QUENCHED INVARIANCE PRINCIPLE FOR SIMPLE RANDOM WALK ON PERCOLATION CLUSTERS

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ABSTRACT. We consider the simple random walk on the (unique) infinite cluster of super-critical percolation in $\mathbb{Z}^d$ with $d \geq 2$. We prove that, for almost every percolation configuration, the path distribution of the walk converges weakly to that of non-degenerate Brownian motion.

1. INTRODUCTION

1.1 Motivation and model.

Consider supercritical bond-percolation on $\mathbb{Z}^d$, $d \geq 2$, and the simple random walk on its infinite cluster. In [20] Sidoravicius and Sznitman asked (and answered, affirmatively, for dimensions $d \geq 4$) the following question: Is it true that for a.e. configuration such that the origin belongs to the infinite cluster, the random walk started at the origin exits the infinite symmetric slab $\{(x_1, \dots, x_d)\colon |x_1| \leq N\}$ through either side with probability tending to $1/2$ as $N \to \infty$? In this paper, we answer this question for dimensions $d = 2$ and 3. As in [20], we will do so by proving a quenched invariance principle for the walk on the super-critical infinite cluster.

Random walk on percolation clusters is only one of many instances of "statistical mechanics in random media" that have been recently studied by both physicists and mathematicians. Other pertinent examples include, e.g., various diluted spin systems, random copolymers [21], spin glasses [5,22], random-graph models [4], etc. From this general perspective, the present problem is interesting for at least two reasons: First, a good handle on simple random walk on a given graph is often a prerequisite for the understanding of more complicated processes, e.g., self-avoiding walk or loop-erased random walk. Second, information about the scaling properties of simple random walk on percolation cluster can, in principle, reveal some new important facts about the structure of the infinite cluster and/or its harmonic properties.

Let us begin developing the mathematical layout of the problem. Let $\mathbb{Z}^d$ be the d-dimensional hypercubic lattice and let $\mathbb{B}_d$ be the set of nearest neighbor edges. We will use b to denote a generic edge, $\langle x, y\rangle$ to denote the edge between x and y, and e to denote the edges from the origin to its nearest neighbors. Let $\Omega = \{0,1\}^{\mathbb{B}_d}$ be the space of all percolation configurations. Here $\omega_b = 1$ indicates that the edge b is occupied and $\omega_b = 0$ implies that it is vacant. Let $\mathcal{B}$ be the Borel σ-algebra on Ω—defined using the product topology—and let $\mathbb{P}$ be an i.i.d. measure such that $\mathbb{P}(\omega_b = 1) = p$ for all $b \in \mathbb{B}_d$. If $x \overset{\omega}{\longleftrightarrow} \infty$ denotes the event that the site x belongs to an infinite self-avoiding path using only occupied bonds in ω, we write $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$ for the set

$$\mathcal{C}_\infty(\omega) = \bigl\{x \in \mathbb{Z}^d \colon x \overset{\omega}{\longleftrightarrow} \infty\bigr\}. \tag{1.1}$$

By Burton-Keane's Uniqueness theorem [6], $\mathcal{C}_\infty$ is connected with $\mathbb{P}$-probability one.

For each $x \in \mathbb{Z}^d$, let $\tau_x \colon \Omega \to \Omega$ be the "shift by x" defined by $(\tau_x\omega)_b = \omega_{x+b}$. Note that $\mathbb{P}$ is τ_x-invariant for all $x \in \mathbb{Z}^d$. Let $p_{\rm c} = p_{\rm c}(d)$ denote the percolation threshold on $\mathbb{Z}^d$ defined as the infimum of all p's for which $\mathbb{P}(0 \in \mathcal{C}_\infty) > 0$. In particular, for $p > p_{\rm c}$ we may define the measure $\mathbb{P}_0$ by

$$\mathbb{P}_0(A) = \mathbb{P}(A|0 \in \mathcal{C}_\infty), \qquad A \in \mathcal{B}. \tag{1.2}$$

This measure is supported on the set $\Omega_0 = \{0 \in \mathcal{C}_\infty\}$. We will use $\mathbb{E}_0$ to denote expectation with respect to $\mathbb{P}_0$.

For each configuration $\omega \in \Omega_0$, let $(X_n)_{n\ge 0}$ be the simple random walk on $\mathcal{C}_\infty(\omega)$ started at the origin. Explicitly, $(X_n)_{n\ge0}$ is a Markov chain with state-space $\mathbb{Z}^d$, whose distribution $P_{0,\omega}$ is defined by the transition probabilities

$$P_{0,\omega}(X_{n+1} = x + e|X_n = x) = \frac{1}{2d}\,1_{\{\omega_e=1\}}\circ\tau_x, \qquad |e| = 1, \tag{1.3}$$

and

$$P_{0,\omega}(X_{n+1} = x|X_n = x) = \sum_{e\colon |e|=1} \frac{1}{2d}\,1_{\{\omega_e=0\}}\circ\tau_x, \tag{1.4}$$

supplemented by the initial condition

$$P_{0,\omega}(X_0 = 0) = 1. \tag{1.5}$$

Thus, at each unit of time, the walk picks a neighbor at random and if the corresponding edge is occupied, the walk moves to this neighbor. If the edge is vacant, the move is suppressed.

1.2 Main results.

Our main result is that for $\mathbb{P}_0$-almost every $\omega \in \Omega_0$, the linear interpolation of (X_n), properly scaled, converges weakly to Brownian motion. For every $T > 0$, let $(C[0,T], \mathcal{W}_T)$ be the space of continuous functions $f \colon [0,T] \to \mathbb{R}$ equipped with the σ-algebra $\mathcal{W}_T$ of Borel sets relative to the supremum topology. The precise statement is now as follows:

Theorem 1.1 *Let $d \ge 2$, $p > p_{\rm c}(d)$ and let $\omega \in \Omega_0$. Let $(X_n)_{n\ge0}$ be the random walk with law $P_{0,\omega}$ and let*

$$\widetilde{B}_n(t) = \frac{1}{\sqrt{n}}\bigl(X_{\lfloor tn\rfloor} + (tn - \lfloor tn\rfloor)(X_{\lfloor tn\rfloor+1} - X_{\lfloor tn\rfloor})\bigr), \qquad t \ge 0. \tag{1.6}$$

Then for all $T > 0$ and for $\mathbb{P}_0$-almost every ω, the law of $(\widetilde{B}_n(t)\colon 0 \le t \le T)$ on $(C[0,T], \mathcal{W}_T)$ converges weakly to the law of an isotropic Brownian motion $(B_t\colon 0 \le t \le T)$ whose diffusion constant, $D = E(|B_1|^2) > 0$, depends only on the percolation parameter p and the dimension d.

The Markov chain $(X_n)_{n\ge0}$ represents only one of two natural ways to define a simple random walk on the supercritical percolation cluster. Another possibility is that, at each unit of time, the walk moves to a site chosen uniformly at random from the *accessible* neighbors, i.e., the walk takes no pauses. In order to define this process, let $(T_k)_{k\geq 0}$ be the sequence of stopping times that mark the moments when the walk $(X_n)_{n\geq 0}$ made a move. Explicitly, $T_0 = 0$ and

$$T_{n+1} = \inf\{k > T_n \colon X_k \neq X_{k-1}\}, \qquad n \geq 0. \tag{1.7}$$

Using these stopping times—which are $P_{0,\omega}$-almost surely finite for all $\omega \in \Omega_0$—we define a new Markov chain $(X'_n)_{n\geq 0}$ by putting

$$X'_n = X_{T_n}, \qquad n \geq 0. \tag{1.8}$$

It is easy to see that $(X'_n)_{n\geq 0}$ has the desired distribution. Indeed, the walk starts at the origin and its transition probabilities are given by

$$P_{0,\omega}(X'_n = x + e | X'_n = x) = \frac{1_{\{\omega_e = 1\}} \circ \tau_x}{\sum_{e' \colon |e'|=1} 1_{\{\omega_{e'}=1\}} \circ \tau_x}, \qquad |e| = 1. \tag{1.9}$$

A simple modification of the arguments leading to Theorem 1.1 allow us to establish a corresponding functional central limit theorem for this random walk as well:

Theorem 1.2 *Let $d \geq 2$, $p > p_c(d)$ and let $\omega \in \Omega_0$. Let $(X'_n)_{n\geq 0}$ be the random walk defined from $(X_n)_{n\geq 0}$ as described in* (1.8) *and let $\widetilde{B}'_n(t)$ be the linear interpolation of $(X'_k)_{0\leq k\leq n}$ defined by* (1.6) *with (X_k) replaced by (X'_k). Then for all $T > 0$ and for $\mathbb{P}_0$-almost every ω, the law of $(\widetilde{B}'_n(t) \colon 0 \leq t \leq T)$ on $(C[0,T], \mathcal{W}_T)$ converges weakly to the law of an isotropic Brownian motion $(B_t \colon 0 \leq t \leq T)$ whose diffusion constant, $D' = E(|B_1|^2) > 0$, depends only on the percolation parameter p and the dimension d.*

De Gennes [11], who introduced the problem of random walk on percolation cluster to the physics community, thinks of the walk as the motion of “an ant in a labyrinth.” From this perspective, the “lazy” walk (X_n) corresponds to a “blind” ant, while the “agile” walk (X'_n) represents a “myopic” ant. While the character of the scaling limit of the two “ants” is the same, there seems to be some distinction in the rate of the limit is approached, cf [15] and references therein. As we will see in the proof, the diffusion constants D and D' are related via $D' = D\Theta^2$, where Θ^{-1} is the expected degree of the origin normalized by $2d$, cf (6.21).

1.3 Discussion and outline.

The subject of simple random walk in random environment has a long history; we refer to, e.g., [5] for a recent overview of (certain parts of) this field. On general grounds, each random-media problem comes in two distinct flavors: *quenched*, corresponding to the situations discussed above where the walk is distributed according to an ω-dependent measure $P_{0,\omega}$, and *annealed*, in which the path distribution of the walk is taken from the averaged measure $A \mapsto \mathbb{E}_0(P_{0,\omega}(A))$. Under suitable ergodicity assumptions, the annealed problem typically corresponds to the quenched problem averaged over the starting point. Yet the distinction is clear: In the annealed setting the slab-exit problem from Sect. 1.1 is trivially true by the symmetries of the averaged measure, while its answer is *a priori* very environment-sensitive in the quenched measure.

An annealed version of our theorems was proved in 1984 by De Masi *et al* [7, 8], based on earlier results of Kozlov [17], Kipnis and Varadhan [16] and others in the context of random walk in a field of random conductances. A number of proofs of quenched invariance principles have

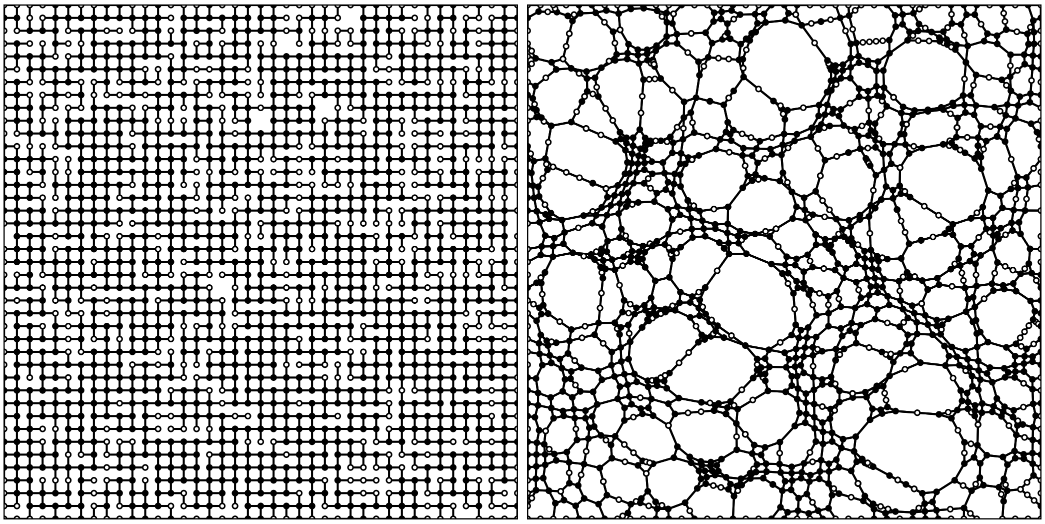

FIGURE 1. A portion of the infinite cluster $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$ before (left) and after (right) the harmonic deformation $x \mapsto x + \chi(x, \omega)$. Here $p = 0.75$ is already so large that all but a few sites in the entire block belong to $\mathcal{C}_\infty$. Upon the deformation, all "holes" (i.e., dual connected components) get considerably stretched and rounded while the "dangling ends" collapse onto the rest of the structure. The two types of bullets encode the coordination number of various nodes: white (1 and 2 neighbors) and black (3 and 4 neighbors). Both pictures are to scale.

appeared in recent years for the cases where an annealed principle was already known. The most relevant paper is that of Sidoravicius and Sznitman [20] which established Theorem 1.2 for simple random walk among random conductances in all $d \geq 1$ and, using a very different method, also for random walk on percolation in $d \geq 4$. Another paper of direct relevance is that of Rassoul-Agha and Seppäläinen [19] where a quenched invariance principle was established for *directed* random walks in (space-time) random environments. The directed setting offers the possibility to use independence more efficiently—every time step the walk enters a new environment—but the price to pay for this is the lack of reversibility. A number of remarkable results on harmonic properties of the infinite cluster have been proved by Barlow [2]; see also references therein.

Let us outline the main steps of our proof of Theorems 1.1 and 1.2. The principal idea—which permeates through the work of Kozlov [17], Kipnis and Varadhan [16], Sidoravicius and Sznitman [20] and many others—is to consider an embedding of $\mathcal{C}_\infty(\omega)$ into the Euclidean space which makes the corresponding simple random walk a martingale. The distance between the natural position of a site $x \in \mathcal{C}_\infty$ and its counterpart in this *harmonic embedding* is expressed in terms of the so-called *corrector* $\chi(x, \omega)$, see Fig. 1. It is clear that the corrector can be defined in any finite volume by solving an appropriate discrete Dirichlet problem (this is how Fig. 1 was drawn); the difficult part—solved beautifully by Kipnis and Varadhan [16] using spectral calculus—is to define the corrector in infinite volume while maintaining the natural (distributional) invariance with respect to shifts of the underlying lattice. Actually, there is an alternative, probabilistic definition of the corrector,

$$\chi(x,\omega)=\lim_{n\to\infty}\bigl(E_{x,\omega}(X_n)-E_{0,\omega}(X_n)\bigr), \tag{1.10}$$

but the only way we know how to prove the existence of the limit is by invoking the spectral calculus from Sect. 2.3.

