Schrödinger operator with a junction of two 1-dimensional periodic potentials

Evgeny Korotyaev *

July 14, 2005

Abstract

The spectral properties of the Schrödinger operator $T_t y = -y'' + q_t y$ in $L^2(\mathbb{R})$ are studied, with a potential $q_t(x) = p_1(x), x < 0$, and $q_t(x) = p(x+t), x > 0$, where p_1, p are periodic potentials and $t \in \mathbb{R}$ is a parameter of dislocation. Under some conditions there exist simultaneously gaps in the continuous spectrum of T_0 and eigenvalues in these gaps. The main goal of this paper is to study the discrete spectrum and the resonances of T_t. The following results are obtained: i) In any gap of T_t there exist $0, 1$ or 2 eigenvalues. Potentials with 0,1 or 2 eigenvalues in the gap are constructed. ii) The dislocation, i.e. the case $p_1 = p$ is studied. If $t \to 0$, then in any gap in the spectrum there exist both eigenvalues ($\leqslant 2$) and resonances ($\leqslant 2$) of T_t which belong to a gap on the second sheet and their asymptotics as $t \to 0$ are determined. iii) The eigenvalues of the half-solid, i.e. $p_1 =$ constant, are also studied. iv) We prove that for any even 1-periodic potential p and any sequences $\{d_n\}_1^\infty$, where $d_n = 1$ or $d_n = 0$ there exists a unique even 1-periodic potential p_1 with the same gaps and d_n eigenvalues of T_0 in the n-th gap for each $n \geqslant 1$.

1 Introduction

We consider the Schrödinger operator $T_t y = -y'' + q_t(x)y$ acting on $L^2(\mathbb{R})$, where the real potential q_t is given by

$$q_t(x) = \begin{cases} p_1(x) & \text{if} \quad x < 0, \quad p_1 \in L^1(\tau\mathbb{T}) \\ p(x+t) & \text{if} \quad x > 0, \quad p \in L^1(\mathbb{T}) \end{cases}, \quad t \in [0,1], \tag{1.1}$$

where $\tau > 0$ is the period of p_1 and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We call such a potential q_t biperiodic. If $p_1 = p$ and $t = 0$, then we obtain the well known periodic case, that is the Hill operator $H = -\frac{d^2}{dx^2} + p$ in $L^2(\mathbb{R})$. It is well known (see [T]) that the spectrum of H is absolutely continuous and consists of intervals $\sigma_n = [\alpha_{n-1}^+, \alpha_n^-]$, where $\alpha_{n-1}^+ < \alpha_n^- \leqslant \alpha_n^+,\ n \geqslant 1$. These

*Institut für Mathematik, Humboldt Universität zu Berlin, Rudower Chaussee 25, 12489, Berlin, Germany, e-mail: evgeny@math.hu-berlin.de

intervals are separated by gaps $\gamma_n(H) = (\alpha_n^-, \alpha_n^+), n \geqslant 1$. We set $\gamma_0(H) = (-\infty, \alpha_0^+)$ and $\gamma(H) = \cup\gamma_n(H)$. If a gap degenerates, i.e. $\gamma_n = \emptyset, n \geqslant 1$, then the corresponding segments σ_n, σ_{n+1} merge. Define the Hill operator $H(t) = -\frac{d^2}{dx^2} + p(x+t)$ in $L^2(\mathbb{R})$. It is clear that $\sigma(H) = \sigma(H(t))$ for any $t \in [0,1]$. Introduce the Hill operator $H_m = -\frac{d^2}{dx^2} + p_m(x)$ in $L^2(\mathbb{R}), m = 1,2$ where here and below $p_2 \equiv p$. In the biperiodic case (see Theorem 2.1) we prove that the spectrum of each $T_t, t \in \mathbb{R}$ has the following form

$$\sigma(T_t) = \sigma_{ac}(T_t) \cup \sigma_d(T_t), \qquad \sigma_{ac}(T_t) = \sigma(H_1) \cup \sigma(H_2), \qquad \sigma_d(T_t) \subset \gamma(H_1) \cap \gamma(H_2). \qquad (1.2)$$

Note that (1.2) implies $\sigma_{sc}(T_t) = \emptyset$ and there are no embedded eigenvalues. The absolutely continuous spectrum $\sigma_{ac}(T_t)$ consists of intervals $\sigma_n(T_t), n \geqslant 1$. These intervals are separated by the gaps $\gamma_n(T_t), n \geqslant 1$. In general, there exist eigenvalues in these gaps and we have (see Theorem 2.1)

$$\#(T_t, \gamma_n(T_t)) \leqslant 2, \quad \text{for each} \quad n \geqslant 0, \qquad (1.3)$$

where $\#(T_t, \omega)$ is the number of eigenvalues of T_t on an interval ω. The basic goal of this paper is to study eigenvalues in these gaps.

In this paper we study in more detail two cases of biperiodic potentials. Firstly, we consider the case of dislocation, that is the operator $T_t^{di} = -\frac{d^2}{dx^2} + p_{(t)}(x)$ in $L^2(\mathbb{R})$, with the potential $p_{(t)}(x) = \chi_-(x)p(x) + \chi_+(x)p(x+t), t \in \mathbb{R}$, here and below $\chi_\pm(x) = 1, \pm x \geqslant 0$ and $\chi_\pm(x) = 0, \pm x < 0$ and the potential p as above. Changing t we get different potentials $p_{(t)}(\cdot)$. It can be shown that the eigenvalues of T_t^{di} are periodic in t (see (1.4)). Actually, they are 2-periodic, not 1-periodic. If $t = 0$, then $p_{(0)}(x) = p(x)$ is periodic and $T_0^{di} = H$ and eigenvalues are absent. But if $t \neq 0$ what are we able to say about the eigenvalues in the gaps $\gamma_n, n \geqslant 0$, of the operator T_t^{di}?

Secondly, we consider the half-solid, i. e. the operator $T_t^s y = -y'' + q_t^s(x)y$ in $L^2(\mathbb{R})$, where $q_t^s(x) = s\chi_-(x) + \chi_+(x)p(x+t)$ and $t, s \in \mathbb{R}$. Changing constants t, s we get different potentials $q_t^s(x)$. The eigenvalues of T_t^s depend on t, s. If the first gap of H is open and $s > \alpha_1^-$, then there exists a gap in the spectrum of T_t^s. What are we able to say on the eigenvalues in these gaps $\gamma_n(T_t^s), n \geqslant 0$?

We describe the main results which are proved in the present paper:

i) relations (1.2-3),

ii) the description of the possible coexistence of eigenvalues and resonances for different classes of potentials,

iii) we prove that for any fixed numbers $m (m = 0,1,2)$ and $N \geqslant 1$ there exists a biperiodic potential such that $\#(T_0, \gamma_n(T_0)) = m$ for all $1 \leqslant n \leqslant N$,

iv) the discrete spectrum of T_t, T_t^{di}, T_t^s is studied. For example, if t is small then there exist eigenvalues of T_t^d in the gaps $\gamma_n, n \geqslant 0$, and their asymptotics are determined as $t \to 0$.

v) The analytic continuation of $(T_0 - \lambda)^{-1}, \operatorname{Im}\lambda > 0$, into the second sheet of the energy surface is obtained and the resonances are studied. For the dislocation, $T_t^{di} = T_t$, we have: λ_1 is an eigenvalue of T_t, iff the number λ_1 considered as a point on the second sheet is a resonance of the operator $\widetilde{T}_t$ given by $\widetilde{T}_t y = -y'' + \widetilde{q}_t(x)y$, where $\widetilde{q}_t(x) = \chi_-(x)p_2(x+t) + \chi_+(x)p_1(x)$.

We briefly describe the proof. Using results of self-adjoint extensions [Kr] we obtain (1.2-3). An eigenvalue of T_t is a zero of the corresponding Wronskian. Using techniques from inverse spectral theory [L], [Tr], [K2] we study how such a zero depends on t. Then we prove (1.3) with $\#(T, \gamma_n(T)) = m, m = 0, 1$, for all $n \geqslant 1$. In order to obtain the case m=2 we use additional results from [GT] (see also [KK1]) on the inverse problem for the Hill operator. In order to determine the asymptotics of the eigenvalues as $t \to 0$ we study the corresponding Wronskian. Here we use the implicit function Theorem, various properties of the fundamental solutions and the quasimomentum.

Tamm [Ta] was the first, who considered the Schrödinger operator with biperiodic potentials (for the Kronig-Penny model) and proved the existence of eigenvalues (the famous surface states). The spectral properties of the Schrödinger operator with biperiodic potentials were studied in various papers (see e.g. [A] [A1], [DS]). The spectral problem with biperiodic potentials arises in non-linear equations. For example, Bikbaev and Sharipov [BS] considered the KdV equation with biperiodic initial data. In this paper [BS] it is assumed that eigenvalues are absent. Anoshchenko [A] studied the inverse problem for the Schrödinger operator with a biperiodic potential plus a decreasing one. But the author does not study the discrete spectrum of the Schrödinger operator with a biperiodic potential. The author [K3] obtained the following result concerning the motion of eigenvalues for the case of the dislocation: in each gap $\gamma_n \neq \emptyset, n \geqslant 1$ there exist two unique "states" (an eigenvalue and a resonance) $\lambda_n^\pm(t)$ of the dislocation operator, such that $\lambda_n^\pm(0) = \alpha_n^\pm$ and the point $\lambda_n^\pm(t)$ runs clockwise around the gap γ_n changing the energy sheet whenever it hits $\alpha_n^\pm$, making $n/2$ complete revolutions in unit time. On the first sheet $\lambda_n^\pm(t)$ is an eigenvalue and on the second sheet $\lambda_n^\pm(t)$ is a resonance. Moreover, the following identities are fulfilled:

$$\lambda_{2n}^\pm(t+1) = \lambda_{2n}^\pm(t), \quad \textit{and} \quad \lambda_{2n+1}^\pm(t+1) = -\lambda_{2n+1}^\pm(t) + \alpha_{2n+1}^- + \alpha_{2n+1}^+, \quad \textit{any } n \geqslant 0, t \in \mathbb{R}. \tag{1.4}$$

We think that the results of the present paper are needed to study the KdV equation, the inverse problem and the spectral properties of $T_t + V(x)$, where $V(x) \to 0$, as $x \to \pm\infty$ or $V = \varepsilon x$ (Stark effect), and junctions of two d-dimensional periodic potentials, $d > 1$.

2 Main results

We recall some properties of the Hill operator $H = -\frac{d^2}{dx^2} + p$ acting in $L^2(\mathbb{R})$, where the potential $p \in L^1(\mathbb{T})$ is real. The spectrum of H is absolutely continuous and consists of intervals $\sigma_n = [\alpha_{n-1}^+, \alpha_n^-]$, where $\alpha_{n-1}^+ < \alpha_n^- \leqslant \alpha_n^+, n \geqslant 1$. We set $\alpha_0^+ = 0$. These intervals are separated by gaps $\gamma_n(H) = (\alpha_n^-, \alpha_n^+), n \geqslant 1$. Let $\varphi(x, z), \vartheta(x, z)$ be the solutions of the equation

$$-y'' + py = \lambda y, \quad \lambda \in \mathbb{C}, \tag{2.1}$$

satisfying $\varphi_x(0, \lambda) = \vartheta(0, \lambda) = 1$, and $\varphi(0, \lambda) = \vartheta_x(0, \lambda) = 0$. We define the Lyapunov function $\Delta(\lambda) = \frac{1}{2}(\varphi_x(1, \lambda) + \vartheta(1, \lambda))$ and the function $a(\lambda) = \frac{1}{2}(\varphi_x(1, \lambda) - \vartheta(1, \lambda))$. Note that $\Delta(\alpha_n^\pm) = (-1)^n$, $n \geqslant 1$. The sequence $\alpha_0^+ < \alpha_1^- \leqslant \alpha_1^+ < \ldots$ is the spectrum of Eq. (2.1) with 2-periodic boundary conditions, that is $y(x + 2) = y(x), x \in \mathbb{R}$. Let $\Psi(x, \alpha_n^\pm)$ be the corresponding real normalized eigenfunctions, i.e. $\int_0^1 \Psi(x, \alpha_n^\pm)^2 dx = 1$. If $\alpha_n^- = \alpha_n^+$, then α_n^- is a double eigenvalue. The lowest eigenvalue α_0^+ is simple, $\Delta(\alpha_0^+) = 1$, and the corresponding eigenfunction has period 1. The eigenfunctions corresponding to $\alpha_n^\pm$ have period 1 when n is even and they are anti-periodic, $\Psi(x+1,\alpha_n^\pm)=-\Psi(x,\alpha_n^\pm),\ x\in\mathbb{R}$, when n is odd. Let $\mu_n(p), n\geqslant 1$, be the Dirichlet spectrum of (2.1) with the boundary condition $y(0)=y(1)=0$. Let $\nu_n(p), n\geqslant 0$, be the Neumann spectrum of (2.1) with the boundary condition $y'(0)=y'(1)=0$. It is well known that $\mu_n,\nu_n\in[\alpha_n^-,\alpha_n^+]$ and $\nu_0\leqslant\alpha_0^+$. The energy Riemann surface $\Lambda_E(H)$ for the Hill operator consists of 2 sheets $\Lambda^{(1)}=\mathbb{C}\setminus\cup\sigma_n$ and $\Lambda^{(2)}$. Each sheet is a copy of the complex plane slit along $\sigma(H)$. The first sheet $\Lambda^{(1)}$ is glued to the sheet $\Lambda^{(2)}$ by identifying "crosswise" all slits in $\sigma(H)$ (if $p\equiv 0$, then we have the Riemann surface of $\sqrt{\lambda}$). We define the quasimomentum $k(\lambda)=\arccos\Delta(\lambda), \lambda\in\mathbb{C}_+$, which is fixed by $k(0)=0$. The function $\sin k(\lambda)$ is analytic on the Riemann surface $\Lambda_E(H)$ (see [M]). The function $k(\lambda)$ has an analytic extension from $\mathbb{C}_+$ into the Riemann surface $\Lambda_E(H)$ without slits $\gamma_n(H), n\geqslant 1$. Recall that if a gap $|\gamma_n|=0$, then $k(\lambda)$ is analytic at $\lambda=\alpha_n^\pm$; if $|\gamma_n|\neq 0$, then $k(\lambda)$ has branch points $\alpha_n^\pm$ and

$$k(\lambda)=\pi n+i\sqrt{-2M_n^\pm(\lambda-\alpha_n^\pm)}(1+\mathrm{o}(1)),\quad \text{as}\quad \lambda\to\alpha_n^\pm,\ \ \lambda\in\gamma_n(H),$$

where $\pm M_n^\pm>0$ is the effective mass; here and below $\sqrt{z}>0, z>0$ (see [KK2]). We introduce the Bloch functions $\psi_\pm(x,\lambda)$ and the Weyl functions $m^\pm(\lambda)$ by

$$\psi_\pm(x,\lambda)=\vartheta(x,\lambda)+m_\pm(\lambda)\varphi(x,\lambda),\qquad m^\pm(\lambda)=\frac{a(\lambda)\pm i\sin k(\lambda)}{\varphi(1,\lambda)}.$$

We define resonances of T_t. There are different kinds of the resonances. Due to (1.2) $\sigma_{ac}(T_t)=\sigma_{ac}(T_0)$ does not depend on $t\in\mathbb{R}$. It has the decomposition

$$\sigma_{ac}(T_t)=\cup_2^4\sigma^{(n)},\quad \sigma^{(2)}=\sigma(H_1)\cap\sigma(H),\quad \sigma^{(3)}=\sigma(H_1)\setminus\sigma(H),\quad \sigma^{(4)}=\sigma(H)\setminus\sigma(H_1). \tag{2.2}$$

In order to describe the resonances we need the energy Riemann surface $\Lambda_E(T_0)$ for the operator T_0. There are 2 cases. In the first case $\sigma_{ac}(T_0)\neq\sigma^{(2)}$, the Riemann surface $\Lambda_E(T_0)$ consists of 4 sheets $\Lambda_0^{(1)},\Lambda_0^{(2)},\Lambda_0^{(3)},\Lambda_0^{(4)}$. Each sheet is a copy of the complex plane slit along $\sigma_{ac}(T_0)$. The first sheet $\Lambda_0^{(1)}$ is glued to the sheet $\Lambda_0^{(n)}$ by identifying the sides of all slits in $\sigma^{(n)}, n=2,3,4$. Similarly, the second sheet $\Lambda_0^{(2)}$ is glued to the sheets $\Lambda_0^{(3)}$ and $\Lambda_0^{(4)}$ by identifying "crosswise" the sides of all slits in $\sigma^{(4)}$ and $\sigma^{(3)}$ respectively. The third sheet $\Lambda_0^{(3)}$ is glued to the sheet $\Lambda_0^{(4)}$ along slits in $\sigma^{(2)}$. For each complex number $\lambda\in\Lambda_0^{(1)}$ the number $\lambda^{(n)}, n=1,2,3,4$ will denote the corresponding point on the sheet $\Lambda_0^{(n)}$. In the second case $\sigma_{ac}(T_0)=\sigma^{(2)}$ the energy Riemann surface $\Lambda_E(T_0)$ consists of 2 sheets $\Lambda_0^{(1)},\Lambda_0^{(2)}$ and coincides with the Riemann surface $\Lambda_E(H)$ for the Hill operator H.

