Phase transition induced by increasing the range of interaction in Potts Model

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Abstract. We consider the Q-state Potts model on $\mathbb{Z}^d$, $Q \geq 3$, $d \geq 2$, with a Kac ferromagnetic interaction, with scaling parameter γ. When $\gamma \to 0$ the range increases as γ^{-1} and the "strength" remains equal to 1. We prove the existence of a first order phase transition for γ small enough (thus with potential range finite). The proof is obtained by a perturbation around mean-field using the Pirogov-Sinai theory. The result is valid in particular for $d = 2$, $Q = 3$, thus providing an example of a system which undergoes a transition from second to first order phase transition when varying the finite range of the interaction.

1. Introduction

The Potts model is among the most interesting models in Statistical Mechanics and beyond. Since its original description by Potts as a simplified version of the clock model[**39**], it has become an ever growing source of interest, in particular in the field of phase transition. Started as the simplest generalization of the Ising Model (classical spins with Q values interacting through alike/dislike interactions), it acquired a widest meaning through the Kasteleyn-Fortuin representation [**21**], [**37**], which allows for both a straightforward generalization to any real positive value of the parameter Q (random cluster model [**22**]-[**9**]), and a direct connection between its partition function and the Tutte dichromatic polynomial [**44**]-[**42**], which have a central meaning in large areas of graph theory. The first instance gave rise to important connections with percolation theory ($Q \to 1$) and resistor networks ($Q \to 0$), while the second emphasizes on the properties of the graph it is defined on.

Though the Potts model is in general not solvable, it has been known since the original work by Potts that it undergoes an order-disorder phase transition. Since that time, a lot of work has been dedicated to the study of its critical properties, beginning with the nature of the transition, which appears to depend both on the number of spin values Q and the dimension of the lattice. In particular it has been proven that the transition is first order for nearest neighbor interactions in two dimensions for $Q > 4$ and continuous for $Q \leq 4$, [**4**]-[**5**]-[**45**].

Besides these exact results, an important guideline in the rigorous analysis of the Potts model has been the mean field theory [**45**], in which the transition is continuous for $Q \leq 2$

and first order for $Q \geq 3$. The Potts model with nearest neighbor interactions on $\mathbb{Z}^d$ gets close to its mean field approximation in two ways, namely in high dimensions, or for large values of Q. Early attempts led to the heuristic determination of a critical value $Q_c(d)$ beyond which the transition become mean-field like, and in particular to a few exact results $Q_c(2) = 4$, $Q_c(4) = 2$, $Q_c(6) = 1$ [**45**]. Rigorous results considered the large values of Q [**29**] or high dimension limit [**28**]while a more recent account gives a proof for d sufficiently large and all $Q \geq 3$, [**7**].

Another possible approach consists in considering a larger range of interaction. The known results include one dimensional systems with $\frac{1}{r^s}$ interactions, $1 < s \leq 2$. In [**1**], the existence of a transition has been proven in one dimension for $s = 2$. Numerical results [**41**] indicate that a transition from continuous to first order behavior occurs when s varies across some Q-dependant critical value. Recently [**8**] extended the previous rigorous results for long range potentials by proving the existence of a first order phase transition for potentials with the property of "reflection positivity" and sufficient spread out. It applies mainly to power law potentials for $d < 3$, and exponentially decaying potentials for $d \geq 3$. In particular this lead to a class of bidimensional systems with $Q = 3$ which undergo a first order phase transition in contradistinction with the nearest neighbor case. This result confirms the behavior suggested by the mean field that the order of the transition may change with the range of interaction.

In the present work, we consider long but finite interaction potentials, and prove that for any $Q \geq 3$, $d \geq 2$, such a system undergoes a first order phase transition provided the range of interactions is long enough. In particular we prove that the transition takes place at a temperature close to the mean field critical temperature and that there is coexistence at the critical point between one disordered and Q ordered phases. As a byproduct, we obtain that for $d = 2$, $Q = 3$ there is a change in the nature of the transition from second order for nearest neighbor interaction to first order beyond some finite value for the interaction range.

Our method relies on the derivation of a Peierls estimate for the system of contours defined on top of coarse grained configurations, where the "mean-field states" play the role of the ground states. This method was first developed in the context of Ising models with Kac interactions [**14**],[**11**],[**12**], and later included an extended Pirogov-Sinai theory [**46**] in order to cope with non symmetrical systems, such as a continuous point particle model [**33**] and a quantum version of it [**2**]. See also [**40**] for a thorough account on this approach. Absence of symmetry between ordered and disordered phases also lead us to use the Pirogov-Sinai theory. This requires in turn a precise estimate of finite volume corrections to the pressures which follows from an extension of the Dobrushin uniqueness criterion ([**16**]) introduced in [**3**].

Our results suggests three possible directions for further work: (1) the proof of the existence of a transition from first to second order when increasing the range of interactions.

(2) the extension of our results to the random cluster model for all real positive values of Q.
(3) the study of the phase diagram in a neighborhood of the critical temperature, applying the techniques developed in [**10**].

The paper is organized as follows: in Section 2, we define the model and state our main result. In Section 3, we explain the strategy of the proof and give some hints on why such approach is successful. In section 4, we define various notations and introduce the scales which appear in the problem. In section 5, we define the coarse grained configurations and prove a "Lebowitz Penrose theorem", introducing a mean field functional. Section 6 contains the definition of contours and states the Peierls bound for their weights. This bound is proven in the remaining sections: in section 7 we prove a factorization theorem for the contour weights and introduce abstract contour models which are at the core of the Pirogov-Sinai in the Zahradnìk approach, [**46**].

In section 8 we identify the value of the inverse temperature β at which the order-disorder phase transition takes place. Section 9 contains an estimate for the finite volume corrections to the pressure, which requires both a control on the decay of correlation and a small deviation estimate. Section 10 gives a large deviation estimate which provides the Peierls constant. Five appendices are added at the end of the paper: Appendix A contains a short review of the mean field theory for the Potts model as well as the derivation of the properties on which we rely in the rest of the paper. Appendix B discuss the local equilibrium properties which are used in various parts of the paper. In Appendix C, we prove the existence of the pressure for the abstract models introduced in section 7. In Appendix D, we prove a lemma needed in section 8. Finally, in Appendix E, we prove a lemma needed in the proof of the small deviation estimate of section 9.

2. Potts model with Kac interactions: definitions and main results.

We first define the Q-state Potts model on $\mathbb{Z}^d$ with long but finite interactions. In the following, $\mathbb{Z}^d$ is considered as naturally embedded in $\mathbb{R}^d$.

For a fixed value of the scaling parameter $\gamma > 0$, we introduce the Kac potential J_γ defined on $\mathbb{R}^d \times \mathbb{R}^d$ as

$$J_\gamma(x,y) = \gamma^d \mathcal{J}(\gamma|x-y|) \tag{2.1}$$

where $\mathcal{J}(r)$ is a symmetric probability kernel with range 1, differentiable with bounded derivative and $|.|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Let denote by a_q, $q = 1, \ldots, Q$, the Q different "colors". We denote by $\tilde{\Omega}_o := \{a_1, \ldots a_Q\}$ and by $\tilde{\Omega} := \tilde{\Omega}_o^{\mathbb{Z}^d}$ the space of configurations of the Q-state Potts model on $\mathbb{Z}^d$ and by

$\tilde{\Omega}_\Lambda = \tilde{\Omega}_o^\Lambda$ its restriction to a finite subset Λ of $\mathbb{Z}^d$. The color at the site i is denoted by $\sigma(i) \in \tilde{\Omega}_o$

For any finite region Λ of $\mathbb{Z}^d$, we define the Potts-Kac Hamiltonian with boundary conditions $\sigma_{\Lambda^c} \in \tilde{\Omega}_{\Lambda^c}$ as

$$H_{\gamma,\Lambda}(\sigma_\Lambda|\sigma_{\Lambda^c}) := -\frac{1}{2}\sum_{\substack{i,j\in\Lambda\\ i\neq j}} J_\gamma(i,j)\mathbf{1}_{\{\sigma_\Lambda(i)=\sigma_\Lambda(j)\}} - \sum_{\substack{i\in\Lambda\\ j\in\Lambda^c}} J_\gamma(i,j)\mathbf{1}_{\{\sigma_\Lambda(i)=\sigma_{\Lambda^c}(j)\}}$$

The Gibbs measure specifications are:

$$\mu_{\gamma,\beta,\Lambda}(\sigma_\Lambda|\sigma_{\Lambda^c}) = \frac{e^{-\beta H_{\gamma,\Lambda}(\sigma_\Lambda|\sigma_{\Lambda^c})}}{Z_{\gamma,\beta,\Lambda}(\sigma_{\Lambda^c})} \tag{2.2}$$

where $Z_{\gamma,\beta,\Lambda}(\sigma_{\Lambda^c})$ is the partition function

$$Z_{\gamma,\beta,\Lambda}(\sigma_{\Lambda^c}) := \sum_{\sigma_\Lambda} e^{-\beta H_{\gamma,\Lambda}(\sigma_\Lambda|\sigma_{\Lambda^c})}$$

In this paper, we will work with an alternative formulation of the Potts model which is more suited for our purpose and prepare for the introduction of coarse graining: Let $\vec{u}_q$, $q = 1, \cdots, Q$ the vectors of $\mathbb{R}^Q$ with components $u_{q,k} = \delta_{q,k}$. We define $\Omega_o := \{\vec{u}_1, \cdots, \vec{u}_Q\}$ and $\Omega := \Omega_o^{\mathbb{Z}^d}$. Then, we associate to each element σ of $\tilde{\Omega}$, a vector configuration $\vec{\xi}$ of Ω as

$$\vec{\xi}(i) = \vec{u}_q \Longleftrightarrow \sigma(i) = a_q \tag{2.3}$$

In these notations, the Hamiltonian (2.2) reads

$$H_{\gamma,\Lambda}(\vec{\xi}_\Lambda|\vec{\xi}_{\Lambda^c}) = -\frac{1}{2}\sum_{\substack{i,j\in\Lambda\\ i\neq j}} J_\gamma(i,j)\vec{\xi}_\Lambda(i)\cdot\vec{\xi}_\Lambda(j) - \sum_{\substack{i\in\Lambda\\ j\in\Lambda^c}} J_\gamma(i,j)\vec{\xi}_\Lambda(i)\cdot\vec{\xi}_{\Lambda^c}(j) \tag{2.4}$$

where "·" denote the scalar product: $\vec{v}\cdot\vec{v}' := \sum_{k=1}^Q v_k v'_k$. In these variables, the Potts model appears as a superposition of Q lattice gases, with configurations $\xi_q \in \{0,1\}^{\mathbb{Z}^d}$, interacting through hard core exclusion, $\sum_q \xi_q(i) = 1$ for all $i \in \mathbb{Z}^d$.

The main result of this paper is stated in the following theorem:

Theorem 2.1. *For any $d \leq 2$ and $Q \geq 3$, there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (0, \bar{\gamma})$, there is a value of the inverse temperature $\beta_c(\gamma)$ at which there exist $Q+1$ distinct DLR measures.*

In particular there are Q DLR measures $\mu^{(q)}_{\gamma,\beta}$, $q = 1, \cdots, Q$, the "ordered" states, where one of Q colors is dominant:

$$\mu^{(q)}_{\gamma,\beta}(\sigma_0 = a_q) > \mu^{(q)}_{\gamma,\beta}(\sigma_0 = a_p) \quad \forall p, q \in \{1, \cdots, Q\} \tag{2.5}$$

The remaining DLR measure is the "disordered" state $\mu_{\gamma,\beta}^{(-1)}$ where all colors appear with the same density:

$$\mu_{\gamma,\beta}^{(-1)}(\sigma_0 = a_q) = \frac{1}{Q}$$

3. Idea of the proof and stability properties of the Free energy functional

The main stream of ideas which leads to the present proof originates in the work by Lebowitz and Penrose [**35**] on the rigorous derivation of the Van der Waals theory of liquid-vapor transition using the inverse range of interaction γ as a control parameter.

The first step consists in the introduction of a continuous approximation of the free energy of the system by a "mean field free energy functional"

$$\mathcal{F}_{\gamma,\beta}(\vec{\rho}) = \int dr\ \Phi_\beta^{\mathrm{mf}}(\vec{\rho}) + \frac{1}{4}\int\int dxdy\ J_\gamma(x,y)(\vec{\rho}(y) - \vec{\rho}(x))^2$$

defined for all $\vec{\rho} \in L^\infty(\mathbb{R}^d, S_Q)$, where

$$S_Q = \big\{\vec{\rho} \in \mathbb{R}_+^Q, \sum_q \rho_q = 1\big\} \tag{3.1}$$

The mean field free energy density ϕ_β^{mf} is defined on S_Q as

$$\phi_\beta^{\mathrm{mf}}(\vec{v}) \quad := \quad \frac{1}{2}(\vec{v}\cdot\vec{v}) - \frac{1}{\beta}\vec{v}\cdot\ln(\vec{v})$$

and finally

$$\Phi_\beta^{\mathrm{mf}}(\vec{v}) \quad := \quad \phi_\beta^{\mathrm{mf}}(\vec{v}) - \inf_{\vec{v}\in S_Q} \phi_\beta^{\mathrm{mf}}(\vec{v})$$

This approximation allows for a perturbative analysis that relies on the stability properties of $\mathcal{F}_{\gamma,\beta}(\vec{\rho})$.

The critical points in $L^\infty(\mathbb{R}^d, S_Q)$ of $\mathcal{F}_{\gamma,\beta}(\cdot)$ identify with the fixed points of the map $\vec{\mathfrak{M}} = (\mathfrak{M}_1, \dots, \mathfrak{M}_Q)$ defined on $L^\infty(\Lambda, S_Q)$ as:

$$\vec{\rho} \to \vec{\mathfrak{M}}(\vec{\rho}) \qquad \text{with} \qquad \mathfrak{M}_k(\vec{\rho}) := \frac{e^{\beta\mathcal{L}_k(\vec{\rho})}}{\sum_l e^{\beta\mathcal{L}_l(\vec{\rho})}} \tag{3.2}$$

where $\vec{\mathcal{L}}(\vec{\rho})(r) = \int dy\ J_\gamma(x,y)\vec{\rho}(y)$. The stability property we are referring to is the fact that the map $\vec{\mathfrak{M}}(\cdot)$ is a contraction in a neighborhood of each fixed point relevant to our analysis. These fixed points are the constant vector functions, which take values in the set of the fixed points of the "mean field map"$\vec{\mathfrak{M}}^{\mathrm{mf}}(\cdot)$ defined on S_Q:

$$\mathfrak{M}_k^{\mathrm{mf}}(\vec{v}) := \frac{e^{\beta v_k}}{\sum_l e^{\beta v_l}}$$

This correspondence uses the fact that $\vec{\mathcal{L}}(\vec{\rho}) = \vec{\rho}$ for any constant profile $\vec{\rho}$ on $\mathbb{R}^d$. In appendix A it is shown that for β in an interval (β_0, Q) there are $Q+1$ solutions of the mean field equation in S_Q:

$$\vec{v} = \mathfrak{M}_k^{\mathrm{mf}}(\vec{v}) \tag{3.3}$$

denoted by $\vec{\rho}^{(\hat{q})} \equiv \vec{\rho}_\beta^{(\hat{q})}$, $\hat{q} \in \{-1, 1, \dots, Q\}$, which are the local minimizers of the mean field free energy density function $\phi_\beta^{\mathrm{mf}}(\vec{v})$ on S_Q (Theorem A.1). Q of them are the "ordered states" $\vec{\rho}^{(q)}$, $q > 0$, in which one of the color dominates, and the remaining one is a "disordered state", $\vec{\rho}^{(-1)}$ in which all colors have the same density. Around each of these minimizers, the mean field map is a contraction (Theorem A.3). and the map $\vec{\mathfrak{M}}$ inherits the same property for β close enough to the mean field critical inverse temperature $\beta_c^{\mathrm{mf}}(Q)$ as shown in section 9.1.

The next step consists in the introduction of a coarse grained level of description, at which one can introduce reference states in which the local empirical mean is everywhere close to one of the constant minimizers of $\mathcal{F}_{\gamma,\beta}(\vec{\rho})$. Our result is then based on the derivation of Peierls bounds for a systems of contours defined on a larger scale, as perturbations of these references states.

The absence of symmetry between ordered and disordered phases lead to use the Pirogov-Sinai theory. We follow the approach of Zahradnìk [**46**] (see also [**40**], [**33**], [**2**]) and define abstract contours models with cut-off weights, supported by a the restricted ensemble of configurations associated to a reference state. The value of the inverse temperature at which the pressures of these models are equal identifies with the critical value at which the transition takes place. In fact, equality between these pressures together with an accurate estimate on their finite volume corrections lead to a Peierls estimate for the "true model". This latter point is obtained by proving for the abstract models both the fast decay of correlations and a small deviation estimate. These proofs depend in a crucial way on the stability properties of the free energy functional $\mathcal{F}_{\gamma,\beta}(\vec{\rho})$. The value of the Peierls constant as well as the control over the various error terms then follows from a large deviation estimate based on the continuous approximation introduced above.

Technically, we first derive a factorization theorem (Theorem 7.1) for the weight of a contour. The proof then splits into two parts: a large deviation estimate gives an estimate for the free energy of a contour (Theorem 7.2) while the control on the other term in the factorization formula follows from the analysis of the abstract contours models (Theorem 7.3). This last theorem uses estimates on decay of correlations and small deviation estimates for these models. Connection with the original model holds when the cut-off weights for the abstract models identify with the true weightswhich occurs for a single value $\beta = \beta_c(\gamma)$.

By these three Theorem (7.1, 7.2 and 7.3) we prove that a Peierls bound holds at the critical temperature $\beta_c(\gamma)$ for γ small enough (Theorem (6.3)). A classical Peierls argument then proves that, at $\beta = \beta_c(\gamma)$, the Q ordered phases coexist with the disordered one (Theorem (2.1)).

4. Notations and scales

Notations

We consider $\mathbb{Z}^d$ as imbedded in $\mathbb{R}^d$ and frequently need to define similar functions both on $\mathbb{Z}^d$ and on $\mathbb{R}^d$. In order to avoid confusion and duplication of all notations, we will stick to the convention that functions are defined on $\mathbb{Z}^d$ or on $\mathbb{R}^d$ according to which of the two following notations is used for their arguments:

x,y are points in $\mathbb{R}^d$.

i, j are points in $\mathbb{Z}^d$.

Accordingly, for any subset $\Lambda \in \mathbb{R}^d$, we will write $\{i \in \Lambda\}$ for the set $\{i \in \Lambda \sqcap \mathbb{Z}^d\}$.