Once we have the corrector under control, the proof splits into two parts: (1) proving that the martingale—i.e., the path of the walk on the deformed graph—converges to Brownian motion and (2) proving that the deformation of the path caused by the change of embedding is negligible on the scale we expect the convergence to Brownian motion to take place. The latter part (which is the one more difficult) amounts to an almost-sure estimate of the corrector $\chi(x,\omega)$ as a function of x. Here, somewhat unexpectedly, our level of control is considerably better in $d=2$ than in $d\ge 3$. In particular, while our proof in $d=2$ avoids using any of the recent sophisticated discrete-harmonic analyses, in $d\ge 3$ we cannot quite do without invoking the main theorem of [2]. The proof is actually carried out along these lines only for the setting in Theorem 1.1; Theorem 1.2 is then extracted by invoking ergodicity of the underlying Markov chain.

Here is an outline of the rest of this paper: In Sect. 2 we introduce the aforementioned corrector and prove some of its basic properties. Sect. 3 collects the needed facts about ergodic properties of the Markov chain "on environments." Both of these sections are based on existing material; the proofs have been included in order to make this paper entirely self-contained. The novel parts of the proof—sublinearity bounds on the corrector—appear in Sects. 4 and 5. The actual proofs of our main theorems are carried out in Sect. 6. The Appendix contains the proof of an upper bound for the transition probabilities of our random walk, and a list of open problems.

2. CORRECTOR—CONSTRUCTION AND HARMONICITY

In this section we will define and study the aforementioned corrector which is the basic instrument of our proofs. The main technical idea is to consider the Markov chain "on environements" (Sect. 2.1); the relevant properties of the corrector are listed in Theorem 2.2 (Sect. 2.2). The proofs are based on spectral calculus (see Sect. 2.3).

2.1 Markov chain "on environments".

As is well known, cf Kipnis and Varadhan [16], the Markov chain $(X_n)_{n\ge0}$ in (1.3–1.5) induces a Markov chain on Ω_0, which can be interpreted as the trajectory of "environments viewed from the perspective of the walk." The transition probabilities of this chain are given by the kernel $Q\colon \Omega_0\times\mathscr{B}\to[0,1]$,

$$Q(\omega,A)=\frac{1}{2d}\sum_{e\colon |e|=1}\bigl(1_{\{\omega_e=1\}}\,1_{\{\tau_e\omega\in A\}}+1_{\{\omega_e=0\}}\,1_{\{\omega\in A\}}\bigr). \tag{2.1}$$

Our basic observations about the induced Markov chain are as follows:

Lemma 2.1 *For every bounded measurable $f\colon\Omega\to\mathbb{R}$ and every e with $|e|=1$,*

$$\mathbb{E}_0\bigl(f\circ\tau_e\,1_{\{\omega_e=1\}}\bigr)=\mathbb{E}_0\bigl(f\,1_{\{\omega_{-e}=1\}}\bigr), \tag{2.2}$$

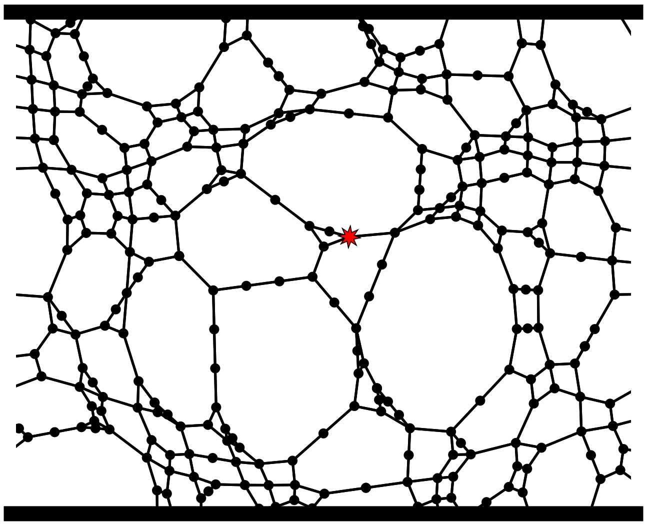

FIGURE 2. The harmonic deformation of a percolation configuration in the symmetric slab $\{(x_1,x_2)\in\mathbb{Z}^2\colon |x_2|\le N\}$. The star denotes the new position of the origin which in the undeformed configuration was right in the center. The relative vertical shift of the origin corresponds to the deviation of $P_{0,\omega}$(top hit before bottom) from one half. The figure also has an interesting electrostatic interpretation: If the bottom and top bars are set to potentials -1 and $+1$, respectively, then the site with *deformed* coordinates (x_1,x_2) has potential x_2/N.

where $-e$ is the bond that is opposite to e. As a consequence, $\mathbb{P}_0$ is reversible and, in particular, stationary for Markov kernel Q.

Proof. First we prove (2.2). Neglecting the normalization by $\mathbb{P}(0\in\mathcal{C}_\infty)$, we need to prove
$$\mathbb{E}\big(f\circ\tau_e\,1_{\{0\in\mathcal{C}_\infty\}}\,1_{\{\omega_e=1\}}\big)=\mathbb{E}\big(f\,1_{\{0\in\mathcal{C}_\infty\}}\,1_{\{\omega_{-e}=1\}}\big).\tag{2.3}$$
This will be a consequence of $1_{\{\omega_e=1\}}=1_{\{\omega_{-e}=1\}}\circ\tau_e$ and the fact that, on $\{\omega_e=1\}$ we have $1_{\{0\in\mathcal{C}_\infty\}}=1_{\{0\in\mathcal{C}_\infty\}}\circ\tau_e$. Indeed, these observations imply
$$f\circ\tau_e\,1_{\{0\in\mathcal{C}_\infty\}}\,1_{\{\omega_e=1\}}=\big(f\,1_{\{0\in\mathcal{C}_\infty\}}\,1_{\{\omega_{-e}=1\}}\big)\circ\tau_e\tag{2.4}$$
and (2.3) then follows by the shift invariance of $\mathbb{P}$.

From (2.2) we immediately have that for any bounded, measurable $f,g\colon\Omega\to\mathbb{R}$,
$$\mathbb{E}_0\big(f(Qg)\big)=\mathbb{E}_0\big(g(Qf)\big),\tag{2.5}$$
where $Qf\colon\Omega\to\mathbb{R}$ is the function given by
$$(Qf)(\omega)=\frac{1}{2d}\sum_{e\colon|e|=1}\big(1_{\{\omega_e=1\}}\,f(\tau_e\omega)+1_{\{\omega_e=0\}}\,f(\omega)\big).\tag{2.6}$$
Indeed, splitting the last sum into two terms, the second part reproduces exactly on both sides of (2.5). For the first part we apply (2.2) and note that averaging over e allows us to neglect the negative sign in front of e on the right-hand side. But (2.5) is the *definition* of reversibility and, setting $f=1$ and noting that $Q1=1$, we also get the stationarity of $\mathbb{P}_0$. $\square$

2.2 Kipnis-Varadhan construction.

Next we will adapt the construction of Kipnis and Varadhan [16] to the present situation. Let $L^2 = L^2(\Omega, \mathcal{B}, \mathbb{P}_0)$ be the space of all Borel-measurable, square integrable functions on Ω. Abusing the notation slightly, we will use "L^2" both for $\mathbb{R}$-valued functions as well as $\mathbb{R}^d$-valued functions. We equip L^2 with the inner product $(f, g) = \mathbb{E}_0(fg)$—with "$fg$" interpreted as inner product of f and g when these functions are vector-valued. Let Q be the operator defined by (2.6). From (2.5) we know

$$(f, Qg) = (Qf, g) \tag{2.7}$$

and so Q is symmetric. An explicit calculation gives us

$$\begin{aligned} \bigl|(f, Qf)\bigr| &\le \frac{1}{2d} \sum_{e\colon |e|=1} \Bigl\{ \bigl|(f, 1_{\{\omega_e=1\}}\, f\circ\tau_e)\bigr| + (f, 1_{\{\omega_e=0\}}\, f) \Bigr\} \\ &\le \frac{1}{2d} \sum_{e\colon |e|=1} \Bigl\{ \bigl|(f, 1_{\{\omega_e=1\}}\, f)\bigr|^{1/2} \bigl|(f, 1_{\{\omega_{-e}=1\}}\, f)\bigr|^{1/2} + (f, 1_{\{\omega_e=0\}}\, f) \Bigr\} \\ &\le \frac{1}{2d} \sum_{e\colon |e|=1} \Bigl\{ (f, 1_{\{\omega_e=1\}}\, f) + (f, 1_{\{\omega_e=0\}}\, f) \Bigr\} = (f, f) \end{aligned} \tag{2.8}$$

and so $\|Q\|_{L^2} \le 1$. In particular, Q is self-adjoint and $\mathrm{spec}(Q) \subset [-1, 1]$.

Let $V\colon \Omega \to \mathbb{R}^d$ be the local drift at the origin, i.e.,

$$V(\omega) = \frac{1}{2d} \sum_{e\colon |e|=1} e\, 1_{\{\omega_e=1\}}\,. \tag{2.9}$$

(We will only be interested in $V(\omega)$ for $\omega \in \Omega_0$, but that is of no consequence here.) Clearly, since V is bounded, we have $V \in L^2$. For each $\epsilon > 0$, let $\psi_\epsilon\colon \Omega \to \mathbb{R}^d$ be the solution of

$$(1 + \epsilon - Q)\psi_\epsilon = V. \tag{2.10}$$

Since $1 - Q$ is a non-negative operator, ψ_ϵ is well-defined and $\psi_\epsilon \in L^2$ for all $\epsilon > 0$. The following theorem is the core of the whole theory:

Theorem 2.2 *There exists a sequence (ϵ_n) of positive numbers tending to zero and a function $\chi\colon \mathbb{Z}^d \times \Omega_0 \to \mathbb{R}^d$ such that for every $x \in \mathbb{Z}^d$,*

$$\lim_{n\to\infty} 1_{\{x\in\mathcal{C}_\infty\}}(\psi_{\epsilon_n}\circ\tau_x - \psi_{\epsilon_n}) = \chi(x, \cdot) \tag{2.11}$$

both $\mathbb{P}_0$-almost surely and in L^2. Moreover, the following properties hold:

(1) *(Shift covariance) For $\mathbb{P}_0$-almost every $\omega \in \Omega_0$,*

$$\chi(x, \omega) - \chi(y, \omega) = \chi\bigl(x - y, \tau_y(\omega)\bigr) \tag{2.12}$$

holds for all $x, y \in \mathcal{C}_\infty(\omega)$.

(2) *(Harmonicity) For $\mathbb{P}_0$-almost every $\omega \in \Omega_0$, the function*

$$x \mapsto \chi(x, \omega) + x \tag{2.13}$$

is harmonic with respect to the transition probabilities (1.3–1.4) of the random walk (X_n).

(3) *(Square integrability) There exists a constant $C<\infty$ such that*

$$\bigl\|[\chi(x+e,\cdot)-\chi(x,\cdot)]\,1_{\{x\in\mathcal{C}_\infty\}}\,1_{\{\omega_e=1\}}\circ\tau_x\bigr\|_2<C \tag{2.14}$$

is true for all $x\in\mathbb{Z}^d$ and all e with $|e|=1$.

The rest of this section is spent on proving Theorem 2.2. The proof is based on spectral calculus and it closely follows the corresponding arguments from [16].

2.3 Spectral calculations.

Let μ_V denote the spectral measure of $Q\colon L^2\to L^2$ associated with function V, i.e., for every bounded, continuous $\Phi\colon[-1,1]\to\mathbb{R}$, we have

$$\bigl(V,\Phi(Q)V\bigr)=\int_{-1}^{1}\Phi(\lambda)\mu_V(\mathrm{d}\lambda). \tag{2.15}$$

Here we used that, since $\operatorname{spec}(Q)\in[-1,1]$, the measure μ_V is supported entirely in $[-1,1]$. The first observation, made already by Kipnis-Varadhan, is stated as follows:

Lemma 2.3 *We have $V\in\operatorname{Ran}(1-Q)^{1/2}$ and*

$$\int_{-1}^{1}\frac{1}{1-\lambda}\mu_V(\mathrm{d}\lambda)<\infty. \tag{2.16}$$

Proof. With some caution concerning the infinite cluster, the proof is a combination of arguments right before Theorem 1.3 and those in the proof of Theorem 4.1 in [16]. Let $f\in L^2$ be a bounded real-valued function and note that, by Lemma 2.1 and the symmetry of the sums,

$$\sum_{e\colon|e|=1}e\,\mathbb{E}_0(f\,1_{\{\omega_e=1\}})=\frac12\sum_{e\colon|e|=1}e\,\mathbb{E}_0\bigl((f-f\circ\tau_e)\,1_{\{\omega_e=1\}}\bigr). \tag{2.17}$$

Hence, for every $a\in\mathbb{R}^d$ we get

$$\begin{aligned}(f,a\cdot V)&=\frac12\frac{1}{2d}\sum_{e\colon|e|=1}(e\cdot a)\mathbb{E}_0\bigl((f-f\circ\tau_e)\,1_{\{\omega_e=1\}}\bigr)\\&\le\frac12\Bigl(\frac{1}{2d}\sum_{e\colon|e|=1}(e\cdot a)^2\mathbb{P}(\omega_e=1)\Bigr)^{1/2}\Bigl(\frac{1}{2d}\sum_{e\colon|e|=1}\mathbb{E}_0\bigl((f-f\circ\tau_e)^2\,1_{\{\omega_e=1\}}\bigr)\Bigr)^{1/2}.\end{aligned} \tag{2.18}$$

The first term on the right-hand side equals a constant times $|a|$, while Lemma 2.1 allows us to rewrite the second term into

$$\begin{aligned}&\frac{1}{2d}\sum_{e\colon|e|=1}\mathbb{E}_0\bigl((f-f\circ\tau_e)^2\,1_{\{\omega_e=1\}}\bigr)\\&\qquad=2\frac{1}{2d}\sum_{e\colon|e|=1}\mathbb{E}_0\bigl(f(f-f\circ\tau_e)\,1_{\{\omega_e=1\}}\bigr)=2\bigl(f,(1-Q)f\bigr).\end{aligned} \tag{2.19}$$

