Instability of Equilibrium States for Coupled Heat Reservoirs at Different Temperatures

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July 8, 2005

Abstract

We consider quantum systems consisting of a "small" system coupled to two reservoirs (called open systems). We show that such a system has no equilibrium states normal with respect to any state of the decoupled system in which the reservoirs are at different temperatures, provided that either the temperatures or the temperature difference divided by the product of the temperatures are not too small.

Our proof involves an elaborate spectral analysis of a general class of generators of the dynamics of open quantum systems, including quantum Liouville operators ("positive temperature Hamiltonians") which generate the dynamics of the systems under consideration.

1 Introduction

It seems obvious that a quantum system consisting of a small subsystem coupled to several reservoirs at different temperatures does not have an equilibrium state. However, such a result (a precise formulation of which we present in Section 3) was proven only recently, in [14] for (two) fermionic heat baths at temperatures T_1 and T_2, under the condition $0 < |g| < C\min\big(T_1, T_2, g_1(\Delta T)\big)$, where g is the interaction strength (coupling constant), $\Delta T = |T_1 - T_2| > 0$, and in [8] for bosonic reservoirs, under the condition $0 < |g| < g_2(\Delta T)$, *uniformly* in $T_1, T_2 \to 0$. Here $g_{1,2}(\Delta T)$ are some (implicit) functions which vanish in the limit $\Delta T \to 0$. One of our goals is to prove absence of equilibria for small coupling constants, uniformly in $T_j \to 0$, *and uniformly in* $\Delta T \downarrow 0$. In this paper we take the first step in this direction by proving non-existence of equilibria under either of the following conditions

- $0 < |g| < c[\min(T_1, T_2)]^{\frac{1}{2+\alpha}}$ (except possibly for a finite set of points) and $|T_1^{-1} - T_2^{-1}| < c'$ for some $c' > 0$,

or

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[§]A part of the Ph.D. requirements, supported by DAAD under grant HSP III.

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[‖]Supported by NSF under grant DMS-0400-526.

– $0 < |g| < c\left[\frac{|\Delta T|^2}{T_1 T_2 + |\Delta T|^2}\right]^{1/\alpha}$,

where $\alpha = \frac{\mu - 1/2}{\mu + 1/2}$. Here, c is an absolute constant and $\mu > 1/2$ is a parameter describing the infra-red behaviour of interactions (see Condition (A) and Remark 2 in Section 3 below, and the next paragraph). In Section 7 we sketch the strategy how to prove the instability of equilibrium states without temperature-dependent restrictions on the coupling strength. The detailed analysis of this is given in [17].

Since the quantum excitations of the heat reservoirs (photons or phonons) are massless we have to deal with an infra-red technical problem. The severity of this problem is determined by the infra-red behaviour of coupling operators $G_j(k)$ entering the interaction term of the Hamiltonian, where $k \in \mathbb{R}^3$ is the momentum of photons (or phonons). Our results hold for $G_j(k)$ proportional, at $|k| \to 0$, to $|k|^p$, where p can take the values $n + 1/2$, with $n = 0, 1, 2, \ldots$ ($p > \mu - 1$, where μ is the parameter in the preceding paragraph). This is the same infra-red condition as in [14], and it presents an improvement of the one in [8], since [8] requires $p > 2$, though with less restrictions on the regularity of $k \mapsto G_j(k)$.

Our approach is based on the characterization of equilibrium states in terms of eigenvectors corresponding to the eigenvalue zero of certain selfadjoint operators L, called *Liouville operators*, which act on the GNS representation Hilbert space (positive temperature Hilbert space) (see [12, 5, 13, 10]).

Parts of our techniques can be viewed as a perturbation theory in the temperatures, around $\delta\beta := |T_1^{-1} - T_2^{-2}| = 0$. This is a *singular perturbation theory* in the sense that the Hilbert spaces representations of the system for $\delta\beta = 0$ and $\delta\beta > 0$ are not normal with respect to each other ([20, 6, 7]).

Our techniques are applicable to a wide class of non-selfadjoint operators K, containing in particular the Liouville operators mentioned above, but also containing non-selfadjoint generators of the dynamics used in the examination of non-equilibrium stationary states ([14, 16]). We thus carry out our analysis for this more general class of operators.

In order to study the spectrum of the operators K, we develop a new type of spectral deformation, $K \mapsto K_\theta$, with a spectral deformation parameter $\theta \in \mathbb{C}^2$, which combines the deformations introduced in [12] and in [5], hence θ is in $\mathbb{C}^2$ rather than in $\mathbb{C}$. (Such a combination was already mentioned in [5].) In order to establish the desired spectral characteristics of the operator family K_θ, we use the method of the Feshbach map, and perform the basic step of the spectral renormalization group approach as developed in [2, 3, 4].

Already a single application of the Feshbach map, considered in this paper, yields the results mentioned above. Adapting ideas of [2, 4, 5] on the full renormalization group approach, the restriction on the temperatures can be removed. We present in [17] a detailed analysis of the RG to the specific model at hand. It relies on [3, 4, 5] and features some simplifications due to the specificity of our problem and some recent developments [2].

In contrast to the case of quantum Hamiltonians for zero temperature systems, the spectral theory of time-translation generators of open quantum systems is at an early stage of its development. Our paper is a contribution to this theory.

This paper is organized as follows. In Section 2 we describe our model and define the dynamics of it. (The definition of the dynamics is a somewhat subtle matter.) In Section 3 we give a precise formulation of our assumptions, state the results and discuss assumptions and results. In Section 4 we present the Araki-Woods construction which we use throughout this paper. In Section 5 we define a spectral deformation of a family of operators K which contains the generator of the evolution, and we establish some basic analyticity and spectral properties of those operators. In Section 6 we carry out a more refined spectral analysis, preparing for a proof of absence of normal invariant states, which is given in Section 7. Finally, in Appendices A–C we collect some technical results.

2 Model and Mathematical Framework

We consider a system consisting of a particle system, described by a Hamiltonian H_p on a Hilbert space $\mathcal{H}_p$, and two (thermal) reservoirs, at inverse temperatures β_1 and β_2, described by the Hamiltonians H_{r1} and H_{r2} acting on Hilbert spaces $\mathcal{H}_{r1}$ and $\mathcal{H}_{r2}$, respectively. The full Hamiltonian is

$$H := H_0 + gv \ , \tag{2.1}$$

acting on the tensor product space $\mathcal{H}_0 := \mathcal{H}_p \otimes \mathcal{H}_{r1} \otimes \mathcal{H}_{r2}$. Here

$$H_0 := H_p \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes H_{r1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes H_{r2} \tag{2.2}$$

is the unperturbed Hamiltonian, v is an operator on $\mathcal{H}_0$ describing the interaction and $g \in \mathbb{R}$ is a coupling constant.

For the moment we just require that H_p is a self-adjoint operator on $\mathcal{H}_p$, with the property that $\mathrm{Tr}\, e^{-\beta H_p} < \infty$ (any $\beta > 0$). The operators H_{rj} describe free scalar (or vector, if wished) quantum fields on $\mathcal{H}_{rj}$, the bosonic Fock spaces over the one-particle space $L^2(\mathbb{R}^3, d^3k)$,

$$H_{rj} = \int \omega(k) a_j^*(k) a_j(k)\, d^3k, \tag{2.3}$$

where $a_j^*(k)$ and $a_j(k)$ are creation and annihilation operators on $\mathcal{H}_{rj}$ and $\omega(k) = |k|$ is the dispersion relation for relativistic massless bosons. The interaction operator is given by

$$v = \sum_{j=1}^{2} v_j \quad \text{with} \quad v_j = a_j(G_j) + a_j^*(G_j). \tag{2.4}$$

Its choice is motivated by standard models of particles interacting with the quantized electromagnetic field or with phonons.

Here, $G_j : k \mapsto G_j(k)$ is a map from $\mathbb{R}^3$ into $\mathcal{B}(\mathcal{H}_p)$, the algebra of bounded operators on $\mathcal{H}_p$, and

$$a_j(G_j) := \int G_j(k)^* \otimes a_j(k)\, d^3k \qquad \text{and} \qquad a_j^*(G_j) := a_j(G_j)^*. \tag{2.5}$$

If the coupling operators G_j are such that

$$g^2 \int_{\mathbb{R}^3} \left(1 + |k|^{-1}\right) \|G_j(k)\|^2 \ dk \qquad \text{is sufficiently small,} \tag{2.6}$$

then the operator H is self-adjoint (see e.g. [5]).

Now we set up a mathematical framework for non-equilibrium statistical mechanics. Operators on the Hilbert space $\mathcal{H}_0$ will be called observables. (Strictly speaking only certain self-adjoint operators on $\mathcal{H}_0$ are physical observables.) As an algebra of observables describing the system we take the C^*-algebra

$$\mathcal{A} = \mathcal{B}(\mathcal{H}_p) \otimes \mathfrak{W}(L^2_0) \otimes \mathfrak{W}(L^2_0), \tag{2.7}$$

where $\mathfrak{W}(L^2_0)$ denotes the Weyl CCR algebra over the space $L^2_0 := L^2(\mathbb{R}^3, (1+|k|^{-1})d^3k)$. States of the system are positive linear functionals, ψ, on the algebra $\mathcal{A}$ normalized as $\psi(\mathbf{1}) = 1$.

The reason we chose $\mathcal{A}$ rather than $\mathcal{B}(\mathcal{H}_0)$ is that the algebra $\mathcal{A}$ supports states in which each reservoir is at a thermal equilibrium at its own temperature. More precisely, consider the evolution for the i-th reservoir given by

$$\alpha^t_{ri}(A) := e^{iH_{ri}t} A e^{-iH_{ri}t}. \tag{2.8}$$

Then there are stationary states on the i-th reservoir algebra of observables, $\mathfrak{W}(L^2_0)$, which describe thermal equilibria. These states are parametrized by the inverse temperature β and their generating functional is given by

$$\omega^{(\beta)}_{ri}\left(W_i(f)\right) = \exp\left\{-\frac{1}{4}\int_{\mathbb{R}^3} \frac{e^{\beta|k|}+1}{e^{\beta|k|}-1}|f(k)|^2 d^3k\right\}, \tag{2.9}$$

where $W_j(f) := e^{i\phi_j(f)}$, with $\phi_j(f) := \frac{1}{\sqrt{2}}\left(a^*_j(f) + a_j(f)\right)$, is a Weyl operator, see e.g. [7]. The choice of the space L^2_0 above is dictated by the need to have the r.h.s. of this functional finite. These states are characterized by the KMS condition and are called (α^t_{ri}, β)-KMS states.

Remark. It is convenient to define states ψ on products $a^\#(f_1)\ldots a^\#(f_n)$ of the creation and annihilation operators, where $a^\#$ is either a or a^*. This is done using s-derivatives of its values on the Weyl operators $W(s_1 f_1)\ldots W(s_n f_n)$ (see [7], Section 5.2.3 and (2.15)).

Consider states (on $\mathcal{A}$) of the form

$$\omega_0 := \omega_p \otimes \omega^{(\beta_1)}_{r1} \otimes \omega^{(\beta_2)}_{r2}, \tag{2.10}$$

where ω_p is a state of the particle system and $\omega^{(\beta)}_{ri}$ is the (α^t_{ri}, β)-KMS state of the i-th reservoir. The set of states which are normal w.r.t. ω_0 is the same for any choice of ω_p. A state ψ which is normal w.r.t. ω_0 (i.e., which is represented by a density matrix ρ in the GNS representation $(\mathcal{H}, \pi, \Omega_0)$ of $(\mathfrak{A}, \omega_0)$, according to $\psi(A) = \mathrm{Tr}(\rho\pi(A))$) will be called a $\beta_1\beta_2$-*normal state.*

In the particular case $\omega_p(\cdot) = \mathrm{Tr}(e^{-\beta_p H_p}\,\cdot)/\mathrm{Tr}(e^{-\beta_p H_p})$ we call ω_0 a *reference state.*

The Hamiltonian H generates the dynamics of observables $A \in \mathcal{B}(\mathcal{H}_0)$ according to the rule

$$A \mapsto \alpha^t(A) := e^{iHt} A e^{-iHt}\ . \tag{2.11}$$

Eqn (2.11) defines a group of *-automorphisms of $\mathcal{B}(\mathcal{H}_0)$. However, α^t is not expected to map the algebra $\mathcal{A}$ into itself. To circumvent this problem we define the interacting evolution of states on $\mathcal{A}$ by using the Araki-Dyson expansion. Namely, for a state ψ on the algebra $\mathcal{A}$ normal w.r.t. the state ω_0, we define the evolution by

$$\psi^t(A) := \lim_{n\to\infty} \sum_{m=0}^{\infty} (ig)^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m \; \psi_n^{t,t_1,\ldots,t_m}(A), \tag{2.12}$$

where the term with $m = 0$ is $\psi(\alpha_0^t(A))$, and, for $m \geq 1$,

$$\psi_n^{t,t_1,\ldots,t_m}(A) := \psi\left([\alpha_0^{t_m}(v_n), [\cdots [\alpha_0^{t_1}(v_n), \alpha_0^t(A)] \cdots]]\right).$$

Here, $v_n \in \mathcal{A}$ is an approximating sequence for the operator v, satisfying the relation

$$\lim_{n\to\infty} \omega_0(A^*(v_n^* - v^*)(v_n - v)A) = 0, \tag{2.13}$$

$\forall A \in \mathcal{A}$ of the form $A = B \otimes W_1(f_1) \otimes W_2(f_2)$ with $B \in \mathcal{B}(\mathcal{H}_0)$, $f_{1,2} \in L_0^2$. Such a sequence is constructed as follows. Let $\{e_m\}$ be an orthonormal basis of L_0^2. We define the approximate creation operators

$$a_{j,n}^*(G_j) = \sum_{m=1}^{M} \langle e_m, G_j \rangle b_{j,\lambda}^*(e_m), \tag{2.14}$$

where $n = (\lambda, M)$, and, for any $f \in L^2(\mathbb{R}^3)$ and $\lambda > 0$,

$$b_{j,\lambda}^*(f) := \frac{\lambda}{\sqrt{2}i} \left\{ W_j(f/\lambda) - \mathbf{1} - iW_j(if/\lambda) + i\mathbf{1} \right\}. \tag{2.15}$$

Similarly we define the approximate annihilation operators $a_{j,n}(G_j)$. Via the above construction we obtain the family of interactions $v_n \in \mathcal{A}$. Using (2.9), one easily shows that (2.13) is satisfied.

