LARGE-DEVIATIONS/THERMODYNAMIC APPROACH TO PERCOLATION ON THE COMPLETE GRAPH

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ABSTRACT. We present a large-deviations/thermodynamic approach to the classic problem of percolation on the complete graph. Specifically, we determine the large-deviation rate function for the probability that the giant component occupies a fixed fraction of the graph. One consequence is an immediate derivation of the "cavity" formula for the fraction of sites in the giant component. As a by-product of our analysis we compute also the large-deviation rate functions for the probabilities of the event that the random graph is connected, the event that it contains no loops and the event that it contains only "small" components.

1. INTRODUCTION

For physical systems, mean-field theory often provides a qualitatively correct description of "realistic behavior." The corresponding analysis usually begins with the derivation of so called mean-field equations which are self-consistent relations involving the physical quantity of primary interest (e.g., the order parameter or condensate fraction) and the various parameters of the model. This approach may be realized and, to some extent, justified mathematically by considering the model on the complete graph where each component interacts with all others.

In the context of percolation the relevant "physical" quantity is the *infinite cluster density*, P_∞, which on infinite transitive graphs coincides with the probability that a given vertex lies in an infinite connected component. The mean-field version of percolation, known under the name The Erdös-Renyi Random Graph, is defined by considering n vertices each pair of which is independently connected by an occupied edge with probability α/n, where $0 \le \alpha < \infty$. In this setting the quantity P_∞ is identified with the limiting fraction of the sites that belong to the "giant component" of the graph. The corresponding mean-field equation,

$$P_\infty = 1 - \mathrm{e}^{-\alpha P_\infty}, \tag{1.1}$$

is readily derived from heuristic "cavity" considerations. As is well known, $P_\infty = 0$ is the only solution for $\alpha \le \alpha_{\mathrm{c}} = 1$, while for $\alpha > \alpha_{\mathrm{c}}$ there is another, strictly positive solution. This solution tends to zero as $\alpha \downarrow \alpha_{\mathrm{c}}$; hence we may speak of a continuous transition.

While (1.1) and its analogues in other system are indeed straightforward to derive, matters at the level of mean-field equations are not always satisfactory; the problem being that equations such as (1.1) typically have multiple solutions. As it turns out, for the percolation model (as well as the k-core percolation) the proper choice is always the *maximal* solution, but prescriptions of

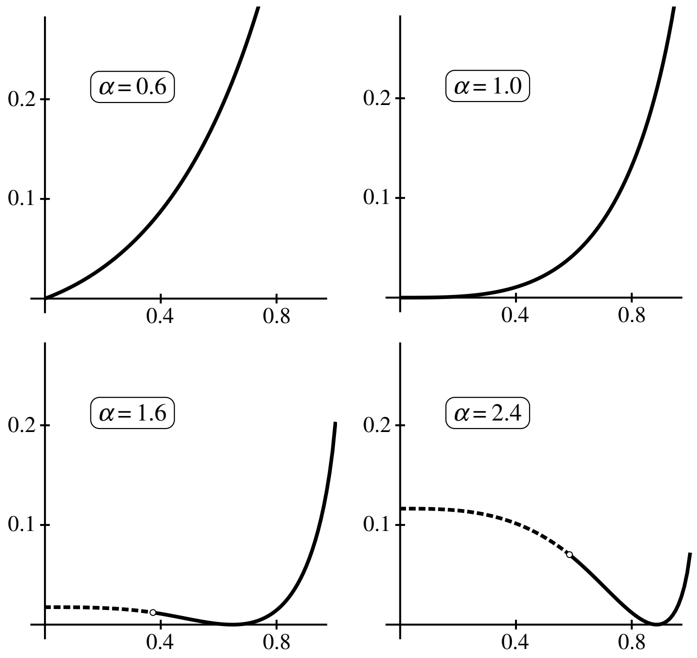

FIGURE 1. The graph of the free energy function $\varrho \mapsto \Phi(\varrho, \alpha)$ for four distinct values of α. For $\alpha \le 1$, the function is minimized by $\varrho = 0$, while for $\alpha > 1$ the unique minimum occurs at $\varrho > 0$. In any case, the minimizer is the maximal positive solution for P_∞ from (1.1). The dashed portion of the graphs for $\alpha = 1.6$ and 2.4 marks the part where the background contribution, $\Psi(\alpha(1-\varrho))$, to $\Phi(\varrho, \alpha)$ in (2.4) is strictly positive. This rules out the zero solution to (1.1) for all $\alpha > 1$.

this sort generically fail, e.g., for systems with a discontinuous transition. Thus, one is in need of an additional principle which determines the physically relevant solutions.

The existing mathematical approach to these difficulties—e.g., for percolation [11], see also [1, 6, 13], or the k-core [18]—is to work with sufficient precision until the mean-field conclusions are rigorously established. The physical approach—which admits some prospects of generality and extendability outside the mean-field context [3, 4]—is to complete the picture by the introduction of the *mean-field free-energy function* $\Phi(\varrho, \alpha)$. For spin systems in the mean-field approximation, $\Phi(\varrho, \alpha)$ represents the free energy of the system when it is *forced* to exhibit a specified condensate fraction ϱ. However, on the complete graph, this object (minus its infimum) is exactly the large-deviation rate function for the probability of observing said fraction ϱ. In any case, this spells the end of the story from the perspective of theoretical physics: One seeks the minimum of the free energy; setting its derivative to zero yields the mean-field equations.

The free-energy approach to mean-field problems has met with success in Ising systems [10] and, to some extent, it has been applied to the Potts and random-cluster models [7, 16, 9]. However, no attempt seems to have been made to extend this technology to "purely geometrical" problems on the complete graph, specifically, ordinary percolation or k-core percolation. The purpose of this note is to derive the large-deviation rate function for the event that the random graph contains a fraction ϱ of sites in "large" components. As we will see, the function has a unique minimum for all α which coincides with the "correct" solution of (1.1). We do not claim that the resultant justification of this equation is easier than which already exists in the literature. However, the picture presented here provides some additional insights into the model while the overall approach indeed admits the possibility of generalizations.