We denote by $|\Lambda|$ the volume of the set Λ and by $|\partial\Lambda|$ its surface.
Furthermore, the Q colors in the microscopic model are denoted $\{a_1, \cdots, a_Q\}$, and we use indices p or q in $\{1, \cdots, Q\}$ to denote them. After coarse graining, these indices refer to the Q ordered states in which one color is dominant, while one extra color a_{-1} is introduced corresponding to a disordered state. Thus we use indices $\hat{p}$ or $\hat{q}$ in $\{-1, 1, \cdots, Q\}$ to designate the colors after coarse graining. Correspondingly there are $Q+1$ local minimizers in the mean field model which we denote by the symbol $\vec{\rho}^{\,(\hat{q})} \equiv \vec{\rho}_\beta^{\,(\hat{q})}$. We will also denote by the same symbol the $Q+1$ density functions in $L^\infty(\mathbb{Z}^d, S_Q)$ (respectively $L^\infty(\mathbb{R}^d, S_Q)$) constantly equal to $\vec{\rho}^{\,(\hat{q})}$ on $\mathbb{Z}^d$ (respectively $\mathbb{R}^d$).

We will use instead k and l as indices in $\{1, \cdots, Q\}$ to designate the k^{th} and l^{th} coordinates of a vector in S_Q. Finally, m is an index in $\{1, \cdots, d\}$.

We equip $\mathbb{R}^d$ with the following two norms: let $x, y \in \mathbb{R}^d$

$$\operatorname{dist}(x, y) := \max_{m\in\{1,\dots,d\}} |x_m - y_m| \tag{4.4}$$

$$|x - y| := \sqrt{\sum_{m=1}^{d} (x_m - y_m)^2} \tag{4.5}$$

We also introduce two norms on S_Q. For all $\vec{v}$ in S_Q, let

$$\|\vec{v}\|_\star := \sup_{k\in\{1,\dots,Q\}} |v_k| \tag{4.6}$$

$$|\vec{v}| := \sqrt{\sum_{k=1}^{Q} v_k^2} \tag{4.7}$$

Finally, we denote by $\mathfrak{J}_\gamma$ the normalization constant of the interaction kernel J_γ on $\mathbb{Z}^d$:

$$\mathfrak{J}_\gamma := \sum_{j\in\mathbb{Z}^d} J_\gamma(0, j) \qquad\qquad \lim_{\gamma\to 0} \mathfrak{J}_\gamma = 1 \tag{4.8}$$

Scales:

In the sequel, we introduce three lengths ℓ_0, $\ell_{-,\gamma}$ and $\ell_{+,\gamma}$ which grow as powers of the potential range, respectively as $\gamma^{-\frac{1}{2}}$, $\gamma^{-(1-\alpha)}$ and $\gamma^{-(1+\alpha)}$, where $\alpha \in (0, \frac{1}{16d})$. The bound

$\frac{1}{16d}$ will be requested in the course of the proofs in order to satisfy various requirements. For $\ell = \ell_0, \ell_{-,\gamma}, \ell_{+,\gamma}$ we construct a partition $\mathcal{D}^\ell$ of $\mathbb{R}^d$ in cubes of side length ℓ. In order that these three partitions are one coarser than the other, we define ℓ_0, (respectively $\ell_{-,\gamma}$ and $\ell_{+,\gamma}$) as the largest power of 2 smaller than $\gamma^{-\frac{1}{2}}$, (respectively $\gamma^{-(1-\alpha)}$ and $\gamma^{-(1+\alpha)}$). In other terms,

$$\ell_0 = 2^{\left[\frac{1}{2}\frac{\ln\gamma^{-1}}{\ln 2}\right]} \tag{4.9}$$

$$\ell_{\pm,\gamma} = 2^{(1\pm\alpha)\left[\frac{\ln\gamma^{-1}}{\ln 2}\right]} \tag{4.10}$$

where $[\cdot]$ denotes the integer part We will also consider γ small enough so that these lengths are ordered as

$$1 < \ell_0 < \ell_{-,\gamma} < \gamma^{-1} < \ell_{+,\gamma} \tag{4.11}$$

5. Coarse graining and mean field functional

Let ℓ a large positive integer and $\mathcal{D}^\ell$ a partition of $\mathbb{R}^d$ in cubes of size ℓ. For all x in $\mathbb{R}^d$, we denote by C_x^ℓ the cube of $\mathcal{D}^\ell$ containing x. We define a coarse grained configuration on $\mathcal{D}^\ell$ as follows: for each configuration $\vec{\xi} \in \Omega$ and any $x \in \mathbb{R}^d$, we define the coarse grained configuration (at scale ℓ) as the Q-dimensional vector,

$$\vec{\xi}^\ell(x) = \ell^{-d} \sum_{i \in C_x^\ell} \vec{\xi}(i) \tag{5.12}$$

where we make use of our notational conventions. The k^{th} component $\xi_k^\ell(x)$ is the empirical density of color a_k in C_x^ℓ. Due to the underlying discretization, $\vec{\xi}^\ell(x)$ takes values in the finite set $M_{\ell^d}^Q$

$$M_{\ell^d}^Q = \big\{(r_1, \cdots, r_Q); (\ell^d r_k) \in \mathbb{N}, \sum_{k=1}^Q r_k = 1\big\}$$

Let Λ a $\mathcal{D}^\ell$-measurable subset of $\mathbb{R}^d$. The set of coarse grained configuration in Λ is denoted by Ω_Λ^ℓ and identifies with the set of $\mathcal{D}^\ell$-measurable functions on Λ with values in $M_{\ell^d}^Q$. We extend the discrete set $M_{\ell^d}^Q$ to the simplex S_Q in $\mathbb{R}^Q$ defined as

$$S_Q = \big\{\vec{\rho} \in \mathbb{R}_+^Q, \sum_q \rho_q = 1\big\} \tag{5.13}$$

Thus all coarse grained configurations in Ω_Λ^ℓ are also elements of $L^\infty(\Lambda, S_Q)$. Conversely we will approximate any function in $L^\infty(\Lambda, S_Q)$ by a coarse grained configuration. For any

$\vec{\rho} \in L^\infty(\Lambda, S_Q)$, we will denote by $\vec{\rho}^\ell$ its $\mathcal{D}^\ell$-measurable approximation

$$\vec{\rho}^\ell(x) = \ell^{-d} \int_{C_x^\ell} \vec{\rho}(y) dy \tag{5.14}$$

for all x in Λ, and by $[\vec{\rho}]^\ell$ the only function in Ω_Λ^ℓ such that

$$-\frac{1}{2\ell^d} < [\vec{\rho}]_k^\ell(x) - \rho_k^\ell(x) \leq \frac{1}{2\ell^d} \tag{5.15}$$

for all k and all x in Λ.

For Λ a finite $\mathcal{D}^\ell$-measurable region of $\mathbb{R}^d$, we define the mean field free energy functional on $L^\infty(\Lambda, S_Q)$ as,

$$F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\rho}_{\Lambda^c}) = V_{\gamma,\Lambda}(\vec{\rho}_\Lambda|\vec{\rho}_{\Lambda^c}) - \frac{1}{\beta} I(\vec{\rho}_\Lambda) \tag{5.16}$$

where $\vec{\rho}_{\Lambda^c} \in L^\infty(\Lambda^c, S_Q)$ defines the boundary conditions. The two functionals $V_{\gamma,\Lambda}(\vec{\rho}_\Lambda|\vec{\rho}_{\Lambda^c})$ and $I(\vec{\rho}_\Lambda)$ are respectively the energy and the entropy of configuration $\vec{\rho}_\Lambda$,

$$V_{\gamma,\Lambda}(\vec{\rho}_\Lambda|\vec{\rho}_{\Lambda^c}) = -\frac{1}{2} \int_\Lambda \int_\Lambda J_\gamma(x,y) \big(\vec{\rho}_\Lambda(x) \cdot \vec{\rho}_\Lambda(y)\big) dxdy - \int_\Lambda \int_{\Lambda^c} J_\gamma(x,y) \big(\vec{\rho}_\Lambda(x) \cdot \vec{\rho}_{\Lambda^c}(y)\big) dxdy$$

$$I(\vec{\rho}_\Lambda) = -\int_\Lambda \sum_{q=1}^{Q} \rho_{\Lambda,q}(x) \ln(\rho_{\Lambda,q}(x)) dx =: -\int_\Lambda \vec{\rho}_\Lambda(x) \cdot \ln \vec{\rho}_\Lambda(x) dx$$

For all $\mathcal{A} \sqsubset \Omega_\Lambda$ we define the constrained partition function $Z_{\gamma,\beta,\Lambda}(\mathcal{A}|\vec{\xi}_{\Lambda^c})$

$$Z_{\gamma,\beta,\Lambda}(\mathcal{A}|\vec{\xi}_{\Lambda^c}) = \sum_{\vec{\xi}_\Lambda \in \mathcal{A}} e^{-\beta H_{\gamma,\Lambda}(\vec{\xi}_\Lambda|\vec{\xi}_{\Lambda^c})}$$

We now state a theorem relating constrained partition functions and mean field free energy:

Theorem 5.1. *There exists a constant $c > 0$ such that for all $\gamma > 0$, $\ell \in (1, \gamma^{-1})$ and all bounded $\mathcal{D}^\ell$-measurable region Λ of $\mathbb{R}^d$, the following inequalities hold:*

For all subsets $\mathcal{A}$ of Ω_Λ^ℓ,

$$\ln Z_{\gamma,\beta,\Lambda}(\{\vec{\xi}_\Lambda^\ell \in \mathcal{A}\}|\vec{\xi}_{\Lambda^c}) + \beta \inf_{\vec{\rho}_\Lambda \in \mathcal{A}} F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\xi}_{\Lambda^c}^\ell) \leq \beta c \epsilon(\gamma,\ell)|\Lambda|$$

and for all $\vec{\rho}_\Lambda \in L^\infty(\Lambda, S_Q)$,

$$\ln Z_{\gamma,\beta,\Lambda}(\{\vec{\xi}_\Lambda : \vec{\xi}_\Lambda^\ell = [\vec{\rho}_\Lambda]^\ell\}|\vec{\xi}_{\Lambda^c}) + \beta F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\xi}_{\Lambda^c}^\ell) \geq -\beta c \epsilon(\gamma,\ell)|\Lambda|$$

where

$$\epsilon(\gamma,\ell) = \gamma\ell + \frac{\ln \ell}{\ell^d}$$

Proof. We first estimate the difference between the energy $H_{\gamma,\Lambda}(\vec{\xi}_\Lambda|\vec{\xi}_{\Lambda^c})$ and its coarse grained approximation $V_{\gamma,\Lambda}(\vec{\xi}_\Lambda^\ell|\vec{\xi}_{\Lambda^c}^\ell)$. Given two cubes C_1 and C_2 of the partition $\mathcal{D}^\ell$, for any two points $i \in C_1$ and $j \in C_2$, we have

$$|J_\gamma(i,j) - \frac{1}{\ell^{2d}} \int_{C_1 \times C_2} J_\gamma(x,y)\ dx\ dy| \leq 2\sqrt{d} \|\nabla \mathcal{J}\|_\infty \gamma^{d+1} \ell \mathbf{1}_{d(C1,C2) \leq \gamma^{-1}}$$

where $d(C1,C2) = \inf_{x \in C_1, y \in C_2} |x-y|$.

Hence

$$
\begin{aligned}
&\big|H_{\gamma,\Lambda}(\vec{\xi}_\Lambda|\vec{\xi}_{\Lambda^c}) - V_{\gamma,\Lambda}(\vec{\xi}^\ell_\Lambda|\vec{\xi}^\ell_{\Lambda^c})\big| \\
&\quad\le \sum_{\substack{C_1\in\mathcal{D}^\ell_\Lambda\\ C_2\in\mathcal{D}^\ell}} \Big| \sum_{\substack{i\in C_1\cap\mathbb{Z}^d\\ j\in C_2\cap\mathbb{Z}^d}} J_\gamma(i,j)\vec{\xi}_\Lambda(i)\cdot\vec{\xi}_\Lambda(j) - \int_{C_1\times C_2} J_\gamma(x,y)\vec{\xi}^\ell_\Lambda(x)\cdot\vec{\xi}^\ell_\Lambda(y)\Big| \\
&\quad\le \sum_{\substack{C_1\in\mathcal{D}^\ell_\Lambda\\ C_2\in\mathcal{D}^\ell}} \Big| \sum_{\substack{i\in C_1\cap\mathbb{Z}^d\\ j\in C_2\cap\mathbb{Z}^d}} \Big(J_\gamma(i,j) - \frac{1}{\ell^{2d}}\int_{C_1\times C_2} J_\gamma(x,y)\;dx\;dy\Big)\vec{\xi}_\Lambda(i)\cdot\vec{\xi}_\Lambda(j)\Big| \\
&\quad\le 2\sqrt{d}\|\nabla\mathcal{J}\|_\infty\gamma^{d+1}\ell\sum_{C_1,C_2}|C_1||C_2|\mathbf{1}_{d(C_1,C_2)\le\gamma^{-1}} \\
&\quad\le c_d\gamma\ell|\Lambda|
\end{aligned}
$$

where c_d is a constant independent on γ, $\ell\le\gamma^{-1}$ and Λ

$$
c_d = 2^{d+1}\sqrt{d}\|\nabla\mathcal{J}\|_\infty \tag{5.17}
$$

Thus for any profile $\vec{\rho}_\Lambda$ in Ω^ℓ_Λ, we have

$$
\Big|\ln\frac{Z_{\gamma,\beta,\Lambda}(\{\vec{\xi}_\Lambda:\vec{\xi}^\ell_\Lambda=\vec{\rho}_\Lambda\}|\vec{\xi}_{\Lambda^c})}{|\{\vec{\xi}_\Lambda:\vec{\xi}_\Lambda^{\,\ell}=\vec{\rho}_\Lambda\}|e^{-\beta V_{\gamma,\Lambda}(\vec{\rho}_\Lambda|\vec{\xi}^\ell_{\Lambda^c})}}\Big| \le c_d\gamma\ell|\Lambda|
$$

The cardinality of $\{\vec{\xi}_\Lambda:\vec{\xi}^\ell_\Lambda=\vec{\rho}_\Lambda\}$ can be related to the entropy of $\vec{\rho}_\Lambda$. The error bounds for the Stirling formula

$$
\frac{1}{12N+1}\le\frac{\ln(N!)}{N(\ln(N)-1)+\ln(\sqrt{2\pi N})}\le\frac{1}{12N} \tag{5.18}
$$

leads to the following estimate

$$
\big|\ln|\{\vec{\xi}_\Lambda:\vec{\xi}_\Lambda^{\,\ell}=\vec{\rho}_\Lambda\}| - I(\vec{\rho}_\Lambda)\big| \le dQ|\Lambda|\frac{\log\ell}{\ell^d}
$$

one gets

$$
\big|\ln Z_{\gamma,\beta,\Lambda}(\{\vec{\xi}_\Lambda:\vec{\xi}^\ell_\Lambda=\vec{\rho}_\Lambda\}|\vec{\xi}_{\Lambda^c}) + \beta F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\xi}^\ell_{\Lambda^c})\big| \le \beta c_d\gamma\ell|\Lambda| + Qd|\Lambda|\frac{\log\ell}{\ell^d}
$$

Now an easy upper bound for the cardinality of Ω^ℓ_Λ gives for all $\mathcal{A}\sqsubset\Omega^\ell_\Lambda$

$$
\ln|\mathcal{A}|\le\ln|\Omega^\ell_\Lambda|\le Qd\frac{|\Lambda|}{\ell^d}\log\ell
$$

where $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$. Combining these two last inequalities gives the first part of Theorem 5.1 with $c=\max(c_d,\frac{2Qd}{\beta})$. On the other side, for all $\vec{\rho}\in L^\infty(\Lambda,S_Q)$, one has

$$
\log Z_{\gamma,\beta,\Lambda}(\vec{\xi}^\ell_\Lambda=[\vec{\rho}_\Lambda]^\ell|\vec{\xi}_{\Lambda^c}) \ge -\beta F_{\gamma,\beta,\Lambda}([\vec{\rho}]^\ell|\vec{\xi}^\ell_{\Lambda^c}) - c\epsilon(\gamma,\ell)|\Lambda|
$$

Now using

$$
|V_{\gamma,\Lambda}(\vec{\rho}_\Lambda|\vec{\xi}^\ell_{\Lambda^c}) - V_{\gamma,\Lambda}(\vec{\rho}^\ell_\Lambda|\vec{\xi}^\ell_{\Lambda^c})| \le c_d\gamma\ell|\Lambda|
$$

and the concavity of the entropy, one gets

$$
F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\xi}^\ell_{\Lambda^c}) \ge F_{\gamma,\beta,\Lambda}(\vec{\rho}^\ell_\Lambda|\vec{\xi}^\ell_{\Lambda^c}) - c_d\gamma\ell|\Lambda|
$$

Furthermore, approximating $\vec{\rho}_\Lambda^\ell$ by $[\vec{\rho}_\Lambda]^\ell$ gives

$$F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda^\ell|\vec{\xi}_{\Lambda^c}^\ell) \geq F_{\gamma,\beta,\Lambda}([\vec{\rho}_\Lambda]^\ell|\vec{\xi}_{\Lambda^c}^\ell) - dQ\frac{|\Lambda|}{\ell^d}$$

we thus get:

$$\log Z_{\gamma,\beta,\Lambda}(\vec{\xi}_\Lambda^\ell = [\vec{\rho}]^\ell|\vec{\xi}_{\Lambda^c}) \geq -\beta F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\xi}_{\Lambda^c}^\ell) - \beta c\epsilon(\gamma,\ell)|\Lambda|$$

which gives the second part of the theorem with the same constant as before. $\square$

We will use theorem 5.1 mostly in the following weaker form:

Corollary 5.2. *There exists a constant $c > 0$ such that for all $\gamma > 0$, ℓ an integer in $(1, \gamma^{-1})$ and all bounded $\mathcal{D}^\ell$-measurable region Λ of $\mathbb{R}^d$,*

$$\Big|\log Z_{\gamma,\beta,\Lambda}(\vec{\xi}_{\Lambda^c}) + \beta\inf_{\vec{\rho}_\Lambda} F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\xi}_{\Lambda^c}^\ell)\Big| \leq \beta c\epsilon(\gamma,\ell)|\Lambda|$$

Theorem 5.1 leads also to the Lebowitz-Penrose limit for the Potts model

Theorem 5.3. *[Lebowitz-Penrose] There exists the limit*

$$\lim_{\gamma\to\infty}\lim_{\Lambda\to\mathbb{R}^d}\frac{\log Z_{\gamma,\beta,\Lambda}}{|\Lambda|} = P_\beta^{\mathrm{mf}}$$

where P_β^{mf} is the mean field pressure.