In Appendix A we show that under condition (2.13) the integrands on the r.h.s. of (2.12) are continuous functions in $t_1, \ldots, t_m$, that the series is absolutely convergent and that the limit exists and is independent of the approximating sequence v_n.

A $\beta_1\beta_2$-normal state ψ is called invariant (under the interacting dynamics), or stationary, if $\psi^t(A) = \psi(A)$ for all $A \in \mathcal{A}$, $t \in \mathbb{R}$, see (2.12). Our goal is to show that, if $\beta_1 \neq \beta_2$, then there are no $\beta_1\beta_2$-normal states which are invariant. In particular, there are no equilibrium states (see Theorem 3.1).

To pass to a Hilbert space framework one uses the GNS representation of $(\mathcal{A}, \omega_0)$, where ω_0 is given in (2.10):

$$(\mathcal{A}, \omega_0) \to (\mathcal{H}, \pi, \Omega_0).$$

Here $\mathcal{H}$, π and Ω_0 are a Hilbert space, a representation of the algebra $\mathcal{A}$ by bounded operators on $\mathcal{H}$, and a cyclic element in $\mathcal{H}$ (i.e. $\overline{\pi(\mathcal{A})\Omega_0} = \mathcal{H}$) s.t.

$$\omega_0(A) = \langle \Omega_0, \pi(A)\Omega_0 \rangle \ .$$

(In this paper we use the Araki-Woods GNS representation with $\omega_p(A) := \mathrm{Tr}(e^{-\beta_p H_p} A)/\mathrm{Tr}(e^{-\beta_p H_p})$ in (2.10), see Section 4.)

With the free evolution $\alpha_0^t(A) := e^{itH_0} A e^{-itH_0}$ one associates the unitary one-parameter group, $U_0(t) = e^{itL_0}$, on $\mathcal{H}$ s.t.

$$\pi(\alpha_0^t(A)) = U_0(t)\pi(A)U_0(t)^{-1} \tag{2.16}$$

and $U_0(t)\Omega_0 = \Omega_0$. Define the *standard Liouville operator*

$$L := L_0 + g\pi(v) - g\pi'(v), \tag{2.17}$$

defined on the dense domain $\mathcal{D}(L_0) \cap \mathcal{D}(\pi(v)) \cap \mathcal{D}(\pi'(v))$. Here, $\pi(v)$ and $\pi'(v)$ can be defined either using explicit formulae for π and π' in the Araki-Woods representation given below, or by using the approximation $v_n \in \mathcal{A}$ of v, constructed above. By the Glimm-Jaffe-Nelson commutator theorem, the operator L is essentially self-adjoint; we denote its self-adjoint closure again by L. The operator L generates the one-parameter group of $*$automorphisms σ^t on the von Neumann algebra $\pi(\mathcal{A})''$ (the weak closure of $\pi(\mathcal{A})$),

$$\sigma^t(B) := e^{itL} B e^{-itL}, \tag{2.18}$$

where $B \in \pi(\mathcal{A})''$. Let ψ be a state on the algebra $\mathcal{A}$ normal w.r.t. the state ω_0, i.e.

$$\psi(A) = \mathrm{Tr}(\rho\pi(A)) \tag{2.19}$$

for some positive trace class operator ρ on $\mathcal{H}$ of trace one. It is shown in Appendix A that for ψ as above the limit on the r.h.s. of (2.12) exists and equals

$$\psi^t(A) = \mathrm{Tr}(\rho\sigma^t(\pi(A))). \tag{2.20}$$

In particular, the limit is independent of the choice of the approximating family v_n.

The following result connects the existence of normal invariant states to spectral properties of the standard Liouvillian L:

Theorem 2.1 ([13, 10]) *A normal σ^t-invariant state on $\pi(\mathcal{A})''$ exists if and only if zero is an eigenvalue of L.*

In order to obtain rather subtle spectral information on the operator L, we develop a new type of spectral deformation, $L \mapsto L_\theta$, with a spectral deformation parameter $\theta \in \mathbb{C}^2$. This deformation has the property that zero is an eigenvalue of L if and only if zero is an eigenvalue of L_θ, for $\theta \in (\mathbb{C}_+)^2$. We then investigate the spectrum of L_θ, using a Feshbach map iteratively.

3 Assumptions and Results

For our analysis we need conditions considerably stronger than (2.6). In order to formulate them, we first introduce some definitions. We refer the reader to the remarks at the end of this section for a discussion of the definitions and conditions.

We define the map $\gamma : L^2(\mathbb{R}^3) \to L^2(\mathbb{R} \times S^2)$,

$$(\gamma f)(u,\sigma) = \sqrt{|u|} \begin{cases} f(u\sigma), & u \geq 0, \\ -\overline{f}(-u\sigma), & u < 0. \end{cases} \tag{3.1}$$

Let $j_\theta(u) = e^{\delta \mathrm{sgn}(u)}u + \tau$ for $\theta = (\delta, \tau) \in \mathbb{C}^2$ and $u \in \mathbb{R}$ (see (B.2.2)) and define $(\gamma_\theta f)(u,\sigma) = (\gamma f)(j_\theta(u),\sigma)$, for $f \in L^2(\mathbb{R} \times S^2)$, $\theta \in \mathbb{R}^2$.

We extend the maps γ and γ_θ to operator valued functions in the obvious way. Now, we are ready to formulate our assumptions.

(A) *Analyticity.* For $j = 1, 2$ and every fixed $(u, \sigma) \in \mathbb{R} \times S^2$, the maps

$$\theta \mapsto (\gamma_\theta G_j)(u, \sigma) \tag{3.2}$$

from $\mathbb{R}^2$ to the bounded operators on $\mathcal{H}_p$ have analytic continuations to

$$\left\{ (\delta, \tau) \in \mathbb{C}^2 \, \middle| \, |\mathrm{Im}\, \delta| < \delta_0, |\tau| < \tau_0 \right\} , \tag{3.3}$$

for some δ_0, $\tau_0 > 0$, $\frac{\tau_0}{\cos \delta_0} \leq \frac{2\pi}{\beta}$, where $\beta = \max(\beta_1, \beta_2)$. Moreover,

$$\|G_j\|_{\mu,\theta} := \sum_{\nu = 1/2, \mu} \left[\int\limits_{\mathbb{R} \times S^2} \left\| \gamma_\theta \left[\frac{\sqrt{|u| + 1}}{|u|^\nu} G_j \right] (u, \sigma) \right\|^2 du d\sigma \right]^{1/2} < \infty, \tag{3.4}$$

for some fixed $\mu > 1/2$.

(B) *Fermi Golden Rule Condition.*

$$\gamma_{0j} := \min_{0 \leq n < m \leq N-1} \int_{\mathbb{R}^3} \delta(|k| - |E_{nm}|) \; |G_j(k)_{nm}|^2 d^3 k > 0, \quad j = 1, 2, \tag{3.5}$$

where $G_j(k)_{mn} := \langle \varphi_m, G_j(k) \varphi_n \rangle$, φ_n are normalized eigenvectors of H_p corresponding to the eigenvalues E_n, $n = 0, \ldots, N-1$, and δ is the Dirac delta distribution.

For some of our results, we impose the additional condition

(C) *Simplicity of spectrum of H_p.* The eigenvalues of the particle Hamiltonian H_p are simple.

Let

$$\sigma := \min \left\{ |\lambda - \mu| \mid \lambda, \mu \in \sigma(H_p), \lambda \neq \mu \right\} . \tag{3.6}$$

Define

$$g_0 := C \sigma^{1/2} \sin(\delta_0) \left[(1 + \beta_1^{-1/2} + \beta_2^{-1/2}) \max_j \sup_{|\theta| \leq \theta_0} \|G_j\|_{1/2, \theta} \right]^{-1} , \tag{3.7}$$

where C is a constant depending only on $\tan \delta_0$, and set

$$g_1 := \min \left((g_0)^{1/\alpha}, [\min(T_1, T_2)]^{\frac{1}{2+\alpha}} \right) . \tag{3.8}$$

Remarks. 1) The map (3.1) has the following origin. In the positive-temperature representation of the CCR (the Araki-Woods representation on a suitable Hilbert space, see Appendix A), the interaction term v_j is represented by $a_j(\widetilde{\gamma}_{\beta_j} G_j) + a_j^*(\widetilde{\gamma}_{\beta_j} G_j)$, where

$$\widetilde{\gamma}_\beta := \sqrt{\frac{u}{1 - e^{-\beta u}}} \; \gamma. \tag{3.9}$$

2) A class of interactions satisfying Condition (A) is given by $G_j(k) = g(|k|) G$, where $g(u) = u^p e^{-u^2}$, with $u \geq 0$, $p = n + 1/2$, $n = 0, 1, 2, \ldots$, and $G = G^* \in \mathcal{B}(\mathcal{H}_p)$. A straightforward estimate gives that the norms (3.4) have the bound

$$\|G_j\|_{\mu,\theta} \leq C ||G||, \tag{3.10}$$

provided $\mu < p+1$, where the constant C does not depend on the inverse temperatures, nor on θ varying in any compact set.

The restriction $p = n + 1/2$ with $n = 0, 1, 2, \ldots$ comes from the requirement of translation analyticity (the τ–component of θ), which appears also in [14].

3) The condition $\tau_0/\cos\delta_0 < 2\pi/\beta$ after (3.3) guarantees that the square root in (3.9) is analytic in translations $u \mapsto u + \tau$.

4) Condition (C) guarantees that the *level-shift operators* of the system have certain technical features which facilitate the analysis (see also Proposition 7.2 and [5]). We believe that this condition can be removed.

Our result on instability of normal stationary states is

Theorem 3.1 *Assume conditions (A), (B) and (C) are obeyed for some* $0 < \beta_1, \beta_2 < \infty$, $\mu > 1/2$, *and set* $\alpha = (\mu - 1/2)/(\mu + 1/2)$. *Assume* $\delta\beta := |\beta_1 - \beta_2| \neq 0$. *There are constants* c, c', c'' *s.t. if either of the two following conditions hold,*

1. $0 < |g| < cg_1$, $\delta\beta < c'$, $\|G_1 - G_2\| < c'$, *and* g *avoids possibly finitely many values in the set* $\{0 < |g| < cg_1\}$, *or*

2. $0 < |g| < c'' \min\left((g_0)^{1/\alpha}, \left[\min_j(\gamma_{0j}) \frac{|\delta\beta|^2}{1+|\delta\beta|^2}\right]^{1/\alpha} \right)$,

then there are no normal σ^t*-invariant states on* $\pi(\mathcal{A})''$.

Remarks. 5) Using Araki's theory of perturbation of KMS states (c.f. [9]) it is not hard to show that if the reservoir-temperatures are equal, then the system has an equilibrium state.

6) By an analyticity argument one can show that the result 1. holds for all but a discrete set of values of $\delta\beta$ and $\|G_1 - G_2\|$.

7) We will remove the "high temperature" restriction $|g| < c[\min(T_1, T_2)]^{\frac{1}{2+\alpha}}$, (3.8), in [17]; see the end of Section 7 for the relevant ideas.

4 Araki-Woods representation and Liouville operators

In this section we present the explicit GNS representation provided by the Araki-Woods construction, which is used in our analysis (see [5, 12, 6, 7] for details and [1, 11] for original papers). In the Araki-Woods GNS representation the (positive temperature) Hilbert space is given by

$$\mathcal{H} = \mathcal{H}^p \otimes \mathcal{H}^r, \tag{4.1}$$

where $\mathcal{H}^p = \mathcal{H}_p \otimes \mathcal{H}_p$ and $\mathcal{H}^r = \mathcal{H}^{r1} \otimes \mathcal{H}^{r2}$ with

$$\mathcal{H}^{rj} = \mathcal{H}_{rj} \otimes \mathcal{H}_{rj}. \tag{4.2}$$

We denote by $a^{\#}_{\ell,j}(f)$ (resp., $a^{\#}_{r,j}(f)$) the creation and annihilation operators which act on the left (resp., right) factor of (4.2). They are related to the zero temperature creation and annihilation operators $a^{\#}_j(f)$ by

$$\pi(a_j(f)) = a_{\ell j}(\sqrt{1+\rho_j}\, f) + a^*_{rj}(\sqrt{\rho_j}\, \bar{f}) \tag{4.3}$$

and

$$\pi'(a_j(f)) = a^*_{\ell j}(\sqrt{\rho_j}\, f) + a_{rj}(\sqrt{1+\rho_j}\, \bar{f}) \tag{4.4}$$

where $\rho_j \equiv \rho_j(k) = (e^{\beta_j \omega(k)} - 1)^{-1}$ with $\omega(k) = |k|$. Finally, we denote $\Omega_r := \Omega_{r1} \otimes \Omega_{r2}$, where $\Omega_{rj} := \Omega_{rj,\ell} \otimes \Omega_{rj,r}$ are the vacua in $\mathcal{H}^{rj}$. Thus, Ω_r is the vacuum in $\mathcal{H}^r$.

Definition (2.10) and our choice of ω_p made at the beginning of this section imply that

$$\Omega_0 = \Omega_p \otimes \Omega_r \quad \text{with} \quad \Omega_p \equiv \Omega^p_{\beta_p} = \frac{\sum_j e^{-\beta_p E_j/2} \varphi_j \otimes \varphi_j}{[\sum_j e^{-\beta_p E_j}]^{1/2}}, \tag{4.5}$$

where, recall, E_j and φ_j are the eigenvalues and normalized eigenvectors of H_p.