2. MAIN RESULTS

Consider the set of vertices $\mathscr{V} = \{1, \dots, n\}$ and let $(\omega_{kl})_{1\le k<l\le n}$ be a collection of i.i.d. random variables taking value one with probability p and zero with probability $1-p$. Let $\mathscr{E} = \mathscr{E}(\omega)$ be the (random) set $\{(k,l)\colon 1 \le k < l \le n,\, \omega_{kl} = 1\}$. In accord with the standard notation, cf [1, 6], we will use $\mathscr{G}(n,p)$ to denote the undirected graph with vertices $\mathscr{V}$ and edges $\mathscr{E}$. Of particular interest are the cases where p decays to zero proportionally to ${}^1\!/_n$. Since these are the only problems we will consider, let us set, for once and all, $p = {}^\alpha\!/_n$ for some fixed $\alpha \in [0,\infty)$. We will denote the requisite probability measure by $P_{n,\alpha}$.

In order to state our main theorems, we need to introduce some notation. First, consider the standard entropy function

$$S(\varrho) = \varrho \log \varrho + (1-\varrho)\log(1-\varrho) \tag{2.1}$$

and let

$$\pi_1(\alpha) = 1 - \mathrm{e}^{-\alpha}. \tag{2.2}$$

In addition, consider the function

$$\Psi(\alpha) = \biggl(\log\alpha - \frac{1}{2}\Bigl[\alpha - \frac{1}{\alpha}\Bigr]\biggr) \wedge 0 \tag{2.3}$$

and note that $\Psi(\alpha) < 0$ if and only if $\alpha > 1$. Finally, let us also define

$$\begin{aligned}\Phi(\varrho,\alpha) = S(\varrho) &- \varrho\log\pi_1(\alpha\varrho)\\ &- (1-\varrho)\log\bigl[1-\pi_1(\alpha\varrho)\bigr] - (1-\varrho)\Psi\bigl(\alpha(1-\varrho)\bigr).\end{aligned} \tag{2.4}$$

Then we have:

Theorem 2.1 *Consider $\mathscr{G}(n, {}^\alpha\!/_n)$ and let $\mathscr{V}_r$ be the set of sites that are in connected components of size larger than r. Then for every $\varrho \in [0,1]$,*

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty} P_{n,\alpha}\bigl(|\mathscr{V}_{\epsilon n}| = \lfloor \varrho n\rfloor\bigr)^{1/n} = \mathrm{e}^{-\Phi(\varrho,\alpha)}. \tag{2.5}$$

Fig. 1 shows the graph of Φ for various values of α which is archetypal of free-energy functions in complete graph setting. The figure indicates a unique global minimum; direct, albeit arduous differentiation of (2.4) yields the fact that all local extrema satisfy the mean-field equation (1.1). The extremum at $\varrho = 0$ is ruled out for $\alpha > 1$ by noting that $\Psi(0,\alpha) > 0$.

The corresponding conclusion may also be extracted from the following probabilistic argument: Let $m = \lfloor \varrho n\rfloor$ and note that $e^{-nS(\varrho)}$ is then the exponential growth-rate of $\binom{n}{m}$. This allows us to write

$$\mathrm{e}^{-n\Phi(\varrho,\alpha)} = e^{o(n)}\binom{n}{m}\bigl[\pi_1(\alpha\varrho)\bigr]^m\bigl[1-\pi_1(\alpha\varrho)\bigr]^{n-m}\,\mathrm{e}^{(n-m)\Psi(\alpha(1-\varrho))}. \tag{2.6}$$

Neglecting the Ψ-term (which provides a lower bound on Φ), one sees a quantity reminiscent of binomial distribution. Well known results on the latter inform us that the right-hand side is exponentially small unless

$$\pi_1(\alpha\varrho) = \frac{m}{n}, \tag{2.7}$$

which is exactly the mean-field equation (1.1). If Ψ is set to zero, there are degenerate minima for $\alpha > 1$; however, the Ψ-function will lift the degeneracy and, in fact, create a local *maximum* at $\varrho = 0$ once $\alpha > 1$. Meanwhile, in the region of the maximal solution, Ψ has vanished and the above mentioned approximation is exact.

Remark 2.2 It is somewhat mysterious that the Ψ-function never appears in the standard rendition of the mean-field equations. Presumably, this is related to complicated effects which emerge under the assumption that the giant component is small. One way to bring these effects out of the realm of exponentially-improbable would be to give each configuration a weight suppressing large components. However, we will not pursue these matters in the present note.

Our results are also not sharp enough to provide detailed description of the critical behavior, i.e., the situations when $\alpha = 1+O(n^{-1/3})$. The corresponding analysis of the scaling phenomena that happen inside the "critical window" has been performed in [5, 14, 15, 17, 8].

Several ingredients enter our proof of Theorem 2.1 which are of independent interest. We state these as separate theorems. The first one concerns the exponential decay rate for the probability that the random-graph is (completely) connected:

Theorem 2.3 *Let K denote the event that $\mathscr{G}(n, \alpha/n)$ is connected. Then*

$$P_{n,\alpha}(K) = (1-e^{-\alpha})^n \mathrm{e}^{O(\log n)}, \qquad n\to\infty, \tag{2.8}$$

where $O(\log n)$ is bounded by a constant times $\log n$ uniformly on compact sets of $\alpha\in[0,\infty)$.

We remark that Theorem 2.3 can be derived on the basis of the result in [2]. However, the union of [2] and the requisite steps are far in excess of the derivation in Sect. 3.

Next we present a result concerning the event that $\mathscr{G}(n, \alpha/n)$ contains no loops:

Theorem 2.4 *Let L be the event that $\mathscr{G}(n, \alpha/n)$ contains no loops. Then*

$$\lim_{n\to\infty} P_{n,\alpha}(L)^{1/n} = \begin{cases} \alpha\exp\left(-\frac{\alpha}{2}+\frac{1}{2\alpha}\right), & \textit{if } \alpha>1,\\ 1, & \textit{otherwise}. \end{cases} \tag{2.9}$$

Strictly speaking, this result is not needed for the proof of our main theorem; it is actually used to derive the exponential decay for the probability of the event that $\mathscr{G}(n, \alpha/n)$ contains only "small" components. Surprisingly, the decay rates for these two events are exactly the same:

Theorem 2.5 *Let L be the event that $\mathscr{G}(n, \alpha/n)$ contains no loops and let B_r be the event that there are no components larger than r. Then*

$$\lim_{r\to\infty} \liminf_{n\to\infty} P_{n,\alpha}(B_r)^{1/n} = \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} P_{n,\alpha}(B_{\epsilon n})^{1/n} = \lim_{n\to\infty} P_{n,\alpha}(L)^{1/n}. \tag{2.10}$$

The motivation underlying the use of the aforementioned theorems comes from our "physical" interpretation of the birth of the giant cluster. We like to picture the giant component as a macroscopic droplet of a dense, e.g., liquid, phase that formed out of oversaturated thin phase, e.g., vapor. To determine the size of the droplet, we will express the thermodynamic weight of each macroscopic configuration: The weight of the droplet itself is just the probability that all sites in the droplet are connected to each other. The "surface" energy of the droplet is then the probability that no site inside is connected to no site outside. Finally, there is the weight of the gaseous phase outside—that amounts to the probability that all components outside are of submacroscopic scale.

