)$ is the family of probability measures on $(X, \mathcal{B}(X))$;
$C_b(X)$ is the space of bounded continuous functions $f : X \to \mathbb{R}$; it is endowed with the norm

$$\|f\|_\infty := \sup_{u \in X} |f(u)|.$$

If $f \in C_b(X)$ and $\mu \in \mathcal{P}(X)$, then we denote by (f, μ) the integral of f over X with respect to μ.
$\mathcal{L}(H)$ is the space of functions $f \in C_b(H)$ such that

$$\|f\|_{\mathcal{L}} := \|f\|_\infty + \sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|_X}.$$

The space of probability measures $\mathcal{P}(X)$ is endowed with the weak* topology, which is equivalent to the topology defined by the dual Lipschitz norm:

$$\|\mu_1 - \mu_2\|_{\mathcal{L}}^* := \sup_{\|f\|_{\mathcal{L}} \le 1} \big|(f, \mu_1) - (f, \mu_2)\big|.$$

We shall also need the total variation norm, which is defined by the formula

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} := \sup_{\Gamma\in\mathcal{B}(X)} |\mu_1(\Gamma) - \mu_2(\Gamma)|.$$

If ξ is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then we denote by $\mathcal{D}(\xi)$ or $\mathbb{P}\{\xi \in \cdot\}$ the distribution of ξ.

1 Main theorem

1.1 Preliminaries

Let $(u_t, \mathbb{P}_u)$, $t \geq 0$, be a Feller family of Markov processes defined on a measurable space $(\Omega, \mathcal{F})$ with range in a separable Banach space X. We shall always assume that the trajectories $u_t(\omega)$ are continuous for all $\omega \in \Omega$. Let $P(t, u, \Gamma)$ be the transition function associated with $(u_t, \mathbb{P}_u)$,

$$P(t, u, \Gamma) = \mathbb{P}_u\{u_t \in \Gamma\}, \quad u \in X, \quad \Gamma \in \mathcal{B}(X),$$

and let P_t and P_t^* be the corresponding Markov operators,

$$P_t f(u) = \int_X P(t, u, dv) f(v), \quad P_t^* \mu(\Gamma) = \int_X P(t, v, \Gamma)\mu(dv),$$

where $f \in C_b(X)$, $\mu \in \mathcal{P}(X)$, and $\Gamma \in \mathcal{B}(X)$. In what follows, we shall need a concept of extension for Markov processes.

Let $\boldsymbol{X} = X \times X$ be the direct product of two copies of X. We write $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ and denote by

$$\Pi\colon \boldsymbol{u} \mapsto u, \quad \Pi'\colon \boldsymbol{u} \mapsto u'$$

the natural projections from $\boldsymbol{X}$ to its components. Let $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$, $t \geq 0$, be a Feller family of Markov processes in $\boldsymbol{X}$ whose trajectories are continuous in time for all values of the random parameter. We denote by $\boldsymbol{P}(t, \boldsymbol{u}, \boldsymbol{\Gamma})$ the transition function of the family $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ and by $\boldsymbol{P}_t$ and $\boldsymbol{P}_t^*$ the corresponding Markov operators.

Definition 1.1. The Markov family $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ is called an *extension of* $(u_t, \mathbb{P}_u)$ if, for all $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ and $t \geq 0$, we have

$$\Pi_* \boldsymbol{P}(t, \boldsymbol{u}, \cdot) = P(t, u, \cdot), \quad \Pi'_* \boldsymbol{P}(t, \boldsymbol{u}, \cdot) = P(t, u', \cdot), \tag{1.1}$$

where $\Pi_*\mu$ and $\Pi'_*\mu$ stand for the images of a measure μ under the projections Π and Π', respectively.

Let us consider a simple example of extension that will be used in Section 3. We denote by $(\Omega, \mathcal{F})$ the measurable space on which the family $(u_t, \mathbb{P}_u)$ is defined and by $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ the direct product of two copies $(\Omega, \mathcal{F})$:

$$\boldsymbol{\Omega} = \Omega \times \Omega, \quad \boldsymbol{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}.$$

The points of $\boldsymbol{\Omega}$ will be denoted by $\boldsymbol{\omega} = (\omega, \omega')$. We now define a process $\boldsymbol{u}_t(\boldsymbol{\omega})$ in the space $\boldsymbol{X}$ by the formula

$$\boldsymbol{u}_t(\boldsymbol{\omega}) = (u_t(\omega), u_t(\omega'))$$

and, for any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$, denote by $\mathbf{P}_{\boldsymbol{u}}$ the unique probability measure on $\boldsymbol{X}$ such that

$$\mathbf{P}_{\boldsymbol{u}}(\Gamma \times \Gamma') = \mathbb{P}_u(\Gamma)\, \mathbb{P}_{u'}(\Gamma') \quad \text{for all } \Gamma, \Gamma' \in \mathcal{B}(X).$$

It is a matter of direct verification to show that the family $(\boldsymbol{u}_t, \mathbf{P}_{\boldsymbol{u}})$ is an extension of $(u_t, \mathbb{P}_u)$. Furthermore, the transition functions satisfy the following relation for any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ and $\Gamma, \Gamma' \in \mathcal{B}(X)$:

$$\boldsymbol{P}(t, \boldsymbol{u}, \Gamma \times \Gamma') = P(t, u, \Gamma)\, P(t, u', \Gamma'). \tag{1.2}$$

In what follows, the extension constructed above will be called *a pair of independent copies of* $(u_t, \mathbb{P}_u)$.

1.2 Formulation of the result

Let $(u_t, \mathbb{P}_u)$, $t \geq 0$, be a Feller family of Markov processes in a separable Banach space X and let $(\boldsymbol{u}_t, \mathbf{P}_{\boldsymbol{u}})$ be an arbitrary extension of $(u_t, \mathbb{P}_u)$ in $\boldsymbol{X} = X \times X$. As before, we assume that u_t and $\boldsymbol{u}_t$ are continuous with respect to t for all values of the random parameters. We shall denote by $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ the measurable space on which the extension is defined and shall write $\boldsymbol{u}_t = (u_t, u_t')$. If $G \subset X$ is a closed subset, then we denote by $\boldsymbol{\tau}(G)$ the first hitting time of $G \times G$ for $\boldsymbol{u}_t$:

$$\boldsymbol{\tau}(G) = \min\{t \geq 0 : u_t \in G, u_t' \in G\}.$$

Suppose that for any integer $m \geq 1$ there is a closed subset $G_m \subset X$ such that the following two properties are satisfied.

($\mathbf{P_1}$) For any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ and $m \geq 1$, we have

$$\mathbf{P}_{\boldsymbol{u}}\{\boldsymbol{\tau}(G_m) < \infty\} = 1.$$

($\mathbf{P_2}$) There is a sequence $\delta_m > 0$ going to zero as $m \to \infty$ such that, for any $\boldsymbol{u} \in G_m \times G_m$, we have

$$\sup_{t \geq 0} \big\| P(t, u, \cdot) - P(t, u', \cdot) \big\|_{\mathcal{L}}^* \leq \delta_m.$$

Theorem 1.2. *Suppose that conditions* (P_1) *and* (P_2) *are satisfied for a sequence of closed subsets $G_m \subset X$. Then, for any $u, u' \in X$, we have*

$$\big\| P(t, u, \cdot) - P(t, u', \cdot) \big\|_{\mathcal{L}}^* \to 0 \quad \textit{as } t \to \infty. \tag{1.3}$$

A proof of this result is given in the next subsection. We now establish a simple corollary from it. Recall that a measure $\mu \in \mathcal{P}(X)$ is said to be *stationary* for the family $(u_t, \mathbb{P}_u)$ if $P_t^*\mu = \mu$ for all $t \geq 0$.

