THE SPECTRUM OF MAGNETIC SCHRÖDINGER OPERATORS AND k-FORM LAPLACIANS ON CONFORMALLY CUSP MANIFOLDS

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ABSTRACT. We consider open manifolds which are interiors of a compact manifold with boundary, and Riemannian metrics asymptotic to a conformally cylindrical metric near the boundary. We show that the essential spectrum of the Laplace operator on functions vanishes under the presence of a magnetic field which does not define an integral relative cohomology class. It follows that the essential spectrum is not stable by perturbation even by a compactly supported magnetic field. We also treat magnetic operators perturbed with electric fields. In the same context we describe the essential spectrum of the k-form Laplacian. This is shown to vanish precisely when the k and $k-1$ de Rham cohomology groups of the boundary vanish. In all the cases when we have pure-point spectrum we give Weyl-type asymptotics for the eigenvalue-counting function. In the other cases we describe the essential spectrum.

INTRODUCTION

There exist complete, noncompact manifolds on which the scalar Laplacian has pure point spectrum (e.g., [6]). One of the goals of this paper is to exhibit the same phenomenon for the Laplacian on differential forms. We study Riemannian manifolds X with ends diffeomorphic to a cylinder $[0,\infty)\times M$, where M is a closed, possibly disconnected Riemannian manifold. The metric on X is the asymptotically conformally cylindrical metric already studied in [21]. The general form of such a metric is given in Eq. (7); for the purpose of this introduction, the reader should bear in mind the toy example

$$g_p = y^{-2p}(dy^2 + h), \qquad y \to \infty \tag{1}$$

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where h is a metric on M and $p > 0$. We first give conditions for the Laplacian on $\Lambda^k X$ to have pure point spectrum.

Theorem 1. *Let $p > 0$, g_p the metric given by (7) and*

$$\Delta_{k,p} : \mathcal{C}_c^\infty(X, \Lambda^k X) \to L^2(X, \Lambda^k X)$$

the Laplacian on k-forms on X, viewed as an unbounded operator in L^2. Assume that the Betti numbers $h^k(M)$ and $h^{k-1}(M)$ both vanish. Then $\Delta_{k,p}$ is essentially self-adjoint and has pure point spectrum.

This result will be restated in larger generality in Theorem 3. Note that X is complete if and only if $p \leq 1$. In that case, it is well-known that the Laplacian on forms is essentially self-adjoint. Without the hypothesis on the cohomology of the end, we can describe the essential spectrum of $\Delta_{k,p}$.

Theorem 2. *Assume that not both $h^k(M)$ and $h^{k-1}(M)$ vanish. Let g_0 be an* exact *cusp metric (see Definition 4), $0 < p \leq 1$ and $g_p = x^{2p} g_0$. Then the essential spectrum of $\Delta_{k,p}$ is that of a direct sum of at most two ordinary differential equations on the real half-line, and*

- *If $0 < p < 1$ then $\sigma_{\mathrm{ess}}(\Delta_{k,p}) = [0, \infty)$.*
- *If $p = 1$ then $\sigma_{\mathrm{ess}}(\Delta_{k,1}) = [c, \infty)$ where*

$$c = \begin{cases} \left(\frac{n-2k-1}{2}\right)^2 & \text{if } h^{k-1}(M) = 0; \\ \left(\frac{n-2k+1}{2}\right)^2 & \text{if } h^k(M) = 0; \\ \min\left\{\left(\frac{n-2k-1}{2}\right)^2, \left(\frac{n-2k+1}{2}\right)^2\right\} & \text{otherwise.} \end{cases}$$

In particular, the Laplacian of g_p on 0-, 1- and n-forms always has non-empty essential spectrum when X is complete, i.e., $p \leq 1$.

For $p > 1$ we have a partial result, namely we show that all self-adjoint extensions of $\Delta_{k,p}$ have pure point spectrum but only for the metric (1) (see Proposition 26).

In a recent paper Antoci [4] studies the essential spectrum of a manifold X with ends as above, with metric on the end

$$e^{-2(a+1)t} dt^2 + e^{-2bt} h. \tag{2}$$

Antoci assumes that X is complete so $a \leq -1$. But with the change of variables $y = e^{(b-a-1)t}$ the metric (2) is a constant multiple of (1) under the assumption that $b-a-1 > 0$ and $b < 0$. Thus our results have some non-empty intersection with [4]; we improve one of Antoci's results by showing that 0 cannot be isolated in the essential spectrum for a general

M. (for $M = S^{n-1}$ this is proved in [4, Theorem 6.1]). This issue is actually important, since the absence of 0 from the essential spectrum is equivalent to the Hodge decomposition

$$L^2(X, \Lambda^k X, g_p) = \ker(\Delta_{k,p}) \oplus \mathrm{Im}(\Delta_{k,p}).$$

We also get somewhat more refined information on the nature of the essential spectrum. Namely for the metric (1) we show the absence of singular continuous spectrum (a classical fact for $k = 0$).

The second topic of this article is the so-called magnetic Laplacian on a manifold X as above. We develop this in Section 3. A magnetic field B is a smooth exact 2-form on X. There exists a real 1-form A, called magnetic potential, satisfying $dA = B$. Set $d_A := d + iA\wedge : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X, T^*X)$. The magnetic Laplacian on $\mathcal{C}^\infty(X)$ is given by

$$\Delta_A := d_A^* d_A \tag{3}$$

When the manifold is complete, this operator is essentially self-adjoint [23]. Given two magnetic potentials A and A' such that $A - A'$ is exact, the two magnetic Laplacians Δ_A and $\Delta_{A'}$ are unitarily equivalent (this property is called gauge invariance). Hence when $H^1(X, \mathbb{R}) = 0$, the spectral properties of the magnetic Laplacian do not depend on the choice of the magnetic potential A.

We have found a strong relationship between the emptiness of the essential spectrum of Δ_A and the cohomology class $[B]$ inside the relative cohomology group $H^2(X, M)$. We show first that if the magnetic field does not vanish on the boundary, then the operator Δ_A has compact resolvent. In comparison, in $\mathbb{R}^n$ with the Euclidean metric this happens when the magnetic field blows up at infinity with a certain control on its derivatives (see [12] and references therein). Furthermore, if B does vanish on M but it defines a non-integral cohomology class $[B]$ inside the relative cohomology group $H^2(X, M)$ (up to a factor of 2π), then the same conclusion holds.

In $\mathbb{R}^n$, a compactly supported magnetic field does not affect the essential spectrum [5, 17] and furthermore the wave operators are asymptotically complete for a certain choice of gauge [24]. In contrast, on a manifold X as above, in dimensions other than 3 we find in particular that a generic magnetic field B with compact support will turn off the essential spectrum of Δ. It seems quite surprising on one hand that we get the (non)integrality condition, and on the other hand that there are no orientable examples in dimension 3. We stress that it is not the size of the magnetic field which kills the essential spectrum, but a more subtle phenomenon which has yet to be interpreted physically.

In the case of $H^1(X,\mathbb{R}) \neq 0$, in quantum mechanics, it is known that the choice of a potential vector has a physical meaning. This is known as the Aharonov-Bohm effect [2]. In $\mathbb{R}^n$ with a bounded obstacle, this phenomenon can be seen through a difference of phase of the waves arising from two non-homotopic paths that circumvent the obstacle. Note that the essential spectrum is independent of this choice. In contrast, in our setting we show that the essential spectrum itself can be turned off by a suitable choice of magnetic potential.

In all these instances when the operators have empty essential spectra, the eigenvalues turn out to satisfy an asymptotic law that we prove using results from [21]. We state here only one such result and we refer to the body of the paper for the others.

Theorem 3. *In the setting of Theorem 1, assume that $h^k(M) = h^{k-1}(M) = 0$. Let $N_{k,p}$ be the counting function of the eigenvalues of $\Delta_{k,p}$. Then*

$$N_{k,p}(\lambda) \approx \begin{cases} C_1\lambda^{n/2} & \text{for } 1/n < p < \infty, \\ C_2\lambda^{n/2}\log\lambda & \text{for } p = 1/n, \\ C_3\lambda^{1/2p} & \text{for } 0 < p < 1/n \end{cases} \tag{4}$$

where

$$\begin{aligned} C_1 &= \binom{n}{k}\frac{\operatorname{Vol}(X,g_p)\operatorname{Vol}(S^{n-1})}{n(2\pi)^n}, \\ C_2 &= \binom{n}{k}\frac{\operatorname{Vol}(M,{a_0}^{1/2}dh_0)\operatorname{Vol}(S^{n-1})}{2(2\pi)^n}. \end{aligned} \tag{5}$$

If moreover we assume that g_0 is an exact cusp metric, then

$$C_3 = \frac{\Gamma\left(\frac{1-p}{2p}\right)\left(\zeta\left(\Delta_k^{h_0},\frac{1}{p}-1\right)+\zeta\left(\Delta_{k-1}^{h_0},\frac{1}{p}-1\right)\right)}{2\sqrt{\pi}\,\Gamma\left(\frac{1}{2p}\right)},$$

where $\Delta_k^{h_0}$ is the Laplacian on k forms on M with respect to the canonical metric h_0 defined in Section 1.

We also find some surprising conditions for the Schrödinger operator on functions to have pure-point spectrum. It is well-known that if the electric potential blows up at infinity then there is no essential spectrum. But we find potentials which are not bounded below for which the same happens.

The result continues to hold in the more general setting of magnetic Schrödinger operators acting on k-forms, for which we also compute the eigenvalue asymptotics. When the magnetic field is non-integral, the potential (which can be endomorphism-valued) may even go uniformly to $-\infty$ at a specified rate. Finally, even if the potential is not self-adjoint, we give some conditions for the absence of the essential spectrum.

Our methods are based on the Melrose cusp calculus of pseudodifferential operators (see e.g., [19]). It would also be interesting to apply in this setting the groupoid techniques of [14].

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1. The Laplacian of a cusp metric

This section follows closely [21], see also [19]. Let $\overline{X}$ be a smooth n-dimensional compact manifold with closed boundary M, and $x : \overline{X} \to [0, \infty)$ a boundary-defining function. A *cusp metric* on $\overline{X}$ is a complete Riemannian metric g_0 on $X := \overline{X} \setminus M$ which in local coordinates near the boundary takes the form

$$g_0 = a_{00}(x,y)\frac{dx^2}{x^4} + \sum_{j=1}^{n-1} a_{0j}(x,y)\frac{dx}{x^2}dy_j + \sum_{i,j=1}^{n-1} a_{ij}(x,y)dy_i dy_j \tag{6}$$

such that the matrix $(a_{\alpha\beta})$ is smooth and non-degenerate down to $x = 0$. For example, if $a_{00} = 1$, $a_{0j} = 0$ and a_{ij} is independent of x, we get a product metric near M. Thus a cusp metric is asymptotically cylindrical, with a precise meaning for the word "asymptotic".