Proof. The free energy functional on $L^\infty(\Lambda, S_Q)$ with boundary conditions $\vec{\rho}_{\Lambda^c}$ can be rewritten as

$$F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\rho}_{\Lambda^c}) = \mathcal{F}_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\rho}_{\Lambda^c}) - \frac{1}{2}\int_\Lambda\int_{\Lambda^c} J_\gamma(x,y)|\vec{\rho}_{\Lambda^c}(y)|^2 dxdy$$

where

$$\begin{aligned}\mathcal{F}_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\rho}_{\Lambda^c}) = \quad & \int_\Lambda \phi^{\mathrm{mf}}(\vec{\rho}_\Lambda(x))dx + \frac{1}{4}\int_\Lambda\int_\Lambda J_\gamma(x,y)|\vec{\rho}_\Lambda(x) - \vec{\rho}_\Lambda(y)|^2 dxdy \\ & + \frac{1}{2}\int_\Lambda\int_{\Lambda^c} J_\gamma(x,y)|\vec{\rho}_\Lambda(x) - \vec{\rho}_{\Lambda^c}(y)|^2 dxdy\end{aligned}$$

with $\phi_\beta^{\mathrm{mf}}(\vec{\rho})$ the mean field free energy density on S_Q (A.2). We have clearly

$$\inf_{\vec{\rho}_\Lambda\in L^\infty(\Lambda,S_Q)} \mathcal{F}_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\rho}_{\Lambda^c}) \geq \inf_{\vec{\rho}_\Lambda\in L^\infty(\Lambda,S_Q)} \int_\Lambda \phi_\beta^{\mathrm{mf}}(\vec{\rho}_\Lambda(x))dx$$

which gives a lower bound for the free energy as

$$\inf_{\vec{\rho}_\Lambda\in L^\infty(\Lambda,S_Q)} F_{\gamma,\beta,\Lambda}(\vec{\rho}_\Lambda|\vec{\rho}_{\Lambda^c}) \geq -P_\beta^{\mathrm{mf}}|\Lambda| - c\gamma^{-1}|\partial\Lambda|$$

with

$$P_\beta^{\mathrm{mf}} = -\inf_{\vec{v}\in S_Q}\phi_\beta^{\mathrm{mf}}(\vec{v}) \tag{5.19}$$

From 5.1 with $\ell = \gamma^{-\frac{1}{2}}$, one gets

$$\frac{\log Z_{\gamma,\beta,\Lambda}}{\beta|\Lambda|} \leq P_\beta^{\mathrm{mf}} + c\epsilon(\gamma,\gamma^{-1/2}) + \gamma^{-1}\frac{|\partial\Lambda|}{|\Lambda|}$$

and hence

$$\liminf_{\gamma\to 0}\liminf_{\Lambda\nearrow\mathbb{R}^d}\frac{\log Z_{\gamma,\beta,\Lambda}}{\beta|\Lambda|}\le P_\beta^{\rm mf}$$

where the limit $\liminf_{\Lambda\nearrow\mathbb{R}^d}$ is taken on a sequence of Van Hove subsets of $\mathbb{R}^d$. On the other side, writing (5.19) for $\vec\rho_\Lambda=\vec\rho^{\hat p}\mathbf{1}_\Lambda$, where $\vec\rho^{\hat p}$ is the absolute minimizer of $\phi_\beta^{\rm mf}$, we get

$$F_{\gamma,\beta,\Lambda}(\vec\rho^{\,*}\mathbf{1}_\Lambda|\vec\rho_{\Lambda^c})\le\mathcal{F}_{\gamma,\beta,\Lambda}(\vec\rho^{\,*}\mathbf{1}_\Lambda|\vec\rho_{\Lambda^c})\le -P_\beta^{\rm mf}|\Lambda|+c\gamma^{-1}|\partial\Lambda|$$

From the second inequality in 5.1 one gets

$$\frac{\log Z_{\gamma,\beta,\Lambda}}{\beta|\Lambda|}\ge P_\beta^{\rm mf}-c\epsilon(\gamma,\gamma^{-1/2})+\gamma^{-1}\frac{|\partial\Lambda|}{|\Lambda|}$$

and finally

$$\limsup_{\gamma\to 0}\limsup_{\Lambda\nearrow\mathbb{R}^d}\frac{\log Z_{\gamma,\beta,\Lambda}}{\beta|\Lambda|}\ge P_\beta^{\rm mf}$$

□

6. Contours

We first define three *phase indicators*: η, θ, Θ. These indicators are block spin variables which take value in $\{0,a_{-1},a_1,\dots,a_Q\}$, ($a_{-1}$ a new "color" which will sign the disordered state). We will define them both on configurations in Ω and on functions in $L^\infty(\mathbb{R}^d,S_Q)$ or in $L^\infty(\mathbb{Z}^d,S_Q)$, using the same notation, as follows: Let $C_x^{\ell_{-,\gamma}}$ the block of the partition $\mathcal{D}^{\ell_{-,\gamma}}$ of size $\ell_{-,\gamma}$ that contains the point $x\in\mathbb{R}^d$.

$$\eta_x(\vec s):=\begin{cases}a_{\hat p} & \text{if } \|\vec s^{\,\ell_{-,\gamma}}(x)-\vec\rho_\beta^{\,(\hat p)}\|_\star<\gamma^a, \quad \hat p\in\{-1,1,\dots,Q\}\\ \\ 0 & \text{otherwise.}\end{cases}\tag{6.1}$$

where $\vec s^{\,\ell_{-,\gamma}}$ is a $\mathcal{D}^{\ell_{-,\gamma}}$-measurable function defined either by (5.12) or by (5.14) and $\vec\rho^{\,(\hat q)}$ are the local minimizers in S_Q of the mean field free energy density.

The value of the exponent $a>0$ defines the "accuracy parameter" γ^a in the right hand side of (6.1). Our proof will require in addition that $a<\alpha$.

Let then denote by $C_y^{\ell_{+,\gamma}}$ the cube of the partition $\mathcal{D}^{\ell_{+,\gamma}}$ that contains the point $y\in\mathbb{R}^d$. We define

$$\theta_x(\vec s):=\begin{cases}\eta_x(\vec s) & \text{if } \eta_y(\vec s)=a_{\hat p}\quad \forall y\in C_x^{\ell_{+,\gamma}},\\ 0 & \text{otherwise.}\end{cases}\tag{6.2}$$

$$\Theta_x(\vec{s}) = \theta_x(\vec{s}) \prod_{C_y^{\ell_{+,\gamma}} : C_y^{\ell_{+,\gamma}} \sim C_x^{\ell_{+,\gamma}}} \mathbf{1}_{\theta_x(\vec{s})=\theta_y(\vec{s})} \tag{6.3}$$

The set $\{C_y^{\ell_{+,\gamma}} : C_y^{\ell_{+,\gamma}} \sim C_x^{\ell_{+,\gamma}}\}$ is the set of cubes in $\mathcal{D}^{\ell_{+,\gamma}}$ $*$-connected with $C_x^{\ell_{+,\gamma}}$, i.e. : $\overline{C_y^{\ell_{+,\gamma}}} \sqcap \overline{C_x^{\ell_{+,\gamma}}} \neq \emptyset$.

Notice that, by definition, if $\Theta_x(\vec{s}) = a_{\hat{p}}$ and $\Theta_y(\vec{s}) = a_{\hat{q}}$ and $\hat{p} \neq \hat{q}$, then $\mathrm{dist}(x, y) > 2\ell_{+,\gamma}$ Thus the regions $\{x : \Theta_x(\vec{s}) = a_{\hat{p}}\}$ and $\{x : \Theta_x(\vec{s}) = a_{\hat{q}}\}$ will be separated by a domain which form the support for the *contours* which we now define with respect to a configurations $\vec{\xi}$. A similar definition with respect to a continuous profile $\vec{\rho}$ will also hold.

Definition 6.1. *A contour* $\Gamma \equiv (\mathrm{sp}(\Gamma), \eta_\Gamma)$ *for a configuration* $\vec{\xi}$ *in* Ω *, is specified by a couple* $(\mathrm{sp}(\Gamma), \eta_\Gamma)$ *where* $\mathrm{sp}(\Gamma)$ *is one of the maximal connected component of the subset of* $\mathbb{R}^d$ $\{x : \Theta_x(\vec{\xi}) = 0\}$ *and* η_Γ *is the coarse grained configuration on* $\mathrm{sp}(\Gamma)$ *on the scale* $\ell_{\gamma,-}$*,* $\eta_\Gamma \equiv \{\eta_x(\vec{\xi})\}_{x\in \mathrm{sp}(\Gamma)}$.

We will prove that a Peierls bound for these contours holds for a range of values in β. Before stating our precise result in a Theorem, we need to introduce some notations related to the notion of contour:

Notations

For any set $B \sqsubset \mathbb{R}^d$, $r \in \mathbb{R}$, we define

$$\delta_{\mathrm{in}}^r[B] := \{x \in B : \mathrm{dist}(x, B^c) \leq r\} \tag{6.4}$$

$$\delta_{\mathrm{out}}^r[B] := \{x \in B^c : \mathrm{dist}(x, B) \leq r\} \tag{6.5}$$

where $\mathrm{dist}(x, B) = \inf_{y\in B} \mathrm{dist}(x, y)$.

We denote by $|\Gamma| \equiv |\mathrm{sp}(\Gamma)|$ the volume of the region $\mathrm{sp}(\Gamma)$. For any bounded contour ($|\Gamma| < \infty$), we denote by $\mathrm{Ext}(\Gamma)$ the (unique) unbounded connected component of $\mathrm{sp}(\Gamma)^c$ and by $\{\mathrm{Int}_i(\Gamma)\}$, $i \in I$ the collection of its bounded connected components:

$$\mathrm{sp}(\Gamma)^c = \mathrm{Ext}(\Gamma) \sqcup_{i\in I} \mathrm{Int}_i(\Gamma) \tag{6.6}$$

By definition of a contour, for any point x in $\delta_{\mathrm{out}}^{\ell_{+,\gamma}}[\mathrm{sp}(\Gamma)]$, $\Theta_x \neq 0$ and we define

$$A^{\hat{q}} \equiv A^{\hat{q}}(\Gamma) := \{x \in \delta_{\mathrm{out}}^{\ell_{+,\gamma}}[\mathrm{sp}(\Gamma)] : \Theta_x = a_q\} \qquad ; \qquad A(\Gamma) := \bigsqcup_{\hat{q}} A^{\hat{q}} = \delta_{\mathrm{out}}^{\ell_{+,\gamma}}[\mathrm{sp}(\Gamma)] \tag{6.7}$$

Note that $A^{\hat{q}}$ depends only on Γ. Correlatively, for each connected component of $\mathrm{sp}(\Gamma)^c$, $\mathrm{Int}_i(\Gamma)$ (respectively $\mathrm{Ext}(\Gamma)$) the phase indicator Θ is constant and nonzero on $\delta_{\mathrm{in}}^{\ell_{+,\gamma}}[\mathrm{Int}_i(\Gamma)]$ (respectively on $\delta_{\mathrm{in}}^{\ell_{+,\gamma}}[\mathrm{Ext}(\Gamma)]$). Thus we write

$$I^{\hat{q}}(\Gamma) := \{i : \mathrm{Int}_i(\Gamma) \sqcap A^{\hat{q}} \neq \emptyset; \} \qquad ; \qquad \mathrm{Int}^{\hat{q}}(\Gamma) := \bigsqcup_{i\in I^{\hat{q}}(\Gamma)} \mathrm{Int}_i(\Gamma) \tag{6.8}$$

We call Γ a "$\hat{p}$-contour" if $\mathrm{Ext}(\Gamma) \sqcap A^{\hat{p}} \neq \emptyset$ and define

$$c(\Gamma) := \mathrm{sp}(\Gamma) \sqcup_{i \in I} \mathrm{Int}_i(\Gamma) \tag{6.9}$$

Weights of contours

Let denote by $\mathcal{E}(\Gamma, \hat{p})$ the event that Γ is a $\hat{p}$-contour:

$$\mathcal{E}(\Gamma, \hat{p}) := \left\{ \vec{\xi} : \eta_x(\vec{\xi}) = \eta_\Gamma \ \forall \ x \in \mathrm{sp}(\Gamma) \ ; \ \substack{\Theta_x(\vec{\xi}) = a_{\hat{q}} \ \forall \ x \in A^{\hat{q}}(\Gamma) \ \forall \hat{q} \in \{-1,1,\dots,Q\} \\ \Theta_x(\vec{\xi}) = a_{\hat{p}} \ \forall \ x \in A(\Gamma) \sqcap \mathrm{Ext}(\Gamma)} \right\}$$

and by $\mathcal{E}(\not\Gamma, \hat{p})$ the event that the phase $\hat{p}$ extends on $\mathrm{sp}(\Gamma) \sqcup A(\Gamma)$

$$\mathcal{E}(\not\Gamma, \hat{p}) := \{ \vec{\xi} : \Theta_x(\vec{\xi}) = a_{\hat{p}} \ \forall \ x \in \mathrm{sp}(\Gamma) \sqcup A(\Gamma) \}$$

We then define the weight of a $\hat{p}$-contour as follows:

$$w^{\hat{p}}_{\gamma,\beta}(\Gamma; \vec{\xi}_{A^{\hat{p}}}) := \frac{\mu_{\gamma,\beta,c(\Gamma) \setminus \mathrm{Int}^{\hat{p}}(\Gamma)}(\mathcal{E}(\Gamma, \hat{p}) | \vec{\xi}_{A^{\hat{p}}})}{\mu_{\gamma,\beta,c(\Gamma) \setminus \mathrm{Int}^{\hat{p}}(\Gamma)}(\mathcal{E}(\not\Gamma, \hat{p}) | \vec{\xi}_{A^{\hat{p}}})} \tag{6.10}$$

Peierls bounds

We say that the $\hat{p}$-contours satisfy a Peierls bound if there exists a constant $\mathfrak{K} > 0$ such that for any $\hat{p}$-contour Γ:

$$w^{\hat{p}}_{\gamma,\beta}(\Gamma; \vec{\xi}_{A^{\hat{p}}}) \leq e^{-\mathfrak{K} \ N_\Gamma} \tag{6.11}$$

where N_Γ is the number of cubes of $\mathcal{D}^{\ell_+,\gamma}$ in $\mathrm{sp}(\Gamma)$:

$$N_\Gamma := \frac{|\Gamma|}{|C_0^{\ell_+,\gamma}|}$$

Theorem 6.2. *If there is $\mathfrak{K}_\gamma$ large enough such that, for any $\hat{p} \in \{-1, 1, \cdots, Q\}$, the weights $w^{\hat{p}}_\gamma(\Gamma; \vec{\xi}_{A^{\hat{p}}})$ satisfy the Peierls bound* (6.11) *then there exist $Q+1$ distinct DLR measures and:*

$$\mu^{(q)}_{\gamma,\beta}(\Theta = a_{-1}) \neq \mu^{(-1)}_{\gamma,\beta}(\Theta = a_{-1}) \qquad \forall q > 0$$
$$\mu^{(q)}_{\gamma,\beta}(\Theta = a_{-1}) \leq e^{-(\mathfrak{K}_\gamma - b)} \qquad \forall q > 0$$

with b a positive constant.

The proof of Theorem 2.1 follows from Theorem 6.2, showing that there exist a value of $\beta = \beta_c(Q, \gamma)$ and a constant $\mathfrak{K} := \mathfrak{K}_\gamma(\gamma, \beta)$ such that the bounds (6.11) hold.

The proof of Theorem 6.2 uses the well known Peierls argument and is omitted.

The key estimate is stated in the following Theorem:

Theorem 6.3. *there exists a value $\bar{\gamma}$ such that for any $\gamma < \bar{\gamma}$ there exists $\beta_c(\gamma)$:*

$$w^{\hat{p}}_{\gamma,\beta_c(\gamma)}(\Gamma;\vec{\xi}_{A^p}) \le \exp\{-\mathfrak{K}_\gamma |N_\Gamma|\} \qquad \text{with } \mathfrak{K}_\gamma = c_p\beta_c(\gamma)\gamma^{2a}|C^{\ell_-,\gamma}| \tag{6.12}$$

with c_d a constant depending only on Q and on the dimension of the space,

$$c_p = \frac{1}{3^{d+1}}(Q - \beta_c^{\mathrm{mf}}(Q)) > 0 \tag{6.13}$$

7. Peierls estimates

Let $\mathcal{X}^{\hat{q}} := \{\vec{s} : \Theta_x(\vec{s}) = a_{\hat{q}}\ \forall x \in \mathbb{R}^d\}$, defined for $\vec{s}$ that can be either in Ω, or in $L^\infty(\mathbb{R}^d, S_Q)$ or in $L^\infty(\mathbb{Z}^d, S_Q)$.

For any $\hat{q} \in \{-1, 1, \dots, Q\}$, and any $\vec{s} \in \mathcal{X}^{\hat{q}} := \{\vec{s} \in L^\infty(\mathbb{Z}^d, S_Q) : \Theta_x(\vec{s}) = a_{\hat{q}}\ \forall x \in \mathbb{R}^d\}$, we define the "diluted partition function" $Z^{\hat{q}}_{\gamma,\beta,\Lambda}(\vec{s}_{\Lambda^c})$, as

$$Z^{\hat{q}}_{\gamma,\beta,\Lambda}(\vec{s}_{\Lambda^c}) \quad := \quad Z_{\gamma,\beta,\Lambda}(\{\vec{\xi}_\Lambda : \theta_x(\vec{\xi}_\Lambda) = a_{\hat{q}} \quad \forall x \in \delta^{\ell_+,\gamma}_{\mathrm{in}}[\Lambda]\}|\vec{s}_{\Lambda^c}) \tag{7.14}$$

where the definition of the Hamiltonian (2.4) have been naturally extended to boundary conditions in $L^\infty(\mathbb{Z}^d, S_Q)$.