The self-adjoint operator L_0 generating the free evolution, $U_0(t)$, defined in (2.16), is of the form $L_0 = L_p \otimes \mathbf{1}^r + \mathbf{1}^p \otimes L_r$ with $L_r = \sum_{j=1}^2 L_{rj}$. The operator L_p has the standard form

$$L_p = H_p \otimes \mathbf{1}_p - \mathbf{1}_p \otimes H_p$$

and the operators L_{rj} are as follows

$$L_{rj} = \int \omega(k) \left(a^*_{\ell,j}(k) a_{\ell,j}(k) - a^*_{r,j}(k) a_{r,j}(k) \right) d^3k.$$

A standard argument shows that the spectrum of the operator L_0 fills the axis $\mathbb{R}$ with the thresholds and eigenvalues located at $\sigma(L_p)$ and with 0 an eigenvalue of multiplicity at least $\dim H_p$ and at most $(\dim H_p)^2$ (depending on the degeneracy of the spectrum of L_p).

5 A class of Liouville operators and their Spectral Deformation

To investigate the point spectrum of the self-adjoint Liouvillian L we perform a complex deformation of the operator L, producing a family of operators L_θ, $\theta \in \mathbb{C}^2$, with the property $L_{\theta=0} = L$ and s.t. L_θ is unitarily equivalent to L for $\theta \in \mathbb{R}^2$. We investigate the spectrum of L_θ for complex θ which we relate to the properties of L that are of interest to us. In this section we construct the family L_θ and establish some global spectral and analyticity properties. In the next section we give a finer description of the spectrum of L_θ.

In fact, the analysis of both this section and the next one works for a general class of operators which are of the form

$$K := L_0 + gI, \quad I := U - W', \tag{5.1}$$

where $U = \pi(u)$ and $W' = \pi'(w)$, with operators u, w of the form

$$u = \sum_{j=1,2} \left\{ a^*_j(G_{j1}) + a_j(G_{j2}) \right\} \tag{5.2}$$

$$w = \sum_{j=1,2} \left\{ a^*_j(G_{j3}) + a_j(G_{j4}) \right\}. \tag{5.3}$$

If

$$G_{jk} = G_j, \quad \text{for } k = 1, \ldots 4 \text{ and } j = 1, 2, \tag{5.4}$$

then the operator K reduces to the standard Liouville operator L, (2.17). We carry out the analysis for the more general class of operators K since they are needed in the construction of non-equilibrium stationary states, [16]. Note that in general, K is not a normal operator.

For the spectral analysis of the operators K we replace condition (A) by condition (AA) below, which reduces to (A) for self-adjoint K. For a scalar function $f(u,\sigma)$ and $k=1,3$, set

$$\gamma(fG_{jk})(u,\sigma) := |u|^{1/2}\begin{cases} f(u,\sigma)G_{jk}(u\sigma), & u\geq 0 \\ -\overline{f}(-u,\sigma)G^*_{j(k+1)}(-u\sigma), & u<0 \end{cases} \tag{5.5}$$

and define $\gamma_\theta(fG_{jk})$ as after (3.1) (if (5.4) holds then (5.5) coincides with $(\gamma G_j)(u,\sigma)$ as defined by (3.1)).

(AA) *Analyticity (non-selfadjoint case).* For $j=1,2$, $k=1,3$, and for every fixed $(u,\sigma)\in\mathbb{R}\times S^2$, the maps

$$\theta\mapsto(\gamma_\theta G_{jk})(u,\sigma) \tag{5.6}$$

from $\mathbb{R}^2$ to the bounded operators on $\mathcal{H}_p$ have analytic continuations to

$$\left\{(\delta,\tau)\in\mathbb{C}^2\,\big|\,|\mathrm{Im}\,\delta|<\delta_0, |\tau|<\tau_0\right\}, \tag{5.7}$$

for some δ_0, $\tau_0>0$, $\frac{\tau_0}{\cos\delta_0}\leq\frac{2\pi}{\beta}$, where $\beta=\max(\beta_1,\beta_2)$. Moreover,

$$\|G_j\|_{\mu,\theta} := \sum_{k=1,3}\sum_{\nu=1/2,\mu}\left[\int_{\mathbb{R}\times S^2}\left\|\gamma_\theta\left[\frac{\sqrt{|u|+1}}{|u|^\nu}G_{jk}\right](u,\sigma)\right\|^2 du d\sigma\right]^{1/2}<\infty, \tag{5.8}$$

for some fixed $\mu>1/2$.

If (5.4) holds then condition (AA) coincides with condition (A). One shows that K is a closed operator on the dense domain $\mathcal{D}(L)\cap\mathcal{D}(U)\cap\mathcal{D}(W')$.

In order to carry out the spectral analysis of the operator K, which we begin in this section, we use the specifics of the Araki-Woods representation. They were not used in an essential way for the developments up to this section.

As a complex deformation we choose a combination of the complex dilation used in [5] and complex translation due to [12] (see [5], Section V.2 for a sketch of the relevant ideas).

First we define the group of dilations. Let $\hat{U}_{d,\delta}$ be the second quantization of the one-parameter group

$$u_{d,\delta}: f(k)\rightarrow e^{3\delta/2}f(e^\delta k)$$

of dilations on $L^2(\mathbb{R}^n)$. This group acts on creation and annihilation operators $a_r^\#(f)$ on the Fock space, $\mathcal{H}_r$, according to the rule

$$\hat{U}_{d,\delta}a_r^\#(f)\hat{U}_{d,\delta}^{-1}=a_r^\#(u_{d,\delta}f), \qquad \hat{U}_{d,\delta}\Omega_{rj}=\Omega_{rj}. \tag{5.9}$$

We lift this group to the positive-temperature Hilbert space, (4.1), according to the formula

$$U_{d,\delta}=\mathbf{1}_p\otimes\mathbf{1}_p\otimes\hat{U}_{d,\delta}\otimes\hat{U}_{d,-\delta}\otimes\hat{U}_{d,\delta}\otimes\hat{U}_{d,-\delta}. \tag{5.10}$$

Note that we could dilate each reservoir by a different amount. However, this does not give us any advantage, so to keep notation simple we use one dilation parameter for both reservoirs. We record for future reference how the group $U_{d,\delta}$ acts on the Liouville operator L_0 and the positive-temperature photon number operator $N := \sum_{j=1}^{2} N_j$, where

$$N_j := \int \left[a^*_{\ell,j}(k)a_{\ell,j}(k) + a^*_{r,j}(k)a_{r,j}(k)\right] \, d^3k, \tag{5.11}$$

and where the operators $a^{\#}_{\{\ell,r\},j}(k)$ were introduced after (4.2). We have (below we do not display the identity operators):

$$U_{d,\delta} L_{rj} U_{d,\delta}^{-1} = \cosh(\delta) L_{rj} + \sinh(\delta)\Lambda_j, \tag{5.12}$$

where Λ_j is the positive operator on the jth reservoir Hilbert space given by

$$\Lambda_j = \int \omega(k) \left(a^*_{\ell,j}(k)a_{\ell,j}(k) + a^*_{r,j}(k)a_{r,j}(k)\right) \, d^3k, \tag{5.13}$$

and

$$U_{d,\delta} N_j U_{d,\delta}^{-1} = N_j. \tag{5.14}$$

Now we define a one-parameter group of translations. It can be defined as one-parameter group arising from transformations of the underlying physical space similarly to the dilation group. This is done in Appendix B. Here we define the translation group by means of the selfadjoint generator $T := \sum_{j=1}^{2} T_j$, where

$$T_j = \int \left[a^*_{\ell,j}(k)\vartheta a_{\ell,j}(k) + a^*_{r,j}(k)\vartheta a_{r,j}(k)\right] \, d^3k. \tag{5.15}$$

Here, $\vartheta = i(\hat{k}\cdot\nabla + \nabla\cdot\hat{k})$ with $\hat{k} = k/|k|$. Since ϑ is a self-adjoint operator on $L^2(\mathbb{R}^3)$, the operators $T_j, j = 1,2$, and therefore the operator T, are self-adjoint as well. We define the one-parameter group of translations as

$$U_{t,\tau} := \mathbf{1}_p \otimes \mathbf{1}_p \otimes e^{i\tau T}. \tag{5.16}$$

Eqns. (5.15) - (5.16) imply the following expressions for the action of this group on the Liouville operators:

$$U_{t,\tau} L_{rj} U_{t,\tau}^{-1} = L_{rj} + \tau N_j. \tag{5.17}$$

Observe that neither the dilation nor the translation group affects the particle vectors, and that $U_{t,\tau} N_j U_{t,\tau}^{-1} = N_j$.

Now we want to apply the product of these transformations to the full operator $K = L_0 + gI$, (5.1). Since the dilation and translation transformations do not commute we have to choose the order in which we apply them. The operator $\Lambda = \sum_j \Lambda_j$ is not analytic under the translations, while the operator N is analytic under dilations. Thus we apply first the translation and then the dilation transformation, and define the combined translation-dilation transformation as

$$U_\theta = U_{d,\delta} U_{t,\tau} \tag{5.18}$$

where $\theta = (\delta, \tau)$. In what follows we will use the notation $|\theta| = (|\delta|, |\tau|)$, $\mathrm{Im}\theta = (\mathrm{Im}\delta, \mathrm{Im}\tau)$, and similarly for $\mathrm{Re}\theta$, and

$$\mathrm{Im}\theta > 0 \quad \Longleftrightarrow \quad \mathrm{Im}\delta > 0 \wedge \mathrm{Im}\tau > 0. \tag{5.19}$$

Now we are ready to define a complex deformation of the operator K. On the set $D(\Lambda) \cap D(N)$ we define for $\theta \in \mathbb{R}$

$$K_\theta := U_\theta K U_\theta^{-1}. \tag{5.20}$$

Recalling the decomposition $K = L_0 + gI$, (5.1), where $L_0 := L_p + L_r$, $L_r := \sum_{j=1}^{2} L_{rj}$ and $I = U - W'$, we have

$$K_\theta = L_{0,\theta} + gI_\theta, \tag{5.21}$$

where the families $L_{0,\theta}$ and I_θ are defined accordingly. Due to Eqns. (5.12), (5.14) and (5.17) we have:

$$L_{0,\theta} = L_p + \cosh(\delta) L_r + \sinh(\delta)\Lambda + \tau N, \tag{5.22}$$

where $\theta = (\delta, \tau)$, and $\Lambda = \sum_{j=1}^{2} \Lambda_j$. An explicit expression for the family I_θ is given in Appendix B.2 (see Eqns (B.2.5) and (B.2.7)).

Of course the operator families above are well defined for real θ. Our task is to define them as analytic families on the strips

$$S_{\theta_0}^{\pm} = \left\{\theta \in \mathbb{C}^2 | 0 < \pm\mathrm{Im}\theta < \theta_0\right\} \tag{5.23}$$

where $\theta_0 = (\delta_0, \tau_0) > 0$ is the same as in Condition (AA). Recall that the inequality $\pm\mathrm{Im}\theta < \theta_0$ is equivalent to the following inequalities: $\pm\mathrm{Im}\delta < \delta_0$ and $\pm\mathrm{Im}\tau < \tau_0$. (The fact that analyticity in a neighbourhood of a fixed $\theta \in S_{\theta_0}^{\pm}$ implies analyticity in the corresponding strip in which $\mathrm{Re}\theta$ is not constraint follows from the explicit formulas (5.22), (B.2.5) and (B.2.7).) The analytic continuations of the operators (if they exist) are denoted by the same symbols.

We define the family K_θ for $\theta \in \{\theta \in \mathbb{C}^2 \,||\mathrm{Im}\,\theta| < \theta_0\}$ by the explicit expressions (5.21), (5.22), (B.2.5) and (B.2.7). Clearly, $D(\Lambda) \cap D(N) \subset D(L_{0\theta})$ and on this domain the family $L_{0\theta}$ is manifestly strongly analytic in $\theta \in \{\theta \in \mathbb{C}^2 \,||\mathrm{Im}\theta| < \theta_0\}$. It is shown in Appendix B that for $|\mathrm{Im}\theta| < \theta_0$ we have $D(\Lambda^{1/2}) \subset D(I_\theta)$ and $I_\theta f$ is analytic $\forall f \in D(\Lambda^{1/2})$. Here Condition (AA) is used. Hence the family K_θ for $\theta \in \{\theta \in \mathbb{C}^2 \,||\mathrm{Im}\,\theta| < \theta_0\}$ is bounded from $D(\Lambda) \cap D(N)$ to $\mathcal{H}$ (and $K_\theta f$ is analytic in $\theta \in \{\theta \in \mathbb{C}^2 \,||\mathrm{Im}\,\theta| < \theta_0\}, \forall f \in \mathcal{D}(\Lambda) \cap \mathcal{D}(N)$). Moreover, for $|\mathrm{Im}\,\theta| > 0$ the operators K_θ are closed on the domain $D(\Lambda) \cap D(N)$.

However, $\{K_\theta | \,|\mathrm{Im}\theta| < \theta_0\}$ is *not* an analytic family in the sense of Kato. The problem here is the lack of coercivity – the perturbation I is not bounded relatively to the unperturbed operator L_0. To compensate for this we have chosen the deformation U_θ in such a way that the operator $M_\theta := \mathrm{Im}L_{0,\theta}$ is coercive for $\mathrm{Im}\theta > 0$, i.e., the perturbation I_θ, as well as $\mathrm{Re}L_{0,\theta}$, are bounded relative to this operator. The problem here is that $M_\theta \to 0$ as $\mathrm{Im}\theta \to 0$ so we have to proceed carefully.

The next result is similar to one in [5], but the proof given below is simpler than that of [5].