Let $\mathcal{I} \subset \mathcal{C}^\infty(\overline{X})$ be the principal ideal generated by the function x. Recall that a *cusp vector field* is a smooth vector field V on $\overline{X}$ such that $dx(V) \in \mathcal{I}^2$. The space of cusp vector fields forms a Lie subalgebra ${}^c\mathcal{V}$ of the Lie algebra of smooth vector fields on $\overline{X}$. Let $E, F \to \overline{X}$ be smooth vector bundles. The space of cusp differential operators $\mathrm{Diff}_c(\overline{X}, E, F)$ is the space of those differential operators which in local trivializations can be written as composition of cusp vector fields and smooth bundle morphisms.

The *normal operator* of $P \in \mathrm{Diff}_c(\overline{X}, E, F)$ is defined by

$$\mathbb{R} \ni \xi \mapsto \mathcal{N}(P)(\xi) := \left(e^{i\xi/x} P e^{-i\xi/x}\right)_{|x=0} \in \mathrm{Diff}(M, E_{|M}, F_{|M}).$$

From the definition, $\ker \mathcal{N} = \mathcal{I} \cdot \mathrm{Diff}_c$, which we denote again by $\mathcal{I}$.

We want to study the Laplacian on k-forms associated to the metric

$$g_p := x^{2p} g_0 \tag{7}$$

where g_0 is a cusp metric given near M by (6). Note that the metric (1) is a particular case of such metric (set $x = 1/y$). By a general principle, $\Delta_{k,p}$ should be a cusp differential operator of type $(2, 2p)$, i.e., $\Delta_{k,p} \in x^{-2p}\mathrm{Diff}^2_c(\overline{X}, \Lambda^k(\overline{X}))$. We show below that this is so, and we compute the normal operator of $x^{2p}\Delta_{k,p}$.

In [21, Lemma 6] it is constructed a canonical metric on M induced from g_0. From now on we fix a product decomposition of $\overline{X}$ near M and we rewrite (6) as

$$g_0 = a\left(\frac{dx}{x^2} + \alpha(x)\right)^2 + h(x) \tag{8}$$

where

$$\begin{aligned} a := a_{00} &\in \mathcal{C}^\infty(\overline{X})\\ \alpha &\in \mathcal{C}^\infty([0,\varepsilon) \times M, \Lambda^1(M))\\ h &\in \mathcal{C}^\infty([0,\varepsilon) \times M, S^2TM). \end{aligned}$$

Then by [21, Lemma 6], the function $a_0 := a(0)$, the metric $h_0 := h(0)$ and the class modulo exact forms of the 1-form $\alpha_0 := \alpha(0)$, defined on M, are independent of the chosen product decomposition and of the boundary-defining function x inside the fixed cusp structure.

Definition 4. The metric g_0 is called *exact* if $a_0 = 1$ and α_0 is an exact 1-form.

If g_0 is exact, by replacing x with another boundary-defining function inside the same cusp structure, we can as well assume that $\alpha = 0$ (see [21]).

Let V_0 be the vector field $x^2\partial_x \in {}^c\mathcal{V}$ (this makes sense since we fixed a product decomposition). Choose local coordinates $(y_i)_{1\leq i\leq n-1}$ on M and set

$$V_i := \partial_{y_i} - \alpha(\partial_{y_i})V_0, \qquad i = 1, \dots, n-1. \tag{9}$$

Notice that V_0 is orthogonal to V_i, $i \geq 1$ with respect to g_0. Let

$$V^0 := x^{-2}dx + \alpha, \qquad V^i = dy^i$$

be the dual basis. Then $(V^j)_{0\leq j\leq n-1}$ form a local basis for ${}^cT^*\overline{X}$ near M. Thus we get an orthogonal decomposition of smooth vector bundles

$$\Lambda^k({}^cT\overline{X}) \simeq \Lambda^k(TM) \oplus V^0 \wedge \Lambda^{k-1}(TM) \tag{10}$$

valid in a neighborhood of M. We mention here that we could have worked with a simpler decomposition corresponding to a "product" cusp metric, if we did not want to evaluate the constants C_2, C_3.

Proposition 5. *The de Rham differential*

$$d : \mathcal{C}^\infty(X, \Lambda^k X) \to \mathcal{C}^\infty(X, \Lambda^{k+1}(X))$$

restricts to a cusp differential operator

$$d : \mathcal{C}^\infty(\overline{X}, \Lambda^k({}^cT\overline{X})) \to \mathcal{C}^\infty(\overline{X}, \Lambda^{k+1}({}^cT\overline{X})).$$

Its normal operator in the decomposition (10) *is*

$$\mathcal{N}(d)(\xi) = \begin{bmatrix} d^M - i\xi\alpha_0\wedge & d^M\alpha_0\wedge \\ i\xi & -(d^M - i\xi\alpha_0\wedge) \end{bmatrix}.$$

Proof. Let $\omega \in \mathcal{C}^\infty(\overline{X}, \Lambda^c T\overline{X})$ and decompose it using (10) into $\omega = \omega_1 + V^0 \wedge \omega_2$, where $V_0 \lrcorner \omega_{1,2} = 0$. Trivialize the bundle $\Lambda^*(\overline{X})$ with the local basis $V^0, \dots, V^{n-1}$ and write for $j = 1, 2$ using (9)

$$\begin{aligned} d\omega_j =& \sum_{i=1}^{n-1} V^i \wedge V_i(\omega_j) + V^0 \wedge V_0(\omega_j) \\ =& d^M\omega_j - \alpha \wedge V_0(\omega_j) + V^0 \wedge V_0(\omega_j) \end{aligned}$$

Now $dV^0 = d\alpha \equiv d^M\alpha$ modulo $\mathcal{I}^2$ (recall that the ideal $\mathcal{I}$ maps to 0 under the normal operator map) so the announced formula holds. □

From the definition, the normal operator map is linear and multiplicative. Moreover, it is a star-morphism, i.e., it commutes with taking the (formal) adjoint. This last property needs some explanation since the volume form used for taking the adjoint on the boundary must be specified.

Lemma 6. *Let $P \in \mathrm{Diff}_c(\overline{X}, E, F)$ be a cusp operator and P^* its adjoint with respect to g_0. Then $\mathcal{N}(P^*)(\xi)$ is the adjoint of $\mathcal{N}(P)(\xi)$ with respect to the metric on $E_{|M}, F_{|M}$ induced by restriction, and the volume form ${a_0}^{1/2} dh_0$.*

Proof. Since $\mathcal{N}$ commutes with products and sums, it is enough to prove the result for a set of local generators of Diff_c. Work in local coordinates y_i on M. Then the vector fields $x^2\partial_x$, ∂_{y_i} and the set of smooth bundle morphisms $\{A \in \mathcal{C}^\infty(\overline{X}, E, F)\}$ form local generators for $\mathrm{Diff}_c(\overline{X}, E, F)$. From the definition of $\mathcal{N}$ it follows that

$$\begin{aligned}\mathcal{N}(x^2\partial_x)(\xi) =& i\xi;\\ \mathcal{N}(\partial_{y_i})(\xi) =& \partial_{y_i|M};\\ \mathcal{N}(A)(\xi) =& A_{|M}.\end{aligned}$$

Clearly, restriction of A to M commutes with taking the adjoint. The volume form of g_0 is $dg_0 = \sqrt{a\det(h_{ij})}x^2dxdy$. Since the derivative of a and of the metric coefficients h_{ij} with respect to $x^2\partial_x$ belong to $\mathcal{I}^2$, we see that $x^2\partial_x^* \equiv -x^2\partial_x + \mathcal{I}^2$, so

$$\mathcal{N}((x^2\partial_x)^*)(\xi) = \mathcal{N}(-x^2\partial_x)(\xi) = -i\xi = (i\xi)^* = (\mathcal{N}(x^2\partial_x)(\xi))^*.$$

Similarly,

$$(\partial_{y_i})^* = -\left(a\det(h_{ij})\right)^{-1/2}\partial_{y_i}(a\det(h_{ij}))^{1/2}$$

restricts to M to the adjoint of ∂_{y_i} on M with respect to the volume form $\sqrt{a\det(h_{ij})}_{|M}dy = {a_0}^{1/2}dh_0$. □

A related result is the conjugation invariance by powers of x. Namely, if $P \in \mathrm{Diff}_c$ and $s \in \mathbb{C}$ then $x^sPx^{-s} \in \mathrm{Diff}_c$, and

$$\mathcal{N}(x^sPx^{-s}) = \mathcal{N}(P).$$

The principal symbol of a cusp operator on X extends as a map on the cusp cotangent bundle down to $x = 0$. In particular a cusp operator of positive order cannot be elliptic. A cusp operator is called (cusp) elliptic if its principal symbol is invertible on ${}^cT^*\overline{X}\setminus\{0\}$ down to $x = 0$. In the rest of the article we refer to this property as ellipticity, as no confusion can arise. A cusp operator is called *fully elliptic* if it is elliptic and moreover its normal operator is invertible for all values of $\xi \in \mathbb{R}$.

Theorem 7. *The Laplacian $\Delta_{k,p}$ belongs to $x^{-2p}\mathrm{Diff}_c^2(\overline{X}, \Lambda^k({}^cT\overline{X}))$. Moreover, $x^{2p}\Delta_{k,p}$ is fully elliptic if and only if the de Rham cohomology groups $H^k_{\mathrm{dR}}(M)$ and $H^{k-1}_{\mathrm{dR}}(M)$ both vanish.*

Proof. The principal symbol of the Laplacian of g_p on Λ^kX is simply g_p times the identity. Since $x^{-2p}g_p = g_0$ (on the cotangent bundle!), it follows that $x^{2p}\Delta_{k,p}$ is elliptic down to $x = 0$.

Let δ_0^k, δ_p^k be the formal adjoint of $d : \Lambda^kX \to \Lambda^{k+1}(X)$ with respect to g_0, resp. g_p. Then $\delta_p^k = x^{(2k-n)p+2}\delta_0x^{(n-2(k+1))p-2}$. What matters is that the sum of the exponents of x equals $-2p$. By conjugation invariance,

$$\mathcal{N}(x^{2p}(d\delta_p + \delta_pd)) = \mathcal{N}(d\delta_0 + \delta_0d).$$

By Hodge theory (essentially, the fact that the range of an elliptic operator is closed), the kernel of $\mathcal{N}(d\delta_0+\delta_0 d)(\xi)$ is isomorphic to the cohomology of the complex $(\Lambda^*({}^c T\overline{X})_{|M}, \mathcal{N}(d)(\xi))$.