Let $\mathcal{B}$ denote the class of bounded operators in $L^2(\mathbb{R})$. For each $\eta\in C_0^\infty(\mathbb{R})$ we introduce the operator-valued function $A:\Lambda_0^{(1)}\to\mathcal{B}$ by $A(\lambda)=\eta R(\lambda)\eta$, where $R(\lambda)=(T_t-\lambda)^{-1}$. Assume that $A(\lambda),\lambda\in\Lambda_0^{(1)}$, has a meromorphic continuation across the set $\sigma^{(m)}$ on "the sheet" $\Lambda_0^{(m)}, m=2,3,4$. Suppose that $A(\lambda)$ has a pole $\lambda_r\in\Lambda_0^{(m)}$ (this does not depend on the choice of η). We call λ_r a resonance of T_t. Let $\#^{(m)}(T_t,\omega)$ be the number of resonances of T_t in an interval $\omega\subset\Lambda_0^{(m)}, m=2,3,4$. We formulate our first result.

Theorem 2.1. *Let $q_t, t\in\mathbb{R}$, be a biperiodic potential in the sense of (1.1). Then*

i) Relations (1.2-3) are fulfilled.
ii) For each $\eta \in C_0^\infty(\mathbb{R})$ the operator-valued function $A : \Lambda_0^{(1)} \to \mathcal{B}$ has a meromorphic continuation into the Riemann surface $\Lambda_E(T_0)$ described above. The number $\lambda_e \in \Lambda_0^{(1)}$ is an eigenvalue of the operator T_t iff the same number considered as an element $\lambda_e^{(2)} \in \Lambda_0^{(2)}$ on the second sheet is a resonance of $\widetilde{T}_t$ given by $\widetilde{T}_t y = -y'' + \widetilde{q}_t(x)y$, where $\widetilde{q}_t = \chi_- p_2(\cdot + t) + \chi_+ p_1$.
iii) The Riemann surface $\Lambda_E(T_t^{di})$ for the dislocation operator coincides with the Riemann surface $\Lambda_E(H)$ for the Hill operator. If $p \not\equiv s$, then the Riemann surface $\Lambda_E(T_t^s)$ for the half-solid case consists of 4 sheets.
Remark. 1) To say anything about all complex resonances on the level of generality of Theorem 2.1 is highly non-trivial. It is not known, for precisely which subclass of potentials there exist non real resonances. Theorem 2.1, however, gives control on all resonances on the second sheet $\Lambda_0^{(2)}$ (but not on the third and fourth), since they are eigenvalues of the self-adjoint operator $\widetilde{T}_t$. In particular, they are real and belong to $\cup\gamma_n(T_t) \subset \Lambda_0^{(2)}$. It is this result which makes the operator $\widetilde{T}_t$ important in Theorem 2.1 (and later on in Lemma 4.1). 2) In the case of the dislocation operator ($p_1 = p_2$) the energy Riemann surface $\Lambda_E(T_t^d)$ consists of only 2 sheets $\Lambda_0^{(1)}, \Lambda_0^{(2)}$. Thus, by Theorem 2.1, all resonances are real in this case. 3) In the case of the half-solid ($p_1 = const$) there are (except for $p_1 = p_2$) 4 sheets and it can be shown that in this case there exist complex resonances [KP].

Introduce the subspaces of even potentials $L^r_{even}(\mathbb{T}) \subset L^r(\mathbb{T})$ given by

$$L^r_{even}(\mathbb{T}) = \Big\{ p \in L^r(\mathbb{T}) : p(x) = p(1-x), 0 < x < 1 \Big\}, \quad r \geqslant 1.$$

It is well known that a potential $p \in L^1_{even}(\mathbb{T})$ iff $|\gamma_n(H)| = |\mu_n(p) - \nu_n(p)|$ for all $n \geqslant 1$. Next we consider a biperiodic potential with even potentials p_1, p_2.

Theorem 2.2. *Let q_0 be a biperiodic potential in the sense of (1.1), where $p_1, p \in L^1_{even}(\mathbb{T})$ and $t = 0$. Assume that some gap of T_0 is given by $\widetilde{\gamma}(T_0) = \widetilde{\gamma}_1 \cap \widetilde{\gamma}_2 \neq \emptyset$ for some gap $\widetilde{\gamma}_n$ of $H_n, n = 1, 2$. In the case $|\widetilde{\gamma}_n| = |\widetilde{\mu}_n - \widetilde{\nu}_n| < \infty, n = 1, 2$ we denote by $\widetilde{\mu}_n, \widetilde{\nu}_n \in \overline{\widetilde{\gamma}_n}$ the corresponding Dirichlet and Neumann eigenvalues. Then*

$$\#(T_0, \widetilde{\gamma}(T_0)) = \begin{cases} 0 & \textit{if} \quad (\widetilde{\mu}_1 - \widetilde{\nu}_1)(\widetilde{\mu}_2 - \nu_2) > 0, \\ 1 & \textit{if} \quad (\widetilde{\mu}_1 - \widetilde{\nu}_1)(\widetilde{\mu}_2 - \widetilde{\nu}_2) < 0. \end{cases}, \qquad \text{if} \quad |\widetilde{\gamma}_1|, |\widetilde{\gamma}_2| < \infty, \tag{2.3}$$

$$\#(T_0, \widetilde{\gamma}(T_0)) = \begin{cases} 0, & \textit{if} \quad \widetilde{\mu}_2 < \widetilde{\nu}_2, \\ 1, & \textit{if} \quad \widetilde{\nu}_2 < \widetilde{\mu}_2 \leqslant 0. \end{cases}, \qquad \text{if} \quad |\widetilde{\gamma}_2| < \infty = |\widetilde{\gamma}_1|. \tag{2.4}$$

Moreover, if $\widetilde{\gamma}(T_0) = (-\infty, 0)$, then $\#(T_0, \widetilde{\gamma}(T_0)) = 0$.

We now consider the problem of the reconstruction of p_1: if we know p plus some spectral data. In fact we consider the inverse problem including the problem of characterization for $T_0 = -\frac{d^2}{dx^2} + q_0$. We introduce the sequence $d = \{d_n\}_1^\infty$, where $d_n \in \{0, 1\} = 0$. Define the spaces

$$Q = \Big\{ q_0 = \chi_- p_1 + \chi_+ p : \ p_1, p \in L^2_{even}(\mathbb{T}), \gamma_n(H_1) = \gamma_n(H), \text{ for all } n \geqslant 0 \Big\},$$

$$P = \Big\{ (p, d) : \ \ p \in L^2_{even}(\mathbb{T}), d = \{d_n\}_1^\infty, \ d_n \in \{0, 1\}, \quad d_n = 0 \ \text{ if } \ |\gamma_n(H)| = 0 \Big\}.$$

Define the mapping $\rho : Q \to P$ given by $\rho(q_0) = (p, d)$, where $d_n = \#(T_0, \gamma_n(T_0)), n \geqslant 1$. Note that (2.3) yields $d_n \in \{0, 1\}, n \geqslant 1$ and (1.2) implies $d_n = 0$ if $|\gamma_n(H)| = 0$. We shall show that any sequence $d = \{d_n\}_1^\infty$ of the number of eigenvalues in the gaps actually occurs. More precisely, we have the following inverse result:

Theorem 2.3. *The mapping $\rho : Q \to P$ is 1-to-1 and onto. Moreover, for $\|p\|^2 \equiv \int_{\mathbb{T}} |p(x)|^2 dx$ the following estimates are fulfilled:*

$$\|p_1\| = \|p\| \leqslant 2G(1 + G^{1/3}), \quad G \leqslant 2\|p\|(1 + \|p\|^{1/3}), \qquad G = (\sum |\gamma_n(T_0)|^2)^{1/2}. \tag{2.5}$$

Remark. i) It follows from assertion of Theorem, that p_1 in the decomposition $q_0 = \chi_- p_1 + \chi_+ p$ is uniquely determined by (p, d). ii) Assume that $q_0 \in Q$ and $|\gamma_n(H)| > 0, n \geqslant 1$ for some $p \in L^2_{even}(\mathbb{T})$. Then the definition of Q and (1.2) yield $\gamma_n(T_0) = \gamma_n(H)$ for all $n \geqslant 1$. If $d_n = 1$ for all $n \geqslant 1$, then Theorem 2.3 yields $\#(T_0, \gamma_n(T_0)) = 1, n \geqslant 1$, and $\#(T_0, \mathbb{R}) = \infty$. If $d_n = 0, n \geqslant 1$, then Theorem 2.3 yields $\#(T_0, \gamma_n(T_0)) = 0, n \geqslant 1$, and $T_0 = H$.

We now consider the **dislocation operator**

$$T_t^{di} = -\frac{d^2}{dx^2} + p_{(t)}(x) \ \text{ in } \ L^2(\mathbb{R}), \quad p_{(t)} = \chi_- p + \chi_+ p(\cdot + t), \ \ p \in L^1(\mathbb{T}), \ t \in \mathbb{R}.$$

Using (1.2-3) we obtain the following relations

$$\sigma(T_t^{di}) = \sigma_{ac}(T_t^{di}) \cup \sigma_d(T_t^{di}), \quad \sigma_{ac}(T_t^{di}) = \sigma(H), \quad \sigma_d(T_t^{di}) \subset \cup \gamma_n(H), \quad \#(T_t^{di}, \gamma_n(T_t^{di})) \leqslant 2. \tag{2.6}$$

We emphasize that $\gamma_n(T_t^{di}) = \gamma_n(H)$ for any $t \in [0, 1], n \geqslant 0$, i.e., $\gamma_n(T_t^{di})$ are independent of t. This follows from Theorem 2.1.i. Then $\sigma_{ac}(T_t^{di}) = \sigma^{(2)}$, and $\sigma^{(3)} = \sigma^{(4)} = \emptyset$, and the Riemann surface $\Lambda_E(T_t^{di})$ coincides with the Riemann surface $\Lambda_E(H)$, and consists of 2 sheets $\Lambda_0^{(1)}, \Lambda_0^{(2)}$. Introduce the function (here and below $\dot{u} = \frac{\partial}{\partial t} u$)

$$L(t, \lambda) = \mp[\dot{\Psi}(t, \lambda)^2 - (p(t) - \lambda)\Psi(t, \lambda)^2], \quad \lambda = \alpha_n^\pm, \quad n \geqslant 0, \quad t \in [0, 1]. \tag{2.7}$$

Let $W_r^2(a, b)$ be the Sobolev space of functions f on the interval such that $f^{(r)} \in L^2(a, b), r \geqslant 0$. We have the following result about eigenvalues of the dislocation operator.

Theorem 2.4. *Let the dislocation potential $p_{(t)} = \chi_- p + \chi_+ p(\cdot + t)$, where $p \in L^2(\mathbb{T}), t \in \mathbb{R}$. Then for each gap $\gamma_n(H) = (\alpha_n^-, \alpha_n^+) \neq \emptyset, n \geqslant 1$ there exists a function $z_n^\pm(\cdot) \in W_1^2(-\varepsilon_n, \varepsilon_n)$, for some $\varepsilon_n > 0$, such that: $z_n^\pm(0) = 0$ and $\lambda_n^\pm(t) \equiv \alpha_n^\pm \mp z_n^\pm(t)^2 \in \gamma_n(H) = \gamma_n(T_t^{di})$ satisfies*
i) If $z_n^\pm(t) > 0$ for some $t \in (-\varepsilon_n, \varepsilon_n)$, then $\lambda_n^\pm(t) \in \gamma_n(H)$ is an eigenvalue of T_t^{di}.
If $z_n^\pm(t) < 0$ for some $t \in (-\varepsilon_n, \varepsilon_n)$, then $\lambda_n^\pm(t) \in \gamma_n(H) \subset \Lambda_0^{(2)}$ is a resonance of T_t^{di}.
Moreover, the following asymptotics are fulfilled:

$$z_n^\pm(t) = \sqrt{|M_n^\pm|/2} \int_0^t L(s, \alpha_n^\pm) ds + O(t^{3/2}), \qquad t \to 0, \tag{2.8}$$

$$z_n^\pm(t) = \mp t \sqrt{|M_n^\pm/2|} \dot{\Psi}(0, \alpha_n^\pm)^2 + O(t^2), \qquad t \to 0, \qquad \text{if} \quad \mu_n(p) = \alpha_n^\pm. \tag{2.9}$$

ii) If, in addition, p is real analytic, then $z_n^\pm(\cdot)$ is real analytic on $(-\varepsilon_n, \varepsilon_n)$.

Remark. i) If in (2.9) we have $\mu_n(p)=\alpha_n^-$, then $z_n^-(t)>0$ for $t>0$ and $\lambda_n^-(t)$ is an eigenvalue; if $\mu_n(p)=\alpha_n^+$, then $z_n^-(t)<0$ for $t>0$ and $\lambda_n^-(t)$ is a resonance. ii) Let p be smooth potential and $n\gg 1$. Then using (2.8) we deduce that $z_n^-(t)>0$, as $t>0$ (we get an eigenvalue) and $z_n^+(t)<0$ (we get a resonance). Hence for fixed $n\gg 1$ and any small $t>0$ we have an eigenvalue near α_n^- and a resonance near α_n^+ on the second sheet. iii) It is possible to formulate Theorem 2.4 for potentials $p\in L^1(\mathbb{T})$ and then the asymptotics in (2.8) has the form $z_n^\pm(t)=\sqrt{|M_n^\pm/2|}\int_0^t L(s,\alpha_n^\pm)ds+O(t)$, as $t\to 0$

Using Theorem 2.4 we construct the dislocation operator T_t^{di}, such that $\#(T_t^{di},\gamma_n(T_t^{di}))=2$ for all $n=1,..,N$ for each fixed $N\geqslant 1$.