Theorem 5.1 *Assume that Condition (AA) holds and let $\theta_0 = (\delta_0, \tau_0)$ be as in that condition. Take an*

$$a > \frac{g^2}{\sin(\mathrm{Im}\delta)} C_0^2 \left(\sum_{j=1,2} \|G_j\|_{1/2,\theta} \right)^2, \tag{5.24}$$

where

$$C_0 := C(1 + \beta_1^{-1/2} + \beta_2^{-1/2}), \tag{5.25}$$

and where C is a constant depending only on $\tan\delta_0$. *Then we have:*

(i) $\{z \in \mathbb{C} | \mathrm{Im}\, z \leq -a\} \subseteq \rho(K_\theta)$ *(the resolvent set of K_θ) if $\theta \in S^+_{\theta_0}$; if in addition $K = K^*$ then we can take $\theta \in \overline{S^+_{\theta_0}}$;*

(ii) *The family K_θ is analytic of type A (in the sense of Kato) in $\theta \in S^+_{\theta_0}$;*

(iii) *If $K = K^*$, then, for any u and v which are U_θ-analytic in a strip $\{\theta \in \mathbb{C}^2 \,|\, 0 \leq \mathrm{Im}\theta < \theta_1\}$, for some $\theta_1 = (\delta_1, \tau_0), \delta_1 \in [0, \min\{\pi/3, \theta_0\})$, the following relation holds:*

$$\left\langle u, (K - z)^{-1} v \right\rangle = \left\langle u_{\overline{\theta}}, (K_\theta - z)^{-1} v_\theta \right\rangle, \tag{5.26}$$

where $u_\theta = U_\theta u$, etc., for $\mathrm{Im}\, z \leq -a$ and $0 < \mathrm{Im}\,\theta < \theta_1/2$.

Similar statements hold also for $-\theta_0 < \mathrm{Im}\theta \leq 0$.

Proof.

(i) This statement is a special case of the following proposition (estimate (5.35) below suffices). Let $C_{a,b}$ be the truncated wedge

$$C_{a,b} := \tag{5.27}$$
$$\left\{ z \in \mathbb{C} \mid \mathrm{Im}\, z > -a/2,\ |\mathrm{Re}\, z| < 2[(\sin b)^{-1} + a/4](\mathrm{Im}\, z + a) + \|L_p\| + 1 \right\}.$$

Proposition 5.2 *Let $\theta \in S^+_{\theta_0}$, and take a as in (5.24). Then $\sigma(K_\theta) \subset C_{a,\mathrm{Im}\,\delta}$, and for $z \in \mathbb{C}\backslash C_{a,\mathrm{Im}\,\delta}$ we have*

$$\|(K_\theta - z)^{-1}\| \leq [\mathrm{dist}\,(z, C_{a,\mathrm{Im}\,\delta})]^{-1}. \tag{5.28}$$

Proof. To keep notation simple we prove the proposition for θ purely imaginary: $\theta = i\theta'$, $\theta' = (\delta', \tau') \in \mathbb{R}^2$. In this case the operator M_θ is of the form

$$M_\theta = \sin\delta' \Lambda + \tau' N. \tag{5.29}$$

The proof below is based on the following bounds on the interaction, which, to simplify the notation, we formulate for the case $\theta = i\theta'$ only.

Lemma 5.3 *Let μ be the same as in Condition (AA) above. We have*

$$\left\|(M_\theta+a)^{-1/2}I_\theta(M_\theta+a)^{-1/2}\right\| \le C_0\frac{\sum_{j=1}^2\|G_j\|_{1/2,\theta}}{\sqrt{a\sin\delta'}}, \tag{5.30}$$

$$\|\chi_{M_\theta\le\rho}I_\theta\chi_{M_\theta\le\rho}\| \le C_0\left(\frac{2\rho}{\sin\delta'}\right)^\mu\sum_{j=1}^2\|G_j\|_{\mu,\theta}\,, \tag{5.31}$$

$$|\langle\psi,I_\theta\psi\rangle| \le \frac{\varepsilon}{\sin\delta'}C_0^2\left(\sum_{j=1}^2\|G_j\|_{1/2,\theta}\right)^2\langle\psi,M_\theta\psi\rangle+\frac{1}{\varepsilon}\|\psi\|^2\,, \tag{5.32}$$

for any $a,\rho,\varepsilon>0$, and where C_0 is given in (5.25). Similar estimates hold also if we replace I_θ by either $\mathrm{Re}I_\theta$ *or* $\mathrm{Im}I_\theta$*.*

This lemma follows from Proposition B.1 of Appendix B.3 and equation (B.3.13) (cf. [4]). The norms on the r.h.s. of (5.30) - (5.32) are defined in (5.8).

Now we use the lemma above to prove Proposition 5.2. First we determine the numerical range, $\mathrm{NR}(K_\theta)$, of the operator K_θ. Let $u\in\mathcal{D}(M_\theta^{1/2})$ and $\|u\|=1$. Recall the notation $|A|:=(A^*A)^{1/2}$ and remember that we assumed that $\theta=(i\delta',i\tau')$. By estimate (5.32) and $|\mathrm{Re}L_{0,\theta}|\le\|L_p\|+\cos\delta'\Lambda$ we have

$$|\mathrm{Re}\left\langle K_\theta\right\rangle_u| \le \left\langle\Lambda+\frac{C_1^2g^2}{\sin\delta'}M_\theta+\|L_p\|+1\right\rangle_u\,, \tag{5.33}$$

where $\langle A\rangle_u:=\langle u,Au\rangle$, and we have set $C_1:=C_0\sum_{j=1}^2\|G_j\|_{1/2,\theta}$. Next, using that $\mathrm{Im}K_\theta=M_\theta+g\mathrm{Im}I_\theta$, we write

$$\mathrm{Im}\left\langle K_\theta+ia\right\rangle_u=\left\langle\overline{M_\theta}^{1/2}(1+R)\overline{M_\theta}^{1/2}\right\rangle_u,$$

where $\overline{M_\theta}=M_\theta+a$ and $R=g\overline{M_\theta}^{-1/2}\mathrm{Im}I_\theta\overline{M_\theta}^{-1/2}$. Using estimate (5.30) we obtain $\|R\|\le\frac{gC_1}{\sqrt{a\sin\delta'}}$. Hence if

$$gC_1<\frac{1}{2}\sqrt{a\sin\delta'}, \tag{5.34}$$

then we have

$$\mathrm{Im}\left\langle K_\theta\right\rangle_u+a\ge\frac{1}{2}\left\langle\overline{M_\theta}\right\rangle_u\ge a/2. \tag{5.35}$$

This shows that $\mathrm{Im}\left\langle K_\theta\right\rangle_u\ge -a/2$. It follows from $M_\theta=\sin\delta'\Lambda+\tau'N$ and (5.33) that $|\mathrm{Re}\left\langle K_\theta\right\rangle_u|\le\left\langle\frac{1+C_1^2g^2}{\sin\delta'}M_\theta+\|L_p\|+1\right\rangle_u$, and hence, by $\left\langle M_\theta\right\rangle_u\le\left\langle\overline{M_\theta}\right\rangle_u$ and (5.35),

$$|\mathrm{Re}\left\langle K_\theta\right\rangle_u|\le 2\frac{1+C_1^2g^2}{\sin\delta'}(\mathrm{Im}\left\langle K_\theta\right\rangle_u+a)+\|L_p\|+1. \tag{5.36}$$

Using the bound (5.34) in the last expression shows that $\mathrm{NR}(K_\theta)\subset C_{a,\delta'}$, where $C_{a,\delta'}$ is the truncated wedge (5.27), provided condition (5.34) is satisfied. In particluar, the spectrum of the operator K_θ is inside $C_{a,\delta'}$. Moreover, for $z\notin C_{a,\delta'}$ and u as above we have the estimate

$$\|(K_\theta-z)u\|\ge|\left\langle K_\theta\right\rangle_u-z|\ge\mathrm{dist}(z,C_{a,\delta'}), \tag{5.37}$$

which, by taking $u=(K_\theta-z)^{-1}v/\|(K_\theta-z)^{-1}v\|$, implies (5.28).

(ii) Estimates $\|u\|\|(K_\theta - z)u\| \geq \mathrm{Im}\,\langle u, (K_\theta - z)u\rangle$ and (5.35) imply for $\mathrm{Im}\, z \leq -a$:

$$\|(K_\theta - z)u\| \geq \frac{\sqrt{a}}{2}\|M_\theta^{1/2}u\| \,. \tag{5.38}$$

The last estimate can be rewritten as

$$\|M_\theta^{1/2}(K_\theta - z)^{-1}\| \leq \frac{2}{\sqrt{a}} \,. \tag{5.39}$$

Similarly we have

$$\|M_\theta^{-1/2}\partial_\theta K_\theta M_\theta^{-1/2}\| \leq C, \tag{5.40}$$

where ∂_θ stands for ∂_δ, ∂_τ. The last two estimates and the computation

$$\partial_\theta(K_\theta - z)^{-1} = -(K_\theta - z)^{-1}\partial_\theta K_\theta(K_\theta - z)^{-1} \tag{5.41}$$

imply that $(K_\theta - z)^{-1}$ is analytic in $\theta \in S^{\pm}_{\theta_0}$, provided $\mathrm{Im}\, z \leq -a$.

(iii) Now to fix ideas we assume that $\mathrm{Im}\,\theta \geq 0$ and $\mathrm{Im}\, z < -a$. For $\alpha > 0$ we define $K^{(\alpha)} := K + i\alpha N$. Then $K_\theta^{(\alpha)} := U_\theta K^{(\alpha)} U_\theta^{-1} = K_\theta + i\alpha N$ and by standard estimates similar to those in Proposition A.1 of Appendix A, $(K_\theta^{(\alpha)} - z)^{-1}$ is analytic for $\mathrm{Im}\,\theta > 0$, uniformly bounded and strongly continuous for $\mathrm{Im}\,\theta \geq 0$. (To prove the latter property it suffices to show that $(K_\theta^{(\alpha)} - z)^{-1}$ is strongly continuous on the dense set $\mathcal{D}(\Lambda)$ which is straightforward.) Let u and v be $U(\theta)$-analytic for $|\mathrm{Im}\,\theta| < \delta_1$ for some $\frac{2}{3}\theta_0 > \delta_1 > 0$. Then in a standard way

$$\langle u, (K^{(\alpha)} - z)^{-1}v\rangle = \langle u_{\overline{\theta}}, (K_\theta^{(\alpha)} - z)^{-1}v_\theta\rangle \tag{5.42}$$

for θ with $\mathrm{Im}\,\theta > 0$. Let now $v \in \mathcal{D}(N)$ (then $v_\theta \in \mathcal{D}(N)$). With a help of the second resolvent equation

$$(K_\theta^{(\alpha)} - z)^{-1} = (K_\theta - z)^{-1} - (K_\theta^{(\alpha)} - z)^{-1} i\alpha N (K_\theta - z)^{-1},$$

we see that both sides of (5.42) converge as $\alpha \to 0$, with (5.26) resulting in the limit. Finally, we remove the constraint $v \in \mathcal{D}(N)$ using a standard density argument. Namely, we approximate the U_θ-analytic vectors u and v by the vectors $(1+\epsilon N)^{-1}u$ and $(1+\epsilon N)^{-1}v$ which belong to $\mathcal{D}(N)$ and, since $U_\theta N U_\theta^{-1} = N$, are U_θ-analytic as well. ■

Remark. The other two complex deformations, [12] and [5], are not suitable technically in the present context due to the following reasons:

- [12] leads to the problem in contour integration for the resolvent representation of the dynamics (see [16])

- [5] leads to a spectrum in which an eigenvalue at 0 is embedded at a “tip” of the continuous spectrum and consequently it is technically more difficult to define the pole approximation in this case (see [16]).

6 Spectral Analysis of K_θ

In this section we describe the spectrum of the operator K_θ, $\mathrm{Im}\theta > 0$, in the half-space

$$S = \left\{ z \in \mathbb{C} \middle| \mathrm{Im}\, z < \frac{\sin(\mathrm{Im}\,\delta)}{4}\rho_0 \right\}, \tag{6.1}$$

where $0 < \mathrm{Im}\delta < \delta_0$, and $\rho_0 \in (0, \sigma/2)$, c.f. (3.6).

Let e be an eigenvalue of L_p and let Λ_e be the operator acting on $\mathrm{Ran}\chi_{L_p=e}$ defined by

$$\Lambda_e := -P_e I (L_0 - e + i0)^{-1} I P_e, \tag{6.2}$$

where $P_e = \chi_{L_p=e} \otimes \chi_{L_r=0}$. Since $\mathrm{Ran}(IP_e)$ is orthogonal to $\mathrm{Null}(L_0 - e)$ this operator can be, at least in principle, defined. To show that it is well defined we consider the operator $P_e I_\theta L_{0\theta}^{-1} I_\theta P_e$ which *is* well-defined since $\mathrm{Ran}(I_\theta P_e)$ is orthogonal to $\mathrm{Null}(L_{0\theta} - e)$ (and e is an isolated eigenvalue of $L_{0\theta}$), is independent of θ and is equal to Λ_e. The operator Λ_e is called the *level shift operator*.

The main result of this section is Theorem 6.1, which shows how the level shift operators Λ_e determine the essential features of the spectrum of K_θ.

For $\rho_0 \in (0, \sigma/2)$ we decompose the half space S into the strips

$$S_e = \{ z \in S \,|\, |\mathrm{Re}\, z - e| \le \rho_0 \} \tag{6.3}$$

where $e \in \sigma(L_p)$, and we set $\overline{S} = S \setminus \bigcup_{e \in \sigma(L_p)} S_e$, so that $S = \bigcup_{e \in \sigma(L_p)} S_e \cup \overline{S}$.