Let Γ a $\hat{p}$-contour and

$$G^{\hat{p}} := \mathrm{sp}(\Gamma) \bigsqcup_{\hat{q}\neq\hat{p}} A^{\hat{q}} \tag{7.15}$$

Theorem 7.1. *There are $\bar{\gamma}, \bar{b}$ and a constant c such that for all $\gamma < \bar{\gamma}$, $|\beta - \beta_c^{\mathrm{mf}}(Q)| < \bar{b}$:*

$$w^{\hat{p}}_{\gamma,\beta}(\Gamma;\vec{\xi}_{A^{\hat{p}}}) \quad \le \quad \frac{\prod_{\hat{q}\neq\hat{p}} Z^{\hat{q}}_{\gamma,\beta,\mathrm{Int}^q(\Gamma)\setminus A^{\hat{q}}}(\vec{\rho}^{\hat{q}})}{\prod_{\hat{q}\neq\hat{p}} Z^{\hat{p}}_{\gamma,\beta,\mathrm{Int}^{\hat{q}}(\Gamma)\setminus A^{\hat{q}}}(\vec{\rho}^{\hat{p}})} \exp\{c\gamma^{\frac{1}{2}}|\Gamma|\} \tag{7.16}$$

$$\cdot \exp\Big\{ -\beta\Big[\inf_{\substack{\vec{\rho}_{G^{\hat{p}}}:\eta(\vec{\rho}_{G^{\hat{p}}})=\eta_\Gamma \\ \vec{\rho}_{A^{\hat{q}}}=\vec{\rho}^{\ \hat{q}}}} F_{\gamma,\beta,G^{\hat{p}}}(\vec{\rho}_{G^{\hat{p}}}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) - \inf_{\substack{\vec{\rho}_{G^{\hat{p}}}:\eta(\vec{\rho}_{G^{\hat{p}}})=a^p \\ \vec{\rho}_{A^{\hat{q}}}=\vec{\rho}^{\ \hat{p}}}} F_{\gamma,\beta,G^{\hat{p}}}(\vec{\rho}_{G^{\hat{p}}}|\vec{\xi}^{(\ell_0)}_{A^p})\Big]\Big\}$$

Proof of Theorem 7.1. The proof of this theorem is a straightforward consequence of the analysis of section 5 and appendix B. We recall the definition of the weight of a $\hat{p}$-contour given in (6.10):

$$w^{\hat{p}}_{\gamma,\beta}(\Gamma;\vec{\xi}_{A^{\hat{p}}}) := \frac{\mu_{\gamma,\beta,c(\Gamma)\setminus\mathrm{Int}^{\hat{p}}(\Gamma)}(\mathcal{E}(\Gamma,\hat{p})|\vec{\xi}_{A^{\hat{p}}})}{\mu_{\gamma,\beta,c(\Gamma)\setminus\mathrm{Int}^{\hat{p}}(\Gamma)}(\mathcal{E}(\not{\Gamma},\hat{p})|\vec{\xi}_{A^{\hat{p}}})}$$

For each set $A^{\hat q}$, $\hat q \neq \hat p$, we denote by:

$$\begin{aligned}
\hat A^{\hat q} &:= A^{\hat q} \sqcup \delta_{\text{out}}^{\ell_+,\gamma}[A^{\hat q}] && (7.17)\\
\check A^{\hat q} &:= A^{\hat q} \sqcup \delta_{\text{out}}^{\ell_+,\gamma/2}[A^{\hat q}] && (7.18)\\
\Sigma^{\hat q,e} &:= \big(\hat A^{\hat q} \setminus \check A^{\hat q}\big) \sqcap \text{sp}(\Gamma)^c && (7.19)\\
\bar\Sigma^{\hat q,e} &:= \big(\check A^{\hat q} \setminus A^{\hat q}\big) \sqcap \text{sp}(\Gamma)^c && (7.20)\\
\Sigma^{e} &:= \sqcup_{\hat q\neq \hat p}\Sigma^{\hat q,e} && (7.21)
\end{aligned}$$

and also:

$$\begin{aligned}
\Delta^{\hat p} &:= \text{sp}(\Gamma) \sqcup_{\hat q \neq \hat p} \check A^{\hat q}\\
V^{\hat q} &:= \text{Int}^{\hat q}(\Gamma) \setminus \hat A^{\hat q}
\end{aligned}$$

We write:

$$w^{\hat p}_{\gamma,\beta}(\Gamma;\vec\xi_{A^{\hat p}}) \quad := \quad \frac{\displaystyle\sum_{\vec\xi_{\Sigma^e}:\vec\xi_{\Sigma^{\hat q,e}}\in\mathcal{X}^{\hat q}} e^{-\beta\sum_{\hat q\neq\hat p} H_\gamma(\vec\xi_{\Sigma^{\hat q,e}})} Z_{\beta,\Delta^{\hat p}}(\mathcal{E}(\Gamma)|\vec\xi_{\Sigma^e};\vec\xi_{A^{\hat p}}) \prod_{\hat q\neq\hat p} Z^{\hat q}_{\beta,V^{\hat q}}(\vec\xi_{\Sigma^{\hat q,e}})}{\displaystyle\sum_{\vec\xi_{\Sigma^e}\in\mathcal{X}^{\hat p}} e^{-\beta H_\gamma(\vec\xi_{\Sigma^e})} Z_{\beta,\Delta^{\hat p}}(\mathcal{E}(\cancel{\Gamma})|\vec\xi_{\Sigma^e};\vec\xi_{A^{\hat p}}) \prod_{\hat q\neq\hat p} Z^{\hat p}_{\beta,V^{\hat q}}(\vec\xi_{\Sigma^{\hat q,e}})}$$

By Theorem 5.1 with $\ell = \ell_0$, we have:

$$w^{\hat p}_{\gamma,\beta}(\Gamma;\vec\xi_{A^{\hat p}}) \leq e^{c\gamma\ell_0|\Gamma|}\,. \tag{7.22}$$

$$\frac{\displaystyle\sum_{\vec\xi_{\Sigma^e}:\vec\xi_{\Sigma^{\hat q,e}}\in\mathcal{X}^{\hat q}} e^{-\beta\sum_{\hat q\neq\hat p} H_\gamma(\vec\xi_{\Sigma^{\hat q,e}})} \exp\{-\beta \inf_{\substack{\rho:\eta(\rho)=\eta_\Gamma\ r\in\text{sp}(\Gamma)\\ \eta(\rho)=a_{\hat q}\ r\in\check A^{\hat q}}} F_{\gamma,\beta,\Delta^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi^{\ell_0}_{A^{\hat p}})\} \prod_{\hat q\neq\hat p} Z^{\hat q}_{\beta,V^{\hat q}}(\vec\xi_{\Sigma^{\hat q,e}})}{\displaystyle\sum_{\vec\xi_{\Sigma^e}\in\mathcal{X}^{\hat p}} e^{-\beta H_\gamma(\vec\xi_{\Sigma^e})} \exp\{-\beta \inf_{\rho:\eta(\rho)=a_{\hat p}} F_{\gamma,\beta,\Delta^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi^{\ell_0}_{A^{\hat p}})\} \prod_{\hat q\neq\hat p} Z^{\hat p}_{\beta,V^{\hat q}}(\vec\xi_{\Sigma^{\hat q,e}})}$$

Using Corollary B.5 with $\Lambda = \check A^{\hat q}$, we can find a lower bound for the free energy term in the numerator by considering density profiles $\vec\rho$ identically equal to $\vec\rho^{\hat q}$ on $A^{\hat q}$,at the expense of a small error term; we have:

$$\begin{aligned}
&\inf_{\substack{\rho:\eta(\rho)=\eta_\Gamma\ r\in\text{sp}(\Gamma)\\ \eta(\rho)=a_{\hat q}\ r\in\check A^{\hat q}}} F_{\gamma,\beta,\Delta^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi^{\ell_0}_{A^{\hat p}}) \geq \inf_{\substack{\rho:\eta(\rho)=\eta_\Gamma\ r\in\text{sp}(\Gamma)\\ \eta(\rho)=a_{\hat q}\ r\in\check A^{\hat q}\\ \vec\rho\equiv\vec\rho^{\hat q}\ r\in A^{\hat q}}} F_{\gamma,\beta,\Delta^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi^{\ell_0}_{A^{\hat p}}) - c_\omega e^{-\omega\gamma\ell_{+,\gamma/2}}|\check A|\\
&= \inf_{\substack{\rho:\eta(\rho)=\eta_\Gamma\ r\in\text{sp}(\Gamma)\\ \vec\rho\equiv\vec\rho^{\hat q}\ r\in A^{\hat q}}} F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi_{A^{\hat p}}) + \sum_{\hat q} \inf_{\rho:\eta(\rho)=a_{\hat q}} F_{\gamma,\beta,\bar\Sigma^{\hat q,e}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\rho^{q}_{A^{\hat q}}) - c_\omega e^{-\omega\gamma\ell_{+,\gamma/2}}|\Gamma|\\
&\geq \inf_{\substack{\rho:\eta(\rho)=\eta_\Gamma\ r\in\text{sp}(\Gamma)\\ \vec\rho\equiv\vec\rho^{\hat q}\ r\in A^{\hat q}}} F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi_{A^{\hat p}}) - \frac{1}{\beta}\sum_{\hat q}\log\big(Z_{\beta,\bar\Sigma^{\hat q,e}}(\vec\xi_{\Sigma^e};\vec\rho^{q}_{A^{\hat q}})\big) - \big(c_\omega e^{-\omega\gamma\ell_{+,\gamma/2}} + c\gamma\ell_0\big)|\Gamma|
\end{aligned}$$

The free energy term in the denominator of (7.22) can be directly bounded from above, as

$$\begin{aligned}
\inf_{\rho:\eta(\rho)=a_{\hat p}} F_{\gamma,\beta,\Delta^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi^{\ell_0}_{A^{\hat p}}) &\le \inf_{\substack{\rho:\eta(\rho)=a_{\hat p}\\ \vec\rho\equiv\vec\rho^{\hat q}\ r\in A^{\hat q}}} F_{\gamma,\beta,\Delta^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi^{\ell_0}_{A^{\hat p}})\\
&= \inf_{\rho:\eta(\rho)=a_{\hat p}} F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi_{A^{\hat p}}) + \sum_{\hat q}\inf_{\rho:\eta(\rho)=a_{\hat p}} F_{\gamma,\beta,\bar\Sigma^{\hat q},e}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\rho^{\hat p}_{A^{\hat q}})\\
&\le \inf_{\rho:\eta(\rho)=a_{\hat p}} F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi_{A^{\hat p}}) - \frac{1}{\beta}\sum_{\hat q}\log\big(Z_{\beta,\bar\Sigma^{\hat q},e}(\vec\xi_{\Sigma^e};\vec\rho^{\hat p}_{A^{\hat q}})\big) + c\gamma\ell_0|\Gamma|
\end{aligned}$$

Inserting both estimates in (7.22) we get the following factorization:

$$\begin{aligned}
& w^{\hat p}_{\gamma,\beta}(\Gamma;\vec\xi_{A^{\hat p}})\\
&\le e^{\beta(3c\gamma\ell_0+c_\omega e^{-\omega\gamma\ell_{+,\gamma}/2})|\Gamma|}\ \frac{\displaystyle\sum_{\vec\xi_{\Sigma^e}:\vec\xi_{\Sigma^{\hat q},e}\in\mathcal X^{\hat q}}\prod_{\hat q\neq\hat p} e^{-\beta H_\gamma(\vec\xi_{\Sigma^{\hat q},e})}Z_{\beta,\bar\Sigma^{\hat q},e}(\vec\xi_{\Sigma^e};\vec\rho^{q}_{A^{\hat q}})Z^{\hat q}_{\beta,V^{\hat q}}(\vec\xi_{\Sigma^{\hat q},e})}{\displaystyle\sum_{\vec\xi_{\Sigma^e}\in\mathcal X^{\hat p}}\prod_{\hat q\neq\hat p} e^{-\beta H_\gamma(\vec\xi_{\Sigma^{\hat q},e})}Z_{\beta,\bar\Sigma^{\hat q},e}(\vec\xi_{\Sigma^e};\vec\rho^{\hat p}_{A^{\hat q}})Z^{\hat p}_{\beta,V^{\hat q}}(\vec\xi_{\Sigma^{\hat q},e})}\\
&\quad\times\exp\Big\{-\beta\Big(\inf_{\substack{\rho:\eta(\rho)=\eta_\Gamma\ r\in\mathrm{sp}(\Gamma)\\ \vec\rho\equiv\vec\rho^{\hat q}\ r\in A^{\hat q}}} F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi_{A^{\hat p}}) - \inf_{\rho:\eta(\rho)=a^p} F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{\ell_0}_{\Sigma^e};\vec\xi_{A^{\hat p}})\Big)\Big\}\\
&= e^{3\beta c\gamma\ell_0|\Gamma|+\beta c_\omega e^{-\omega\gamma\ell_{+,\gamma}/2}}\ \frac{\prod_{\hat q\neq\hat p} Z^{\hat q}_{\gamma,\beta,\mathrm{Int}^q(\Gamma)\setminus A^{\hat q}}(\vec\rho^{\hat q})}{\prod_{\hat q\neq\hat p} Z^{\hat p}_{\gamma,\beta,\mathrm{Int}^{\hat q}(\Gamma)\setminus A^{\hat q}}(\vec\rho^{\hat p})}\\
&\cdot\exp\Bigg\{-\beta\Bigg[\inf_{\substack{\vec\rho_{G^{\hat p}}:\eta(\vec\rho_{G^{\hat p}})=\eta_\Gamma\\ \vec\rho_{A^{\hat q}}=\vec\rho^{\ \hat q}}} F_{\gamma,\beta,G^{\hat p}}(\vec\rho_{G^{\hat p}}|\vec\xi^{(\ell_0)}_{A^{\hat p}}) - \inf_{\substack{\vec\rho_{G^{\hat p}}:\eta(\vec\rho_{G^{\hat p}})=a^p\\ \vec\rho_{A^{\hat q}}=\vec\rho^{\ \hat p}}} F_{\gamma,\beta,G^{\hat p}}(\vec\rho_{G^{\hat p}}|\vec\xi^{(\ell_0)}_{A^p})\Bigg]\Bigg\}
\end{aligned}$$

□

Using Theorem 7.1, the proof of the Theorem 6.3 follows from estimates on the terms appearing in (7.16), which are provided by two theorems, 7.2 and 7.3.

Notice that the first factor in (7.16):

$$\frac{\prod_{\hat q\neq\hat p} Z^{\hat q}_{\gamma,\beta,\mathrm{Int}^{\hat q}(\Gamma)\setminus A^{\hat q}}(\vec\rho^{(\hat q)})}{\prod_{\hat q\neq\hat p} Z^{\hat p}_{\gamma,\beta,\mathrm{Int}^{\hat q}(\Gamma)\setminus A^{\hat q}}(\vec\rho^{(\hat p)})}$$

would be identically equal to 1 in presence of symmetry between all the phases. In a non-symmetric case this factor is close to one only for special values of the parameters. Our case is somehow an intermediate one since there is a symmetry (by color permutation) relating the ordered phases, and we need only to distinguish between ordered ($\hat q > 0$) and disordered ($\hat q < 0$) phases. The proof of the Peierls bound requires an accurate estimate of the partition functions in this ratio, which means to go beyond the bulk contribution and get also some control over boundary terms.

We now state the two Theorems which allow us to control the terms appearing in the factorization formula of Theorem 7.1:

Theorem 7.2. *For all β such that $|\beta-\beta_c^{\rm mf}(Q)|\le c_b\gamma^{1/2}$*

$$\inf_{\substack{\vec\rho_{G^{\hat p}}:\eta(\vec\rho_{G^{\hat p}})=\eta_\Gamma\\ \vec\rho_{A^{\hat q}}=\vec\rho^{\hat q}}} F_{\gamma,\beta,G^{\hat p}}(\vec\rho_{G^{\hat p}}|\bar\xi^{(\ell_0)}_{A^p})-\inf_{\substack{\vec\rho_{G^{\hat p}}:\eta(\vec\rho_{G^{\hat p}})=a_{\hat p}\\ \vec\rho_{A^{\hat q}}=\vec\rho^{\hat p}}} F_{\gamma,\beta,G^{\hat p}}(\vec\rho_{G^{\hat p}}|\bar\xi^{(\ell_0)}_{A^p})$$
$$\ge c_f\gamma^{2a}\ell^d_{-,\gamma}|\Gamma|+\sum_{\hat q\neq\hat p}\mathcal{I}_{A^{\hat q}}(|\vec\rho_{\hat q}|^{\,2}-|\vec\rho_{\hat p}|^{\,2})$$

where

$$\mathcal{I}_{A^{\hat q}}=\frac{1}{2}\int_{A^{\hat q}}\int_{(\mathrm{Int}^{\hat q}(\Gamma)\setminus A^{\hat q})}J_\gamma(x,y)dxdy$$

and the constant c_b is fixed by theorem 8.1 and c_f reads

$$c_f=\frac{1}{3^{d+1}}(\frac{Q}{\beta}-1) \tag{7.23}$$

Theorem 7.3. *There is $\bar\gamma$, and a constant κ_1 such that for any $\gamma<\bar\gamma$, there is a value of β, $\beta_c(\gamma)$:*

$$\left|\ln\Big(\frac{Z^{(q)}_{\gamma,\beta_c(\gamma),\Lambda}(\vec\rho^{(q)})e^{\mathcal{I}_{A^{\hat q}}|\vec\rho^{(q)}|^{\,2}}}{Z^{(-1)}_{\gamma,\beta_c(\gamma),\Lambda}(\vec\rho^{(-1)})e^{\mathcal{I}_{A^{\hat q}}|\vec\rho^{(-1)}|^{\,2}}}\Big)\right|\le\kappa_1\gamma^{1/8}|\Gamma| \qquad \forall q>0 \tag{7.24}$$

The proof of Theorem 7.2 is postponed until Section 10, while Theorem 7.3 will be proved in the next sections.

In order to prove Theorem 7.3 we will introduce $Q+1$ “abstract contour models” and estimate the ratio between their finite volume partition functions. The relation between the partition functions of the “abstract contour models” and the corresponding diluted partition functions of the “true model” (see 7.14), will be stated for suitable values of β in Theorem 7.4 below.

We will prove that there exists a value of β, $\beta_c(\gamma)$, so that the pressures of these abstract systems are all equal. For this value of the temperature we will be also able to get an estimate on the “finite volume correction to these pressures”, hence on the value of finite volume partition functions of the abstract models. This result is obtained by estimates on the decay of the correlations and a control on the probability of “small deviations” for the measures of these abstract contour models.

Abstract contour models

For any $\hat p$ we define:

$$\mathcal{X}^{\hat p}:=\{\vec\xi:\theta_x(\vec\xi)\equiv a_{\hat p},\ x\in\mathbb{Z}^d\} \tag{7.25}$$
$$\{\Gamma\}^{\hat p}:=\ \text{the collection of all } \hat p \text{ contours s.t } |\mathrm{sp}(\Gamma)|<\infty \tag{7.26}$$
$$\mathcal{B}^{\hat p}:=\big\{\underline\Gamma=(\Gamma_1,..,\Gamma_n):n\in\mathbb{N}_+,\ \Gamma_i\in\{\Gamma\}^{\hat p},\ \overline{\mathrm{sp}(\Gamma_i)}\sqcap\overline{\mathrm{sp}(\Gamma_j)}=\emptyset,\ i\neq j\big\} \tag{7.27}$$
$$\mathcal{B}^{\mathrm{Ext},\hat p}:=\{\underline\Gamma\in\mathcal{B}^{\hat p}:\mathrm{sp}(\Gamma)\sqcap\mathrm{Int}(\Gamma')=\emptyset\ \forall\Gamma,\Gamma'\in\underline\Gamma\} \tag{7.28}$$

where $\overline{A}$ denotes the closure of the set A.

$\{\Gamma\}^{\hat p}_{\Lambda}$, $\mathcal{B}^{\hat p}_{\Lambda}$ and $\mathcal{B}^{\mathrm{Ext},\hat p}_{\Lambda}$ are defined similarly, but with the condition that all contours should have spatial support in Λ.