Theorem 2.5. *For any finite sequences $d=\{d_n\}_1^N,\{s_n\}_1^N,d_n\in\{0,1\},s_n>0,N\geqslant 1$ there exists a potential $p\in L^2_{even}(\mathbb{T})$ and $\varepsilon>0$ such that the each dislocation operator $T_t^{di},t\in(0,\varepsilon)$ has gaps with lengths $|\gamma_n(T_t^{di})|=s_n$, and $\#(T_t^{di},\gamma_n(T_t^{di}))=2d_n$ for any $n=1,2,..,N$.*

We now consider a **half-solid**, that is the Schrödinger operator

$$T_t^s=-\frac{d^2}{dx^2}+q_t^s(x)\quad\text{in}\quad L^2(\mathbb{R}),\qquad q_t^s(x)=s\chi_-(x)+\chi_+(x)p(x+t),$$

where $s,t\in\mathbb{R}$ and $p\in L^1(\mathbb{T})$ is real. Theorem 2.1 yields :

$$\sigma(T_t^s)=\sigma_{ac}(T_t^s)\cup\sigma_d(T_t^s),\qquad \sigma_{ac}(T_t^s)=\sigma(H)\cup[s,\infty),\tag{2.10}$$

$$\gamma_n(T_t^s)=\gamma_n(H)\cap(-\infty,s),\qquad \#(T_t^s,\gamma_n(T_t^s))\leqslant 2,\quad n\geqslant 0,\ t\in[0,1].\tag{2.11}$$

Hence Theorem 2.1 yields if $p\not\equiv s$, then the Riemann surface $\Lambda_E(T_t^s)$ consists of 4 sheets. Note that if $s\leqslant\alpha_1^-(H)$, then $\sigma_{ac}(T_t^s)=(s_-,\infty)$, where $s_-=\min(0,s)$. If $s>\alpha_1^-(H)$, then there exists a gap in the spectrum of T_t^s. Our goal is to study the eigenvalues in the gaps $\gamma_n(T_t^s),n\geqslant 0$, and to find how these eigenvalues depend on t,s. It is clear that they depend on t periodically. Define sets $Q_N=\Big\{q_0^s=s\chi_-+\chi_+p_2:\quad p\in L^2_{even}(\mathbb{T}),\ \ \alpha_N^+<s<\alpha_{N+1}^-\Big\}$ and $P_N=\Big\{(r,d,\varepsilon):\{r_n\}_1^\infty\in\ell^2,d=\{d_n\}_1^N$, where for any $n\leqslant N$ the number $r_n\geqslant 0$ and $d_n=0$ if $r_n=0,\varepsilon\in(0,1)\Big\}$ for some integer $N\geqslant 0$. Define the mapping $\omega:Q_N\to P_N$ by $\omega(q_0^s)=(r,d,\varepsilon)$, where

$$r_n=|\gamma_n(T_0^s)|,\ d_n=\#(T_0^s,\gamma_n(T_0^s)),\ n\leqslant N,\quad r_n=|\sigma_n^1(T_0^s)|\,\mathrm{sign}(\mu_n(p)-\nu_n(p)),\quad n>N,$$

$\varepsilon=\frac{\alpha_{N+1}^--s}{\alpha_{N+1}^--\alpha_N^+}$ and $\sigma_n^1(T_0^s)=(s,\infty)\cap\gamma_n(H)$ is a segment of the spectrum of T_0^s with multiplicity one. We consider a half-solid with an even potential p.

Theorem 2.6 *i) Let $T_0^s=-\frac{d^2}{dx^2}+q_0^s(x)$ acting in $L^2(\mathbb{R})$, where $q_0^s=s\chi_-+\chi_+p$ and $p\in L^2_{even}(\mathbb{T})$. Then $\#(T_0^s,\gamma_0(T_0^s))=0$. Let, in addition, $s>a_m^+(p)$ for some $m\geqslant 0$. Then $\sigma^{(4)}\neq\emptyset$, and for each $\gamma_n(T_0^s)\neq\emptyset,n=1,2,..,m$, the following identities are fulfilled:*

$$\#^{(4)}(T_0^s,\gamma_n(T_0^s))+\#(T_0^s,\gamma_n(T_0^s))=1,\quad \#(T_0^s,\gamma_n(T_0^s))=\begin{cases}1 & \textit{if}\quad \mu_n>\nu_n,\\ 0 & \textit{if}\quad \nu_n>\mu_n.\end{cases}\tag{2.12}$$

ii) Each mapping $\omega:Q_N\to P_N,N\geqslant 0$, is 1-to-1 and onto.

In this Theorem we proved that for any $\{r_n\}_1^\infty \in \ell^2, d = \{d_n\}_1^N$, and $\varepsilon \in (0,1)$ (such that $r_n \geqslant 0$ and $d_n = 0$ if $r_n = 0, 1 \leqslant n \leqslant N$), there exists a unique half-solid potential $q_0^s = s\chi_- + \chi_+ p$ with $p \in L^2_{even}(\mathbb{T})$.

For fixed $\gamma_n(T_t^s) \neq \emptyset, n \geqslant 0,\ s > \alpha_n^\pm$ we introduce the equation

$$\Psi_y(y, \alpha_n^\pm) = \sqrt{s - \alpha_n^\pm}\Psi(y, \alpha_n^\pm), \qquad y \in [0,1]. \tag{2.13}$$

For each $n \geqslant 1$ there exist $N \geqslant n$ roots $y_1, .., y_N \in [0,1]$ of Eq. (2.13) on the interval $[0,1]$, see below Lemma 3.4. Let $m^\pm(\lambda, t)$ be the Weyl function for the potential $p(x+t)$ and let $\alpha_0^- \equiv -\infty$.

Theorem 2.7. *Let $T_t^s y = -y'' + q_t^s y$ be an operator acting in $L^2(\mathbb{R})$, where $q_t^s = s\chi_- + \chi_+ p(\cdot + t)$ and $p \in L^1(\mathbb{T}), t \in \mathbb{R}$. Then*

i) Suppose that $\alpha_n^- < \alpha_n^+ < s$ for some $n \geqslant 0$ and $y \in [0,1]$ is some root of (2.13). Then there exists a unique function $z_n^\pm(\cdot) \in C(-\varepsilon, \varepsilon), z_n^\pm(0) = 0$ for some $\varepsilon > 0$, such that :

if $z_n^\pm(t) > 0$ for some $t \in (-\varepsilon, \varepsilon)$, then $\lambda_n^\pm(t) \equiv \alpha_n^\pm \mp z_n^\pm(t)^2 \in \gamma_n$ is an eigenvalue of T_{y+t}^s,

if $z_n^\pm(t) < 0$ for some $t \in (-\varepsilon, \varepsilon)$, then $\lambda_n^\pm(t) \in \gamma_n \subset \Lambda_0^{(4)}$ is a resonance of T_{y+t}^s.

Moreover, the following asymptotics is fulfilled:

$$z_n^\pm(t) = \sqrt{|M_n^\pm/2|}\Psi(y, \alpha_n^\pm)^2 \int_0^t [p(y+\tau) - \alpha_n^\pm] d\tau + O(t), \quad t \to 0. \tag{2.14}$$

ii) Suppose that $\alpha_n^- < s < \alpha_n^+$, for some $n \geqslant 1$, or $\nu_0^- < s \leqslant \alpha_0^+$. Let y be a root of the equation $m^+(s, y) = 0, 0 \leqslant y \leqslant 1$. Then there exists a unique function $z(\cdot) \in W_1^2(-\varepsilon, \varepsilon), z(0) = 0$ for some $\varepsilon > 0$ such that

if $z(t) > 0$ for some $t \in (-\varepsilon, \varepsilon)$ then $\lambda(t) \equiv s + z(t)^2 \in \gamma_n$ is an eigenvalue of T_{y+t}^s,

if $z(t) < 0$ for some $t \in (-\varepsilon, \varepsilon)$ then $\lambda(t) \in \gamma_n \subset \Lambda_0^{(3)}$ is a resonance of T_{y+t}^s.

Moreover, the following asymptotic estimates are fulfilled:

$$z(t) = \int_0^t [p(y+\tau) - s] d\tau + O(t), \qquad t \to 0. \tag{2.15}$$

3 Hill operator and translations

Let $\varphi(x, \lambda, t), \vartheta(x, \lambda, t)$ be the solutions of the equation

$$-y'' + p(x+t)y = \lambda y, \quad \lambda \in \mathbb{C}, \quad t \in \mathbb{R}, \tag{3.1}$$

satisfying $\varphi_x(0, \lambda, t) = \vartheta(0, \lambda, t) = 1$, and $\varphi(0, \lambda, t) = \vartheta_x(0, \lambda, t) = 0$. Remark that the Lyapunov function $\Delta(\lambda)$ for (3.1) coincides with the Lyapunov function for (2.1) (see [L]). Then we define the quasimomentun k for (3.1) and again $k(\lambda)$ does not depend on t. Let $\mu_n(p, t), n \geqslant 1$, be the Dirichlet spectrum of $p(x+t)$, i.e the spectrum of (3.1) with the boundary condition $y(0) = y(1) = 0$ and let $\nu_n(p, t), n \geqslant 0$, be the Neumann spectrum of $p(x+t)$, that is the spectrum of (3.1) with the boundary condition $y'(0) = y'(1) = 0$. We need some results on Eq. (3.1) (see [L], [PTr], [T],[K2]). It is well known that $\varphi(1, \lambda, t)$ is an entire function of λ of order $\frac{1}{2}$ for fixed t. The zeros of $\varphi(1,\lambda,t)$ coincide with the Dirichlet eigenvalues $\mu_n(p,t), n \geqslant 1$, and the following asymptotics are fulfilled:

$$\mu_n(p,t) = (\pi n)^2 + \int_0^1 p(x)dx - \int_0^1 p(x+t)\cos 2\pi nx dx + O(1/n), \qquad \text{as} \qquad n \to \infty, \quad (3.2)$$

uniformly on bounded subsets of $[0,1] \times L^1(0,1)$. It is well known that $\vartheta_x(1,\lambda,t)$ is an entire function of λ of order $1/2$ for fixed t. The zeros of $\vartheta_x(1,\lambda,t)$ coincide with the Neumann eigenvalues $\nu_n(p,t), n \geqslant 0$. The functions $\mu_n(p,t), \nu_n(p,t)$ are 1-periodic. If the parameter t runs through the interval $[0,1]$, then $\mu_n(p,t), \nu_n(p,t)$ run through the gap $\gamma_n = (\alpha_n^-, \alpha_n^+), n \geqslant 1$. If the gap $\gamma_n = \emptyset$, then $\mu_n(p,t), \nu_n(p,t)$ don't move and $\mu_n(p,t) = \nu_n(p,t) = \alpha_n^\pm$. The eigenvalue $\nu_0(p,t) \leqslant \alpha_0^+, t \in [0,1]$. In the book [PTr] there are the asymptotics of the solutions $\varphi(x,\lambda), \vartheta(x,\lambda)$ as $|\lambda| \to \infty$. Repeating it for Eq. (3.1) we obtain the asymptotics for $\varphi(x,\lambda,t), \vartheta(x,\lambda,t)$ as $|\lambda| \to \infty$. For example,

$$\varphi(x,\lambda,t) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + O\Big(\frac{\exp \operatorname{Im}\sqrt{\lambda}x}{\lambda}\Big) \qquad \text{as} \qquad x,t \in [0,1], \quad |\lambda| \to \infty. \qquad (3.3)$$

These asymptotics can be differentiated with respect to x, t and /or λ and are uniform on $[0,1] \times [0,1] \times L^1(0,1)$. We have the Trubowitz identity (see [Tr])

$$\varphi(1,\lambda,t) = \varphi(1,\lambda)\psi_+(t,\lambda)\psi_-(t,\lambda) = \prod_{n\geqslant 1} \frac{(\mu_n(p,t) - \lambda)}{(\pi n)^2}, \quad \lambda \in \mathbb{C}, \quad t \in \mathbb{R}, \qquad (3.4)$$

and the equality $\varphi(1,\lambda)\psi_+(t,\lambda)\psi_-(t,\lambda) = -2\Delta'(\lambda)\Psi(t,\lambda)^2$ at $\lambda = \alpha_n^\pm$, yields

$$\varphi(1,\alpha_n^\pm,t) = (-1)^n 2M_n^\pm \Psi(t,\alpha_n^\pm)^2, \quad t \in \mathbb{R}, \quad n \geqslant 0. \qquad (3.5)$$

where the eigenfunction $\Psi(t,\alpha_n^\pm)$ is defined at the start of Section 2. Define a function $\phi_n(\lambda,t)$ by $\varphi(1,\lambda,t) \equiv (-1)^n(\lambda - \mu_n(p,t))\phi_n(\lambda,t)$. Then we have

$$\phi_n(\lambda,t) > 0, \quad \lambda \in \gamma_n, \quad \text{and} \quad (-1)^n\varphi(1,\lambda,t) > 0, \quad \lambda > \mu_n(t), \quad \lambda \in \gamma_n, \qquad (3.6)$$

Introduce a function $b(\lambda) = -i\sin k(\lambda)$. It is known that

$$k(\lambda) \equiv \pi + iv(\lambda), \quad v(\lambda) > 0, \quad b(\lambda) = (-1)^n\sqrt{\Delta^2(\lambda) - 1}, \qquad \lambda \in \gamma_n \in \Lambda, \qquad (3.7)$$

where $\sqrt{\Delta(\lambda)^2 - 1} > 0$ as $\lambda \in \gamma_n \subset \Lambda$. Below we need the identity (see [KK1])

$$M_n^\pm = -\Delta(\alpha_n^\pm)\Delta'(\alpha_n^\pm), \quad n \geqslant 0, \qquad (3.8)$$

and using (3.7-8) we obtain

$$b(\lambda) = (-1)^n z(\sqrt{|2M_n^\pm|} + O(z^2)), \;\; \text{as} \;\; \lambda \in \gamma_n \neq \emptyset, \;\; \lambda = \alpha_n^\pm \mp z^2, \;\; z \to 0. \qquad (3.9)$$

The Weyl function $m^\pm$ for the potential $p(x+t)$ has the form

$$m^\pm(\lambda,t) = \frac{a(\lambda,t) \pm i\sin k(\lambda)}{\varphi(1,\lambda,t)} = \frac{a(\lambda,t) \mp b(\lambda)}{\varphi(1,\lambda,t)}, \qquad a(\lambda,t) \equiv \frac{\varphi_x(1,\lambda,t) - \vartheta(1,\lambda,t)}{2}. \quad (3.10)$$

Below we need the identity

$$a^2(\lambda,t)+1-\Delta^2(\lambda)=a^2(\lambda,t)-b^2(\lambda)=-\varphi(1,\lambda,t)\vartheta_x(1,\lambda,t). \tag{3.11}$$

Recall that $\dot u=\frac{\partial}{\partial t}u$. We have the equations

$$\dot\vartheta_x(1,\lambda,t)=(\lambda-p(t))\dot\varphi(1,\lambda,t),\quad \dot\varphi(1,\lambda,t)=2a(\lambda,t),$$
$$\dot a(\lambda,t)=-\vartheta_x(1,\lambda,t)-(\lambda-p(t))\varphi(1,\lambda,t), \tag{3.12}$$

(see [L]) and the following identities

$$\varphi(x,\lambda,t)=\vartheta(t,\lambda)\varphi(x+t,\lambda)-\vartheta(x+t,\lambda)\varphi(t,\lambda), \tag{3.13}$$

$$\vartheta(x,\lambda,t)=\vartheta(x+t,\lambda)\varphi_x(t,\lambda)-\vartheta_x(t,\lambda)\varphi(x+t,\lambda). \tag{3.14}$$

Let $C^m(\mathbb{T}),m\geqslant 0$, be the space of m times continuously differentiable real-valued 1-periodic functions. Suppose that $p\in C^1(\mathbb{T})$, then for any $t\in[0,1]$ the identity (the trace formula)

$$p(t)=\alpha_0^+ +\sum_{n\geqslant 1}\Big(\alpha_n^- +\alpha_n^+ -2\mu_n(p,t)\Big), \tag{3.15}$$

holds, where the series converges absolutely and uniformly (see [L]). We need the following result on the Dirichlet spectrum (see [K2]).

