Approximate controllability of three-dimensional Navier–Stokes equations

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Abstract

The paper is devoted to studying the problem of controllability for 3D Navier–Stokes equations in a bounded domain. We develop the method introduced by Agrachev and Sarychev in the 2D case and establish a sufficient condition under which the problem in question is approximately controllable by a finite-dimensional force. In the particular case of a torus, it is shown that our sufficient condition is fulfilled for a control of low dimension.

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0 Introduction

In the pioneering article [AS05a], Agrachev and Sarychev introduced a new method for studying controllability properties of PDE's perturbed by a finite-dimensional control force. They considered the 2D Navier–Stokes (NS) equations

$$\dot{u} + (u, \nabla)u - \nu\Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0, \tag{0.1}$$

where $x \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, $\nu > 0$ is the viscosity, $u(t, x)$ is the velocity field, $p(t, x)$ is the pressure, and $\eta(t, x)$ is a control function that takes on values in a *finite-dimensional* space $E \subset L^2(\mathbb{T}^2, \mathbb{R}^2)$. One of the main results in [AS05a] states that if E contains sufficiently many Fourier modes, then for any $T > 0$ and $\nu > 0$ Eq. (0.1) is approximately controllable in time T. Without going into details, let us explain two key ideas that enable one to prove approximate controllability (AC) of (0.1). [1]

Introduce the space

$$H = \left\{u \in L^2(\mathbb{T}^2, \mathbb{R}^2) : \operatorname{div} u \equiv 0\right\} \tag{0.2}$$

and denote by $\Pi : L^2(\mathbb{T}^2, \mathbb{R}^2) \to H$ the orthogonal projection in $L^2(\mathbb{T}^2, \mathbb{R}^2)$ onto the subspace H. Projecting (0.1) to H, we obtain the following evolution equation in H, which is equivalent to (0.1):

$$\dot{u} + \nu Lu + B(u) = \eta(t). \tag{0.3}$$

Here $L = -\Pi\Delta$, $B(u) = \Pi\{(u, \nabla)u\}$, and we use the same notation for the right-hand side η and its projection to H. It is well known that the Cauchy problem for (0.3) is well posed in appropriate functional spaces, and the corresponding solutions defined on the positive half-line are continuous functions of time with range in H. Recall that Eq. (0.3) is said to be *approximately controllable in time T by an E-valued control* (where $E \subset H$ is a finite-dimensional subspace) if for any $u_0, \hat{u} \in H$ and any $\varepsilon > 0$ there is an essentially bounded function $\eta : [0, T] \to E$ such that

$$\|u(T) - \hat{u}\| < \varepsilon,$$

where $u(t)$ denotes the solution of (0.3) issued from u_0 and $\|\cdot\|$ stands for the L^2-norm.

[1] The scheme presented below is not entirely accurate and differs slightly from the one used in the original paper [AS05a].

Along with (0.3), let us consider the control system

$$\dot{u} + \nu L(u + \zeta(t)) + B(u + \zeta(t)) = \eta(t). \tag{0.4}$$

Here η and ζ are E-valued control functions. It turns out that control systems (0.3) and (0.4) are equivalent. Namely, we have the following property, which is an analogue for PDE's of a more general result established in [AS86] for the case of ODE's (see also Sections 6.1 and 12.4 in [AS05a]):

(P$_1$) Equation (0.3) with $\eta \in E$ is AC in time $T > 0$ if and only if so is Eq. (0.4) with $\eta, \zeta \in E$.

We now compare (0.4) with a control system of the form (0.3) in which the control function takes on values in a space $E_1 \supset E$. More precisely, for any subset $\mathcal{A} \subset H$, denote by $\operatorname{co}\mathcal{A}$ the convex hull of $\mathcal{A}$, that is, the set of vectors $v \in H$ that are representable in the form

$$v = \sum_{i=1}^{k} \lambda_i u_i,$$

where $k \geq 1$ is an integer depending on v, $u_i \in \mathcal{A}$ for $i = 1, \dots, k$, and $\lambda_i > 0$ are some constants whose sum is equal to 1. Let $E_1 \subset H$ be the largest vector space such that

$$B(u) + E_1 \subset \operatorname{co}\bigl\{B(u + \zeta) + \nu L\zeta + \eta : \eta, \zeta \in E\bigr\} \quad \text{for any } u \in H. \tag{0.5}$$

Consider the control system

$$\dot{u} + \nu L u + B(u) = \eta_1(t), \tag{0.6}$$

where η_1 is an E_1-valued control. The following property is a version of the well-known convexification principle (for instance, see Theorem 8.2 in [AS04] or Theorem 7 in [Jur97, Chapter 3]).

(P$_2$) Suppose that Eq. (0.6) with $\eta_1 \in E_1$ is AC in time $T > 0$. Then so is Eq. (0.4) with $\eta, \zeta \in E$.

Note that, in a general situation, the subspace E_1 may coincide with E. However, if E is a proper subset of E_1, then properties (P$_1$) and (P$_2$) enable one to reduce the question of AC for Eq. (0.3) to a similar problem with a larger control space. Iterating this argument, for any initial space E one can construct a non-decreasing sequence of subspaces $E_1 \subset E_2 \subset \cdots$ such that the following property holds.

(P) Equation (0.3) with $\eta \in E$ is AC in time $T > 0$ if and only if so is Eq. (0.3) with $\eta \in E_k$ for some $k \geq 1$.

Now let $\{e_j\}$ be an orthonormal basis in H formed of trigonometric polynomials and let $H_N \subset H$ be the vector space spanned by $e_1, \dots, e_N$. It was shown by

Agrachev and Sarychev [AS05a] that if $E \supset H_{N_0}$ for a sufficiently large $N_0 \geq 1$, then there is a sequence $N_k \to \infty$ such that $H_{N_k} \subset E_k$ for any $k \geq 1$. This property combined with (P) implies that (0.3) is AC.

The Agrachev–Sarychev approach is rather general and does not use any particular property of 2D NS equations other than well-posedness of the Cauchy problem in appropriate functional spaces. It can be applied to various controlled PDE's, including the 2D Euler system and nonlinear Schrödinger equation [AS05b]. Moreover, combining some refined versions of properties (P_1) and (P_2) with a degree theory argument, it was shown in [AS05a] that the 2D NS system on the torus possesses the property of exact controllability in observed projections.

The aim of this paper is to develop the Agrachev–Sarychev method in such a way that it can be applied to equations for which the well-posedness of the Cauchy problem is not known to hold. Namely, we consider the 3D Navier–Stokes system on a torus $\mathbb{T}^3$. Let H be the space of divergence-free vector fields on $\mathbb{T}^3$ (cf. (0.2)) and let $V = H^1(\mathbb{T}^3, \mathbb{R}^3) \cap H$. As in the 2D case, one can reduce the problem in question to an evolution equation in H of the form (0.3). Let $E \subset H$ be a finite-dimensional subspace. We shall say that the 3D NS system (0.3) with $\eta \in E$ is *approximately controllable in time* T if for any $u_0, \hat{u} \in V$ and any $\varepsilon > 0$ there is an essentially bounded function $\eta : [0, T] \to E$ and a strong solution $u(t)$ of (0.3) such that

$$u(0) = u_0, \quad \|u(T) - \hat{u}\|_V \leq \varepsilon.$$

The following theorem is a simplified version of the main result of this paper (see Section 2 for more details).

Main Theorem. *There is a finite-dimensional subspace $E \subset H$ such that for any $T > 0$ and $\nu > 0$ the 3D Navier–Stokes system* (0.3) *with $\eta \in E$ is approximately controllable in time T.*

To prove this result, we show that properties (P_1) and (P_2) remain valid for the 3D NS system. Their proof, however, is different from that in the 2D case and relies substantially on a perturbative result on existence of strong solutions for 3D NS equations (see Section 1.4). We note that even in the 2D case the approach of this paper contains some new elements compared with the proofs in [AS05a]. We also hope that our presentation will help the readers not familiar with the geometric control theory of ODE's to gain a better understanding of the Agrachev–Sarychev method.

It should be mentioned that the problem of controllability for Navier–Stokes and Euler equations was studied by many authors during the last fifteen years, and a number of deep results have been obtained (see the papers [Lio90, Fur95, Cor96, CF96, Ima98, FE99, Cor99, FC99, Gla00, Zua02] and the references therein). In particular, it was proved that NS equations possess the property of exact controllability (both in 2D and 3D) by a force supported in any given domain (see [Cor96, CF96, Ima98, FE99]). Furthermore, feedback stabilisation properties of NS and Euler equations were studied in [Cor99, BS01, Fur01, Fur04, BT04]. To the best of my knowledge, this paper provides a first result on approximate controllability of 3D NS equations by a control of finite-dimension not depending on the viscosity. In conclusion, we note that our arguments can be used to prove the property of exact controllability in observed projections for 3D NS system; we shall address this question in a subsequent publication.

The paper is organised as follows. In Section 1, we have compiled some preliminaries on Navier–Stokes equations. The main results of the paper are presented in Section 2. We establish a sufficient condition under which the 3D NS system is controllable by a finite-dimensional force and then show that it is satisfied in the case of periodic boundary conditions. Section 3 is devoted to the proofs.

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Notation. We use standard functional spaces arising in the theory of Navier–Stokes equations; they are defined in Section 1.1. For a separable Banach space X and a compact interval $J \subset \mathbb{R}$, we introduce the following notation.

$B_X(R)$ is the closed ball in X of radius R centred at the origin.

$L^p(J,X)$ is the space of measurable functions $f : J \to X$ such that

$$\|f\|_{L^p(J,X)} := \left(\int_J \|f(t)\|_X^p \, dt\right)^{1/p} < \infty, \tag{0.7}$$

where $\|\cdot\|_X$ stands for the norm in X. In the case $p = \infty$, condition (0.7) is replaced by

$$\|f\|_{L^\infty(J,X)} := \operatorname*{ess\,sup}_{t \in J} \|f(t)\|_X < \infty,$$

$C(J,X)$ is the space of continuous functions $f : J \to X$ endowed with the norm

$$\|f\|_{C(J,X)} := \max_{t \in J} \|f(t)\|_X .$$

$\mathcal{L}(X)$ denotes the space of continuous linear operators in X with the usual operator norm $\|\cdot\|_{\mathcal{L}(X)}$.

If X is a Hilbert space and $E \subset X$ is a closed subspace, then $E^\perp$ stands the orthogonal complement of E in X. In this case, we denote by $\mathsf{P} = \mathsf{P}_E$ and $\mathsf{Q} = \mathsf{Q}_E$ the orthogonal projections in X onto the subspaces E and $E^\perp$, respectively.

Throughout the paper, we denote by C_i, $i = 1, 2, \dots$, unessential positive constants and by J_T the time interval $[0,T]$.

1 Preliminaries on 3D Navier–Stokes equations

In this section, we have compiled some auxiliary results on 3D Navier–Stokes equations. Although the methods used in their proofs are well known, we present a rather detailed justification of all statements, since they will play an essential role in Sections 2 and 3.

1.1 Functional spaces and Leray projection

Let $D \subset \mathbb{R}^3$ be a bounded domain with C^2-smooth boundary ∂D. Denote by $H^s = H^s(D, \mathbb{R}^3)$ the space of vector functions $u = (u_1, u_2, u_3)$ whose components belong to the Sobolev space of order s and by $\|\cdot\|_s$ the corresponding norm. In the case $s = 0$, we shall write $L^2 = L^2(D, \mathbb{R}^3)$ and $\|\cdot\|$, respectively. If $s > 1/2$, then $H_0^s(D, \mathbb{R}^3)$ stands for the space of functions $u \in H^s$ vanishing on ∂D. Let

$$H = \big\{u \in L^2(D, \mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } D,\ (u, \boldsymbol{n})|_{\partial D} = 0\big\},$$

where $\boldsymbol{n}$ is the outward unit normal to ∂D. Introduce the spaces

$$V = H_0^1(D, \mathbb{R}^3) \cap H, \quad U = H^2(D, \mathbb{R}^2) \cap V,$$

endowed with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively.