We thus get that there exists a constant $C < \infty$ such that for all bounded $f \in L^2$,

$$\bigl|(f, a \cdot V)\bigr| \le C|a|\bigl(f, (1-Q)f\bigr)^{1/2}. \tag{2.20}$$

Applying (2.20) for f of the form $f = e \cdot \Psi(Q)V$ and invoking (2.15), we find that for every bounded continuous $\Psi\colon [-1,1] \to \mathbb{R}$, we have

$$\left|\int \Psi(\lambda)\mu_V(\mathrm{d}\lambda)\right| \le C\biggl(\int (1-\lambda)\Psi(\lambda)^2\mu_V(\mathrm{d}\lambda)\biggr)^{1/2}. \tag{2.21}$$

Substituting $\Psi_\epsilon(\lambda) = (1/\epsilon) \wedge \frac{1}{1-\lambda}$ for Ψ and noting that $(1-\lambda)\Psi_\epsilon(\lambda) \le 1$, we get

$$\int \Psi_\epsilon(\lambda)\mu_V(\mathrm{d}\lambda) \le C\biggl(\int \Psi_\epsilon(\lambda)\mu_V(\mathrm{d}\lambda)\biggr)^{1/2} \tag{2.22}$$

and so

$$\int \Psi_\epsilon(\lambda)\mu_V(\mathrm{d}\lambda) \le C^2. \tag{2.23}$$

The Monotone Convergence Theorem now implies

$$\int \frac{1}{1-\lambda}\mu_V(\mathrm{d}\lambda) = \sup_{\epsilon>0} \int \Psi_\epsilon(\lambda)\mu_V(\mathrm{d}\lambda) \le C^2 < \infty. \tag{2.24}$$

The fact that $V \in \mathrm{Ran}(1-Q)^{1/2}$ follows from (2.24). □

Using spectral calculus we will now prove:

Lemma 2.4 *Let ψ_ϵ be as defined in* (2.10). *Then*

$$\lim_{\epsilon\downarrow 0} \epsilon\|\psi_\epsilon\|_2^2 = 0. \tag{2.25}$$

Moreover, for e with $|e| = 1$ let $G_e^{(\epsilon)} = 1_{\{0\in\mathcal{C}_\infty\}}\, 1_{\{\omega_e=1\}}(\psi_\epsilon \circ \tau_e - \psi_\epsilon)$. Then for all $x \in \mathbb{Z}^d$ and all e with $|e| = 1$,

$$\lim_{\epsilon_1,\epsilon_2\downarrow 0} \bigl\|G_e^{(\epsilon_1)} \circ \tau_x - G_e^{(\epsilon_2)} \circ \tau_x\bigr\|_2 = 0 \tag{2.26}$$

Proof. The main ideas are again more or less directly taken from the proof of Theorem 1.3 in [16]; some caution is necessary concerning the containment in the infinite cluster in the proof of (2.26). In light of the definition of ψ_ϵ and the fact that we have

$$\epsilon\|\psi_\epsilon\|_2^2 = \int_{-1}^{1} \frac{\epsilon}{(1+\epsilon-\lambda)^2}\mu_V(\mathrm{d}\lambda). \tag{2.27}$$

The integrand is dominated by $\frac{1}{1-\lambda}$ and tends to zero as $\epsilon \downarrow 0$ for every λ in the support of μ_V. From here (2.25) follows by the Dominated Convergence Theorem.

The second part of the claim is proved similarly: First we get rid of the x-dependence by noting that, due to the fact that $G_e^{(\epsilon)} \circ \tau_x$ enforces $x \in \mathcal{C}_\infty$, the translation invariance of $\mathbb{P}$ implies

$$\bigl\|G_e^{(\epsilon_1)} \circ \tau_x - G_e^{(\epsilon_2)} \circ \tau_x\bigr\|_2 \le \bigl\|G_e^{(\epsilon_1)} - G_e^{(\epsilon_2)}\bigr\|_2. \tag{2.28}$$

Next we square the right-hand side and average over all e. Using that G_e also enforces $\omega_e = 1$ and applying (2.17), we thus get

$$\frac{1}{2d}\sum_{e\colon |e|=1} \bigl\|G_e^{(\epsilon_1)} - G_e^{(\epsilon_2)}\bigr\|_2^2 = 2\bigl(\psi_{\epsilon_1}-\psi_{\epsilon_2},(1-Q)(\psi_{\epsilon_1}-\psi_{\epsilon_2})\bigr). \tag{2.29}$$

Now we calculate

$$\bigl(\psi_{\epsilon_1}-\psi_{\epsilon_2},(1-Q)(\psi_{\epsilon_1}-\psi_{\epsilon_2})\bigr) = \int_{-1}^{1}\frac{(\epsilon_1-\epsilon_2)^2(1-\lambda)}{(1+\epsilon_1-\lambda)^2(1+\epsilon_2-\lambda)^2}\mu_V(\mathrm{d}\lambda). \tag{2.30}$$

The integrand is again bounded by $\frac{1}{1-\lambda}$, for all $\epsilon_1,\epsilon_2>0$, and it tends to zero as $\epsilon_1,\epsilon_2\downarrow 0$. The desired claim follows by the Dominated Convergence Theorem. □

Now we are ready to prove Theorem 2.2:

Proof of Theorem 2.2. Let $G_e^{(\epsilon)}\circ\tau_x$ be as in Lemma 2.4. Using (2.26) we know that $G_e^{(\epsilon)}\circ\tau_x$ converges in L^2 as $\epsilon\downarrow 0$ and, restricting ϵ to a suitable sequence (ϵ_n), we can even ensure that the convergence happens almost surely. If x and y are nearest neighbors, with $y=x+e$, we will denote the limit by $G_{x,y}$. Note that $G_{x,y}(\omega)=0$ if $\omega_{\langle x,y\rangle}=0$.

Since $\|\psi_\epsilon\circ\tau_x\,1_{\{x\in\mathcal{C}_\infty\}}\|_2\le\|\psi_\epsilon\|_2$, Lemma 2.4 implies that $\sqrt{\epsilon}\psi_\epsilon$ tends to zero in L^2 as $\epsilon\downarrow 0$. Reducing to a subsequence, we may again assume that $\sqrt{\epsilon_n}\psi_{\epsilon_n}$ tends to zero almost surely. We conclude that there exists $\Omega_0^\star\in\mathcal{B}$ with $\Omega_0^\star\subset\Omega_0$ and $\mathbb{P}_0(\Omega_0^\star)=1$ such that for all $\omega\in\Omega_0^\star$, all $x\in\mathbb{Z}^d$ and all e with $|e|=1$,

$$\lim_{n\to\infty} G_e^{(\epsilon_n)}(\tau_x\omega) = G_{x,x+e}(\omega) \quad\text{and}\quad \lim_{n\to\infty}\sqrt{\epsilon_n}\,\psi_{\epsilon_n}(\omega)=0. \tag{2.31}$$

Both limits exist in L^2 as well. For convenience we will also require that $\mathcal{C}_\infty(\omega)$ is connected (i.e., the infinite cluster is unique) for all $\omega\in\Omega_0^\star$.

Pick $\omega\in\Omega_0^\star$ and $x\in\mathcal{C}_\infty(\omega)$, and let $(x_0,x_1,\dots,x_n)$ be a nearest-neighbor path on $\mathcal{C}_\infty(\omega)$ connecting $x_0=0$ to $x_n=x$. Writing

$$\psi_{\epsilon_n}\circ\tau_x(\omega)-\psi_{\epsilon_n}(\omega)=\sum_{k=0}^{n-1}G_{x_k,x_{k+1}}^{(\epsilon_n)}(\omega) \tag{2.32}$$

we see that the difference on the left converges to

$$\chi(x,\omega)\overset{\text{def}}{=}\sum_{k=0}^{n-1}G_{x_k,x_{k+1}}(\omega) \tag{2.33}$$

as $n\to\infty$. The functions $G_{x,x+e}$ form a gradient field on $\mathcal{C}_\infty$ and so we have $G_{x,x+e}(\omega)+G_{x+e,x}(\omega)=0$ and, more generally, $\sum_{k=0}^n G_{x_k,x_{k+1}}=0$ whenever $(x_0,\dots,x_n)$ is a closed loop on $\mathcal{C}_\infty$. This shows that the definition of χ is independent of the initial choice of the path. The same argument also proves the shift covariance in (2.12).

In light of shift covariance, to prove the harmonicity of $x\mapsto x+\chi(x,\omega)$ it suffices to show for all $\omega\in\Omega_0^\star$,

$$\frac{1}{2d}\sum_{e\colon |e|=1}\bigl[\chi(0,\omega)-\chi(e,\omega)\bigr]\,1_{\{\omega_e=1\}}=V(\omega). \tag{2.34}$$

Noting that $\chi(e,\omega)-\chi(0,\omega)=G_{0,e}(\omega)$, the left hand side is the $n\to\infty$ limit of

$$\frac{1}{2d}\sum_{e\colon |e|=1}\bigl[\psi_{\epsilon_n}-\psi_{\epsilon_n}\circ\tau_e\bigr]\,1_{\{\omega_e=1\}}=(1-Q)\psi_{\epsilon_n}. \tag{2.35}$$

The definition of ψ_ϵ tells us that $(1-Q)\psi_{\epsilon_n}=-\epsilon_n\psi_{\epsilon_n}+V$. From here we get (2.34) by recalling that $\epsilon_n\psi_{\epsilon_n}(\omega)$ tends to zero for all $\omega\in\Omega_0^\star$.

To prove the square integrability in part (3) we note that, by the construction of the corrector,

$$\bigl[\chi(x+e,\cdot)-\chi(x,\cdot)\bigr]\,1_{\{x\in\mathcal{C}_\infty\}}\,1_{\{\omega_e=1\}}\circ\tau_x=G_{x,x+e}. \tag{2.36}$$

But $G_{x,x+e}$ is the L^2-limit of L^2-functions $G_e^{(\epsilon_n)}\circ\tau_x$ whose L^2-norm is bounded by that of $G_e^{(\epsilon)}$. Hence the bound (2.14) follows with $C=\max_{e\colon |e|=1}\|G_{0,e}\|_2$. □

3. Ergodic-theory input

Here we will establish some basic claims whose common feature is the use of ergodic theory. Modulo some care for the containment in the infinite cluster, all of these results are quite standard and their proofs (cf Sect. 3.2) can be skipped on a first reading. The principal conclusions of this section are Theorems 3.1 and 3.2.

3.1 Statements.

Our first result concerns the ergodicity of the Markov chain. The claim that will suffice for our later needs is as follows:

Theorem 3.1 *Let $f\in L^1(\Omega,\mathcal{B},\mathbb{P}_0)$. Then for $\mathbb{P}_0$-almost all $\omega\in\Omega$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\circ\tau_{X_k}(\omega)=\mathbb{E}_0(f),\qquad P_{0,\omega}\text{-almost surely}. \tag{3.1}$$

Similarly, if $f\colon\Omega^2\to\mathbb{R}$ is a measurable function such that $\mathbb{E}_0(E_{0,\omega}|f(\omega,\tau_{X_1}\omega)|)<\infty$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(\tau_{X_k}\omega,\tau_{X_{k+1}}\omega)=\mathbb{E}_0\bigl(E_{0,\omega}(f(\omega,\tau_{X_1}\omega))\bigr) \tag{3.2}$$

for $\mathbb{P}_0$-almost all ω and $P_{0,\omega}$-almost all trajectories of $(X_k)_{k\ge0}$.

The next principal result of this section will be the ergodicity of the "induced shift" on Ω_0. To define this concept, let e be a vector with $|e|=1$ and, for every $\omega\in\Omega_0$, let

$$n(\omega)=\min\bigl\{k>0\colon ke\in\mathcal{C}_\infty(\omega)\bigr\}. \tag{3.3}$$

By Birkhoff's Ergodic Theorem we know that $\{k>0\colon ke\in\mathcal{C}_\infty\}$ has positive density in $\mathbb{N}$ and so $n(\omega)<\infty$ almost surely. Therefore we can define the map $\sigma_e\colon\Omega_0\to\Omega_0$ by

$$\sigma_e(\omega)=\tau_{n(\omega)e}\,\omega. \tag{3.4}$$

We call σ_e the *induced shift*. Then we claim:

Theorem 3.2 *For every e with $|e| = 1$, the induced shift $\sigma_e \colon \Omega_0 \to \Omega_0$ is $\mathbb{P}_0$-preserving and ergodic with respect to $\mathbb{P}_0$.*

Both theorems are consequences of ergodicity of the underlying Markov chain on environments (see Proposition 3.5). For finite-state Markov chains this is a standard textbook material, but our state space is somewhat large and so alternative arguments are necessary. Since we could not find appropriate versions of these claims in the literature, we included complete proofs.

3.2 Proofs.

We begin by Theorem 3.2 which will follow from a more general statement, Lemma 3.3, below. Let $(\mathcal{X}, \mathscr{X}, \mu)$ be a probability space, and let $T \colon \mathcal{X} \to \mathcal{X}$ be invertible, measure preserving and ergodic with respect to μ. Let $A \in \mathscr{X}$ be of positive measure, and define $n \colon A \to \mathbb{N} \cup \{\infty\}$ by $n(x) = \min\{k > 0 \colon T^k(x) \in A\}$. The Poincaré Recurrence Theorem tells us that $n(x) < \infty$ almost surely. Therefore we can define, up to a set of measure zero, the map $S \colon A \to A$ by

$$S(x) = T^{n(x)}(x), \qquad x \in A. \tag{3.5}$$

Then we have:

Lemma 3.3 *S is measure preserving and ergodic with respect to $\mu(\cdot|A)$. It is also almost surely invertible with respect to the same measure.*

Proof. (1) S is measure preserving: For $j = 1, 2, \dots$, let $A_j = \{x \in A \colon n(x) = j\}$. Then the A_j's are disjoint and $\mu(A \setminus \bigcup_{j\ge1} A_j) = 0$. First we show that for every $i \neq j$,

$$S(A_i) \cap S(A_j) = \emptyset. \tag{3.6}$$

To do this, we use the fact that T is invertible. Indeed, if $x \in S(A_i) \cap S(A_j)$ for $1 \le i < j$, then $x = T^i(y) = T^j(z)$ for some $y, z \in A$ with $n(y) = i$ and $n(z) = j$. But the fact that T is invertible implies that $y = T^{j-i}(z)$, i.e., $n(z) \le j - i < j$, a contradiction. To see that S is measure preserving, we note that the restriction of S to A_j is T^j, which is measure preserving. Hence, S is measure preserving on A_j and, by (3.6), on the disjoint union $\bigcup_{j\ge1} A_j$.