We write $\mathcal{N}(d)(\xi) = A(\xi) + B(\xi)$ where

$$A(\xi) = \begin{bmatrix} 0 & 0 \\ i\xi & 0 \end{bmatrix}, \qquad B(\xi) = \begin{bmatrix} d^M - i\xi\alpha_0\wedge & d^M\alpha_0\wedge \\ 0 & -(d^M - i\xi\alpha_0\wedge) \end{bmatrix}.$$

We claim that for $\xi \neq 0$ the cohomology of $\mathcal{N}(d)(\xi)$ vanishes. The idea is to use again Hodge theory but with respect to the volume form dh_0 on M. Then $B(\xi)^*$ anticommutes with $A(\xi)$ and similarly $A(\xi)^*B(\xi) + B(\xi)A(\xi)^* = 0$. Therefore

$$\mathcal{N}(d)(\xi)\mathcal{N}(d)(\xi)^* + \mathcal{N}(d)(\xi)^*\mathcal{N}(d)(\xi) = \xi^2 I + B(\xi)^*B(\xi) + B(\xi)B(\xi)^*$$

where I is the 2×2 identity matrix. So for $\xi \neq 0$ the Laplacian of $\mathcal{N}(d)(\xi)$ is a strictly positive elliptic operator, hence it is invertible.

Let us turn to the case $\xi = 0$. We claim that the cohomology of $(\Lambda^*M \oplus \Lambda^{*-1}M, \mathcal{N}(d)(0))$ is isomorphic to $H^*_{\mathrm{dR}}(M) \oplus H^{*-1}_{\mathrm{dR}}(M)$. Indeed, it is enough to notice that

$$\mathcal{N}(d)(0) = \begin{bmatrix} 1 & -\alpha_0\wedge \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d^M & 0 \\ 0 & -d^M \end{bmatrix} \begin{bmatrix} 1 & \alpha_0\wedge \\ 0 & 1 \end{bmatrix}.$$

In other words, the differential $\mathcal{N}(d)(0)$ is conjugated to the diagonal de Rham differential, so they have isomorphic cohomology. □

2. Proof of Theorem 3

Proof of Theorem 3. By Theorem 7, the hypothesis implies that $\Delta_{k,p}$ is fully elliptic. By [21, Theorem 17], the spectrum of $\Delta_{k,p}$ is pure-point and accumulates towards infinity according to (4), modulo identification of the correct coefficients. To identify these coefficients we use [21, Proposition 14] and Delange's theorem ([21, Lemma 16]. Note that $\sigma_2(\Delta_{k,p})$ is identically 1 on the cosphere bundle. The dimension of the form bundle equals $\binom{n}{k}$. This gives C_1 and C_2 using the above cited results. If we assume that g_0 is exact (i.e., the 1-form α_0 is exact and the function a_0 equals 1) then using a different function x inside the cusp structure we can assume that $\alpha_0 = 0$. Thus for exact metrics the computation of $\mathcal{N}(d)$ gives

$$\mathcal{N}(x^{2p}\Delta_{k,p})(\xi) = \begin{bmatrix} \xi^2 + \Delta_k^M & 0 \\ 0 & \xi^2 + \Delta_{k-1}^M \end{bmatrix}. \tag{11}$$

This allows us to compute the integral from [21, Proposition 14] in terms of the zeta functions of the form Laplacians on M with respect to h_0. □

3. The magnetic Laplacian

3.1. **The magnetic Laplacian on a Riemannian manifold.** A *magnetic field* B on the Riemannian manifold (X, g) is simply an exact real valued 2-form. A *magnetic potential* A associated to B is a 1-form such that $dA = B$. We form the *magnetic Laplacian* acting on $\mathcal{C}^\infty(X)$:

$$\Delta_A := d_A^* d_A.$$

This formula makes sense for complex-valued 1-forms A. Note that when A is real, d_A is a metric connection on the trivial bundle $\underline{\mathbb{C}}$ with the canonical metric, and Δ_A is the connection Laplacian.

If we alter A by adding to it a real exact form, say $A' = A + dw$, the resulting magnetic Laplacian satisfies

$$\Delta_{A'} = e^{-iw}\Delta_A e^{iw}$$

so it is unitarily equivalent to Δ_A in $L^2(X, g)$. Therefore if $H^1_{\mathrm{dR}}(X) = 0$ (for instance if $\pi_1(X)$ is finite) then Δ_A depends, up to unitary equivalence, only on the magnetic field B. This property is called *gauge invariance.*

In the literature one usually encounters gauge invariance as a consequence of 1-connectedness (i.e., $\pi_1 = 0$). But in dimensions at least 4, every finitely generated group (in particular, every finite group) can be realized as π_1 of a compact manifold. Thus the hypothesis $\pi_1 = 0$ is unnecessarily restrictive.

While the properties of Δ_A in $\mathbb{R}^n$ with the flat metric are quite well understood, the (absence of) essential spectrum of magnetic Laplacians on other manifolds has not been much studied so far. One exception is the case of bounded geometry, studied in [13]. However our manifolds are not of bounded geometry because the injectivity radius tends to 0 at infinity.

3.2. **The cusp magnetic Laplacian.** Let X be like in Section 1. We assume that the metric is of the form (7) near ∂X, and moreover that $A \in \mathcal{C}^\infty(\overline{X}, {}^cT^*\overline{X})$, in other words A extends to a smooth section of ${}^cT^*\overline{X}$. We have seen in the introduction that the scalar Laplacian (corresponding to $A = 0$) always has continuous spectrum, at least for the warped product metrics (1). We will show in Section 6 that this fact continues to hold for general conformally cusp metrics.

Nevertheless, for most other A, Δ_A has pure-point spectrum and we can describe its eigenvalue asymptotics. Note that A below can be complex-valued.

Theorem 8. *Let $p > 0$, g_p a metric on X given by (7) near ∂X and $A \in \mathcal{C}^\infty(X, T^*X)$ a complex-valued 1-form satisfying near ∂X*

$$A = \varphi(x)\frac{dx}{x^2} + \theta(x) \tag{12}$$

where $\varphi \in \mathcal{C}^\infty(\overline{X})$ and $\theta \in \mathcal{C}^\infty([0, \varepsilon) \times M, \Lambda^1(M))$. Assume that, on each connected component M_0 of M, either $\varphi_0 := \varphi(0)$ is not constant, or that $\theta_0 := \theta(0)$ is not closed, or the cohomology class $[\theta_0] \in H^1_{\mathrm{dR}}(M_0)$ does not belong to the image of

$$2\pi H^1(M_0, \mathbb{Z}) \to H^1(M_0, \mathbb{C}) \simeq H^1_{\mathrm{dR}}(M_0) \otimes \mathbb{C}.$$

Then Δ_A has pure-point spectrum. Its counting function satisfies

$$N_{A,p}(\lambda) \approx \begin{cases} C_1\lambda^{n/2} & \text{for } 1/n < p < \infty, \\ C_2\lambda^{n/2}\log\lambda & \text{for } p = 1/n, \\ C_3\lambda^{1/2p} & \text{for } 0 < p < 1/n \end{cases}$$

in the limit $\lambda \to \infty$, where C_1, C_2 are given by (5) with $k = 0$.

In particular, we note that C_1, C_2 do not depend on A. The hypotheses on A is another way of saying that A is a smooth cusp 1-form on $\overline{X}$.

Proof. As before, we show that $x^{2p}\Delta_A$ is fully elliptic. To follow the proof, the reader may as well assume that $\alpha_0 = 0$; the price to pay for this simplification is losing the information on C_2. We write $A = \varphi \cdot (dx/x^2 + \alpha) + (\theta - \varphi\alpha)$ in order to use the decomposition (10). Using Proposition 5 on functions, we get

$$\mathcal{N}(d_A)(\xi) = \begin{bmatrix} d^M - i\xi\alpha_0 + i(\theta_0 - \varphi_0\alpha_0) \\ i(\xi + \varphi_0) \end{bmatrix}.$$

Assume that there exists $u \in \ker(\mathcal{N}(x^{2p}\Delta_A)(\xi))$ different from 0. By elliptic regularity u is smooth. We replace M by one of its connected components on which u does not vanish identically, so we can suppose that M is connected. Using Lemma 6, by integration by parts with respect to the volume form ${a_0}^{1/2}dh_0$ on M and the metric h_0 on $\Lambda^1(M)$, we see that $u \in \ker(\mathcal{N}(\Delta_A)(\xi))$ implies $u \in \ker(\mathcal{N}(d_A)(\xi))$. Set $\omega := \theta_0 - \alpha_0(\varphi_0 + \xi) \in \Lambda^1(M)$. Then

$$\begin{aligned} (\xi + \varphi_0)u =&0, \\ (d^M + i\omega)u =&0, \end{aligned} \tag{13}$$

so u is a global parallel section in the trivial bundle $\underline{\mathbb{C}}$ over M, with respect to the connection $d^M + i\omega$. This implies

$$0 = (d^M)^2 u = d^M(-iu\omega) = -i(d^M u) \wedge \omega - iud^M\omega = -iud^M\omega.$$

By uniqueness of solutions to ordinary differential equations, u is never 0, so $d^M\omega = 0$. Furthermore, from Eq. (13), we see that φ_0 equals the constant function $-\xi$. Therefore $\omega = \theta_0$. In conclusion θ_0 is closed and φ_0 is constant. It remains to show the assertion about the cohomology class $[\theta_0]$.

Let $\tilde{M}$ be the universal cover of M. Denote by $\tilde{u}$, $\tilde{\theta}_0$ the lifts of u, θ_0 to $\tilde{M}$. The identity $(d^M + i\theta_0)u = 0$ lifts to

$$(d^{\tilde{M}} + i\tilde{\theta}_0)\tilde{u} = 0. \tag{14}$$

The 1-form $\tilde{\theta}_0$ is closed on the simply connected manifold $\tilde{M}$, hence it is exact (recall that by the universal coefficients formula, $H^1(\tilde{M}, \mathbb{C}) = H_1(\tilde{M}, \mathbb{C}) = H_1(\tilde{M}, \mathbb{Z}) \otimes \mathbb{C}$, and $H_1(\tilde{M}, \mathbb{Z})$ vanishes as it is the abelianization of $\pi_1(\tilde{M})$). Let $v \in \mathcal{C}^\infty(\tilde{M})$ be a primitive of $\tilde{\theta}_0$, i.e., $d^{\tilde{M}}v = \tilde{\theta}_0$. Then $\tilde{u}$ can be explicitly computed from (14):

$$\tilde{u} = Ce^{-iv}$$

for some constant $C \neq 0$.

The fundamental group $\pi_1(M)$ acts to the right on $\tilde{M}$ via deck transformations. The condition that $\tilde{u}$ be the lift of u from M is the invariance under the action of $\pi_1(M)$, in other words

$$\tilde{u}(y) = \tilde{u}(y[\gamma])$$

for all closed loops γ in M. This is obviously equivalent to

$$v(y[\gamma]) - v(y) \in 2\pi\mathbb{Z}, \qquad\qquad \forall y \in \tilde{M}.$$

Let $\tilde{\gamma}$ be the lift of γ starting in y. Then

$$v(y[\gamma]) - v(y) = \int_{\tilde{\gamma}} d^{\tilde{M}}v = \int_{\tilde{\gamma}} \tilde{\theta}_0 = \int_\gamma \theta_0.$$

Thus the solution $\tilde{u}$ is $\pi_1(M)$-invariant if and only if the cocycle θ_0 evaluates to an integer multiple of 2π on each closed loop γ. These loops span $H_1(M, \mathbb{Z})$, so $[\theta_0]$ lives in the image of $H^1(M, \mathbb{Z})$ inside $H^1(M, \mathbb{C}) = \operatorname{Hom}(H_1(M, \mathbb{Z}), \mathbb{C})$. Therefore the solution u cannot be different from 0 unless φ_0 is constant, θ_0 is closed and $[\theta_0] \in 2\pi H^1(M,\mathbb{Z})$.