Let $\Pi : L^2 \to H$ be the Leray projection, that is, the orthogonal projection in L^2 onto the closed subspace H. The following result is a straightforward consequence of the Hodge–Kodaira decomposition, elliptic regularity, and complex interpolation (for instance, see [Soh01]).

Proposition 1.1. *The projection Π satisfies the inequality*

$$\|\Pi u\|_s \le C\,\|u\|_s \quad \text{for any } u \in H^s(D, \mathbb{R}^3),$$

where $0 \le s \le 2$ and $C > 0$ is a constant not depending on $u(x)$ and s.

1.2 Parabolic semigroups generated by the Stokes operator

Let L be the Stokes operator, that is, the operator $-\Pi\Delta$ with the domain U. It is well known that L is a positive self-adjoint operator in H with discrete spectrum (for instance, see [CF88, Chapter 4]). We shall use sometimes the following equivalent norms on U and V:

$$\|u\|_U = (Lu, Lu)^{1/2}, \quad \|u\|_V = (Lu, u)^{1/2},$$

where $(\cdot,\cdot)$ stands for the scalar product in L^2.

Consider the problem

$$\dot u + Lu = h(t), \tag{1.1}$$

$$u(0) = u_0. \tag{1.2}$$

For any $T>0$, we set $J_T=[0,T]$ and define the space

$$\mathcal{X}_T=C(J_T,V)\cap L^2(J_T,U)$$

endowed with the norm

$$\|u\|_{\mathcal{X}_T}=\max_{0\le t\le T}\|u(t)\|_V+\left(\int_0^T\|u(t)\|_U^2\,dt\right)^{1/2}.$$

The following result is a consequence of the above-mentioned properties of L (for instance, see [Hen81, Section 1.3]).

Proposition 1.2. *For any $h\in L^2(J_T,H)$ and $u_0\in V$, problem* (1.1), (1.2) *has a unique solution $u\in\mathcal{X}_T$, which satisfies the inequality*

$$\|u(t)\|_V^2+\int_0^t\|u(s)\|_U^2ds\le\|u_0\|_V^2+\int_0^t\|h(s)\|^2ds,\quad t\in J_T. \tag{1.3}$$

We now consider the projections of problem (1.1), (1.2) to a subspace of finite codimension. Let $E\subset U$ be a finite-dimensional subspace and let $E^\perp$ be its orthogonal complement in H. We denote by P_E and Q_E the orthogonal projections in H onto the subspaces E and $E^\perp$, respectively. Consider the problem

$$\dot w+L_Ew=f(t), \tag{1.4}$$

$$w(0)=w_0, \tag{1.5}$$

where $L_E=\mathsf{Q}_EL$. Define the space

$$\mathcal{X}_T(E):=C(J_T,V\cap E^\perp)\cap L^2(J_T,U\cap E^\perp),$$

endowed with the norm $\|\cdot\|_{\mathcal{X}_T}$.

Proposition 1.3. *For any $f\in L^2(J_T,E^\perp)$ and $w_0\in V\cap E^\perp$, problem* (1.4), (1.5) *has a unique solution $w\in\mathcal{X}_T(E)$, which satisfies the inequality*

$$\|w\|_{\mathcal{X}_T}\le C\left(\|w_0\|_V+\|f\|_{L^2(J_T,H)}\right), \tag{1.6}$$

where $C>0$ is a constant depending only on E and T.

Proof. Step 1. We first prove the uniqueness of solution. To this end, suppose that $w\in\mathcal{X}_T(E)$ is a solution of problem (1.4), (1.5) with $f=0$ and $w_0=0$. Then

$$\frac{d}{dt}\|w(t)\|^2=2(w(t),\dot w(t))=-2(w(t),L_Ew(t))\le 0,$$

whence it follows that $w\equiv 0$.

Step 2. We now prove the existence of solution. Without loss of generality, we shall assume that $T>0$ is sufficiently small; the general case can be reduced to the former by iteration. Let us set $\mathcal{Y}=L^2(J_T,E^\perp)$. We claim that there is a continuous operator $\mathcal{S}_E : \mathcal{Y} \to \mathcal{Y}$ with the following property: if $u \in \mathcal{X}_T$ is the solution of problem (1.1), (1.2) with $h = \mathcal{S}_E f$ and $u_0 = w_0$, then the function $\mathsf{Q}_E u$ belongs to $\mathcal{X}_T(E)$ and satisfies Eqs. (1.4), (1.5). If this assertion is proved, then inequality (1.6) is a straightforward consequence of (1.3).

To construct the operator $\mathcal{S}_E$, suppose that $u \in \mathcal{X}_T$ is a solution of (1.4), (1.5) with $u_0 = w_0$ and some function $h \in \mathcal{Y}$ and let $w = \mathsf{Q}_E u$. Since $E \subset U$ and $\dim E < \infty$, the projection Q_E is continuous in the spaces U and V. This implies that $w \in \mathcal{X}_T(E)$. Moreover, it follows from (1.2) that (1.5) holds. Applying Q_E to (1.1), we derive

$$\dot{w} + L_E w = h - \mathsf{Q}_E L \mathsf{P}_E u.$$

Thus, w is a solution of (1.4) if and only if

$$h - \mathsf{Q}_E L \mathsf{P}_E u = f. \tag{1.7}$$

Let us denote by $K : L^2(J_T, H) \to \mathcal{X}_T$ the operator that takes each function h to the solution in $\mathcal{X}_T$ of problem (1.1), (1.2) with $u_0 = 0$:

$$Kh(t) = \int_0^t e^{-(t-s)L} h(s)ds. \tag{1.8}$$

Then the solution of (1.1), (1.2) with $u_0 = w_0$ can be written in the form

$$u = v + Kh, \quad v(t) = e^{-tL} w_0.$$

Substituting this expression for u in the left-hand side of (1.7) and denoting by I the identity operator, we obtain the following functional equation for h:

$$(I - \mathsf{Q}_E L \mathsf{P}_E K)h = f + \mathsf{Q}_E L \mathsf{P}_E v.$$

The right-hand side of this equation belongs to $\mathcal{Y}$. Therefore, the required assertion will be established if we show that

$$\|\mathsf{Q}_E L \mathsf{P}_E K\|_{\mathcal{L}(\mathcal{Y})} \le \tfrac{1}{2} \quad \text{for sufficiently small } T > 0, \tag{1.9}$$

where $\mathcal{L}(\mathcal{Y})$ stands for the space of continuous linear operators in $\mathcal{Y}$.

Step 3. Let us prove (1.9). Since

$$\|e^{-tL}\|_{\mathcal{L}(H)} = e^{-\alpha_1 t},$$

where $\alpha_1 > 0$ is the first eigenvalue of L, it follows from (1.8) that

$$\|Kh\|_{\mathcal{Y}} \le C_1 T \|h\|_{\mathcal{Y}}, \tag{1.10}$$

where $C_1 > 0$ does not depend on T. Using again the fact that $E \subset U$ is finite-dimensional, we see that

$$\|\mathsf{Q}_E L \mathsf{P}_E g\| \le C_2 \|g\| \quad \text{for any } g \in H, \tag{1.11}$$

where $C_2 > 0$ depends only on E. Combining (1.10) and (1.11), we arrive at (1.9). The proof of Proposition 1.3 is complete. $\square$

Remark 1.4. It is clear that inequality (1.6) remains valid if we replace T by any $T' < T$, and the corresponding constant C in the right-hand side will be independent of T'. In particular, we obtain the estimate

$$\|w(t)\|_V^2 + \int_0^t \|w(s)\|_U^2 ds \le C\left(\|w_0\|_V^2 + \int_0^t \|f(s)\|^2 ds\right), \quad t \in J_T, \tag{1.12}$$

where $C > 0$ does not depend on w_0 and f.

We now consider a particular case of (1.4) in which E is a subspace generated by eigenfunctions of the Stokes operator. Let $\{e_j\}$ be an orthonormal basis in H formed of the eigenfunctions of L and let $\{\alpha_j\}$ be the corresponding sequence of eigenvalues indexed in an increasing order. Let us denote by H_N the vector space spanned by $e_1, \dots, e_N$ and by $H_N^\perp$ its orthogonal complement in H. We write P_N and Q_N for the orthogonal projections in H onto the subspaces H_N and $H_N^\perp$, respectively. In what follows, we shall need a refinement of inequality (1.6) for the case $E = H_N$.

Corollary 1.5. *Suppose that the conditions of Proposition 1.3 are fulfilled with $E = H_N$, where $N \ge 1$ is an integer, and that $f \in L^2(J_T, H^r)$ for some $r \in (0, 1/2)$. Then there is a constant $C > 0$ not depending on N and r such that the solution $w \in \mathcal{X}_T(H_N)$ of problem (1.4), (1.5) with $w_0 = 0$ satisfies the inequality*

$$\|w\|_{\mathcal{X}_T} \le C\, \alpha_{N+1}^{-r/2} \|f\|_{L^2(J_T, H^r)}. \tag{1.13}$$

Proof. Let $D(L^r)$ be the domain of the operator L^r:

$$D(L^r) = \left\{u = \sum_{j=1}^{\infty} u_j e_j \in H : \sum_{j=1}^{\infty} \alpha_j^{2r} u_j^2 < \infty\right\}.$$

It is well known that (see [Tay97, Chapter 17])

$$D(L^{r/2}) = H^r \cap H \quad \text{for } r \in (0, 1/2). \tag{1.14}$$

Therefore, using the Poincaré inequality and the fact that $f(t) \in H_N^\perp$ for almost every $t \in J_T$, we derive

$$\|f(t)\|_r \ge C_1 \|L^{r/2} f(t)\| \ge \alpha_{N+1}^{r/2} \|f(t)\| \quad \text{almost surely.}$$

Combining this with (1.6), we arrive at (1.13). ☐

1.3 Linearised Navier–Stokes system

For any $u, v \in H^2$, we have $(u, \nabla)v \in L^2$, and therefore we can define a bilinear operator by the formula

$$B(u, v) = \Pi\{(u, \nabla)v\}. \tag{1.15}$$

The following proposition, which establishes some continuity properties for B, can be proved with the help of Proposition 1.1, Sobolev embedding theorems, and interpolation inequalities (cf. [CF88, Chapter 6]).

Proposition 1.6. *There are positive constants C_1 and C_2 such that, for any $u, v \in H^2$, we have*

$$\|B(u,v)\| \le C_1 \min\bigl\{ \bigl(\|u\|_1\|u\|_2\bigr)^{1/2}\|v\|_1, \bigl(\|v\|_1\|v\|_2\bigr)^{1/2}\|u\|_1 \bigr\}, \tag{1.16}$$

$$\|B(u,v)\|_1 \le C_2 \bigl(\|u\|_1\|u\|_2\bigr)^{1/2}\|v\|_2. \tag{1.17}$$

In particular, the function $B(u) = B(u,u)$ is continuous from H^2 to $H^1 \cap H$.

We now fix a finite-dimensional subspace $E \subset H$ and consider the equation

$$\dot w + L_E w + \mathsf{Q}B(v_1(t), w) + \mathsf{Q}B(w, v_2(t)) = f(t), \tag{1.18}$$

where v_1 and v_2 are given functions and Q denotes the orthogonal projection in H onto $E^\perp$.