(2) S is almost surely invertible: $S^{-1}(\{x\}) \cap \{S \text{ is well defined}\}$ is a one-point set by the fact that T is itself invertible.

(3) S is ergodic: Let $B \in \mathscr{X}$ be such that $B \subseteq A$ and $0 < \mu(B) < \mu(A)$. Assume that B was S-invariant. Then for every $x \in B$ and every $n \ge 1$ we have $S^n(x) \notin A \setminus B$. This means that for every $x \in B$ and every $k \ge 1$ such that $T^k(x) \in A$, we have $T^k(x) \notin A \setminus B$. If follows that $C = \bigcup_{k\ge1} T^k(B)$ is (almost-surely) T-invariant and $\mu(C) \in (0, 1)$, a contradiction with the ergodicity of T. $\square$

Proof of Theorem 3.2. We know that the shift τ_e is invertible, measure preserving and ergodic with respect to $\mathbb{P}$. By Lemma 3.3 the induced shift $\sigma_e \colon \Omega_0 \to \Omega_0$ is $\mathbb{P}_0$-preserving, almost-surely invertible and ergodic with respect to $\mathbb{P}_0$. $\square$

In the present circumstances, Theorem 3.2 has one important consequence:

Lemma 3.4 *Let $B \in \mathscr{B}$ be a subset of Ω_0 such that for almost all $\omega \in B$,*

$$P_{0,\omega}(\tau_{X_1}\omega \in B) = 1. \tag{3.7}$$

Then B is a zero-one event under $\mathbb{P}_0$.

Proof. The Markov property and (3.7) imply that $P_{0,\omega}(\tau_{X_n}\omega \in B) = 1$ for all $n \geq 1$ and $\mathbb{P}_0$-almost every $\omega \in B$. We claim that $\sigma_e(\omega) \in B$ for $\mathbb{P}_0$-almost every $\omega \in B$. Indeed, let $\omega \in B$ be such that $\tau_{X_n}\omega \in B$ for all $n \geq 1$, $P_{0,\omega}$-almost surely. Let $n(\omega)$ be as in (3.3) and note that we have $n(\omega)e \in \mathcal{C}_\infty$. By the uniqueness of the infinite cluster, there is a path of finite length connecting 0 and $n(\omega)e$. If ℓ is the length of this path, we have $P_{0,\omega}(X_\ell = n(\omega)e) > 0$. This means that $\sigma_e(\omega) = \tau_{n(\omega)e}(\omega) \in B$, i.e., B is almost surely σ_e-invariant. By the ergodicity of the induced shift, B is a zero-one event. □

Our next goal will be to prove that the Markov chain on environments is ergodic. Let us fix $\mathcal{X} = \Omega^{\mathbb{Z}}$ and let $\mathscr{X}$ be the product σ-algebra, $\mathscr{X} = \mathscr{B}^{\otimes\mathbb{Z}}$. The space $\mathcal{X}$ is a space of two-sided sequences $(\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$—note that the index on ω is an index in the sequence which is unrelated to the value of the configuration at a point. Let μ be the measure on $(\mathcal{X}, \mathscr{X})$ such that for any $B \in \mathscr{B}^{2n+1}$,

$$\mu\big((\omega_{-n}, \dots, \omega_n) \in B\big) = \int_B \mathbb{P}_0(\mathrm{d}\omega_{-n}) Q(\omega_{-n}, \mathrm{d}\omega_{-n+1}) \cdots Q(\omega_{n-1}, \mathrm{d}\omega_n), \tag{3.8}$$

where Q is the Markov kernel defined in (2.1). (Since $\mathbb{P}_0$ is preserved by Q, these finite-dimensional measures are consistent and μ exists and is unique by Kolmogorov's Theorem.) Let $T\colon X \to X$ be the shift defined by $(T\omega)_n = \omega_{n+1}$. Then T is measure preserving.

Proposition 3.5 *T is ergodic with respect to μ.*

Proof. Let E_μ denote expectation with respect to μ. Pick $A \subseteq X$ that is measurable and T-invariant. We need to show that

$$\mu(A) \in \{0, 1\}. \tag{3.9}$$

Let $f\colon \Omega \to \mathbb{R}$ be defined as

$$f(\omega_0) = E_\mu(1_A \,|\omega_0). \tag{3.10}$$

First we claim that $f = 1_A$ almost surely. Indeed, since A is T-invariant, there exist $A_+ \in \sigma(\omega_k\colon k > 0)$ and $A_- \in \sigma(\omega_k\colon k < 0)$ such that A and $A_\pm$ differ only by null sets from one another. (This follows by approximation of A by finite-dimensional events and using the T-invariance of A.) Now conditional on ω_0, the event A_+ is independent of $\sigma(\omega_k\colon k < 0)$ and so Lévy's Martingale Convergence Theorem gives us

$$\begin{aligned} E_\mu(1_A \,|\omega_0) &= E_\mu(1_{A_+} \,|\omega_0) = E_\mu(1_{A_+} \,|\omega_0, \omega_{-1}, \dots, \omega_{-n}) \\ &= E_\mu(1_{A_-} \,|\omega_0, \omega_{-1}, \dots, \omega_{-n}) \underset{n\to\infty}{\longrightarrow} 1_{A_-} = 1_A, \end{aligned} \tag{3.11}$$

with all equalities valid μ-almost surely.

Next let $B \subset \Omega$ be defined by $B = \{\omega_0\colon f(\omega_0) = 1\}$. Clearly, B is $\mathscr{B}$-measurable and, since the ω_0-marginal of μ is $\mathbb{P}_0$,

$$\mu(A) = E_\mu(f) = \mathbb{P}_0(B). \tag{3.12}$$

Hence, to prove (3.9), we need to show that

$$\mathbb{P}_0(B) \in \{0, 1\}. \tag{3.13}$$

But A is T-invariant and so, up to sets of measure zero, if $\omega_0 \in B$ then $\omega_1 \in B$. This means that B satisfies condition (3.7) of Lemma 3.4 and so B is a zero-one event under $\mathbb{P}_0$. $\square$

Now we can finally prove Theorem 3.1:

Proof of Theorem 3.1. First note that $(\tau_{X_k}(\omega))_{k\ge 0}$ has the same law in $\mathbb{P}_0 \otimes P_{0,\omega}$ as $(\omega_0, \omega_1, \dots)$ has in μ. Hence, if $g(\dots, \omega_{-1}, \omega_0, \omega_1, \dots) = f(\omega_0)$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{\infty} f \circ \tau_{X_k} \overset{\mathcal{D}}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{\infty} g \circ T^k. \tag{3.14}$$

The latter limit exists by Birkhoff's Ergodic Theorem and, in light of Proposition 3.5, equals $E_\mu(g) = \mathbb{E}_0(f)$ almost surely. The second part is proved analogously. $\square$

4. Sublinearity along coordinate directions

Equipped with the tools from the previous two sections, we can start addressing the main problem of our proof: the sublinearity of the corrector. Here we will prove the corresponding claim along the coordinate directions in $\mathbb{Z}^d$.

Let $n(\omega)$ be as defined in (3.3) and define inductively $n_1(\omega) = n(\omega)$ and $n_{k+1}(\omega) = n_k(\sigma_e(\omega))$. The numbers (n_k), which are well-defined and finite on a set of full $\mathbb{P}_0$-measure, represent the successive "arrivals" of $\mathcal{C}_\infty$ to the positive part of the coordinate axis in direction e. The main goal of this section is to prove the following theorem:

Theorem 4.1 *For all e with $|e| = 1$ and $\mathbb{P}_0$-almost all $\omega \in \Omega_0$,*

$$\lim_{k\to\infty} \frac{\chi(n_k(\omega)e, \omega)}{k} = 0. \tag{4.1}$$

The proof is based on the following facts about the moments of $\chi(n_k(\omega)e, \omega)$:

Proposition 4.2 *Let e obey $|e| = 1$ and let $v_e = v_e(\omega) = n(\omega)e$, where $n(\omega)$ is the smallest positive integer such that $n(\omega)e \in \mathcal{C}_\infty(\omega)$. Let χ be the corrector defined in Theorem 2.2. Then*

(1) $\mathbb{E}_0(|\chi(v_e, \cdot)|) < \infty$.
(2) $\mathbb{E}_0(\chi(v_e, \cdot)) = 0$.

The proof of this proposition will in turn be based on a bound on the tails of the length of the shortest path connecting the origin to v_e. We begin by showing that $|v_e|$ has exponential tails:

Lemma 4.3 *For each $p > p_c$ there is $a = a(p) > 0$ such that for all e with $|e| = 1$,*

$$\mathbb{P}_0\big(|v_e| > n\big) \le \mathrm{e}^{-an}, \qquad n \ge 1. \tag{4.2}$$

Proof. The proof uses a different argument in $d = 2$ and $d \ge 3$. In dimensions $d \ge 3$, we will use the fact that the slab-percolation threshold coincides with p_c, as was proved by Grimmett and Marstrand [14]. Indeed, given $p > p_c$, let $K \ge 1$ be so large that $\mathbb{Z}^{d-1} \times \{1, \dots, K\}$ contains an infinite cluster almost surely. By the uniqueness of the percolation cluster in $\mathbb{Z}^d$, this slab-cluster is almost surely a subset of $\mathcal{C}_\infty$. Our bound in (4.2) is derived as follows: Let A_K be the event that at least one of the sites in $\{je\colon j = 1, \dots, K\}$ is contained in the infinite connected component in $\mathbb{Z}^{d-1} \times \{1, \dots, K\}$. Then $\{|v_e| \ge Kn\} \cap \{0 \in \mathcal{C}_\infty\} \subset \bigcap_{\ell \le n} \tau_{\ell K e}(A)$. Since the events $\tau_{\ell K e}(A)$, $\ell = 1, \dots, n$, are independent, letting $p_K = \mathbb{P}(A_K)$ we have

$$\mathbb{P}(|v_e| \ge Kn,\, 0 \in \mathcal{C}_\infty) \le (1 - p_K)^n, \qquad n \ge 1. \tag{4.3}$$

From here (4.2) follows by choosing a appropriately.

In dimension $d = 2$, we will instead use a duality argument. Let Λ_n be the box $\{1, \dots, n\} \times \{1, \dots, n\}$. On $\{|v_e| \ge n\} \cap \{0 \in \mathcal{C}_\infty\}$, none of the boundary sites $\{je\colon j = 1, \dots, n\}$ are in $\mathcal{C}_\infty$. So either at least one of these boundary sites is in a finite component of size larger than n or there exists a dual crossing of Λ_n in the direction of e. By the exponential decay of truncated connectivities (Theorem 8.18 of Grimmett [13]) as well as dual connectivities (Theorem 6.75 of Grimmett [13]), the probability of each of these events decays exponentially with n. ☐

Our next lemma concerns the length of the shortest path between the origin and v_e:

Lemma 4.4 *Let $L = L(\omega)$ be the length of the shortest path from 0 to v_e. Then there exist a constant $C < \infty$ and $a > 0$ such that for every $n \ge 1$,*

$$\mathbb{P}_0(L > n) < C\mathrm{e}^{-an}. \tag{4.4}$$

Proof. Let $\mathrm{d}_\omega(0, x)$ be the length of the shortest path from 0 to x in configuration ω. Pick $\epsilon > 0$ such that ϵn is an integer. Then

$$\{L > n\} \subset \bigl\{|v_e| \ge \epsilon n\bigr\} \cup \bigcup_{k=1}^{\epsilon n} \bigl\{\mathrm{d}_\omega(0, ke) > n;\, 0, ke \in \mathcal{C}_\infty\bigr\}. \tag{4.5}$$

In light of Lemma 4.3, the claim will follow once we show that the probability of all events in the giant union on the right-hand side is bounded by $\mathrm{e}^{-a'n}$ with some $a' > 0$.

We will use the following large-deviation result from Theorem 1.1 of Antal and Pisztora [1]: There exist constants $a, \rho < \infty$ such that

$$\mathbb{P}\bigl(\mathrm{d}_\omega(0, x) > \rho|x|\bigr) \le \mathrm{e}^{-a|x|} \tag{4.6}$$

once $|x|$ is sufficiently large. Unfortunately, we cannot use this bound in (4.5) directly, because ke can be arbitrarily close to 0 (in Euclidean metric on $\mathbb{Z}^d$). To circumvent this problem, let w_e be the site $-me$ such that $m = \min\{m' > \epsilon n\colon -m'e \in \mathcal{C}_\infty\}$ and let $A_{x,y} = \{\mathrm{d}_\omega(x, y) \ge {}^n\!/\!_2,\, x, y \in \mathcal{C}_\infty\}$. Then either $|w_e| > 2\epsilon n$ or at least one site between $-2\epsilon n$ and $-\epsilon n$ is connected to either 0 or ke by a path longer than ${}^n\!/\!_2$. Since on $\{|w_e| > 2\epsilon n\}$ we must have $|v_{-e} \circ \sigma_{-e}^m| > \epsilon n$ for at least one $m = 1, \dots \epsilon n$, we have

$$\begin{aligned}&\bigl\{\mathrm{d}_\omega(0, ke) > n;\, 0, ke \in \mathcal{C}_\infty\bigr\} \\ &\qquad\subset \Bigl(\bigcup_{m=1}^{\epsilon n} \sigma_{-e}^m\bigl(\{|v_{-e}| \ge \epsilon n\}\bigr) \cup \bigcup_{\epsilon n \le \ell \le 2\epsilon n} \bigl(A_{0,-\ell e} \cup A_{ke,-\ell e}\bigr)\Bigr).\end{aligned} \tag{4.7}$$

Now all events in the first giant union have the same probability, which is exponentially small by Lemma 4.3. As to the second union, by (4.6) we know that

$$\mathbb{P}_0(A_{0,-\ell e}) \le \mathrm{e}^{-a\ell} \le \mathrm{e}^{-a\epsilon n} \tag{4.8}$$

whenever ϵ is so small that $4\epsilon\rho \le 1$, and a similar bound holds for $A_{ke,-\ell e}$ as well (except that here we need that $6\epsilon\rho \le 1$). The various unions then contribute a linear factor in n, which is easily absorbed into the exponential once n is sufficiently large. $\square$