Theorem 3.1. *Let a real potential $p\in L^1(\mathbb{T})$ and $\mu_n(t)=\mu_n(p,t),n\geqslant 1$. Then*
i) Each $\mu_n(\cdot)\in C^2(\mathbb{T}),n\geqslant 1$, and $\mu_n'''\in L^1(\mathbb{T})$. Let in addition $p\in L^2(\mathbb{T})$ (or $p\in C^m(\mathbb{T}),m\geqslant 0$). Then $\mu_n'''\in L^2(\mathbb{T})$ (or $\mu_n(\cdot)\in C^{m+3}(\mathbb{T}),m\geqslant 0$).
ii) There exists a function $y_n\in C^1(\mathbb{R})$, such that $\mu_n(t)=\alpha_n^-+|\gamma_n|\sin^2 y_n(t)$, where $\mu_n(0)=\alpha_n^-+|\gamma_n|\sin^2 y_n(0)$, and uniformly on $t\in[0,1]$ the following asymptotics are fulfilled:

$$y_n(t)=y_n(0)+\pi nt+O(1/n),\quad \dot y_n(t)=\pi n+o(1),\qquad n\to\infty. \tag{3.16}$$

iii) Suppose that $\mu_n(t_0)=\alpha_n^-$ or $\mu_n(t_0)=\alpha_n^+$, for some $t_0\in[0,1]$, and $n\geqslant 1$. Then the following asymptotics is fulfilled:

$$\mu_n(t_0+t)=\mu_n(t_0)+t^2\ddot\mu_n(t_0)/2+o(t^2),\quad \ddot\mu_n(t_0)=-4M_n^\pm/\varphi_\lambda'(\mu_n(t_0),t_0)^2. \tag{3.17}$$

Remark. i) In other words we have the following result. Slit the n-th gap $\gamma_n\neq\emptyset$ and place μ_n on the upper or lower lip according to the signature of $\sinh q(\mu_n)$, i.e., on the upper when positive and on the lower when negative. Then $\mu_n(t)$ runs clockwise around the "circle ", changing lips when it hits $\alpha_n^\pm$, making n complete revolutions in unit time. Then (roughly speaking) $\mu_n(t)=\alpha_n^-+|\gamma_n|\sin^2\pi nt$, when t runs through the interval $[0,1]$.

In order to study the eigenvalues in gaps we need properties of the function $\zeta(\lambda,t)=\dot\varphi(1,\lambda,t)/(2\varphi(1,\lambda,t))$.

Lemma 3.2. *For each $(t,p,\lambda)\in F=[0,1]\times L^1(\mathbb{T})\times\mathbb{C}\setminus\{\mu_n(p,t),n\geqslant 1\}$ the following identities are fulfilled:*

$$\zeta(\lambda,t)\equiv\frac{\dot\varphi(1,\lambda,t)}{2\varphi(1,\lambda,t)}=\frac12\sum_{n\geqslant 1}\frac{\dot\mu_n(p,t)}{\mu_n(p,t)-\lambda}, \tag{3.18}$$

$$\dot m^{\pm}(\lambda,t)=(p(t)-\lambda)-m^{\pm}(\lambda,t)^2, \tag{3.19}$$

$$\dot\zeta(\lambda,t)=(p(t)-\lambda)-\zeta^2(\lambda,t)-\frac{b^2(\lambda)}{\varphi^2(1,\lambda,t)}, \tag{3.20}$$

$$\int_0^1\zeta(\lambda,t)dt=0,\quad \lambda<\alpha_1^-,\qquad \int_0^1\zeta(\alpha_0^+,t)^2dt=\int_0^1 p(t)dt-\alpha_0^+, \tag{3.21}$$

$$\varphi(1,\lambda,t)\dot\zeta(\lambda,t)=\pm(-1)^n 2M_n^{\pm}L(t,\lambda),\quad if\ \lambda=\alpha_n^{\pm}\neq\mu_n(p,t). \tag{3.22}$$

where the series converges absolutely and uniformly on compact sets in F and L is given by (2.7).

Proof. Let $\mu_n(t)=\mu_n(p,t)$. The functions $\varphi(1,\lambda,t)$ and $\dot\varphi(1,\lambda,t)$ are entire in λ. xThe zeros of $\varphi(1,\lambda,t)$ have the asymptotics (3.2). Using (3.2-3) we get the asymptotics $\zeta(\lambda,t)=O(1/\sqrt{\lambda})$ as $|\lambda|\to\infty, |\sqrt{\lambda}-\pi n|\geqslant 1/4$. Then (3.4) yields

$$\zeta(\lambda,t)-\sum_{n\geqslant 1}^{N}\frac{\dot\mu_n(t)}{\mu_n(t)-\lambda}=\frac{1}{2\pi i}\int_{|\sqrt{z}|=\pi(2N+1)/2}\frac{\zeta(z,t)dz}{z-\lambda},$$

since the residue of the function $\zeta(z,t)$ at the simple pole $\mu_n(t)$ has the form: $\mathrm{Res}\zeta(\lambda,t)=\dot\varphi(1,\mu_n(t),t)/\varphi'(1,\mu_n(t),t)=\dot\mu_n(t)$. Then as $N\to\infty$ we get (3.18). Remark that (3.16) yields $\dot\mu_n(t)=O(n|\gamma_n|)$ as $n\to\infty$ uniformly on $t\in[0,1]$. Then by (3.2), series (3.18) converges absolutely and uniformly on compact sets.

We will obtain the equations for the function $m^+(\lambda,t)$, the proof for $m^-(\lambda,t)$ and (3.20) is similar. Using (3.12), (3.10-12) we deduce that

$$\dot m^+=\frac{\dot a}{\varphi}-\frac{(a-b)\dot\varphi}{\varphi^2}=(p-\lambda)-\frac{\vartheta_x}{\varphi}-\frac{2(a-b)a}{\varphi^2}=(p-\lambda)+\frac{a^2-b^2}{\varphi^2}-\frac{2am^+}{\varphi},$$

which yields (3.19). Integrating $\zeta=\dot\varphi/2\varphi$ and since $\varphi(\lambda,t)>0$ if $\lambda<\alpha_1^-, t\in[0,1]$, we obtain the first identity in (3.21). Integrating (3.20) implies the second identity in (3.21). Substituting (3.5) into (3.20) we have (3.22). ■

Now we will obtain more exact estimates concerning $L(t,\alpha_n^{\pm})$ defined in (2.7). These results are used to prove the existence of two eigenvalues in gaps (see Theorem 2.5). Recall that if $p\in L^2_{even}(0,1)$, then $\nu_0<\mu_1$ and $\gamma_n(H)=(\mu_n,\nu_n)$ or $\gamma_n(H)=(\nu_n,\mu_n), n\geqslant 1$ [GT].

Lemma 3.3. *i) Let $p\in C^2(\mathbb{T})$ be even. Assume that for some $N\geqslant 1$ and for all $n>N$ the gap $\gamma_n(H)$ are given by $\gamma_n(H)=(\mu_n(p),\nu_n(p))$. Then*

$$p(0)-\alpha_N^{\pm}\geqslant-(\pi N)^2+\sum_{m\geqslant N+1}|\gamma_m|-2\sum_1^N|\gamma_m|. \tag{3.23}$$

ii) For any finite sequences $\{s_n\}_1^N,\{d_n\}_1^N$, where $s_n>0, d_n\in\{0,1\}$, there exists $p\in L^2_{even}(\mathbb{T})$ with gap lengths $|\gamma_n(H)|=s_n$ and $(-1)^{d_n}L(0,\alpha_n^{\pm})<0$ for all $n=1,\dots,N$.

Remark. Roughly speaking the result of ii) is the effect of a big gap γ_{N+1} such that $|\gamma_{N+1}|>(\pi N)^2+2(|\gamma_1|+...+|\gamma_N|)$. It is important that we have 2 types of gaps: 1) the gap

$\gamma_n(H) = (\mu_n(p), \nu_n(p))$ for all $n > N$, 2) $\gamma_n(H) = (\mu_n, \nu_n)$ or $\gamma_n(H) = (\nu_n, \mu_n)$, which depends on d_n for $1 \leqslant n \leqslant N$.
Proof. i) Using the trace formula (3.14) and the identity $\alpha_n^+ = \sum_{m\geqslant 1}^n (|\sigma_m| + |\gamma_m|)$ we obtain

$$p(0) - \alpha_N^+ = \sum_{m\geqslant 1}(\alpha_m^+ + \alpha_m^- - 2\mu_m(p)) - \sum_{m=1}^N (|\sigma_m| + |\gamma_m|) \geqslant \sum_{m\geqslant N+1} |\gamma_m| - 2\sum_{m\geqslant 1}^N |\gamma_m| - \sum_{m=1}^n |\sigma_m|$$

and the estimate $|\sigma_m| < \pi^2(2m-1)$ (see [Mos]) yields (3.23).
ii) We need a result from [GT]. For an even periodic potential we define a signed gap length $l_n = \mu_n - \nu_n, n \geqslant 1$, and the corresponding sequence $l = \{l_n\}_1^\infty$. For any sequence $\{t_n\}_1^\infty \in \ell^2$ there exists a unique even periodic potential p such that the signed gap length $l_n = t_n, n \geqslant 1$. Using this result we fix the number N and due to (3.23) we take a sequence of signed gap lengths $\{l_n\}_1^\infty$ such that $p(0) - \alpha_N^+ > 0$. Hence for all $n = 1, .., N$ we get $p(0) - \alpha_n^\pm > 0$. Thus for each $1 \leqslant n \leqslant N$ we obtain:

If $d_n = 1$ we take $\mu_n = \alpha_n^-$ and $\nu_n = \alpha_n^+$, then $L(0, \alpha_n^\pm) > 0$.

If $d_n = 0$ we take $\nu_n = \alpha_n^-$ and $\mu_n = \alpha_n^+$, then $L(0, \alpha_n^\pm) < 0$. ■

We need some results about the roots of the equation $m^\pm(\lambda, t) = \omega, t \in [0,1]$, and some formulas for even potentials.

Lemma 3.4. *i) Let $p \in L^1(\mathbb{T})$ and a gap $\gamma_n(H) \neq \emptyset$ for some $n \geqslant 1$. Then for any fixed $(\lambda, \omega) \in [\alpha_n^-, \alpha_n^+] \times \mathbb{R}$, there exist $N_\pm \geqslant n$ roots of the equation $m^\pm(\lambda, t) = \omega, t \in [0,1]$.*
ii) For any even potential $p \in L^1(\mathbb{T})$ the following identities are fulfilled:

$$a(\lambda, 0) \equiv 0, \qquad \Delta(\lambda) \equiv \vartheta(1, \lambda), \qquad m^\pm(\lambda) \equiv \frac{\pm i \sin k(\lambda)}{\varphi(1, \lambda, 0)}. \tag{3.24}$$

Proof. Let $\mu_n(t) = \mu_n(p, t), n \geqslant 1$. i) We consider m^+ and firstly let $\lambda = \alpha_n^-$, the proof for m^+ and $\lambda = \alpha_n^+$ is similar. By Theorem 3.1, $\mu_n(\cdot) \in C^2(\mathbb{T})$ and there exist points $\tau_r \in [0,1), r = 1, .., n$, such that $\mu_n(\tau_r) = \alpha_n^-$ and $\dot\mu_n(\tau_r) = 0, \ddot\mu_n(\tau_r) > 0$. By (3.18),

$$m^+(\lambda, t) = \zeta(\lambda, t) = \frac{\dot\mu_n(t)}{\mu_n(t) - \alpha_n^-} + \sum_{m\neq n} \frac{\dot\mu_m(t)}{\mu_m(t) - \alpha_n^-}.$$

Then $m^+(\lambda, \cdot)$ maps the interval $I_r = (\tau_r, \tau_{r+1})$ onto the real line $\mathbb{R}$. Therefore, for any number $\omega \in \mathbb{R}$ there exist $N \geqslant n$ roots of the equation $m^+(\lambda, t) = \omega$.

Secondly, let $\lambda \in (\alpha_n^-, \alpha_n^+)$ and $b(\lambda) > 0$, the proof for $b(\lambda) < 0$ is similar. We get $2a(\lambda, t) = \dot\varphi(1, \lambda, t) = \dot\mu_n(t)(-1)^n \phi(\lambda, t)$ at $\lambda = \mu_n(t)$. Since $\mu_n(t)$ crosses the point λ exactly $2n$ times there exist points $\tau_r \in [0,1)$ $r = 1, 2, 3, .., n$, such that $\mu_n(\tau_r) = \lambda$ and $(-1)^n \dot\mu_n(\tau_r) < 0$. Then $a(\lambda, \tau_r) - b(\lambda) \neq 0$, and we have

$$m^+(\lambda, t) = \frac{a(\lambda, t) - b(\lambda)}{\varphi(1, \lambda, t)} = \frac{a(\lambda, t) - b(\lambda)}{(-1)^n (\lambda - \mu_n(t)) \phi_n(\lambda, t)}. \tag{3.25}$$

Recall that each function $\phi_n(\lambda, \cdot), \lambda \in \gamma_n$ has no zero on the interval $[0,1]$. By the properties of $\mu_n(t)$ from Theorem 3.1, the function $m^+(\lambda, \cdot)$ maps the interval $I_p = (\tau_p, \tau_{p+1})$ onto the real line $\mathbb{R}$. Then for any $\omega \in \mathbb{R}$ there exist $N_\pm \geqslant n$ roots of the equation $m^+(\lambda, t) = \omega$.

ii) It is well known that for an even potentials the points $\mu_n(0)$ and $\nu_n(0)$ lie on the endpoints of the gap γ_n(see [GT]) . Then by Theorem 3.1, $\dot\mu_n(0)=0$, and using (3.18) we get $\zeta(\lambda,0)=0$ for all $\lambda\in\mathbb{C}$. Hence relations (3.10), (3.12) imply (3.24). ■

We consider the equation $\zeta(0,t)=r, t\in[0,1)$, for fixed $r\in\mathbb{R}$. Let $\psi(t)$ be any smooth 1-periodic positive function. Then we take $\Psi(t,0)=\psi(t)$ and the potential $p(t)=\ddot\psi(t)/\psi(t)$. In this case the function $\zeta(0,t)$ is periodic smooth with $\int_0^1\zeta(0,t)dt=0$ see (3.21). We are able to get a function ψ, when the equation $\zeta(0,t)=r$ has any number of roots (depending on ψ). In order to study the ground state we need some properties of the Weyl function.
Lemma 3.5. *Let $p\in L^1(0,1)$. Then $\pm m^\pm(\lambda)\to-\infty$ as $\lambda\to-\infty$, and*

$$\pm m^\pm(\lambda)<0 \quad \text{if} \quad \lambda<\nu_0, \quad \text{and} \quad m^+(\lambda)m^-(\lambda)>0, \quad \text{if} \quad \nu_0<\lambda<0, \tag{3.26}$$

Moreover, if $\pm m^+(0)>0$, then $\pm m^+(\lambda)>0$ for $\nu_0<\lambda<0$.
Proof. (4.5) yields the asymptotics $\pm m^\pm(\lambda)\to-\infty$ as $\lambda\to-\infty$. We have the identity $a^2-b^2=-\varphi\vartheta_x$. It is well known that: $\varphi(\lambda)>0, \lambda<\mu_1$, and $\vartheta_x(\lambda)>0, \lambda<\nu_0$ and $\vartheta_x(\lambda)<0, \nu_0<\lambda<0$. Hence we get (3.26) since $m^\pm$ has no zero on the intervals $(-\infty,\nu_0)$ and $(\nu_0,0)$.