Corollary 1.3. *Under the conditions of Theorem 1.2, if μ is a stationary measure for the family $(u_t, \mathbb{P}_u)$, then it is unique, and for any $\lambda \in \mathcal{P}(X)$ we have*

$$\left\| P_t^*\lambda - \mu \right\|_{\mathcal{L}}^* \to 0 \quad \textit{as } t \to \infty. \tag{1.4}$$

Proof. The uniqueness is a straightforward consequence of (1.4). To prove (1.4), we take any function $f \in \mathcal{L}(H)$ and write

$$\begin{aligned} |(f, P_t^*\lambda - \mu)| &= |(f, P_t^*\lambda - P_t^*\mu)| \\ &= \left| \iint_{X\times X} \big(P_t f(u) - P_t f(u')\big)\, \lambda(du)\, \mu(du') \right| \\ &\leq \iint_{X\times X} \left| P_t f(u) - P_t f(u') \right| \lambda(du)\, \mu(du'). \end{aligned}$$

Taking into account (1.3) and using the Lebesgue theorem on dominated convergence, we see that

$$(f, P_t^*\lambda - \mu) \to 0 \quad \text{as } t \to +\infty \text{ for any } f \in \mathcal{L}(H).$$

In view of Theorem 11.3.3 in [Dud02], this is equivalent to (1.4). ☐

1.3 Proof of Theorem 1.2

Let us set

$$\boldsymbol{\tau}(m,t) = \boldsymbol{\tau}(G_m) \wedge t, \quad p(\boldsymbol{u}, m, t) = \mathbb{P}_{\boldsymbol{u}}\{\boldsymbol{\tau}(G_m) > t\}.$$

Applying the strong Markov property (SMP) to the family $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ and taking into account the first relation in (1.1), for any integer $m \geq 1$ we write

$$\begin{aligned} P(t, u, \Gamma) = \boldsymbol{P}(t, \boldsymbol{u}, \Gamma \times X) &= \mathbb{E}_{\boldsymbol{u}} \boldsymbol{P}(t - \boldsymbol{\tau}(m,t), \boldsymbol{u}_{\tau(m,t)}, \Gamma \times X) \\ &= \mathbb{E}_{\boldsymbol{u}} P(t - \boldsymbol{\tau}(m,t), u_{\tau(m,t)}, \Gamma). \end{aligned}$$

A similar relation holds for $P(t, u', \Gamma)$. It follows that

$$\left\| P(t, u, \cdot) - P(t, u', \cdot) \right\|_{\mathcal{L}}^* \leq \mathbb{E}_{\boldsymbol{u}} g_m(t - \boldsymbol{\tau}(m,t), \boldsymbol{u}_{\tau(m,t)}), \tag{1.5}$$

where for $s \geq 0$ and $\boldsymbol{z} = (z, z') \in \boldsymbol{X}$, we set

$$g_m(s, \boldsymbol{z}) = \left\| P(t, z, \cdot) - P(t, z', \cdot) \right\|_{\mathcal{L}}^*.$$

Now note that, by hypothesis (P_2), we have

$$\sup_{s\geq 0} g_m(s, \boldsymbol{z}) \leq \delta_m \quad \text{for } \boldsymbol{z} \in G_m \times G_m. \tag{1.6}$$

On the other hand, the definition of $\boldsymbol{\tau}(m,t)$ and $p(\boldsymbol{u},m,t)$ implies that

$$\mathbb{P}_{\boldsymbol{u}}\big\{\boldsymbol{u}_{\boldsymbol{\tau}(m,t)} \notin G_m \times G_m\big\} \le p(\boldsymbol{u},m,t). \tag{1.7}$$

Combining (1.5) – (1.7), we see that

$$\big\|P(t,u,\cdot) - P(t,u',\cdot)\big\|_{\mathcal{L}}^{*} \le \delta_m + p(\boldsymbol{u},m,t).$$

In view of hypotheses (P_1) and (P_2), the right-hand side of this inequality can be made arbitrarily small by choosing first m and then t sufficiently large. This completes the proof of Theorem 1.2.

2 Mixing for the two-dimensional Navier–Stokes system

2.1 Initial-boundary value problem

Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary ∂D. We shall denote by $\boldsymbol{n}$ the outward unit normal to ∂D. For any $s \in \mathbb{R}$, let $H^s(D,\mathbb{R}^2)$ be the space of vector functions (u_1,u_2) on D whose components belong to the Sobolev space of order s. In the case $s=0$, we shall write $L^2(D,\mathbb{R}^2)$.

Let us consider the 2D Navier–Stokes system in D. After projecting to the space of divergence-free square integrable vector fields on D whose normal component vanishes on ∂D, we obtain the following evolution equation:

$$\dot{u} + \nu L u + B(u,u) = h + \eta(t). \tag{2.1}$$

Here $\nu > 0$ is the viscosity, $L = -\Pi\Delta$ and $B(u,v) = \Pi(u,\nabla)v$, where Π is the orthogonal projection from $L^2(D,\mathbb{R}^2)$ to the subspace

$$H = \Big\{u \in L^2(D,\mathbb{R}^2) : \operatorname{div} u = 0 \text{ in } D, (u,\boldsymbol{n})\big|_{\partial D} = 0\Big\}.$$

Concerning the right-hand side of (2.1), we assume that $h \in H$ and η is a random process of the form

$$\eta(t,x) = \frac{\partial}{\partial t}\zeta(t,x), \quad \zeta(t,x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x), \tag{2.2}$$

where $\{e_j\}$ is a complete set of normalised eigenfunctions of the operator L supplemented by the Dirichlet boundary condition, $\{\beta_j\}$ is a sequence of independent standard Brownian motions defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and $b_j \ge 0$ are some constants such that

$$B_0 := \sum_{j=1}^{\infty} b_j^2 < \infty. \tag{2.3}$$

Roughly speaking, this means that the function $\zeta(t)$ belongs to H for almost all values of the random parameter.

Let us consider the Cauchy problem for (2.1), (2.2):

$$u(0) = u_0, \tag{2.4}$$

where $u_0 = u_0(x)$ is an H-valued random variable on Ω independent of $\{\beta_j\}$. We first recall the concept of solution for (2.1), (2.4).

For a closed interval $I \subset \mathbb{R}$ and a Banach space X, we denote by $C(I, X)$ the space of continuous functions $f : I \to X$ and by $L^2_{\text{loc}}(I, X)$ the space of Bochner-measurable functions $f : I \to X$ such that $\int_J \|f(t)\|_X^2 \, dt < \infty$ for any compact interval $J \subset I$.

Definition 2.1. An H-valued random process $u(t, x)$, $t \geq 0$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *strong solution* of the problem (2.1), (2.2), (2.4) if it is progressively measurable with respect to the filtration generated by u_0 and $\{\beta_j\}$ and satisfies the two properties below:

(i) Almost every realization of $u(t, x)$ belongs to the space

$$\mathcal{X} := C(\mathbb{R}_+, H) \cap L^2_{\text{loc}}(\mathbb{R}_+, V),$$

where $\mathbb{R}_+ = [0, +\infty)$ and $V = H \cap H^1(D, \mathbb{R}^2)$.

(ii) Almost every realization of $u(t, x)$ satisfies the relation

$$u(t) + \int_0^t (\nu L u + B(u, u) - h) \, ds = u_0 + \zeta(t), \quad t \geq 0,$$

where the left- and right-hand sides are regarded as elements of $H^{-1}(D, \mathbb{R}^2)$.