Abstract contour models are systems defined on the product spaces $\mathcal{X}^{\hat p}\times\mathcal{B}^{\hat p}$. The partition functions read:

$$Z^{\hat p}_{\mathrm{abs},\beta,\Lambda}(\vec{\underline{\xi}}_{\Lambda^c}) = \sum_{\underline{\Gamma}\in\mathcal{B}^{\hat p}_{\Lambda}}\sum_{\vec\xi_\Lambda\in\mathcal{X}^{\hat p}_{\Lambda}}\prod_{\Gamma\in\underline{\Gamma}} W^{\hat p}(\Gamma;\vec\xi)e^{-\beta H_{\gamma,\Lambda}(\vec\xi_\Lambda|\vec{\underline{\xi}}_{\Lambda^c})} \tag{7.29}$$

where the weights $W^{\hat p}(\Gamma;\vec\xi)$ are assigned with an arbitrary rule and may depend on $\vec\xi\in\mathcal{X}^{\hat p}$, but only through its restriction on $\mathrm{sp}(\Gamma)\bigsqcup_{\hat p}A^{\hat p}$. The abstract contours are given by specifying $(\mathrm{sp}(\Gamma),\eta_\Gamma)$, where η_Γ is an arbitrary $\mathcal{D}^{\ell_-,\gamma}$-measurable function with value in $\{0,a_{-1},a_1,\cdots,a_Q\}$, with a constant value different from zero on each connected component of the boundaries $\delta^{\ell_+,\gamma}_{\mathrm{in}}[\mathrm{sp}(\Gamma)]$, and compatible with the condition $\Gamma\in\{\Gamma\}^{\hat p}$.

Then, for any $\vec{\underline{\xi}}\in\mathcal{X}^{\hat p}$, we introduce the "dilute", finite volume Gibbs measures on $\mathcal{X}^{\hat p}_{\Lambda}\times\mathcal{B}^{\hat p}_{\Lambda}$, Λ a bounded $\mathcal{D}^{\ell_+,\gamma}$ measurable region, by setting

$$\mu^{\hat p}_{\mathrm{abs},\Lambda}(\vec\xi',\underline{\Gamma}|\vec{\underline{\xi}}_{\Lambda^c}) := \frac{\mathbf{1}_{\vec\xi'\in\mathcal{X}^{\hat p}}}{Z^{\hat p}_{\mathrm{abs},\beta,\Lambda}(\vec{\underline{\xi}}_{\Lambda^c})}\prod_{\Gamma\in\underline{\Gamma}} W^{\hat p}(\Gamma,\vec\xi')e^{-H_{\gamma,\Lambda}(\vec\xi'_\Lambda|\vec{\underline{\xi}}_{\Lambda^c})} \tag{7.30}$$

Remarks. Notice that the elements of a pair $(\vec\xi,\underline{\Gamma})$ in an abstract model are totally unrelated, in fact the configuration $\vec\xi$ is in $\mathcal{X}^{\hat p}$ and therefore has no contour.

We will consider the $Q+1$ "abstract contour models" , for any $\hat p\in\{-1,1,\dots,Q\}$, defined by the following choice of "cut-off weights" $W^{\hat p}(\Gamma|\vec\xi)$,

$$\hat W^{\hat p}_{\gamma}(\Gamma;\vec\xi) := \min\{w^{\hat p}_{\mathrm{abs},\beta}(\Gamma;\vec\xi), e^{-\frac{\mathfrak{K}_\gamma}{2}N_\Gamma}\} \tag{7.31}$$

where $\mathfrak{K}_\gamma$ is given by (6.12) and $w^{\hat p}_{\mathrm{abs},\beta}(\Gamma;\vec\xi)$ are defined by an inductive way on the volume, as:

$$w^{\hat p}_{\mathrm{abs},\beta}(\Gamma;\vec\xi_{A^{\hat p}}) := \frac{\sum_{\vec\xi_{G^{\hat p}}} e^{-\beta H_{\gamma,G^{\hat p}}(\vec\xi_{G^{\hat p}}|\vec\xi_{A^{\hat p}})}\mathbf{1}_{\eta(\vec\xi_{\mathrm{sp}(\Gamma)})=\eta_\Gamma}\prod_{\hat q\neq\hat p}\mathbf{1}_{\eta_{\vec\xi_{A^{\hat q}}}=a_{\hat q}}Z^{\hat q}_{\mathrm{abs},\beta,\mathrm{Int}^{\hat q}(\Gamma)\setminus A^{\hat q}}(\vec\xi_{A^{\hat q}})}{\sum_{\vec\xi_{G^{\hat p}}} e^{-\beta H_{\gamma,G^{\hat p}}(\vec\xi_{G^{\hat p}}|\vec\xi_{A^{\hat p}})}\mathbf{1}_{\eta(\vec\xi_{G^{\hat p}})=a_{\hat p}}\prod_{\hat q\neq\hat p}Z^{\hat q}_{\mathrm{abs},\beta,\mathrm{Int}^{\hat q}(\Gamma)\setminus A^{\hat q}}(\vec\xi_{A^{\hat q}})} \tag{7.32}$$

We point out that the sum $\sum_{\vec\xi_{G^{\hat p}}}$ in the definition (7.32), that is a sum on the whole space $\Omega_{G^{\hat p}}$, enter only in the definition of the weights, without any relations with the configurations $\vec\xi\in\mathcal{X}^{\hat p}$ of the abstract model that we are considering.

Recalling the definition of diluted partition function (7.14) we state the following theorem:

Theorem 7.4. *If for any $\hat{p}$ and any $\hat{p}$-contour Γ the weights $\hat{W}^{\hat{p}}_{\gamma}(\Gamma;\vec{\xi})$, defined by* (7.31)*, satisfy*

$$\hat{W}^{\hat{p}}_{\gamma}(\Gamma;\vec{\xi}) < e^{-\frac{\mathfrak{K}_\gamma}{2}N_\Gamma} \tag{7.33}$$

then

$$W^{\hat{p}}_{\gamma}(\Gamma;\vec{\xi}) = w^{\hat{p}}_{\gamma,\beta}(\Gamma;\vec{\xi}) \qquad ; \qquad Z^{\hat{p}}_{\text{abs},\beta,\Lambda}(\underline{\vec{\xi}}^{c}_{\Lambda}) = Z^{\hat{p}}_{\beta,\Lambda}(\underline{\vec{\xi}}^{c}_{\Lambda}) \tag{7.34}$$

Proof. We first rewrite the diluted partition function of the "true model", $Z^{\hat{p}}_{\beta,\Lambda}(\underline{\vec{\xi}}_{\Lambda^c})$, as

$$Z^{\hat{p}}_{\beta,\Lambda}(\underline{\vec{\xi}}_{\Lambda^c}) = \sum_{\underline{\Gamma}\in\mathcal{B}^{\text{Ext},\hat{p}}_{\Lambda}} \sum_{\vec{\xi}_{R(\underline{\Gamma})}\in\mathcal{X}^{\hat{p}}} \prod_{\Gamma\in\underline{\Gamma}} w^{\hat{p}}_{\gamma,\beta}(\Gamma;\vec{\xi}) e^{-\beta H_{\gamma,\Lambda}(\vec{\xi}_{R(\underline{\Gamma})}|\underline{\vec{\xi}}_{\Lambda^c})} \prod_{\hat{q}} \prod_{i\in I_{\hat{q}}} Z^{\hat{p}}_{\text{abs},\beta,\text{Int}^{\hat{q}}_i(\Gamma)\setminus A^{\hat{q}}(\Gamma)}(\vec{\xi}_{R(\underline{\Gamma})})$$

where

$$R(\underline{\Gamma}) := \Lambda \setminus \left(\sqcup_{\Gamma\in\underline{\Gamma}} \sqcup_{i\in I_\Gamma} (\text{Int}^{\hat{q}}_i(\Gamma)\setminus A^{\hat{q}}(\Gamma))\right)$$

Proceeding by iteration on $Z^{\hat{p}}_{\text{abs},\beta,\text{Int}^{\hat{q}}(\Gamma)\setminus A^{\hat{q}}}(\vec{\xi}_{R(\underline{\Gamma})})$ one gets:

$$Z^{\hat{p}}_{\beta,\Lambda}(\underline{\vec{\xi}}_{\Lambda^c}) = \sum_{\underline{\Gamma}\in\mathcal{B}^{\hat{p}}_{\Lambda}} \sum_{\vec{\xi}_{\Lambda}\in\mathcal{X}^{\hat{q}}_{\Lambda}} \prod_{\Gamma\in\underline{\Gamma}} w^{\hat{p}}_{\gamma,\beta}(\Gamma;\vec{\xi}) e^{-\beta H_{\gamma,\Lambda}(\vec{\xi}_{\Lambda}|\underline{\vec{\xi}}_{\Lambda^c})}$$

The proof of the theorem 7.4 is then obtained by induction on the volume (see for instance [**40**], or [**3**]).

Let Δ_n a sequence of increasing $\mathcal{D}^{\ell_{+,\gamma}}$-measurable regions with $\Delta_1 : |\Delta_1| = 3^d|C^{\ell_{+,\gamma}}|$ (the minimal size for the support of a contour); and $\Delta_{n+1} := \delta^{2\ell_{+,\gamma}}_{\text{out}}[\Delta_n]$. Let $\mathcal{T}_n := \{\Gamma \in \mathcal{B}^{\hat{q}} : |\text{Int}_i(\Gamma)| \le |\Delta_n| \forall i\}$. Suppose then that for any $\Gamma\in\mathcal{T}_1$, the weights $\hat{W}^{\hat{p}}_{\gamma}(\Gamma;\vec{\xi})$, defined by (7.31), satisfy the hypothesis (7.33), that means $\hat{W}^{\hat{p}}_{\gamma}(\Gamma;\vec{\xi}) = w^{\hat{p}}_{\text{abs},\beta}(\Gamma;\vec{\xi})$ (see (7.31)). Since contours in $\mathcal{T}_1$ cannot contain any smaller contour in their interior, it follows that:

$$Z^{\hat{q}}_{\text{abs},\beta,\text{Int}(\Gamma)\setminus A^{\hat{q}}}(\vec{\xi}_{A^{\hat{q}}}) = Z^{\hat{q}}_{\beta,\text{Int}(\Gamma)\setminus A^{\hat{q}}}(\vec{\xi}_{A^{\hat{q}}}) \qquad \forall\, \Gamma\in\mathcal{T}_1$$

and then, recalling (7.32), $w^{\hat{p}}_{\text{abs},\beta}(\Gamma;\vec{\xi}) = w^{\hat{p}}_{\gamma,\beta}(\Gamma;\vec{\xi})$. As a consequence the diluted partition functions on Δ_2 are equal :

$$Z^{\hat{q}}_{\text{abs},\beta,\Delta_2} = Z^{\hat{q}}_{\beta,\Delta_2}$$

because all contours in $\mathcal{B}^{\hat{p}}_{\Delta_2\setminus\delta^{\ell_{+,\gamma}}_{\text{in}}[\Lambda_1]}$ are in $\mathcal{T}_1$. This equality induces, via (7.32), equality of the weights $w^{\hat{p}}_{\text{abs},\beta}(\Gamma;\vec{\xi}) = w^{\hat{p}}_{\gamma,\beta}(\Gamma;\vec{\xi})$ for contours in $\mathcal{T}_2$. The iteration of this argument proves the Theorem.

□

An immediate corollary of Theorem 7.4 is:

Corollary 7.5. *If for any $\hat{q}$, any $\hat{q}$-contour Γ satisfies:*

$$w^{\hat{q}}_{\mathrm{abs},\beta}(\Gamma;\vec{\xi}) < e^{-\mathfrak{K}_\gamma N_\Gamma} \tag{7.35}$$

then the Peierls bound (6.12) *holds for any $\hat{q}$ contour.*

In fact in this case (7.33) holds, and then $w^{\hat{q}}_{\gamma,\beta}(\Gamma;\vec{\xi}) = w^{\hat{q}}_{\mathrm{abs},\beta}(\Gamma;\vec{\xi}) < e^{-\mathfrak{K}_\gamma N_\Gamma}$. The proof of the phase transition is then obtained by proving the existence of a value of β such that (7.35) holds.

A statement similar to Theorem 7.1 holds analogously for the weights $w^{\hat{q}}_{\mathrm{abs},\beta}(\Gamma;\vec{\xi})$ defined in (7.32). Hence the proof of (7.35) will follow by Theorem 7.2 and from the analogous of Theorem 7.3 for the abstract partition functions, that also implies, via Theorem 7.4, the validity of (7.24) for the dilute partition functions.

The proof of the version of Theorem 7.3 for abstract partition functions requires estimates on the pressures of the abstract models and their finite volume corrections that will given respectively in sections 8 and 9. The proof of Theorem 7.2 is given in section 10.

Working with the abstract models has many advantages, since Peierls bounds hold by definition (7.31). We will show that Peierls bounds for the cut-off weights $\hat{W}^{\hat{p}}_\gamma(\Gamma;\vec{\xi})$ together with the stability properties discussed in section 3 allow to prove uniqueness of the Gibbs measure $\mu^{\hat{q}}_{\mathrm{abs},\beta}$ of the abstract models. From a technical point of view the we will take advantage of (7.31) making a partial cluster expansion of the partition function that gives an extra interaction, an "effettive hamiltonian " $\mathcal{H}^{\hat{p}}_{\gamma,\Lambda}$, with infinite range but fast decay. In this way the space of the configurations can be restricted to $\mathcal{X}^{\hat{p}}$ while the "effective Hamiltonian" $\mathcal{H}^{\hat{p}}_{\gamma,\Lambda}(\vec{\xi})$ takes into account the sum over contours as a Gibbs factor,

$$\sum_{\underline{\Gamma}\in\mathcal{B}^{\hat{p}}_\Lambda} \hat{W}^{\hat{p}}_\gamma(\underline{\Gamma};\vec{\xi}) = e^{-\beta\mathcal{H}^{\hat{p}}_{\gamma,\Lambda}(\vec{\xi}_\Lambda)} \tag{7.36}$$

The validity of the cluster expansion leading to the Hamiltonians $\mathcal{H}^{\hat{p}}_{\gamma,\Lambda}(\vec{\xi})$ follows here by the Peierls bound that the abstract weights satisfy by definition (7.31) (for γ small enough):

Theorem 7.6. *If the weights $W^{\hat{p}}(\Gamma,\vec{\xi})$ satisfy the Peierls bounds with a constant $\mathfrak{K}_\gamma$ large enough then, for any bounded, $\mathcal{D}^{\ell_{+,\gamma}}$-measurable region Λ and any $\vec{\xi}\in\mathcal{X}^{\hat{p}}$,*

$$Z^{\hat{p}}_{\mathrm{abs},\beta,\Lambda}(\underline{\vec{\xi}}_{\Lambda^c}) = \sum_{\underline{\Gamma}\in\mathcal{B}^{\hat{p}}_\Lambda}\sum_{\vec{\xi}'\in\mathcal{X}^{\hat{p}}_\Lambda} \hat{W}^{\hat{p}}_\gamma(\underline{\Gamma},;\vec{\xi}')e^{-H_{\gamma,\Lambda}(\vec{\xi}'_\Lambda|\vec{\xi}_{\Lambda^c})} = \sum_{\vec{\xi}'\in\mathcal{X}^{\hat{p}}_\Lambda} e^{-\tilde{H}^{\hat{p}}_{\gamma,\Lambda}(\vec{\xi}'_\Lambda|\vec{\xi}_{\Lambda^c})} \tag{7.37}$$

$$\tilde{H}^{\hat{p}}_{\gamma,\Lambda}(\vec{\xi}'_\Lambda|\vec{\xi}_{\Lambda^c}) := H_{\gamma,\Lambda}(\vec{\xi}'_\Lambda|\vec{\xi}_{\Lambda^c}) + \mathcal{H}^{\hat{p}}_{\gamma,\Lambda}(\vec{\xi}'_\Lambda) \tag{7.38}$$

$$\mathcal{H}^{\hat{p}}_{\gamma,\Lambda}(\vec{\xi}) = \sum_{\Delta\sqsubset\Lambda} U^{\hat{p}}_{\Delta}(\vec{\xi}) \tag{7.39}$$

where the potentials $U^{\hat{p}}_{\Delta}(\vec{\xi}_\Delta)$ *are defined by* (7.45) *below and satisfy:*

$$U^{\hat{p}}_{\Delta} = 0 \text{ if } \Delta \text{ is not connected} \tag{7.40}$$

$$\|U^{\hat{p}}_{\Delta}(\cdot)\|_\infty = \sup_{\vec{\xi}_\Delta} |U^{\hat{p}}_{\Delta}(\vec{\xi}_\Delta)| \le e^{-(\frac{\mathfrak{K}_\gamma}{2}-b)N_\Delta} \tag{7.41}$$

where N_Δ *is the number of* $\mathcal{D}^{\ell_+,\gamma}$*-cubes in* Δ*;* $b > 0$ *a dimension dependent constant, such that* $\sum_{\Delta\ni x} e^{-bN_\Delta} \le 1$*.*

The Gibbs measures relative to $\tilde{H}^{\hat{p}}_{\gamma,\Lambda}(\vec{\xi}'_\Lambda|\vec{\xi}_{\Lambda^c})$ *on* $\mathcal{X}^{\hat{p}}$*,*

$$\tilde{\mu}^{\hat{p}}_{\mathrm{abs},\Lambda}(\vec{\xi}_\Lambda|\underline{\vec{\xi}}_{\Lambda^c}) := \frac{e^{-\beta\tilde{H}^{\hat{p}}_{\gamma,\Lambda}(\vec{\xi}_\Lambda|\vec{\xi}_{\Lambda^c})}}{Z^{\hat{p}}_{\mathrm{abs},\beta,\Lambda}(\underline{\vec{\xi}}_{\Lambda^c})} \tag{7.42}$$

are the marginals on $\mathcal{X}^{\hat{p}}$ *of the measures* $\mu^{\hat{p}}_{\mathrm{abs},\Lambda}(\vec{\xi},\underline{\Gamma}|\underline{\vec{\xi}}_{\Lambda^c})$*.*

Proof. (7.37) follows from (7.29) and (7.38), by setting

$$e^{-\mathcal{H}^{\hat{p}}_{\Lambda}(\vec{\xi}_\Lambda)} = \sum_{\underline{\Gamma}\in\mathcal{B}^+_\Lambda} W^{\hat{p}}(\underline{\Gamma},\vec{\xi}_\Lambda) \tag{7.43}$$

To prove the remaining statements we use a cluster expansion to express the energy $\mathcal{H}^{\hat{p}}_{\Lambda}(\vec{\xi}_\Lambda)$ in terms of a sum of weights of polymers, which will then identify the many-body potentials $U^{\hat{p}}_{\Delta}(\vec{\xi}_\Delta)$.