Theorem 6.1 *Assume condition (AA) holds. Take $0 < |g| < \sqrt{\rho_0}\, g_0$ (c.f. (6.1), (3.7)), and $e \in \sigma(L_p)$. Let $\alpha = (\mu - 1/2)/(\mu + 1/2)$, where $\mu > 1/2$ is given in Condition (AA).*

1. *We have $\sigma(K_\theta) \subset \bigcup_{e \in \sigma(L_p)} S_e$.*

2. *Choose $\rho_0 = |g|^{2-2\alpha}$. Suppose $\mathrm{Im}\Lambda_e := \frac{1}{2i}(\Lambda_e - \Lambda_e^*) \ge \gamma_e > 0$. If $|g|^\alpha \ll \gamma_e$, then*

$$\sigma(K_\theta) \cap S_e \subset \{ z \in \mathbb{C} \,|\, \mathrm{Im}\, z \ge \tfrac{1}{2} g^2 \gamma_e \}. \tag{6.4}$$

3. *Choose $\rho_0 = |g|^{2-2\alpha}$. Suppose that Λ_e has a simple eigenvalue λ_e, and that $\mathrm{Im}\left(\sigma(\Lambda_e) \backslash \{\lambda_e\}\right) \ge \mathrm{Im}\lambda_e + \delta_e$, for some $\delta_e > 0$. There is a $C > 0$ s.t. if $0 < |g| < C g_2$, where*

$$g_2 := \min[(\delta_e)^{1/\alpha}, (\tau')^{\frac{1}{2+\alpha}}], \tag{6.5}$$

then

$$\sigma(K_\theta) \cap S_e \subset \{z_0\} \cup \{ z \in \mathbb{C} \,|\, \mathrm{Im}\, z \ge g^2 \mathrm{Im}\lambda_e + \tfrac{1}{2} \min(g^2 \delta_e, \tau') \}, \tag{6.6}$$

where z_0 is a simple isolated eigenvalue of K_θ, satisfying $|z_0 - e - g^2 \lambda_e| = O(|g|^{2+\alpha})$. Moreover, $g \mapsto z_0(g)$ is analytic in an open complex neighbourhood of the set $0 < |g| < \min[(g_0)^{1/\alpha}, g_2] \subset \mathbb{R}$.

Remark. The analysis leading to Theorem 6.1 works also for infinite dimensional particle systems. We need $\dim \mathcal{H}_p < \infty$ in order to verify the assumptions $\gamma_e > 0$, $\delta_e > 0$, see Proposition 7.2 and Assumption (B), (3.5).

Proof of Theorem 6.1. 1. We use the operator $M_\theta := \mathrm{Im} L_{0,\theta} > 0$ and the representation

$$K_\theta - z = (M_\theta + a)^{1/2}(A + B)(M_\theta + a)^{1/2}, \tag{6.7}$$

where $a = \frac{\sin\delta'}{2}\rho_0$, $A := (M_\theta + a)^{-1/2}(L_{0,\theta} - z)(M_\theta + a)^{-1/2}$ and $B = g(M_\theta + a)^{-1/2} I_\theta (M_\theta + a)^{-1/2}$. For $z = x - iy \in \overline{S}$, the operator A has a spectral gap independent of the coupling constant g. Specifically, we claim that

$$\|Au\| \geq \frac{1}{10}\|u\|. \tag{6.8}$$

To prove this claim we observe first that the operators M_θ and $L_{0,\theta}$ commute and that A is a normal operator. Next, since $\mathrm{Im} L_{0,\theta} = M_\theta$ we have that $\mathrm{Im} A = (M_\theta + a)^{-1}(M_\theta + y)$. On the subspace $\{M_\theta \geq 2a\}$ we have, for $z \in \overline{S}$ (thus $y > -a/2$), $M_\theta + y \geq \frac{1}{2}(M_\theta + a)$ and therefore $|A| \geq \mathrm{Im} A \geq \frac{1}{2}$. On the subspace $\{M_\theta \leq 2a\}$ we estimate

$$|A| \geq |\mathrm{Re} A| \geq (1/3a)|L_p + \cos\delta' L_r - x| \ . \tag{6.9}$$

Now, recall that $\theta = (i\delta', i\tau')$ and use (5.22) to conclude that $M_\theta = \sin\delta'\Lambda + \tau' N$. Hence $|L_r| \leq (\sin\delta')^{-1} M_\theta \leq 2a/\sin\delta'$. Since $L_{0,\theta} = L_p + \cos\delta' L_r + i\sin\delta'\Lambda + i\tau' N$, we have for $z = x - iy \in \overline{S}$

$$|A| \geq \frac{1}{3a} \min_{e \in \sigma(L_p)} \{|e - x| - |\cos\delta' L_r|\} \geq \frac{1}{3a}(\rho_0 - 2a\cos\delta') \, .$$

Remembering that $a = \frac{\sin\delta'}{2}\rho_0$ we conclude that on the subspace $\{M_\theta \leq 2a\}$

$$|A| \geq \frac{2}{3\sin\delta'}(1 - \cos\delta' \sin\delta') \geq \frac{2(\sqrt{2}-1)}{3\sqrt{2}\sin\delta'} > \frac{1}{10\sin\delta'} > 1/10. \tag{6.10}$$

We used that $\sin\delta'\cos\delta' \leq 2^{-1/2}$. Consequently, $|A| \geq \frac{1}{10}$ which implies (6.8).

On the other hand, for $|g| < \sqrt{\rho_0}\, g_0$, the operator B is small (see Lemma 5.3):

$$\|B\| \leq C_0|g| \frac{\max_j \|G_j\|_{1/2,\theta}}{\sqrt{a\sin\delta'}} \leq C_0|g| \frac{\max_j \|G_j\|_{1/2,\theta}}{\sqrt{\rho_0}\sin\delta'} < 1/20. \tag{6.11}$$

Hence the operator $K_\theta - z\mathbf{1}$ is invertible for $z \in \overline{S}$ and for $|g| < \sqrt{\rho_0} g_0$. This completes the proof of 1.

2. To analyze the spectrum of K_θ inside S_e we use Feshbach maps introduced in [3, 4], and extended in [2]. We review quickly the definitions and some properties of these maps referring the reader to [4, 2] for more detail. For simplicity we present here the original version, [3, 4], though the refined one, [2], the smooth Feshbach map, is easier to use from a technical point of view. Let X be a Banach space and P be a projection on X. Define $\overline{P} := \mathbf{1} - P$ and let $H_{\overline{P}} := \overline{P} H \overline{P}$ and $R_{\overline{P}}(H) := \overline{P} H_{\overline{P}}^{-1} \overline{P}$ if $H_{\overline{P}}$ is invertible on $\mathrm{Ran}\overline{P}$. We define the Feshbach map F_P by

$$F_P(H) := P\,(H - H R_{\overline{P}}(H) H)\,P \tag{6.12}$$

on the domain

$$\mathcal{D}(F_P) \;=\; \{H : X \to X | H_{\overline{P}} \text{ is invertible}, \\ \mathrm{Ran}P \subseteq \mathcal{D}(H) \text{ and } \mathrm{Ran}R_{\overline{P}}(H) \subseteq \mathcal{D}(PH\overline{P})\}. \tag{6.13}$$

A key property of the maps F_P is given in the following statement proven in [4]:

Theorem 6.2 (Isospectrality Theorem) (i) $0 \in \sigma(H) \iff 0 \in \sigma(F_P(H))$,

(ii) $H\psi = 0 \iff F_P(H)\varphi = 0$ *with* $\varphi = P\psi$ *("⇒") and* $\psi = (\mathbf{1} - R_{\overline{P}}(H)H)\varphi$ *("⇐").*

Thus, Feshbach maps have certain isospectrality properties while reducing operators from the original space X to the smaller space $\mathrm{Ran}P$.

Now, we use Feshbach maps $F_{P_{e\rho}}$ with projections $P_{e\rho}$ defined as

$$P_{e\rho} := \chi_{L_p=e} \otimes \chi_{M_\theta \le \rho}. \tag{6.14}$$

Here, recall, $\chi_{L_p=e}$ is the eigenprojection for the operator L_p corresponding to an eigenvalue $e \in \sigma(L_p)$ and $\chi_{M_\theta \le \rho}$ is the spectral projection for the self-adjoint operator M_θ corresponding to the spectral interval $[0, \rho]$ (remember that M_θ is a positive operator).

Lemma 6.3 *Assume Condition (AA) holds. Let* $|g| < \sqrt{\rho_0}\, g_0$. *If* $z \in S_e$ *then* $K_{\theta z} := K_\theta - z \in \mathcal{D}(F_{P_{e\rho_0}})$, *and the operator* $K_{\theta z}^{(1)} := F_{P_{e\rho_0}}(K_{\theta z})$ *acting on* $\mathrm{Ran}P_{e\rho_0}$ *is of the form*

$$K_{\theta z}^{(1)} = (e - z)\mathbf{1} + L_{r\theta} + g^2 \Lambda_e + O(\epsilon(g, \rho_0)), \tag{6.15}$$

where $\rho_0 \in (0, \sigma/2)$, *the remainder is estimated in operator norm, and for any* $|g|, \rho > 0$,

$$\epsilon(g, \rho) := |g|\rho^\mu + |g|^3 \rho^{-1/2} + |g|^2 \rho^{2\mu - 1}. \tag{6.16}$$

We give here a short proof of Lemma 6.3. Another proof is obtained by an easy translation of Theorem V.6 and Lemma V.9 of [5].

Proof of Lemma 6.3. In this proof we write ρ for ρ_0. In order to prove that $K_{\theta z} \in \mathcal{D}(F_{P_{e\rho}})$ we let $W := M_\theta + \rho$, note that W commutes with $L_{0\theta}$, and write

$$\overline{P}_{e\rho} K_{\theta z} \overline{P}_{e\rho} = \overline{P}_{e\rho} W^{1/2} [A + B] W^{1/2} \overline{P}_{e\rho}, \tag{6.17}$$

where $A := W^{-1}(L_{0\theta} - z)$ and $B := gW^{-1/2} I_\theta W^{-1/2}$. Using that the operator A is normal and that its spectrum on $\mathrm{Ran}\overline{P}_{e\rho}$ can be explicitly found, one can show easily that it is invertible on $\mathrm{Ran}\overline{P}_{e\rho}$, and $\|A^{-1}\| \le C$, uniformly in ρ and g (cf. the proof of (6.7) given after that equation). By Lemma 5.3,

$$\|B\| \le C_0 |g| \frac{\max_j \|G_j\|_{1/2,\theta}}{\sqrt{\rho} \sin \delta'}. \tag{6.18}$$

Hence, for $|g| < \sqrt{\rho}\, g_0$, the operator $A + B$ is invertible and therefore so is $\overline{P}_{e\rho} K_{\theta z} \overline{P}_{e\rho}$ on $\mathrm{Ran}\overline{P}_{e\rho}$. It is easy to see that the other conditions in the definition of $D(F_{P_{e\rho}})$ (see Eqn (6.13)) are satisfied and therefore $K_{\theta z} \in \mathcal{D}(F_{P_{e\rho}})$.

Next, in view of definition (6.12) we compute

$$P_{e\rho}K_{\theta z}P_{e\rho} = (e-z)\mathbf{1} + L_{r\theta} + gP_{e\rho}I_\theta P_{e\rho} \tag{6.19}$$

acting on $\mathrm{Ran}P_{e\rho}$. By (5.31) and with μ as in Condition (AA)

$$gP_{e\rho}I_\theta P_{e\rho} = O(g\rho^\mu). \tag{6.20}$$

Using (6.17), expanding $\overline{P}_{e\rho}(\overline{P}_{e\rho}K_{\theta z}\overline{P}_{e\rho})^{-1}\overline{P}_{e\rho}$ in the Neumann series in B, and using that $\|B\| \le C|g|\rho^{-1/2}$, we find

$$-g^2 P_{e\rho}I_\theta R_{\overline{P}_{e\rho}}(K_{\theta z})I_\theta P_{e\rho} = g^2\Lambda_{e\rho\theta} + O(g^3\rho^{-1/2}), \tag{6.21}$$

where $\Lambda_{e\rho\theta} := P_{e\rho}I_\theta\overline{P}_{e\rho}L_{0\theta}^{-1}\overline{P}_{e\rho}I_\theta P_{e\rho}$.

To estimate the operator $\Lambda_{e\rho\theta}$ we use the expression of I_θ in terms of creation and annihilation operators, pull through the annihilation operators to the right until they either contract or hit the projections $P_{e\rho}$, and use estimates (B.3.4) and (B.3.8) for $a_{j\ell,r}(k)P_{e\rho}$ and $P_{e\rho}a^*_{j\ell,r}(k)$. As a result we obtain

$$\Lambda_{e\rho\theta} = \Lambda_e P_{e\rho} + O(\rho^{2\mu-1}), \tag{6.22}$$

where Λ_e acts nontrivially only on the particle Hilbert space (see Appendix C for more detail). Using relations (6.19) – (6.22) in the expression for $F_{P_{e\rho}}(K_{\theta z})$ (see (6.12)) we arrive at (6.15).

This finishes the proof of Lemma 6.3. ■

We now complete the proof of Theorem 6.1, parts 2 and 3. By the isospectrality of the map $F_{P_{e\rho_0}}$ and Lemma 6.3, we have

$$\sigma(K_\theta)\cap S_e = \Big(\sigma\left(L_{r\theta} + g^2\Lambda_e + O(\epsilon(g,\rho_0))\right) + e\Big)\cap S_e. \tag{6.23}$$

2. Since $\mathrm{Im}(L_{r\theta} + g^2\Lambda_e) \ge g^2\gamma_e$, and $\epsilon(g,\rho_0) \le 3|g|^{2+\alpha}$, the numerical range of $L_{r\theta} + g^2\Lambda_e + O(\epsilon(g,\rho_0))$ is a subset of $\{\mathrm{Im}z \ge \frac{1}{2}g^2\gamma_e\}$, provided $|g|^\alpha \ll \gamma_e$. The desired result follows from the fact that the spectrum of $L_{r\theta} + g^2\Lambda_e + O(\epsilon(g,\rho_0))$ is contained in the closure of the numerical range, and from (6.23).