When the leading-order exponential decay rate of all of these contributions is extracted using the above theorems, we get a quantity that only depends on the fraction of sites taken by the droplet. Thus, summing over all possible configurations amounts to multiplying the existing weight by the number of ways to choose the sites that will end up in the droplet. The resulting expression is the one on the right-hand side of (2.6).

3. Everybody connected

The goal of this section is to prove Theorem 2.3. Our proof is based on showing that the probability in (2.8) is exactly the same probability in a related, directed graph problem.

For a collection of vertices $\mathscr{V}_n = \{1, \dots, n\}$ and a set of bond probabilities $(p_{kl})_{1\le k<l\le n}$, let $\mathscr{G}$ be the inhomogeneous undirected random graph over $\mathscr{V}_n$. Similarly, let $\vec{\mathscr{G}}$ denote the inhomogeneous *directed* complete random graph with the restriction that the two possible (directed) bonds between k and l occur both with probability p_{kl}. To keep our notation distinct from the special case $p_{kl} = \alpha/n$ treated throughout this paper, we will denote the corresponding probability by P.

Definition 3.1 *A labelled directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is said to be* grounded *at vertex $v \in \mathscr{V}$ if for every $w \in \mathscr{V}$ there exists a (directed) path from v to w in $\mathscr{E}$.*

The identification of the two problems is now stated as follows:

Lemma 3.2 *Let K be the event that $\mathscr{G}$ is connected and let G be the event that $\vec{\mathscr{G}}$ is grounded at vertex "1." Then $P(K) = P(G)$.*

Proof. We use induction on the total number of edges incident with vertex "n." Indeed, if $p_{kn} = 0$ for all $k = 1, \dots, n-1$, then $P(K) = P(G)$ because both probabilities are zero. Now let us suppose that $P(K) = P(G)$ when $p_{\ell n} = 0$ for all $\ell = k, \dots, n-1$ and let us prove that it also for $p_{kn} > 0$. It clearly suffices to show that the partial derivatives of $P(K)$ and $P(G)$ with respect to p_{kn} are equal for all $p_{kn} \in [0, 1]$.

Notice first that both K and G are increasing events. Invoking Russo's formula, see [19] or [12, Theorem 2.25], we obtain

$$\frac{\partial}{\partial p_{kn}} P(G) = P\big((n,k) \text{ is pivotal for } G\big), \tag{3.1}$$

where the event $\{(n,k)$ is pivotal for $G\}$ means that if (n,k) is occupied, the event G occurs and if not, it does not. (Note that (n,k) denotes the edge going from "n" to "k.") The conditions under which this event occurs are straightforward: The set $\mathscr{V}_n = \{1,\dots,n\}$ splits into two disjoint components, one rooted at "1" and the other at "n," such that no site in the component associated with vertex "n" has an oriented bond to the other component. Similarly, we have

$$\frac{\partial}{\partial p_{kn}} P(K) = P\big((k,n) \text{ is pivotal for } K\big). \tag{3.2}$$

Here $\{(k,n)$ is pivotal for $K\}$ simply means that, if the edge (k,n) is disconnected, $\mathscr{V}_n$ consist of two connected components, one containing "1" and the other containing "n."

To see the equality of partial derivatives, we split both "pivotal" events according to the component containing the vertex "n." If $\mathscr{W}$ are the sites in this component, let $\mathscr{G}_{n,\mathscr{W}}$ and $\mathscr{G}_{1,\mathscr{W}}$ be the restrictions of $\mathscr{G}$ to $\mathscr{W}$, and $\mathscr{V}_n \setminus \mathscr{W}$, respectively. Similarly, let $\vec{\mathscr{G}}_{n,\mathscr{W}}$ and $\vec{\mathscr{G}}_{1,\mathscr{W}}$ be the corresponding "components" of the oriented graph. Let $K_{n,\mathscr{W}}$ and $K_{1,\mathscr{W}}$ be the events that $\mathscr{G}_{n,\mathscr{W}}$ and $\mathscr{G}_{1,\mathscr{W}}$ are connected and let $G_{n,\mathscr{W}}$ and $G_{1,\mathscr{W}}$ be the events that $\vec{\mathscr{G}}_{n,\mathscr{W}}$ is grounded at "n" and that $\vec{\mathscr{G}}_{1,\mathscr{W}}$ is grounded at "1," respectively. Since these pairs of events are independent, we have

$$P\big((n,k) \text{ is pivotal for } G\big) = \sum_{\substack{\mathscr{W}\colon 1\in\mathscr{W} \\ n\notin\mathscr{W}}} P(G_{1,\mathscr{W}})P(G_{n,\mathscr{W}})P(C_{\mathscr{W}})\big|_{p_{kn}=0}, \tag{3.3}$$

where $C_{\mathscr{W}}$ is the event that no site in $\mathscr{W}$ has a (directed) edge to $\mathscr{V}_n \setminus \mathscr{W}$. But the induction assumption tells us that $P(G_{1,\mathscr{W}}) = P(K_{1,\mathscr{W}})$ and $P(G_{n,\mathscr{W}}) = P(K_{n,\mathscr{W}})$, and the symmetry of bond probabilities for the directed graph tells us that $P(C_{\mathscr{W}})$ is the probability that $\mathscr{G}_{n,\mathscr{W}}$, and $\mathscr{G}_{1,\mathscr{W}}$ are not connected by a bond in $\mathscr{G}$. Substituting these into (3.3), we get the right-hand side of (3.2). This completes the induction step. □

From now on, let K and G pertain to the specific random graphs $\mathscr{G}(n, {}^{\alpha}/_{n})$ and $\vec{\mathscr{G}}(n, {}^{\alpha}/_{n})$. We begin with upper and lower bounds on $P_{n,\alpha}(K)$:

Lemma 3.3 $P_{n,\alpha}(K) \le \Big(1-(1-{}^{\alpha}/_{n})^{n-1}\Big)^{n-1}$.