Polymers are functions $I : \{\Gamma\}^{\hat{p}} \to \mathbb{N}$ such that the collection $\{\Gamma : I(\Gamma) > 0\}$ is finite and connected, where two elements Γ and Γ' in $\{\Gamma\}^{\hat{p}}$ are connected if $\overline{\mathrm{sp}(\Gamma)} \sqcap \overline{\mathrm{sp}(\Gamma')} \ne \emptyset$. Denote by $\mathcal{P}^{\hat{p}}$ the collection of all polymers and by $\mathcal{P}^{\hat{p}}_\Lambda$ those made by contours in $\{\Gamma\}^{\hat{p}}_\Lambda$. It then follows from Kotecki and Preis, [**30**], that, if the Peierls constant is large enough, there are numbers $\varpi(I,\underline{s})$, such that

$$\ln \sum_{\underline{\Gamma}\in\mathcal{B}^{\hat{p}}_\Lambda} W^{\hat{p}}(\underline{\Gamma},\vec{\xi}) = \sum_{I\in\mathcal{P}^{\hat{p}}_\Lambda} \varpi^{\hat{p}}(I,\vec{\xi}) \tag{7.44}$$

Calling $\mathrm{sp}(I) = \bigsqcup_{\Gamma:I(\Gamma)>0} \mathrm{sp}(\Gamma)$, we then set

$$-U^{\hat{p}}_{\Delta}(\vec{\xi}_\Delta) = \sum_{I\in\mathcal{P}^{\hat{p}}_\Lambda,\,\mathrm{sp}(I)=\Delta} \varpi^{\hat{p}}(I,\vec{\xi}_\Delta) \tag{7.45}$$

(7.41) follows from (7.45) and [**30**].

The proof of (7.42) follows from (7.30) in a similar way. Theorem 7.6 is proved.

□

8. Equality of the pressures

Since all the measures with $\hat{q} > 0$ are equivalent (equal up to a permutation of the indices), (7.29)-(7.31) define actually only two distinct "abstract models", the ordered and the disordered one, relative respectively to values $\hat{q} > 0$, $\hat{q} < 0$. In the next two sections we will use the superscripts $\pm$ instead of $\hat{p}$ in order to distinguish only between disordered and ordered phases.

Let denote by

$$P^{\pm}_{\text{abs},\Lambda,\gamma,\beta}(\underline{\vec{\xi}}_{\Lambda^c}) := \frac{1}{\beta|\Lambda|} \ln Z^{\pm}_{\text{abs},\beta,\Lambda}(\underline{\vec{\xi}}_{\Lambda^c})$$

the "finite volume pressures".

The proof of Theorem 7.3 requires the proof of the following theorem to control the bulk contribution to the ratio in (7.24):

Theorem 8.1. *Let $\{\Lambda_n\}$ a sequence of sets in $\mathbb{R}^d$ of side $2^n \ell_{+,\gamma}$. There exist the two limits:*

$$P^{\pm}_{\text{abs},\gamma,\beta} := \lim_{n\to\infty} \frac{1}{\beta|\Lambda_n|} \ln P^{\pm}_{\text{abs},\Lambda,\gamma,\beta}(\vec{\rho}^{\,\pm}_{\beta}) \tag{8.1}$$

that are continuous in β, moreover there are constants c_b, $\bar{\gamma}$ s.t. for any $\gamma < \bar{\gamma}$ there is a value of β, $\beta_c(\gamma)$, s.t.

$$P^{+}_{\text{abs},\gamma,\beta_c(\gamma)} = P^{-}_{\text{abs},\gamma,\beta_c(\gamma)} \qquad |\beta^{\text{mf}}_c(Q) - \beta_c(\gamma)| < c_b \gamma^{1/2} \tag{8.2}$$

The proof of the Theorem 8.1 relies on the well known mean field result on the existence of a first order phase transition for any $Q > 2$, (see [**45**]), that we recall for completeness in appendix A. In particular we prove that there exists an inverse temperature $\beta^{\text{mf}}_c(Q)$ such that the mean field free energy function satisfies:

$$\phi_{\beta^{\text{mf}}_c(Q)}(\vec{\rho}^{\,+}_{\beta^{\text{mf}}_c(Q)}) = \phi_{\beta^{\text{mf}}_c(Q)}(\vec{\rho}^{\,-}_{\beta^{\text{mf}}_c(Q)}) = \inf_{\vec{\rho}\in S_Q} \phi_{\beta^{\text{mf}}_c(Q)}(\vec{\rho}) \tag{8.3}$$

and

$$\frac{d}{d\beta}\left[\phi^{\text{mf}}_{\beta}(\vec{\rho}_{\beta}^{\,+}) - \phi^{\text{mf}}_{\beta}(\vec{\rho}_{\beta}^{\,-})\right]\Big|_{\beta=\beta^{\text{mf}}_c(Q)} \neq 0 \tag{8.4}$$

Proof of Theorem 8.1. The proof of existence of the two pressures $P^{\pm}_{\text{abs},\gamma,\beta_c(\gamma)}$ and their continuity in β is given in Appendix C, while the proof of (8.2) is an immediate consequence the following lemma :

Lemma 8.2. *There are constants κ and $\bar\gamma$ such that for any $\gamma < \bar\gamma$ and for any β : $|\beta - \beta_c^{\rm mf}(Q)| \le \gamma^{2a}$:*

$$|P^{\pm}_{{\rm abs},\gamma,\beta} + \phi^{\rm mf}_{\beta}(\vec\rho_\beta^{\ \pm})| \le \frac{\kappa}{\beta}\gamma^{1/2} \tag{8.5}$$

that implies:

$$|P^{+}_{{\rm abs},\gamma,\beta} - P^{-}_{{\rm abs},\gamma,\beta} - (\phi^{\rm mf}_{\beta}(\vec\rho^{\ -}) - \phi^{\rm mf}_{\beta}(\vec\rho^{\ +}))| < 2\frac{\kappa}{\beta}\gamma^{1/2} \tag{8.6}$$

(8.6), (8.4), (8.3), and the continuity in β of the pressures prove (8.2) and complete the proof of the Theorem 8.1 □

PROOF OF LEMMA 8.2. The proof of (8.5) could be obtained as a byproduct of a more detailed analysis contained in the next section but since a direct proof is quite shorter we sketch it here. We first prove an upper bound for $P^{\pm}_{{\rm abs},\gamma,\beta}$.

$$P^{\pm}_{{\rm abs},\gamma,\beta} \quad = \quad \lim_{n\to\infty}\frac{1}{\beta|\Lambda_n|}\ln Z^{\pm}_{{\rm abs},\beta,\Lambda}(\vec\rho^{\pm}_\beta)$$

and denoting by $\hat Z^{\pm}_{{\rm abs},\beta,\Lambda}(\vec\rho^{\pm}_\beta)$ the abstract partition function with interactions $H_{\gamma,\Lambda}(\vec\xi_\Lambda|\vec\xi_{\Lambda^c})$, we get:

$$P^{\pm}_{{\rm abs},\gamma,\beta} \quad \le \quad \lim_{n\to\infty}\frac{1}{\beta|\Lambda_n|}\ln \hat Z^{\pm}_{{\rm abs},\beta,\Lambda}(\vec\rho^{\pm}_\beta) + \sup_{\vec\xi\in\mathcal{X}^\pm}\sum_{\Delta:\Delta\ni 0}|U^{\pm}_{\Delta}(\vec\xi)|$$

By (7.41) the last term is bounded as $e^{-\frac{\mathfrak{K}_\gamma}{2}+2b}$ and by Theorem 5.1 we have:

$$P^{\pm}_{{\rm abs},\gamma,\beta} \le -\lim_{n\to\infty}\inf_{\vec\rho\in\mathcal{X}_{\Lambda_n}}\frac{F_{\beta,\gamma,\Lambda_n}(\vec\rho|\vec\rho^{\pm}_\beta)}{\beta|\Lambda_n|} + c_d\gamma^{1/2}$$

we postpone at the end of this section the proof of the following bound that follows by the concavity of the entropy:

$$P^{\pm}_{{\rm abs},\gamma,\beta} \le c_d\gamma^{1/2} - \lim_{n\to\infty}\inf_{\vec\rho\in\mathcal{X}_{\Lambda_n}}\frac{1}{|\Lambda_n|}\int_{\Lambda_n}\phi_\beta\Big(J_\gamma * (\vec\rho\mathbf{1}_{\Lambda_n} + \vec\rho^{\pm}_\beta\mathbf{1}_{\Lambda_n^c})\Big)dr \tag{8.7}$$

where $\mathbf{1}_A := \mathbf{1}_{\{x\in A\}}$. Let

$$J_\gamma^{(\ell_{-,\gamma})}(x,y) := \frac{1}{(\ell_{-,\gamma})^d}\int_{z\in C_y^{\ell_{-,\gamma}}} J_\gamma(x,z)$$

where $C_y^{\ell_{-,\gamma}}$ is the cube of the partition $\mathcal{D}^{\ell_{-,\gamma}}$ containing the point y. Then for any $\vec s\in\mathcal{X}^\pm$:

$$\begin{aligned}\|J_\gamma * \vec s(r) - \vec\rho^{\pm}\|_\star \le \|\int J_\gamma^{(\ell_{-,\gamma})}(r,r')(\vec s(r') - \vec\rho^{\pm})\|_\star \\ +\|\int [J_\gamma^{(\ell_{-,\gamma})}(r,r') - J_\gamma(r,r)](\vec s(r') - \vec\rho^{\pm})\|_\star\end{aligned}$$

By the assumptions on J_γ (see (2.1)-(4.8)) the second term is bounded as :

$$\| \int [J_\gamma^{(\ell_{-,\gamma})}(r,r') - J_\gamma(r,r')](\vec{s}(r') - \vec{\rho}^{\pm})\|_\star \le \kappa_3 \gamma \ell_{-,\gamma} = \kappa_3 \gamma^\alpha$$

while the first term, since $\vec{s} \in \mathcal{X}\pm$ and $J^{(\ell_{-,\gamma})}(r,r')$ is constant w.r.t. the second variable in each cube of $\mathcal{D}^{\ell_{-,\gamma}}$:

$$\| \int J_\gamma^{(\ell_{-,\gamma})}(r,r')(\vec{s}(r') - \vec{\rho}^{\pm})\|_\star \le \gamma^a$$

going back to (8.7) we have:

$$P^{\pm}_{\mathrm{abs},\gamma,\beta} \le c\gamma^{1/2} - \inf_{\|\vec{s}-\vec{\rho}^{\pm}\|_\star < \gamma^a + \kappa_3\gamma^\alpha} \phi_\beta(\vec{s}) \tag{8.8}$$

In appendix A, is shown that $\inf_{\vec{s}} \phi_{\beta_c^{\mathrm{mf}}(Q)}(\vec{s}) = \phi_{\beta_c^{\mathrm{mf}}(Q)}(\vec{\rho}^{\,\hat{q}}_{\beta_c^{\mathrm{mf}}(Q)})$, for any $\hat{q} \in \{-1,1,\dots Q\}$. By continuity for $|\beta - \beta_c^{\mathrm{mf}}(Q)| < \gamma^{2a}$, γ small enough $\phi_\beta(\vec{s})$ has $Q+1$ local minima $\vec{\rho}^{\,\hat{q}}_\beta$ s.t. $|\vec{\rho}^{\,\hat{q}}_\beta - \vec{\rho}^{\,\hat{q}}_{\beta_c^{\mathrm{mf}}(Q)}| < c|\beta - \beta_c^{\mathrm{mf}}(Q)| < c\gamma^{2a}$. The are respectively then absolute minimizers in the sets $\|\vec{s} - \vec{\rho}^{\pm}\|_\star < \gamma^a + \kappa_3\gamma^\alpha$. We then get:

$$P^{\pm}_{\mathrm{abs},\gamma,\beta} \le c\gamma^{1/2} - \phi_\beta(\vec{\rho}^{\pm}_\beta) \tag{8.9}$$

We now prove a lower bound for the pressures. By Theorem 5.1 with $\ell = \ell_0$, we have for any $\vec{\rho}^{(n)} \in \mathcal{X}^{\pm}_{\Lambda_n}$:

$$P^{\pm}_{\mathrm{abs},\gamma,\beta} \ge -\lim_{n\to\infty} \frac{F_{\beta,\gamma,\Lambda_n}(\vec{\rho}^{(n)}|\vec{\rho}^{\pm}_\beta)}{\beta|\Lambda_n|} - c\gamma^{1/2}$$

and for γ small enough the same argument as before shows that $\rho^{\pm}_\beta \mathbf{1}_{\Lambda_n} \in \mathcal{X}^{\pm}_{\Lambda_n}$. We then get:

$$P^{\pm}_{\mathrm{abs},\gamma,\beta} \ge -c\gamma^{1/2} - \phi_\beta(\vec{\rho}^{\pm}_\beta) \tag{8.10}$$

□

Proof of (8.7):

$$\begin{aligned}
F_{\beta,\gamma,\Lambda_n}(\vec{\rho}|\vec{\rho}^{\pm}_\beta) &= F_{\beta,\gamma}(\vec{\rho}\mathbf{1}_{\Lambda_n} + \vec{\rho}^{\pm}_\beta \mathbf{1}_{\Lambda_n^c}) - F_{\beta,\gamma}(\vec{\rho}^{\pm}_\beta \mathbf{1}_{\Lambda_n^c}) \\
&= \int_{\Lambda_n \sqcup \delta^{\gamma^{-1}}_{\mathrm{out}}[\Lambda_n]} \phi_\beta\big(J_\gamma * (\vec{\rho}\mathbf{1}_{\Lambda_n} + \vec{\rho}^{\pm}_\beta \mathbf{1}_{\Lambda_n^c})\big) dr \\
&\quad + \frac{1}{\beta}\int_{\Lambda_n \sqcup \delta^{\gamma^{-1}}_{\mathrm{out}}[\Lambda_n]} \Big[I\big(J_\gamma * (\vec{\rho}\mathbf{1}_{\Lambda_n} + \vec{\rho}^{\pm}_\beta \mathbf{1}_{\Lambda_n^c})\big) - J_\gamma * I(\vec{\rho}\mathbf{1}_{\Lambda_n} + \vec{\rho}^{\pm}_\beta \mathbf{1}_{\Lambda_n^c}) \Big] dr \\
&\quad - \int_{\delta^{\gamma^{-1}}_{\mathrm{out}}[\Lambda_n]} \phi_\beta\big(J_\gamma * (\vec{\rho}^{\pm}_\beta \mathbf{1}_{\Lambda_n^c})\big) dr - \frac{1}{\beta}\int_{\delta^{\gamma^{-1}}_{\mathrm{out}}[\Lambda_n]} \Big[I(J_\gamma * \vec{\rho}^{\pm}_\beta \mathbf{1}_{\Lambda_n^c}) - J_\gamma * I(\vec{\rho}^{\pm}_\beta \mathbf{1}_{\Lambda_n^c}) \Big] dr
\end{aligned}$$

where we have used the fact that $\phi_\beta(0) = I(0) = 0$. By concavity of $I(\cdot)$ the second term is non negative and since $|\phi_\beta(\vec{s})|$ is bounded in $\vec{s} \in S_Q$ we have:

$$F_{\beta,\gamma,\Lambda_n}(\vec{\rho}|\vec{\rho}^{\pm}_{\beta}) \geq \int_{\Lambda_n} \phi_\beta\big(J_\gamma*(\vec{\rho}\mathbf{1}_{\Lambda_n}+\vec{\rho}^{\pm}_{\beta}\mathbf{1}_{\Lambda_n^c})\big)dr - c|\delta_{\text{out}}^{\gamma^{-1}}[\Lambda_n]|$$

9. finite volume corrections to the pressure

The proof of the Theorem 7.3, is obtained by proving the same bound as in (7.24) for the abstract partition functions:

$$|\log\Big(\frac{Z^{+}_{\text{abs},\beta,\Lambda}(\vec{\rho}^{(+1)})e^{\mathcal{I}_{A\hat{q}}|\vec{\rho}^{(+1)}|^{2}}}{Z^{-}_{\text{abs},\beta,\Lambda}(\vec{\rho}^{(-1)})e^{\mathcal{I}_{A\hat{q}}|\vec{\rho}^{(-1)}|^{2}}}\Big)| \leq \kappa_1\gamma^{1/8}|\Gamma| \tag{9.1}$$

for $|\beta-\beta_c^{\text{mf}}(Q)|<c_b\gamma^{1/2}$ and the conclusion, at the inverse temperature $\beta_c(\gamma)$, comes from (7.34) once proved (7.33).

In the regime when Theorem 8.1 holds, the proof of (9.1) requires a control of the finite volume corrections to the pressure.

We define $R^{\pm}_{\text{abs},\Lambda}$ the finite volume correction to the thermodynamical pressure $P^{\pm}_{\text{abs},\gamma}$ as:

$$R^{\pm}_{\text{abs},\Lambda} := \log Z^{\pm}_{\text{abs},\beta,\Lambda}(\underline{\vec{\xi}}) - \beta|\Lambda|P^{\pm}_{\text{abs},\gamma} \qquad \lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{R^{\pm}_{\text{abs},\Lambda}}{|\Lambda|}=0$$

In this section we will study the partition functions $Z^{\pm}_{\text{abs},\beta,\Lambda}(\underline{\vec{\xi}})$, with $\underline{\vec{\xi}}\in\mathcal{X}^{\pm}$ an ordered-disordered boundary condition, Λ a $\mathcal{D}^{(\ell_+,\gamma)}$ bounded region, and prove the following theorem.

Theorem 9.1. *There is a constant $c>0$ so that*

$$\Big|\ln Z^{\pm}_{\text{abs},\beta,\Lambda}(\underline{\vec{\xi}}) - \{\beta|\Lambda|P^{\pm}_{\text{abs},\gamma} + \frac{\beta}{2}\sum_{\substack{i\in\Lambda\\ j\in\Lambda^c}} J_\gamma(i,j)\,(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm})\}\Big|$$
$$\leq \beta\sum_{i\in\Lambda}|\sum_{j\in\Lambda^c}J_\gamma(i,j)\,(\underline{\vec{\xi}}(i)-\vec{\rho}^{\pm})| + c\gamma^{1/8}|\delta_{\text{in}}^{\ell_+,\gamma}[\Lambda]| \tag{9.2}$$

The proof of Theorem 9.1 is obtained in two steps, the first one is a bound on the decay of correlations, that allows to control the contribution of the bulk to the surface corrections, and the second step is a small deviation estimate to control the contribution coming from regions near the boundary.