3. We start with the following result.

Lemma 6.4 *Let A be a normal operator on a Hilbert space $\mathcal{H}_1$, and let B be an operator on a Hilbert space $\mathcal{H}_2$, $\dim\mathcal{H}_2 = d < \infty$. Then*
(i) $\sigma(A\otimes\mathbb{1} + \mathbb{1}\otimes B) = \sigma(A) + \sigma(B)$,
(ii) *for $z \notin \sigma(A) + \sigma(B)$ we have*

$$\left\|(A\otimes\mathbb{1} + \mathbb{1}\otimes B - z)^{-1}\right\| \le C\left[\mathrm{dist}(\sigma(A)+\sigma(B), z)\right]^{-n}, \tag{6.24}$$

where $1 \le n \le d$ is the largest degree of nilpotency of the eigenvalues of B.
(iii) *Let c be an isolated eigenvalue of $A\otimes\mathbb{1} + \mathbb{1}\otimes B$. There is a p, $1 \le p \le d$, s.t. for $i = 1,\dots,p$ we have $c = a_i + b_i$, where the a_i are isolated eigenvalues of A and the b_i are eigenvalues of B. The (Riesz) projection onto c is $\sum_{j=1}^p \chi_{A=a_j}\otimes\chi_{B=b_j}$, where $\chi_{A=a}$ and $\chi_{B=b}$ are the (Riesz) projections onto a and b, respectively.*

We prove part 3 of Theorem 6.1 using Lemma 6.4 and refer to the end of this section for a proof of the lemma. We approximate the operator Λ_e by a family of operators $\Lambda_e^{(\eta)}$, satisfying $\|\Lambda_e - \Lambda_e^{(\eta)}\| \leq \eta$, where $\eta > 0$ is arbitrarily small, and where $\Lambda_e^{(\eta)}$ has semisimple spectrum, with a simple eigenvalue at λ_e, and with $\mathrm{Im}\left(\sigma(\Lambda_e^{(\eta)})\backslash\{\lambda_e\}\right) \geq \mathrm{Im}\lambda_e + \delta_e$. A possible realization of $\Lambda_e^{(\eta)}$ is as follows. Let $\Lambda_e = \sum_j (D_j + N_j)$ be the Jordan decomposition of Λ_e, i.e., $D_j = \ell_j \mathbb{1}$ (here the ℓ_j are the eigenvalues of Λ_e), $N_j^{m_j} = 0$. Define

$$\Lambda_e^{(\eta)} := \sum_j \left(D_j^{(\eta)} + N_j\right), \tag{6.25}$$

where (for ℓ_j non-semisimple) $D_j^{(\eta)} := \mathrm{diag}(\ell_j, \ell_{j,1}(\eta), \ldots, \ell_{j,m_j-1}(\eta))$, and where the $\ell_{j,k}(\eta)$ are arbitrary distinct complex numbers with imaginary part $\geq \mathrm{Im}\lambda_e + \delta_e$, satisfying $|\ell_j - \ell_{j,k}(\eta)| \leq \eta$.

Choosing $A = L_{r\theta}$, $B = g^2\Lambda_e^{(\eta)}$, we see from Lemma 6.4 (i), (iii) that the operator $L_{r\theta} + g^2\Lambda_e^{(\eta)}$ has a simple eigenvalue at $g^2\lambda_e$ and the rest of the spectrum is located in $\{z \in \mathbb{C} \,|\, \mathrm{Im}\, z \geq g^2\mathrm{Im}\lambda_e + \min(g^2\delta_e, \tau')\}$.

We use relation (6.23) to investigate the spectrum of K_θ inside S_e. The error term in (6.23) satisfies $O(\epsilon(g,\rho_0)) = O(|g|^{2+\alpha})$. From (6.24) (with $n = 1$) and an elementary Neumann series estimate it follows that the spectrum of $L_{r\theta} + g^2\Lambda_e + O(\epsilon(g,\rho_0))$ lies in a neighbourhood of order $O(|g|^{2+\alpha} + g^2\|\Lambda_e^{(\eta)} - \Lambda_e\|) = O(|g|^{2+\alpha})$ of the spectrum of $L_{r\theta} + g^2\Lambda_e^{(\eta)}$ (for η small enough). Moreover, since by our assumptions

$$|g|^{2+\alpha} \ll \min(g^2\delta_e, \tau') \tag{6.26}$$

(see (6.5)), one easily proves, using Riesz projections, that $L_{r\theta} + g^2\Lambda_e + O(\epsilon(g,\rho_0))$ has a simple eigenvalue z_0 in an $O(|g|^{2+\alpha})$-neighbourhood of $g^2\lambda_e$. The rest of the spectrum of $L_{r\theta} + g^2\Lambda_e + O(\epsilon(g,\rho_0))$ is located in $\{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > g^2\mathrm{Im}\lambda_e + \frac{1}{2}\min(g^2\delta_e, \tau')\}$. The result (6.6) follows from the isospectrality, (6.23).

Fix an arbitrary g', $0 < |g'| < \min[(g_0)^{1/\alpha}, g_2]$. By the Kato-Rellich Theorem, $g \mapsto z_0(g)$ is analytic in a complex neighbourhood of g'. This completes the proof of Theorem 6.1, point 3, and hence the entire proof of Theorem 6.1. ■

Proof of Lemma 6.4. By using the spectral representation of A and the normal form of the operator B, [15] I.5.3, one obtains

$$(A \otimes \mathbb{1} + \mathbb{1} \otimes B - z)^{-1} = \sum_j \sum_{n=0}^{m_j-1} (-1)^n (A + b_j - z)^{-n-1} \otimes Q_j^{(n)}, \tag{6.27}$$

where b_j are the eigenvalues of B, $Q_j^{(0)} = \chi_{B=b_j}$ is the projection (Riesz integral) onto the eigenvalue b_j, and, for $n \geq 1$, $Q_j^{(n)} = N_j^n$, with $N_j = Q_j^{(0)} N_j = N_j Q_j^{(0)}$ a nilpotent matrix, $N_j^{m_j} = 0$. Assertions (i), (ii) follow.

Let C be a circle of radius $r < \mathrm{dist}\,[c, (\sigma(A)+\sigma(B))\backslash\{c\}]$ around c. From (6.27),

$$\frac{1}{2\pi i}\oint_C dz(A\otimes \mathbb{1} + \mathbb{1}\otimes B - z)^{-1} = \frac{1}{2\pi i}\oint_C dz \sum_j \sum_{n=0}^{m_j-1} (-1)^n$$
$$\times \left[(c-z)^{-n-1}\chi_{A=a_j}\otimes Q_j^{(n)} + (A+b_j-z)^{-n-1}(1-\chi_{A=a_j})\otimes Q_j^{(n)}\right]. \qquad (6.28)$$

The first term on the r.h.s. of (6.28) contributes only for $n=0$ (for each j fixed), while the second term does not contribute at all. This concludes the proof of Lemma 6.4. ■

7 Absence of $\beta_1\beta_2$-normal stationary states

In this section we prove Theorem 3.1. Let $L = L_0 + g\pi(v) - g\pi'(v)$ be the standard (self-adjoint) Liouville operator, (2.17), and let L_θ be its U_θ-deformation. If Condition (C) is satisfied then the operator $\Lambda_0 = i\Gamma_0$ is anti-selfadjoint, with $\Gamma_0 \geq 0$ (see also Proposition 7.2 below, and [5]). Let $\gamma_0 \geq 0$ be the lowest eigenvalue of Γ_0, and let $\delta_0 > 0$ denote the distance of γ_0 to the rest of the spectum of Γ_0.

Theorem 7.1 *Assume that conditions (A), (B) and (C) are obeyed for some $0 < \beta_1, \beta_2 < \infty$, $\mu > 1/2$, and set $\alpha = (\mu - 1/2)/(\mu + 1/2)$. Assume $\gamma_0 > 0$. There is a constant $C > 0$ s.t. if $0 < |g| < Cg_3$, where*

$$g_3 := \min\left((g_0)^{1/\alpha}, (\delta_0)^{1/\alpha}, [\min(T_1, T_2)]^{\frac{1}{2+\alpha}}\right), \qquad (7.1)$$

then L_θ has a simple isolated eigenvalue $z_0(g) \in S_0$, satisfying $|z_0(g) - ig^2\gamma_0| = O(|g|^{2+\alpha})$, and the rest of the spectrum of L_θ inside S_0 lies in the region $\{z \in \mathbb{C}\,|\,\mathrm{Im}\,z \geq \frac{1}{2}\min(g^2\delta_0, \tau')\}$.

Moreover, we have $\mathrm{Im}\,z_0(g) > 0$, for all $0 < |g| < Cg_3$, except possibly for finitely many values of g in $\{C'(\gamma_0)^{1/\alpha} < |g| < Cg_3\}$, for some constant $C' > 0$.

Remark. The assertion $|z_0(g) - ig^2\gamma_0| = O(|g|^{2+\alpha})$ of the first part of Theorem 7.1 shows that $\mathrm{Im}z_0(g) > 0$ provided $|g|^\alpha \ll \gamma_0$. However, γ_0 depends on the difference of the reservoir temperatures, and it vanishes when both reservoirs are at the same temperature (see also the proof of Proposition 7.2), and thus, $|g|^\alpha \ll \gamma_0$ is a too restrictive condition. The second part of Theorem 7.1 resolves this difficulty, yielding a result for values of the coupling parameter g uniform in the temperature difference of the reservoirs.

Proof of Theorem 7.1. We apply Theorem 6.1, part 3, with $e = 0$. We have $\lambda_0 = i\gamma_0$ and $\tau' = c\min(T_1, T_2)$ for some $c > 0$, see after (3.3), so the conditions $0 < |g| < \sqrt{\rho_0}g_0$ and $0 < |g| < Cg_2$ of Theorem 6.1, part 3, reduce to $0 < |g| < Cg_3$.

We must have $\mathrm{Im}\,z_0(g) \geq 0$, for otherwise, the selfadjoint operator L would have an eigenvalue in the lower complex plane.

To complete the proof of Theorem 7.1 it remains to show that $\mathrm{Im}z_0(g) > 0$, for all $0 < |g| < g_3$, except possibly for a discrete set of values. Let J be the open interval $J =]0, g_3[$. For any $g \in J$ there exists a complex disc $B(g)$ centered at g, s.t. $z_0(g)$ is analytic for $g \in B(g)$ (see also the proof of Theorem 6.1, part 3). Suppose that there is a sequence $g_n \to g'$, s.t. $g_n, g' \in J$, and s.t. $\mathrm{Im}\, z_0(g_n) = \mathrm{Im}\, z_0(g') = 0$. By expanding $z_0(g)$ in a Taylor series around g' it is readily seen that $\mathrm{Im}\, z_0(g) = 0$ for all $g \in B(g') \cap J$. Given any closed interval $J_1 \subset J$ one easily sees that $\inf_{g\in J_1} |B(g)| > 0$, where $|B(g)|$ is the radius of the disc $B(g)$. Therefore, again by Taylor series expansion, it follows that $\mathrm{Im}\, z_0(g) = 0$ for all $g \in J_1$.

However, Theorem 6.1, part 2, shows that there is a $C' > 0$ s.t. if $0 < |g| < C'(\gamma_0)^{1/\alpha}$, then we have $\mathrm{Im}\, z_0(g) \geq \frac{1}{2} g^2 \gamma_0 > 0$. Consequently there cannot exist any accumulation point g' inside J. The only possible such accumulation point is thus $g' = 0$ or $g' = g_3$. The former is ruled out again due to Theorem 6.1, part 2. By choosing a possibly smaller value of the constant C we achieve that $\mathrm{Im}\, z_0(g) > 0$, except possibly for finitely many values of g in $\{C'(\gamma_0)^{1/\alpha} < |g| < Cg_3\}$. ■

Proposition 7.2 *Assume Conditions (B), (C). Then*

(a) $\gamma_0 \geq C \min_j(\gamma_{0j}) \frac{|\delta\beta|^2}{1+|\delta\beta|^2}$, *where* $\delta\beta = |\beta_2 - \beta_1|$, $C > 0$ *is independent of* β_1, β_2, *and where* γ_{0j} *are the constants given in (3.5).*

(b) *There is a constant* $c' > 0$ *s.t. if* $\delta\beta < c'$ *and* $\|G_1 - G_2\| < c'$ *(see (2.4)), then* $\delta_0 \geq \gamma_{01}$.

Proof. Condition (C) ensures that the level shift operator $\Lambda_0 : \mathrm{Ran}\chi_{L_p=0} \to \mathrm{Ran}\chi_{L_p=0}$ is given by the expression $\Lambda_0 := \sum_{j=1}^2 \Lambda_{0j}$ with the operators $\Lambda_{0j} = i\mathrm{Im}\Lambda_{0j} =: i\Gamma_{0j}$ given as in (6.2) with $e = 0$, and with I replaced by $I_j = \pi(v_j) - \pi'(v_j)$, see also (2.4), [5]. Moreover, we know from [5] that $\Gamma_{0j} \geq 0$, that Γ_{0j} has a simple eigenvalue at 0 with eigenvector $\Omega^p_{\beta_j}$, and that on the complement of $\mathbb{C}\Omega^p_{\beta_j}$, $\Gamma_{0j} \geq \gamma_{0j}$. By Condition (B), $\Gamma_{0j} > 0$. Consequently, for $\beta_1 \neq \beta_2$, $\Gamma_0 := \sum_{j=1}^2 \Gamma_{0j} > 0$.