The following result on existence and uniqueness of solution for the Cauchy problem for the 2D Navier–Stokes system is well known (see [VF88, Fla94]):

Proposition 2.2. *Suppose that $h \in H$ and condition (2.3) holds. Then for any H-valued random variable u_0 that is independent of $\{\beta_j\}$ there is a unique solution of the problem (2.1), (2.2), (2.4).*

2.2 Uniqueness of stationary measure and mixing

It is well known that the solutions of (2.1), (2.2), (2.4) corresponding to all possible deterministic initial functions $u_0 \in H$ form a Feller family of Markov processes in H (see [VF88, Fla94]). We denote this family by $(u_t, \mathbb{P}_u)$, where u_t stands for the trajectory of the process and $\mathbb{P}_u$ is the probability measure corresponding to the initial condition $u \in H$. Let $P_t(u, \Gamma) = \mathbb{P}_u\{u_t \in \Gamma\}$ be the transition function associated with the family $(u_t, \mathbb{P}_u)$ and let $P_t : C_b(H) \to C_b(H)$ and $P_t^* : \mathcal{P}(H) \to \mathcal{P}(H)$ be the corresponding Markov operators. The following theorem is the main result of this paper:

Theorem 2.3. *Suppose that $h \in H$, condition (2.3) is satisfied, and*

$$b_j \neq 0 \quad \text{for all } j \geq 1. \tag{2.5}$$

Then for any $\nu > 0$ the family $(u_t, \mathbb{P}_u)$ associated with the problem (2.1), (2.2) has a unique stationary measure $\mu \in \mathcal{P}(H)$. Moreover, the measure μ is mixing in the sense that convergence (1.4) holds for any initial measure $\lambda \in \mathcal{P}(H)$.

A proof of this theorem is given in the next section. Here we make two remarks.

Remark 2.4. In the case of periodic boundary conditions, it is possible to establish the same result under more general assumptions. Namely, one can replace (2.5) with the requirement that sufficiently many coefficients b_j are non-zero. Indeed, the fact that all b_j are non-zero is used to show that the probability of transition from any initial point to an arbitrary open ball is positive. The proof of this assertion is based on the approximate controllability of Eq. (2.1) (see Lemmas 3.1 and 3.2). It was established recently by Agrachev and Sarychev [AS04] that the NS system on the 2D torus possesses the property of approximate controllability for any positive value of the viscosity, provided that sufficiently many Fourier modes are controlled.

Remark 2.5. As was mentioned in the Introduction, there are many papers devoted to the problem of ergodicity for the NS system in the case when the right-hand side is white noise in time and smooth in x (see [EMS01, BKL02, Mat02, KS02]). In all of them, it was assumed that

$$h \equiv 0, \quad \sum_{j=1}^{\infty} \alpha_j b_j^2 < \infty,$$

where α_j is the eigenvalue of L corresponding to the eigenfunction e_j (cf. (2.3)). The first of the above conditions implies that the dynamics of the unperturbed equation is trivial (all solutions converge to zero exponentially fast), and the second ensures that random force $\eta(t)$ belongs to the space V almost surely. Thus, Theorem 2.3 gives more general conditions for ergodicity.

3 Proofs

3.1 Proof of Theorem 2.3

Existence of a stationary measure is well known (see [VF88, Fla94]). To prove the uniqueness and mixing, we first note that the claim of the theorem concerns only the Markov semigroups corresponding to the family $(u_t, \mathbb{P}_u)$. The properties of these semigroups do not depend on the probability space on which the Markov family is defined. Therefore, if we prove Theorem 2.3 for one particular choice of the probability space, it will remain true in the general case as well.

From now on, we make the following assumptions on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

- Ω coincides with the Fréchet[1] space $C_0(\mathbb{R}_+, H)$ of continuous functions from $\mathbb{R}_+$ to H vanishing at $t = 0$;
- $\mathcal{F}$ is the Borel σ-algebra of Ω;
- $\mathbb{P}$ is a coloured Wiener process on Ω with a covariance operator $K : H \to H$ defined by the relation $Ke_j = b_j^2 e_j$ for all $j \geq 1$; in other words, $\mathbb{P}$ is the distribution in $C_0(\mathbb{R}_+, H)$ of the process $\zeta(t, x)$ defined in (2.2).

We shall assume that the random process $\zeta(t) = \zeta_\omega(t)$ entering the right-hand side of (2.1) is defined on the above probability space and has the form

$$\zeta_\omega(t) = \omega_t \quad \text{for } \omega \in \Omega.$$

To prove the theorem, it suffices to show that the family $(u_t, \mathbb{P}_u)$ associated with the problem (2.1), (2.2) possesses an extension $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ satisfying properties (P_1) and (P_2) introduced in Section 1.2 (see Theorem 1.2 and Corollary 1.3). We shall show that a pair of independent copies of $(u_t, \mathbb{P}_u)$ (see the end of Section 1.1) satisfies (P_1) and (P_2) with $G_m = B_H(1/m)$. The proof of this fact is divided into several steps. In what follows, we assume that $\nu = 1$; the proof in the general case is similar.

Step 1. We first show that Condition (P_1) is satisfied. Namely, we claim that for any $r > 0$ there are positive constants C and γ such that

$$\mathbb{E}_{\boldsymbol{u}} \exp(\gamma \boldsymbol{\tau}_r) \leq C(1 + \|\boldsymbol{u}\|^2) \quad \text{for any } \boldsymbol{u} = (u, u') \in \boldsymbol{H}, \tag{3.1}$$

where $\boldsymbol{\tau}_r = \boldsymbol{\tau}(B_H(r))$ denotes the first hitting time of $\boldsymbol{B}_H(r) := B_H(r) \times B_H(r)$ for the process $\boldsymbol{u}_t$ and $\|\boldsymbol{u}\|^2 = \|u\|^2 + \|u'\|^2$.

A well-known argument based on stopping time techniques[2] shows that (3.1) will be established if we prove the following two assertions:

(i) There are positive constants R, K, and α such that

$$\mathbb{E}_{\boldsymbol{u}} \exp(\alpha \boldsymbol{\sigma}_R) \leq 1 + K\|\boldsymbol{u}\|^2 \quad \text{for any } \boldsymbol{u} = (u, u') \in \boldsymbol{H}, \tag{3.2}$$

where $\boldsymbol{\sigma}_R$ denotes the first hitting time of $\boldsymbol{B}_V(R) := B_V(R) \times B_V(R)$ for the process $\boldsymbol{u}_t$.

(ii) For any $T > 0$ and $R > r > 0$ there is $p > 0$ such that

$$\boldsymbol{P}(T, \boldsymbol{u}, \boldsymbol{B}_H(r)) \geq p \quad \text{for any } \boldsymbol{u} \in \boldsymbol{B}_V(R). \tag{3.3}$$

We shall see that the particular choice of the underlying probability space will not play any role in the proof of these assertions.

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[1]The space $C_0(\mathbb{R}_+, H)$ is endowed with the topology of uniform convergence on compact intervals of $\mathbb{R}_+$.

[2]For instance, see Sections III.7 and IV.2 in [Has80], Section 13 in [Ver00] or Proposition 2.3 in [Shi04].