Proposition 1.7. *For any functions $v_1, v_2 \in L^4(J_T, V)$, $f \in L^2(J_T, E^\perp)$ and $w_0 \in V \cap E^\perp$, problem (1.18), (1.5) has a unique solution $w \in \mathcal{X}_T(E)$. Moreover, there is a constant $C > 0$ depending only on $\max\bigl\{\|v_i\|_{L^4(J_T,V)}, i = 1,2\bigr\}$ such that*

$$\|w\|_{\mathcal{X}_T} \le C\,\bigl(\|w_0\|_V + \|f\|_{L^2(J_T,H)}\bigr). \tag{1.19}$$

Proof. Step 1. Let us show that if $w \in \mathcal{X}_T(E)$ is a solution of (1.18), (1.5), then it satisfies inequality (1.19). This will imply, in particular, the uniqueness of solution.

It follows from (1.16) that the function

$$\hat f(t) = f(t) - \mathsf{Q}\bigl(B(v_1(t), w(t)) + B(w(t), v_2(t))\bigr)$$

belongs to the space $L^2(J_T, E^\perp)$ and satisfies the inequality

$$\begin{aligned}
\|\hat f(t)\|^2 &\le 2\|f(t)\|^2 + C_1\bigl(\|v_1(t)\|_1^2 + \|v_2(t)\|_1^2\bigr)\|w(t)\|_V\|w(t)\|_U \\
&\le 2\|f(t)\|^2 + \delta\,\|w(t)\|_U^2 + C_2\bigl(\|v_1(t)\|_1^4 + \|v_2(t)\|_1^4\bigr)\|w(t)\|_V^2
\end{aligned}$$

for any $t \in J_T$, where $\delta > 0$ is an arbitrary constant and $C_2 > 0$ depends only on δ. Combining this with inequality (1.12) in which f is replaced by $\hat f$ and choosing $\delta > 0$ sufficiently small, we obtain

$$\begin{aligned}
&\|w(t)\|_V^2 + \frac{1}{2}\int_0^t \|w(s)\|_U^2\,ds \\
&\qquad \le C_3\biggl(\|w_0\|_V^2 + \int_0^t \bigl(\|v_1\|_1^4 + \|v_2\|_1^4\bigr)\|w\|_V^2\,ds + \|f\|_{L^2(J_T,H)}^2\biggr).
\end{aligned} \tag{1.20}$$

Ignoring the integral on the left-hand side and applying the Gronwall inequality, we obtain

$$\sup_{0\le t\le T}\|w(t)\|_V \le C_4\bigl(\|w_0\|_V + \|f\|_{L^2(J_T,H)}\bigr).$$

Combining this with (1.20), we obtain a similar upper bound for $\|w\|_{L^2(J_T,U)}$.

Step 2. We now construct a solution with the help of contraction mapping principle. Namely, we shall prove the following assertion: there is a constant $\varepsilon > 0$ such that if $v_i \in L^4(J_S, H^1)$, $i = 1, 2$, for some $S \le T$ and

$$\|v_1\|_{L^4(J_S,H^1)} + \|v_2\|_{L^4(J_S,H^1)} \le \varepsilon, \tag{1.21}$$

then for any $w_0 \in V \cap E^\perp$ and $f \in L^2(J_S, E^\perp)$ problem (1.18), (1.5) has a solution $w \in \mathcal{X}_S(E)$. Once this claim is established, existence of solution on J_T will follow by a simple iteration argument.

Let us consider an operator $\mathcal{F}$ that takes each function $\widehat{w} \in \mathcal{X}_S(E)$ to the solution of the equation

$$\dot w + L_E w = g, \quad g := f - \mathsf{Q}\big(B(v_1, \widehat{w}) + B(\widehat{w}, v_2)\big), \tag{1.22}$$

supplemented with the initial condition (1.5). Using Proposition 1.1, we easily show that $g \in L^2(J_S, E^\perp)$. Thus, in view of Proposition 1.3, the operator $\mathcal{F}$ is well defined. Let us show that $\mathcal{F}$ is a contraction. Indeed, if $\widehat{w}_i \in \mathcal{X}_S(E)$, $i = 1, 2$, and $w_i = \mathcal{F}(\widehat{w}_i)$, then the function $w = w_1 - w_2$ is a solution of problem (1.4), (1.5) with

$$f = -\mathsf{Q}\big(B(v_1, \widehat{w}) + B(\widehat{w}, v_2)\big),$$

where $\widehat{w} = \widehat{w}_1 - \widehat{w}_2$. Repeating literally the arguments used in Step 1, we can show that

$$\|f\|_{L^2(J_S,E^\perp)} \le C_5 \|\widehat{w}\|_{\mathcal{X}_S} \big(\|v_1\|_{L^4(J_S,H^1)} + \|v_2\|_{L^4(J_S,H^1)}\big).$$

Proposition 1.3 and Remark 1.4 imply that if (1.21) is satisfied, then

$$\|w\|_{\mathcal{X}_S} = \|\mathcal{F}(\widehat{w}_1) - \mathcal{F}(\widehat{w}_2)\|_{\mathcal{X}_S} \le C_6 \varepsilon \, \|\widehat{w}_1 - \widehat{w}_2\|_{\mathcal{X}_S}.$$

It follows that $\mathcal{F}$ is a contraction for sufficiently small ε. Its unique fixed point $w \in \mathcal{X}_S(E)$ is a solution of problem (1.18), (1.5). The proof is complete. $\square$

1.4 Strong solutions of the Navier–Stokes system

In this subsection, we establish two perturbative results on solvability of the 3D Navier–Stokes system. Let us fix a finite-dimensional subspace $E \subset U$ and consider the problem

$$\dot w + L_E w + \mathsf{Q}\big(B(w) + B(v, w) + B(w, v)\big) = f(t), \tag{1.23}$$

$$w(0) = w_0, \tag{1.24}$$

where $v \in L^4(J_T, H^1)$, $f \in L^2(J_T, E^\perp)$, and $w_0 \in V \cap E^\perp$ are given functions.

Theorem 1.8. *For any $R > 0$ there are positive constants ε and C such that the following assertions hold.*

(i) *Let $\hat{v} \in L^4(J_T, H^1)$, $\hat{f} \in L^2(J_T, E^\perp)$, and $\widehat{w}_0 \in V \cap E^\perp$ be some functions such that problem (1.23), (1.24) with $v = \hat{v}$, $f = \hat{f}$, $w_0 = \widehat{w}_0$ has a solution $\widehat{w} \in \mathcal{X}_T(E)$. Suppose that*

$$\|\hat{v}\|_{L^4(J_T,H^1)} \le R, \quad \|\hat{f}\|_{L^2(J_T,E)} \le R, \quad \|\widehat{w}\|_{\mathcal{X}_T} \le R. \tag{1.25}$$

Then, for any triple (v, f, w_0) satisfying the inequalities

$$\|v - \hat{v}\|_{L^4(J_T,H^1)} \le \varepsilon, \quad \|f - \hat{f}\|_{L^2(J_T,E)} \le \varepsilon, \quad \|w_0 - \widehat{w}_0\|_V \le \varepsilon, \tag{1.26}$$

problem (1.23), (1.24) has a unique solution $\widehat{w} \in \mathcal{X}_T(E)$.

(ii) *Let*

$$\mathcal{R} : L^4(J_T, H^1) \times L^2(J_T, E^\perp) \times (V \cap E^\perp) \to \mathcal{X}_T(E)$$

be an operator that is defined on the set of functions (v, f, w_0) satisfying (1.26) and takes each triple (v, f, w_0) to the solution $w \in \mathcal{X}_T(E)$ of (1.23), (1.24). Then $\mathcal{R}$ is uniformly Lipschitz continuous, and its Lipschitz constant does not exceed C.

Proof. We shall use a refined version of the implicit function theorem (IFT). Its exact formulation is given in the Appendix (see Section 4.1).

Step 1. In view of Proposition 1.3, problem (1.4), (1.5) is well posed in the space $\mathcal{X}_T(E)$. Therefore, we can define continuous operators

$$K_E : L^2(J_T, E^\perp) \to \mathcal{X}_T(E), \quad M_E : V \cap E^\perp \to \mathcal{X}_T(E) \tag{1.27}$$

by the following rule: K_E takes each function $f \in L^2(J_T, E^\perp)$ to the solution $w \in \mathcal{X}_T(E)$ of problem (1.4), (1.5) with $w_0 = 0$ (cf. (1.8)) and M_E takes each function $w_0 \in V \cap E^\perp$ to the solution $w \in \mathcal{X}_T(E)$ of problem (1.4), (1.5) with $f = 0$. In what follows, we shall omit the subscript E to simplify notation.

Let us define the spaces

$$\mathcal{H} = L^4(J_T, H^1) \times L^2(J_T, E^\perp) \times (V \cap E^\perp), \quad \mathcal{Y} = L^2(J_T, E^\perp).$$

We seek a solution of (1.23), (1.24) in the form

$$w = Mw_0 + Kg, \tag{1.28}$$

where $g \in \mathcal{Y}$ is an unknown function. Substituting (1.28) for w in (1.23), we obtain the following functional equation in the space $\mathcal{Y}$:

$$g + \mathsf{Q}\big(B(Mw_0 + Kg) + B(v, Mw_0 + Kg) + B(Mw_0 + Kg, v)\big) - f = 0. \tag{1.29}$$

Let us set $\boldsymbol{u} = (v, f, w_0)$ and denote by $\mathcal{F}(\boldsymbol{u}, g)$ the left-hand side of (1.29). It is a matter of direct verification to show that the operator $\mathcal{F} : \mathcal{H} \times \mathcal{Y} \to \mathcal{Y}$ is twice continuously differentiable. Furthermore, setting

$$\hat{\boldsymbol{u}} = (\hat{v}, \hat{f}, \widehat{w}_0), \quad \hat{g} = \hat{f} - \mathsf{Q}\big(B(\widehat{w}) + B(\hat{v}, \widehat{w}) + B(\widehat{w}, \hat{v})\big) \tag{1.30}$$

we see that $\mathcal{F}(\hat{\boldsymbol{u}}, \hat{g}) = 0$. In view of Proposition 4.1, the desired assertion will be established if we show that for any $R > 0$ there is $\rho(R) > 0$ such that the following three statements hold for any $(\hat{v}, \hat{f}, \widehat{w}_0, \widehat{w})$ satisfying (1.25).

(a) The functions $\hat{\boldsymbol{u}}$ and $\hat{g}$ defined by (1.30) satisfy the inequality

$$\|\hat{\boldsymbol{u}}\|_{\mathcal{H}} + \|\hat{g}\|_{\mathcal{Y}} \le \rho(R).$$

(b) The norm of the second derivative of $\mathcal{F}$ is bounded uniformly in $(\boldsymbol{u}, g)$.

(c) Let $\mathcal{F}'(\boldsymbol{u}, g)$ be the derivative of $\mathcal{F}$ with respect to g. Then $\mathcal{F}'(\hat{\boldsymbol{u}}, \hat{g})$ is an invertible linear operator in $\mathcal{Y}$, and its norm satisfies the inequality

$$\big\| \big(\mathcal{F}'(\hat{\boldsymbol{u}}, \hat{g})\big)^{-1} \big\|_{\mathcal{L}(\mathcal{Y})} \le \rho(R).$$

Step 2. To prove (a), we first note that (1.16) implies the inequality

$$\|B(a,b)\|_{\mathcal{Y}} + \|B(b,a)\|_{\mathcal{Y}} \le C_1 \|a\|_{L^4(J_T, H^1)} \|b\|_{\mathcal{X}_T}. \tag{1.31}$$

It follows that

$$\|\hat{g}\|_{\mathcal{Y}} \le \|\hat{f}\|_{\mathcal{Y}} + C_2 \|\widehat{w}\|_{\mathcal{X}_T} \big(\|\widehat{w}\|_{L^4(J_T, H^1)} + \|\hat{v}\|_{L^4(J_T, H^1)} \big) \le C_3(R).$$

A similar inequality for $\hat{\boldsymbol{u}}$ is obvious.