Notice that if the boundary condition $\underline{\vec{\xi}}$ satisfy:

$$\|\sum_{j\in\Lambda^c}J_\gamma(i,j)\,(\underline{\vec{\xi}}(j)-\vec{\rho}^{\pm})\| < c\gamma^{1/8}|\delta_{\text{in}}^{\ell_+,\gamma}[\Lambda]| \tag{9.3}$$

then the leading contribution to $R^{\pm}_{\text{abs},\Lambda}$ is the finite volume correction to the mean field pressure (with $\rho^{\pm}$ b.c.)

$$R^{\text{mf},\pm}_{\gamma,\Lambda} := \frac{\beta}{2} \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c}} J_\gamma(i,j)\,(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm}) \tag{9.4}$$

Since the proof of the bounds in 9.1 only requires to consider "perfect boundary conditions"(through Theorem 7.1) we restrict the analysis to this case. In the sequel, we will consider boundary conditions $\underline{\vec{\xi}} = \vec{\rho}^{\pm}$ and denote by:

$$P^{\pm}_{\text{abs},\gamma,\Lambda} := \ln Z^{\pm}_{\text{abs},\beta,\Lambda}(\vec{\rho}^{\pm})$$

Following Dobrushin [**16**] we write the pressures of the abstract models in terms of correlations functions introducing the interpolating Hamiltonians $h^{\pm}_u(\vec{\xi})$, $u\in[0,1]$ defined as follows: let $u\in[0,1]$, and:

$$h^{\pm}_u(\vec{\xi}) := u\tilde{H}^{\pm}(\vec{\xi}|\vec{\rho}^{\pm}) + (1-u)\mathfrak{H}^{\pm}_\gamma(\vec{\xi})$$

where $\mathfrak{H}^{\pm}$ are the one body "mean field" Hamiltonians,

$$\mathfrak{H}^{\pm}_{\gamma,\Lambda}(\vec{\xi}) := -\sum_{i\in\Lambda}(\delta\vec{\xi}_i\cdot\vec{\mathcal{L}}_\gamma(\vec{\rho}^{\pm};i)) + H_{\gamma,\Lambda}(\vec{\rho}^{\pm}|\vec{\rho}^{\pm}) \tag{9.5}$$

with $\vec{\mathcal{L}}_\gamma(\vec{\xi};i)$ the "convolution":

$$\vec{\mathcal{L}}_\gamma(\vec{\xi};i) := \sum_{j\neq i} J_\gamma(i,j)\vec{\xi}(j) \tag{9.6}$$

and

$$\delta\vec{\xi} := \vec{\xi} - \vec{\rho}^{\pm} \tag{9.7}$$

The choice of the one body hamiltonians $\mathfrak{H}^{\pm}_{\gamma,\Lambda}(\vec{\xi})$ is done in such a way that the expectations of the densities with respect to the corresponding Gibbs measures converges to the mean field values:

$$\lim_{\gamma\to 0}\langle\xi_q\rangle_{\mathfrak{H}^{\pm}_{\gamma,\Lambda}} = \lim_{\gamma\to 0}\frac{e^{\beta\mathcal{L}_q(\rho^{\pm})}}{\sum_i e^{\beta\mathcal{L}_i(\rho^{\pm})}} = \rho^{\pm}_q \; q\in\{1,\dots,Q\} \tag{9.8}$$

The additive constant appearing in (9.5) is added to get the mean field pressure in the limit when γ goes to 0:

$$\begin{aligned} P^{\pm}_{\text{abs},\gamma,0} &:= \lim_{\Lambda\nearrow\infty}\frac{1}{\beta|\Lambda|}\ln Z^{\pm}_{\text{abs},\beta,\Lambda;0} \\ &\equiv \frac{1}{\beta|C_0|}\ln\sum_{\vec{\xi}\in\mathcal{X}^{\pm}_{C_0}}\prod_{i\in C_0} e^{\beta\left(\delta\vec{\xi}_i\cdot\vec{\mathcal{L}}(\rho^{\pm};i)\right)} + \frac{1}{2}\mathfrak{J}_\gamma(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm}) \end{aligned} \tag{9.9}$$

where C_0 the cube of the partition $\mathcal{D}^{\ell_+,\gamma}$ that contains 0 and

$$\mathfrak{J}_\gamma := \sum_j J_\gamma(i,j)$$

The pressure relative to the one body interaction $\mathfrak{H}^{\pm}_{\gamma,\Lambda}(\vec{\xi})$ represents the first order approximation of $P^{\pm}_{\text{abs},\gamma}$.

The proof of (9.9) and the proof of the convergence of the pressures $P^{\pm}_{\text{abs},\gamma,0}$ to the mean field free energy is given in Appendix D. Estimates on the finite volume corrections to the pressures $P^{\pm}_{\text{abs},\gamma,0}$ are also given in Appendix D.

Let denote by $\tilde{\mu}^{\pm}_{\text{abs},\Lambda;u}$ the [finite volume] Gibbs measure with hamiltonian $h^{\pm}_u(\vec{\xi})$ on $\mathcal{X}^{\pm}_\Lambda$. Following Dobrushin [**16**], we write the "finite volume pressure" in terms of correlation functions:

$$\begin{aligned} P^{\pm}_{\text{abs},\gamma,\Lambda} := & \quad \frac{1}{\beta|\Lambda|} \ln Z^{\pm}_{\text{abs},\beta,\Lambda;1}(\vec{\rho}^{\pm}) \\ = & \quad \frac{1}{\beta|\Lambda|} \ln Z^{\pm}_{\text{abs},\beta,\Lambda;0} - \frac{1}{|\Lambda|}\int_0^1 du \, \left\langle \tilde{H}^{\pm}_\Lambda(\vec{\xi}_\Lambda|\vec{\rho}^{\pm}) - \mathfrak{H}^{\pm}_\Lambda(\vec{\xi}_\Lambda)\right\rangle_{\tilde{\mu}^{\pm}_{\text{abs},\Lambda;u}} \end{aligned} \tag{9.10}$$

Proposition 9.2. *The exist $g_{\gamma,i,\Lambda}$, $g_{\gamma,i} : \mathbb{Z}^d \to \mathbb{R}$:*

$$\left[\tilde{H}^{\pm}_\Lambda(\vec{\xi}_\Lambda|\vec{\rho}^{\pm}) - \mathfrak{H}^{\pm}_\Lambda(\vec{\xi}_\Lambda)\right] = \sum_{i\in\Lambda} g_{\gamma,i,\Lambda} \tag{9.11}$$

$$\lim_{\Lambda\nearrow\mathbb{R}^d} g_{\gamma,i,\Lambda} = g_{\gamma,i} \tag{9.12}$$

Proof.

$$\tilde{H}^{\pm}_\Lambda(\vec{\xi}_\Lambda|\vec{\rho}^{\pm}) - \mathfrak{H}^{\pm}_\Lambda(\vec{\xi}_\Lambda) = +\frac{1}{2}\sum_{i\in\Lambda}\delta\vec{\xi}_i \sum_{\substack{j\in\Lambda\\ j\neq i}} J_\gamma(i,j)\delta\vec{\xi}_j + \mathcal{H}^{\pm}_{\gamma,\Lambda}(\vec{\xi}_\Lambda)$$

Recalling (7.39):

$$\mathcal{H}^{\pm}_\Lambda(\vec{\xi}) = \sum_{\Delta\sqsubset\Lambda} U^{\pm}_\Delta(\vec{\xi}) = \sum_{i\in\Lambda}\sum_{\substack{\Delta\sqsubset\Lambda\\ \Delta\ni i}} \frac{1}{|\Delta|} U^{\pm}_\Delta(\vec{\xi})$$

Defining:

$$g_{\gamma,i,\Lambda}(\vec{\xi}) := +\frac{1}{2}\delta\vec{\xi}_i \sum_{\substack{j\in\Lambda:\\ j\neq i}} J_\gamma(i,j)\delta\vec{\xi}_j + \sum_{\substack{\Delta\ni i\\ \Delta\sqsubset\Lambda}} \frac{1}{|\Delta|} U^{\pm}_\Delta(\vec{\xi}) \tag{9.13}$$

we can write:

$$\tilde{H}^{\pm}_\Lambda(\vec{\xi}_\Lambda|\vec{\rho}^{\pm}) - \mathfrak{H}^{\pm}_\Lambda(\vec{\xi}_\Lambda) = \sum_{i\in\Lambda} g_{\gamma,i,\Lambda}(\vec{\xi})$$

Recalling (7.41), the limit (9.12) exists:

$$\lim_{\Lambda\nearrow\mathbb{R}^d} g_{\gamma,i,\Lambda}(\vec{\xi}) = -\frac{1}{2}\delta\vec{\xi}_i \sum_{j:j\neq i} J_\gamma(i,j)\delta\vec{\xi}_j + \sum_{\Delta\ni i}\frac{1}{|\Delta|}U^{\pm}_{\Delta}(\vec{\xi}) =: g_{\gamma,i}(\vec{\xi}) \tag{9.14}$$

□

Suppose now that the Gibbs measure $\tilde{\mu}^{\pm}_{\mathrm{abs},\Lambda;u}$ has a unique limit when $\Lambda \nearrow \mathbb{R}^d$ that we denoted by $\tilde{\mu}^{\pm}_{\mathrm{abs};u}$, then taking the limit of (9.10), we would have :

$$P^{\pm}_{\mathrm{abs},\gamma} = P^{\pm}_{\mathrm{abs},\gamma,0} - \int_0^1 du\ \langle g_{\gamma,0}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs};u}} \tag{9.15}$$

and we can rewrite (9.10) as:

$$\begin{aligned} P^{\pm}_{\mathrm{abs},\gamma,\Lambda} = P^{\pm}_{\mathrm{abs},\gamma} &+ \left[\frac{1}{\beta|\Lambda|}\ln Z^{\pm}_{\mathrm{abs},\beta,\Lambda;0} - P^{\pm}_{\mathrm{abs},\gamma,0}\right] \\ &- \int_0^1 du\ \left[\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\langle g_{\gamma,i,\Lambda}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs},\Lambda;u}} - \langle g_{\gamma,0}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs};u}}\right] \end{aligned}$$

The finite volume corrections are then given by:

$$R^{\pm}_{\mathrm{abs},\Lambda} = R^{\pm}_{\mathrm{abs},\Lambda,0} - \beta\int_0^1 du \sum_{i\in\Lambda}\left[\langle g_{\gamma,i,\Lambda}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs},\Lambda;u}} - \langle g_{\gamma,i}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs};u}}\right] \tag{9.16}$$

where

$$R^{\pm}_{\mathrm{abs},\Lambda,0} := \ln Z^{\pm}_{\mathrm{abs},\beta,\Lambda;0} - \beta|\Lambda|P^{\pm}_{\mathrm{abs},\gamma,0}$$

In appendix D it is proven that:

$$R^{\pm}_{\mathrm{abs},\Lambda,0} = R^{\mathrm{mf},\pm}_{\gamma,\Lambda}$$

and then the proof of the Theorem 9.1 follows by estimating the remaining terms in (9.16). In the next subsections we will prove that there are positive constant c,ω such that:

$$\left|\langle g_{\gamma,i,\Lambda}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs},\Lambda;u}} - \langle g_{\gamma,i}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs};u}}\right| \le c\ e^{-\omega\gamma\,\mathrm{dist}(i,\Lambda)} \tag{9.17}$$

$$\sum_{i\in\delta^{\ell_+,\gamma}_{\mathrm{in}}[\Lambda]} \{|\langle g_{\gamma,i}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs};u}}| + |\langle g_{\gamma,i,\Lambda}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs},\Lambda;u}}|\} \le c\gamma^{1/8}|\delta^{\ell_+,\gamma}_{\mathrm{in}}[\Lambda]| \tag{9.18}$$

The proof of (9.17) and the existence of the limit of $\tilde{\mu}^{\pm}_{\mathrm{abs},\Lambda;u}$ when $\Lambda \nearrow \mathbb{R}^d$ is given in the next subsection as a consequence of the exponentially decay of the correlations, while the estimate (9.18) is a consequence of the small deviations estimates proved in the last subsection.

9.1. Decay of the correlations. In this subsection we prove the following Theorem:

Theorem 9.3. *There are c and ω positive so that for any bounded sets Λ and Δ, Λ $\mathcal{D}^{\ell_+,\gamma}$-measurable, and any $\underline{\vec{\xi}} \in \mathcal{X}^{(\pm)}$, there is a coupling Q_u of $\tilde{\mu}^{(\pm)}_{\text{abs},\Lambda;u}(\cdot|\underline{\xi})$ and $\tilde{\mu}^{(\pm)}_{\text{abs};u}(\cdot)$ such that*

$$Q_u(\vec{\xi}_\Delta \neq \vec{\xi}'_\Delta) \leq c|\Delta| e^{-\omega\gamma\, dist(\Delta,\Lambda^c)} \tag{9.19}$$

It follows that:

Corollary 9.4. *For any $u \in [0,1]$ there is a unique DLR measure $\tilde{\mu}^{\pm}_{\text{abs};u}$ with hamiltonian $h^{\pm}_u$ and for any local function f with support in Δ :*

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \langle f \rangle_{\tilde{\mu}^{\pm}_{\text{abs},\Lambda;u}} = \langle f \rangle_{\tilde{\mu}^{\pm}_{\text{abs};u}}$$

and, for any $\Delta \sqsubset \Lambda$, there are positive constants c, ω:

$$\left| \langle f \rangle_{\tilde{\mu}^{\pm}_{\text{abs},\Lambda;u}} - \langle f \rangle_{\tilde{\mu}^{\pm}_{\text{abs};u}} \right| \leq c\ |\Delta| \sup_{x \in \Delta}\{f(x)\}\ e^{-\omega\gamma \text{dist}(\Delta,\Lambda^c)}$$

Corollary 9.4 applied to our case, proves inequality (9.17):

$$\left| \langle g_{\gamma,x,\Lambda} \rangle_{\tilde{\mu}^{\pm}_{\text{abs},\Lambda;u}} - \langle g_{\gamma,x} \rangle_{\tilde{\mu}^{\pm}_{\text{abs};u}} \right| \leq \kappa_4\ e^{-\omega\gamma \text{dist}(x,\Lambda^c)}$$

Proof. The proof of Theorem 9.3 requires an extension of the Dobrushin high temperature uniqueness Theorem. In fact the classical Dobrushin Theorem does not apply to this case because of the restriction to the set $\mathcal{X}^{\pm}$, that corresponds to an hard-core interaction.

We recall that the classical Dobrushin uniqueness criterium requires that there is a function $r_{ij} : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+$, $r_{ii} = 0$ such that for all $i \in \mathbb{Z}^d$: and all $\vec{\xi}^{(1)}, \vec{\xi}^{(2)} \in \mathcal{X}^{\pm}$:

$$R_i(\mu_i(\cdot|\vec{\xi}^{(1)}_{i^c}), \mu_i(\cdot|\vec{\xi}^{(2)}_{i^c})) \leq \sum_{j \neq i} r_{ij}\ \text{dist}_j(\vec{\xi}^{(1)}_j, \vec{\xi}^{(2)}_j) \tag{9.20}$$

$$\sup_j \sum_{j \neq i} r_{ij} < \delta \qquad\qquad 0 < \delta < 1 \tag{9.21}$$

where $R_i(\mu_i(\cdot|\vec{\xi}^{(1)}_{i^c}), \mu_i(\cdot|\vec{\xi}^{(2)}_{i^c}))$ is the Vaserstein distance between the two single spin measures $\mu_i(\cdot|\vec{\xi}^{(1)}_{i^c}), \mu_i(\cdot|\vec{\xi}^{(2)}_{i^c})$.

In our case this criterium cannot be satisfied uniformly in $\vec{\xi}^{(1)}_{i^c}, \vec{\xi}^{(2)}_{i^c}$ because there is a subset of configurations $\underline{\vec{\xi}}_{i^c}$ that are "critical" in the sense that the conditional measures $\mu^{\hat{p}}_{\text{abs},i;u}(\cdot|\underline{\vec{\xi}}_{i^c})$ have support only a strict subset of Ω_i (the subset of values of $\vec{\xi}_x$ for which $\underline{\vec{\xi}}_y \mathbf{1}_{y\neq x} + \vec{\xi}_x \mathbf{1}_{y=x}$ is in $\mathcal{X}^{\pm}$), while the Vaserstein distance between measures with different supports is not small.

In reference [**3**], the Dobrushin uniqueness criterium is extended to the case when the "classical Dobrushin condition" is not satisfied uniformly in the boundary conditions, but only for "most of the configurations". Uniqueness of the measure can be proven here along the same line as in the original Dobrushin proof, using a criterium analogous to (9.20)-(9.21), where the single spin measures are replaced by measures on Ω_{C_n}, with C_n a cube of a partition $\mathcal{D}^\ell$ of Z^d. With such a choice, the uniqueness Dobrushin criterium would read as:

$$R_{C_n}(\mu_C(\cdot|\vec{\xi}^{(1)}_{C_n^c}), \mu_{C_n}(\cdot|\vec{\xi}^{(2)}_{C_n^c})) \leq \sum_{m\in(\ell\mathbb{Z})^d} r_{nm}\mathrm{dist}(\vec{\xi}^{(1)}_{C_m}, \vec{\xi}^{(2)}_{C_m}) \tag{9.22}$$

$$\sup_n \sum_{m\neq n} r_{nm} < \delta \qquad\qquad 0<\delta<1 \tag{9.23}$$

with $r_{nm} : (\ell\mathbb{Z})^d \times (\ell\mathbb{Z})^d \to \mathbb{R}^+$, $r_{nn} \equiv 0$.

Here we use an alternative line of proof [**3**] inspired by [**26**] and based on the Kantorovich-Rubinstein duality in order to avoid measurability problems. This formulation has the advantage of being less restrictive and in particular the [small] probability of "bad configurations" can be "adsorbed" in r_{nm}, but the nice situation of having an easily checkable criterium is lost. (In the context of spin systems, the Vaserstein distance between single spin measures can be explicitly calculated).

In reference [**3**], a set of five assumptions on the measures is provided which validity imply (9.22)-(9.23) and can be proven more easily that a direct check of the criterium. Theorem 9.3 will be proved through the demonstration that these five assumptions hold for the two abstract models $\tilde{\mu}^{(\pm)}_{\mathrm{abs};u}(d\vec{\xi}')$.

In our context, recalling the definition of $\mathcal{X}^\pm$, the "natural" choice for the partition in (9.22) is $\mathcal{D}^{\ell_{-,\gamma}}$ and the five assumptions stated below will imply the validity of the criterium (9.22)-(9.23) with $\ell = \ell_{-,\gamma}$.

In the sequels we prove assumptions 1 and 2 which require a specific treatment in the present case. Assumptions 3-5 can be verified along the same lines as in the Ising case (section 9 of [**3**]) and will be stated here without proofs.