We proved that under the hypotheses of the theorem, Δ_A, which belongs to $x^{-2p}\mathrm{Diff}_c^2(M)$, is fully elliptic. Then by [21, Theorem 17] we get the advertised eigenvalue asymptotics. The constants C_1, C_2 are identified exactly as in the proof of Theorem 3 (note that from [21, Proposition 14], they depend only on the principal symbol, which is unaffected by A). □

Conversely, if φ_0 is constant, θ_0 is closed and $[\theta_0] \in 2\pi H^1(M,\mathbb{Z})$ then $\tilde{u} = e^{-iv}$ like above is $\pi_1(M)$-invariant, so it is the lift of some $u \in \mathcal{C}^\infty(M)$ which belongs to $\ker(\mathcal{N}(x^{2p}\Delta_A)(\xi))$ for $-\xi$ equal to the constant value of φ. Thus Δ_A is not fully elliptic, hence it is not Fredholm between cusp Sobolev spaces. Unfortunately, this fact alone does not allow us to conclude anything about its self-adjoint extension(s) in L^2, and more work is needed. In case of exact cusp metrics and real magnetic potentials we give a converse to Theorem 8 (and also to Theorem 1) in Section 6.

We have also had difficulties in extending the above proof to the magnetic Laplacian on k-forms.

Note that the hypotheses of Theorem 8 are independent of the choice of the magnetic potential A in the class of cusp 1-forms. Indeed, assume that $A' = A + dw$ for some $w \in \mathcal{C}^\infty(X)$ is again a cusp 1-form. Then dw must be itself a cusp form, so

$$dw = x^2\partial_x w \frac{dx}{x^2} + d^M w \in \mathcal{C}^\infty(\overline{X}, {}^cT^*\overline{X}).$$

Write this as $dw = \varphi' \frac{dx}{x^2} + \theta'_x$. For each $x > 0$ the form θ'_x is exact. By the Hodge decomposition theorem, the space of exact forms on M is closed, so the limit θ'_0 is also exact. Now dw is an exact cusp form, in particular it is closed. This implies that $x^2\partial_x\theta' = d^M\varphi'$. Setting $x = 0$ we deduce $d^M(\varphi')_{x=0} = 0$, or equivalently $\varphi'_{|x=0}$ is constant. Hence the conditions from Theorem 8 on the vector potential are satisfied simultaneously by A and A'.

3.3. **On the non-stability of the essential spectrum.** In $\mathbb{R}^n$, with the flat metric, it is shown in [17] that the essential spectrum of a magnetic Laplacian (for a bounded magnetic field) is determined by the restriction of the magnetic field to the boundary of a suitable compactification of $\mathbb{R}^n$. In other words, only the behavior of the magnetic field at infinity plays a rôle in the computation of the essential spectrum.

Moreover, [13, Theorem 4.1] states that the non-emptiness of the essential spectrum is preserved by the addition of a bounded magnetic field, again for the Euclidean metric on $\mathbb{R}^n$.

In contrast with these examples, Theorem 8 indicates that in general the essential spectrum is not stable (and may even vanish) under compact perturbation of the magnetic field. This phenomenon is quite remarkable.

3.4. **The case $H^1(X) \neq 0$.** The gauge invariance property does not hold in this case. Then the operators Δ_A and $\Delta_{A'}$ may have different essential spectra even when $d(A - A') = 0$. We give below an example where the essential spectrum of $\Delta_{A'}$ is empty, while that of Δ_A is not.

Example 9. Consider the manifold $\overline{X} = (S^1)^{n-1} \times [0, 1]$ with a metric g_p as in (7) near the two boundary components. Let θ_j be variables on the torus $(S^1)^{n-1}$, so $e^{i\theta_j} \in S^1$. Choose the vector potentials A to be 0, and A' to be the closed form $\mu d\theta_1$ for some $\mu \in \mathbb{R}$. In this example the magnetic field B vanishes. We have mentioned in the introduction that the Laplacian on functions has non-empty essential spectrum. However, the class $[i_M^*(A')]$ is an integer multiple of 2π if and only if $\mu \in \mathbb{Z}$. Therefore by Theorem 8, for $\mu \in \mathbb{R} \setminus \mathbb{Z}$ the essential spectrum of $\Delta_{A'}$ is empty.

This effect is known as the Aharonov-Bohm effect [2]. The phenomenon we point out is surprising in light of what is known from the case of $\mathbb{R}^n$ with an bounded obstacle. Indeed, the most you can expect in this case is a phase difference of the waves due to the choice of two non-homotopic paths that circumvent the obstacle. Note that the essential spectrum will not depend on the choice of A in the case.

3.5. **The case $H^1(X) = 0$.** Assume that $H^1(X, \mathbb{R}) = 0$ so Δ_A is determined by B up to unitary equivalence (by gauge invariance). Assume that B is a smooth 2-form on $\overline{X}$. which is closed on X and such that its restriction to X is exact. Then there exists $A \in \mathcal{C}^\infty(\overline{X}, \Lambda^1(\overline{X}))$ such that $B = dA$ (since the cohomology of the de Rham complex on $\overline{X}$ equals the singular cohomology of $\overline{X}$, hence that of X). Assume moreover that the pull-back of B to M vanishes. Then B defines a relative cohomology class in $H^2(\overline{X}, M, \mathbb{R})$. Besides, the vanishing of $i_M^*(B)$ means that $i_M^*(A)$ is closed. It is easy to see that B determines a 1-cohomology class $[i_M^*(A)]$ on M. Notice that $\partial[i_M^*(A)] = [dA] = [B]$, where $\partial : H^1(M, \mathbb{R}) \to H^2(\overline{X}, M, \mathbb{R})$ is the connecting homomorphism in the cohomology long exact sequence of the pair $(\overline{X}, M)$ with real coefficients. Consider the same map but for integer coefficients, and the natural arrows from $\mathbb{Z}$- to $\mathbb{R}$-cohomology, which are just tensoring with $\mathbb{R}$.

$$\begin{array}{ccc} H^1(M,\mathbb{R}) & \xrightarrow{\partial} & H^2(X,M,\mathbb{R}) \\ \uparrow{\scriptstyle \otimes\mathbb{R}} & & \uparrow{\scriptstyle \otimes\mathbb{R}} \\ H^1(M,\mathbb{Z}) & \xrightarrow{\partial} & H^2(X,M,\mathbb{Z}) \end{array} \tag{15}$$

We see that $[B]$ lives in the image of $\otimes\mathbb{R}$ if $[A]$ does (conversely, if we assume additionally that $H_1(X,\mathbb{Z}) = 0$, which is the case for instance if $\pi_1(X) = 0$, then $[B]$ is integer if and only if $[A]$ is; this holds because $H^2(X,\mathbb{Z})$ is torsion-free). Thus from Theorem 8 we get the following

Corollary 10. *Assume that the first Betti number of X vanishes. Let B be a smooth exact 2-form on $\overline{X}$. Assume that either $i_M^* B$ does not vanish identically, or that $i_M^* B = 0$ and the cohomology class $[B] \in H^2(\overline{X}, M, \mathbb{R})$ is not an integer multiple of 2π (in the sense described above). Then the essential spectrum of the associated magnetic Laplacian with respect to the metric (7) is empty.*

Example 11. Suppose that $B = df \wedge dx/x^2$ where f is a function on X smooth down to the boundary M of X. Assume that f is not constant on M. Then the essential spectrum of the magnetic operator (which is well-defined by B if $H^1(X) = 0$) is empty.

Note that in this example the pull-back to the border of the magnetic field is null. For such magnetic fields in flat $\mathbb{R}^n$, the essential spectrum would be $[0, \infty)$.

Example 12. There exist Riemannian manifolds X with gauge invariance (i.e., $H^1(X,\mathbb{R}) = 0$) and a smooth magnetic field B with compact support in X such that the essential spectrum of the magnetic Laplacian is empty. We choose X to be the interior of a smooth manifold with boundary. Assume that $H^1(X) = 0$, for instance X is simply connected. Assume moreover that $H^1(M,\mathbb{R}) \neq 0$ where M is the boundary of X. Note that these two assumptions are impossible to fulfill simultaneously for orientable manifolds in dimension 3 (see Remark 14).

We construct A like in (12) satisfying the hypotheses of Theorem 8. We take φ_0 to be constant. Let $\psi \in \mathcal{C}^\infty([0,\varepsilon))$ be a cut-off function such that $\psi(x) = 0$ for $x \in [3\varepsilon/4, \varepsilon)$ and $\psi(x) = 1$ for $x \in [0, \varepsilon/2)$. Since $H^1(M) \neq 0$, there exists a closed 1-form β on M which is not exact. Up to multiplying β by a real constant, we can assume that the cohomology class $[\beta] \in H^1_{\mathrm{dR}}(M_0)$ does not belong to the image of $2\pi H^1(M_0, \mathbb{Z}) \to H^1(M_0, \mathbb{R}) \simeq H^1_{\mathrm{dR}}(M_0)$. Choose θ_x to be $\psi(x)\beta$ for $\varepsilon > x > 0$ and extend it by 0 to X.

By Theorem 8, Δ_A has pure point spectrum. On the other hand, the magnetic field $B = dA = (d\psi/dx) \wedge \beta$ has compact support in X.

Example 13. The simplest particular case of Example 12 is $X = \mathbb{R}^2$, but of course not with its Euclidean metric. The metric (7) could be for instance (in polar coordinates)

$$r^{-2p}(dr^2 + d\sigma^2).$$

Here M is the circle at infinity and $x = 1/r$ for large r.

The product of this manifold with a closed, connected, simply connected manifold Y of dimension k yields an example of dimension $2+k$. Indeed, by the Künneth formula, the first cohomology group of $\mathbb{R}^2 \times Y$ vanishes, while $H^1(S^1 \times Y) \simeq H^1(S^1) = \mathbb{Z}$.

Clearly k cannot be 1 since the only closed manifold in dimension 1 is the circle. Thus dimension 3 is actually exceptional.

Remark 14. For orientable X of dimension 3 (the most interesting for Physical purposes) the assumptions $H^1(X) = 0$ and $H^1(M) \neq 0$ cannot be simultaneously fulfilled. Indeed, we have the following long exact sequence (valid actually regardless of the dimension of X)

$$H^1(X) \to H^1(M) \to H^2(X, M).$$

If $\dim(X) = 3$, the spaces $H^1(X)$ and $H^2(X, M)$ are dual by Poincaré duality, hence $H^1(X) = 0$ implies $H^2(X, M) = 0$ and so (by exactness) $H^1(M) = 0$.