Step 2. Let us prove (i). The definition of the process $\boldsymbol{u}_t = (u_t, u_t')$ implies that its components satisfy the stochastic equations

$$\partial_t u_t + Lu_t + B(u_t, u_t) = h + \eta, \quad \partial_t u_t' + Lu_t' + B(u_t', u_t') = h + \eta', \tag{3.4}$$

where $\eta = \partial_t \zeta$ and $\eta' = \partial_t \zeta'$ are (independent) processes of the form (2.2) defined on the same probability space. Let us apply the Itô formula to the functional $V(t, \boldsymbol{u}) = e^{\alpha_1 t}\|\boldsymbol{u}\|^2$, where α_1 is the first eigenvalue of L. Setting $V_1(t, \boldsymbol{u}) = e^{\alpha_1 t}\|\boldsymbol{u}\|_1^2$, where $\|\boldsymbol{u}\|_1^2 = \|u\|_1^2 + \|u'\|_1^2$, and carrying out some simple transformations, we obtain[3]

$$\begin{aligned} dV(t, \boldsymbol{u}_t) = \big\{\alpha_1 V(t, \boldsymbol{u}_t) - 2V_1(t, \boldsymbol{u}_t) + 2e^{\alpha_1 t}\big(B_0 + (u_t, h) + (u_t', h)\big)\big\}\, dt \\ + 2e^{\alpha_1 t}\big\{(u, d\zeta) + (u', d\zeta')\big\}. \end{aligned} \tag{3.5}$$

Taking any constants $T > 0$ and $R > 0$, integrating (3.5) with respect to $t \in (0, T \wedge \boldsymbol{\sigma}_R)$, and taking the mean value, we derive

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{u}} V(T \wedge \boldsymbol{\sigma}_R, \boldsymbol{u}(T \wedge \boldsymbol{\sigma}_R)) = \|\boldsymbol{u}\|^2 \\ &+ \mathbb{E}_{\boldsymbol{u}} \int_0^{T\wedge\boldsymbol{\sigma}_R} \big\{\alpha_1 V(t, \boldsymbol{u}_t) - 2V_1(t, \boldsymbol{u}_t) + 2e^{\alpha_1 t}\big(B_0 + (u_t, h) + (u_t', h)\big)\big\}\, dt. \end{aligned} \tag{3.6}$$

To estimate the integral on the right-hand side of (3.6), we note that

$$V_1(t, \boldsymbol{u}_t) \geq \alpha_1 V(t, \boldsymbol{u}_t), \quad \big|(u_t, h) + (u_t', h)\big| \leq \tfrac{\alpha_1}{4}\|\boldsymbol{u}_t\|^2 + \tfrac{2}{\alpha_1}\|h\|^2.$$

Substituting these inequalities into (3.6), we obtain

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{u}} V(T \wedge \boldsymbol{\sigma}_R, \boldsymbol{u}(T \wedge \boldsymbol{\sigma}_R)) \leq \|\boldsymbol{u}\|^2 \\ &\qquad + \mathbb{E}_{\boldsymbol{u}} \int_0^{T\wedge\boldsymbol{\sigma}_R} \big\{-\tfrac{1}{2}V_1(t, \boldsymbol{u}_t) + 2e^{\alpha_1 t}\big(B_0 + 2\alpha_1^{-1}\|h\|^2\big)\big\}\, dt. \end{aligned} \tag{3.7}$$

We now assume that $R^2 \geq 8(B_0 + 2\alpha_1^{-1}\|h\|^2)$. Then the definition of $\boldsymbol{\sigma}_R$ implies that

$$V_1(t, \boldsymbol{u}_t) \geq R^2 e^{\alpha_1 t} \geq 8e^{\alpha_1 t}(B_0 + 2\alpha_1^{-1}\|h\|^2) \quad \text{for } 0 \leq t < \boldsymbol{\sigma}_R. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain the inequality

$$\mathbb{E}_{\boldsymbol{u}} \left(\frac{R^2}{4} \int_0^{T\wedge\boldsymbol{\sigma}_R} e^{\alpha_1 t} dt\right) \leq \|\boldsymbol{u}\|^2,$$

whence it follows that

$$\mathbb{E}_{\boldsymbol{u}} \exp\{\alpha_1(T \wedge \boldsymbol{\sigma}_R)\} \leq 1 + 4\alpha_1 R^{-2}\|\boldsymbol{u}\|^2.$$

[3]We confine ourselves to formal calculations; their justification can be obtained by usual arguments (for instance, see [Par79, KR77, Shi02]).

Passing to the limit as $T \to \infty$ and using the Fatou lemma, we arrive at (3.2) with $\alpha = \alpha_1$ and $K = 4\alpha_1 R^{-2}$.

Step 3. We now prove assertion (ii) (see Step 1). Let us fix arbitrary $T > 0$ and $R > r > 0$. In view of (1.2), it suffices to show that

$$P(T, u, B_H(r)) \geq \sqrt{p} \quad \text{for any } u \in B_V(R).$$

In other words, we need to show that

$$\mathbb{P}\{\|u(T; u_0)\| \leq r\} \geq \sqrt{p} \quad \text{for any } u_0 \in B_V(R), \tag{3.9}$$

where $u(t; u_0)$ stands for the solution of the problem (2.1), (2.4).

To prove (3.9), we shall use a modification of an argument in [FM95, Section 5]. We seek $u(t; u_0)$ in the form $u = z + v$, where z is the solution of the linear problem

$$\dot{z} + Lz = \eta(t), \quad z(0) = 0. \tag{3.10}$$

Thus, z is the Ornstein–Uhlenbeck process defined by the formula

$$z(t) = \int_0^t e^{-(t-s)L} d\zeta(s). \tag{3.11}$$

It is a matter of simple calculation to show that the unknown function v must be a solution of the problem

$$\dot{v} + Lv + B(v + z, v + z) = h, \quad v(0) = u_0. \tag{3.12}$$

Let us define the Banach spaces

$$\begin{aligned}
\mathcal{X}_T &:= C(0, T; H) \cap L^2(0, T; V), \\
\dot{\mathcal{X}}_T &:= \{u \in \mathcal{X}_T : u(0) = 0\}, \\
\mathcal{Y}_T &:= \{u \in \mathcal{X}_T : \partial_t u \in L^2(0, T; V^*)\}
\end{aligned}$$

and endow them with the natural norms. Standard methods similar to those used in the theory of 2D Navier–Stokes equations (see [Lio69, CF88]) enable one to show that for any $u_0 \in H$ and $z \in \mathcal{X}_T$ the problem (3.12) has a unique solution $v \in \mathcal{Y}_T$. Moreover, the operator $F : H \times \mathcal{X}_T \to \mathcal{Y}_T$ that takes (u_0, z) to v is uniformly Lipschitz continuous on bounded subsets of $H \times \mathcal{X}_T$. We denote by $F_t(u_0, z)$ the restriction of $F(u_0, z)$ to the time t. We shall need the following lemma, whose proof is given in the next subsection.

Lemma 3.1. *For any $T > 0$ there is a continuous operator $\mathcal{Z} : H \to \dot{\mathcal{X}}_T$ such that the unique solution $v \equiv F(u_0, z) \in \mathcal{Y}_T$ of the problem* (3.12) *with $z = \mathcal{Z}(u_0)$ satisfies the relation*

$$\big(v + \mathcal{Z}(u_0)\big)\big|_{t=T} = 0. \tag{3.13}$$

Moreover, the operator $\mathcal{Z}$ is uniformly Lipschitz on bounded subsets of H.