Proof. Let E be the event—concerning the graph $\vec{\mathscr{G}}(n, {}^{\alpha}/_{n})$—that every vertex except number "1" has at least one outgoing edge. Then $G \subset E$ and so

$$P_{n,\alpha}(G) \le P_{n,\alpha}(E) = \Big(1-(1-{}^{\alpha}/_{n})^{n-1}\Big)^{n-1}. \tag{3.4}$$

Invoking Lemma 3.2, this proves the desired upper bound. □

Lemma 3.4 $P_{n,\alpha}(K) \ge \Big(1-(1-{}^{\alpha}/_{n})^{n-1}\Big)^{n-1}\frac{1}{n}$.

Proof. Consider the following events for directed random graph $\vec{\mathscr{G}}(n, \alpha/n)$: Let E be the event that every vertex, except vertex number "1," has at least one outgoing edge and let F be the event every such vertex has *exactly* one outgoing edge. Since $G \subset E$, we have

$$P_{n,\alpha}(G) = P_{n,\alpha}(E)P_{n,\alpha}(G|E). \tag{3.5}$$

We claim that

$$P_{n,\alpha}(G|E) \geq P_{n,\alpha}(G|F). \tag{3.6}$$

Indeed, let us pick an outgoing edge for each vertex different from "1," uniformly out of all edges going out of that vertex, and let us color these edges red. Let G' be the event that G occurs using only the red edges. The distribution of red edges conditional on E is the same as conditional on F. Hence $P_{n,\alpha}(G|E) \geq P_{n,\alpha}(G'|E) = P_{n,\alpha}(G'|F)$. But, on F, every available edge is red and so $P_{n,\alpha}(G'|F) = P_{n,\alpha}(G|F)$. Combining these inequalities, (3.6) is proved.

The number of configurations that $\vec{\mathscr{G}}(n, \alpha/n)$ can take on F is exactly $(n-1)^{n-1}$. On the other hand, the number of configurations which result in $\vec{\mathscr{G}}(n, \alpha/n)$ being grounded is $a_n = n^{n-2}$—the number of labelled trees with n vertices. Hence

$$P_{n,\alpha}(G|F) \geq \frac{n^{n-2}}{(n-1)^{n-1}} \geq \frac{1}{n}. \tag{3.7}$$

Using that $P_{n,\alpha}(E) = (1-(1-\alpha/n)^{n-1})^{n-1}$ the desired bound follows. □

Proof of Theorem 2.3. The claim is proved by noting

$$\lim_{n\to\infty} \frac{\Bigl(1-(1-\alpha/n)^{n-1}\Bigr)^{n-1}}{(1-e^{-\alpha})^{n-1}} = \exp\left((1-\alpha/2)\frac{\alpha e^{-\alpha}}{1-e^{-\alpha}}\right) \tag{3.8}$$

and using the results of Lemmas 3.3 and 3.4. □

4. ONLY TREES

Here we will assemble the necessary ingredients for the proof of Theorem 2.4. The proof is based on somewhat detailed combinatorial estimates and arguments using generating functions.

Recall that L denotes the event that $\mathscr{G}(n, \alpha/n)$ contains no loops and that B_r denotes the event that all components of $\mathscr{G}(n, \alpha/n)$ have no more than r sites. We begin by a combinatorial representation of the probability $P_{n,\alpha}(L \cap B_r)$:

$$\begin{aligned} P_{n,\alpha}(L\cap B_r) &= \sum_{\substack{\sum m_\ell \ell = n\\ m_\ell=0\,\forall \ell>r}} \frac{n!}{\prod_\ell \bigl[m_\ell!(\ell!)^{m_\ell}\bigr]}\left(\prod_{\ell\geq 1}\left[a_\ell\left(\frac{\alpha}{n}\right)^{\ell-1}\right]^{m_\ell}\right)\left(1-\frac{\alpha}{n}\right)^{\binom{n}{2}-n+\sum m_\ell} \\ &= n!\left(\frac{\alpha}{n}\right)^n\left(1-\frac{\alpha}{n}\right)^{\binom{n}{2}-n}\sum_{k=1}^n\left(\frac{\alpha}{n}\right)^{-k}\left(1-\frac{\alpha}{n}\right)^k Q_{n,k,r}. \end{aligned} \tag{4.1}$$

Here a_ℓ denotes the number of labeled trees on ℓ vertices and $Q_{n,k,r}$ abbreviates the sum

$$Q_{n,k,r} = \sum_{\substack{\sum m_\ell \ell = n\\ \sum m_\ell = k\\ m_\ell = 0\,\forall \ell > r}} \prod_{\ell \ge 1} \Bigl(\frac{a_\ell}{\ell!}\Bigr)^{m_\ell} \frac{1}{m_\ell!}. \tag{4.2}$$

We begin by isolating the large-n,k behavior of this quantity:

Proposition 4.1 *Consider the polynomial*

$$F_r(s) = \sum_{\ell=1}^r \frac{s^\ell a_\ell}{\ell!} \tag{4.3}$$

Then for all $n,k,r \ge 1$,

$$Q_{n,k,r} \le \frac{1}{k!} \inf_{s>0} \frac{F_r(s)^k}{s^n}. \tag{4.4}$$

Moreover, for each $\eta > 0$, there is $n_0 < \infty$ and a sequence $(c_r)_{r\ge1}$ of positive numbers for which

$$Q_{n,k,r} \ge \frac{c_r}{\sqrt{n}} \frac{1}{k!} \inf_{s>0} \frac{F_r(s)^k}{s^n} \tag{4.5}$$

holds for all $n \ge n_0$, all $k \ge 1$ and all $r \ge 2$ such that $k < (1-\eta)n$ and $rk > n(1+\eta)$.