Step 3. The definition of $\mathcal{F}$ implies that the operator

$$\mathcal{F}_1(\boldsymbol{u}, g) = \mathcal{F}(\boldsymbol{u}, g) - g + f$$

is a sum of bilinear forms with respect to (v, g, w_0). Therefore, the second derivative of $\mathcal{F}$ coincides with the symmetrization of $\mathcal{F}_1$. Thus, to prove (b), it suffices to show that $\mathcal{F}_1$ is continuous in appropriate functional spaces. This assertion is a straightforward consequence of (1.31) and the continuity of operators (1.27).

Step 4. We now prove (c). Let us set $a = M\widehat{w}_0 + K\hat{g} \in \mathcal{X}_T$. We wish to show that for any $\xi \in \mathcal{Y}$ the equation

$$\mathcal{F}'(\hat{\boldsymbol{u}}, \hat{g})h := h + \mathsf{Q}B(a + \hat{v}, Kh) + \mathsf{Q}B(Kh, a + \hat{v}) = \xi$$

has a unique solution $h \in \mathcal{Y}$, whose norm satisfies the inequality

$$\|h\|_{\mathcal{Y}} \le \rho(R)\, \|\xi\|_{\mathcal{Y}}. \tag{1.32}$$

Setting $\zeta = Kh$, we obtain the following problem for $\zeta \in \mathcal{X}_T(E)$:

$$\dot{\zeta} + L_E \zeta + \mathsf{Q}B(a(t) + \hat{v}(t), \zeta) + \mathsf{Q}B(\zeta, a(t) + \hat{v}(t)) = \xi, \quad \zeta(0) = 0.$$

In view of Proposition 1.7, this problem has a unique solution $\zeta \in \mathcal{X}_T(E)$, which satisfies the inequality

$$\|\zeta\|_{\mathcal{X}_T} \le C_4(R)\, \|\xi\|_{\mathcal{Y}}. \tag{1.33}$$

Since

$$h = \dot{\zeta} + L_E \zeta = \xi - \mathsf{Q}\big(B(a(t) + \hat{v}(t), \zeta) + B(\zeta, a(t) + \hat{v}(t))\big)$$

inequality (1.32) follows from (1.33) and (1.31). The proof of Theorem 1.8 is complete. □

We now consider problem (1.23), (1.24) in which $E = H_N$.

Proposition 1.9. *For any $R > 0$ there is an integer $N \geq 1$ such that if functions $v \in \mathcal{X}_T$, $f \in L^2(J_T, H_N^\perp)$, and $w_0 \in H_N^\perp \cap V$ satisfy the inequalities*

$$\|v\|_{\mathcal{X}_T} \leq R, \quad \|f\|_{L^2(J_T,H)} \leq R, \quad \|w_0\|_V \leq R, \tag{1.34}$$

then problem (1.23), (1.24) with $E = H_N$ has a unique solution $w \in \mathcal{X}_T(H_N)$, which satisfies the inequality

$$\|w\|_{\mathcal{X}_T} \leq C(R), \tag{1.35}$$

where the constant $C(R) > 0$ depends only on R.

Proof. The uniqueness of solution can be established by a standard argument. We shall use the contraction mapping principle to construct a solution.

Let us denote by B_ρ the set of functions $w \in \mathcal{X}_T(H_N)$ such that $\|w\|_{\mathcal{X}_T} \leq \rho$ and $w(0) = w_0$. Consider an operator $\mathcal{F} : B_\rho \to \mathcal{X}_T(H_N)$ that takes each function $\widehat{w} \in B_\rho$ to the solution $w \in \mathcal{X}_T(H_N)$ of the problem

$$\dot{w} + L_N w = \mathsf{Q}_N(f - B(v + \widehat{w})), \tag{1.36}$$

$$w(0) = w_0, \tag{1.37}$$

where $L_N = \mathsf{Q}_N L$. We claim that for any $R > 0$ there is a constant $\rho > 0$ and an integer $N \geq 1$ such that $\mathcal{F}$ is a contraction of the set B_ρ into itself.

Step 1. We first show that $\mathcal{F}(B_\rho) \subset B_\rho$ for an appropriate choice of ρ and N. Let us fix any $r \in (0, 1/2)$. In view of Proposition 1.3 and Corollary 1.5, the solution $w \in \mathcal{X}_T(H_N)$ of (1.36), (1.37) satisfies the inequality

$$\|w\|_{\mathcal{X}_T} \leq C_1\big(\|w_0\|_V + \|f\|_{L^2(J_T,H)}\big) + C_2\alpha_{N+1}^{-r/2}\,\|B(v + \widehat{w})\|_{L^2(J_T,H^r)}. \tag{1.38}$$

It follows from (1.16), (1.17), and an interpolation inequality that

$$\|B(v + \widehat{w})\|_r \leq \|B(v + \widehat{w})\|_{1/2} \leq C_3\|v + \widehat{w}\|_1\|v + \widehat{w}\|_2,$$

whence we see that

$$\|B(v + \widehat{w})\|_{L^2(J_T,H^r)} \leq C_4\big(\|v\|_{\mathcal{X}_T}^2 + \|\widehat{w}\|_{\mathcal{X}_T}^2\big). \tag{1.39}$$

Combining (1.38) and (1.39), we derive

$$\|\mathcal{F}(\widehat{w})\|_{\mathcal{X}_T} \leq C_5(R) + C_6\alpha_{N+1}^{-r/2}\,\rho^2.$$

Hence, if $\rho = 2C_5(R)$ and N is so large that $\alpha_{N+1} \geq (4C_6C_5(R))^{1/r}$, then $\mathcal{F}(B_\rho) \subset B_\rho$.

Step 2. We now show that $\mathcal{F}$ is a contraction. Indeed, if $\widehat{w}_1, \widehat{w}_2 \in B_\rho$ and $w_i = \mathcal{F}(\widehat{w}_i)$, $i = 1, 2$, then the difference $w = w_1 - w_2 \in \mathcal{X}_T(H_N)$ is a solution of problem (1.4), (1.5) with $E = H_N$, $w_0 = 0$ and

$$f = \mathsf{Q}_N\big(B(v + \widehat{w}_2) - B(v + \widehat{w}_1)\big) = \mathsf{Q}_N\big(B(\hat{u}_2, \widehat{w}) + B(\widehat{w}, \hat{u}_1)\big),$$

where $\hat{u}_i = v + \widehat{w}_i$, $i = 1, 2$, and $\widehat{w} = \widehat{w}_2 - \widehat{w}_1$. It follows from (1.16) and (1.17) that

$$\|f\| \le C_7\big\{\big(\|\hat{u}_1\|_1\|\hat{u}_1\|_2\big)^{1/2} + \big(\|\hat{u}_2\|_1\|\hat{u}_2\|_2\big)^{1/2}\big\}\|\widehat{w}\|_1,$$
$$\|f\|_1 \le C_8\big\{\big(\|\hat{u}_2\|_1\|\hat{u}_2\|_2\big)^{1/2}\|\widehat{w}\|_2 + \big(\|\widehat{w}\|_1\|\widehat{w}\|_2\big)^{1/2}\|\hat{u}_1\|_2\big\}.$$

Combining these estimates with an interpolation inequality, we see that

$$\int_0^T \|f(t)\|_{1/2}^2 dt \le C_9\|\widehat{w}\|_{\mathcal{X}_T}^2\big(\|\hat{u}_1\|_{\mathcal{X}_T}^2 + \|\hat{u}_2\|_{\mathcal{X}_T}^2\big) \le C_{10}(R^2+\rho^2)\,\|\widehat{w}_1 - \widehat{w}_2\|_{\mathcal{X}_T}^2.$$

Applying Corollary 1.5, we arrive at the inequality

$$\begin{aligned}\|\mathcal{F}(\widehat{w}_1) - \mathcal{F}(\widehat{w}_2)\|_{\mathcal{X}_T} = \|w\|_{\mathcal{X}_T} &\le C\,\alpha_{N+1}^{-r/2}\,\|f\|_{L^2(J_T,H^r)}\\ &\le C_{11}\alpha_{N+1}^{-r/2}\,(R^2+\rho^2)^{1/2}\|\widehat{w}_1 - \widehat{w}_2\|_{\mathcal{X}_T}.\end{aligned}$$

It follows that the operator $\mathcal{F}$ is a contraction for sufficiently large N and, hence, has a unique fixed point $w \in B_\rho$, which is a solution of (1.23), (1.24). Since $\rho = 2C_5(R)$, we see that w satisfies (1.35). The proof is complete. □

2 Main results

In this section, we present our main results on approximate controllability of NS equations. To simplify notation, we shall confine ourselves to the case $\nu = 1$. All the results are valid for any positive viscosity, and the proofs remain literally the same.

2.1 Approximate controllability

Let $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H)$ be the space of measurable functions $h : \mathbb{R}_+ \to H$ whose restriction to any interval J_T belongs to $L^\infty(J_T, H)$. Consider the controlled Navier–Stokes system

$$\dot{u} + Lu + B(u) = h(t) + \eta(t), \tag{2.1}$$
$$u(0) = u_0, \tag{2.2}$$

where $h \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H)$ and $u_0 \in V$ are given functions and $\eta(t)$ is a control taking on values in a finite-dimensional subspace $E \subset U$. Let us recall the concept of approximate controllability.

Definition 2.1. Let $T > 0$ be a constant. Equation (2.1) is said to be *approximately controllable in time* T if for any $\varepsilon > 0$ and any points $u_0, \hat{u} \in V$ there is a control function $\eta \in L^\infty(J_T, E)$ and a solution $u \in \mathcal{X}_T = C(J_T, V) \cap L^2(J_T, U)$ of problem (2.1), (2.2) such that

$$\|u(T) - \hat{u}\|_1 < \varepsilon. \tag{2.3}$$

To formulate the main result of this paper, we introduce some notation. In view of Proposition 1.6, the nonlinear operator $B(u)$ is continuous from U to $H^1 \cap H$. For any finite-dimensional subspace $G \subset U$, we denote by $\mathcal{F}(G)$ the largest vector space $F \subset U$ such that for any $\eta_1 \in F$ there are vectors $\eta, \zeta^1, \dots, \zeta^k \in G$ and positive constants $\alpha_1, \dots, \alpha_k$ satisfying the relation

$$\eta_1 = \eta - \sum_{j=1}^{k} \alpha_j B(\zeta^j).$$

We emphasise that the integer $k \geq 1$ may depend on η_1. It is not difficult to see that $\mathcal{F}(G)$ is well defined and that $G \subset \mathcal{F}(G)$. Moreover, taking into account the fact that $B(u)$ is a bilinear form on U, we conclude that $\dim \mathcal{F}(G) < \infty$. We now set

$$E_0 = E, \quad E_k = \mathcal{F}(E_{k-1}) \quad \text{for } k \geq 1, \quad E_\infty = \bigcup_{k=1}^{\infty} E_k. \tag{2.4}$$

The following theorem is the main result of this paper.

Theorem 2.2. *Let $h \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H)$ and let $E \subset U$ be a finite-dimensional subspace such that E_∞ is dense in H. Then for any $T > 0$ the Navier–Stokes system* (2.1) *is approximately controllable in time T.*

The proof of Theorem 2.2 is based on an auxiliary result which is of independent interest (cf. property (P) in the Introduction). To formulate it, we introduce the following definition.