Let $\pm m^+(0)>0$. Due to (3.11), the function m^+ has no zero on the interval $(\nu_0,0)$. Then for the case $\lambda\in(\nu_0,0)$ we have $m^+(\lambda)>0$, if $m^+(0)>0$, and $m^+(\lambda)<0$, if $m^+(0)<0$. ■

4 Biperiodic potentials

In this section we study the spectrum of T_t at $t=0$, i.e., $T_0=-\frac{d^2}{dx^2}+q_0$, where $q_0=p_1\chi_-+\chi_+p$ and $p\in L^1(\mathbb{T}), p_1\in L^1(\tau\mathbb{T})$. We now determine the eigenfunctions of T_0. Let $u(x,\lambda), v(x,\lambda)$ be the solutions of the equation

$$-y''+q_0y=\lambda y, \quad \lambda\in\mathbb{C}, \tag{4.1}$$

satisfying $u'(0,\lambda)=v(0,\lambda)=1$, and $u(0,\lambda)=v'(0,\lambda)=0$. Let $\varphi_j(x,\lambda),\vartheta_j(x,\lambda), j=1,2$, be the solutions of the equation

$$-y''+p_jy=\lambda y, \quad \lambda\in\mathbb{C},$$

satisfying $\varphi_j'(0,\lambda)=\vartheta_j(0,\lambda)=1$, and $\varphi_j(0,\lambda)=\vartheta_j'(0,\lambda)=0$. Remark that $\vartheta,\varphi,\vartheta_j,\varphi_j$ are entire functions of $\lambda\in\mathbb{C}$, and are real on the real line. Then we obtain

$$v(x,\lambda)=\begin{cases}\vartheta_1(x,\lambda) & \text{if} \quad x<0,\\ \vartheta_2(x,\lambda) & \text{if} \quad x>0,\end{cases} \qquad u(x,\lambda)=\begin{cases}\varphi_1(x,\lambda) & \text{if} \quad x<0,\\ \varphi_2(x,\lambda) & \text{if} \quad x>0,\end{cases} \tag{4.2}$$

Let $\Delta_j(\lambda)$ be the Lyapunov function for the potential $p_j, j=1,2$. We need also the quasi-momentum $k_j(\lambda)=\arccos\Delta_j(\lambda)$, which is analytic in the domain $\Lambda_j=\mathbb{C}\setminus\sigma(H_j)$. Introduce the Bloch functions $\psi_j^\pm(\cdot,\lambda)\in L^2(\mathbb{R}_\pm), \lambda\in\Lambda_j$, for the operator $H_j=-\frac{d^2}{dx^2}+p_j, j=1,2$, by

$$\psi_j^\pm(x,\lambda)=\vartheta_j(x,\lambda)+m_j^\pm(\lambda)\varphi_j(x,\lambda), \qquad m_j^\pm(\lambda)=\frac{a_j(\lambda)\pm i\sin k_j(\lambda)}{\varphi_j(\lambda)}. \tag{4.3}$$

Then for $\lambda \in \Lambda_j, j = 1, 2$ we have

$$\varphi_j(x,\lambda) = \frac{\psi_j^+(x,\lambda) - \psi_j^-(x,\lambda)}{w_j(\lambda)}, \quad \vartheta_j(x,\lambda) = \frac{m_j^+(\lambda)\psi_j^-(x,\lambda) - m_j^-(\lambda)\psi_j^+(x,\lambda)}{w_j(\lambda)}. \tag{4.4}$$

The function $e^{\mp i k_j(\lambda) x}\psi_j^\pm(x,\lambda)$ is τ_j-periodic in x for any λ, where $\tau_1 = \tau, \tau_2 = 1$. Introduce the domain $\Lambda_j(\delta) = \{\lambda \in \Lambda_j, |\sqrt{\lambda} - \pi n/\tau_j| \geqslant \delta, n \geqslant 1\}, \delta > 0$. Below we need the asymptotics

$$\psi_j^\pm(x,\lambda) = e^{\pm i\sqrt{\lambda}x}(1 + O(\lambda^{-1/2})), \qquad m_j^\pm(\lambda) = \pm i\sqrt{\lambda}(1 + O(\lambda^{-1/2})), \quad |\lambda| \to \infty, \tag{4.5}$$

$\lambda \in \Lambda_j(\delta)$, uniformly on $[0, \tau_j]$ (see [T]). We introduce $\Psi^\pm(\cdot,\lambda) \in L^2(\mathbb{R}_\pm), \lambda \in \Lambda_0^{(1)} = \mathbb{C} \setminus \sigma_c(T_0)$, which are solutions of Eq. (4.1) and having the form

$$\Psi^+(x,\lambda) = \psi_2^+(x,\lambda), \quad x > 0; \qquad \Psi^-(x,\lambda) = \psi_1^-(x,\lambda), x < 0, \quad \lambda \in \Lambda_0^{(1)}.$$

We determine these functions on the real line. First we find the Wronskians

$$w(\lambda) = \{\Psi_-, \Psi_+\} = m_2^+(\lambda) - m_1^-(\lambda), \quad \lambda \in \Lambda_0^{(1)}, \tag{4.6}$$

$$w_j(\lambda) = \{\psi_j^-, \psi_j^+\} = m_j^+(\lambda) - m_j^-(\lambda), \quad \lambda \in \Lambda_0^{(1)}, \quad j = 1, 2. \tag{4.7}$$

Hence we get

$$w - w_2 = (m_2^+ - m_1^-) - (m_2^+ - m_2^-) = m_2^- - m_1^-, \tag{4.8}$$

$$w - w_1 = (m_2^+ - m_1^-) - (m_1^+ - m_1^-) = m_2^+ - m_1^+. \tag{4.9}$$

The definition of the Weyl functions yields

$$w(\lambda) = \Big(\frac{a_2(\lambda)}{\varphi_2(1,\lambda)} - \frac{a_1(\lambda)}{\varphi_1(1,\lambda)}\Big) + i\Big(\frac{\sin k_2(\lambda)}{\varphi_2(1,\lambda)} + \frac{\sin k_1(\lambda)}{\varphi_1(1,\lambda)}\Big), \quad \lambda \in \Lambda_0^{(1)}. \tag{4.10}$$

Relations (4.6), (4.3) yield: if $\lambda \in \sigma(H) \setminus (\partial\sigma(H) \cup \{\mu_n(p_j), j = 1, 2, \ n \geqslant 1\})$, then we get $\pm \mathrm{Im}\ w(\lambda \pm i0) > 0$. Using (4.1-2) we have

$$\Psi^+(x,\lambda) = \psi_2^+(x,\lambda) = \vartheta_2(x,\lambda) + m_2^+(\lambda)\varphi_2(x,\lambda) = v(x,\lambda) + m_2^+(\lambda)u(x,\lambda), \ \ x \geqslant 0, \ \ \lambda \in \Lambda_0^{(1)},$$

and (4.2) implies $\Psi^+(x,\lambda) = \vartheta_1(x,\lambda) + m_2^+(\lambda)\varphi_1(x,\lambda), \quad x < 0, \lambda \in \Lambda_0$. Then identities (4.3-9) yield for $x < 0, \ \lambda \in \Lambda_0^{(1)}$:

$$\Psi^+ = \frac{m_1^+\psi_1^- - m_1^-\psi_1^+}{w_1} + m_2^+\frac{\psi_1^+ - \psi_1^-}{w_1} = \frac{m_1^+ - m_2^+}{w_1}\psi_1^- + \frac{m_2^+ - m_1^-}{w_1}\psi_1^+ = \frac{w_1 - w}{w_1}\psi_1^- + \frac{w}{w_1}\psi_1^+,$$

and therefore,

$$\Psi^+(x,\lambda) = \begin{cases} \psi_2^+(x,\lambda) & \text{if} \quad x > 0, \\ \frac{w_1(\lambda) - w(\lambda)}{w_1(\lambda)}\psi_1^-(x,\lambda) + \frac{w(\lambda)}{w_1(\lambda)}\psi_1^+(x,\lambda) & \text{if} \ \ x < 0 \end{cases}, \quad \lambda \in \Lambda_0^{(1)}, \tag{4.11}$$

and similarly we have

$$\Psi^-(x,\lambda) = \begin{cases} \frac{w_2(\lambda)-w(\lambda)}{w_2(\lambda)}\psi_2^+(x,\lambda) + \frac{w(\lambda)}{w_2(\lambda)}\psi_2^-(x,\lambda) & \text{if } \ x>0, \\ \psi_1^-(x,\lambda) & \text{if } \ x<0 \end{cases}, \qquad \lambda \in \Lambda_0^{(1)}. \tag{4.12}$$

Substituting (4.5) into (4.11-12) we obtain the asymptotics

$$\Psi^\pm(x,\lambda) = \exp(\pm i\sqrt{\lambda}x)(1+O(\lambda^{-1/2})), \quad \text{as} \quad |\lambda|\to\infty, \ \lambda\in\Lambda_1(\delta)\cap\Lambda_2(\delta), \tag{4.13}$$

uniformly on $[-\tau,1]$, for some $\delta>0$. Since the functions $\Psi_\pm(x,\lambda), x\in\mathbb{R}$, are real on the half-line $\lambda\in(-\infty,0)$ we have the identities

$$\bar{\Psi}^\pm(x,\lambda) = \Psi^\pm(x,\bar{\lambda}), \qquad \lambda\in\Lambda_0^{(1)}. \tag{4.14}$$

The kernel $R(x,x',\lambda)$ of the operator $(T_0-\lambda)^{-1}, \lambda\in\Lambda$, has the form

$$R(x,x',\lambda) = \frac{1}{w(\lambda)}\begin{cases} \Psi_+(x,\lambda)\Psi_-(x',\lambda) & \text{if } \ x>x' \\ \Psi_-(x,\lambda)\Psi_+(x',\lambda) & \text{if } \ x<x' \end{cases}, \qquad \lambda\in\Lambda_0^{(1)}. \tag{4.15}$$

We prove the first main theorem concerning T_t.

Proof of Theorem 2.1 i) We shall use standard tools from the theory of self-adjoint extensions and the Weyl Theorem. For the sake of the reader, we shall briefly mention all arguments. We need the following result of Krein [Kr]. Let A be a closed symmetric operator in a separable Hilbert space $\mathcal{H}$. Define the deficiency index $n_\pm = dim[ker(A^*\pm i)]$. Remark that by the Neumann extension theory, every closed symmetric operator with $n_+ = n_-$ admits a self-adjoint extension. In accordance with Krein an open interval (α^-,α^+), is called a gap of A if

$$\|(A-\frac{\alpha^-+\alpha^+}{2})f\| \geqslant \frac{\alpha^+-\alpha^-}{2}\|f\|, \quad \text{for} \quad \text{all} \quad f\in D(A).$$

Assume that: a) $n_-=n_+=n\geqslant 0$, b) the operator A has a gap (α^-,α^+), c) there exists a self-adjoint extension A_e of A. Then for any self-adjoint extension A_0 of A we have: $\sigma_{ess}(A_0)=\sigma_{ess}(A_e)$, and the spectrum of A_0 is discrete inside the gap and consists of at most n eigenvalues counting multiplicities.

We need the simple fact (see Zheludev [Z]). Define the operator $H_m^\pm y = -y''+p_m y$ in $L^2(\mathbb{R}_\pm)$, with the Dirichlet boundary condition $y(0)=0$ for $m=1,2,$. Furthermore, $\sigma(H_m^\pm)=\sigma_{ac}(H_m^\pm)\cup\sigma_d(H_m^\pm)$, $\sigma_{ac}(H_m^\pm)=\sigma(H_m)$, and the eigenvalues of $H_m^\pm$ coincide with some $\mu_n(p_m), n\geqslant 1$. Define the symmetric operator $Af=T_0 f$, $f\in C_0^\infty(\mathbb{R}\setminus\{0\})$. Then $n_-(A)=n_+(A)=2$ and there exists a special self-adjoint extension $A_e=H_1^-\oplus H_2^+$ of A in the space $L^2(\mathbb{R}_-)\oplus L^2(\mathbb{R}_+)$. Note that the operator $(T_0-i)^{-1}-(A_e-i)^{-1}$ is compact with rank 2, since $n_-(A)=n_+(A)=2$ (see [Kr]).

Hence using the results of Krein and Zheludev, (and the Weyl Theorem about the invariance of the essential spectrum) we obtain: a) $\sigma_{ess}(T_0)=\sigma(H_1)\cup\sigma(H_2)$, b) there exist gaps $\gamma_{nm}(T_0)=\gamma_n(H_1)\cap\gamma_m(H_2)$, for some $n,m\geqslant 0$, in the essential spectrum of T_0, c) for each

$n, m \geqslant 0$, the spectrum of T_0 is discrete inside the gap $\gamma_{nm}(T_0) \neq \emptyset$ and consists of at most two eigenvalues.

Moreover, using the properties of $w, \Psi^{\pm}$ we deduce that the function $R(x, x', \lambda \pm i0), \lambda \in \mathbb{R}$, is real analytic in $\lambda \in \mathbb{R}$ away from the discrete set of the zeros of the functions $w(\lambda), \varphi_m(1, \lambda), m = 1, 2$, which implies $\sigma_{sc}(T_0) = \emptyset$.