(a) By analyzing the explicit form of the level shift operators, it is easy to show that $\Gamma_0 \geq C \min_j(\gamma_{0j}) \frac{|\delta\beta|^2}{1+|\delta\beta|^2}$. (In fact, $\Gamma_0 \geq C \min_j(\gamma_{0j})(\delta\beta)^2[1 - Z(\beta_1 + \beta_2)/Z(\beta_1/2 + \beta_2/2)]$, where $Z(\beta) = \mathrm{Tr}(e^{-\beta H_p})$.)

(b) We view the gap δ_0 as a function of the inverse temperatures $\beta_{1,2}$ and of the coupling operators $G_{1,2}$. Then we have $\delta_0(\beta_1 = \beta_2, G_1 = G_2) = 2\gamma_{01}$. The result follows from the continuity of the operator Λ_0 in G_j and β_j. ■

Proof of Theorem 3.1. 1. The conditions on g, $\delta\beta$, $\|G_1 - G_2\|$ in Theorem 3.1, part 1, and Proposition 7.2, (b), imply that Theorem 7.1 is applicable. The latter theorem shows that $\sigma(L_\theta) \cap \mathbb{R} \cap S_0 = \emptyset$. Hence the spectrum of non-deformed standard Liouville operator L, inside $\mathbb{R} \cap S_0$, is purely absolutely continuous. The result follows from Theorem 2.1.

2. In the same way as for 1, combine Proposition 7.2, (a), Theorem 6.1, part 2 (for $e = 0$), and Theorem 2.1.

Removing the high temperature condition $|g| \ll [\min(T_1, T_2)]^{\frac{1}{2+\alpha}}$ *in (3.8),* [17]. The origin of this condition lies in Theorem 7.1, where we use the bound

$$O(\epsilon(g, \rho_0)) = O(|g|^{2+\alpha}) \ll \min(g^2\delta_0, \tau')$$

(see also (7.1)) in order to be able to trace the simple isolated eigenvalue z_0 (c.f. (6.26), in the setting of Theorem 7.1, where $|g|^{2+\alpha}$ represents the error term $O(\epsilon(g,\rho_0))$ in (6.15)). If this condition fails then we use the Feshbach map iteratively until the error term in the equation for the final iteration (corresponding to (6.15) in the above case) is $\ll \tau' \approx \min(T_1, T_2)$. Applying Theorems V.17 and V.18 of [5] we conclude that the spectrum of the operator L_θ inside S_0, $\operatorname{Im}\theta > 0$, consists of a simple isolated eigenvalue at some point z_0 with the rest of the spectrum lying in the half space $\{z \in \mathbb{C} \;\mid \operatorname{Im} z \geq \operatorname{Im} z_0 + \tau'/2\}$. The arguments in the proof of Theorem 7.1 then show that L_θ does not have any real eigenvalues inside S_0, for all $0 < |g| < C\min((g_0)^{1/\alpha}, (\delta_0)^{1/\alpha})$, except possibly for finitely many values of g.

Acknowledgements. The authors are grateful to V. Bach, G. Elliott, J. Fröhlich, V. Jakšić and especially C.-A. Pillet for useful discussions. Part of this work was done while the first author was visiting the University of Toronto, the third author ETH Zürich, and the second and third authors ESI Vienna. During work on this paper the second author was at the University of Toronto on a DAAD fellowship. The authors are grateful to these places for hospitality.

A Proof of existence of dynamics

In this appendix we prove existence of the dynamics (2.12). Recall the definition of the operator $L^{(\ell)} := L_0 + g\pi(v)$ and of the one parameter group $\sigma^t(B) := e^{itL^{(\ell)}} B e^{-itL^{(\ell)}}$, $B \in \pi(\mathcal{A})''$.

Proposition A.1 *Assume the operators $v_n \in \mathcal{A}$ satisfy (2.13). Then the integrands on the r.h.s. of (2.12) are continuous functions, the series is absolutely convergent, the limit exists and equals*

$$\psi^t(A) = \operatorname{Tr}(\rho\sigma^t(\pi(A))) \tag{A.1}$$

and, consequently, is independent of the approximating operators.

Proof. Let $v_n \in \mathcal{A}$ be an approximating sequence for the operator v satisfying (2.13). We define the selfadjoint operators $L_n^{(\ell)} := L_0 + g\pi(v_n)$ on the dense domain $\mathcal{D}(L_0)$. Let the one parameter group $\sigma_{(n)}^t$ on $\pi(\mathcal{A})$ be given by

$$\sigma_{(n)}^t(B) := e^{itL_n^{(\ell)}} B e^{-itL_n^{(\ell)}}. \tag{A.2}$$

Set $\sigma_0^t(\pi(A)) := \pi(\alpha_0^t(A))$ and let ψ be an ω_0-normal state on $\mathcal{A}$, i.e.

$$\psi(A) = \operatorname{Tr}(\rho\pi(A)) \tag{A.3}$$

for some positive, trace class operator ρ on $\mathcal{H}$ of trace 1. Then using the definition $V_n = \pi(v_n)$ we find

$$\psi([\alpha_0^{t_m}(v_n), \cdots [\alpha_0^{t_1}(v_n), \alpha_0^t(A)] \cdots]) = \operatorname{Tr}(\rho[\sigma_0^{t_m}(V_n), \cdots [\sigma_0^{t_1}(V_n), \sigma_0^t(A)] \cdots]). \tag{A.4}$$

Clearly the r.h.s. is continuous in $t_1, \cdots, t_m$ and therefore the integrals in (2.12) are well defined and, by a standard estimate, the series on the r.h.s. of (2.12) converges absolutely. In fact, using the Araki-Dyson series

$$\begin{aligned}\sigma^t_{(n)}(\pi(A)) \quad &= \quad \sum_{m=0}^{\infty}(ig)^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m\ [\sigma_0^{t_m}(\pi(v_n)), \cdots \\ &\qquad [\sigma_0^{t_1}(\pi(v_n)), \sigma_0^t(\pi(A))] \cdots],\end{aligned} \tag{A.5}$$

one can easily see that this series is nothing but the Araki-Dyson expansion of the function $\mathrm{Tr}(\rho\sigma^t_{(n)}(\pi(A)))$. Thus we have shown that the r.h.s. of (2.12) is equal to $\lim_{n\to\infty} \mathrm{Tr}(\rho\sigma^t_{(n)}(\pi(A)))$.

Now, V_n converges to V strongly on the dense set $\mathrm{Span}\{\pi(B \otimes W_1(f_1) \otimes W_2(f_2))\Omega_0 | B \in \mathcal{B}(\mathcal{H}_0), f_{1,2} \in L^2_0\}$ as follows from (2.13) and the relation

$$\|(V_n - V)\pi(A)\Omega_0\|^2 = \omega_0(A^*(v_n^* - v^*)(v_n - v)A). \tag{A.6}$$

Hence $L_n^{(\ell)}$ converges to $L^{(\ell)}$ strongly on the same set. Since this set is a core for $L_n^{(\ell)}$ and $L^{(\ell)}$ we conclude that $L_n^{(\ell)}$ converge to $L^{(\ell)}$ in the strong resolvent sense as $n \to \infty$ ([19], Theorem VIII.25), and therefore, $e^{itL_n^{(\ell)}} \to e^{itL^{(\ell)}}$ strongly. Hence the functions $\mathrm{Tr}(\rho\sigma^t_{(n)}(\pi(A))$ converge to $\mathrm{Tr}(\rho\sigma^t(\pi(A)))$ which, in particular, shows (A.1). ■

B Positive Temperature Representation and Relative Bounds

B.1 Jakšić-Pillet Gluing

In this appendix, we represent the Hilbert space $\mathcal{H}$ in a form which is well suited for a definition of the translation transformation. This representation is due to [12].

Consider the Fock space

$$\mathcal{F} := \mathcal{F}(L^2(X \times \{1,2\})), \quad X = \mathbb{R} \times S^2 \tag{B.1.1}$$

and denote $x = (u, \sigma) \in X$. The vacuum in $\mathcal{F}$ is denoted by $\tilde{\Omega}_r$. The smeared-out creation operator $a^*(F)$, $F \in L^2(X \times \{1,2\})$ is given by

$$a^*(F) = \sum_\alpha \int_X F(x,\alpha) a^*(x,\alpha)$$

and analogously for annihilation operators. The CCR read

$$[a(x,\alpha), a^*(x',\alpha')] = \delta_{\alpha,\alpha'}\delta(x - x').$$

Following [12], we introduce the unitary map

$$U : \left[\mathcal{F}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}(L^2(\mathbb{R}^3))\right] \otimes \left[\mathcal{F}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}(L^2(\mathbb{R}^3))\right] \to \mathcal{F}(L^2(X \times \{1,2\})) \tag{B.1.2}$$

defined by

$$U\left([\Omega_{r1} \otimes \Omega_{r1}] \otimes [\Omega_{r2} \otimes \Omega_{r2}]\right) := \tilde{\Omega}_r \tag{B.1.3}$$

and

$$
\begin{aligned}
U\Big([a^*(f_1)\otimes \mathbf{1}+\mathbf{1}\otimes a^*(g_1)]\otimes\mathbf{1}\otimes\mathbf{1}& \\
+\mathbf{1}\otimes\mathbf{1}\otimes[a^*(f_2)\otimes\mathbf{1}+\mathbf{1}\otimes a^*(g_2)]\Big)U^{-1} \quad &:= \quad a^*(f\oplus g),
\end{aligned}
\tag{B.1.4}
$$

where, for $x=(u,\sigma)\in X$,

$$
[f\oplus g]\,(u,\sigma,\alpha) := \begin{cases} u\, f_\alpha(u\sigma), & u\geq 0,\\ u\, g_\alpha(-u\sigma), & u<0.\end{cases}
\tag{B.1.5}
$$

This map is extended to the Hilbert space $\mathcal{H}=\mathcal{H}^p\otimes\mathcal{F}$ in the obvious way. We keep the same notation for its extension.

The operators $L_{r1}\otimes\mathbf{1}_{r2}+\mathbf{1}_{r1}\otimes L_{r2}$ and $N_{r1}\otimes\mathbf{1}_{r2}+\mathbf{1}_{r1}\otimes N_{r2}$ are mapped under U to the (total) free field Liouvillian and number operator given by

$$
\begin{aligned}
L_f &= \mathrm{d}\Gamma(u)=\sum_\alpha\int_X a^*(x,\alpha)ua(x,\alpha),\\
N &= \mathrm{d}\Gamma(\mathbb{1})=\sum_\alpha\int_X a^*(x,\alpha)a(x,\alpha).
\end{aligned}
$$

Moreover, the interaction I in the operator K takes the form (c.f. (5.1))

$$
UIU^{-1}=a^*(F_1)+a(F_2)
\tag{B.1.6}
$$

where the $F_j\in L^2(X\times\{1,2\},\mathcal{B}(\mathcal{H}_p\otimes\mathcal{H}_p))$ are explicitly given by ($x=(u,\sigma)\in X=\mathbb{R}\times S^2$)

$$
\begin{aligned}
&F_1(u,\sigma,\alpha)= \\
&\quad\sqrt{\frac{u}{1-e^{-\beta_\alpha u}}}\;|u|^{1/2}\begin{cases} G_{\alpha 1}(u\sigma)\otimes\mathbb{1}_p-e^{-\beta_\alpha u/2}\mathbb{1}_p\otimes\overline{G_{\alpha 4}}^*(u\sigma), & u>0\\ -G^*_{\alpha 2}(-u\sigma)\otimes\mathbb{1}_p+e^{-\beta_\alpha u/2}\mathbb{1}_p\otimes\overline{G_{\alpha 3}}(-u\sigma), & u<0\end{cases}
\end{aligned}
\tag{B.1.7}
$$

$$
\begin{aligned}
&F_2(u,\sigma,\alpha)= \\
&\quad\sqrt{\frac{u}{1-e^{-\beta_\alpha u}}}\;|u|^{1/2}\begin{cases} G_{\alpha 2}(u\sigma)\otimes\mathbb{1}_p-e^{-\beta_\alpha u/2}\mathbb{1}_p\otimes\overline{G_{\alpha 3}}^*(u\sigma), & u>0\\ -G^*_{\alpha 1}(-u\sigma)\otimes\mathbb{1}_p+e^{-\beta_\alpha u/2}\mathbb{1}_p\otimes\overline{G_{\alpha 4}}(-u\sigma), & u<0\end{cases}
\end{aligned}
\tag{B.1.8}
$$

Thus the operator $\tilde{K}:=UKU^{-1}$ can be written as

$$
\tilde{K}=\tilde{L}_0+g\tilde{I}
$$

where $\tilde{I}=UIU^{-1}$ is given in (B.1.6) and $\tilde{L}_0:=UL_0U^{-1}$ is of the form

$$
\tilde{L}_0=L_p\otimes\mathbf{1}_f+\mathbf{1}_p\otimes L_f.
$$

B.2 Complex Deformation

Now we express the complex deformation operators U_θ introduced in Section 5 in the Jakšić-Pillet glued Hilbert space. For a function $F\in L^2(X\times\{1,2\})$ and $\theta=(\delta,\tau)$, $x=(u,\sigma)\in X$, define

$$
[\tilde{u}_\theta F]\,(u,\sigma,\alpha)=e^{\frac{1}{2}\delta\operatorname{sgn}(u)}F(j_\theta(u),\sigma,\alpha),
\tag{B.2.1}
$$

where

$$j_\theta(u) = e^{\delta \mathrm{sgn}(u)} u + \tau, \tag{B.2.2}$$

and sgn is the sign function, $\mathrm{sgn}(u) = 1$ if $u \geq 0$, $\mathrm{sgn}(-u) = -\mathrm{sgn}(u)$. Next, we lift the operator family $\tilde{u}_\theta$ from $L^2(X \times \{1,2\})$ to the operator family, $\tilde{U}_\theta$, on $\mathcal{H}^p \otimes \mathcal{F}(L^2(X \times \{1,2\}))$ in a standard way (cf. (5.9)). The family $\tilde{U}_\theta$ is related to the family U_θ introduced in Section 5 as

$$U_\theta = U \tilde{U}_\theta U^{-1}.$$

The operator $\tilde{K}$ becomes after spectral deformation

$$\tilde{K}_\theta := \tilde{U}_\theta K \tilde{U}_\theta^{-1} = \tilde{L}_{0,\theta} + g \tilde{I}_\theta \tag{B.2.3}$$

where

$$\begin{aligned} \tilde{L}_{0,\theta} &= L_p + \cosh\delta\ L_f + \sinh\delta\ \Lambda_f + \tau N, && \text{(B.2.4)} \\ \Lambda &= \mathrm{d}\Gamma(|u|) = \sum_\alpha \int_X a^*(x,\alpha)|u| a(x,\alpha), && \\ \tilde{I}_\theta &= a^*(F_{1,\theta}) + a(F_{2,\theta}) \quad \text{with} \quad F_{j,\theta} = \tilde{u}_\theta F_j. && \text{(B.2.5)} \end{aligned}$$