Example 15. In the setting of Example 13, the relative cohomology group $H^2(X, M)$ is one-dimensional. The integrality condition becomes particularly nice in this case. Namely, let B be a compactly supported magnetic field (or more generally, a smooth exact 2-form on $\overline{X}$ vanishing at M) which is not zero in $H^2(X, M)$. Then λB is integral precisely for λ in an infinite discrete subgroup of $\mathbb{R}$. Thus the multiples of B for which the magnetic Laplacian has non-empty essential spectrum are discrete and periodic. The fact that there is essential spectrum for integral $[B]$ follows from Proposition 27.

4. Schrödinger operators

We now consider Schrödinger operators with magnetic potentials on k-forms. An easy and well-known fact, which follows from Proposition 31, is the compactness of the resolvent of (16) for any self-adjoint potential V such that $V \to +\infty$ at infinity. Using the cusp algebra we get much stronger results. We will apply our methods for V belonging to $x^{-2p}\mathcal{C}^\infty(\overline{X})$ but not necessarily positive. Unlike in the Euclidean case, for $p \geq 1/n$ the eigenvalue asymptotics do not depend on V. The result holds moreover for endomorphism-valued potentials V.

Proposition 16. *Let $p > 0$, g_p the metric on X given by* (7) *near ∂X and $V \in x^{-2p}\mathcal{C}^\infty(\overline{X}, \operatorname{End}(\Lambda^k\overline{X}))$ a self-adjoint potential. Set $V_0 := (x^{2p}V)_{|M} \in \mathcal{C}^\infty(M, \operatorname{End}(\Lambda^k\overline{X})_{|M})$. Assume that $V_0 \geq 0$ (i.e., V_0 is semi-positive definite), and in each connected component of M there exists z with $V_0(z) > 0$. Let A be any vector potential of the form* (12) *near $M = \partial X$. Then the magnetic Schrödinger Laplacian*

$$\Delta_{A,V} := d_A^* d_A + d_A d_A^* + V \tag{16}$$

is essentially self-adjoint and has pure-point spectrum. Its eigenvalue counting function satisfies

$$N_{A,V}(\lambda) \approx \begin{cases} C_1\lambda^{n/2} & \text{for } 1/n < p < \infty, \\ C_2\lambda^{n/2}\log\lambda & \text{for } p = 1/n \\ C_3\lambda^{\frac{1}{2p}} & \text{for } p < 1/n \end{cases}$$

with C_1, C_2 given by (5).

Remark 17. Note that for $p \geq 1/n$, the asymptotic behavior of the eigenvalues depends neither on V nor on A, i.e., C_1, C_2 depend only on the metric!

Proof. We claim that the operator $\Delta_{A,V}$ is fully elliptic. Indeed, we write first $\mathcal{N}(x^{2p}\Delta_{A,V}) = \mathcal{N}(x^{2p}\Delta_A) + V_0$. For all $\xi \in \mathbb{R}$, the operators $\mathcal{N}(x^{2p}\Delta_A)(\xi)$ and V_0 are non-negative, so a solution of $\mathcal{N}(x^{2p}\Delta_{A,V})(\xi)$ must satisfy

$$\mathcal{N}(x^{2p}\Delta_A)(\xi)\phi = 0, \qquad V_0\phi = 0.$$

By unique continuation, solutions of the elliptic operator $\mathcal{N}(x^{2p}\Delta_A)(\xi)$ which are not identically zero on a given connected component of M do not vanish anywhere on that component. But then $V_0\phi = 0$ and the assumption on V_0 show that there are no non-zero solutions, in other words $\Delta_{A,V}$ is fully elliptic.

By [21, Lemma 10 and Corollary 13], it follows that $\Delta_{A,V}$ is essentially self-adjoint with domain $x^{2p}H^2_c(M)$, and has pure-point spectrum. Note that $\Delta_{A,V}$ is not necessarily positive since V may take negative values. Write $V = V^+ + V^-$, where $V_0^+ = V_0$ while $V^- \in x^{1-2p}\mathcal{C}^\infty(\overline{X})$. Then V^- is compact from $H^{2,2p} = \mathcal{D}(\Delta_{A,V^+})$ to L^2, so $\Delta_{A,V}$ is a compact perturbation of a strictly positive operator and therefore it is bounded below.

By [21, Proposition 14], for $p \geq 1/n$, the asymptotic behaviour of the function $N(\Delta_{A,V})$ as $\lambda \to \infty$ is given by (5), in particular it only depends on the principal symbol.

For $0 < p < 1/n$, again by [21, Proposition 14], the coefficient C_3 in the spectral asymptotics of $\Delta_{A,V}$ only depends on $\mathcal{N}(\Delta_{A,V})$. □

The coefficient C_3 can be computed if we assume that the metric is exact and A for instance vanishes. It then depends on the zeta function of $\Delta^M + V_0$. The computation is very similar to Theorem 3 and we leave it as an exercise to the interested reader.

Specialize now to the case of functions. Then we can have pure-point spectrum and asymptotic laws for potentials which are unbounded below in some, but not all directions towards infinity.

Example 18. Let X be a Riemannian manifold with metric given by (7) near infinity, and $p > 0$. Let $V \in x^{-2p}\mathcal{C}^\infty(\overline{X})$ be such that $V_0 = (x^{2p}V)_{|M}$ is non-negative and not identically zero on M. Then Proposition 16 implies that the Schrödinger operator $\Delta + V$ has pure-point spectrum and obeys a generalized Weyl law. There exist such V which tend to $-\infty$ towards some part of M. For instance, assume $\mathrm{supp}(V_0)$ does not cover M; this implies that near $M \backslash \mathrm{supp}(V_0)$, V/x^{1-2p} has a limit, which can be very well negative. Now if $p > 1/2$, then x^{1-2p} goes to ∞ so V tends to $-\infty$ as we approach $M \setminus \mathrm{supp}(V_0)$.

Remark 19. This example is quite surprising in light of what happens in Euclidean $\mathbb{R}^n$. Let $\overline{\mathbb{R}^n}$ be its spherical compactification, and $V \in \mathcal{C}(\mathbb{R}^n; \mathbb{R})$ a potential which extends continuously to $\overline{\mathbb{R}^n}$ with values in $\overline{\mathbb{R}}$. Then $\sigma_{\text{ess}}(\Delta + V) = \emptyset$ if and only if $V(x) = +\infty$ for $x \notin \mathbb{R}^n$. One can nicely see this in the setting of C^*-algebras using the formalism from [10]. The same remains true for the Laplacian on a tree with potentials that extend to the hyperbolic compactification of the tree (see [11]), and for the natural generalization of these operators on a total Fock space [9].

Example 20. Assume that the magnetic potential A satisfies the hypotheses of Theorem 8. Then we can control the essential spectrum of the magnetic Schrödinger operator $\Delta_{A,V}$ even for potentials which tend to $-\infty$ at infinity in *all* directions. Namely, there exists a strictly positive constant c depending on g_p and i_M^*A, such that for any real potential $V \in x^{-2p}\mathcal{C}^\infty(\overline{X})$ satisfying

$$|V_0| < c,$$

the operator $\Delta_{A,V}$ on functions has pure-point spectrum and satisfies the conclusion of Proposition 16. Indeed, the constant c is the infimum over $\xi \in \mathbb{R}$ of the first eigenvalue of $\mathcal{N}(x^{2p}\Delta_{A,V})(\xi)$, which can be seen to be strictly positive. The proof is again the same as in Proposition 16: under the above hypotheses, the operator $\Delta_{A,V}$ is a fully elliptic cusp operator of order $(2, 2p)$, so the conclusion follows by the above cited results of [21].

Remark 21. Consider a potential $V \in x^{-2p}\mathcal{C}^\infty(\overline{X}, \mathrm{End}(\Lambda^k(\overline{X}))$ *not necessarily self-adjoint* (e.g., a possibly imaginary function for $k = 0$). Let A be as in Proposition 16. Then the Schrödinger operator $\Delta_{A,V}$ has pure-point spectrum, provided that the eigenvalues of the limiting endomorphism $(x^{2p}V)_{|x=0}$ belong to $\mathbb{C} \setminus (-\infty, 0]$ for all points on M. For functions, if A satisfies the hypotheses of Theorem 8 (so Δ_A is fully elliptic) then it is enough to ask that the eigenvalues of $(x^{2p}V)_{|x=0}$ do not live in $(-\infty, -c)$ for some $c > 0$. The Weyl-type formulae do not hold however when V is not self-adjoint.

5. Finite multiplicity of L^2 eigenvalues

We will need in section 6 a technical lemma that guarantees that the deficiency indices of the Laplacian in the non-complete case are finite. In fact we can prove much more, namely that for an exact cusp metric and all $p > 0$, the multiplicity of every eigenvalue of the maximal extension of the Laplacian on forms and of the magnetic Laplacian is finite. This may seem surprising since in the non-fully elliptic case the operators are not Fredholm; on the other hand, this phenomenon was already noted in other Melrose-type algebras (see for instance [18]).

Lemma 22. *Assume that the metric g_0 is exact (see Definition 4). Then the multiplicity of all complex L^2 eigenvalues, in the sense of distributions, of the Laplacian $\Delta_{k,p}$ and of the magnetic Laplacian Δ_A is finite.*

Proof. Let Δ denote one of our operators. We start by noticing that Δ can be seen as an unbounded operator in a larger L^2 space. Namely, let L^2_ε be the completion of $\mathcal{C}^\infty_c(X, \Lambda^* X)$ with respect to the volume form $e^{-\frac{2\varepsilon}{x}} dg_p$ for some $\varepsilon > 0$. Clearly then L^2_ε contains L^2. The operator Δ is no longer symmetric in L^2_ε, however a distributional solution of $\Delta - \lambda$ in L^2 is evidently also a distributional solution of $\Delta - \lambda$ in L^2_ε. Thus the conclusion will follow by showing that $\Delta - \lambda$ has in L^2_ε a unique closed extension, which moreover is Fredholm. The strategy for this is by now clear. First we conjugate $\Delta - \lambda$ through the isometry

$$L^2_\varepsilon \to L^2, \qquad\qquad \phi \mapsto e^{-\frac{\varepsilon}{x}} \phi.$$

We get an unbounded operator $e^{-\frac{\varepsilon}{x}} \Delta e^{\frac{\varepsilon}{x}}$ in L^2 which is unitarily equivalent to Δ (acting in L^2_ε). Essentially from the definition,

$$\mathcal{N}(x^{2p} e^{-\frac{\varepsilon}{x}} (\Delta - \lambda) e^{\frac{\varepsilon}{x}})(\xi) = \mathcal{N}(x^{2p}\Delta)(\xi + i\varepsilon)$$

(the term containing λ vanishes under the normal operator since $p > 0$). If the metric is exact, Eq. (11) (for $\Delta = \Delta_{k,p}$) and a direct computation (for $\Delta = \Delta_A$) give

$$\mathcal{N}(x^{2p}\Delta)(\xi + i\varepsilon) = (\xi + i\varepsilon)^2 + Q$$

for a certain explicit non-negative elliptic operator Q on M. Since $\varepsilon \neq 0$, $\mathcal{N}(x^{2p}\Delta)(\xi + i\varepsilon)$ is clearly invertible for $\xi \neq 0$. At $\xi = 0$ we get $Q - \varepsilon^2$ which is also invertible, provided $\varepsilon^2 \notin \mathrm{Spec}(Q)$. It is enough to choose ε^2 outside the spectrum of Q, which is pure-point since M is closed. For such ε the operator $\Delta - \lambda$ is unitarily equivalent to a fully elliptic cusp operator of order $(2, 2p)$. It is then a general fact about the cusp algebra [21, Theorem 17] that such an operator has a unique closed extension and has pure-point spectrum, in particular its eigenvalues have finite multiplicity. $\square$

6. Non emptiness of the essential spectrum in the non fully-elliptic case

To complete our investigation it remains to compute the essential spectrum when the operators are not fully-elliptic. We restrict ourselves to metrics which are of the form (1) near M.