Let us choose a constant $\rho > 0$ so large that the image of $B_V(R)$ under $\mathcal{Z}$ is contained in the ball $B_{\dot{\mathcal{X}}_T}(\rho)$. Since F_T is uniformly Lipschitz on bounded subsets, there is $C \geq 1$ such that, for any $u_0 \in B_V(R)$ and $z_0, z_1 \in B_{\dot{\mathcal{X}}_T}(\rho)$, we have

$$\|F_T(u_0, z_0) - F_T(u_0, z_1)\| \leq C\,\|z_0 - z_1\|_{\dot{\mathcal{X}}_T}.$$

In particular, taking $z_1 = \mathcal{Z}(u_0)$ with $u_0 \in B_V(R)$, we see that

$$\|F_T(u_0, z_0) - F_T(u_0, \mathcal{Z}(u_0))\| \leq \tfrac{r}{2} \quad \text{for } \|z_0 - \mathcal{Z}(u_0)\|_{\dot{\mathcal{X}}_T} \leq \tfrac{r}{2C}. \tag{3.14}$$

Recalling that $u = z + v$, where z is the Ornstein–Uhlenbeck process defined by (3.11), and using (3.13), we see that

$$\begin{aligned}\|u(T, u_0)\| &\leq \|z(T) + F_T(u_0, z)\| \\ &\leq \|z - \mathcal{Z}(u_0)\|_{\dot{\mathcal{X}}_T} + \|F_T(u_0, z) - F_T(u_0, \mathcal{Z}(u_0))\|.\end{aligned}$$

Combining this inequality with (3.14), we conclude that

$$\mathbb{P}\{\|u(T, u_0)\| \leq r\} \geq \mathbb{P}\{\|z - \mathcal{Z}(u_0)\|_{\dot{\mathcal{X}}_T} < \tfrac{r}{2C}\} \quad \text{for any } u_0 \in B_V(R). \tag{3.15}$$

We wish to estimate the right-hand side of (3.15) from below. To this end, we need the following lemma whose proof is given in the next subsection.

Lemma 3.2. *Let $\mathcal{O}$ and $\mathcal{K}$ be, respectively, open and compact sets in $\dot{\mathcal{X}}_T$. If conditions* (2.3) *and* (2.5) *are fulfilled, then there is $\beta > 0$ such that process* (3.11) *satisfies the inequality*

$$\inf_{\hat{z} \in \mathcal{K}} \mathbb{P}\{z - \hat{z} \in \mathcal{O}\} \geq \beta.$$

Let us set $\mathcal{K} = \mathcal{Z}(B_V(R))$ and denote by $\mathcal{O} \subset \dot{\mathcal{X}}_T$ the open ball of radius $\frac{r}{2C}$ centred at zero. Since $B_V(R)$ is compact in H, its image under the continuous operator $\mathcal{Z}$ is also compact, and we can apply Lemma 3.2. We conclude that

$$\mathbb{P}\{\|z - \hat{z}\|_{\dot{\mathcal{X}}_T} < \tfrac{r}{2C}\} \geq \beta \quad \text{for any } \hat{z} \in \mathcal{Z}(B_V(R)).$$

Combining this with (3.15), we arrive at (3.9).

Step 4. We now show that condition (P_2) is also fulfilled. To this end, we first explain the main idea. Let us fix a small constant $\varepsilon > 0$ and two initial conditions $u, u' \in B_H(\varepsilon)$ and denote by $u_t(\boldsymbol{\omega})$ and $u'_t(\boldsymbol{\omega})$ the corresponding solutions of (3.4). Suppose that we have constructed two H-valued random processes $\hat{u}_t(\boldsymbol{\omega})$ and $\hat{u}'_t(\boldsymbol{\omega})$ with the same distribution and two measurable transformations

$$\Phi : \boldsymbol{\Omega} \to \boldsymbol{\Omega}, \quad \Phi' : \boldsymbol{\Omega} \to \boldsymbol{\Omega}$$

such that, for some constant $\delta > 0$,

$$\begin{aligned}&\sup_{t \geq 0} \mathbb{P}\big\{\|u_t(\boldsymbol{\omega}) - \hat{u}_t(\Phi(\boldsymbol{\omega}))\| \geq \delta\big\} \leq \delta, \\ &\sup_{t \geq 0} \mathbb{P}\big\{\|u'_t(\boldsymbol{\omega}) - \hat{u}'_t(\Phi(\boldsymbol{\omega}))\| \geq \delta\big\} \leq \delta,\end{aligned}$$

where $\mathbb{IP}$ denotes the direct product of two copies of $\mathbb{P}$. Then it can be shown (see Lemma 3.4 below) that, for any $t \geq 0$, we have

$$\big\|\mathcal{D}(u_t) - \mathcal{D}(u_t')\big\|_{\mathcal{L}}^* \leq 6\delta + 2\,\|\mathbb{IP} - \Phi_*(\mathbb{IP})\|_{\mathrm{var}} + 2\,\|\mathbb{IP} - \Phi'_*(\mathbb{IP})\|_{\mathrm{var}}. \qquad (3.16)$$

Thus, condition (P_2) will be verified if we show that the constant δ and transformations Φ and Φ' can be chosen in such a way that the right-hand side of (3.16) goes to zero with ε uniformly with respect to $u, u' \in B_H(\varepsilon)$.

To construct Φ and Φ', we first consider the original problem (2.1), (2.4) and denote by $\tilde{u}_t(\omega)$ its solution corresponding to the initial function $u_0 = 0$. For any $u \in B_H(\varepsilon)$ with sufficiently small $\varepsilon > 0$, we shall construct a measurable transformation $\Psi_u : \Omega \to \Omega$ such that

$$\sup_{t\geq 0} \mathbb{P}\big\{\|u_t(\omega) - \tilde{u}_t(\Psi_u(\omega))\| \geq \delta\big\} \leq \delta, \qquad (3.17)$$

$$\|\mathbb{P} - \Psi_{u*}(\mathbb{P})\|_{\mathrm{var}} \leq \delta, \qquad (3.18)$$

where $u_t(\omega)$ denotes the solution of (2.1), (2.4) with $u_0 = u$ and $\delta > 0$ is a function of $\varepsilon > 0$ vanishing as $\varepsilon \to 0$. If such a family of transformations is constructed, then for any $\boldsymbol{\omega} = (\omega, \omega') \in \boldsymbol{\Omega}$ we can set

$$\hat{u}_t(\boldsymbol{\omega}) = \tilde{u}_t(\omega), \qquad \hat{u}_t'(\boldsymbol{\omega}) = \tilde{u}_t(\omega'), \qquad (3.19)$$

$$\Phi(\boldsymbol{\omega}) = (\Psi_u(\omega), \omega'), \qquad \Phi'(\boldsymbol{\omega}) = (\omega, \Psi_{u'}(\omega')). \qquad (3.20)$$

All required properties of Φ and Φ' will follow from similar properties for Ψ_u.

Step 5. Let us turn to the accurate proof. We need the proposition below.

Proposition 3.3. *Let $\tilde{u}_t(\omega)$ be the solution of the problem* (2.1), (2.4) *with $u_0 = 0$. Then for any $\delta > 0$ there is $\varepsilon > 0$ such that for any $u \in B_H(\varepsilon)$ one can find a measurable transformation $\Psi_u : \Omega \to \Omega$ for which* (3.17) *and* (3.18) *hold.*

Taking this proposition for granted, let us show that condition (P_2) is fulfilled. We fix an integer $m \geq 1$ and two initial functions $u, u' \in B_H(1/m)$. Let $u_t(\boldsymbol{\omega})$ and $u_t'(\boldsymbol{\omega})$ be the solutions of (3.4) corresponding to the initial functions u and u', respectively. We claim that

$$\sup_{u,u'\in B_H(1/m)} \sup_{t\geq 0} \|\mathcal{D}(u_t) - \mathcal{D}(u_t')\|_{\mathcal{L}}^* \to 0 \quad \text{as } m \to \infty. \qquad (3.21)$$

To show this, we shall need the following lemma, whose proof is given in the next subsection.