Lemma 4.5 *Let N be a random variable taking values in positive integers and suppose that $N \in L^p$ for some $p > 2$. Let $X_1, X_2, \dots$ be random variables such that $\sup_{j\ge1} \|X_j\|_2 < \infty$. Then*

$$\sum_{j=1}^{N} X_j \in L^1. \tag{4.9}$$

Proof. By Cauchy-Schwarz inequality and the uniform bound on $\|X_j\|_2$ we get

$$\begin{aligned} E\Bigl|\sum_{j=1}^{N} X_j\Bigr| &= \sum_{n\ge1} E\biggl(\Bigl|\sum_{j=1}^{n} X_j\Bigr| 1_{\{N=n\}}\biggr) \\ &\le \sum_{n\ge1} \Bigl\|\sum_{j=1}^{n} X_j\Bigr\|_2 P(N=n)^{1/2} \\ &\le \bigl(\sup_{j\ge1} \|X_j\|_2\bigr) \sum_{n\ge1} \sqrt{n}\, P(N=n)^{1/2}. \end{aligned} \tag{4.10}$$

Under the assumption that N has the p-th moment, we get

$$\sum_{n\ge1} \sqrt{n}\, P(N=n)^{1/2} \le \biggl(\sum_{n\ge1} \frac{1}{n^{p-1}}\biggr)^{1/2} \bigl(E(N^p)\bigr)^{1/2} \tag{4.11}$$

by invoking Cauchy-Schwarz inequality one more time. $\square$

Proof of Proposition 4.2. Let ψ_ϵ be as defined in (2.10) and recall the corrector is given by the almost-sure limit (2.11). Recall the notation $G_{x,x+e}$ for the gradient $\chi(x+e,\cdot) - \chi(x,\cdot)$ whenever $x \in \mathcal{C}_\infty$ and the bond between x and $x+e$ is occupied. As in Lemma 4.4, let $L = L(\omega)$ be the length of the shortest path from 0 to v_e. It is easy to see that

$$|\chi(v_e,\omega)| \le \sum_{x\colon |x|_\infty \le L(\omega)} \; \sum_{e\colon |e|=1} |G_{x,x+e}(\omega)| \tag{4.12}$$

But Theorem 2.2(3) ensures that $\|G_{x,x+e}\|_2 < C$ uniformly in x and e and the number of terms in the sum is not larger than $N = 2d(2L+1)^d$. By Lemma 4.4, the random variable N has all moments and so Lemma 4.5 implies that $\chi(v_e,\cdot) \in L^1$.

In order to prove part (2), we note that $\chi(v_e,\cdot)$ is an almost sure, L^2 and hence also L^1-limit of $\psi_\epsilon \circ \tau_{v_e} - \psi_\epsilon$ (for ϵ varying along a sequence ϵ_n). But the definition of induced shift σ_e implies $\psi_\epsilon \circ \tau_{v_e} = \psi_\epsilon \circ \sigma_e$ and Theorem 3.2 gives

$$\mathbb{E}_0(\psi_\epsilon \circ \sigma_e - \psi_\epsilon) = 0. \tag{4.13}$$

From here (2) follows by taking L^1 limit $\epsilon \downarrow 0$. $\square$

Proof of Theorem 4.1. Let $f(\omega) = \chi(n_1(\omega)e, \omega)$, and let σ_e be the induced shift in the direction of e. Then we can write

$$\chi\bigl(n_k(\omega)e, \omega\bigr) = \sum_{\ell=0}^{k-1} f \circ \sigma_e^{\ell}(\omega). \tag{4.14}$$

By Proposition 4.2, $f \in L^1$ and $\mathbb{E}_0(f) = 0$. Since Theorem 3.2 ensures that σ_e is $\mathbb{P}_0$-preserving and ergodic, the claim follows by Birkhoff's Ergodic Theorem. □

5. Sublinearity everywhere

Here we will prove the principal technical estimates of this work. The level of control is different in $d = 2$ and $d \geq 3$, so we treat these cases separately.

5.1 Sublinearity in two dimensions.

We begin with an estimate of the corrector in large boxes in $\mathbb{Z}^2$:

Theorem 5.1 *Let $d = 2$ and let χ be the corrector defined in Theorem 2.2. Then for $\mathbb{P}_0$-almost every $\omega \in \Omega_0$,*

$$\lim_{n\to\infty} \max_{\substack{x\in\mathcal{C}_\infty(\omega)\\ |x|_\infty \leq n}} \frac{|\chi(x,\omega)|}{n} = 0. \tag{5.1}$$

The proof will be based on the following concept:

Definition 5.2 *Given $K > 0$ and $\epsilon > 0$, we say that a site $x \in \mathbb{Z}^d$ is* K, ϵ-good *(or just* good*) in configuration $\omega \in \Omega$ if $x \in \mathcal{C}_\infty(\omega)$ and*

$$\bigl|\chi(y,\omega) - \chi(x,\omega)\bigr| < K + \epsilon|x - y| \tag{5.2}$$

holds for every $y \in \mathcal{C}_\infty(\omega)$ of the form $y = \ell e_j$, where $\ell \in \mathbb{Z}$ and $j = 1, \dots, d$. We will use $\mathcal{G}_{K,\epsilon} = \mathcal{G}_{K,\epsilon}(\omega)$ to denote the set of K, ϵ-good sites in configuration ω.

On the basis of Theorem 4.1 it is clear that for each $\epsilon > 0$ there exists a $K < \infty$ such that the $\mathbb{P}_0(0 \in \mathcal{G}_{K,\epsilon}) > 0$. Our first goal is to estimate the size of the largest interval free of good points in blocks $[-n, n]$ on the coordinate axes:

Lemma 5.3 *Let e be one of the principal lattice vectors in $\mathbb{Z}^2$ and, given $\epsilon > 0$, let K be so large that $\mathbb{P}_0(0 \in \mathcal{G}_{K,\epsilon}) > 0$. For all $n \geq 1$ and $\omega \in \Omega$, let $y_0 < \dots < y_r$ be the ordered set of all integers from $[-n, n]$ such that $y_i e \in \mathcal{G}_{K,\epsilon}(\omega)$. Let*

$$\triangle_n(\omega) = \max_{j=1,\dots,r} (y_j - y_{j-1}). \tag{5.3}$$

(If no such y_i exists, we define $\triangle_n(\omega) = \infty$.) Then

$$\lim_{n\to\infty} \frac{\triangle_n}{n} = 0, \qquad \mathbb{P}\text{-almost surely}. \tag{5.4}$$

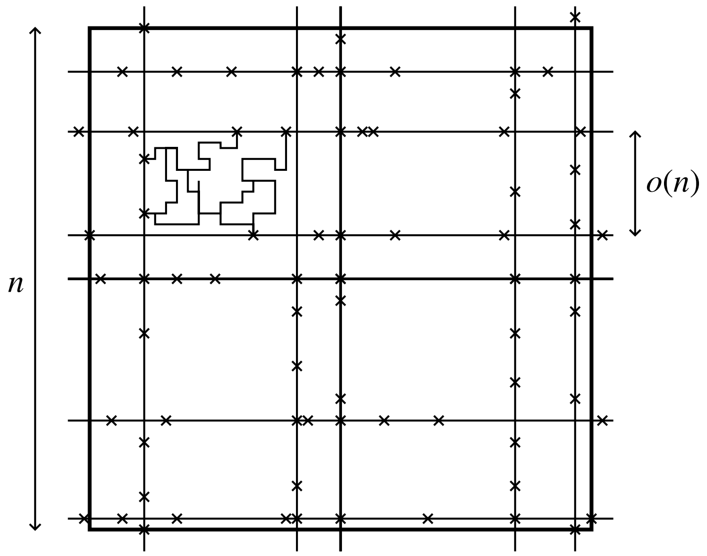

FIGURE 3. An illustration of the main idea of the proof of Theorem 5.1. Here a square of side n is intersected by a grid $\mathbb{G}$ of *good lines* "emanating" from the good points on the x and y axes. The crosses represent the points on these lines which are in $\mathcal{C}_\infty$. Along the good lines the corrector grows slower than linear and so anywhere on $\mathbb{G}$ sublinearity holds. For the part of $\mathcal{C}_\infty$ that is not on $\mathbb{G}$, the maximum principle for $x \mapsto x + \chi(x,\omega)$ lets us bound the corrector by the values on the parts of the grid that surround it, modulo an error of order $o(n)$.

Proof. Since $\mathbb{P}$ is τ_e invariant and τ_e is ergodic, we have

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} 1_{\{0\in\mathcal{G}_{K,\epsilon}\}} \circ \tau_e^k = \mathbb{P}(0 \in \mathcal{G}_{K,\epsilon}) \tag{5.5}$$

$\mathbb{P}$-almost surely. But if $\triangle_n/n$ does not tend to zero, the limit could not exist. $\square$

Proof of Theorem 5.1. Fix $\epsilon \in (0, 1/2)$ and let K_0 be such that $\mathbb{P}(0 \in \mathcal{G}_{K,\epsilon}) > 0$ for all $K \geq K_0$ (we are using that $\mathcal{G}_{K,\epsilon}$ increases with K). Let $\Omega_0^\star \subset \Omega_0$ be the set of configurations such that the conclusion of Lemma 5.3 applies for both x and y-axes. We will show that for every $\omega \in \Omega_0^\star$ the limit in (5.1) is less than 6ϵ almost surely.

Let e_1 and e_2 denote the coordinate vectors in $\mathbb{Z}^2$. Fix $\omega \in \Omega_0^\star$ and adjust $K \geq K_0$ so that $0 \in \mathcal{G}_{K,\epsilon}$. (This is possible by the definition of $\Omega_0^\star$.) Then we define $(x_k)_{k\in\mathbb{Z}}$ to be the increasing two-sided sequence of all integers such that

$$x_k e_1 \in \mathcal{G}_{K,\epsilon}(\omega), \qquad k \in \mathbb{Z}. \tag{5.6}$$

If $\triangle_n$ be the maximal gap between consecutive x_j's that lie in $[-n,n]$, cf (5.3), we define $n_1(\omega)$ be the least integer such that $\triangle_n/n < \epsilon$ for all $n \geq n_1(\omega)$. Similarly we identify a two-sided sequence $(y_n)_{n\in\mathbb{Z}}$ of integers such that

$$y_k e_2 \in \mathcal{G}_{K,\epsilon}(\omega), \qquad k \in \mathbb{Z}, \tag{5.7}$$

and let $n_2(\omega)$ be the quantity corresponding to $n_1(\omega)$ in this case.

Let $n_0 = \max\{n_1, n_2\}$. We claim that for all $n \ge n_0(\omega)$,

$$\max_{\substack{x\in\mathcal{C}_\infty(\omega)\\ |x|_\infty\le n}} |\chi(x,\omega)| \le 2K + 6\epsilon n. \tag{5.8}$$

To prove this, let us consider the grid $\mathbb{G} = \mathbb{G}(\omega)$ of *good lines*

$$\{y_k e_1 + n e_2 \colon n \in \mathbb{Z}\}, \qquad k \in \mathbb{Z}, \tag{5.9}$$

and

$$\{n e_1 + x_k e_2 \colon n \in \mathbb{Z}\}, \qquad k \in \mathbb{Z}, \tag{5.10}$$

see Fig. 3. As a first step we will use the harmonicity of $x \mapsto x+\chi(x,\omega)$ to deal with $x \in \mathcal{C}_\infty \setminus \mathbb{G}$. Indeed, any such x is enclosed between two horizontal and two vertical grid lines and every path on $\mathcal{C}_\infty$ connecting x to "infinity" necessarily intersects one of these lines at a point which is also in $\mathcal{C}_\infty$. Applying the maximum (and minimum) principle for harmonic functions we get

$$\max_{\substack{x\in\mathcal{C}_\infty\setminus\mathbb{G}\\ |x|_\infty\le n}} |\chi(x,\omega)| \le 2\epsilon n + \max_{\substack{x\in\mathcal{C}_\infty\cap\mathbb{G}\\ |x|_\infty\le 2n}} |\chi(x,\omega)|. \tag{5.11}$$

Here we used that the enclosing lines are not more than $\frac{\epsilon}{1-\epsilon}n \le 2\epsilon n \le n$ apart and, in particular, they all intersect the block $[-2n, 2n] \times [-2n, 2n]$.

To estimate the maximum on the grid, we pick, say, a horizontal grid line with y-coordinate y_k and note that, by (2.12), for every $x \in \mathcal{C}_\infty$ on this line,

$$\chi(x,\omega) - \chi(y_k e_2,\omega) = \chi(x - y_k e_2, \tau_{y_k e_2}\omega). \tag{5.12}$$

By (5.7) and the fact that $x - y_k e_2 \in \mathcal{C}_\infty(\tau_{y_k e_2}\omega)$ we have

$$\bigl|\chi(x,\omega) - \chi(y_k e_2,\omega)\bigr| \le K + 2\epsilon n \tag{5.13}$$

whenever x is such that $|x|_\infty \le 2n$. Applying the same argument to the vertical line through the origin, and x replaced by $y_k e_2$, we get

$$\bigl|\chi(x,\omega)\bigr| \le 2K + 4\epsilon n \tag{5.14}$$

for every $x \in \mathcal{C}_\infty \cap \mathbb{G}$ with $|x|_\infty \le 2n$. Combining this with (5.11), the estimate (5.8) and the whole claim are finally proved. $\square$

5.2 Three and higher dimensions.

In $d \ge 3$ we have the following weaker version of Theorem 5.1:

Theorem 5.4 *Let $d \ge 3$. Then for all $\epsilon > 0$ and $\mathbb{P}_0$-almost all ω,*

$$\limsup_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{\substack{x\in\mathcal{C}_\infty(\omega)\\ |x|\le n}} 1_{\{|\chi(x,\omega)|\ge\epsilon n\}} = 0. \tag{5.15}$$

The proof will use induction on the dimension of the box over which we take average in (5.15). For each $\nu = 1, \dots, d$, let Λ_n^ν be the ν-dimensional box

$$\Lambda_n^\nu = \bigl\{k_1 e_1 + \dots + k_\nu e_\nu \colon k_i \in \mathbb{Z},\, |k_i| \le n \ \forall i = 1, \dots, \nu\bigr\}. \tag{5.16}$$

For each $\omega \in \Omega$, we also define the quantity

$$\varrho_\nu(\omega) = \lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} \inf_{y \in \mathcal{C}_\infty(\omega) \cap \Lambda_n^1} \frac{1}{|\Lambda_n^\nu|} \sum_{x \in \mathcal{C}_\infty(\omega) \cap \Lambda_n^\nu} 1_{\{|\chi(x,\omega) - \chi(y,\omega)| \ge \epsilon n\}} . \tag{5.17}$$

Note that the infimum is taken only over sites in one-dimensional box Λ_n^1. The induction step is encapsulated into the following lemma:

Lemma 5.5 *Let $1 \le \nu < d$. If $\varrho_\nu = 0$, $\mathbb{P}$-almost surely, then also $\varrho_{\nu+1} = 0$, $\mathbb{P}$-almost surely.*

Before we start the formal proof, let us discuss its main idea: Suppose that $\varrho_\nu = 0$ for some $\nu < d$, $\mathbb{P}$-almost surely. Pick $\epsilon > 0$. Then for $\mathbb{P}$-almost all ω and all sufficiently large n, there exists a set of sites $\Delta \subset \Lambda_n^\nu \cap \mathcal{C}_\infty$ such that

$$\bigl|(\Lambda_n^\nu \cap \mathcal{C}_\infty) \setminus \Delta\bigr| \le \epsilon |\Lambda_n^\nu| \tag{5.18}$$

and

$$\bigl|\chi(x,\omega) - \chi(y,\omega)\bigr| \le \epsilon n, \qquad x, y \in \Delta. \tag{5.19}$$

Moreover, recall our notation $\mathcal{G}_{K,\epsilon}(\omega)$ for the set of sites in $\mathcal{C}_\infty(\omega)$ that are K,ϵ-good in configuration ω. Then for K sufficiently large, Δ could be picked so that $\Delta \cap \Lambda_n^1 \ne \emptyset$ and

$$\Delta \subset \mathcal{G}_{K,\epsilon} \tag{5.20}$$

provided n is large enough. This is seen by a direct application of the Ergodic Theorem which tells us that $\mathcal{G}_{K,\epsilon} \uparrow \mathcal{C}_\infty$ as $K \to \infty$, $\mathbb{P}$-almost surely.