Remark 23. In the case of complete *exact* cusp metrics, we can assume without loss of generality that the metric is the "toy metric" (1). Indeed, for an exact cusp metric (see Definition 4), by changing the function x inside its cusp structure, we can assume that $\alpha_{|M} = 0$ (see [18] and [21]). Moreover, using Theorem 28 and Proposition 29, if the metric is complete then, as far as the essential spectrum is concerned, we may replace $h(x)$ in (8) by the metric $h_0 := h(0)$ on M, extended to a symmetric 2-tensor constant in x near M.

6.1. **The Laplacian acting on forms.** We will use the decomposition principle (Proposition 31) to localize the computation of the essential spectrum to the end $X' := (0,\varepsilon) \times M \subset X$. Set

$$\mathscr{K}_{k,\varepsilon} := L^2\left((0,\varepsilon), x^{(n-2k)p-2}dx\right).$$

Using (10), we get:

$$L^2(X', \Lambda^k X) = \mathscr{K}_{k,\varepsilon} \otimes \left(L^2(M, \Lambda^k M) \oplus L^2(M, \Lambda^{k-1} M)\right) \tag{17}$$

(the $k-1$ forms on M appear as multiples of dx/x^2).

Now, we set $\mathscr{H}_0 := \mathscr{K}_{k,\varepsilon} \otimes \ker(\Delta_k^M)$ and $\mathscr{H}_1 := \mathscr{K}_{k,\varepsilon} \otimes \ker(\Delta_{k-1}^M)$. Then (17) becomes

$$L^2(X', \Lambda^k X) = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_h \tag{18}$$

where (following [16]) the space of "high energy forms" $\mathscr{H}_h$ is by definition the orthogonal complement of $\mathscr{H}_0 \oplus \mathscr{H}_1$.

Proposition 24. *The Laplacian acting on k-forms $\Delta_{k,p}$ stabilizes the decomposition* (18) *of $L^2(X', \Lambda^k X)$. Let $\Delta_{k,p}^0$, $\Delta_{k,p}^1$ and $\Delta_{k,p}^h$ be the Friedrichs extensions of the restrictions of $\Delta_{k,p}$ to these spaces, respectively. The essential spectrum of $\Delta_{k,p}$ is the superposition of the essential spectra of these three operators. Moreover, $\Delta_{k,p}^h$ has compact resolvent, and*

$$\Delta_{k,p}^0 = \left(D^*D + c_0^2 x^{2-2p}\right) \otimes 1, \qquad \Delta_{k,p}^1 = \left(D^*D + c_1^2 x^{2-2p}\right) \otimes 1,$$

where

$$c_0 := ((2k+2-n)p-1)/2, \qquad c_1 := ((2k-2-n)p+1)/2 \tag{19}$$

and $D := x^{2-p}\partial_x - c_0 x^{1-p}$ acts in $\mathscr{K}_{k,\varepsilon}$.

Note that the essential spectra of $\Delta_{k,p}^0$, $\Delta_{k,p}^1$ are computed in Proposition 26.

Proof. Since the de Rham operator and the Hodge star stabilize (18), so does the Laplacian. We will show that $\sigma_{\mathrm{ess}}(\Delta_{k,p})$ only arises from $\mathscr{H}_0$ and $\mathscr{H}_1$. For this we use the cusp algebra to perturb $\Delta_{k,p}$ into a fully elliptic cusp operator.

Let P denote the orthogonal projection in $L^2(M, \Lambda^k M \oplus \Lambda^{k-1} M)$ onto the space $\ker(\Delta_k^M) \oplus \ker(\Delta_{k-1}^M)$ of harmonic forms. Choose a real

Schwartz cut-off function $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi(-C) = 1$. Assume moreover that the Fourier transform $\hat{\psi}$ has compact support (the reason for this assumption will appear later). Then $\psi(\xi)P$ defines a suspended operator of order $-\infty$ (see e.g., [20, Section 2]). From (11) we see that

$$\mathcal{N}(x^{2p}\Delta_{k,p})(\xi) + \psi^2(\xi)P$$

is strictly positive, hence invertible for all $\xi \in \mathbb{R}$. By the surjectivity of the normal operator, there exists $R \in \Psi_c^{-\infty}(X, \Lambda^k X)$ such that in the decomposition (10) over M,

$$\mathcal{N}(R)(\xi) = \psi(\xi)P.$$

Fix a cut-off function ϕ with compact support in X (this function appears in the proof of Proposition 31), and another cut-off function η on $\overline{X}$ which is 1 near M and such that $\eta\phi = 0$. By multiplying R to the left and to the right by η we can assume that $R\phi = \phi R = 0$, without changing $\mathcal{N}(R)$. In fact we can give an explicit formula for the Schwartz kernel of R:

$$\kappa_R(x, x', z, z') = \eta(x)\hat{\psi}\left(\frac{x - x'}{x^2}\right)\eta(x')\kappa_P(z, z') \tag{20}$$

where κ_P is the Schwartz kernel of P on M^2. We assumed that $\hat{\psi}$ has compact support, thus R acts on $\mathcal{C}_c^\infty(X', \Lambda^k X)$. Notice that R preserves the decomposition (18), and acts by 0 on the space $\mathscr{H}_h$ of high energy forms. Let

$$R_p := x^{-p}R \in \Psi_c^{-\infty,p}(X, \Lambda^k X). \tag{21}$$

Then $R_p^* R_p \in \Psi_c^{-\infty,2p}(X, \Lambda^k X)$ is symmetric on $\mathcal{C}_c^\infty(X, \Lambda^k X)$ with respect to dg_p. Moreover, by the above discussion, $\Delta_{k,p} + R_p^* R_p$ is fully elliptic, so by [21, Theorem 17], it is essentially self-adjoint in $L^2(M, dg_p)$ with domain $x^{2p}H^2(M)$, and has pure-point spectrum.

We apply now the decomposition principle to the non-negative operator $\Delta_{k,p} + R_p^* R_p$. By Remark 32, the spectrum of $\Delta_{k,p} + R_p^* R_p$ can be computed on $X' = (0, \varepsilon) \times M$. Thus the Friedrichs extension of $\Delta_{k,p} + R_p^* R_p$ on X' also has pure-point spectrum; there, $\Delta_{k,p} + R_p^* R_p$ preserves the decomposition (18), and, by the construction of R_p, it acts on the high energy forms as $\Delta_{k,p}$. Thus, it follows that over X', the Friedrichs extension of $\Delta_{k,p}$ with domain $\mathcal{C}_c^\infty(X')$ has pure point spectrum on $\mathscr{H}_h$. Proving this fact has been the reason for introducing the operator R_p.

Again by the decomposition principle (Proposition 31), $\sigma_{\text{ess}}(\Delta_{k,p})$ can be computed on the end X'. But there $\Delta_{k,p}$ also preserves (18), and we have just seen that its Friedrichs extension has pure-point spectrum on the high energy forms. It follows that $\sigma_{\mathrm{ess}}(\Delta_{k,p})$ equals the superposition of the essential spectra of the Friedrichs extensions of $\Delta^0_{k,p}$ and $\Delta^1_{k,p}$. A straightforward computation (see also [4]) shows that

$$\Delta^0_{k,p} = -\, x^{(2k-n)p} x^2 \partial_x x^{(n-2k-2)p} x^2 \partial_x \otimes 1 \tag{22}$$

acting in $\mathscr{H}_0$. Similarly,

$$\Delta^1_{k,p} = -\, x^2 \partial_x x^{(2k-2-n)p} x^2 \partial_x x^{(n-2k)p} \otimes 1 \tag{23}$$

acting in $\mathscr{H}_1$. Expanding D^*D from the definition, we get the announced expressions for $\Delta^j_{k,p}$. □

We now compute the essential spectrum of D^*D:

Lemma 25. *Let $D = x^{2-p}\partial_x - c_0 x^{1-p}$ act in $L^2\left((0,\varepsilon), x^{(n-2k)p-2}dx\right)$. For $p \leq 1$, the essential spectrum of the operator D^*D is $[0,\infty)$.*

Proof. We conjugate D^*D through the unitary transformation

$$L^2\left(x^{(n-2k)p-2}dx\right) \to L^2\left(x^{p-2}dx\right) \qquad\qquad \phi \mapsto x^\alpha \phi$$

where $\alpha = (n-2k-1)p/2$. Let

$$L(x) = \begin{cases} -\ln(x) & \text{for } p = 1, \\ -\frac{x^{p-1}}{p-1} & \text{for } p < 1. \end{cases}$$

Under the change of variables $z := L(x)$, D^*D becomes the Euclidean Laplacian on $L^2((c,\infty), dz)$ for a certain c, plus a potential that goes to 0 at infinity. □

We are now able to compute the essential spectrum of $\Delta_{k,p}$, and we obtain Theorem 2 as a corollary of the next proposition. We introduce the following set of *thresholds*:

$$\text{for } p<1,\ \mathcal{T} = \begin{cases} \emptyset, & \text{if } h^k(M) = h^{k-1}(M) = 0, \\ \{0\}, & \text{otherwise.} \end{cases}$$

$$\text{for } p=1,\ \mathcal{T} = \left\{c_i^2 \in \{c_0^2, c_1^2\}; h^{k-i}(M) \neq 0\right\}.$$

$$\text{for } p>1,\ \mathcal{T} = \emptyset$$

where c_0, c_1 are defined in (19).

Proposition 26. *Assume either that the metric g_0 is exact and $p \leq 1$, or that g_0 is the metric* (1) *and $p > 1$. Then the essential spectrum of $\Delta_{k,p}$ is given by*

$$\sigma_{\mathrm{ess}}(\Delta_{k,p}) = [\inf(\mathcal{T}), \infty)$$

where $\mathcal{T}$ is the set of thresholds. In the incomplete case (i.e., $p > 1$), this holds for all self-adjoint extensions of the operator $\Delta_{k,p}$.