We prove the absence of eigenvalues in the continuous spectrum by contradiction. Let $\lambda \in \sigma_c(T_0), f$ be an eigenvalue and a corresponding eigenfunction of T_0. Hence $\lambda \in \sigma(H_1)$ or $\lambda \in \sigma(H_2)$. Let $\lambda \in \sigma(H_2)$, the proof for $\lambda \in \sigma(H_1)$ is similar. Then $-f'' + p_2 f = \lambda f, x > 0$. But it is well known that in this case the function $f \notin L^2(\mathbb{R}_+)$ (see [T] or [CL]). Hence we have a contradiction.
The results of ii) will be proved in Lemma 4.1-3.

iii) The Wronskian for the dislocation has the form $w(\lambda, t) = m^+(\lambda, t) - m^+(\lambda, 0)$, which follows from (4.10). The Wronskian for the half-solid has the form $w(\lambda, t) = m^+(\lambda, t) - \sqrt{s - \lambda}$, which follows from (4.10). Using these identities and the definitions of $m^+(\lambda, t)$ (see(3.10)) we obtain the statement iii). ■

Lemma 4.1. *Let q_0 be a biperiodic potential in the sense of (1.1). Then*
i) $\sigma^{(2)} = \sigma(H_1) \cap \sigma(H) \neq \emptyset$ and for each $x, x' \in \mathbb{R}$ the functions $R(x, x', \cdot), w, w_1, w_2, \Psi^{\pm}(x, \cdot)$ have a meromorphic continuation from $\Lambda_0^{(1)}$ across the set $\sigma^{(2)}$ to the second sheet $\Lambda_0^{(2)}$, where the following identities are fulfilled:

$$w_n(\lambda^{(2)}) = -w_n(\lambda), \qquad m_n^{\pm}(\lambda^{(2)}) = m_n^{\mp}(\lambda), \qquad \psi_n^{\pm}(\cdot, \lambda^{(2)}) = \psi_n^{\mp}(\cdot, \lambda), \quad n = 1, 2, \tag{4.16}$$

$$w(\lambda^{(2)}) = m_2^-(\lambda) - m_1^+(\lambda), \tag{4.17}$$

$$\Psi^+(x, \lambda^{(2)}) = \begin{cases} \psi_2^-(x, \lambda) & \text{if} \quad x > 0, \\ \Psi^+(x, \lambda) + w_2(\lambda)\varphi_1(x, \lambda) & \text{if} \quad x < 0. \end{cases} \tag{4.18}$$

$$\Psi^-(x, \lambda^{(2)}) = \begin{cases} \Psi^-(x, \lambda) + w_1(\lambda)\varphi_2(v, \lambda) & \text{if} \quad x > 0 \\ \psi_1^+(x, \lambda) & \text{if} \quad x < 0. \end{cases} \tag{4.19}$$

ii) The number λ_e is an eigenvalue of T_0 iff the number λ_e considered as a point on the second sheet $\lambda_e^{(2)} \in \Lambda_0^{(2)}$ is a resonance of $\widetilde{T}_0$. All resonances of T_0 on the sheet $\Lambda_0^{(2)}$ lie on the gaps $\gamma_n(T_0), n \geqslant 0$ and $\#^{(2)}(T_0, \gamma_n) \leqslant 2$ for all $n \geqslant 0$.

Proof. i) In order to get an analytic continuation we have to study the function $\sin k_j(\lambda), j = 1, 2$. The function $\sin k_j(\lambda)$ is real on $\sigma(H_j)$. Then k_j has a meromorphic continuation across $\sigma(H_j)$ by the formula $\bar{k}_j(\lambda) = k_j(\bar{\lambda})$. Using (3.7) we have $k_j(\lambda) = \pi n + iv(\lambda), \lambda \in \gamma_n \subset \Lambda_0^{(1)}$ and $k_j(\lambda) = \pi n - iv(\lambda), \lambda \in \gamma_n \subset \Lambda_0^{(2)}$, where $v = \operatorname{Im} k$. Then we get $\sin k_j(\lambda) = -\sin k_j(\lambda^{(2)}), \lambda \in \gamma_n$, and using (4.3), (4.6-7) we obtain (4.16-17); (4.16-17) and (4.11-12) imply (4.18-19). Relations (4.16-19) yield a meromorphic continuation of $R(x, x', \lambda)$ into the second sheet $\Lambda_0^{(2)}$.

ii) Let $\widetilde{w}$ be the Wronskian for $\widetilde{T}_0$. Identity (4.17) yields $w(\lambda^{(2)}) = -\widetilde{w}(\lambda)$. Then for any eigenvalue λ_e of T_0 the number $\lambda_e^{(2)}$ is a resonance of $\widetilde{T}_0$ and inversely. Hence all resonances on the sheet $\Lambda_0^{(2)}$ of T_0 are real and lie on the gaps $\gamma_n(T_0), n \geqslant 0$, of this sheet. Moreover, (1.3) yields $\#^{(2)}(T_0, \gamma_n) \leqslant 2$. ■

We formulate the results about an analytic continuation into the sheet $\Lambda_0^{(3)}$.

Lemma 4.2 *Let q_0 be a biperiodic potential in the sense of (1.1) and let $\sigma^{(3)} = \sigma(H_1) \setminus \sigma(H_2) \neq \emptyset$. Then for each $x, x' \in \mathbb{R}$ the functions $R(x, x', \cdot), w, w_1, w_2, \Psi^{\pm}(x, \cdot)$ have meromorphic continuations from $\Lambda_0^{(1)}$ across the set $\sigma^{(3)}$ to the sheet $\Lambda_0^{(3)}$, where the following identities are fulfilled:*

$$m_1^{\pm}(\lambda^{(3)}) = m_1^{\mp}(\lambda), \quad \psi_1^{\pm}(\cdot, \lambda^{(3)}) = \psi_1^{\mp}(\cdot, \lambda), \quad w(\lambda^{(3)}) = m_2^{+}(\lambda) - m_1^{+}(\lambda), \tag{4.20}$$

$$\Psi^{-}(x, \lambda^{(3)}) = \begin{cases} \Psi^{-}(x,\lambda) - w_1(\lambda)\varphi_2^{+}(x,\lambda), & \textit{if} \quad x > 0, \\ \psi_1^{+}(x,\lambda) & \textit{if} \quad x < 0, \end{cases} \tag{4.21}$$

$w_1(\lambda^{(3)}) = -w_1(\lambda)$, *and the functions $\Psi^{+}(x, \cdot), w_2, m_2^{\pm}, \psi_2^{\pm}$ don't change.*

We formulate the results about an analytical continuation into the sheet $\Lambda^{(4)}$.

Lemma 4.3 *Let q_0 be a biperiodic potential in the sense of (1.1) and let $\sigma^{(4)} = \sigma(H_2) \setminus \sigma(H_1) \neq \emptyset$. Then for each $x, x' \in \mathbb{R}$ the functions $R(x, x', \cdot), w, w_1, w_2, \Psi^{\pm}(x, \cdot)$ have meromorphic continuation from $\Lambda_0^{(1)}$ across the set $\sigma^{(4)}$ to the sheet $\Lambda_0^{(4)}$ and the following identities are fulfilled:*

$$m_2^{\pm}(\lambda^{(3)}) = m_2^{\mp}(\lambda), \quad \psi_2^{\pm}(x, \lambda^{(3)}) = \psi_2^{\mp}(x, \lambda), \quad w(\lambda^{(4)}) = m_2^{-}(\lambda) - m_1^{-}(\lambda), \tag{4.22}$$

$$\Psi^{+}(x, \lambda^{(3)}) = \begin{cases} \psi_2^{-}(x,\lambda) & \textit{if} \quad x > 0, \\ \Psi^{+}(x,\lambda) - w_2(\lambda)\varphi_1(x,\lambda) & \textit{if} \quad x < 0 \end{cases} \tag{4.23}$$

$w_2(\lambda^{(4)}) = -w_2(\lambda)$, *and the functions $\Psi^{-}(x, \cdot), w_1, m_1^{\pm}, \psi_1^{\pm}$ don't change.*

The proof of Lemmas 4.2-3 repeats the proof of Lemma 4.1.

5 Eigenvalues and resonances

We consider the eigenvalues of the operator T_0 with even potentials p_1, p. Recall that for an even potential the function $a(\cdot) \equiv 0$ and the Weyl function $m^{\pm} = \pm i \sin k / \varphi(1, \cdot)$.

Proof of Theorem 2.2 i) If $\widetilde{\mu}_1 < \widetilde{\nu}_1$ and $\widetilde{\mu}_2 < \widetilde{\nu}_2$, then the gaps have the form $\widetilde{\gamma}_1 = (\widetilde{\mu}_1, \widetilde{\nu}_1), \widetilde{\gamma}_2 = (\widetilde{\mu}_2, \widetilde{\nu}_2)$. Using (3.6-7), (3.24) we have $m_2^{+} < 0$ and $m_1^{-} > 0$ in the gap $\widetilde{\gamma}(T_0)$. Then we deduce that the Wronskian $w < 0$ in this gap and $\#(T_0, \widetilde{\gamma}(T_0)) = 0$.

Assume $\widetilde{\mu}_1 > \widetilde{\nu}_1$ and $\widetilde{\mu}_2 < \widetilde{\nu}_2$ and let $\widetilde{\gamma}(T_0) = (\alpha^{-}, \alpha^{+})$. The function w is real analytic on (α^{-}, α^{+}) and relations (3.6-7), (3.24) imply $w(\alpha^{-}) < 0, w(\alpha^{+}) > 0$. Then the estimate $\#(T_0, \widetilde{\gamma}(T_0)) \leqslant 2$ gives $\#(T_0, \widetilde{\gamma}(T_0)) = 1$.

ii) Using (3.6-7), (3.25) we deduce that $m_1^{-} > 0$ on the interval $\widetilde{\gamma}(T_0)$. Let $\widetilde{\mu}_2 < \widetilde{\nu}_2$. Then (3.6-7), (3.25) yield $m_2^{+} < 0$ on $(\widetilde{\mu}_2, \widetilde{\nu}_2)$, and the function $w < 0$ on the interval $\widetilde{\gamma}(T_0)$. Let $\widetilde{\nu}_2 < \widetilde{\mu}_2 \leqslant 0$. Then $w(\widetilde{\nu}_2) < 0$ and $w(\widetilde{\mu}_2) > 0$. Then since $\#(T_0, \widetilde{\gamma}(T_0)) \leqslant 2$, we get $\#(T_0, \widetilde{\gamma}(T_0)) = 1$.

iii) We consider the ground state in the gap $\widetilde{\gamma}_0(T_0)$. Using (3.6-7), (3.24) we obtain $m_2^{+} < 0$ and $m_1^{-} > 0$ in the gap $\widetilde{\gamma}_0(T_0)$. Then the function $w < 0$ on the interval $\widetilde{\gamma}_0(T_0)$ and $\#(T_0, \widetilde{\gamma}_0(T_0)) = 0$. ■

Next we obtain biperiodic potentials with a prescribed number of eigenvalues.

Proof of Theorem 2.3. Let $q_0 \in Q$ and let $p \in L^2_{even}(\mathbb{T})$. Then due to Theorem 2.2 we have the sequence $\{d_n\}_1^\infty$, $d_n = \#(T_0, \gamma_n(T_0)) \in \{0,1\}, n \geqslant 1$. Note that Theorem 2.1 yields $d_n = 0$ if $|\gamma_n(T_0)| = 0$. Hence we have a mapping $q_0 \to (p,d) \in P$.
Let $(p,d) \in P$. For fixed $p \in L^2_{even}(\mathbb{T})$ the gaps have the forms: $\gamma_n(H) = (\mu_n(p), \nu_n(p))$ or $\gamma_n(H) = (\nu_n(p), \mu_n(p))$ for all $n \geqslant 1$ (see [GT]). In order to construct p_1 we need the following result from [GT]: for any $p \in L^2_{even}(\mathbb{T})$ with gaps $\gamma_n(H) = (\alpha_n^-, \alpha_n^+)$ there exists unique $p_1 \in L^2_{even}(\mathbb{T})$ such that $\gamma_n(H) = \gamma_n(H_1)$ and $\mu_n(p_1) \in \{\alpha_n^-, \alpha_n^+\}$ for all $n \geqslant 1$.

In order to determine p_1, firstly, we set $\gamma_n(H) = \gamma_n(H_1)$. Secondly, using the number $d_n \in \{0,1\}$, according to Theorem 2.2, we choose the position of $\mu_n(p_1)$ by the following: if $d_n = 1$, then $\mu_n(p_1) = \nu_n(p)$ and if $d_n = 0$, then $\mu_n(p_1) = \mu_n(p)$. Thus, we know the gaps $\gamma_n(H_1)$ and the position of $\mu_n(p_1)$ for all $n \geqslant 1$ and by the result of [GT], Theorem 2.2, there exists a unique isospectral $p_1 \in L^2_{even}(\mathbb{T})$ such that $\#(T_0, \gamma_n(T_0)) = d_n$ for all $n \geqslant 1$. Hence the mapping $q_0 \to (p,d)$ from Q into P is 1-to-1 and onto.

For isospectral potentials p, p_1 the identity $\|p_1\| = \|p\|$ holds (see [M] or [K1]) and the estimates (2.5) are fulfilled (see [K1]). ■

We consider the eigenvalues for the general case. In order to prove Theorem 5.1 we need the following definitions. Suppose that a gap of T_t has the form $\widetilde{\gamma}(T_t) = (\alpha^-, \alpha^+) = \gamma_n(H_2) \cap \gamma_m(H_1) \neq \emptyset$ for some gaps $\gamma_k(H_j) = (\alpha_{k,j}^-, \alpha_{k,j}^+), j = 1,2, k = n,m$. By Lemma 3.4, the equation $m_2^+(\alpha^-, y) = m_1^-(\alpha^-), y \in [0,1]$ has $N \geqslant n$ roots $y_1, \dots, y_N$. Assume for simplicity $y_1 = 0$ and introduce the function

$$\mathcal{L}(t,\alpha^-) = \begin{cases} \sqrt{2|M_{n,2}^-|}L_2(t,\alpha^-), & \text{if} \quad \alpha_{m,1}^- < \alpha_{n,2}^-, \\ \sqrt{2|M_{n,2}^-|}L_2(t,\alpha^-)/r(\alpha^-,t), & \text{if} \quad \alpha_{m,1}^- = \alpha_{n,2}^-, \\ \sqrt{2|M_{m,1}^-|}[\dot\Psi_1(0,\alpha^-)^2 - (p_2(t)-\alpha^-)\Psi_1(0,\alpha^-)^2], & \text{if} \quad \alpha_{m,1}^- > \alpha_{n,2}^-, \end{cases} \tag{5.1}$$

where $L_j(t,\lambda) = \mp[\dot\Psi_j(t,\lambda)^2 - (p_j(t)-\lambda)\Psi_j(t,\lambda)^2]$ for $\lambda = \alpha_{k,j}^\pm, \ k \geqslant 0, \ t \in [0,1]$. Here $\Psi_j(t,\alpha_{k,j}^\pm)$ is the corresponding real normalized eigenfunction for 2-periodic problem $-y'' + p_j y = \lambda y$, and $\pm M_{k,j}^\pm > 0$ are the corresponding effective masses for $H_j, j = 1,2$ and

$$r(\lambda,t) = 1 + \sqrt{|M_{n,2}^-/M_{m,1}^-|}\Psi_2(t,\lambda)^2/\Psi_1(t,\lambda)^2, \qquad \lambda = \alpha_{m,1}^- = \alpha_{n,2}^-.$$

Theorem 5.1. *Let q_t be a biperiodic potential given by (1.1), where $p_1, p_2 \in L^2_{loc}(\mathbb{R})$. Suppose that a gap $\widetilde{\gamma}(T_t) = (\alpha^-, \alpha^+) = \gamma_n(H_2) \cap \gamma_m(H_1) \neq \emptyset$ for some gaps $\gamma_k(H_j) = (\alpha_{k,j}^-, \alpha_{k,j}^+), j = 1,2, k = n,m$. Assume that $\mu_{m,1}(p_1) \in (\alpha^-, \alpha_{m,1}^+]$ and let $m_2^+(\alpha^-,0) = m_1^-(\alpha^-)$. Then there exists a unique function $z(\cdot) \in W_1^2(-\varepsilon,\varepsilon), z^-(0) = 0$ for some $\varepsilon > 0$, such that*
i) If $z(t) > 0$, then $\lambda(t) \equiv \alpha^- + z(t)^2 \in \widetilde{\gamma}(T_t)$ is an eigenvalue of T_t.
If $z(t) < 0$, then $\lambda(t) \in \widetilde{\gamma}(T_t) \subset \Lambda_0^{(m_0)}$ is a resonance of T_t. Here $m_0 = 2$ if $\alpha_{m,1}^- = \alpha_{n,2}^-$; $m_0 = 3$ if $\alpha_{m,1}^- > \alpha_{n,2}^-$ and $m_0 = 4$ if $\alpha_{m,1}^- < \alpha_{n,2}^-$. Moreover, the following asymptotics is fulfilled:

$$z(t) = \int_0^t \mathcal{L}(t,\alpha^-)dt + O(t^{3/2}) \quad as \quad t \to 0. \tag{5.2}$$

ii)If p_1, p_2 are continuous then $z(\cdot) \in C^1(-\varepsilon, \varepsilon)$ and $z(t) = \int_0^t \mathcal{L}(t, \alpha^-)dt + O(t^2), t \to 0$.
iii) If p_1, p_2 are analytic functions, then $z(\cdot)$ is real analytic on $(-\varepsilon, \varepsilon)$.
Remark. *Result for an eigenvalue in some neighborhood of $\alpha_n^+, n \geqslant 0$, is similar.*
Proof. i) There are 3 cases: $\alpha_{n,2}^- < \alpha_{m,1}^-, a_{n,2}^- = a_{m,1}^-, a_{n,2}^- > a_1^-$. Recall $b_2(\lambda) = \sqrt{\Delta_2^2(\lambda) - 1}$, and

$$\zeta_2(\lambda, t) = \frac{\dot{\varphi}_2(1, \lambda, t)}{2\varphi_2(1, \lambda, t)}, \qquad m_2^+(\lambda, t) = \zeta_2(\lambda, t) - \frac{b_2(\lambda)}{\varphi_2(1, \lambda, t)}, \qquad \lambda \in \gamma_2, \quad t \in [0, 1). \tag{5.3}$$