Lemma 3.4. *Let X be a separable Banach space and let $u(\omega)$ and $\hat{u}(\omega)$ be two X-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that there is a measurable transformation $\Psi : \Omega \to \Omega$ such that*

$$\mathbb{P}\big\{\|u(\omega) - \hat{u}(\Psi(\omega))\| \geq \delta\big\} \leq \delta, \qquad (3.22)$$

where $\delta > 0$ is a constant. Then

$$\|\mathcal{D}(u) - \mathcal{D}(\hat{u})\|_{\mathcal{L}}^* \leq 3\delta + 2\,\|\mathbb{P} - \Psi_*(\mathbb{P})\|_{\mathrm{var}}. \qquad (3.23)$$

The random variables $\hat{u}_t$ and $\hat{u}'_t$ defined by (3.19) have the same distribution. Therefore,

$$\|\mathcal{D}(u_t)-\mathcal{D}(u'_t)\|^*_{\mathcal{L}} \le \|\mathcal{D}(u_t)-\mathcal{D}(\hat{u}_t)\|^*_{\mathcal{L}} + \|\mathcal{D}(u'_t)-\mathcal{D}(\hat{u}'_t)\|^*_{\mathcal{L}}. \tag{3.24}$$

Furthermore, it follows from (3.17), (3.18) and the definition of $\hat{u}_t$ and Φ that

$$\sup_{t\ge 0} \mathbb{P}\{\|u_t(\boldsymbol{\omega})-\hat{u}_t(\Phi(\boldsymbol{\omega}))\| \ge \delta_m\} \le \delta_m,$$

$$\|\mathbb{P}-\Phi_*(\mathbb{P})\|_{\mathrm{var}} \le \delta_m,$$

where $\delta_m \to 0$ as $m\to\infty$. Applying Lemma 3.4, we see that

$$\sup_{t\ge 0}\|\mathcal{D}(u_t)-\mathcal{D}(\hat{u}_t)\|^*_{\mathcal{L}} \le 3\delta_m + 2\,\|\mathbb{P}-\Phi_*(\mathbb{P})\|_{\mathrm{var}} \le 5\delta_m.$$

A similar inequality is valid for u'_t, $\hat{u}'_t$ and Φ'. Combining these inequalities with (3.24), we obtain (3.21).

Step 6. To complete the proof, it remains to establish Proposition 3.3. It is here where the particular choice of the underlying probability space plays its role (see the beginning of the proof).

We fix an integer $N\ge 1$ and denote by H_N the vector space spanned by $e_1,\dots,e_N$ and by $H_N^\perp$ its orthogonal complement in H. Let $\mathsf{P}_N : H \to H_N$ and $\mathsf{Q}_N : H \to H_N^\perp$ be the corresponding projections. Consider the following auxiliary problem in $H_N^\perp$:

$$\dot{w} + Lw + \mathsf{Q}_N B(v+w) = \mathsf{Q}_N h + \mathsf{Q}_N \eta, \tag{3.25}$$

$$w(0) = w_0. \tag{3.26}$$

Here $w_0 \in H_N^\perp$ is an initial condition, $v\in C(\mathbb{R}_+, H_N)$ is a given function, and we set $B(u) = B(u,u)$. The following proposition can be established using standard methods of the theory of NS equations (see [VF88, CF88, Fla94]).

Proposition 3.5. *Let η be defined by (2.2), where the constants $b_j \ge 0$ satisfy (2.3). Then there is a Borel subset $\Omega_0\subset\Omega$ of full measure such that the following assertions hold for any $\omega\in\Omega_0$ and $h\in H$.*

- **(i)** *For any $v\in C(\mathbb{R}_+, H_N)$ and $w_0\in H_N^\perp$, the problem (3.25), (3.26) has a unique solution $w\in\mathcal{X} := C(\mathbb{R}_+, H_N^\perp)\cap L^2_{\mathrm{loc}}(\mathbb{R}_+, H_N^\perp\cap V)$.*
- **(ii)** *The function $w(t)$ depends only on the restrictions of v and $\mathsf{Q}_N\omega$ to the interval $[0,t]$.*
- **(iii)** *For any $\omega\in\Omega$, let $\mathcal{W}(\cdot,\cdot,\omega) : C(\mathbb{R}_+, H_N)\times H_N^\perp \to \mathcal{X}$ be the operator defined by the formula*

$$\mathcal{W}(v,w_0,\omega) = \begin{cases} w & \text{if } \omega\in\Omega_0, \\ 0 & \text{if } \omega\in\Omega\setminus\Omega_0, \end{cases}$$

where $w\in\mathcal{X}$ is the solution of (3.25), (3.26). Then $\mathcal{W}$ is uniformly Lipschitz continuous in (v,w_0) on bounded subsets for any fixed $\omega\in\Omega$ and is measurable[4] with respect to (v,w_0,ω).

[4] All Polish spaces are endowed with their Borel σ-algebra.

We denote by $\mathcal{W}_t(v, w_0, \omega)$ the value of the function $\mathcal{W}(v, w_0, \omega) \in \mathcal{X}$ at the time t. We shall sometimes write $\mathcal{W}_t(v_\cdot, w_0, \omega)$ to emphasize that v is a function of t.

We can now define the transformation $\Psi_u : \Omega \to \Omega$. Let $\theta \in C^\infty(\mathbb{R})$ be an arbitrary function such that $\theta_t = 1$ for $t \le 0$ and $\theta_t = 0$ for $t \ge 1$ and let $\Theta_t = \int_0^t \theta_s \, ds$. We fix any function $u \in H$ and set

$$v_t(\omega) = \mathsf{P}_N u_t(\omega), \quad w_t(\omega) = \mathsf{Q}_N u_t(\omega),$$

where $u_t(\omega)$ denotes the solution of (2.1), (2.4) with $u_0 = u$. Let $\Psi_u(\omega)$ be defined by the relations

$$\mathsf{P}_N \Psi_u(\omega) = \mathsf{P}_N\bigl(\omega - \theta u - \Theta L u\bigr) + \int_0^t D_s(u, \omega)\, ds, \quad \mathsf{Q}_N \Psi_u(\omega) = \mathsf{Q}_N \omega,$$

where we set

$$D_s(u,\omega) = \mathsf{P}_N\bigl\{B\bigl(v_s - \theta_s \mathsf{P}_N u + \mathcal{W}_s(v_\cdot - \theta_\cdot \mathsf{P}_N u, 0, \omega)\bigr) - B\bigl(v_s + \mathcal{W}_s(v_\cdot, \mathsf{Q}_N u, \omega)\bigr)\bigr\}.$$

It is clear that $\Psi_u(\omega)$ is measurable with respect to $(u, \omega) \in H \times \Omega$. We claim that if $N \ge 1$ is sufficiently large, then for every $\delta > 0$ there is $\varepsilon > 0$ such that inequalities (3.17) and (3.18) hold for any $u \in B_H(\varepsilon)$.

Step 7. Let us prove (3.17). To this end, we first show that, for a.a $\omega \in \Omega$,

$$\mathsf{P}_N \tilde{u}_t(\Psi_u(\omega)) = v_t(\omega) - \theta_t \mathsf{P}_N u, \quad t \ge 0. \tag{3.27}$$

Indeed, the definition of the operator $\mathcal{W}$ implies that the problem (2.1), (2.4) is equivalent to the following system for the components (v_t, w_t) of u_t:

$$\dot v + L v + \mathsf{P}_N B\bigl(v + \mathcal{W}_t(v_\cdot, \mathsf{Q}_N u_0, \omega)\bigr) = \mathsf{P}_N h + \mathsf{P}_N \dot\omega, \tag{3.28}$$
$$\dot w + L w + \mathsf{Q}_N B(v + w) = \mathsf{Q}_N h + \mathsf{Q}_N \dot\omega, \tag{3.29}$$
$$v_0 = \mathsf{P}_N u_0, \quad w_0 = \mathsf{Q}_N u_0. \tag{3.30}$$

It follows from (3.28) and (3.30) that the left-hand side of (3.27), which we denote by $p_t(\omega)$, is a solution of the problem

$$\dot p + L p + \mathsf{P}_N B(p + \mathcal{W}_t(p_\cdot, 0, \omega)) = \mathsf{P}_N h + \varphi_t, \tag{3.31}$$
$$p_0 = 0. \tag{3.32}$$

where we set

$$\varphi_t = \mathsf{P}_N(\dot\omega - \dot\theta u - \theta L u) + D_t(u, \omega).$$

It is a matter of direct verification to show that the function $v_t(\omega) - \theta_t \mathsf{P}_N u$ also satisfies (3.31) and (3.32). Thus, relation (3.27) will be established if we show that the finite-dimensional problem (3.31), (3.32) admits at most one solution. This is a straightforward consequence of the Gronwall lemma and the local Lipschitz property of the operators entering (3.31).