Definition 2.3. Let T, R, and ε be positive constants and let $E \subset U$ be a subspace. Equation (2.1) is said to be (ε, R)-*controllable in time* T if for any $u_0 \in B_V(R)$ and $\hat{u} \in B_U(R)$ there is a control function $\eta \in L^\infty(J_T, E)$ and a solution $u \in \mathcal{X}_T$ of problem (2.1), (2.2) such that (2.3) holds.

Theorem 2.4. *Let T, R, and ε be positive constants, let $E \subset U$ be a finite-dimensional subspace, let $E_1 = \mathcal{F}(E)$, and let $h \in L^2(J_T, H)$. Then Eq.* (2.1) *with $\eta \in E$ is (ε, R)-controllable in time T if and only if so is the equation*

$$\dot{u} + Lu + B(u) = h(t) + \eta_1(t), \quad \eta_1 \in E_1. \tag{2.5}$$

A proof of Theorem 2.4 will be given in Section 3. Here we show that Theorem 2.4 implies the approximate controllability of the Navier–Stokes system and that the hypothesis of Theorem 2.2 is fulfilled for the case of a torus in $\mathbb{R}^3$.

2.2 Proof of Theorem 2.2: reduction to ε-controllability

The required assertion will be established if we show that, for any positive constants T, R, and ε, Eq. (2.1) is (ε, R)-controllable in time T. From now on, we fix T, R, and ε and we shall say that a system is ε-*controllable* if it is (ε, R)-*controllable in time* T.

Step 1. Recall that the subspaces H_N and $H_N^\perp$ were introduced in Section 1.2. We first show that there is an integer $N \geq 1$ such that Eq. (2.1) with $\eta \in H_N$ is ε-controllable, and the control function $\eta \in L^\infty(J_T, H_N)$ can be chosen so that

$$\|\eta\|_{L^\infty(J_T,H)} \leq K, \tag{2.6}$$

where $K > 0$ is a constant that depends only on R, T, and ε.

We fix arbitrary points $u_0 \in B_V(R)$ and $\hat{u} \in B_U(R)$ and set

$$v_N(t) = T^{-1}\mathsf{P}_N\bigl(t\hat{u} + (T-t)e^{-tL}u_0\bigr) \quad \text{for } 0 \leq t \leq T. \tag{2.7}$$

Consider the problem

$$\dot{w} + \mathsf{Q}_N L(w + v_N) + \mathsf{Q}_N B(w + v_N) = \mathsf{Q}_N h(t), \quad w(0) = \mathsf{Q}_N u_0. \tag{2.8}$$

Since

$$\mathsf{Q}_N L v_N \equiv 0, \quad \|\mathsf{Q}_N u_0\|_V \leq C\|u_0\|_V, \quad \|\mathsf{Q}_N h(t)\| \leq C\|h(t)\|,$$

Theorem 1.8 implies that (2.8) has a unique solution $w_N \in \mathcal{X}_T(H_N)$. It follows that the function $u_N = v_N + w_N$ belongs to the space $\mathcal{X}_T$ and satisfies Eqs. (2.1) and (2.2) with

$$\eta(t) = \dot{v}_N + \mathsf{P}_N\bigl(Lu_N + B(u_N) - h(t)\bigr). \tag{2.9}$$

Moreover, it follows from (2.7) that

$$\|u_N(T) - \hat{u}\|_1 = \|\mathsf{Q}_N(u_N(T) - \hat{u})\|_1 \leq \|w_N(T)\|_1 + \|\mathsf{Q}_N\hat{u}\|_1. \tag{2.10}$$

The second term in the right-hand side of (2.10) goes to zero as $N \to \infty$ uniformly with respect to $\hat{u} \in B_U(R)$. Therefore, the ε-controllability of (2.1) with $\eta \in H_N$ will be established if we show that

$$\sup_{u_0,\hat{u}} \|w_N(T)\|_1 \to 0 \quad \text{as } N \to \infty, \tag{2.11}$$

where the supremum is taken over $u_0 \in B_V(R)$, $\hat{u} \in B_U(R)$.

To prove (2.11), we take the scalar product in H of the function $2Lw_N$ and the first equation in (2.8). This results in

$$\partial_t\|w_N\|_V^2 + 2\|w_N\|_U^2 = 2(h, Lw_N) - 2(B(u_N), Lw_N). \tag{2.12}$$

Let us estimate the right-hand side of this relation. By the Cauchy inequality and (1.16), we have

$$|(h, Lw_N)| \leq \tfrac{1}{4}\|w_N\|_U^2 + \|h\|^2,$$
$$|(B(u_N), Lw_N)| \leq \tfrac{1}{4}\|w_N\|_U^2 + \|B(u_N)\|^2 \leq \tfrac{1}{4}\|w_N\|_U^2 + C_1\|u_N\|_1^3\|u_N\|_2.$$

Substituting these estimates into (2.12) and using the Poincaré inequality, we derive

$$\partial_t\|w_N\|_V^2 + \alpha_{N+1}\|w_N\|_V^2 \leq 2\|h\|^2 + 2C_1\|u_N\|_1^3\|u_N\|_2.$$

Applying the Gronwall and Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned}\|w_N(T)\|_V^2 &\le e^{-\alpha_{N+1}T}\|u_0\|_V^2 + C_2\int_0^T e^{-\alpha_{N+1}(T-s)}\big(\|h\|^2 + \|u_N\|_1^3\|u_N\|_2\big)\,ds\\ &\le e^{-\alpha_{N+1}T}\|u_0\|_V^2 + C_3\alpha_{N+1}^{-1}\|h\|^2_{L^\infty(J_T,H)} + C_4\alpha_{N+1}^{-1/2}\|u_N\|^4_{\mathcal{X}_T}.\end{aligned}\tag{2.13}$$

The first two terms on the right-hand side of (2.13) go to zero as $N\to\infty$ uniformly with respect to $u_0\in B_V(R)$. If we show that

$$\sup_{N\ge1}\|u_N\|_{\mathcal{X}_T}\le C_5(R,T),\tag{2.14}$$

then (2.11) will follow from (2.13).

To prove (2.14), note that the definition of v_N implies that

$$\sup_{N\ge1}\|v_N\|_{\mathcal{X}_T}\le C_6(R,T).\tag{2.15}$$

It follows from Proposition 1.9 (see (1.35)) that the functions $w_N\in\mathcal{X}_T$ satisfy a similar upper bound. Combining it with (2.15), we arrive at (2.14).

Step 2. We now show that, for sufficiently large $k\ge1$, Eq. (2.1) with $\eta\in E_k$ is ε-controllable. Indeed, let us chose an integer $N\ge1$ and a constant $K>0$ such that for any points $u_0\in B_V(R)$ and $\hat u\in B_U(R)$ and an appropriate control function $\eta_N\in L^\infty(J_T,H_N)$ satisfying (2.6) there is a unique solution $u_N\in\mathcal{X}_T$ of (2.1), (2.2) with $\eta=\eta_N$, and it satisfies the inequality

$$\|u_N(T)-\hat u\|_1<\varepsilon/2,\tag{2.16}$$

By Theorem 1.8, there is $\delta_0>0$ not depending on N such that, for any function $\eta\in L^\infty(J_T,H)$ verifying the condition

$$\|\eta-\eta_N\|_{L^\infty(J_T,H)}\le\delta_0,$$

problem (2.1), (2.2) has a unique solution $u\in\mathcal{X}_T$, which satisfies the inequality

$$\|u-u_N\|_{\mathcal{X}_T}\le C\,\|\eta-\eta_N\|_{L^\infty(J_T,H)},\tag{2.17}$$

where $C>0$ does not depend on N. Since E_∞ is dense in H and H_N is finite-dimensional, for any $\delta>0$ we can find $k\ge1$ such that $B_H(K)$ is contained in the δ-neighbourhood of E_k. It follows that for any function $\eta_N\in L^\infty(J_T,H_N)$ satisfying (2.6) there is $\eta\in L^\infty(J_T,E_k)$ such that

$$\|\eta-\eta_N\|_{L^\infty(J_T,H)}\le\delta.$$

Let us choose $\delta\in(0,\delta_0)$ so small that $2C\delta<\varepsilon$. Then (2.17) and (2.16) imply that (2.3) holds. Thus, Eq. (2.1) with $\eta\in E_k$ is ε-controllable for a sufficiently large k.

Step 3. We now show that Eq. (2.1) with $\eta\in E$ is ε-controllable. Indeed, since Eq. (2.1) with $\eta\in E_k$ is ε-controllable, applying Theorem 2.4 in which $E=E_{k-1}$, we see that so is Eq. (2.1) with $\eta\in E_{k-1}$. Repeating this argument k times, we arrive at the required result. The proof of Theorem 2.2 is complete.

2.3 Navier–Stokes equations on a torus

In this subsection, we study controlled Navier–Stokes equations with periodic boundary conditions. More precisely, let us fix a vector $q = (q_1, q_2, q_3)$ with positive components and set $\mathbb{T}_q^3 = \mathbb{R}^3/2\pi\mathbb{Z}_q^3$, where

$$\mathbb{Z}_q^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i/q_i \in \mathbb{Z} \text{ for } i = 1, 2, 3\}.$$

Consider the Navier–Stokes system

$$\dot{u} + (u, \nabla)u - \nu\Delta u + \nabla p = h(t, x) + \eta(t, x), \quad \operatorname{div} u = 0, \tag{2.18}$$

where $x = (x_1, x_2, x_3) \in \mathbb{T}_q^3$. In other words, we assume that all functions are periodic of period $2\pi q_i$ with respect to x_i, $i = 1, 2, 3$. To simplify notation, we shall assume, without loss of generality, that the mean values of u, h, and η with respect to x are zero. As in the case of a bounded domain with Dirichlet boundary condition, one can reduce (2.18) to an evolution equation in an appropriate Hilbert space. Namely, we set

$$H = \left\{u \in L^2(\mathbb{T}_q^3, \mathbb{R}^3) : \operatorname{div} u \equiv 0, \int_{\mathbb{T}_q^3} u(x)\, dx = 0\right\}$$

and denote by $\Pi : L^2(\mathbb{T}_q^3, \mathbb{R}^3) \to H$ the orthogonal projection in $L^2(\mathbb{T}_q^3, \mathbb{R}^3)$ onto the subspace H. Define the spaces

$$V = H^1(\mathbb{T}_q^3, \mathbb{R}^3) \cap H, \quad U = H^2(\mathbb{T}_q^3, \mathbb{R}^3) \cap H,$$

endowed with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Projecting (2.18) to the space H and taking $\nu = 1$, we obtain Eq. (2.1) in which $L = -\Delta$ is the Stokes operator with the domain $D(L) = U$ and $B(u) = \Pi\{(u, \nabla)u\}$. Theorem 2.2, which was formulated for the Dirichlet boundary condition, remains valid in this case as well. Our aim is to describe explicitly a finite-dimensional subspace $E \subset U$ for which the hypothesis of Theorem 2.2 is fulfilled.