Note that, by the construction of the set Δ, we have that

$$\bigl|\chi(z,\omega) - \chi(x,\omega)\bigr| \le K + \epsilon n \tag{5.21}$$

for any $x \in \Delta$ and any $z \in \Lambda_n^{\nu+1} \cap \mathcal{C}_\infty$ of the form $x + je_{\nu+1}$. This allows us to "connect" any $r, s \in \mathcal{C}_\infty \cap \Lambda_n^{\nu+1}$ by broken-line path and use (5.19) and (5.21) to estimate

$$|\chi(r,\omega) - \chi(s,\omega)| \le 2K + 3\epsilon n, \tag{5.22}$$

see Fig. 4 for an illustration.

Unfortunately, since each such z has to have an $x \in \Lambda_n^\nu \cap \mathcal{C}_\infty$ underneath, the set of r's we have so far under control will occupy at most $P_\infty = \mathbb{P}(0 \in \mathcal{C}_\infty)$ fraction of all sites in $\Lambda_n^{\nu+1}$. To fix this problem, we will have to carry out the same argument in a whole "stack" of translates of Λ_n^ν in the $(\nu+1)$-st direction—not depicted on the right of Fig. 4—i.e., using the sets

$$\Lambda_{n,j}^\nu = \tau_{je_{\nu+1}}(\Lambda_n^\nu), \qquad j = 0, \dots, L-1. \tag{5.23}$$

Here L is a deterministic number chosen so that, for a given $\delta > 0$, the set

$$\Delta_0 = \bigl\{x \in \Lambda_n^\nu \colon \exists j \in \{0, \dots, L-1\},\ x + je_{\nu+1} \in \Lambda_{n,j}^\nu \cap \mathcal{C}_\infty\bigr\} \tag{5.24}$$

is so large that

$$|\Delta_0| \ge (1-\delta)|\Lambda_n^\nu| \tag{5.25}$$

once n is sufficiently large.

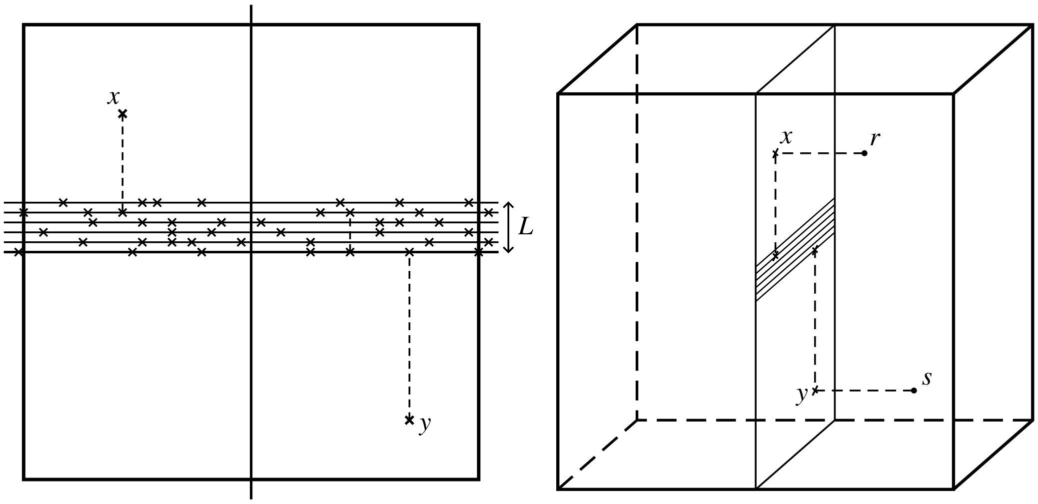

FIGURE 4. The main idea underlying the proof of Theorem 5.4. The figure on the left represents an $n \times n$ square in a two-dimensional plane in $\mathbb{Z}^3$; the crosses stand for good points. Here L is chosen so that $(1-\delta)$-fraction of all vertical lines find a cross on the intersection with one of the L horizontal lines; n is assumed so large that every pair of these lines has two good points "above" each other. Any two good points x and y in the square are connected by broken-line path that uses at most 4 good points in between. The dashed lines indicate the vertical pieces of one such path. The figure on the right indicates how this is used to control the difference of the corrector at two general points $r, s \in \mathcal{C}_\infty$ in an $n \times n \times n$ cube in $\mathbb{Z}^3$.

Proof of Lemma 5.5. Suppose that $\varrho_\nu(\omega) = 0$ for some $\nu < d$ and $\mathbb{P}$-almost all ω. Fix δ with $0 < \delta < \frac{1}{2}P_\infty^2$ and let L be as indicated above. Choose $\epsilon > 0$ so that

$$L\epsilon + \delta < \frac{1}{2}P_\infty^2 \tag{5.26}$$

and let K and n be so large that in each $\Lambda_{n,j}^\nu$ there exists $\Delta_j \subset \Lambda_{n,j}^\nu \cap \mathcal{C}_\infty$ satisfying (5.18–5.20). Let Λ be the set of sites in $\Lambda_n^{\nu+1} \cap \mathcal{C}_\infty$ whose projection onto the linear subspace $\mathbb{H} = \{k_1e_1 + \cdots + k_\nu e_\nu \colon k_i \in \mathbb{Z}\}$ belongs to the corresponding projection of $\Delta_1 \cup \cdots \cup \Delta_L$. Note that, since Δ_0 can be chosen so that $\Delta_0 \cap \Lambda_n^\nu \neq \emptyset$, we have $\Lambda \cap \Lambda_n^1 \neq \emptyset$.

Note that the projections of the Δ_j's, $j = 0, \dots, L-1$, onto $\mathbb{H}$ "fail to cover" at most $L\epsilon|\Lambda_n^\nu|$ sites in Δ_0, and so at most $(\delta + L\epsilon)|\Lambda_n^\nu|$ sites in Λ_n^ν do not find a site in one of Δ_j's "above." It follows that

$$\big|(\Lambda_n^{\nu+1} \cap \mathcal{C}_\infty) \setminus \Lambda\big| \le (\delta + L\epsilon)|\Lambda_n^{\nu+1}|, \tag{5.27}$$

i.e., Λ contains all but at most $(L\epsilon + \delta)$-fraction of all sites in $\Lambda_n^{\nu+1}$ that we care about. Next we note that if K is large (but fixed and non-random), then for every $0 \le i < j \le L-1$, the set $\mathbb{H}$ contains a density at least $\frac{1}{2}P_\infty^2$ of sites x such that

$$z_i = x + ie_\nu \in \mathcal{G}_{K,\epsilon} \quad \text{and} \quad z_j = x + je_\nu \in \mathcal{G}_{K,\epsilon}. \tag{5.28}$$

Since we assumed (5.26), once $n \gg 1$, for each pair (i,j) with $0 \le i < j \le L-1$ such z_i and z_j can be found so that $z_i \in \Delta_i$ and $z_j \in \Delta_j$. But the Δ_j's satisfy (5.19) and so via these pairs of sites we now show that

$$\big|\chi(y,\omega)-\chi(x,\omega)\big|\le K+\epsilon L+2\epsilon n \tag{5.29}$$

for every $x,y\in\Delta_1\cup\dots\cup\Delta_L$; see again (the left part) of Fig. 4.

From (5.19) and (5.29) we now conclude that for all $r,s\in\Lambda$,

$$\big|\chi(r,\omega)-\chi(s,\omega)\big|\le 3K+\epsilon L+4\epsilon n<5\epsilon n, \tag{5.30}$$

provided that n is already so large that $3K+\epsilon L<\epsilon n$. If $\varrho_{\nu,\epsilon}$ denotes the right-hand side of (5.17) before taking $\epsilon\downarrow 0$, the bounds (5.27) and (5.30) and $\Lambda\cap\Lambda_n^1\ne\emptyset$ yield

$$\varrho_{\nu+1,5\epsilon}(\omega)\le\delta+L\epsilon, \tag{5.31}$$

for $\mathbb{P}$-almost all ω. But the left-hand side of this inequality increases as $\epsilon\downarrow 0$ while the right-hand side decreases. Thus, taking $\epsilon\downarrow 0$ and $\delta\downarrow 0$ proves that $\rho_{\nu+1}(\omega)=0$ for $\mathbb{P}$-almost every ω. □

Proof of Theorem 5.4. The proof is an easy consequence of Lemma 5.5. First, by Theorem 4.1 we know that $\varrho_1(\omega)=0$ for $\mathbb{P}_0$-almost every ω. Using induction on dimension, Lemma 5.5 then tells us that $\varrho_d(\omega)=0$ for $\mathbb{P}_0$-almost every ω. Let us now focus on $\omega\in\Omega_0$. By Theorem 4.1, for each $\epsilon>0$ there is $n_0=n_0(\omega)$ with $\mathbb{P}_0(n_0(\omega)<\infty)=1$ such that for all $n\ge n_0(\omega)$,

$$\big|\chi(x,\omega)\big|\le\epsilon n,\qquad\forall x\in\Lambda_n^1\cap\mathcal{C}_\infty(\omega). \tag{5.32}$$

Using this to estimate away the infimum in (5.17), the fact that $\varrho_d=0$ now immediately implies (5.15) for all $\epsilon>0$. □

6. PROOF OF MAIN RESULTS

Here we will finally prove our main theorems. First, in Sect. 6.1, we will show the convergence of the "lazy" walk on the deformed graph to Brownian motion and then, in Sect. 6.2, we use our previous results on corrector growth to extend this to the walk on the original graph. This separation will allow us to treat the parts of the proof common for $d=2$ and $d\ge 3$ in a unified way. Theorem 1.2, which concerns the "agile" walk, is proved in Sect. 6.3.

6.1 Convergence on deformed graph.

We begin with a simple observation that will drive all underlying derivations:

Lemma 6.1 *Fix $\omega\in\Omega_0$ and let $x\mapsto\chi(x,\omega)$ be the corrector. Given a path of random walk $(X_n)_{n\ge0}$ with law $P_{0,\omega}$, let*

$$M_n^{(\omega)}=X_n+\chi(X_n,\omega),\qquad n\ge 0. \tag{6.1}$$

Then $(M_n^{(\omega)})_{n\ge0}$ is an L^2-martingale for the filtration $(\sigma(X_0,\dots,X_n))_{n\ge0}$. Moreover, conditional on $X_{k_0}=x$, the increments $(M_{k+k_0}^{(\omega)}-M_{k_0}^{(\omega)})_{k\ge0}$ have the same law as $(M_k^{(\tau_x\omega)})_{k\ge0}$.

Proof. Since X_n is bounded, $\chi(X_n,\omega)$ is bounded and so $M_n^{(\omega)}$ is square integrable with respect to $P_{0,\omega}$. Since $x\mapsto x+\chi(x,\omega)$ is harmonic with respect to the transition probabilities of the random walk (X_n) with law $P_{0,\omega}$, we have

$$E_{0,\omega}\big(M_{n+1}^{(\omega)}\big|\sigma(X_n)\big) = M_n^{(\omega)}, \qquad n \ge 0, \tag{6.2}$$

$P_{0,\omega}$-almost surely. Since $M_n^{(\omega)}$ is $\sigma(X_n)$-measurable, $(M_n^{(\omega)})$ is a martingale. $\square$

Next we will establish the convergence of the above martingale to Brownian motion. The precise statement is as follows:

Theorem 6.2 *Let $d \ge 2$, $p > p_{\rm c}$ and $\omega \in \Omega_0$. Let $(X_n)_{n\ge0}$ be the random walk with law $P_{0,\omega}$ and let $(M_n^{(\omega)})_{n\ge0}$ be as defined in* (6.1)*. Let $(\widehat{B}_n^{(\omega)}(t)\colon t \ge 0)$ be defined by*

$$\widehat{B}_n^{(\omega)}(t) = \frac{1}{\sqrt{n}}\big(M_{\lfloor tn\rfloor}^{(\omega)} + (tn - \lfloor tn\rfloor)(M_{\lfloor tn\rfloor+1}^{(\omega)} - M_{\lfloor tn\rfloor}^{(\omega)})\big), \qquad t \ge 0. \tag{6.3}$$

Then for all $T > 0$ and $\mathbb{P}_0$-almost every ω, the law of $(\widehat{B}_n(t)\colon 0 \le t \le T)$ on $(C[0,T], \mathcal{W}_T)$ converges weakly to the law of the isotropic Brownian motion $(B_t\colon 0 \le t \le T)$ with diffusion constant D, i.e., $E(B_t^2) = Dt$, where

$$D = \mathbb{E}_0\Big(E_{0,\omega}\big|X_1 + \chi(X_1,\omega)\big|^2\Big) \in (0,\infty). \tag{6.4}$$

Proof. Without much loss of generality, we may confine ourselves to the case when $T = 1$. Let $\mathcal{F}_k = \sigma(X_0, X_1, \dots, X_k)$ and fix a vector $a \in \mathbb{R}^d$. We will show that (the piece-wise linearization) of $t \mapsto a \cdot M_{\lfloor tn\rfloor}^{(\omega)}$ scales to one-dimensional Brownian motion. For $m \le n$, consider the random variable

$$V_{n,m}^{(\omega)}(\epsilon) = \frac{1}{n}\sum_{k=0}^{m} E_{0,\omega}\Big(\big[a\cdot(M_{k+1}^{(\omega)} - M_k^{(\omega)})\big]^2\, 1_{\{|a\cdot(M_{k+1}^{(\omega)}-M_k^{(\omega)})|\ge\epsilon\sqrt{n}\}}\,\Big|\mathcal{F}_k\Big). \tag{6.5}$$

In order to apply the Lindeberg-Feller Functional CLT for martingales (Theorem 7.7.3 of Durrett [9]), we need to verify that

(1) $V_{n,\lfloor tn\rfloor}^{(\omega)}(0) \to Ct$ in $P_{0,\omega}$-probability for all $t \in [0,1]$ and some $C \in (0,\infty)$.