Proof of upper bound. Let us consider the generating function

$$\hat{Q}_r(s,z) = 1 + \sum_{n=1}^\infty \sum_{k=1}^n Q_{n,k,r} z^k s^n = \exp\bigl\{zF_r(s)\bigr\}, \tag{4.6}$$

where we used Fubini-Tonelli to derive the second equality. Since F_r is a polynomial, the Cauchy integral formula yields

$$Q_{n,k,r} = \frac{1}{(2\pi\mathrm{i})^2} \oint \mathrm{d}s \oint \mathrm{d}z \frac{\exp\{zF_r(s)\}}{s^{n+1}z^{k+1}} = \frac{1}{2\pi\mathrm{i}} \frac{1}{k!} \oint \mathrm{d}s \frac{F_r(s)^k}{s^{n+1}}, \tag{4.7}$$

where all integrals are over a circle of positive radius centered at the origin of $\mathbb{C}$. Since all coefficients of F_r are non-negative, $\theta \mapsto |F_r(s\mathrm{e}^{\mathrm{i}\theta})|$ for $s > 0$ is maximized at $\theta = 0$. Bounding the integrand by its maximal modulus, computing the integral and optimizing over $s > 0$ then yields the upper bound in (4.4). □

Proof of lower bound. As is common in Tauberian arguments, the lower bound will require somewhat more effort. First let us note that under the conditions $k < (1-\eta)n$ and $rk > n(1+\eta)$ the function $s \mapsto F_r(s)^k/s^n$, for $s > 0$, blows up both at 0 and ∞. Its minimum is thus achieved at an interior point; for the rest of this proof we will fix s to a minimizer of this function. Since $|F_r(s\mathrm{e}^{\mathrm{i}\theta})| < F_r(s)$ for all $\theta \in (-\pi,\pi] \setminus \{0\}$, the part of the integral in (4.7) corresponding to $|\theta| > \epsilon$ is exponentially small (in n) compared to the infimum in (4.5). We thus need to show the lower bound only for the portion of the integral over θ with $|\theta| \le \epsilon$, for some fixed $\epsilon > 0$.

Since F_r has positive coefficients, $F_r \ne 0$ in the (complex) ϵ-neighborhood of s. This allows us to define the function

$$g(\theta) = \log \frac{F_r(s\mathrm{e}^{\mathrm{i}\theta})^\varrho}{s\mathrm{e}^{\mathrm{i}\theta}}, \qquad |\theta| \le \epsilon, \tag{4.8}$$

where ϱ plays the role of k/n. The function g is analytic in a small neighborhood of the origin. The choice of s implies that $g'(0) = 0$ which is equivalent to

$$\frac{sF_r'(s)}{F_r(s)} = \frac{1}{\varrho}. \tag{4.9}$$

For the second derivative we get $g''(0) = -\varrho\mathrm{Var}(X)$, where X is the random variable with law

$$P(X = \ell) = \frac{1}{F_r(s)}\frac{a_\ell s^\ell}{\ell!}, \qquad \ell = 1, \dots, r. \tag{4.10}$$

In particular, since our restrictions on ϱ between $\frac{1}{r}(1+\eta)$ and $1-\eta$ imply that s is bounded away from zero and infinity by η-dependent constants, same is true for $|g''(0)|$.

A straightforward calculation now shows that, for $\epsilon \ll 1$, the function g has a uniformly bounded third derivative and so we have

$$g(\theta) = g(0) - A\theta^2 + O(\theta^3). \tag{4.11}$$

where $A = A(r, \varrho)$ is positive uniformly in the allowed range of ϱ's and $O(\theta^3)$ is a quantity bounded by $|\theta|^3$ times a constant depending only on r, ϵ and η. (In particular, we may assume that $O(\theta^3)$ is dominated by $\frac{1}{2}A\theta^2$ for $|\theta| \le \epsilon$.)

We will split the integral over $\theta \in [-\epsilon, \epsilon]$ into two more parts. Let $\delta > 0$ and note that $ng(0)$ is the logarithm of the infimum in (4.5). Then for θ with $\delta n^{-1/3} \le |\theta| \le \epsilon$ we have

$$n\,\mathrm{Re}\,g(\theta) \le ng(0) - \frac{1}{2}A\delta^2 n^{1/3} \tag{4.12}$$

which shows that even this portion of the integral brings a contribution that is negligible compared to the right-hand side of (4.5). But for $|\theta| \le \delta n^{-1/3}$ we have $nO(\theta^3) = O(\delta)$ and so for $\delta \ll 1$, the Taylor remainder will always have imaginary part between, say, $-\pi/4$ and $\pi/4$. This means that

$$\mathrm{Re}\int_{-\delta n^{-1/3}}^{\delta n^{-1/3}} \mathrm{e}^{ng(\theta)}\,\mathrm{d}\theta \ge \frac{1}{2}\mathrm{e}^{ng(0)}\int_{-\delta n^{-1/3}}^{\delta n^{-1/3}} \mathrm{e}^{-nA\theta^2}\,\mathrm{d}\theta \ge \frac{c}{\sqrt{n}}\,\mathrm{e}^{ng(0)} \tag{4.13}$$

for some constant $c > 0$ which may depend on r and η but not on ϱ and n. Combined with the previous estimates, this proves the lower bound (4.5). $\square$

In light of the above lemma, the k-th term in the sum on the extreme right of (4.1) becomes

$$\alpha^{-k}n^k\mathrm{e}^{-\alpha\frac{k}{n}}Q_{n,k,r} = \mathrm{e}^{o(n)}\inf_{s>0}\exp\bigl\{n\Theta_r(s, k/n)\bigr\}, \tag{4.14}$$

where

$$\Theta_r(s,\varrho) = -\varrho\log\alpha - \varrho\log\varrho + \varrho + \varrho\log F_r(s) - \log s. \tag{4.15}$$

Here we should interpret (4.14) as an upper bound for $r = n$ and a lower bound for fixed r. It is clear that, regardless of r, the sum is dominated by the k's for which $\varrho \mapsto \inf_{s>0}\Theta_r(s,\varrho)$ is maximal. Such values are characterized as follows:

Lemma 4.2 *Let $\alpha > 0$ and $r \ge 2$. Then there is a unique $(s_r, \varrho_r) \in [0,\infty] \times [1/r, 1]$ for which*

$$\Theta_r(s_r, \varrho_r) = \sup_{1/r \le \varrho \le 1}\inf_{s>0}\Theta_r(s,\varrho). \tag{4.16}$$