To this end, we first construct an orthogonal basis in H formed of the eigenfunctions of L. For $x, y \in \mathbb{R}^3$, let

$$\langle x, y\rangle_q = \sum_{i=1}^{3} q_i^{-1} x_i y_i, \quad (x, y) = \sum_{i=1}^{3} x_i y_i, \quad |x| = \sum_{i=1}^{3} |x_i|.$$

Define $\mathbb{Z}_*^3 = \mathbb{Z}^3 \setminus \{0\}$ and, for $m \in \mathbb{Z}_*^3$, denote by $m^\perp$ the two-dimensional subspace in $\mathbb{R}^3$ defined by the equation $\langle x, m\rangle_q = 0$. Note that $m^\perp = (-m)^\perp$. Let us choose unit vectors $\ell(m) \in m^\perp$ so that $\{\ell(m), \ell(-m)\}$ is an orthonormal basis in $m^\perp$ with respect to the scalar product $(\cdot, \cdot)$. We now set

$$c_m(x) = \ell(m)\cos\langle m, x\rangle_q, \quad s_m(x) = \ell(m)\sin\langle m, x\rangle_q \quad \text{for } m \in \mathbb{Z}_*^3.$$

It is a matter of direct verification to show that c_m and s_m are eigenfunctions of L and that $\{c_m, s_m, m \in \mathbb{Z}_*^3\}$ is an orthogonal basis in H. For a finite family of functions $\mathcal{A}$, we denote by $\operatorname{span}\mathcal{A}$ the vector space spanned by $\mathcal{A}$.

Theorem 2.5. *For any vector $q = (q_1, q_2, q_3)$ with positive components there is an integer $d \geq 4$ such that if*

$$E = \operatorname{span}\{c_m, s_m, |m| \leq d\},$$

then the vector space E_∞ defined in (2.4) *is dense in H.*

Theorems 2.2 and 2.5 imply the following result on approximate controllability of the NS system by a force of finite dimension.

Corollary 2.6. *Let $E \subset U$ be the finite-dimensional subspace defined in Theorem 2.5. Then for any $T > 0$ the Navier–Stokes system* (2.1) *with $\eta \in E$ is approximately controllable in time T.*

Remark 2.7. In the particular case when $q = (1,1,1)$, it is possible to give a more precise description of a subspace $E \subset U$ for which E_∞ is dense in H. Namely, let E be the vector space that is spanned by the functions c_m and s_m with indices $m = (m_1, m_2, m_3) \in \mathbb{Z}_*^3$ such that either $|m| \leq 2$ or $|m| = 3$ and $m_i \neq 0$ for $i = 1,2,3$. Repeating the proof of Theorem 2.5 (see below), it is easy to see that the subspace E_∞ defined in (2.4) is dense in H. A simple computation shows that $\dim E = 64$. Thus, for any $T > 0$ and $\nu > 0$ the 3D Navier–Stokes system on the standard torus $\mathbb{T}^3$ is approximately controllable by a 64-dimensional control.

Proof of Theorem 2.5. For any integer $k \geq 1$, set $\mathcal{H}_k = \operatorname{span}\{c_m, s_m, |m| \leq k\}$, so that $E = \mathcal{H}_d$. We shall show by induction that the sequence of subspaces defined in (2.4) satisfies the inclusion

$$E_{2k} \supset \mathcal{H}_{k+d} \quad \text{for any } k \geq 0. \tag{2.19}$$

Since the base of induction is obvious, we shall prove inclusion (2.19) for $k \geq 1$ assuming that it is true for any $k' < k$.

Step 1. Let us endow $\mathbb{R}^3$ with the Euclidean scalar product $(\cdot,\cdot)$ and denote by P_m, $m \in \mathbb{Z}_*^3$, the orthogonal projection in $\mathbb{R}^3$ onto the subspace $m^\perp$. Define the two-dimensional subspaces

$$\mathcal{A}_m := \operatorname{span}\{c_m, c_{-m}\}, \quad \mathcal{B}_m := \operatorname{span}\{s_m, s_{-m}\}, \quad m \in \mathbb{Z}_*^3,$$

and note that any functions $f_m \in \mathcal{A}_m$ and $g_n \in \mathcal{B}_n$ can be represented in the form

$$f_m(x) = \tilde{f}_m \cos\langle m, x\rangle_q, \quad g_n(x) = \tilde{g}_n \sin\langle n, x\rangle_q, \tag{2.20}$$

where $\tilde{f}_m$ and $\tilde{g}_n$ are some functions such that $\langle \tilde{f}_m, m\rangle_q = \langle \tilde{g}_n, n\rangle_q = 0$.

Let us show that the following relations hold for any $m, n \in \mathbb{Z}_*^3$:

$$B(f_m, g_n) = A_{mn}(f_m)\big(\cos\langle m-n, x\rangle_q P_{m-n} + \cos\langle m+n, x\rangle_q P_{m+n}\big)\tilde{g}_n, \tag{2.21}$$

$$B(f_m, f_n) = A_{mn}(f_m)\big(\sin\langle m-n, x\rangle_q P_{m-n} - \sin\langle m+n, x\rangle_q P_{m+n}\big)\tilde{f}_n, \tag{2.22}$$

$$B(g_m, f_n) = A_{mn}(g_m)\big(\cos\langle m+n, x\rangle_q P_{m+n} - \cos\langle m-n, x\rangle_q P_{m-n}\big)\tilde{f}_n, \tag{2.23}$$

where $f_l \in \mathcal{A}_l$ and $g_l \in \mathcal{B}_l$, $l = m, n$, are arbitrary functions, P_0 stands for the zero operator in $\mathbb{R}^3$, and

$$A_{mn}(f_m) = \frac{1}{2}\langle \tilde{f}_m, n\rangle_q, \quad A_{mn}(g_m) = \frac{1}{2}\langle \tilde{g}_m, n\rangle_q. \tag{2.24}$$

We shall confine ourselves to the proof of (2.21), since the other relations can be established in a similar way.

It is a matter of direct verification to show that

$$\Pi\{a\cos\langle l, x\rangle_q\} = (P_l a)\,\cos\langle l, x\rangle_q, \quad \Pi\{a\sin\langle l, x\rangle_q\} = (P_l a)\,\sin\langle l, x\rangle_q \tag{2.25}$$

for any $a \in \mathbb{R}^3$ and $l \in \mathbb{Z}^3_*$. Combining (2.25) and (2.20), we obtain

$$\begin{aligned} B(f_m, g_n) &= \Pi\{\tilde{g}_n \cos\langle m, x\rangle_q (\tilde{f}_m, \nabla)\sin\langle n, x\rangle_q\} \\ &= \Pi\{\tilde{g}_n \langle \tilde{f}_m, n\rangle_q \cos\langle m, x\rangle_q \cos\langle n, x\rangle_q\} \\ &= \tfrac{\langle \tilde{f}_m, n\rangle_q}{2}\,\Pi\{\tilde{g}_n(\cos\langle m-n, x\rangle_q + \cos\langle m+n, x\rangle_q)\} \\ &= \tfrac{\langle \tilde{f}_m, n\rangle_q}{2}\{\cos\langle m-n, x\rangle_q P_{m-n} + \cos\langle m+n, x\rangle_q P_{m+n}\}\tilde{g}_n. \end{aligned}$$

Step 2. To prove (2.19), we first show that

$$E_{2k-1} \supset \mathcal{H}^-_{k+d}, \tag{2.26}$$

where $\mathcal{H}^-_p \subset \mathcal{H}_p$ denotes the subspace spanned by the functions c_l and s_l with indices $l \in \mathbb{Z}^3_*$ such that either $|l| \le p-1$ or $|l| = p$ and there are at least two non-zero components of l. The proof of (2.26) is based on the following proposition.

Proposition 2.8. *For any vector $q = (q_1, q_2, q_3)$ with positive components there is a constant $\varepsilon_q > 0$ such that if $m, n, l \in \mathbb{Z}^3_*$ satisfy the conditions*

$$l = m+n, \quad m \text{ and } n \text{ are not parallel}, \quad |n| \le \varepsilon_q |m|, \tag{2.27}$$

then for any $f \in \mathcal{A}_l$ and $g \in \mathcal{B}_l$ there are $a, b \in \operatorname{span}\{\mathcal{A}_m, \mathcal{A}_n, \mathcal{B}_m, \mathcal{B}_n\}$ such that

$$B(a) + f, B(b) + g \in \operatorname{span}\{\mathcal{A}_{m-n}, \mathcal{B}_{m-n}\}. \tag{2.28}$$

Postponing the proof of Proposition 2.8 until the end of this subsection, let us prove (2.26). Take any vector $l \in \mathbb{Z}^3_*$ of length $|l| = k+d$ with at least two non-zero components. Let us choose non-parallel vectors $m, n \in \mathbb{Z}^3_*$ such that

$$l = m+n, \quad |m| = k+d-1, \quad |n| = 1, \quad |m-n| = k+d-2. \tag{2.29}$$

For instance, if $l = (l_1, l_2, l_3)$ and $l_1 \ge 2$, then we can take $m = (l_1 - 1, l_2, l_3)$ and $n = (1, 0, 0)$. If $d \ge 4$ is sufficiently large, then the second and third relations in (2.29) imply that $|n| \le \varepsilon_q |m|$. Therefore, by Proposition 2.8, for any $f \in \mathcal{A}_l$ and $g \in \mathcal{B}_l$ we can find functions $a, b \in \mathcal{H}_{k+d-1}$ such that

$$B(a) + f, B(b) + g \in \operatorname{span}\{\mathcal{A}_{m-n}, \mathcal{B}_{m-n}\} \subset \mathcal{H}_{k+d-2}. \tag{2.30}$$

The definition of $\mathcal{F}(E_{2k-2})$ implies $\mathcal{A}_l, \mathcal{B}_l \subset E_{2k-1}$. Since l was arbitrary, we obtain (2.26).

Step 3. We can now prove (2.19) using the same argument as in the previous step. In view of (2.26), it suffices to show that $\mathcal{A}_l, \mathcal{B}_l \subset \mathbb{E}_{2k}$ for any vector $l \in \mathbb{Z}^3_*$ of length $|l| = k + d$ with only one non-zero component. To this end, we choose non-parallel vectors $m, n \in \mathbb{Z}^3_*$ such that (cf. (2.29))

$$l = m + n, \quad |m| = k + d, \quad |n| = 2, \quad |m - n| = k + d,$$

and the vectors m, n, and $m - n$ have at least two non-zero components. For instance, if $l = (l_1, 0, 0)$ and $l_1 \geq 2$, then we can take $m = (l_1 - 1, 1, 0)$ and $n = (1, -1, 0)$. If d is sufficiently large, then $|n| \leq \varepsilon_q |m|$, and using again Proposition 2.8, for any $f \in \mathcal{A}_l$ and $g \in \mathcal{B}_l$ we can construct $a, b \in \mathcal{H}^-_{k+d}$ such that

$$B(a) + f, B(b) + g \in \operatorname{span}\{\mathcal{A}_{m-n}, \mathcal{B}_{m-n}\} \subset \mathcal{H}^-_{k+d}.$$

Recalling the definition of $\mathcal{F}(E_{2k-1})$, we see that $\mathcal{A}_l, \mathcal{B}_l \subset E_{2k-1}$, and, hence, (2.19) holds. $\square$

Proof of Proposition 2.8. We shall confine ourselves to the proof of existence of a vector $a \in \operatorname{span}\{\mathcal{A}_m, \mathcal{B}_n\}$ such that

$$B(a) + f \in \operatorname{span}\{\mathcal{A}_{m-n}, \mathcal{B}_{m-n}\}. \tag{2.31}$$

Step 1. We seek a in the form

$$a = f_m + g_n, \quad f_m \in \mathcal{A}_m, \quad g_n \in \mathcal{B}_n. \tag{2.32}$$

Representing f_m and g_n in the form (2.20) and using relations (2.21) and (2.23), we derive

$$\begin{aligned} B(f_m + g_n) &= B(f_m, f_m) + B(f_m, g_n) + B(g_n, f_m) + B(g_n, g_n) \\ &= \cos\langle m - n, x\rangle_q P_{m-n}\big(A_{mn}(f_m)\tilde{g}_n - A_{nm}(g_n)\tilde{f}_m\big) \\ &\quad + \cos\langle m + n, x\rangle_q P_{m+n}\big(A_{mn}(f_m)\tilde{g}_n + A_{nm}(g_n)\tilde{f}_m\big). \end{aligned}$$

Taking into account (2.24), we see that the desired assertion will be established if we show that any vector $c \in (m + n)^\perp$ can be represented in the form

$$c = P_{m+n}\big(\langle \tilde{f}_m, n\rangle_q \tilde{g}_n + \langle \tilde{g}_n, m\rangle_q \tilde{f}_m\big), \tag{2.33}$$

where $\tilde{f}_m \in m^\perp$ and $\tilde{g}_n \in n^\perp$.