Relation (3.27) implies that, if $u \in B_H(\varepsilon)$, then for a.a. $\omega \in \Omega$ we have

$$\sup_{t\geq 0} \|\mathsf{P}_N\big(u_t(\omega) - \tilde{u}_t(\Psi_u(\omega))\big)\| \leq \|u\| \leq \varepsilon. \tag{3.33}$$

We now estimate the norm of the projection of $u_t(\omega) - \tilde{u}_t(\Psi_u(\omega))$ to the subspace $H_N^\perp$.

Let $q_t(\omega) = \mathsf{Q}_N u_t(\Psi_u(\omega))$ and let $z_t = w_t - q_t$. It follows from (3.27), (3.29), and (3.30) that z is the solution of the problem

$$\dot{z}_t + Lz_t + \mathsf{Q}_N\big(B(u_t) - B(u_t - \theta_t \mathsf{P}_N u - z_t)\big) = 0, \quad z_0 = \mathsf{Q}_N u. \tag{3.34}$$

Taking the scalar product in H of the function $2z_t$ with the first equation in (3.34) and using the fact that $(B(g, z_t), z_t) = 0$ for any $g \in H$, we obtain

$$\partial_t \|z_t\|^2 + 2\|z_t\|_1^2 = \theta_t \chi_t(u,\omega) + \rho_t(u,\omega), \tag{3.35}$$

where we set

$$\begin{aligned}
\chi_t(u,\omega) &= 2\left(B(u_t, \mathsf{P}_N u) + B(\mathsf{P}_N u, u_t - \theta_t \mathsf{P}_N u), z_t\right),\\
\rho_t(u,\omega) &= 2\big(B(z_t, u_t - \theta_t \mathsf{P}_N u), z_t\big).
\end{aligned}$$

Standard estimates for the trilinear form $(B(u,v),w)$ imply that, for $u \in B_H(\varepsilon)$,

$$\begin{aligned}
|\chi_t(u,\omega)| &\leq \frac{1}{2}\|z_t\|_1^2 + \varepsilon^2 C_N\big(\|u_t\|^2 + 1\big),\\
|\rho_t(u,\omega)| &\leq \frac{1}{2}\|z_t\|_1^2 + \big(C\|u_t\|_1^2 + \varepsilon^2 C_N\big),
\end{aligned}$$

where C and C_N are some positive constants. Substituting these estimates into (3.35) and using the inequality $\|z_t\|_1^2 \geq \alpha_{N+1}\|z_t\|^2$, we derive

$$\partial_t \|z_t\|^2 + \left(\alpha_{N+1} - C\|u_t\|^2 - \varepsilon^2 C_N\right)\|z_t\|^2 \leq C_N \varepsilon^2 \theta_t \big(\|u_t\|^2 + 1\big). \tag{3.36}$$

We now need the following lemma, whose proof can be obtained by an unessential modification of the argument in [KS02, Lemma 2.3].

Lemma 3.6. *For any $R > 0$ and any initial condition $u \in H$, let*

$$\Omega_R(u) := \Big\{\omega \in \Omega : \int_0^t \|u_s(\omega)\|_1^2 ds \leq B_0 t + \|u\|^2 + R\Big\},$$

where $u_t(\omega)$ is the solution of (2.1) such that $u_0 = u$ and B_0 is defined by (2.3). Then

$$\sup_{u\in H} \mathbb{P}\big\{\Omega_R(u)\big\} \to 0 \quad \textit{as } R \to +\infty. \tag{3.37}$$

Let us choose an integer $N \geq 1$ and a constant $\varepsilon > 0$ such that

$$\alpha_{N+1} \geq CB_0 + 1, \quad \varepsilon^2 C_N \leq 1.$$

Applying the Gronwall inequality to (3.36), for any $u \in B_H(\varepsilon)$ and $\omega \in \Omega_R(u)$ we obtain

$$\begin{aligned}\|z_t\|^2 &\le \|z_0\|^2 \exp\biggl\{\int_0^t \bigl(-\alpha_{N+1} + C\|u_t\|^2 + C_N\varepsilon^2\bigr)\,ds\biggr\} \\ &\quad + C_N\varepsilon^2 \int_0^1 \exp\biggl\{\int_s^t \bigl(-\alpha_{N+1} + C\|u_\tau\|^2 + C_N\varepsilon^2\bigr)\,d\tau\biggr\}\theta_s\bigl(\|u_s\|^2+1\bigr)\,ds \\ &\le K_R\varepsilon^2,\end{aligned}$$

where $K_R > 0$ is a constant not depending on ε and u. Choosing $R > 0$ so large that $\mathbb{P}\bigl\{\Omega_R(u)\bigr\} \le \delta$ for $u \in H$ (cf. (3.37)), we see that

$$\mathbb{P}\Bigl\{\sup_{t\ge 0} \|\mathsf{Q}_N\bigl(u_t(\omega) - \tilde{u}_t(\Psi_u(\omega))\bigr)\| \ge K_R\varepsilon^2\Bigr\} \le \delta.$$

Combining this inequality with (3.33), we obtain the required estimate (3.17) for a sufficiently small $\varepsilon > 0$.

To complete the proof of Proposition 3.3, it remains to establish inequality (3.18). Its proof is based on the Girsanov theorem. In the context of randomly forced PDE's, this type of arguments was first used in [EMS01] for the case of finite-dimensional perturbations. The proof of (3.18) almost literally repeats that of Lemma 4.3 in [KS02], and therefore we omit it. In the second part of this work, we shall discuss the corresponding arguments in detail for the complex Ginzburg–Landau equation, which is technically more complicated.

3.2 Proof of auxiliary assertions

Proof of Lemma 3.1. Let us fix a constant $T > 0$ and set

$$\hat{u}(t) = \chi(t)e^{-tL}u_0, \quad 0 \le t \le T, \tag{3.38}$$

where $\chi \in C^\infty(\mathbb{R})$ is a function such that $0 \le \chi \le 1$, $\chi(t) = 1$ for $t \le 0$ and $\chi(t) = 1$ for $t \ge T$. It is a matter of direct verification to show that $\hat{u} \in \mathcal{Y}_T$, whence it follows that

$$\hat{\eta} := \partial_t\hat{u} + L\hat{u} + B(\hat{u},\hat{u}) - h \in L^2(0,T;V^*).$$

We now define $\mathcal{Z}(u_0) \in \dot{\mathcal{X}}_T$ as the unique solution of the problem (3.10) with the deterministic right-hand side $\eta = \hat{\eta}$. Such a solution can easily be constructed with the help of decomposition in the eigenbasis $\{e_j\}$ (see also [Hen81, Section 3.2] or [CF88, Sections 2 and 3]). Moreover, it is a straightforward consequence of the definition that $\mathcal{Z}(u_0)$ is uniformly Lipschitz on bounded subsets of H. Finally, the construction of $\mathcal{Z}(u_0)$ implies that $v + \mathcal{Z}(u_0) = \hat{u}$, and therefore (3.13) follows from (3.38). □