Note that Theorem 1 does not follow from Proposition 26 because in the former we do not assume that the metric is exact.

Proof. Proposition 24 identifies $\sigma_{\mathrm{ess}}(\Delta_{k,p})$ with the superposition of the essential spectra of $\Delta^i_{k,p}$ on X'. For $p < 1$, $\sigma_{\mathrm{ess}}(\Delta^i_{k,p}) = \sigma_{\mathrm{ess}}(D^*D)$ since the potential part tends to 0 at infinity. For $p = 1$ the potential in $\Delta^j_{k,p}$ is constant equal to c_j^2. Therefore Lemma 25 gives the result for $p \leq 1$.

Let $p > 1$. By Lemma 22 and by the Krein formula, all self-adjoint extensions have the same essential spectrum so it is enough to consider the Friedrichs extension of $\Delta_{k,p}$. We now use Proposition 24. The operator D^*D is non-negative the spectrum of $\Delta^j_{k,p}$ is contained in $[\varepsilon^{2-2p}c_j^2, \infty)$, for $j = 0, 1$. By Proposition 31, the essential spectrum does not depend on the choice of ε. Now we remark that $p > 1$ implies $c_0, c_1 \neq 0$. Indeed, in both cases the equality would imply that $1/p \in \mathbb{Z}$, which is impossible. Thus by letting $\varepsilon \to \infty$ we conclude that the essential spectrum is empty. □

Note that for $p > 1$, the "toy metric" g_p given in (1) is essentially of *metric horn* type [15].

6.2. **The magnetic Laplacian.** To further simplify the assumptions, let A be given by (12) and define $A_0 := \varphi_0 dx/x^2 + \theta_0$. Since Δ_{A_0} is a perturbation of Δ_A by a first-order operator small at infinity, by ellipticity it follows that we can compute the essential spectrum of Δ_A using the metric g_p given near M by Eq (1), and the vector potential A_0 instead of A.

We have shown that the essential spectrum of the magnetic Laplacian Δ_A is empty unless φ_0 is constant, θ_0 is closed and the cohomology class $[\theta_0] \in H^1(M)$ is an integer multiple of 2π. One can guess that in this last case, the essential spectrum is not empty. Let us prove that this is so.

Proposition 27. *Let (X, g_p) be a Riemannian manifold with an exact cusp metric, and $A \in \mathcal{C}^\infty(\overline{X}, {}^cT^*\overline{X})$ a vector potential given by (12) near M. Assume that φ_0 is constant, θ_0 is closed and the cohomology* *class* $[\theta_0] \in H^1(M)$ *is an integer multiple of* 2π. *Then*

$$\sigma_{\mathrm{ess}}(\Delta_A) = \begin{cases} [0, \infty) & \text{for } p < 1, \\ \left[\left(\frac{n-1}{2}\right)^2, \infty\right) & \text{for } p = 1, \\ \emptyset & \text{for } p > 1. \end{cases}$$

Proof. As shown above, we can assume that near M we have

$$g_p = x^{2p}\left(\frac{dx^2}{x^4} + h_0\right), \qquad A = C\,dx/x^2 + \theta_0.$$

We decompose the space of 1-forms as the direct sum (10), where now $V^0 = x^{-2}dx$. Recall that δ_M is the adjoint of d^M with respect to h_0. Set $d_{\tilde{\theta}} := d^M + i\theta_0\wedge$. We compute

$$\begin{aligned} d_A &= \begin{bmatrix} d^M + i\theta_0\wedge \\ x^2\partial_x + iC \end{bmatrix} \\ d_A^* &= x^{-np}\left[\delta_M - i\theta_0\lrcorner \quad -(x^2\partial_x + iC)\right] x^{(n-2)p} \\ \Delta_A &= x^{-2p}(d_{\tilde{\theta}}{}^* d_{\tilde{\theta}} - (x^2\partial_x + iC)^2 - (n-2)px(x^2\partial_x + iC)) \end{aligned} \tag{24}$$

Since $d^M\theta_0 = 0$, the operator $d_{\tilde{\theta}}$ on $\mathcal{C}^\infty(M, \Lambda^*(M))$ defines a differential complex (i.e., $d_{\tilde{\theta}}{}^2 = 0$); thus by Hodge theory, we decompose orthogonally

$$\mathcal{C}^\infty(M) = \ker(d_{\tilde{\theta}}) \oplus \operatorname{Im}(d_{\tilde{\theta}}{}^*)$$

and so

$$\mathcal{C}_c^\infty(M \times (0, \varepsilon)) = \mathcal{C}_c^\infty(0, \varepsilon) \otimes \ker(d_{\tilde{\theta}}) \oplus \mathcal{C}_c^\infty(0, \varepsilon) \otimes \operatorname{Im}(d_{\tilde{\theta}}{}^*) \tag{25}$$

Functions in the second term of this decomposition are called *high energy functions*. It is clear from (24) that Δ_A preserves the decomposition (25). Following the proof of Proposition 24, we will show that $\sigma_{\mathrm{ess}}(\Delta_A)$ only arises from the first space in this decomposition.

Let P be the orthogonal projection on $\ker d_{\tilde{\theta}}$ in $L^2(M)$. Choose a Schwartz cut-off function ψ with $\psi(-C) = 1$, whose Fourier transform $\hat{\psi}$ has compact support Then $\psi(\xi)P$ defines a suspended operator of order $-\infty$. We claim that

$$\mathcal{N}(x^{2p}\Delta_A)(\xi) + \psi^2(\xi)P$$

is invertible for all ξ; indeed, this is a sum of non-negative operators for all ξ, the first of which is positive for $\xi \neq -C$ while at $\xi = -C$ we get $d_{\tilde{\theta}}{}^* d_{\tilde{\theta}} + P$ which is clearly invertible.

Define R by Eq. (20) where now κ_P is the Schwartz kernel of the above projector P in $L^2(M)$, and consider the operator $R_p \in \Psi_c^{-\infty,p}(X)$

defined by (21). We assumed that $\hat{\psi}$ has compact support, thus R acts on $\mathcal{C}_c^\infty(X)$. Notice that R preserves the decomposition (25), and acts by 0 on the space of high energy functions. Then $\Delta_A + R_p^* R_p$ is fully elliptic and symmetric, so by [21, Theorem 17] it is essentially self-adjoint with pure-point spectrum.

By the decomposition principle (Remark 32), the essential spectrum of $\Delta_A + R_p^* R_p$ can be computed on $X' = (0, \varepsilon) \times M$. Thus $\Delta_A + R_p^* R_p$ on X' has also pure-point spectrum. Since R_p is 0 on the high energy functions, it follows that the Friedrichs extension of Δ_A with domain $\mathcal{C}_c^\infty(X')$ has pure point spectrum on the second factor of (25).

This is the place to mention that the hypotheses on θ_0 imply that the kernel of $d_{\tilde{\theta}}$ is 1-dimensional. Indeed, let v be a primitive of θ_0 on the universal cover of M. Then on different sheets of the cover, v changes by $2\pi\mathbb{Z}$ so e^{iv} is a well-defined function on M. It is not hard to see that this function spans $\ker(d_{\tilde{\theta}})$.

Again by the decomposition principle (Proposition 31), $\sigma_{\mathrm{ess}}(\Delta_A)$ can be computed on the end X'. But there Δ_A also preserves (25), and we have just seen that its Friedrichs extension has pure-point spectrum on the high energy functions. It follows that $\sigma_{\mathrm{ess}}(\Delta_A)$ equals the essential spectrum of the ordinary differential operator

$$-x^{-2p}((x^2\partial_x + iC)^2 + (n-2)px(x^2\partial_x + iC))$$

in $L^2((0, \varepsilon), x^{np}dx)$. By the unitary transformation $u \mapsto e^{iC/x}u$, we can suppose that $C = 0$, i.e., we reduce to the case without magnetic field. Therefore $\sigma_{\mathrm{ess}}(\Delta_A)$ equals the essential spectrum of the scalar second-order differential operator

$$-x^{-2p}((x^2\partial_x)^2 + (n-2)px^3\partial_x).$$

in $L^2((0, \varepsilon), x^{np-2}dx)$. This is a particular case of the computation for the Laplacian on forms. Namely, it is exactly the operator from (22) for $k = 0$, whose essential spectrum was computed in Proposition 26. □

7. The nature of the essential spectrum

We consider the complete metric (1) for $p \leq 1$. The properties of the continuous spectrum of the Laplacian acting on functions are well known in this case. We now describe the nature of the essential spectrum of Laplacian acting on k-forms.

For this metric, using Proposition 24, the study of the operator $\Delta_{k,p}$ is reduced to the analysis of (22) and (23). We have already shown that the essential spectrum of $\Delta_{k,p}$ is given by $[\inf(\mathcal{T}), \infty)$ where $\mathcal{T}$ is the set of thresholds.

To pursue the analysis, one may use the so-called Mourre estimate introduced by E. Mourre in [22]. We refer to [1] for a clear exposition of the theory. The major consequence of this estimate is to give a limit absorption principle. This allows one to deduce deep spectral results.

By a decomposition principle, the study of the Laplacian can be reduced to the operators (22) and (23) on the half-line. The Mourre analysis of these operators is well understood. We refer to [7] for some general results. We deduce immediately the following corollaries:

- The singular continuous spectrum is empty.
- For each open set $J \subset \mathbb{R}$ that does not contain a threshold, the number of eigenvalues is finite and of finite multiplicity.
- The eigenvalues can accumulate only towards a threshold or towards infinity.
- The multiplicity of the absolutely continuous spectrum is $h^k(M)1_{[c_0^2,\infty)} + h^{k-1}(M)1_{[c_1^2,\infty)}$.

Note that we proved in Section 5 that all eigenvalues (including possibly the thresholds) have finite multiplicity.

Compared to [3, 4], our approach based on the cusp calculus avoided the analysis of the system of ordinary differential equations which appears in the general case (the operator Δ^p_{M3} from loc. cit.). This system could in principle allow $\{0\}$ in the essential spectrum, and renders difficult the analysis of absolutely/singular continuous spectrum. For instance, for the metric (1), i.e. in the case $a \leq -1$ and $b > 0$ in the notation of [3, 4], we can show that the singular continuous spectrum $\sigma_{\mathrm{sc}}(\Delta_{k,p})$ is empty. We also do not have to restrict ourself to the case $\partial X = S^{n-1}$. Finally, we compute the essential spectrum even for incomplete metrics g_p, but we stress once more that we do not cover all the cases studied in loc. cit.

Appendix A. On the stability of the essential spectrum

We review some conditions under which the essential spectrum of the Laplacian is stable by perturbation of the metric and by cutting a compact part of the manifold (the so-called decomposition principle).

A.1. **Perturbation of the metric.** We recall some general results from [8, Section 5]. We stress that we suppose X to be complete.