Step 2. To establish (2.33), we first show that the image of the bilinear operator

$$\Gamma : m^\perp \times n^\perp \to \mathbb{R}^3, \quad (\tilde{f}, \tilde{g}) \mapsto \langle \tilde{f}, n\rangle_q \tilde{g} + \langle \tilde{g}, m\rangle_q \tilde{f},$$

coincides with $(m - n)^\perp$. Indeed, a simple calculation implies that

$$\langle \Gamma(\tilde{f}, \tilde{g}), m - n\rangle_q = 0 \quad \text{for any } \tilde{f} \in m^\perp,\ \tilde{g} \in n^\perp,$$

and therefore $\Gamma(m^\perp, n^\perp) \subset (m-n)^\perp$. To prove the converse inclusion, it suffices to show that $\Gamma(m^\perp, n^\perp)$ contains a two-dimensional affine subspace. To this end, let us choose a vector $\tilde{g}_0 \in n^\perp$ such that $\langle \tilde{g}_0, m\rangle_q = 1$; this can be done because m and n are not parallel. Then

$$\Gamma(\tilde{g}_0, \tilde{f}) = \langle \tilde{f}, n\rangle_q \tilde{g}_0 + \tilde{f}.$$

Since $\tilde{g}_0 \notin m^\perp$, the above relation implies that the affine subspace $\Gamma(\tilde{g}_0, m^\perp)$ is two-dimensional.

Step 3. To conclude the proof of Proposition 2.8, we shall need the following simple lemma; its proof is obvious.

Lemma 2.9. *Let $a, b \in \mathbb{R}^3$ be two nonzero vectors and let $\mathcal{N}_b \subset \mathbb{R}^3$ be the orthogonal complement of b for the scalar product $(\cdot,\cdot)$. Then $\dim P_a(\mathcal{N}_b) = 2$ if and only if $(a, b) \neq 0$.*

Since the image of the bilinear application Γ coincides with $(m-n)^\perp$, representation (2.33) will be established if we show that

$$P_{m+n}(m-n)^\perp = (m+n)^\perp. \tag{2.34}$$

To prove (2.34), we denote by $S : \mathbb{R}^3 \to \mathbb{R}^3$ the linear operator such that

$$\langle a, b\rangle_q = (Sa, b) \quad \text{for any } a, b \in \mathbb{R}^3. \tag{2.35}$$

Obviously, such an operator exists and is invertible. It follows from (2.35) that for any vector $a \in \mathbb{R}^3$ the subspace $a^\perp$ coincides with the orthogonal complement of Sa with respect to the scalar product $(\cdot,\cdot)$. Therefore, in view of Lemma 2.9, relation (2.34) holds if and only if

$$K_{mn} := (S(m-n), S(m+n)) \neq 0. \tag{2.36}$$

Since all the norms in $\mathbb{R}^3$ are equivalent and S is an invertible continuous operator, we can find a constant $C_q > 0$ such that

$$K_{mn} = (Sm, Sm) - (Sn, Sn) \geq C_q^{-1}|m|^2 - C_q|n|^2.$$

Therefore, if $|m| \geq 2C_q|n|$, then (2.36) holds. The proof is complete. □

3 Proof of Theorem 2.4

3.1 Scheme of the proof

Let us fix constants R, T, and ε. As in the proof of Theorem 2.2, we shall say that a system is ε-controllable if it is (ε, R)-controllable in time T. We need to show that if (2.5) is ε-controllable, then so is (2.1).

Along with (2.1) and (2.5), let us consider the equation

$$\dot{u} + L(u + \zeta(t)) + B(u + \zeta(t)) = h(t) + \eta(t), \tag{3.1}$$

where η and ζ are control functions. Suppose we can prove the following two propositions.

Proposition 3.1. *Let $u \in \mathcal{X}_T$ be a solution of (3.1) with $\eta, \zeta \in L^\infty(J_T, E)$. Then there are sequences of controls $\eta_k \in L^\infty(J_T, E)$ and of solutions $u_k \in \mathcal{X}_T$ for Eq. (2.1) with $\eta = \eta_k$ such that*

$$u_k(0) = u(0) \quad \textit{for all } k \geq 1, \tag{3.2}$$
$$\|u_k(T) - u(T)\|_V \to 0 \quad \textit{as } k \to \infty. \tag{3.3}$$

Proposition 3.2. *Let $u \in \mathcal{X}_T$ be a solution of (2.5) with $\eta_1 \in L^\infty(J_T, E_1)$, where $E_1 = \mathcal{F}(E)$. Then there are sequences of controls $\eta_k, \zeta_k \in L^\infty(J_T, E)$ and of solutions $u_k \in \mathcal{X}_T$ for Eq. (3.1) with $\eta = \eta_k$ and $\zeta = \zeta_k$ such that (3.2) holds and*

$$\|u_k - u\|_{C(J_T, V)} \to 0 \quad \textit{as } k \to \infty. \tag{3.4}$$

Propositions 3.1 and 3.2 imply the following results relating the control systems (2.1), (3.1), (2.5) (cf. properties (P_1) and (P_2) in the Introduction).

Extension: Equation (2.1) with $\eta \in E$ is ε-controllable if and only if so is Eq. (3.1) with $\eta, \zeta \in E$.

Convexification: Equation (3.1) with $\eta, \zeta \in E$ is ε-controllable if and only if so is Eq. (3.1) with $\eta \in E_1$, where $E_1 = \mathcal{F}(E)$.

The claim of Theorem 2.4 is a straightforward consequence of the above assertions. Thus, to establish Theorem 2.4, it suffices to prove Propositions 3.1 and 3.2. Their proofs are given in the next two subsections.

3.2 Proof of Proposition 3.1

Recall that P and Q stand for the orthogonal projections in H onto the subspaces E and $E^\perp$, respectively. Let us set

$$v(t) = \mathsf{P}u(t), \quad w(t) = \mathsf{Q}u(t) \quad \text{for } t \in J_T.$$

It is clear that $v \in C(J_T, E)$ and $w \in \mathcal{X}_T(E)$. Moreover, the function w is a solution of the equation

$$\dot{w} + L_E w + \mathsf{Q}\big(B(w) + B(v + \zeta, w) + B(w, v + \zeta)\big) = f(t),$$

where we set

$$f = \mathsf{Q}\big(h - B(v) - L(v + \zeta)\big).$$

Let us choose a sequence $v_k \in C^1(J_T, E)$ such that

$$\|v_k - (v + \zeta)\|_{L^4(J_T, V)} \to 0 \quad \text{as } k \to \infty, \tag{3.5}$$
$$v_k(0) = v(0), \quad v_k(T) = v(T) \quad \text{for all } k \geq 1. \tag{3.6}$$

Consider the equation

$$\dot{z} + L_E z + \mathsf{Q}\big(B(z) + B(v_k, z) + B(z, v_k)\big) = f_k(t), \tag{3.7}$$

where
$$f_k = \mathsf{Q}\big(h - B(v_k) - Lv_k\big).$$
Using (3.5) and (1.16), it is easy to show that
$$\|f_k - f\|_{L^2(J_T,H)} \to 0 \quad \text{as } k \to \infty. \tag{3.8}$$
Theorem 1.8 combined with (3.5) and (3.8) implies that, for sufficiently large $k \geq 1$, Eq. (3.7) has a unique solution $w_k \in \mathcal{X}_T(E)$ that satisfies the initial condition
$$w_k(0) = w(0). \tag{3.9}$$
Moreover, since the resolving operator associated with (3.7) is Lipschitz continuous, we see that
$$\|w_k - w\|_{\mathcal{X}_T} \to 0 \quad \text{as } k \to \infty. \tag{3.10}$$

We now set $u_k = v_k + w_k$. The construction implies that the function w_k belongs to the space $\mathcal{X}_T$ and satisfies Eq. (2.1) with the function
$$\eta(t) = \eta_k(t) := \dot{v}_k(t) - \mathsf{P}\big(Lu_k(t) + B(u_k(t))\big),$$
which belongs to $L^\infty(J_T, E)$. Furthermore, it follows from (3.9) and the first relation in (3.6) that the initial condition (3.2) is also verified. Finally, the second relation in (3.6) and convergence (3.10) imply that
$$\|u_k(T) - u(T)\|_V = \|w_k(T) - w(T)\|_V \leq \|w_k - w\|_{\mathcal{X}_T} \to 0 \quad \text{as } k \to \infty.$$
The proof of Proposition 3.1 is complete.

3.3 Proof of Proposition 3.2

Step 1. Without loss of generality, we can assume that $\eta_1(t)$ is piecewise constant. Indeed, suppose that Proposition 3.2 is proved in this case, and let $u \in \mathcal{X}_T$ be a solution of (2.5), (2.2) with some $\eta_1 \in L^\infty(J_T, E_1)$. Then there is a sequence $\eta^m \in L^\infty(J_T, E_1)$ of piecewise constant functions such that
$$\|\eta^m - \eta_1\|_{L^2(J_T,H)} \to 0 \quad \text{as } m \to \infty.$$
Applying Theorem 1.8 with $E = \{0\}$, we see that, for sufficiently large $m \geq 1$, problem (2.5), (2.2) with η_1 replaced by η^m has a unique solution $u^m \in \mathcal{X}_T$, which converges to u in $\mathcal{X}_T$ as $m \to \infty$. In particular, for any $\varepsilon > 0$ there is a piecewise constant function $\tilde{\eta}_1 \in L^\infty(J_T, E_1)$ and a solution $\tilde{u} \in \mathcal{X}_T$ of problem (2.5), (2.2) with $\eta_1 = \tilde{\eta}_1$ such that
$$\|\tilde{u} - u\|_{\mathcal{X}_T} < \varepsilon/2. \tag{3.11}$$

By assumption, Proposition 3.2 is true for the piecewise constant function η_1. Therefore there are sequences of control functions $\eta_k, \zeta_k \in L^\infty(J_T, E)$ and of solutions $u_k \in \mathcal{X}_T$ for problem (2.1), (2.2) with $\zeta = \zeta_k$ and $\eta = \eta_k$ such that

$$\|u_k - \tilde{u}\|_{\mathcal{X}_T} \to 0 \quad \text{as } k \to \infty.$$

Combining this with (3.11), for any $\varepsilon > 0$ we can find $\eta_\varepsilon, \zeta_\varepsilon \in L^\infty(J_T, E)$ and a solution $u_\varepsilon \in \mathcal{X}_T$ of problem (2.1), (2.2) with $\zeta = \zeta_\varepsilon$ and $\eta = \eta_\varepsilon$ such that

$$\|u_\varepsilon - u\|_{\mathcal{X}_T} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the required assertion.

Step 2. We now prove the proposition for piecewise constant functions $\eta_1(t)$. A simple iteration argument combined with Theorem 1.8 shows that it suffices to consider the case in which there is only one interval of constancy. Thus, we assume that $u \in \mathcal{X}_T$ is a solution of (2.5), (2.2) with $\eta(t) \equiv \eta_1 \in E_1$.