Moreover, we always have $s_r \in (0,\infty)$ and $\varrho_r \in (1/r, 1)$ and, furthermore,

$$\lim_{r\to\infty} \Theta_r(s_r,\varrho_r) = \begin{cases} 1 + \alpha/2 - \log\alpha, & \textit{if } \alpha \le 1, \\ 1 + \frac{1}{2\alpha}, & \textit{if } \alpha > 1. \end{cases} \tag{4.17}$$

Proof. We begin by ruling out the "boundary values" of s and ϱ. First, if $\varrho = 1/r$, then the infimum over s is actually achieved by $s = \infty$. In that case $F_r(s) = \infty$ and the (one-sided) derivative with respect to ϱ is infinite, i.e., $\varrho = 1/r$ is a strict local minimum of $\varrho \mapsto \inf_{s>0}\Theta_r(s,\varrho)$. Similarly, for $\varrho = 1$ the infimum over $s > 0$ is achieved at $s = 0$ but then the ϱ-derivative of $\varrho \mapsto \inf_{s>0}\Theta_r(s,\varrho)$ is negative infinity, i.e., also $\varrho = 1$ is a strict local minimum. It follows that any (s_r,ϱ_r) satisfying (4.16) necessarily lies in $(0,\infty)\times(1/r,1)$.

A simple calculation now shows that any minimizing pair is the solution of the equations

$$F_r(s) = \alpha\varrho \quad \text{and} \quad sF_r'(s) = \alpha. \tag{4.18}$$

In light of monotonicity of $s \mapsto sF_r'(s)$, the solution is actually unique. To figure out the asymptotic as $r \to \infty$, we note that for $s \le 1/\mathrm{e}$,

$$sF_r'(s) = \sum_{\ell=1}^{r} a_\ell \frac{s^\ell}{(\ell-1)!} \underset{r\to\infty}{\longrightarrow} W(s). \tag{4.19}$$

Here W—which is closely related to the survival probability of the Galton-Watson branching process with Poisson offspring distribution—is the unique number in $[0, 1/\mathrm{e}]$ such that $W\mathrm{e}^{-W} = s$. If $s > 1/\mathrm{e}$, then $sF_r'(s) \to \infty$ as $r \to \infty$. Using the relation between $sF_r(s)$ and α, we thus get

$$s_r \underset{r\to\infty}{\longrightarrow} \begin{cases} \alpha\mathrm{e}^{-\alpha}, & \text{if } \alpha \le 1, \\ 1/\mathrm{e}, & \text{if } \alpha > 1. \end{cases} \tag{4.20}$$

Integrating the derivative of F_r now shows that $F_r(s_r) \to \alpha(1-\alpha/2)$ for $\alpha \le 1$. Using that $F_r'(s)$ is bounded for $s \le s_r$, we also find that $F_r(s_r) \to 1/2$ for $\alpha \ge 1$. This yields

$$\varrho_r \underset{r\to\infty}{\longrightarrow} \begin{cases} 1 - \alpha/2, & \text{if } \alpha \le 1, \\ \frac{1}{2\alpha}, & \text{if } \alpha > 1. \end{cases} \tag{4.21}$$

Noting that $\Theta(s_r,\varrho_r) = \varrho_r - \log s_r$ we now get (4.17). □

Proof of Theorem 2.4. By the fact that the supremum over ϱ in (4.16) is achieved at an interior point, we can control the difference between the maximizing k/n and its continuous counterpart ϱ. Thence

$$P_{n,\alpha}(L \cap B_r) = q_{n,r}\, n! \left(\frac{\alpha}{n}\right)^n \mathrm{e}^{-\alpha n/2} \exp\bigl\{n\Theta_r(s_r,\varrho_r)\bigr\}, \tag{4.22}$$

where

$$\frac{\tilde c_r}{\sqrt{n}} \le q_{n,r} \le n \tag{4.23}$$

for some positive constants $\tilde c_r$ which may depend on r and α. Since B_n contains every realization of $\mathscr{G}(n, \alpha/n)$, taking $r = n$ and applying Lemma 4.2 directly shows that $P_{\alpha,n}(L) \le \mathrm{e}^{n\Psi(\alpha)+o(n)}$.

To get a corresponding lower bound, we fix $r \geq 2$ and apply $P_{\alpha,n}(L) \geq P_{\alpha,n}(L \cap B_r)$. Taking $1/n$-th power and letting $n \to \infty$ then yields

$$\lim_{n\to\infty} P_{n,\alpha}(L \cap B_r)^{1/n} = \alpha e^{-1-\alpha/2+\Theta_r(s_r,\varrho_r)}. \tag{4.24}$$

As we have just checked, the right-hand side tends to $e^{\Psi(\alpha)}$ as $r \to \infty$. □

Corollary 4.3 *We have*

$$\lim_{r\to\infty} \lim_{n\to\infty} P_{\alpha,n}(B_r \cap L)^{1/n} = \lim_{n\to\infty} P_{\alpha,n}(L)^{1/n}. \tag{4.25}$$

Proof. This summarizes the last step of the previous proof. □

5. No big = no loops

Here we will prove that absence of large component has a comparable cost to absence of loops, at least on an exponential scale. To achieve this goal, apart from Corollary 4.3, we will need the following upper bound:

Lemma 5.1 *Let B_r be the event that $\mathscr{G}(n, \alpha/n)$ has no components larger than r and let L be the event that all connected components of $\mathscr{G}(n, \alpha/n)$ are trees. Then for all $r \geq 1$,*

$$P_{n,\alpha}(B_r) \leq P_{n,\alpha}(L)\Bigl(1-\frac{\alpha}{n}\Bigr)^{-\frac{1}{2}rn}. \tag{5.1}$$

Proof. Let C be the restriction of $\mathscr{G}(n, \alpha/n)$ to a set $S \subset \{1,\dots,n\}$. Let T be a tree on S. Then

$$\frac{P_{n,\alpha}(C = T)}{P_{n,\alpha}(C \supset T)} = \Bigl(1-\frac{\alpha}{n}\Bigr)^{\binom{|S|}{2}-|S|+1} \geq \Bigl(1-\frac{\alpha}{n}\Bigr)^{\frac{1}{2}|S|^2}. \tag{5.2}$$

Hence

$$P_{n,\alpha}(C \text{ is connected}) \leq \sum_T P_{n,\alpha}(C \supset T) \leq \Bigl(1-\frac{\alpha}{n}\Bigr)^{-\frac{1}{2}|S|^2} P_{n,\alpha}(C \text{ is a tree}). \tag{5.3}$$

Now, if L_r is the event that no component of $\mathscr{G}(n, \alpha/n)$ of size larger than r has loops, then $B_r \subset L_r$ and so $P_{n,\alpha}(B_r) \leq P_{n,\alpha}(L_r)$. Let $\{S_j\}$ be a partition of $\{1,\dots,n\}$ and let $P_{n,\alpha}(\{S_j\})$ denote the probability that $\{S_j\}$ are the connected components of $\mathscr{G}(n, \alpha/n)$. Then

$$P_{n,\alpha}(L_r) = \sum_{\{S_j\}} P_{n,\alpha}(\{S_j\}) P_{n,\alpha}(L_r|\{S_j\}), \tag{5.4}$$

where $P_{n,\alpha}(L_r|\{S_j\})$ is the conditional probability of L_r given that $\{S_j\}$ are the connected components of $\mathscr{G}(n, \alpha/n)$.