(2) $V_{n,n}^{(\omega)}(\epsilon) \to 0$ in $P_{0,\omega}$-probability for all $\epsilon > 0$.

Both of these conditions will be implied by Theorem 3.1. Indeed, by the second conclusion of Lemma 6.1 we may write

$$V_{n,m}^{(\omega)}(\epsilon) = \frac{1}{n}\sum_{k=0}^{m} f_{\epsilon\sqrt{n}} \circ \tau_{X_k}(\omega), \tag{6.6}$$

where

$$f_K(\omega) = E_{0,\omega}\Big(\big[a\cdot M_1^{(\omega)}\big]^2\, 1_{\{|a\cdot M_1^{(\omega)}|\ge K\}}\Big). \tag{6.7}$$

Now if $\epsilon = 0$, Theorem 3.1 tells us that, for $\mathbb{P}_0$-almost all ω,

$$\lim_{n\to\infty} V_{n,n}^{(\omega)}(0) = \mathbb{E}_0\Big(E_{0,\omega}\big([a\cdot M_1^{(\omega)}]^2\big)\Big) = \frac{1}{2}D|a|^2, \tag{6.8}$$

where we used the symmetry of the joint expectations under rotations by $90°$. From here (1) follows by scaling out the t-dependence first and working with tn instead of n. On the other hand, when $\epsilon > 0$, we have $f_{\epsilon\sqrt{n}} \le f_K$ once n is sufficiently large and so, $\mathbb{P}_0$-almost surely,

$$\limsup_{n\to\infty} V_{n,n}^{(\omega)}(\epsilon) \le \mathbb{E}_0\Bigl(E_{0,\omega}\bigl([a\cdot M_1^{(\omega)}]^2\, 1_{\{|a\cdot M_1^{(\omega)}|\ge K\}}\bigr)\Bigr) \underset{K\to\infty}{\longrightarrow} 0, \tag{6.9}$$

where we used the fact that $M_1^{(\omega)}$ is square integrable.

Hence, the limits in (1) and (2) hold—in fact, even $P_{0,\omega}$-almost surely. Applying the Martingale functional CLT and the Cramér-Wald device, we conclude that, for $\mathbb{P}_0$-almost all ω, the linear interpolation of $(M_k^{(\omega)}/\sqrt{n})_{k=1,\dots,n}$ converges to isotropic Brownian motion with covariance matrix $D\,1$. □

6.2 Correction on the corrector.

It remains to estimate the influence of the harmonic deformation on the path of the walk. As already mentioned, while our proof in $d = 2$ is completely self-contained, for $d \ge 3$ we rely heavily on (a discrete version of) the sophisticated Theorem 1 of Barlow [2].

Let us first dismiss the two-dimensional case of Theorem 1.1:

Proof of Theorem 1.1 ($d = 2$). We need to extend the conclusion of Theorem 6.2 to the linear interpolation of (X_n). Since the corrector is an additive perturbation of $M_n^{(\omega)}$, it clearly suffices to show that, for $\mathbb{P}_0$-almost all ω,

$$\max_{1\le k\le n} \frac{\chi(X_k,\omega)}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0, \qquad \text{in } P_{0,\omega}\text{-probability}. \tag{6.10}$$

By Theorem 5.1 we know that for every $\epsilon > 0$ there exists a $K = K(\omega) < \infty$ such that

$$|\chi(x,\omega)| \le K + \epsilon|x|_\infty, \qquad x \in \mathcal{C}_\infty(\omega). \tag{6.11}$$

If $\epsilon < 1/2$, then this implies

$$|\chi(X_k,\omega)| \le 2K + 2\epsilon|M_k^{(\omega)}|. \tag{6.12}$$

But the above CLT for (M_k) tells us that $\max_{k\le n}|M_k^{(\omega)}|/\sqrt{n}$ converges in law to the maximum of a Brownian motion $B(t)$ over $t\in[0,1]$. Hence, if P denotes the distribution of the Brownian motion, Portmanteau Theorem (Theorem 2.1 of [3]) allows us to conclude

$$\limsup_{n\to\infty} P_{0,\omega}\bigl(\max_{k\le n}|\chi(X_k,\omega)| \ge \delta\sqrt{n}\bigr) \le P\Bigl(\max_{0\le t\le 1}|B(t)| \ge \frac{\delta}{2\epsilon}\Bigr). \tag{6.13}$$

The right-hand side tends to zero as $\epsilon \downarrow 0$ for all $\delta > 0$. □

In order to prove the same result in $d \ge 3$, we will need the following upper bound on the transition probability of our random walk:

Theorem 6.3 *(1) There exists a constant $c_1 \in (0,\infty)$ and a random variable $N_x = N_x(\omega)$ such that for all $x, y \in \mathcal{C}_\infty$ and all $n \ge N_x(\omega)$,*

$$P_{x,\omega}(X_n = y) \le c_1 n^{-d/2}. \tag{6.14}$$

(2) There exist constants $c_2, c_3 \in (0,\infty)$ and a random variable $N_x = N_x(\omega)$ such that for all $x \in \mathcal{C}_\infty$, all $R \ge 1$ and all $n \ge N_x(\omega)$,

$$P_{x,\omega}\bigl(|X_n - x| > R\bigr) \le c_2 \exp\bigl\{-c_3 R^2/n\bigr\}. \tag{6.15}$$

For a continuous-time version of our walk, these are the content of Theorem 1 of [2]. Unfortunately, to derive Theorem 6.3 from Barlow's Theorem 1, one needs to invoke various non-trivial facts about percolation and/or mixing of Markov chains. In Appendix A we list these facts and show how to assemble all ingredients together to establish the above upper bounds.

Proof of Theorem 1.1 ($d \ge 3$). We will adapt (the easier part of) the proof of Theorem 1.1 in Sidoravicius and Sznitman [20]. First we show that the laws of $(\widetilde{B}_n(t)\colon t \le T)$ on $(C[0,T], \mathcal{W}_T)$ are tight. To this end we need to show (e.g., by Theorem 8.6 of Ethier-Kurtz [10]) that if $\mathcal{S}_n$ is the class of bounded stopping times of the filtration $(\sigma(\{\widetilde{B}_n(s)\colon s \le t\}))_{t\ge 0}$, then

$$\limsup_{\epsilon\downarrow 0}\ \limsup_{n\to\infty}\ \sup_{\tau\in\mathcal{S}_n} E_{0,\omega}\bigl(|\widetilde{B}_n(\tau+\epsilon)-\widetilde{B}_n(\tau)|^2\bigr)=0. \tag{6.16}$$

As in [20], this is done by replacing the stopping time τ by its integer-valued approximation $\lfloor n\tau\rfloor + 1$, applying that $|\widetilde{B}_n(\tau) - X_{\lfloor n\tau\rfloor+1}/\sqrt{n}| \le 1/\sqrt{n}$, and similarly for $\widetilde{B}_n(\tau+\epsilon)$, and noting that Theorem 6.3 and the strong Markov property imply

$$E_{0,\omega}\bigl(|X_{\tau+\delta n}-X_\tau|^2\bigr)\le c_3\,\delta n, \qquad n > N_x(\omega)/\delta, \tag{6.17}$$

for any (integer-valued) stopping time τ of $(\sigma(X_0,\dots,X_n))_{n\ge 0}$. Here c_3 is a finite constant, δ is proportional to ϵ in (6.16) and N_x is the random variable from Theorem 6.3. Then (6.16) is proved by taking $n\to\infty$ followed by $\delta\downarrow 0$.

Once we know that the laws of $(\widetilde{B}_n(t)\colon t\le T)$ are tight, it suffices to show the convergence of finite-dimensional distributions. In light of Theorem 6.2 (and the Markov property of the walk), for that it is enough to prove that for all $t>0$ and $\mathbb{P}_0$-almost all ω,

$$\frac{\chi(X_{\lfloor tn\rfloor},\omega)}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in } P_{0,\omega}\text{-probability.} \tag{6.18}$$

Without much loss of generality, we need to do this only for $t=1$. By Theorem 6.3, the random variable X_n is with probability $1-\epsilon$ in the block $[-M\sqrt{n}, M\sqrt{n}]^d\cap\mathbb{Z}^d$, provided M sufficiently large. Using the bound in Theorem 6.3 to estimate $P_{0,\omega}(X_n=x)$ for x inside this block, we have

$$P_{0,\omega}\bigl(|\chi(X_n,\omega)|>\delta\sqrt{n}\bigr)\le \epsilon + c_1\mathrm{e}^{-c_2M^2}\,\frac{1}{n^{d/2}}\sum_{\substack{x\in\mathcal{C}_\infty\\ |x|\le M\sqrt{n}}} 1_{\{|\chi(x,\omega)|>\delta\sqrt{n}\}}\,. \tag{6.19}$$

But Theorem 5.4 tells us that, for all $\delta, M>0$ and $\mathbb{P}_0$-almost all ω, the second term tends to zero as $n\to\infty$. This proves (6.18) and the whole claim. □

6.3 Extension to "agile" walk.

It remains to prove Theorem 1.2 for the "agile" version of simple random walk on $\mathcal{C}_\infty$. Since the proof uses only Theorem 1.1, we will resume a unified treatment of all $d\ge 2$. First we will make the observation that the times of the two walks run proportional to each other:

Lemma 6.4 *Let $(T_k)_{k\ge 0}$ be the stopping times defined in* (1.7). *Then for all $t\ge 0$ and $\mathbb{P}_0$-almost every ω,*

$$\frac{T_{\lfloor tn\rfloor}}{n} \underset{n\to\infty}{\longrightarrow} \Theta t, \qquad P_{0,\omega}\text{-almost surely,} \tag{6.20}$$

where

$$\frac{1}{\Theta}=\mathbb{E}_0\Bigl(\frac{1}{2d}\sum_{e\colon |e|=1}1_{\{\omega_e=1\}}\Bigr). \tag{6.21}$$

Proof. This is an easy consequence of the second part of Theorem 3.1 and the fact that for $\mathbb{P}_0$-almost all ω we have $\tau_x\omega\neq\omega$ once $x\neq 0$. Indeed, let $f(\omega,\omega')=1_{\{\omega\neq\omega'\}}$. For $t=0$ the statement holds trivially so let us assume that $t>0$. If n is so large that $T_{\lfloor nt\rfloor}>0$, we have

$$\frac{n}{T_{\lfloor tn\rfloor}}=\frac{1}{T_{\lfloor tn\rfloor}}\sum_{k=1}^{T_{\lfloor tn\rfloor}}f(\tau_{X_{k-1}}\omega,\tau_{X_k}\omega). \tag{6.22}$$

Since $T_{\lfloor tn\rfloor}\to\infty$ as $n\to\infty$, by Theorem 3.1 the right hand side converges to the expectation of $f(\omega,\tau_{X_1}\omega)$ in $\mathbb{P}_0\otimes P_{0,\omega}$. A direct calculations shows that this expectation equals Θ. □

Proof of Theorem 1.2. The proof is based on a standard approximation argument for stochastic processes. Let $\widetilde{B}_n(t)$ be as in Theorem 1.1 and recall that $\widetilde{B}_n'(t)$ is a linear interpolation of the values $\widetilde{B}_n(T_k/n)$ for $k=0,\dots,n$. The path-continuity of the processes $\widetilde{B}_n(t)$ as well as the limiting Brownian motion implies that for every $\epsilon>0$ there is a $\delta>0$ such that

$$P_{0,\omega}\Bigl(\sup_{\substack{t,t'\le T\\ |t-t'|<\delta}}\bigl|\widetilde{B}_n(t)-\widetilde{B}_n(t')\bigr|<\epsilon\Bigr)>1-\epsilon \tag{6.23}$$

once n is sufficiently large. Similarly, Lemma 6.4, the continuity of $t\mapsto\Theta t$ and the monotonicity of $k\mapsto T_k$ imply that for n sufficiently large,

$$P_{0,\omega}\biggl(\sup_{t\le T}\Bigl|\frac{T_{\lfloor tn\rfloor}}{n}-\Theta t\Bigr|<\delta\biggr)>1-\epsilon. \tag{6.24}$$

On the intersection of these events, the equality $\widetilde{B}_n'(k/n)=\widetilde{B}_n(T_k/n)$ gives us that

$$\max_{0\le k\le\lfloor Tn\rfloor}\bigl|\widetilde{B}_n'({}^{k}\!/_{n})-\widetilde{B}_n({}^{\Theta k}\!/_{n})\bigr|<\delta. \tag{6.25}$$

In light of piece-wise linearity this shows that, with probability at least $1-2\epsilon$, the paths $t\mapsto\widetilde{B}_n'(t)$ and $t\mapsto\widetilde{B}_n(\Theta t)$ are within a multiple of δ in the supremum norm of each other. In particular, if B_t denotes the weak limit of the process $(B_n(t)\colon t\le T)$, then $(\widetilde{B}_n'(t)\colon t\le T)$ converges in law to $(B_{\Theta t}\colon t\le T)$. The latter is an isotropic Brownian motion with diffusion constant $D\Theta^2$. □

APPENDIX A: HEAT-KERNEL UPPER BOUNDS

Let $(Z_t)_{t\ge0}$ denote the continuous-time random walk which makes jumps to nearest-neighbors of its current position at rate one (regardless of the number of neighbors). Let $q_t^\omega(x,y)$ denote the probability that Z_t started at x is at y at time t. In his paper [2], Barlow proved the following statement: There exist constants $C_1,C_2\in(0,\infty)$ and, for each $x\in\mathcal{C}_\infty(\omega)$, a random variable $S(x)=S(x,\omega)\in(0,\infty)$ such that for all $t>S(x)$ and all $y\in\mathcal{C}(\omega)$,

$$q_t^\omega(x,y)\le C_1t^{-d/2}\exp\bigl\{-C_2|x-y|^2/t\bigr\}. \tag{A.1}$$

Moreover, $S(x)$ has uniformly stretched-exponential tails and so $\max_{|x|\le R} S(x)$ is bounded by a power of $\log R$ almost surely for all $R > 0$. (Barlow provides also a corresponding, and significantly harder-to-prove lower bound.)