Proof of Lemma 3.2. Conditions (2.3) and (2.5) imply that the distribution of z is a non-degenerate Gaussian measure in $\dot{\mathcal{X}}_T$. In particular, the measure of any open subset is positive (see Section 7.4 in [DZ96]). It follows that for any $\hat{z} \in \dot{\mathcal{X}}_T$ and any open subset $O \subset \dot{\mathcal{X}}_T$ the probability $p(\hat{z}) = \mathbb{P}\{z - \hat{z} \in O\}$ is positive. Now note that the distribution of the random variable $z - \hat{z}$ is continuous with respect to $\hat{z}$ in the weak* topology. It follows that the function $p(\hat{z}) > 0$ is lower semi-continuous and, hence, is separated from zero on any compact subset. □

Proof of Lemma 3.4. We have

$$\|\mathcal{D}(u) - \mathcal{D}(\hat{u})\|_{\mathcal{L}}^* \le \|\mathcal{D}(u) - \mathcal{D}(\hat{u} \circ \Psi)\|_{\mathcal{L}}^* + \|\mathcal{D}(\hat{u} \circ \Psi) - \mathcal{D}(\hat{u})\|_{\mathcal{L}}^*. \tag{3.39}$$

Inequality (3.22) readily implies that

$$\|\mathcal{D}(u) - \mathcal{D}(\hat{u} \circ \Psi)\|_{\mathcal{L}}^* \le 3\delta. \tag{3.40}$$

Furthermore, since

$$\|\mu_1 - \mu_2\|_{\mathcal{L}}^* \le 2\,\|\mu_1 - \mu_2\|_{\mathrm{var}} \quad \text{for any } \mu_1, \mu_2 \in \mathcal{P}(X),$$

we conclude that

$$\begin{aligned} \|\mathcal{D}(\hat{u} \circ \Psi) - \mathcal{D}(\hat{u})\|_{\mathcal{L}}^* &\le 2 \sup_{\Gamma \in \mathcal{B}(X)} \bigl|\mathbb{P}\{\hat{u} \circ \Psi \in \Gamma\} - \mathbb{P}\{\hat{u} \in \Gamma\}\bigr| \\ &= 2 \sup_{\Gamma \in \mathcal{B}(X)} \bigl|\Psi_*(\mathbb{P})\{\hat{u} \in \Gamma\} - \mathbb{P}\{\hat{u} \in \Gamma\}\bigr| \\ &\le 2\,\|\Psi_*(\mathbb{P}) - \mathbb{P}\|_{\mathrm{var}}. \end{aligned} \tag{3.41}$$

Combining (3.39) – (3.41), we arrive at (3.23). □

References

[AS04] A. A. Agrachev and A. V. Sarychev, *Navier–stokes equations: controllability by means of low modes forcing*, J. Math. Fluid Mech. **6** (2004), to appear.

[BKL02] J. Bricmont, A. Kupiainen, and R. Lefevere, *Exponential mixing for the 2D stochastic Navier–Stokes dynamics*, Comm. Math. Phys. **230** (2002), no. 1, 87–132.

[Bri02] J. Bricmont, *Ergodicity and mixing for stohastic partial differential equations*, Proceedings of the International Congress of Mathematicians (Beijing), vol. 1, Higher Ed. Press, Beijing, 2002.

[CF88] P. Constantin and C. Foias, *Navier-Stokes Equations*, University of Chicago Press, Chicago, 1988.

[Dud02] R. M. Dudley, *Real Analysis and Probability*, Cambridge University Press, Cambridge, 2002.

[DZ96] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, Cambridge, 1996.

[EH01] J.-P. Eckmann and M. Hairer, *Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise*, Comm. Math. Phys. **219** (2001), 523–565.

[EMS01] W. E, J. C. Mattingly, and Ya. G. Sinai, *Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation*, Comm. Math. Phys. **224** (2001), 83–106.

[Fla94] F. Flandoli, *Dissipativity and invariant measures for stochastic Navier-Stokes equations*, NoDEA Nonlinear Differential Equations Appl. **1** (1994), no. 4, 403–423.

[FM95] F. Flandoli and B. Maslowski, *Ergodicity of the 2D Navier-Stokes equation under random perturbations*, Comm. Math. Phys. **172** (1995), 119–141.

[Hai02] M. Hairer, *Exponential mixing properties of stochastic PDE's through asymptotic coupling*, Probab. Theory Related Fields **124** (2002), 345–380.

[Has80] R. Z. Has′minskiĭ, *Stochastic Stability of Differential Equations*, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[Hen81] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981.

[KPS02] S. B. Kuksin, A. Piatnitski, and A. Shirikyan, *A coupling approach to randomly forced nonlinear PDE's. II*, Comm. Math. Phys. **230** (2002), no. 1, 81–85.

[KR77] N. V. Krylov and B. L. Rozovskiĭ, *The Cauchy problem for linear stochastic partial differential equations*, Math. USSR-Izv. **41** (1977), no. 6, 1267–1284.

[KS00] S. B. Kuksin and A. Shirikyan, *Stochastic dissipative PDE's and Gibbs measures*, Comm. Math. Phys. **213** (2000), no. 2, 291–330.

[KS01] S. B. Kuksin and A. Shirikyan, *A coupling approach to randomly forced nonlinear PDE's. I*, Comm. Math. Phys. **221** (2001), 351–366.

[KS02] S. B. Kuksin and A. Shirikyan, *Coupling approach to white-forced nonlinear PDE's*, J. Math. Pures Appl. **81** (2002), 567–602.

[Kuk02] S. B. Kuksin, *Ergodic theorems for 2D statistical hydrodynamics*, Rev. Math. Phys. **14** (2002), 585–600.

[Lin92] T. Lindvall, *Lectures on the Coupling Methods*, John Willey & Sons, New York, 1992.

[Lio69] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Gauthier-Villars, Paris, 1969.

[Mat02] J. Mattingly, *Exponential convergence for the stochastically forced Navier–Stokes equations and other partially dissipative dynamics*, Comm. Math. Phys **230** (2002), 421–462.

[MY02] N. Masmoudi and L.-S. Young, *Ergodic theory of infinite dimensional systems with applications to dissipative parabolic PDE's*, Comm. Math. Phys. (2002), 461–481.

[Par79] E. Pardoux, *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics **3** (1979), no. 2, 127–167.

[Shi02] A. Shirikyan, *Analyticity of solutions for randomly forced two-dimensional Navier–Stokes equations*, Russian Math. Surveys **57** (2002), 785–799.

[Shi03] A. Shirikyan, *Some mathematical problems of statistical hydrodynamics*, Proceedings of the XIV International Congress of Mathematical Physics, World Scientific, 2003, to appear.

[Shi04] A. Shirikyan, *Exponential mixing for 2D Navier–Stokes equations perturbed by an unbounded noise*, J. Math. Fluid Mech. **6** (2004), no. 2, 169–193.

[Ver87] A. Yu. Veretennikov, *Estimates of the mixing rate for stochastic equations*, Teor. Veroyatnost. i Primenen. **32** (1987), no. 2, 299–308.

[Ver00] A. Yu. Veretennikov, *Parametric and Non-Parametric Estimation of Markov Chains*, Moscow State University Press, Moscow, 2000.

[VF88] M. I. Vishik and A. V. Fursikov, *Mathematical Problems in Statistical Hydromechanics*, Kluwer, Dordrecht, 1988.