Letting C_j represent the restriction of $\mathscr{G}(n, {}^{\alpha}\!/_{n})$ to S_j, the bound (5.3) tells us that

$$\begin{aligned} P_{n,\alpha}(L_r|\{S_j\}) &= \prod_{j\colon |S_j|\ge r} P_{n,\alpha}(C_j \text{ is a tree}|C_j \text{ is connected}) \\ &\le \prod_{j} P_{n,\alpha}(C_j \text{ is a tree}|C_j \text{ is connected}) \prod_{j\colon |S_j|<r} \Bigl(1-\frac{\alpha}{n}\Bigr)^{-\frac12|S_j|^2} \end{aligned} \tag{5.5}$$

Using that $|S_j| < r$ for every S_j contributing to the second product and applying that the sum of $|S_j|$ over the components with $|S_j| < r$ gives at most n, we then get

$$P_{n,\alpha}(L_r|\{S_j\}) \le P_{n,\alpha}(L|\{S_j\})\Bigl(1-\frac{\alpha}{n}\Bigr)^{-\frac12 rn}. \tag{5.6}$$

Plugging this back in (5.4), the desired bound follows. □

Proof of Theorem 2.5. By Lemma 5.1 we have

$$\limsup_{n\to\infty} P_{n,\alpha}(B_{\epsilon n})^{1/n} \le \mathrm{e}^{\epsilon/2} \lim_{n\to\infty} P_{n,\alpha}(L)^{1/n}. \tag{5.7}$$

On the other hand, the inclusion $B_r \supset B_r \cap L$ and Corollary 4.3 yield

$$\liminf_{n\to\infty} P_{n,\alpha}(B_r)^{1/n} \ge \liminf_{n\to\infty} P_{n,\alpha}(B_r\cap L)^{1/n} \underset{r\to\infty}{\longrightarrow} \lim_{n\to\infty} P_{n,\alpha}(L). \tag{5.8}$$

Since $P_{n,\alpha}(B_r) \le P_{n,\alpha}(B_{\epsilon n})$ eventually for any fixed $r \ge 1$ and $\epsilon > 0$, all limiting quantities are equal provided we take $r \to \infty$ and/or $\epsilon \downarrow 0$ after $n \to \infty$. □

6. PROOF OF MAIN RESULT

Before we start proving the main theorem, we need to ensure that if a large component is present in the graph, then it is unique. The statement we need is as follows:

Lemma 6.1 *Let $K_{\epsilon,2}$ be the event that $\mathscr{G}(n, {}^{\alpha}\!/_{n})$ is either connected or has exactly two connected components, each of which is of size at least ϵn, and recall that K is the event that $\mathscr{G}(n, {}^{\alpha}\!/_{n})$ is connected. Then for all $\alpha_0 > 0$ and $\epsilon_0 > 0$ there exists $c_1 = c_1(\alpha_0, \epsilon_0) < 1$ such that for all $\epsilon \ge \epsilon_0$ and all $\alpha \le \alpha_0$,*

$$\limsup_{n\to\infty} P_{\alpha,n}(K^{\mathrm{c}}|K_{\epsilon,2})^{1/n} < c_1. \tag{6.1}$$

Proof. It clearly suffices to show that the ratio of $P_{\alpha,n}(K_{\epsilon,2}\setminus K)$ and $P_{\alpha,n}(K)$ decays to zero exponentially with n, with a rate that is uniformly bounded in $\epsilon \ge \epsilon_0$ and $\alpha \le \alpha_0$. In light of Theorem 2.3 and the fact that $K_{\epsilon,2}$ only admits components that grow linearly with n, we have

$$\frac{P_{\alpha,n}(K_{\epsilon,2}\setminus K)}{P_{\alpha,n}(K)} = \mathrm{e}^{o(n)} \sum_{\epsilon n\le k\le n-\epsilon n} \binom{n}{k} \frac{\pi_1(\alpha\, {}^{k}\!/_{n})^k\, \pi_1\bigl(\alpha(1-{}^{k}\!/_{n})\bigr)^{n-k}}{\pi_1(\alpha)^n} \Bigl(1-\frac{\alpha}{n}\Bigr)^{k(n-k)}, \tag{6.2}$$

where $o(n)/n$ tends to zero uniformly in $\epsilon \ge \epsilon_0$ and $\alpha \le \alpha_0$. Writing ϱ for ${}^{k}\!/_{n}$, the expression under the sum can be bounded by $\mathrm{e}^{n[\Xi(\varrho)-\Xi(0)]}$, where

$$\Xi(\varrho) = -S(\varrho) + \varrho\log\pi_1(\alpha\varrho) + (1-\varrho)\log\pi_1\bigl(\alpha(1-\varrho)\bigr) - \alpha\varrho(1-\varrho). \tag{6.3}$$

Since ϱ is restricted to the interval $[\epsilon, 1-\epsilon]$, the right-hand side of (6.2) will be exponentially small if we can show $\Xi(\varrho) < \Xi(0)$ for all $\varrho \in (0,1)$ and all α.