This spectral deformation can be translated to the original space $\mathcal{H}$ as

$$K_\theta := U^{-1} \tilde{K}_\theta U^{-1} = L_{0,\theta} + g I_\theta \tag{B.2.6}$$

where $L_{0,\theta} := U^{-1} \tilde{L}_{0,\theta} U$ is given by (5.22) and

$$I_\theta = U^{-1} \tilde{I}_\theta U. \tag{B.2.7}$$

B.3 Relative Bounds

We prove the bounds which imply Lemma 5.3. We will from now on fix $\delta = i\delta'$ and $\tau = i\tau'$ for some $\delta', \tau' > 0$. Recall that the operator M_θ is given by

$$M_\theta := \mathrm{Im}\, \tilde{L}_{0,\theta} = \sin\delta' \Lambda + \tau' N \geq 0.$$

Proposition B.1 *For a function $F : X \times \{1,2\} \to \mathcal{B}(\mathcal{H}_p \otimes \mathcal{H}_p)$ set $F_\theta(x,\alpha) = e^{\mathrm{sgn(u)}\delta/2} F(j_\theta(u), \sigma, \alpha)$, where $x = (u,\sigma)$ and $j_\theta(u)$ is given in (B.2.2), with $\theta = (i\delta', i\tau')$, $\delta', \tau' > 0$. Suppose that the function F satisfies*

$$||F||_\rho := \left(\sum_\alpha \int_{\sin(\delta')|u| + \tau' \leq \rho} \frac{||F_\theta(x,\alpha)||^2}{|j_\theta(u)|} du d\sigma \right)^{1/2} < \infty \tag{B.3.1}$$

for some $0 < \rho \leq \infty$. Then we have the bounds

$$\begin{aligned} ||a(F_\theta) M_\theta^{-1/2}|| &\leq \frac{1}{\sqrt{\sin\delta'}} ||F||_\infty, && \text{(B.3.2)} \\ ||a^*(F_\theta) M_\theta^{-1/2}|| &\leq ||F_\theta||_{L^2} + \frac{1}{\sqrt{\sin\delta'}} ||F||_\infty && \text{(B.3.3)} \\ ||a(F_\theta) \chi_{M_\theta \leq \rho}|| &\leq \frac{1}{\sqrt{\sin\delta'}} \rho^{1/2} ||F||_\rho, && \text{(B.3.4)} \\ \left|\langle \psi, a^\#(F_\theta)\psi \rangle\right| &\leq \frac{1}{\sqrt{\sin\delta'}} ||F||_\infty ||\psi||\, ||M_\theta^{1/2}\psi||, && \text{(B.3.5)} \end{aligned}$$

for all $\psi \in \mathcal{D}(M_\theta^{1/2})$, and where $a^\#$ denotes either a or a^*. In particular, (B.3.2) – (B.3.5) (together with (B.3.13) below) imply Lemma 5.3.

Proof. Note that (B.3.3) follows from Eqn (B.3.2) and the relation

$$\|a^*(G)\psi\|^2 \leq \|G\|^2 \ \|\psi\|^2 + \|a(G)\psi\|^2. \tag{B.3.6}$$

We prove only (B.3.4). Bound (B.3.2) is obtained in a similar way (see [3], Lemma I.6) and bound (B.3.5) follows from (B.3.2). Set for short $P_\rho = \chi_{M_\theta \leq \rho}$. We have for any ψ

$$\|a(F_\theta)P_\rho\psi\|^2 \leq \left[\sum_\alpha \int_X \|F_\theta(x,\alpha)\| \ \|a(x,\alpha)P_\rho\psi\|\right]^2. \tag{B.3.7}$$

Using the pull-through formula

$$a(x,\alpha)M_\theta = (M_\theta + \sin\delta'|u| + \tau')a(x,\alpha),$$

where $x = (u,\sigma)$, we obtain

$$a(x,\alpha)P_\rho = \chi_{M_\theta + \sin\delta'|u| + \tau' \leq \rho} \ a(x,\alpha).$$

Because $M_\theta \geq 0$, the integration in (B.3.7) is restricted to the domain

$$X_\rho := \{u \in \mathbb{R} \mid \ \sin\delta'|u| + \tau' \leq \rho\} \times S^2.$$

Using Hölder's inequality, we obtain from (B.3.7)

$$\|a(F_\theta)P_\rho\psi\|^2 \leq \left(\sum_\alpha \int_{X_\rho} \frac{\|F_\theta(x,\alpha)\|^2}{|j_\theta(u)|}\right) \left\langle P_\rho\psi, \sum_\alpha \int_{X_\rho} a^*(x,\alpha)|j_\theta(u)|a(x,\alpha)P_\rho\psi \right\rangle.$$

Since $|j_\theta(u)| \leq |u| + \tau'$, it is clear that the scalar product on the right side is bounded from above by

$$\langle P_\rho\psi, (\Lambda + \tau' N)P_\rho\psi\rangle \leq \frac{\rho}{\sin\delta'}\|P_\rho\psi\|^2.$$

Then, (B.3.4) follows from definition (B.3.1). ∎

Observe that we have, for any $\nu > 1/2$,

$$\|F\|_\rho \leq \left(\frac{2\rho}{\sin\delta'}\right)^{\nu - 1/2} |||F|||_\nu, \tag{B.3.8}$$

and

$$\|F\|_\infty = |||F|||_{1/2}, \tag{B.3.9}$$

where we define

$$|||F|||_\nu := \left(\sum_\alpha \int_{\mathbb{R}\times S^2} \frac{\|F_\theta(x,\alpha)\|^2}{|j_\theta(u)|^{2\nu}} du d\sigma\right)^{1/2}. \tag{B.3.10}$$

A bound on the norms $|||F_{1,2}|||_\nu^2$, where $F_{1,2}$ are given in (B.1.7), (B.1.8), in terms of $\|G_{1,2}\|_{\mu,\theta}$, (5.8), is obtained as follows. First one sees that for $z = j_\theta(u) = e^{\delta \mathrm{sgn}(u)}u + \tau$, $|\mathrm{Im}\,\delta| < \delta_0$, $|\tau| < \tau_0$, $\tau_0/\cos\delta_0 < 2\pi/\beta$ (where $\beta = \max(\beta_1, \beta_2)$), one has

$$\frac{|z|}{|e^{\beta' z} - 1|} \leq 2|z| + \frac{C}{\beta'}, \tag{B.3.11}$$

for all $\beta' \leq \beta$, and where C is a constant which depends only on $\tan\delta_0$. Using this bound in (B.1.7) gives

$$\begin{aligned} &\|F_1(j_\theta(u), \sigma, \alpha)\|^2 \\ &\quad \leq C(1 + 1/\beta_\alpha) \max_{k=1,\ldots,4} \left\| \gamma \left[\sqrt{|u|+1}\; G_{\alpha k} \right] (j_\theta(u), \sigma) \right\|^2, \end{aligned} \tag{B.3.12}$$

where we recall that γ was defined in (5.5). Estimate (B.3.12) implies

$$\begin{aligned} |||F_1|||_\nu^2 &\leq C \sum_{j=1,2} \sum_{k=1,3} (1 + 1/\beta_j) \int_{\mathbb{R}\times S^2} du d\sigma \left\| \gamma_\theta \left[\frac{\sqrt{|u|+1}}{|u|^\nu} G_{jk} \right] (u, \sigma) \right\|^2 \\ &\leq C \sum_{j=1,2} (1 + 1/\beta_j) \|G_j\|_{\nu,\theta}^2, \end{aligned} \tag{B.3.13}$$

where $\|G_j\|_{\nu,\theta}$ is given in (5.8). The same bound is obtained for $|||F_2|||_\nu^2$.

C Level Shift Operator

We prove estimate (6.22). We pass to the Jakšić-Pillet glued Hilbert space representation (see Appendices B.1 and B.2) and omit the tilde over the operators. In the definition

$$\Lambda_{e\rho\theta} := P_{e\rho} I_\theta \overline{P}_{e\rho} L_{0\theta}^{-1} \overline{P}_{e\rho} I_\theta P_{e\rho} \tag{C.1}$$

we substitute expression (B.2.5) for the operator I_θ and, using the pull-through formulae, pull the annihilation operators to the right and the creation operators to the left until they stand next to the operators $P_{e\rho}$. As a result we obtain the decomposition

$$\Lambda_{e\rho\theta} = \Lambda_{e\rho\theta}^{\text{contracted}} + R\ , \tag{C.2}$$

where $\Lambda_{e\rho\theta}^{\text{contracted}} := P_{e\rho} \left\langle I_\theta \overline{P}_{e\rho} L_{0\theta}^{-1} I_\theta \right\rangle P_{e\rho}$ is the contracted term and the term R consists of remaining terms. Here, we use the notation

$$\langle I_\theta f(\Lambda, L_r) I_\theta \rangle = \langle I_\theta f(\Lambda + \lambda, L_r + \ell) I_\theta \rangle_\Omega \,|_{\lambda=\Lambda, \ell=L_r},$$

where $\langle\cdot\rangle_\Omega = \mathrm{Tr}_{\mathcal{F}}(\cdot P_\Omega)$, P_Ω is the projection onto $\mathbb{C}\Omega$ (the vacuum sector in $\mathcal{F}$), and where f is a function of two variables.

The remaining terms, R, are estimated using (B.3.4) and (B.3.7) and $\|P_{e\rho} L_{0\theta}^{-1} P_{e\rho}\| \leq c\rho^{-1}$. For instance one of the terms appearing in R is of the form

$$P_{e\rho} a^*(F_{i\theta}) \overline{P}_{e\rho} L_{0\theta}^{-1} \overline{P}_{e\rho} a(F_{j\theta}) P_{e\rho} \tag{C.3}$$

which is bounded by (see (B.3.4) and (B.3.7))

$$
\begin{aligned}
&\|P_{e\rho} a^*(F_{i\theta})\| \, \|\overline{P}_{e\rho} L_{0\theta}^{-1} \overline{P}_{e\rho}\| \, \|a(F_{j\theta}) P_{e\rho}\| \\
&\leq \left(\frac{c}{\rho \sin \delta'}\right)^{1/2} \|F_i\|_p \; c\rho^{-1} \left(\frac{c}{\rho \sin \delta'}\right)^{1/2} \|F_j\|_\rho \\
&\leq \left(\frac{c}{\rho \sin \delta'}\right)^{1/2} \left(\frac{c}{\sin \delta'}\right)^{\mu - 1/2} \|F_i\|_\mu c\rho^{-1} \left(\frac{c}{\rho \sin \delta'}\right)^{1/2} \left(\frac{c}{\sin \delta'}\right)^{\mu - 1/2} \|F_j\|_\mu .
\end{aligned}
$$

Similarly, we estimate other terms in R to obtain $R = O(g^{2\mu - 1})$. Now, using $\overline{P}_{e\rho} = \mathbf{1} - P_{e\rho}$ we write the operator $\Lambda_{e\rho\theta}^{\text{contracted}}$ as

$$
\Lambda_{e\rho\theta}^{\text{contracted}} = \Lambda'_{e\rho\theta} + \Lambda''_{e\rho\theta} \tag{C.4}
$$

where $\Lambda'_{e\rho\theta} := P_{e\rho} \left\langle I_\theta L_{0\theta}^{-1} I_\theta \right\rangle P_{e\rho}$ and

$$
\Lambda''_{e\rho\theta} = -P_{e\rho} \left\langle I_\theta P_{e\rho} L_{0\theta}^{-1} I_\theta \right\rangle P_{e\rho} \ . \tag{C.5}
$$

Note that both terms on the r.h.s. of (C.4) are well-defined since $I_\theta(\psi \otimes \Omega)$ is orthogonal to $\mathrm{Null}(L_{0\theta})$, for all $\psi \in \mathcal{H}_p \otimes \mathcal{H}_p$. A simple computation shows that $\Lambda''_{e\rho\theta}$ is equal to $P_{e\rho}$ times an integral over $\omega \leq \rho$ of the trace of the product of two coupling functions $F_{j\theta}$ divided by a function of the form $\pm \cosh \delta\omega + \sinh \delta\omega + \tau$ which is bounded below by $c \sin \delta' \omega$. Hence that integral is bounded by $c\rho^{2\mu-1} \left(\sum_j \|G_j\|_{\mu,\theta}\right)^2$ and, consequently, $\Lambda''_{e\rho\theta} = O(\rho^{2\mu-1})$

A simple consideration shows that $\left\langle I_\theta L_{0\theta}^{-1} I_\theta \right\rangle$ is independent of θ, and $\Lambda'_{e\rho\theta} - \Lambda_e P_{e\rho}$ is of order $O(\rho^{2\mu-1})$ as well. Hence,

$$
\Lambda_{e\rho\theta} = \Lambda_e P_{e\rho} + O(\rho^{2\mu-1}) \ . \tag{C.6}
$$

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