In the remarks after his Theorem 1, Barlow mentions that appropriate modifications to his arguments yield the corresponding discrete time estimates. Here we present these modifications which are needed to make our proof of the invariance principles in Theorems 1.1 and 1.2 complete. Notice that we do not re-prove Barlow's bounds in their full generality, just the minimum necessary for our purposes.

A.1 Uniform bound.

There will be two kinds of bounds on the heat-kernel as a function of the terminal position of the walk after n steps: a uniform bound by a constant times $n^{-d/2}$ and a non-uniform, Gaussian bound on the tails. We begin with the statement of the uniform upper bound:

Proposition A.1 *Let $d \ge 2$ and let $p > p_c(d)$. There exists a constant $c_1 \in (0,\infty)$, a measurable set $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega_1) = 1$ and a collection of random variables $(N_x)_{x\in\mathbb{Z}^d}$ such that the following holds:*

(1) *$N_x(\omega) < \infty$ for each $\omega \in \Omega_1$ and each $x \in \mathcal{C}_\infty(\omega)$.*
(2) *For all $\omega \in \Omega_1$, all $x, y \in \mathcal{C}_\infty$ and all $n > N_x$,*

$$P_{x,\omega}(X_n = y) \le c_1 n^{-d/2}. \tag{A.2}$$

The proof will be based on the isoperimetric bound from Barlow [2]:

Lemma A.2 (Isoperimetric bound) *There exists a constant $c \in (0,\infty)$ such that for $\mathbb{P}_0$-almost all ω and all R sufficiently large,*

$$\frac{|\partial\Lambda|}{|\Lambda|} \ge c|\Lambda|^{-1/d} \tag{A.3}$$

for all $\Lambda \subset \mathcal{C}_\infty \cap [-R,R]^d$ such that $|\Lambda| > R^{0.01}$.

Proof. This is a direct consequence of Proposition 2.11 on page 3042, and Lemma 2.13 on page 3045 of Barlow's paper [2]. □

This isoperimetric bound will be combined with the technique of *evolving sets*, developed by Morris and Peres [18], whose salient features we will now recall. Consider a Markov chain on a countable state-space V, let $p(x,y)$ be the transition kernel and let π be a stationary measure. Let $Q(x,y) = \pi(x)p(x,y)$ and for each $S_1, S_2 \subset V$, let $Q(S_1,S_2) = \sum_{x\in S_1}\sum_{y\in S_2} Q(x,y)$. For each set $S \subset V$ with finite non-zero total measure $\pi(S)$ we define the *conductance* Φ_S by

$$\Phi_S = \frac{Q(S,S^c)}{\pi(S)}. \tag{A.4}$$

For sufficiently large r, we also define the function

$$\Phi(r) = \inf\bigl\{\Phi_S \colon \pi(S) \le r\bigr\}. \tag{A.5}$$

The following is the content of Theorem 2 in Morris and Peres [18]: Suppose that $p(x,x) \geq \gamma$ for some $\gamma \in (0, 1/2]$ and all $x \in V$. Let $\epsilon > 0$ and pick $x, y \in V$. If n is so large that

$$n \geq 1 + \frac{(1-\gamma)^2}{\gamma^2} \int_{4[\pi(x)\wedge\pi(y)]}^{4/\epsilon} \frac{4}{u\Phi(u)^2} \mathrm{d}u, \tag{A.6}$$

then

$$p^n(x,y) \leq \epsilon\pi(y). \tag{A.7}$$

Equipped with this powerful result, we are now ready to complete the proof of Proposition A.1:

Proof of Proposition A.1. First we will prove the desired bound for even times. Fix $\omega \in \Omega$ and let $Y_n = X_{2n}$ be the random walk on $\mathcal{C}_\infty(\omega)$ observed only at even times. For each $x, y \in \mathcal{C}_\infty(\omega)$, let $p(x,y)$ be the transition probability $P_{x,\omega}(Y_1 = y)$. Let $\pi(x)$ denote the degree of x on $\mathcal{C}_\infty(\omega)$. Then π is an invariant measure of this chain. Moreover, by our restriction to even times we have $p(x,x) \geq (2d)^{-2} > 0$ and so (A.6–A.7) can be applied.

By Lemma A.2 we have that $\Phi_S \geq c\pi(S)^{-\frac{1}{d}}$ for some $c = c(\omega) > 0$. Hence $\Phi(r) \leq c\,r^{-\frac{1}{d}}$. Plugging into the integral (A.6) and using that π is bounded, we find that if $n \geq \tilde{c}\,\epsilon^{-\frac{2}{d}}$, then (A.7) holds. Here $\tilde{c}$ is a positive constant that may depend on ω. Choosing the minimal n possible, and applying $p^n(x,y) = P_{x,\omega}(Y_n = y)$, the bound (A.7) proves the desired claim for all even times. To extend the result to odd times, we apply the Markov property at time one. □

A.2 Gaussian tails.

Next we will attend to the Gaussian-tail bound. The relevant statement is as follows:

Proposition A.3 *Let $d \geq 2$ and $p > p_c(d)$. There exist constants $c_1, c_2 \in (0,\infty)$, a measurable set $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega_1) = 1$ and a collection of random variables $(N_x)_{x\in\mathbb{Z}^d}$ such that the following is true:*

(1) *$N_x(\omega) < \infty$ for each $\omega \in \Omega_1$ and each $x \in \mathcal{C}_\infty(\omega)$.*
(2) *For all $\omega \in \Omega_1$, all $x \in \mathcal{C}_\infty(\omega)$, all $R > 0$ and all $n > N_x(\omega)$,*

$$\sum_{y\colon |y-x|>R} P_{x,\omega}(X_n = y) < c_1 \exp\bigl\{-c_2 R^2/n\bigr\}. \tag{A.8}$$

Proof. The proof is an adaptation of Barlow's Theorem 1 to the discrete setting. Let (X_n) be the discrete time random walk, and let $(Z_t)_{t\geq 0}$ be the continuous time random walk with jumps occurring at rate 1, both started at x. We consider the coupling of the two walks such that they make the same moves. We will use P and E to denote the coupling measure and the corresponding expectation, respectively.

Let A_n be the event that $|X_n - x| > R$. Pick a large constant $K > 1$ and let

$$I_n = \int_0^{4n} 1_{\{|Z_t - x| > R/K\}}\ \mathrm{d}t \tag{A.9}$$

be the amount of time that Z_t spends at distance larger than R/K from x. By

$$\mathbb{P}(A_n) \leq \frac{\mathbb{E}(I_n)}{\mathbb{E}(I_n|A_n)}, \tag{A.10}$$

it suffices to derive an (appropriate) upper bound on $E(I_n)$ and a lower bound on $E(I_n|A_n)$.

First we will derive an upper bound on $E(I_n)$. The bound (A.1) yields the estimate

$$\begin{aligned}\mathbb{E}(I_n) &\le \mathrm{e}^{-\frac{R}{KS(x)}} + \int_{S(x)}^{2n} \sum_{y\colon |y-x|>R/K} q_t(x,y)\,\mathrm{d}t \\ &\le \mathrm{e}^{-\frac{R}{KS(x)}} + C_3 \int_{S(x)}^{2n} \mathrm{e}^{-C_4\frac{R^2}{t}}\,\mathrm{d}t \le C_5 n \mathrm{e}^{-C_4\frac{R^2}{n}}.\end{aligned} \tag{A.11}$$

Here we first split off the event that the walk makes at least R/K jumps before time $S(x)$; the resulting probability is bounded by $\mathrm{e}^{-\frac{R}{KS(x)}}$. Then we applied (A.1). The constant C_4 depends on K.

It remains to produce an appropriate lower bound on the conditional expectation. We claim that there exists a constant $C_6 > 0$ such that

$$\mathbb{E}(I_n|A_n) \ge C_6\, n. \tag{A.12}$$

To derive this inequality, let us recall that the transitions of Z_t happen at rate one, and they are independent of the path of the walk. Hence, if B_n is the event that Z_t made at least n jumps by time $2n$, then $P(B_n|A_n) = P(B_n) \ge 1/2$ once n is sufficiently large. Therefore, it suffices to prove that $E(I_n|A_n \cap B_n) \ge C_6 n$.

Let T be the first time after n when the walk $(Z_t)_{t\ge n}$ is farther from x than R. On $A_n \cap B_n$, this happens before time $2n$. Let $Q_{R/K} = [-R/K, R/K] \cap \mathbb{Z}^d$ and suppose that $K > 2$. Then for values z that Z_T can take, the bound (A.1) tells us

$$\sum_{y\in Q_{R/K}} q_t^\omega(z,y) \le C_1 \left(\frac{2R}{K}\right)^d \max_{s>0} \Bigl\{ s^{-d/2} \exp(-\tfrac14 C_2 R^2/s) \Bigr\} \le C_7 K^{-d}, \tag{A.13}$$

provided that $t > S(z)$. But $S(z)$ is less than a power of $\log R$ and R is at most a constant times n, so in light of the fact that $T < 2n$ on $A_n \cap B_n$, (A.13) actually holds for all t such that $T + t \in [3n, 4n]$. Plugging Z_T for z on the left-hand side and taking expectation gets us the probability $P(Z_t \in Q_{R/K}|A_n \cap B_n)$. Hence,

$$E(I_n|A_n \cap B_n) \ge \int_{3n}^{4n} P(Z_t \notin Q_{R/K}|A_n \cap B_n) \ge n(1 - C_7 K^{-d}). \tag{A.14}$$

Choosing K sufficiently large, the right-hand side grows linearly in n. □

Proof of Theorem 6.3. This is now a direct consequence of Propositions A.1 and A.3. □

APPENDIX B: SOME QUESTIONS AND CONJECTURES

While our control of the corrector in $d \ge 3$ is sufficient to push the proof of the functional CLT through, it is not sufficient to provide the *conceptually correct* proof of the kind we have constructed for $d = 2$. However, we do not see any reason why $d \ge 3$ should be different from $d = 2$; so our first conjecture is:

Conjecture B.1 *Theorem 5.1 is true in all $d \ge 2$.*

Our proof of Theorem 5.1 in $d = 2$ hinged on the fact that the corrector plus the position is a harmonic function on the percolation cluster. Of interest is the question whether harmonicity is an essential ingredient or just mere convenience. Yuval Peres suggested the following generalization of Conjecture B.1:

Question B.2 *Let $f\colon \mathbb{Z}^d \to \mathbb{R}$ be a shift invariant, ergodic process on $\mathbb{Z}^d$ whose gradients are in L^1 and have expectation zero. Is it true that*

$$\lim_{n\to\infty} \frac{1}{n} \max_{x\in\mathbb{Z}^d\cap[-n,n]^d} \bigl|f(x)\bigr| = 0 \tag{B.1}$$

almost surely?

The harmonic embedding of $\mathcal{C}_\infty$ has been indispensable for our proofs, but it also appears to be a very interesting object in its own right. This motivates many questions about the corrector $\chi(x,\omega)$. For instance, we would like to know how $\chi(x,\omega)$ scales with x and whether it has a well-defined scaling limit. One might also ask about the size of the "holes" and their distribution. Unfortunately, at the moment it is not even clear what properties make the corrector unique. The following question has been asked by Scott Sheffield:

Question B.3 *Is it true that, for a.e. $\omega \in \Omega_0$, there exists only one function $x \mapsto \chi(x,\omega)$ on $\mathcal{C}_\infty(\omega)$ such that $x \mapsto x + \chi(x,\omega)$ is harmonic on $\mathcal{C}_\infty(\omega)$, $\chi(0,\omega) = 0$ and $\chi(x,\omega)/|x| \to 0$ as $|x| \to \infty$?*

If this question is answered in the affirmative, we could generate the corrector by its finite-volume approximations (this would also fully justify Fig. 1). If we restrict ourselves to functions that have the shift-covariance property (2.12), uniqueness can presumably be shown using the "electrostatic methods" from [12]. However, it is not clear whether (2.12) holds for the corrector defined by the thermodynamic limit from finite boxes.

Despite the emphasis on the harmonic embedding of $\mathcal{C}_\infty$, our proofs used, quite significantly, the underlying group structure of $\mathbb{Z}^d$; e.g., in Sect. 4. Presumably this will not prevent application of our method to other regular lattices, but for more irregular graphs, e.g., Voronoi percolation in $\mathbb{R}^2$, significant changes will be necessary. Another direction of generalizations are the models of *long-range* percolation with power-law decay of bond probabilities. Here we conjecture:

Conjecture B.4 *Consider long-range percolation obtained by adding to $\mathbb{Z}^d$ a bond between distinct sites $x, y \in \mathbb{Z}^d$ independently with probability proportional to $|x-y|^{-\alpha}$. If $\alpha \in (0, 2d)$, then the corresponding walk scales to a symmetric α-stable Levy process in $\mathbb{R}^d$.*

Our final remark concerns the local CLT. The fact that our walk scales to Brownian motion with covariance matrix $D\,1$ suggests that Barlow's heat-kernel estimates can be strengthened to the following form

$$P_{x,\omega}(X_n = y) \sim n^{-\,d/2} \exp\Bigl\{-\frac{|x-y|^2}{2Dn}\Bigr\}. \tag{B.2}$$

Here the implicit constants in front of the polynomial term in n are independent of n but one may need to take n larger than some N_x' and N_y' to account for local irregularity about these points. Our functional CLT implies the above local CLT but only in the regime where it is not very interesting: namely, only for $|x-y| = O(\sqrt{n})$.

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