Define a new parameter z by the formula $z = z(\lambda) = \sqrt{\lambda - \alpha^-}$, where $z(\lambda) > 0, \lambda \in \Lambda$. Firstly, let $\alpha_{n,2}^- = \alpha^- > \alpha_{m,1}^-$. By (3.9), the function $b_{02}(z) \equiv b_2(\alpha^- + z^2)$ is analytic in some neighborhood of zero and we get $b_{02}(z) = z\beta_2^-(1 + O(z^2))$, as $z \to 0$, where $\beta_2^- \equiv \Delta_2(\alpha^-)\sqrt{2|M_{n,2}^-|} > 0$. The function $1/\varphi_2(1, \alpha^- + z^2, t)$ is smooth in the parameter (z, t) in some neighborhood of zero. Therefore, $w(\alpha^- + z^2, t)$ is a smooth function of the parameter (z, t) in some neighborhood of zero and $\dot{w} = \dot{m}_2^+$ (see (3.19)). Moreover, $w_z(\alpha^- + z^2, 0)|_{z=0} = -\beta_2^-/\varphi_2(1, \alpha^-, 0) \neq 0$. Then by the implicit function Theorem there exists a function $z(\cdot) \in W_1^2(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ such that $w(\alpha^- + z^2(t), t) = 0$ and $z(0) = 0$. We prove (5.2). Let $\Omega = 1 + O(|z| + |t|)$. Relations (3.19-20), (3.9) yield

$$\dot{w}(\lambda, t) = \dot{m}_2^+(\lambda, t) = \dot{\zeta}_2(\lambda, t) + O(z) = \dot{\zeta}_2(\alpha^-, t) + O(z), \qquad w_z(\lambda, t) = -\frac{\beta_2^-\Omega}{\varphi_2(1, \alpha^-, t)}, \tag{5.4}$$

as $t \to 0$. Using (5.4), (3.22), we obtain

$$\dot{z}(t) = -\frac{\dot{w}(\lambda(t), t)}{w_z(\lambda(t), t)} = \frac{\varphi(1, \alpha^-, t)\dot{\zeta}_2(\alpha^-, t) + O(z)}{\beta_2^-\Omega} = (\mathcal{L}(t, \alpha^-) + O(z))\Omega. \tag{5.5}$$

Let $z_m = \max|z(y)|, 0 \leqslant y \leqslant t$. Then integration implies

$$z(t) = \int_0^t \dot{z}(s)ds = \int_0^t \mathcal{L}(s, \alpha^-)ds + \sqrt{t}\|p_2\|O(|z_m| + |t|) + O(t^2 + |tz_m|), \tag{5.6}$$

hence $z(t) = O(\sqrt{t})$ and substituting this into (5.6) we get (5.2).
Secondly, let $\alpha_{n,2}^- = \alpha^- = \alpha_{m,1}^-$. By (3.9), the function $b_{0m}(z) \equiv b_m(\alpha^- + z^2), m = 1, 2$, is analytic in some neighborhood of zero and we get $b_{0m}(z) = z\beta_m^-(1 + O(z^2)), z \to 0$, where $\beta_m^- \equiv \Delta_2(\alpha_-)\sqrt{|2M_{m,1}^-|} \neq 0$. The function $1/\varphi_2(1, \alpha^- + z^2, t)$ is smooth in the parameter (z, t) in some neighborhood of zero. Therefore, $w(\alpha^- + z^2, t)$ is a smooth function of the parameter (z, t) in some neighborhood of zero. Moreover,

$$\frac{\partial w(\alpha^- + z^2, 0)}{\partial z}|_{z=0} = -\frac{\beta_2^-}{\varphi_2(1, \alpha^-, 0)} - \frac{\beta_1^-}{\varphi_1(1, \alpha^-)} = -\frac{\beta_2^- r(\alpha^-, 0)}{\varphi_2(1, \alpha^-, 0)} \neq 0, \tag{5.7}$$

since $\beta_1^-\varphi_2/(\beta_2^-\varphi_1) = (\beta_2^-\Psi_2)^2/(\beta_1^-\Psi_1)^2)$. Then by the implicit function Theorem there exists a function $z(\cdot) \in W_1^2(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, such that $w(\alpha^- + z^2(t), t) = 0$ and $z(0) = 0$. We find the asymptotics of $z(t)$ as $t \to 0$. Using (3.19-20) we deduce that

$$\dot{w}(\lambda, t) = \dot{m}_2^+(\lambda, t) = \dot{\zeta}_2(\lambda, t) + O(z) = \dot{\zeta}_2(\alpha^-, t) + O(z), \quad t \to 0, \tag{5.8}$$

and by (5.7),

$$w_z(\lambda_n(t),t)=\Big[\frac{-\beta_2^-}{\varphi_2(1,\alpha^-,0)}+\frac{-\beta_1^-}{\varphi_1(1,\alpha^-)}\Big]\Omega=\frac{-\beta_2^- r(0,\alpha^-)}{\varphi_2(1,\alpha^-,t)}\Omega. \tag{5.9}$$

Then (5.8-9), (3.22) yield

$$\dot z(t)=-\frac{\dot w(\lambda(t),t)}{w_z(\lambda(t),t)}=\frac{\varphi_2(1,\alpha^-,t)\dot\zeta_2(\lambda_n(t),t)+O(z)}{r(\alpha^-,0)\beta_2^-\Omega}=(\mathcal{L}(t)+O(z))\Omega. \tag{5.10}$$

Then as above we get (5.2).
Next, we consider the last case $\alpha_{n,2}^-<\alpha^-=\alpha_{m,1}^-$. By (3.9), the function $b_{01}(z)\equiv b_1(\alpha^-+z^2)$ is analytic in some neighborhood of zero and we get $b_{01}(z)=z\beta_1^-(1+O(z^2))$, as $z\to 0$. The function $1/\varphi_2(\alpha^-+z^2,t)$ is smooth in the parameter (z,t) in some neighborhood of zero. Therefore, $w(\alpha^-+z^2,t)$ is a smooth function of the parameter (z,t) in some neighborhood of zero. Moreover, $w_z(\alpha^-+z^2,0)|_{z=0}=-\beta_1^-/\varphi_1(1,\alpha^-,0)\neq 0$. Then by the implicit function Theorem there exists a function $z(\cdot)\in W_1^2(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$ such that $w(\alpha^-+z^2(t),t)=0$ and $z(0)=0$. We find the asymptotics of $z(t)$ as $t\to 0$. By (3.19-20),

$$\dot w(\lambda,t)=\dot m_2^+(\lambda,t)=(p_2(t)-\lambda)-m_2^+(\lambda,t)^2=(p_2(t)-\lambda)-\zeta_1^+(\alpha^-)^2+O(z),\quad t\to 0, \tag{5.11}$$

and by (3.9),

$$w_z(\lambda,t)=-\beta_1^-\varphi_1(1,\alpha^-,0)^{-1}\Omega. \tag{5.12}$$

Hence the asymptotics (5.11-12) and the identity (3.22) imply

$$\dot z(t)=-\frac{\dot w(\lambda_n(t),t)}{w_z(\lambda_n(t),t)}=\frac{\varphi_1(1,\alpha^-)[(p_2(t)-\lambda)-\zeta_1(\alpha^-)^2+O(z)]}{\beta_1^-\Omega}=(\mathcal{L}(t,\alpha^-)+O(z))\Omega. \tag{5.13}$$

Then, as above we get (5.2).
ii) If $p_,p_2$ is continuous, then using (5.5), (5.10), (5.13) we obtain the needed estimates.
iii) If p_1,p_2 are analytic functions, then by the implicit function Theorem, $z(\cdot)$ is real analytic on $(-\varepsilon,\varepsilon)$. If $z(t)>0$ there exists an eigenvalue $\lambda(t)$ with the asymptotics (5.2). But if $z(t)<0$, then there is no an eigenvalue of T_t in the interval $[\alpha^-,\alpha^--\varepsilon]$ and we have a resonance. ■

We study an eigenvalue λ_0 of T_0 in the 0-th gap $\gamma_0(T_0)$.
Proposition 5.2 *Let q_0 be a biperiodic potential given by (1.1). Let $\gamma_0(H_j)=(-\infty,\alpha_{0,j}^+)$ be the infinite gaps of $H_j, j=1,2$ and $\min\{\alpha_{0,1}^+,\alpha_{0,2}^+\}=0$. Assume that $\nu_{0,j}\in\gamma_0(H_j)$ is the corresponding Neumann eigenvalue. Then $\#(T_0,(-\infty,\nu^0])=0$, where $\nu^0=\min\{\nu_{0,1},\nu_{0,2}\}$. Let in addition $w(0)>0$, then $\#(T_0,\gamma_0(T_0))=1$.*
Proof. If $\lambda<\nu^0$, then by Lemma 3.5, $w(\lambda)=m_2^+(\lambda)-m_1^-(\lambda)<0$, which yields the identity $\#(T,(-\infty,\nu^0])=0$. Let $w(0)>0$. Then by (4.5), $w(\lambda)\to-\infty$ as $\lambda\to-\infty$, and there exist a zero of $w(\lambda),\lambda<0$. But there is no other eigenvalue in the gap since if we have 2, then we get 3 eigenvalues in the gap. This is impossible. ■

6 Dislocations and half-solid

1. Dislocations. We consider the operator $T_t^{di} = -\frac{d^2}{dx^2} + p_{(t)}$ acting in $L^2(\mathbb{R})$, where the dislocation potential $p_{(t)} = \chi_- p + \chi_+ p(\cdot + t), t \in \mathbb{R}$ and $p \in L^1(\mathbb{T})$. Recall that due to (2.6-7) we have only two sheets $\Lambda_0^{(1)}, \Lambda_0^{(2)}$ of the energy surface. Identities (3.10), (4.10) yield

$$w(\lambda, t) = m^+(\lambda, t) - m^-(\lambda, 0) = \frac{a(\lambda, t) - b(\lambda)}{\varphi(1, \lambda, t)} - \frac{a(\lambda, 0) - b(\lambda)}{\varphi(1, \lambda, 0)}, \qquad \lambda \in \Lambda_E(T_0). \tag{6.1}$$

Proof of Theorem 2.4. We consider the case α_n^- and let n be even. The proof for α_n^+ or for odd n is similar. Let $\mu_n(t) = \mu_n(p, t)$. Firstly, we consider the simple case $\alpha_n^- \neq \mu_n(0)$. Define a new parameter z by the formula $z = z(\lambda) = \sqrt{\lambda - \alpha^-}$, where $z(\lambda) > 0, \lambda \in \Lambda_0$. Recall that the function $b_0(z) \equiv b(\alpha_n^- + z^2)$ is analytic in some neighborhood of zero and has asymptotics (3.9). The function $1/\varphi(1, \alpha_n^- + z^2, t)$ is smooth in the parameter (z, t) in some neighborhood of zero. Therefore, $w(\alpha_n^- + z^2, t)$ is a smooth function of the parameter (z, t) in some neighborhood of zero. Moreover, $w_z(\alpha_n^- + z^2, 0)|_{z=0} = -2\beta_n/\varphi(1, \alpha_n^-, 0) \neq 0$. Then by the implicit function Theorem there exists a function $z(\cdot) \in W_1^2(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ such that $w(\alpha_n^- + z^2(t), t) = 0, t \in (-\varepsilon, \varepsilon)$ and $z(0) = 0$. We prove (2.8). By (3.19-20),

$$\dot{w}(\lambda_n(t), t) = \dot{m}^+(\lambda_n(t), t) = \dot{\zeta}(\lambda_n(t), t) + O(z) = \dot{\zeta}(\alpha_n^-, t) + O(z), \tag{6.2}$$

$$w_z(\lambda_n(t), t) = -2\beta_n/\varphi(1, \alpha_n^-, t)\Omega, \quad \Omega = 1 + O(|z| + |t|), \tag{6.3}$$

as $t \to 0$. Recall $L(t, \lambda) = \dot{\Psi}(t, \lambda)^2 - (p(t) - \lambda)\Psi(t, \lambda)^2$. Using (6.2-3), (3.22) we obtain

$$\dot{z}(t) = -\frac{\dot{w}(\lambda_n(t), t)}{w_z(\lambda_n(t), t)} = \frac{\varphi(1, \alpha_n^-, t)\dot{\zeta}(\lambda_n(t), t) + O(z)}{2\beta_n\Omega} = \frac{\beta_n}{2} L(t, \alpha_n^-)\Omega + O(z). \tag{6.4}$$

Let $z_m = \max|z(y)|, 0 \leqslant y \leqslant |t|$, then integration yields

$$z(t) = \int_0^t \dot{z}(s)ds = (\beta_n/2)\int_0^t L(s, \alpha_n^-)ds + \|p\|\sqrt{t}O(|z_m| + |t|) + O(t^2 + |tz_m|). \tag{6.5}$$

Hence $z(t) = O(|t|^{1/2})$ and then $z_m = O(|t|^{1/2})$ and (6.6) yields (2.8).