Let X be a non-compact $\mathcal{C}^1$ manifold, and μ the density of a Riemannian metric g on X. Let $\mathscr{H} := L^2(X, \mu)$, and let $\mathscr{K}$ be the completion of the space of continuous sections of T^*X with compact support under the natural norm

$$\|v\|^2_{\mathscr{K}} = \int_X \|v(x)\|^2_x d\mu(x).$$

Let $d : \mathcal{C}^1_c(X) \to \mathscr{K}$ be a closable first order operator. It could be the de Rham differential, or a magnetic de Rham differential, i.e. $d_{dR} + iA\wedge$ where A is a $\mathcal{C}^1$ 1-form, for instance. We keep the notation d for its closure. Its domain $\mathscr{G}$ is the natural first order Sobolev space $\mathscr{H}^1_0$ defined in this context as the closure of $\mathcal{C}^1_c(X)$ under the norm

$$\|u\|^2_{\mathscr{H}^1} = \int_X \Big(|u(x)|^2 + \|du(x)\|^2_x\Big) d\mu(x).$$

Suppose that the injection from $\mathscr{H}^1_0(\mathscr{O})$ to $\mathscr{H}(\mathscr{O})$ is compact, for all open bounded subsets $\mathscr{O} \subset X$. The Laplacian of d with respect to g is given by d^*d.

Theorem 28. *Let X be a C^1 non-compact manifold endowed with a locally measurable metric g_1. Let g_2 be another locally measurable metric, such that*

$$\alpha(x)g_2(x) \leq g_1(x) \leq \beta(x)g_2(x),$$

with $\lim_{x\to\infty} \alpha(x) = \lim_{x\to\infty} \beta(x) = 1$. Suppose X to be complete for one (thus also for the other) metric. Then the essential spectra of the Laplacians of d with respect to g_1 and g_2 (acting on functions) are the same.

If one supposes moreover that $\mu(X) = \infty$, the convergence of α and β can be assumed to hold in a weaker sense.

We now give a version for the Laplacian acting on forms suitable for our purpose in the context of cusp metrics.

Proposition 29. *Let X be a the interior of a compact smooth manifold. Let g_1 and g_2 be two cusp metrics on it. Suppose that:*

$$\alpha(x)g_2(x) \leq g_1(x) \leq \beta(x)g_2(x)$$

such that $\lim_{x\to\infty} \alpha(x) = \lim_{x\to\infty} \beta(x) = 1$. Suppose X to be complete for one (thus for the other) metric. Then the essential spectra of the

Laplacians acting on forms, with respect to the two metrics are the same.

The proof follows directly from the proof given at the end of [8][Section 5] in the case of forms. Indeed, in the notation of [8], for a complete cusp metric one checks easily that $a^{\pm 1}\mathscr{G} \subset \mathscr{G}$. Note that the hypotheses of this proposition can be significantly weakened.

A.2. **Removing a compact set.** It is well-known that the essential spectrum of an elliptic differential operator on a complete manifold can be computed by cutting out a compact part and studying the Dirichlet extension of the remaining operator on the non-compact part (see e.g., [6]). This result is obvious using Zhislin sequences, but the approach from loc. cit. fails in the non-complete case. For completeness, we give below a proof which has the advantage to hold in a wider context (for cusp pseudodifferential operators).

We start with a general and easy lemma. We recall that a Weyl sequence for a couple (H, λ) with H a self-adjoint operator and $\lambda \in \mathbb{R}$, is a sequence $\varphi_n \in \mathcal{D}(H)$ such that $\|\varphi_n\| = 1$, $\varphi_n \rightharpoonup 0$ (weakly) and such that $(H - \lambda)\varphi_n \to 0$, as n goes to infinity. The importance of this notion comes from the fact that $\lambda \in \sigma_{\mathrm{ess}}(H)$ if and only if there is Weyl sequence for (H, λ).

Lemma 30. *Let H be a self-adjoint operator in a Hilbert space $\mathscr{H}$. Let φ_n be a Weyl sequence for the couple (H, λ). Suppose that there is a closed operator Φ in $\mathscr{H}$ such that:*

1. $\Phi\mathcal{D}(H) \subset \mathcal{D}(H)$,
2. $\Phi(H + i)^{-1}$ *is compact,*
3. $[H, \Phi]$ *extends to a bounded operator in* $\mathcal{B}(\mathcal{D}(H), \mathscr{H})$, *and* $[H, \Phi](H + i)^{-1}$ *is compact.*

Then there exists $\widetilde{\varphi}_n \in \mathcal{D}(H)$ such that $(1 - \Phi)\widetilde{\varphi}_n$ is a Weyl sequence for (H, λ).

Proof. First we note that $\Phi\varphi_n$ goes to 0. Indeed, we have

$$\begin{aligned} \Phi\varphi_n &= \Phi(H + i)^{-1}(H + i)\varphi_n \\ &= \Phi(H + i)^{-1}\big((H - \lambda)\varphi_n + (i + \lambda)\varphi_n\big), \end{aligned}$$

the bracket goes weakly to 0 and $\Phi(H + i)^{-1}$ is compact by (2). Similarly, using (3) we get that $[H, \Phi]\varphi_n \to 0$ too.

There is N such that for $n \geq N$, $\|(1-\Phi)\varphi_n\| \geq 1/2$ then we set $\widetilde{\varphi}_n = \varphi_n/\|(1-\Phi)\varphi_n\|$. We directly see that $(1-\Phi)\widetilde{\varphi}_n \rightharpoonup 0$. It remains to show that $(H-\lambda)(1-\Phi)\widetilde{\varphi}_n \to 0$ but this follows the fact that $[H,\Phi]\varphi_n \to 0$ and that φ_n is a Weyl sequence for (H,λ). □

Let us now state the decomposition principle. Let X be non-compact smooth Riemannian manifold, and K a compact sub-manifold with border of the same dimension. We endow $X' := X \setminus K$ with the Riemannian structure induced from X.

Proposition 31. *Consider the Friedrichs extension of the magnetic Laplacians Δ, Δ' defined on compactly supported forms on X and X', respectively. Then $\sigma_{\mathrm{ess}}(\Delta) = \sigma_{\mathrm{ess}}(\Delta')$.*

Proof. Denote by $\eth$ the closure of the operator $d_A + d_A^*$ acting on $\mathcal{C}_c^\infty(X)$, and by $\eth'$ the closure of the same differential operator acting on $\mathcal{C}_c^\infty(X')$. We view $L^2(X', \Lambda^* X')$ as a closed subspace of $L^2(X, \Lambda^* X)$. Set $\mathscr{Q}(f,g) = \langle \eth f, \eth g\rangle$. We recall that domain of Δ is given by

$$\mathcal{D}(\Delta) = \{f \in \mathcal{D}(\eth) \mid g \mapsto \mathscr{Q}(f,g) \text{ is continuous for the } L^2 \text{ norm}\}.$$

Let $\phi \in \mathcal{C}_c^\infty(X)$ be a non-negative cut-off function such that $\phi|_K = 1$. Let $\tilde{\phi} = 1 - \phi$. We claim that $f \in \mathcal{D}(\Delta)$ implies that $\tilde{\phi} f \in \mathcal{D}(\Delta')$. Note first that the commutator $[\eth, \phi]$ is a smooth endomorphism with compact support (and so are all its derivatives). Thus $f \in \mathcal{D}(\eth)$ implies that $\tilde{\phi} f \in \mathcal{D}(\eth')$, and $\eth'(\tilde{\phi} f) = [\eth, \tilde{\phi}] f + \tilde{\phi} \eth f$. Therefore

$$\begin{aligned}\langle \eth u, \eth(\tilde{\phi} f)\rangle &= \langle \eth u, [\eth, \tilde{\phi}] f + \tilde{\phi}\eth f\rangle = \langle u, \eth^*([\eth, \tilde{\phi}] f)\rangle + \langle \tilde{\phi}\eth u, \eth f\rangle \\ &= \langle u, \eth^*([\eth, \tilde{\phi}] f)\rangle + \langle \eth(\tilde{\phi} u), \eth f\rangle - \langle [\eth, \tilde{\phi}] u, \eth f\rangle\end{aligned}$$

which proves the claim. Similarly, we claim that $f \in \mathcal{D}(\Delta')$ implies $\tilde{\phi} f \in \mathcal{D}(\Delta)$. In fact, the previous argument shows that $\tilde{\phi} f \in \mathcal{D}(\Delta')$. Now taking χ another positive cut-off smooth function such $\chi = 1$ on K and 0 on the support of $\tilde{\phi}$, for $u \in \mathcal{C}_c^\infty(X)$ we notice that $\langle \eth u, \eth(\tilde{\phi} f)\rangle = \langle \eth(1-\chi)u, \eth(\tilde{\phi} f)\rangle$. This gives us the required continuity and then that $\tilde{\phi} f \in \mathcal{D}(\Delta)$. Therefore if $f \in \mathcal{D}(\Delta) \cup \mathcal{D}(\Delta')$, then $\tilde{\phi} f$ belongs to $\mathcal{D}(\Delta) \cap \mathcal{D}(\Delta')$. Moreover $\Delta \tilde{\phi} f = \Delta' \tilde{\phi} f$ since they are clearly both equal to $\Delta(\tilde{\phi} f)$ in the sense of distributions.

From the definition of the Friedrichs extension, the domain of Δ, Δ' is contained in $H_0^1(X) \cap H_{\mathrm{loc}}^2(X)$, respectively in $H_0^1(X') \cap H_{\mathrm{loc}}^2(X')$. Let Φ be the operator of multiplication by ϕ. By the Rellich-Kondrakov lemma, the hypotheses of Lemma 30 are satisfied. We apply Lemma 30 once for $H = \Delta$ and once for $H = \Delta'$ and since $\widetilde{\varphi}_n \in \mathcal{D}(\Delta) \cup \mathcal{D}(\Delta')$, we have $\Delta(1-\phi)\widetilde{\varphi}_n = \Delta'(1-\phi)\widetilde{\varphi}_n$. This proves the double inclusion of the essential spectra. $\square$

Remark 32. More generally, consider a cusp operator of the form $\tilde{\Delta} = \Delta + R_p^* R_p$, where $R_p \in \Psi_c^{-\infty,p}(X)$ preserves $\mathcal{C}_c^\infty(X')$ and is supported far from K in the sense

$$\Phi R_p = R_p \Phi = 0.$$

Such operators were explicitly constructed in the proofs of Propositions 24 and 27. Note that $\tilde{\Delta}$ preserves $\mathcal{C}_c^\infty(X')$, and that the domains of $\tilde{\Delta}$, $\tilde{\Delta}'$ are still contained in $H_0^1 \cap H_{\mathrm{loc}}^2$. Then the proof of Proposition 31 holds for the Friedrichs extension of $\tilde{\Delta}$. Namely if we set

$$\mathscr{Q}(f,g) = \langle \eth f, \eth g\rangle + \langle R_p f, R_p g\rangle,$$

the rest of the proof remains unchanged.

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