We claim that there is a function $\eta \in E$ and a sequence $\zeta_k \in L^\infty(J_T, E)$ such that problem (3.1), (2.2) with $\zeta = \zeta_k$ has a unique solution $u_k \in \mathcal{X}_T$, which satisfies (3.4). We shall need the following lemma whose proof is given in the Appendix (see Section 4.2).

Lemma 3.3. *Let $E \subset U$ be a finite-dimensional space and $E_1 = \mathcal{F}(E)$. Then for any $\eta_1 \in E_1$ there are vectors $\zeta^1, \dots, \zeta^m, \eta \in E$ and positive constants $\lambda_1, \dots, \lambda_m$ whose sum is equal to 1 such that*

$$B(u) - \eta_1 = \sum_{j=1}^m \lambda_j \big(B(u + \zeta^j) + L\zeta^j\big) - \eta \quad \textit{for any } u \in V. \tag{3.12}$$

Relation (3.12) implies that the function $u \in \mathcal{X}_T$ satisfies the equation

$$\partial_t u + Lu + \sum_{j=1}^m \lambda_j \big(B(u + \zeta^j) + L\zeta^j\big) = h(t) + \eta. \tag{3.13}$$

Following a classical idea in the theory of control, we now fix an integer $k \geq 1$ and consider the function

$$\zeta_k(t) = \zeta(kt/T), \tag{3.14}$$

where $\zeta(t)$ is a 1-periodic function defined by the relation

$$\zeta(s) = \zeta^j \quad \text{for } 0 \leq s - (\lambda_1 + \cdots + \lambda_{j-1}) < \lambda_j,\ j = 1, \dots, m.$$

Equation (3.13) can be rewritten as

$$\partial_t u + L(u + \zeta_k(t)) + B(u + \zeta_k(t)) = h(t) + \eta + f_k(t), \tag{3.15}$$

where

$$f_k(t) = B(u(t) + \zeta_k(t)) - \sum_{j=1}^m \lambda_j B(u(t) + \zeta^j). \tag{3.16}$$

It follows from (1.17) that the functions f_k belong to $L^2(J_T, H^1)$ and satisfy the inequality

$$\sup_{k\geq 1} \|f_k\|_{L^2(J_T,H^1)} < \infty. \tag{3.17}$$

Setting $\hat{u}_k = u - Kf_k$, where the operator K is defined by (1.8), we conclude from (3.15) that $\hat{u}_k \in \mathcal{X}_T$ is a solution of the equation

$$\partial_t \hat{u}_k + L(\hat{u}_k + \zeta_k) + B(\hat{u}_k + \zeta_k) + B(\hat{u}_k + \zeta_k, Kf_k) + B(Kf_k, \hat{u}_k + \zeta_k) \\ = h + \eta - B(Kf_k). \tag{3.18}$$

We wish to consider (3.1) as a perturbation of (3.18) and to apply Theorem 1.8. To this end, we note that

$$\|\hat{u}_k\|_{\mathcal{X}_T} + \|\zeta_k\|_{L^\infty(J_T,E)} + \|Kf_k\|_{\mathcal{X}_T} \leq R,$$

where $R > 0$ does not depend on k. Therefore, by Theorem 1.8, there is $\varepsilon > 0$ depending only R such that if functions $v \in L^4(J_T, H^1)$ and $f \in L^2(J_T, H)$ satisfy the inequalities

$$\|v - Kf_k\|_{L^4(J_T,H^1)} \leq \varepsilon, \quad \|f + B(Kf_k)\|_{L^2(J_T,H)} \leq \varepsilon, \tag{3.19}$$

then the equation

$$\partial_t z + L(z + \zeta_k) + B(z + \zeta_k) + B(z + \zeta_k, v) + B(v, z + \zeta_k) = h + \eta + f \tag{3.20}$$

has a unique solution $z \in \mathcal{X}_T$ satisfying the initial condition $z(0) = u_0$. Suppose we have shown that

$$\|Kf_k\|_{C(J_T,V)} + \|B(Kf_k)\|_{L^2(J_T,H)} \to 0 \quad \text{as } k \to \infty. \tag{3.21}$$

In this case, the functions $v \equiv 0$ and $f \equiv 0$ satisfy condition (3.19) for sufficiently large k, and we can conclude that problem (3.1), (2.2) with $\zeta = \zeta_k$ has a unique solution $u_k \in \mathcal{X}_T$, and

$$\|u_k - \hat{u}_k\|_{\mathcal{X}_T} \to 0 \quad \text{as } k \to \infty. \tag{3.22}$$

Since $\|Kf_k\|_{C(J_T,V)} \to 0$ as $k \to \infty$ (see (3.21)), convergence (3.22) and the definition of $\hat{u}_k$ imply that (3.4) holds. Thus, it remains to prove (3.21).

Step 3. To prove (3.21), we note that (1.16) implies the inequality

$$\|B(Kf_k)\|_{L^2(J_T,H)} \leq C_1 \|Kf_k\|_{L^6(J_T,H^1)}^{3/2} \|Kf_k\|_{L^2(J_T,H^2)}^{1/2}.$$

Since the sequence $\{f_k\}$ is bounded in $L^2(J_T, H)$ (see (3.17)), Proposition 1.2 implies that $\|Kf_k\|_{L^2(J_T,H^2)}$ is bounded by a constant not depending on k. Therefore convergence (3.21) will be established if we show that

$$\|Kf_k\|_{C(J_T,V)} \to 0 \quad \text{as } k \to \infty. \tag{3.23}$$

Step 4. To prove (3.23), let us set

$$F_k(t) = \int_0^t f_k(s)\, ds \quad \text{for } t \in J_T. \tag{3.24}$$

Suppose we have shown that

$$\|F_k\|_{C(J_T,H^1)} \to 0 \quad \text{as } k \to \infty. \tag{3.25}$$

The interpolation inequality $\|v\|_V^2 \le \|v\|\,\|v\|_U$ implies that

$$\|Kf_k\|^2_{C(J_T,V)} \le \|Kf_k\|_{C(J_T,H)}\|Kf_k\|_{C(J_T,U)}. \tag{3.26}$$

Using again the fact that $\{f_k\}$ is bounded in $L^2(J_T, H)$, we see that the second factor on the right-hand side of (3.26) is bounded. To estimate the first factor, we integrate by parts and write

$$Kf_k(t) = F_k(t) - G_k(t), \quad G_k(t) = \int_0^t Le^{-(t-s)L}F_k(s)\, ds. \tag{3.27}$$

It is a matter of straightforward verification that

$$\|L^r e^{-tL}\|_{\mathcal{L}(H)} \le C_2 t^{-r} \quad \text{for } r \ge 0,\, t > 0.$$

Combining this inequality with (1.14), we derive

$$\begin{aligned} \|G_k(t)\| &\le \int_0^t \|L^{3/4}e^{-(t-s)L}\|_{\mathcal{L}(H)}\|L^{1/4}F_k(s)\|\, ds \\ &\le C_3 \int_0^t (t-s)^{-3/4}\|F_k(s)\|_1 ds. \end{aligned}$$

Comparing this inequality with (3.25) – (3.27), we arrive at (3.23).

Step 5. We now prove (3.25). We shall show that for any piecewise constant H^2-valued function u on J_T, the sequence $\{F_k\}$ defined by (3.16) and (3.24) converges to zero in the space $C(J_T, H^1)$. If this assertion is established, then a simple approximation argument combined with the continuity of the bilinear operator $B : H^2 \to H^1 \cap H$ shows (3.25) is true for any $u \in \mathcal{X}_T$.

Convergence (3.25) will be established if we prove the following assertions:

(i) The family $\{F_k\} \subset C(J_T, H^1)$ is relatively compact.

(ii) For any $t \in J_T$, the sequence $\{F_k(t)\}$ goes to zero in H as $k \to \infty$.

To prove (i), note that, in view of (3.17), the family $\{F_k\}$ is uniformly equicontinuous on J_T. Therefore, by the Arzelà–Ascoli theorem, it suffices to show that there exists a compact set $\mathcal{K} \subset H^1$ such that

$$F_k(t) \in \mathcal{K} \quad \text{for all } t \in J_T,\ k \ge 1.$$

This assertion follows from the fact that, for piecewise constant functions u, the image of f_k is contained in a finite set not depending on k.

We now prove (ii). Let us denote by $J_q = [t_{q-1}, t_q]$, $q = 1, \dots, L$, the intervals of constancy of u. We fix any integer r, $1 \le r \le L$, and for any $t \in J_{r+1}$ write

$$F_k(t) = \int_0^t f_k(s)\,ds = \sum_{q=1}^{r} \int_{t_{q-1}}^{t_q} f_k(s)\,ds + \int_{t_r}^{t} f_k(s)\,ds.$$

Thus, to prove (ii), it suffices to show that, for any q, $q = 1, \dots, L$, and $t \in J_q$, we have

$$\int_{t_{q-1}}^{t} f_k(s)\,ds \to 0 \quad \text{as } k \to \infty.$$

This can be done by a straightforward computation (cf. [Jur97, Chapter 3]). The proof of Proposition 3.2 is complete.

4 Appendix

4.1 A version of the implicit function theorem

Let X and Y be Banach spaces and let $Z = X \times Y$. We denote by $B_X(x, \delta)$ the closed ball in X of radius δ centred in x and by $B_Z(z)$ the closed ball in Z of radius 1 centred at z. Let $F : Z \to Y$ be a C^2 function. We write $F'_y(z)$ for its Fréchet derivative with respect to y at a point z and denote by $|F|_z$ the C^2 norm of the restriction of F to $B_Z(z)$. The following result can be established by repeating the arguments used in a standard proof of the implicit function theorem (for instance, see [Tay97, Chapter 1]).

Proposition 4.1. *For any $R > 0$ there is $\delta > 0$ such that the following statements hold:*

(i) *Let $\hat{z} = (\hat{x}, \hat{y}) \in Z$ be any point such that the linear operator $F'_y(\hat{z})$ is invertible and*

$$|F|_{\hat{z}} \le R, \quad \left\| \left(F'_y(\hat{z})\right)^{-1} \right\|_{\mathcal{L}(Y)} \le R.$$

Then there is a unique C^2 function $f : B_X(\hat{x}, \delta) \to Y$ such that

$$F(x, f(x)) = 0 \quad \text{for } x \in B_X(\hat{x}, \delta).$$

(ii) *The function f satisfies the inequality*

$$\|f(x_1) - f(x_2)\|_Y \le C\,\|x_1 - x_2\|_X \quad \text{for } x_1, x_2 \in B_X(\hat{x}, \delta).$$

4.2 Proof of Lemma 3.3

In view of the definition $\mathcal{F}(E)$, there are vectors $\tilde{\zeta}^1, \dots, \tilde{\zeta}^k, \tilde{\eta} \in E$ and constants $\alpha_j > 0$, $j = 1, \dots, k$, such that

$$\eta_1 = \tilde{\eta} - \sum_{j=1}^{k} \alpha_j B(\tilde{\zeta}^j).$$

Let us set $m = 2k$, $\eta = \tilde{\eta}$,

$$\begin{aligned} \lambda_j &= \tfrac{\alpha_j}{2\alpha}, & \zeta^j &= \sqrt{\alpha}\,\tilde{\zeta}^j & &\text{for } j = 1, \dots, k, \\ \lambda_j &= \tfrac{\alpha_{j-k}}{2\alpha}, & \zeta^j &= -\sqrt{\alpha}\,\tilde{\zeta}^{j-k} & &\text{for } j = k+1, \dots, m, \end{aligned}$$

where $\alpha = \alpha_1 + \dots + \alpha_k$. It is a matter of direct verification to show that (3.12) holds.

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