Secondly we consider the more complicated case $\mu_n(0) = \alpha_n^-$. Recall $\mu_n(t) = \mu_n(p, t)$. Let $a^0 = a(\lambda, 0), \vartheta_x^0 = \vartheta_x(1, \lambda, 0)$. If we multiply (6.1) by $(a + b)$ and $(a^0 - b)$ and use (3.11) we obtain $-\vartheta_x(a^0 - b) + \vartheta_x^0(a + b) = 0$. Therefore, we have the following equation for the eigenvalues:

$$\Phi(\lambda, t) \equiv b(\vartheta_x + \vartheta_x^0) + (a\vartheta_x^0 - a^0\vartheta_x) = 0. \tag{6.6}$$

Recall that $\varphi(1, \lambda, t) = (\lambda - \mu_n(t))\phi(\lambda, t)$. Using (3.17) we have

$$\dot{\varphi} = -\dot{\mu}_n\phi + (\lambda - \mu_n)\dot{\phi}, \qquad \ddot{\varphi} = -\ddot{\mu}_n\phi - 2\dot{\mu}_n\dot{\phi} + (\lambda - \mu_n)\ddot{\phi}, \tag{6.7}$$

$$\varphi(1, \lambda, t) = O(|z|^2 + t^2), \qquad \dot{\varphi}(1, \lambda, t) = O(|z|^2 + |t|), \quad \text{as } \ t \to 0, \tag{6.8}$$

(6.8) yields $\Phi_z(\alpha_n^-, 0) = 2\beta_n\vartheta_x(1, \alpha_n^-, 0) \neq 0$ and the function $\Phi(z, t)$ is analytic in z and continuous in t. Then by the implicit function Theorem, there exists a function $z(\cdot) \in$

$W_1^2(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$ such that $\Phi(z(t),t)=0$ and $z(0)=0$. We will prove (2.9) and we have

$$\Phi_z(z,t)=2\beta_n\vartheta_x(1,\alpha_n^-,0)+O(|z|+|t|),\qquad \text{as}\quad t\to 0, \tag{6.9}$$

and due to (3.12) we get

$$\dot\Phi(z,t)=(\beta-a^0)\dot\vartheta_x+\dot a\vartheta_x^0=(\beta-a^0)(\lambda-p)\dot\varphi-\vartheta_x\vartheta_x^0-(\lambda-p)\varphi\vartheta_x, \tag{6.10}$$

$$\dot\Phi(z,t)=-\vartheta_x(1,\alpha_n^-)^2+p(t)O(t^2+z^2)+O(|t|+|z|) \tag{6.11}$$

Note that $\vartheta_x(\mu_n(0),0)\neq 0$ and by (3.11), $-(b^2)_\lambda=-\vartheta_x(1,\lambda,0)\varphi_\lambda(1,\lambda,0)$, at $\lambda=\alpha_n^-$. Then (3.17) yields

$$\vartheta_x(1,\alpha_n^-,0)=-2M_n^-/\varphi_\lambda(1,\alpha_n^-,0)=\ddot\mu_n(0)\phi/2>0. \tag{6.12}$$

Using (3.12), we have $\ddot\varphi(1,\alpha_n^-,0)=-2\beta_n^2\dot\Psi(0,\alpha_n^-)^2$ and (6.9) yields $\ddot\varphi(1,\alpha_n^-,0)=-\ddot\mu_n(0)\phi$. Therefore, we get $\vartheta_x^0/2\beta_n=(\beta_n/2)\dot\Psi(0,\alpha_n^-)^2$. Using (6.8-12) we obtain

$$z(t)=-\int_0^t\frac{\dot\Phi(\lambda_n(t),t)}{\Phi_z(\lambda_n(t),t)}dt=-t\frac{\vartheta_x^0}{2\beta_n}+\int_0^t[|p(t)|O(t^2+z^2)+O(|t|+|z|)]dt \tag{6.13}$$

and since $p\in L^2(0,1)$, (6.13) implies

$$z(t)=t(\beta_n/2)\dot\Psi(0,\alpha_n^-)^2+\sqrt{|t|}(\|p\|)O(|z_m|^2+t^2))+O(|z_mt|+t^2). \tag{6.14}$$

Therefore, $z(t)=t(\beta_n/2)\dot\Psi(0,\alpha_n^-)^2+\sqrt{|t|}O(|z_m|)$, and $z_m(t)=O(|t|^{1/2})$. Substituting these into (6.14) we have (2.9). Note that $\lambda_n(t)=\alpha^-+z(t)^2$ while $z(\cdot)$ has the asymptotics (2.9) and the function $a+b$ has a zero $\mu_n(t)$ in the neighborhood of zero. It is important that $\lambda_n(t)$ and $\mu_n(t)$ have different asymptotics, see (2.9) and (3.17).

Assume that p is an analytic function. Hence by the implicit function theorem, there exists a unique analytic function $z_n^-(t)$ of the equation $w(t,z)=0$ for all $t\in(-\varepsilon,\varepsilon)$ for some small $\varepsilon>0$. Then $z(t)=rt^m(1+O(t))$, for some $r\neq 0, m\geqslant 1$. ■

We prove the existence of two eigenvalues in the gap for some potentials.

Proof of Theorem 2.5. In Lemma 3.4 we have proved that for any finite sequences $\{d_n\}^N,\{s_n\}^N, d_n\in\{0,1\}, s_n>0, N\geqslant 1$ there exists an even potential $p\in C^2(\mathbb{T})$ such that $\gamma_n(p)=s_n$, and $L(0,\alpha_n^\pm)>0$, if $d_n=1$ and $L(0,\alpha_n^\pm)<0$, if $d_n=0$ for $n=1,...,N$. We take the dislocation potential in the form $p_{(t)}=\chi_-p+\chi_+p(\cdot+t)$. Then by Theorem 2.4, there exists $\varepsilon>0$ such that $|\gamma_n(T_t^{di})|=s_n$, and $\#(T_t^{di},\gamma_n(T_t^{di}))=2d_n$ for all $n=1,2,..,N$, and each $t\in(0,\varepsilon)$. ■

2. Half-solid. We consider the half-solid operator $T_t^s=-\frac{d^2}{dx^2}+q_t^s(x)$ acting in $L^2(\mathbb{R})$, where the potential $q_t^s(x)=s\chi_-+\chi_+p(\cdot+t)$ and $s,t\in\mathbb{R}$, $p\in L^1(\mathbb{T})$ is real. Assume $\alpha_0^+(H)=0$. We have the following simple results concerning $\sigma(T_t^s)$. If $s\leqslant\alpha_1^-(H)$, then $\sigma_{ac}(T_t^s)=(s_-,\infty)$, where $s_-=\min(0,s)$. If $s>\alpha_1^-$ then there is a gap in the spectrum of T_t^s. Our goal is to study the eigenvalues in the gaps $\gamma_n(T_t^s), n\geqslant 0$, and to find how these eigenvalues depend on t,s. It is clear that they depend on t periodically. Using (4.6) we rewrite the Wronskian $w(z,t)$ in the form

$$w(\lambda,t)=m^+(\lambda,t)-\sqrt{s-\lambda}=\frac{a(\lambda,t)-b(\lambda)}{\varphi(1,\lambda,t)}-\sqrt{s-\lambda},\quad \lambda<s, \tag{6.15}$$

where $\sqrt{s-\lambda} > 0$ if $\lambda < s, \lambda \in \Lambda_0^{(1)}$.

Proof of Theorem 2.6. i) Using Theorem 2.2 with $p_1 = s, p_2 = p$, we obtain (2.12) and the identity $\#(T_0^s, \gamma_0(T_0^s)) = 0$. By Lemma 4.3, the Wronskian w on the second sheet $\Lambda^{(4)}$ has the form $w(\lambda^{(4)}) = b(\lambda)\varphi(1,\lambda)^{-1} - \sqrt{s-\lambda}$. Repeating the proof of Theorem 2.2 we get $\#^{(4)}(T_0^s, \gamma_n(T_0^s)) = 1 - \#(T_0^s, \gamma_n(T_0^s))$. Moreover, (2.4) yields (2.12).

ii) Let $q_0^s \in Q_N, N \geqslant 0$ and $p \in L^2(\mathbb{T})$ be even. Then due to Theorem 2.2 we have the sequence $\{d_n\}_1^N$, $d_n = \#(T_0^s, \gamma_n(T_0^s)), n \geqslant 1$, and $\varepsilon \in (0,1)$ and $r \in \ell^2$. Note that Theorem 2.1 yields $d_n = 0$ if $|\gamma_n(T_0)| = 0$. Hence we have a mapping $q_0^s \to (r,d,\varepsilon) \in P$.

Let $(r,d,\varepsilon) \in P_N, N \geqslant 0$. Define the signed gap length $l_n(p) = (\mu_n(p), \nu_n(p)), n \geqslant 1$ (see [GT]). For the sequence $r \in \ell^2$, there exists an even potential $p \in L^2(\mathbb{T})$ such that $|l_n(p)| = r_n \geqslant 0, n = 1,..,N,\ l_n(p) = r_n, n \geqslant N+1$. In order to get uniqueness of p we need to determine $\operatorname{sign} l_n(p), n = 1,..,N$, (see [GT]).

Fix a number $n = 1,..,N$ and the gap $\gamma_n(H) = (\alpha_n^-, \alpha_n^+) \neq \emptyset$. We have 2 cases. First, if $d_n = 1$, then using (2.12) we obtain $l_n(p) > 0$. Second, if $d_n = 0$, then using (2.12) we obtain $l_n(p) < 0$. Hence we have $\{l_n\} \in \ell^2$ and there exists a unique $p \in L^2(\mathbb{T})$. Using $\varepsilon \in (0,1)$ and the definition of ε we obtain s, which yields a unique $q_0^s \in Q_N$. ■

By Lemma 3.4, for $n \geqslant 1$, Eq. (2.13) has $N \geqslant n$ roots $y_1, y_2, \ldots, y_N \in [0,1)$. The case $n \geqslant 1$ arises for the gap $\gamma_n(T_t(s)), n \geqslant 1$, and $n = 0$ corresponds the basic gap $\gamma_0(T_t^s)$.

Proof of Theorem 2.7 i) We consider the case $\alpha_n^- < s$. The proof for $\alpha_n^+ < s$ is similar. The proof follows from i) of Theorem 5.1. In this case the Weyl function has the form (6.16) and $\mathcal{L}(y,\lambda) = -[\Psi'(y,\lambda)^2 + (\lambda - p(y))\Psi(y,\lambda)^2], \lambda = \alpha_n^-$. The identity (3.5) yields $\zeta(y,\alpha_n^-) = \Psi'(y,\alpha_n^-)/\Psi(y,\alpha_n^-)$. Together with (6.16) this implies $w(\alpha_n^-, y) = 0$. All conditions of Theorem 5.1, i) are fulfilled and we get $z_n^\pm(t)$ with the needed properties. By (2.13), $\Psi_y(y,\lambda) = \sqrt{s-\lambda}\Psi(y,\lambda)$, then $2\mathcal{L}(y,\alpha_n^-) = (p(y)-s)\Psi(y,\alpha_n^-)^2$ and (5.2) yields (2.14).

ii) In this case we use i) of Theorem 5.1, when s is the end of the gap (α_n^-, s) of the operator $T_y(s)$. We have $\Psi_1 \equiv 1, \Psi_1' \equiv 0, M_1^+ = 1/2$, then $2\mathcal{L}(y,s) = (p(y)-s)$ and Theorem 5.1 (see the remark after this Theorem) implies the needed results including (2.15). ■

Now we consider eigenvalues (the ground state) in the gap $\gamma_0(T_t^s)$. The existence of eigenvalue is connected with some property of the function $m^+(0,t), t \in [0,1]$, when $b > 0, \varphi > 0$ in the gap $\gamma_0(T_t^s)$. We have the following result.

Theorem 6.1 *Let $T_0^s = -\frac{d^2}{dx^2} + s\chi_-(x) + \chi_+(x)p(x)$, where $p \in L^1(\mathbb{T})$ is real and let $\alpha_0^+ = 0$. If $m^+(0) < 0$, then $\#(T_0^s, \gamma_0(T_0^s)) = 0$ and if $m^+(0) > 0$, then*

$$\#(T_0^s, \gamma_0(T_0^s)) = \begin{cases} 0, & \text{if} \quad s \leqslant \nu_0 \text{ or } s \geqslant m^+(0)^2 \\ 1, & \text{if} \quad \nu_0 < s < m^+(0)^2 \end{cases}. \tag{6.16}$$

Proof. Let $m^+(0) < 0$, then Lemma 3.5 yields $w(\lambda) < 0$ for $\lambda < \min\{s,0\}$. Hence eigenvalues in $\gamma_0(T_0^s)$ are absent.

Let $m^+(0) > 0$. Assume $\lambda < s \leqslant \nu_0$. Again by Lemma 3.5, $m^+(\lambda) < 0$ for $\lambda < \nu_0$. Then $w(\lambda) < 0$ for $\lambda < s \leqslant \nu_0$, and eigenvalues in $\gamma_0(T_0^s)$ are absent. Now, assume $m^+(0)^2 \leqslant s$. Let the function w have a simple zero $\lambda_1 \in (\nu_0, 0)$. Then for some $s_0 > s$ the function $m^+(\lambda) - \sqrt{s_0 - \lambda}$ has a multiple root $\lambda_1 \in (\nu_0, 0)$. But this is impossible. Then the first line in (6.17) is true. Consider the case $\nu_0 < s < m^+(0)^2$. Then by Lemma 3.5, $w(\nu_0) < 0$ and $w(\lambda_1) > 0, \lambda_1 = \min\{s,0\}$. Hence Proposition 5.2 yields the second line in (6.17). ■

References

[A] Anoshchenko, O.: The inverse scattering problem for the Schrödinger equation with a potential that has periodic asymptotics. J. Soviet Math. 48 (1990), no. 6, 662–668

[A1] Anoshchenko, O.: Eigenfunction expansion of the Schrödinger equation with a potential that has periodic asymptotics. J. Soviet Math. 49 (1990), no. 6, 1237-1241

[BS] Bikbaev R., Sharipov R.: Asymptotics as $t \to \infty$ of the Cauchy problem for the Korteveg-de Vries equation in the class of potentials with finite-gap behavior at $x \to \pm\infty$, Theoret. and Math. Phys. 78, 1989.

[CL] Coddington, E., Levinson, N.: Theory of ordinary differential equations. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[DS] Davies E., Simon B. Scattering theory for systems with different spatial asymptotics on the left and right, Commun. Math. Phys. 63, 277-301, 1978

[GT] Garnett J., Trubowitz E.: Gaps and bands of one dimensional periodic Schrödinger operator. Comment. Math. Helv. 59, 258-312 (1984).

[KK1] Kargaev P., Korotyaev E.: The Inverse Problem for the Hill Operator, the Direct Approach, Invent. Math., 129, no. 3, 567-593 (1997)

[KK2] Kargaev P., Korotyaev E.: Effective masses and conformal mappings. Commum. Math. Phys. 169, 597-625 (1995).

[K1] Korotyaev E.: Estimates for the Hill operator, I. J. Diff. Eq. 162, 2000, 1-26

[K2] Korotyaev E.: Inverse Problem and the trace formula for the Hill Operator, II, Math. Z., 231, 345-368 (1999)

[K3] Korotyaev E.: Lattice dislocations in 1-dimensional model. Commun. Math. Phys. 213, 471-489, 2000

[KP] Korotyaev E., Pokrovski, A.: One dimensional half - crystal (in preparation)

[Kr] Krein M.: Theory of selfadjoint extensions of semi- bounded Hermitian operators and its applications. I. Mat. Sbornik 20, No. 1, 1-95, 1947.(Russian)

[L] Levitan B.M.: The Inverse Sturm-Liouville Problems. Moscow, Nauka, 1984.(Russian)

[M] Marchenko V.: Sturm-Liouville operator and applications. Basel: Birkhäuser 1986.

[Mos] Moser J. : An Example of a Scrödinger operator with almost periodic potential and nowhere dense spectrum. Comment. Math. Helv. 56, 198-224 (1981).

[Ta] Tamm I. Phys.Z. Sowjet.1, 1932, 733.

[T] Titchmarsh E.: Eigenfunction expansions associated with second-order differential equations 2, Clerandon Press, Oxford, 1958.

[Tr] Trubowitz E.: The Inverse problem for Periodic Potentials . Commun. on Pure and Applied Math. V. 30, 321-337, 1977.

[PTr] Pöschel, P.; Trubowitz E.: Inverse Spectral Theory. Boston: Academic Press, 1987.

[Z] Zheludev V. On the spectrum of Schrödinger operator with periodic potentials on the half-line. Trudy kafedry mat. anal. Kaliningarad Univ. 1969.