As is easy to check, the function $\varrho \mapsto \Xi(\varrho)$ is symmetric about the point $\varrho = 1/2$. Hence, if we can prove that it is strictly convex throughout $[0,1]$, then it is maximized at the endpoints. Introducing the function

$$G(\eta) = \eta \log \frac{\pi_1(\eta)}{\eta} \tag{6.4}$$

we have

$$\alpha \Xi(\eta/\alpha) = G(\eta) + G(\alpha - \eta) + \eta(\alpha - \eta). \tag{6.5}$$

In order to prove strict convexity of Ξ, it thus suffices to show that $G''(\eta) + 1 > 0$ for all $\eta > 0$. Introducing yet another abbreviation $q(\eta) = \eta/(1-\mathrm{e}^{-\eta})$, a tedious but straightforward differentiation yields

$$G''(\eta) + 1 = \frac{1}{q}(q' - q)(q\mathrm{e}^{-\eta} - 1). \tag{6.6}$$

A direct evaluation now shows that both $q' - q$ and $q\mathrm{e}^{-\eta} - 1$ are negative once $\eta > 0$. $\square$

We will use the above lemma via the following simple conclusion:

Lemma 6.2 *Let N_r denote the number of connected components of $\mathscr{G}(n, \alpha/n)$ of size at least r and let $\mathscr{V}_r$ be the set of sites contained in these components. Then for all $\varrho > \epsilon > 0$ and all α,*

$$P_{\alpha,n}\bigl(|\mathscr{V}_{\epsilon n}| = \lfloor \varrho n \rfloor \;\&\; N_{\epsilon n} = 1\bigr) \ge \frac{1}{2} P_{\alpha,n}\bigl(|\mathscr{V}_{\epsilon n}| = \lfloor \varrho n \rfloor\bigr) \tag{6.7}$$

once n is sufficiently large.

Proof. Clearly, (6.7) will follow if we can prove that

$$P_{\alpha,n}\bigl(|\mathscr{V}_{\epsilon n}| = \lfloor \varrho n \rfloor \;\&\; N_{\epsilon n} > 1\bigr) \le \frac{1}{2} P_{\alpha,n}\bigl(|\mathscr{V}_{\epsilon n}| = \lfloor \varrho n \rfloor\bigr). \tag{6.8}$$

Let $\mathscr{V}(x)$ denote the connected component of $\mathscr{G}(n, \alpha/n)$ containing the site x and let $x \nleftrightarrow y$ denote the event that $x, y \in \mathscr{V}_{\epsilon n}$ but $\mathscr{V}(x) \cap \mathscr{V}(y) = \emptyset$. Then (6.8) will be proved once we show that for some $c = c(\epsilon, \varrho) > 0$,

$$P_{\alpha,n}\bigl(|\mathscr{V}_{\epsilon n}| = \lfloor \varrho n \rfloor \;\&\; x \nleftrightarrow y\bigr) \le \mathrm{e}^{-cn} P_{\alpha,n}\bigl(|\mathscr{V}_{\epsilon n}| = \lfloor \varrho n \rfloor\bigr). \tag{6.9}$$

(Indeed, the sum over x, y adds only a multiplicative factor of n^2 on the right-hand side.) By conditioning on the set $\mathscr{V}_{\epsilon n}$ *and* the set $\mathscr{V}(x) \cup \mathscr{V}(y)$, this inequality will in turn follow from

$$P_{\alpha,n}\bigl(x \nleftrightarrow y \;\&\; \mathscr{V}(x) \cup \mathscr{V}(y) = \mathscr{V}\bigr) \le \mathrm{e}^{-cn} P_{\alpha,n}\bigl(\mathscr{V}(x) \cup \mathscr{V}(y) = \mathscr{V}\bigr). \tag{6.10}$$

(Multiply both sides by the probability that $\mathscr{V}$ is disconnected from the rest of the graph and that all components—in the complement of $\mathscr{V}$—of size at least ϵn take the total volume $\lfloor \varrho n \rfloor - |\mathscr{V}|$. Then sum over all admissible $\mathscr{V}$.) We will reduce (6.10) to Lemma 6.1.

Recall that K is the event that the graph is connected and $K_{\epsilon,2}$ is the event that it has at most two components, each of which is of size at least ϵn. We will now use these events to the restriction of $\mathscr{G}(n, \alpha/n)$ to $\mathscr{V}$. Let $m = |\mathscr{V}|$, $\tilde{\alpha} = \alpha \frac{m}{n}$ and $\tilde{\epsilon} = \epsilon \frac{n}{m}$. Then

$$\{x \nleftrightarrow y\} \cap \bigl\{\mathscr{V}(x) \cup \mathscr{V}(y) = \mathscr{V}\bigr\} \subset K^{\mathrm{c}} \cap K_{\tilde{\epsilon},2}, \tag{6.11}$$

while for the event on the right-hand side of (6.10) we simply get

$$\bigl\{\mathscr{V}(x)\cup\mathscr{V}(y)=\mathscr{V}\bigr\}=K_{\tilde\epsilon,2}. \tag{6.12}$$

By Lemma 6.1 and the fact that $\tilde\alpha\le\alpha$ and $\tilde\epsilon\ge\epsilon$,

$$P_{\tilde\alpha,m}(K^{\mathrm{c}}|K_{\tilde\epsilon,2})\le \mathrm{e}^{-c_1 m}, \tag{6.13}$$

once n is sufficiently large. But $m\ge 2\epsilon n$ and so (6.10) holds with $c=2\epsilon c_1$. □

Now we have finally amassed all ingredients needed for the proof of our main result.

Proof of Theorem 2.1. The case $\varrho=0$ is quickly reduced to Theorems 2.4–2.5 while $\varrho=1$ boils down to Theorem 2.3. Thus, we are down to the cases $\varrho\in(0,1)$. Let $\epsilon\in(0,\varrho)$. By Lemma 6.2, we can focus on the situations with $N_{\epsilon n}=1$. To make our notation simple, let us assume that ϱn is an integer. Then we have

$$P_{\alpha,n}\bigl(|\mathscr{V}_{\epsilon n}|=\varrho n\,\&\,N_{\epsilon n}=1\bigr)=\binom{n}{\varrho n}P_{\varrho n,\alpha\varrho}(K)P_{n-\varrho n,\alpha(1-\varrho)}(B_{\epsilon n})\Bigl(1-\frac{\alpha}{n}\Bigr)^{\varrho n(1-\varrho)n}. \tag{6.14}$$

The terms on the right-hand side represent the following: the number of ways to choose the unique component of size ϱn, the probability that this component is connected, the probability that the complement contains no component of size larger than ϵn and, finally, the probability that the two parts of the graph do not have any bond between them. Invoking Stirling's formula and Theorems 2.3–2.5, the result quickly follows. □

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