9.1.1. *First assumption:*

The first assumption requires to prove that for any $i \in \mathbb{Z}^d$ there is a measurable set $G_i^\pm \sqsubset \mathcal{X}^\pm$ depending only on $\{\underline{\vec{\xi}}_j,\ j \in C_i^{\ell_{-,\gamma}} \setminus i\}$, such that there exists $b(i,j)$ with the following properties:

$$b(i,j) \geq 0 \;\;;\;\; b(i,i) = 0$$

$$\sup_{i\in\mathbb{Z}^d} \sum_{j\in C_i^{\ell_{-,\gamma}}} b(i,j) < \delta < 1 \tag{9.24}$$

$$\mathcal{R}(\mu^\pm_{\mathrm{abs},i;u}(\vec{\xi}_i|\underline{\vec{\xi}}), \mu^\pm_{\mathrm{abs},i;u}(\vec{\xi}_i|\underline{\vec{\xi}}')) \leq \sum_j b(i,j)\ \mathrm{dist}(\underline{\vec{\xi}}_j, \underline{\vec{\xi}}'_j) \qquad \text{for any } \underline{\vec{\xi}}, \underline{\vec{\xi}}' \in G_i$$

where $\mathrm{dist}(\vec{\xi}_j, \vec{\xi}'_j)$ is a distance defined on the configuration space and $\mathcal{R}(\mu^{\pm}_{\mathrm{abs},i;u}(\vec{\xi}_i|\underline{\xi}), \mu^{\pm}_{\mathrm{abs},i;u}(\vec{\xi}_i|\underline{\xi}'))$ is the associated Vaserstein distance. Here, we consider the following distance between configurations:

$$\mathrm{dist}(\vec{\xi}_i, \vec{\xi}'_i) := \frac{1}{2}\sum_i |\xi_i(i) - \xi'_i(i)| \tag{9.25}$$

and define G_i as :

$$G_i^{\pm} := \{\vec{\xi} \in \mathcal{X}^{(\pm)} : \vec{\xi}^{(i,q)} \in \mathcal{X}^{(\pm)} \forall q\} \tag{9.26}$$

where we have denoted

$$\vec{\xi}_j^{(i,q)} = \begin{cases} \vec{\xi}_j & j \neq i, \\ \vec{u}_q & j = i. \end{cases} \tag{9.27}$$

Remark: $G_i^{\pm}$ is the set of configurations $\vec{\xi}$ which belong to $\mathcal{X}^{(\pm)}$ independently of the value of $\vec{\xi}_i$ and is measurable on $\vec{\xi}_{i^c}$. When $\underline{\xi}, \underline{\xi}'$ are not in $G_i^{\pm}$, the probability measures for $\vec{\xi}_i$, have support on a strict subset of Ω and the Vaserstein distance can be larger than the bound in (9.24).

Theorem 9.5. *There are $\bar{\gamma}, \varsigma$, so that for $\gamma < \bar{\gamma}$, and for any β : $|\beta - \beta_c^{\mathrm{mf}}(Q)| < c_b \gamma^{\frac{1}{2}}$, there exists $b(i,j)$ satisfying the relations* (9.24) *with G_i as in* (9.26)*. $b(i,j)$ has the expression:*

$$b(i,j) = r\left[J_\gamma(i,j) + e^{-(\frac{\mathfrak{K}_\gamma}{2} - 2b)N_\gamma(i,j)}\right]$$

where $r < 1$, $\mathfrak{K}_\gamma$ and b as in (7.41) *and $N_\gamma(i,j)$:*

$$N_\gamma(i,j) = 1 + \textit{integer part}\left[\frac{|i-j|}{\ell_{+,\gamma}}\right]$$

Proof. The Vaserstein distance between the two measures $\mu_i(\vec{\xi}_i|\vec{\xi}_1), \mu_i(\vec{\xi}_i|\vec{\xi}_2)$ is defined as

$$\mathcal{R}(\mu_i(\vec{\xi}_i|\vec{\xi}_1), \mu_i(\vec{\xi}_i|\vec{\xi}_2)) := \inf_{\mathrm{II}} E_{\mathrm{II}}(\mathrm{dist}(\vec{\xi}_i, \vec{\xi}'_i))$$

where the infimum is taken over all couplings between $\mu_i(\vec{\xi}_i|\vec{\xi}_1), \mu_i(\vec{\xi}_i|\vec{\xi}_2)$. Recalling (4.4), the infimum is realized on the couplings which have the maximal mass on the diagonal

$$\mathrm{II}(\vec{\xi}_i = \vec{\xi}'_i = \vec{u}_q) = \min\{\mu_i(\vec{u}_q|\vec{\xi}_1), \mu_i(\vec{u}_q|\vec{\xi}_2)\}$$

Let $\amalg$ be such a coupling for $\mu_{\mathrm{abs},i;u}(\vec{\xi}_i|\vec{\xi}_1), \mu_{\mathrm{abs},i;u}(\vec{\xi}_i|\vec{\xi}_2)$. We have:

$$\begin{aligned}
\mathcal{R}(\mu_{\mathrm{abs},i;u}(\vec{\xi}_i|\vec{\xi}_1), \mu_{\mathrm{abs},i;u}(\vec{\xi}_i|\vec{\xi}_2)) &= E_{\amalg}\left(\mathrm{dist}(\vec{\xi}_i, \vec{\xi}'_i)\right) = \amalg(\vec{\xi}_i \neq \vec{\xi}'_i) \\
&= 1 - \sum_q \min\{\mu_{\mathrm{abs},i;u}(\vec{u}_q|\vec{\xi}_1), \mu_{\mathrm{abs},i;u}(\vec{u}_q|\vec{\xi}_2)\} \\
&= \frac{1}{2}\sum_q \big(\mu_{\mathrm{abs},i;u}(\vec{u}_q|\vec{\xi}_1) + \mu_{\mathrm{abs},i;u}(\vec{u}_q|\vec{\xi}_2) \\
&\qquad -2\min\{\mu_{\mathrm{abs},i;u}(\vec{u}_q|\vec{\xi}_1), \mu_{\mathrm{abs},i;u}(\vec{u}_q|\vec{\xi}_2)\}\big) \\
&= \frac{1}{2}\sum_q |\mu_{\mathrm{abs},i;u}(\vec{u}_q|\vec{\xi}_1) - \mu_{\mathrm{abs},i;u}(\vec{u}_q|\vec{\xi}_2)|
\end{aligned}$$

Since $\underline{\vec{\xi}}$ and $\underline{\vec{\xi}}'$ are in G_i, the constraint of being in $\mathcal{X}^{\pm}$ does not affect the values of $\vec{\xi}_i$, and the conditional measures $\mu^{\pm}_{\mathrm{abs},i;u}(\cdot|\vec{\xi})$ are given by the Gibbs conditional measures. Using (7.42), we get,

$$\begin{aligned}
\mathcal{R}(\mu^{\pm}_{\mathrm{abs},i;u}(\vec{\xi}_i|\underline{\vec{\xi}}), \mu^{\pm}_{\mathrm{abs},i;u}(\vec{\xi}_i|\underline{\vec{\xi}}')) &= \frac{1}{2}\sum_q \left| \frac{e^{\beta k_q^{\pm;u}(\underline{\vec{\xi}})}}{\sum_{p=1}^{Q} e^{\beta k_p^{\pm;u}(\underline{\vec{\xi}})}} - \frac{e^{\beta k_q^{\pm;u}(\underline{\vec{\xi}}')}}{\sum_{p=1}^{Q} e^{\beta k_p^{\pm;u}(\underline{\vec{\xi}}')}} \right| \\
&= \frac{1}{2}\sum_q |g_q(\vec{k}^{\pm;u}(\underline{\vec{\xi}})) - g_q(\vec{k}^{\pm;u}(\underline{\vec{\xi}}'))|
\end{aligned}$$

Where $\vec{g}(\cdot)$ is defined in (A.12) and $\vec{k}_i^{\pm;u}(\underline{\vec{\xi}}) \equiv \vec{k}_i^{\pm;u}(\underline{\vec{\xi}}_{i^c})$ is the vector

$$\begin{aligned}
\vec{k}_i^{\pm}(\underline{\vec{\xi}}) &:= u\vec{\mathcal{L}}_\gamma(\vec{\xi};i) + (1-u)\vec{\mathcal{L}}_\gamma(\vec{\rho}^{\pm};i) - u\sum_q \big(\mathcal{H}^{\pm}(\underline{\vec{\xi}}^{(i,q)}) - \mathcal{H}^{\pm}(\underline{\vec{\xi}}_{i^c})\big)\vec{u}_q \\
&= u\sum_{j\neq i} J_\gamma(i,j)\vec{\xi}_j + (1-u)\sum_{j\neq i} J_\gamma(i,j)\vec{\rho}_j^{\pm} - u\sum_q\sum_{\Delta\ni i} U_\Delta^{\pm}(\vec{\xi}^{(i,q)})\vec{u}_q
\end{aligned}$$

Using (7.41), we have:

$$\lim_{\gamma\to 0}\sup_{\underline{\vec{\xi}}\in G_i} \|\vec{k}^{\pm}(\underline{\vec{\xi}}) - \vec{\rho}^{\pm}\| \leq \lim_{\gamma\to 0}\left(c\gamma^a + \sum_{\Delta\ni i} Q\|U_\Delta^{\pm}(\cdot)\|_\infty\right) = 0 \tag{9.28}$$

Hence , for γ small enough, we can use theorem A.3 and get

$$\mathcal{R}\left(\mu^{\pm}_{\mathrm{abs},i;u}(\vec{\xi}_i|\underline{\vec{\xi}}), \mu^{\pm}_{\mathrm{abs},i;u}(\vec{\xi}_i|\underline{\vec{\xi}}')\right) \leq \frac{1}{2}(1-\frac{1}{2Q})\sum_q |k_q^{\pm;u}(\underline{\vec{\xi}}) - k_q^{\pm;u}(\underline{\vec{\xi}}')|$$

On the other hand, we have the following bound for all $\Delta \ni x$

$$|U_\Delta^{\pm}(\vec{\xi}^{(i,q)}) - U_\Delta^{\pm}(\vec{\xi}'^{(i,q)})| \leq 2\|U_\Delta^{\pm}(\cdot)\|_\infty \sum_{j\in\Delta\setminus i} \mathrm{dist}(\underline{\vec{\xi}}_j, \underline{\vec{\xi}}'_j)$$

and thus

$$\sum_q |k_q^{\pm;u}(\underline{\vec{\xi}}) - k_q^{\pm;u}(\underline{\vec{\xi}}')| \le 2\sum_{j\neq i}\left(uJ_\gamma(i,j) + \sum_{\Delta\ni i,j} Q\|U_\Delta^\pm(\cdot)\|_\infty\right) \ \mathrm{dist}(\underline{\vec{\xi}}_j, \underline{\vec{\xi}}'_j) \tag{9.29}$$

Recalling (7.41) we bound the last term in (9.29) as:

$$\sum_{\Delta\ni i,j} \|U_\Delta^\pm(\cdot)\|_\infty \le \sum_{\Delta\ni i,j} e^{-(\frac{\mathfrak{K}_\gamma}{2}-b)N_\Delta} \le e^{-(\frac{\mathfrak{K}_\gamma}{2}-2b)N_\gamma(i,j)} \sum_{\Delta\ni i} e^{-bN_\Delta} \le e^{-(\frac{\mathfrak{K}_\gamma}{2}-2b)N_\gamma(i,j)} \tag{9.30}$$

Since $\mathfrak{K}_\gamma = c\gamma^{-(1-\alpha)d+2a}$,

$$\lim_{\gamma\to 0} \sum_{\Delta\ni i,j} Q\|U_\Delta^\pm(\cdot)\|_\infty = 0$$

hence, we take for all u,

$$b(i,j) := (1-\frac{1}{2Q})\left(uJ_\gamma(i,j) + e^{-(\frac{\mathfrak{K}_\gamma}{2}-2b)N_\gamma(i,j)}\right) \tag{9.31}$$

We have

$$\lim_{\gamma\to 0}\sup_{i\in\mathbb{Z}^d} \sum_{j\in C_i^{\ell_-,\gamma}\setminus i} b(i,j) = 0 \tag{9.32}$$

so that (9.24) holds for all γ small enough. □

9.1.2. *Second Assumption:*

The second assumption is a bound on the probability of $G_i^{\pm c}$, where $G_i^\pm$ is defined in (9.26). Here we deal with a bounded state space and the assumption may be written as:

$$\sup_{\vec{\xi}^*\in\mathcal{X}^\pm} \mu^{(\pm)}_{\mathrm{abs},D,u}(G_i^{\pm c}|\vec{\xi}^{\,*}) \le e^{-c\gamma^{2a}\ell^d_{-,\gamma}} \tag{9.33}$$

c a positive constant and $D = C_i^{\ell_-,\gamma}$. The proof of (9.33) requires a Peierls estimate inside the restricted set $\mathcal{X}^\pm$. Let $\zeta' > 0$, we define the "bad" set

$$\mathcal{A}_i^\pm := \{\vec{\xi}\in\mathcal{X}^{(\pm)} : \|\vec{\xi}^{(\ell_-,\gamma)}(i) - \vec{\rho}^{\,\pm}\|_\star > (1-\zeta')\gamma^a\}$$

Notice that for any $\zeta' > 0$ and γ small enough, we have for all i

$$G_i^{\pm c} \sqsubset \mathcal{A}_i^\pm$$

so that $\mathcal{A}_i^\pm$ does not depend on $\vec{\xi}_i$, and we have

$$\sup_{\vec{\xi}^{\,*}\in\mathcal{X}^\pm} \mu^{(\pm)}_{\mathrm{abs},D,u}(G_i^{\pm c}|\vec{\xi}^{\,*}) \le \sup_{\vec{\xi}^{\,*}\in\mathcal{X}^\pm} \mu^{(\pm)}_{\mathrm{abs},D,u}(\mathcal{A}_i^\pm|\vec{\xi}^{\,*})$$

Let

$$\begin{aligned} F_{\gamma,D,u} \quad := \quad & -\frac{u}{2}\int_{D\times D} dr\ dr'\ J_\gamma(r,r')\vec\rho_D(r)\cdot\vec\rho_D(r') - (1-u)\int_D dr\ \vec\rho_D(r)\cdot\vec\rho^{\pm} \\ & -\int_{D\times D^c} dr\ J_\gamma(r,r')\vec\rho_D(r)\cdot\vec\rho_{D^c}(r') + \frac{1}{\beta}\int_D dr\vec\rho_D(r)\cdot\ln\vec\rho_D(r) \end{aligned}$$

The proof of the second assumption is thus based on the following proposition whose proof is given at the end of subsection 9.2:

Proposition 9.6. *There is a constant $c>0$ so that for all γ small enough, any x:*

$$\begin{aligned} \sup_{\vec\xi^*\in\mathcal{X}^{(\pm)}} \ln\ \mu_{\text{abs},D,u}(\mathcal{A}^{\pm}|\vec\xi^{\,*}) \le -\inf_{\vec\rho_{D^c}\in\mathcal{X}^{\pm}_{D^c}} \Bigg(& \inf_{\substack{\vec\rho_D\in\mathcal{X}^{\pm}_D,\\ \left|\vec\rho^{(\ell_-,\gamma)}(r)-\vec\rho^{\pm}\right|>(1-\zeta')\zeta}} F_{\gamma,D,u}(\vec\rho_D|\vec\rho_{D^c}) \\ & - \inf_{\vec\rho_D\in\mathcal{X}^{\pm}_D} F_{\gamma,D,u}(\vec\rho_D|\vec\rho_{D^c})\Bigg) + c\gamma^{1/2}\ell^d_{-,\gamma} \end{aligned} \tag{9.34}$$

where, by an abuse of notation, we have denoted by the same symbol $\mathcal{X}^{\pm}$, the restricted ensemble:

$$\mathcal{X}^{\pm} := \left\{\vec\rho\in L^\infty(\mathbb{R}^d,S_Q) : \|\vec\rho^{(\ell_-,\gamma)}(r)-\vec\rho^{\pm}\|_\star \le \gamma^a,\ \forall r\in\mathbb{R}^d\right\} \tag{9.35}$$

and denote by $\mathcal{X}^{\pm}_\Lambda$ the above expression (9.35) when the constrain is imposed on Λ, with $\Lambda\sqsubset\mathbb{R}^d$, a $\mathcal{D}^{(\ell_-,\gamma)}$-measurable set.

Proof. Using a result similar to Theorem 5.1 but with a slightly different functional, one gets for all $\vec\xi^*\in\mathcal{X}^{(\pm)}$

$$\begin{aligned} &\ln\ \mu_{\text{abs},D,u}(\mathcal{A}^{\pm}|\vec\xi^{\,*}) = \ln Z_{\text{abs},D,u}(\mathcal{A}^{\pm}|\vec\xi^{\,*}) - \ln Z_{\text{abs},D,u}(\vec\xi^{\,*}) \\ &\le \inf_{\substack{\vec\rho_D\in\mathcal{X}^{\pm}_D,\\ \left|\vec\rho^{(\ell_-,\gamma)}(r)-\vec\rho^{\pm}\right|>(1-\zeta')\zeta}} F_{\gamma,D,u}(\vec\rho_D|(\vec\xi^{\,*})^{(\ell_0)}) - \inf_{\vec\rho_D\in\mathcal{X}^{\pm}_D} F_{\gamma,D,u}(\vec\rho_D|(\vec\xi^{\,*})^{(\ell_0)}) + c\gamma^{-1}\ell_0|D| \\ &\le -\inf_{\vec\rho_{D^c}\in\mathcal{X}^{\pm}_{D^c}}\Bigg(\inf_{\substack{\vec\rho_D\in\mathcal{X}^{\pm}_D,\\ \left|\vec\rho^{(\ell_-,\gamma)}(r)-\vec\rho^{\pm}\right|>(1-\zeta')\zeta}} F_{\gamma,D,u}(\vec\rho_D|\vec\rho_{D^c}) - \inf_{\vec\rho_D\in\mathcal{X}^{\pm}_D} F_{\gamma,D,u}(\vec\rho_D|\vec\rho_{D^c})\Bigg) + c\gamma^{1/2}\ell^d_{-,\gamma} \end{aligned}$$

□

The proof of the validity of the second assumption (9.33) follows from an estimate for the r.h.s of (9.3). We define:

$$F^0_{\gamma,D,u}(\vec{\rho}_D|\vec{\rho}_{D^c}) := -\int_D dr\ \left(\vec{\mathcal{L}}^u(r,\vec{\rho}_{D^c})\cdot\vec{\rho}_D - \frac{1}{\beta}\vec{\rho}_D\cdot\ln\vec{\rho}_D\right)$$

where $\vec{\mathcal{L}}^u(r,\vec{\rho}_{D^c})$ is the external field:

$$\vec{\mathcal{L}}^u(r,\vec{\rho}_{D^c}) := (1-u)\vec{\rho}^{\pm} + u\int_{D^c} J_\gamma(r,r')\vec{\rho}_{D^c}(r')dr' \tag{9.36}$$

Since $F^0_{\gamma,\beta,D}(\vec{\rho}_D|\vec{\rho}_{D^c})$ differs from $F_{\gamma,\beta,D}(\vec{\rho}_D|\vec{\rho}_{D^c})$ by the self interaction energy, which is bounded proportionally to $|D|^2$, there is a constant $c'>0$ such that:

$$|F_{\gamma,D,u}(\vec{\rho}_D|\vec{\rho}_{D^c}) - F^0_{\gamma,D,u}(\vec{\rho}_D|\vec{\rho}_{D^c})| \le c' u\gamma^d|D|^2 \tag{9.37}$$

$F^0_{\gamma,D}(\vec{\rho}_D|\vec{\rho}_{D^c})$ is a convex functional on $L^\infty(D,S_Q)$, and has thus a unique minimizer that we denote by $\vec{\rho}^{\,*}(r;\vec{\rho}_{D^c})\equiv\vec{\rho}^{\,*}(r)$, whose components are given by:

$$\rho_k^*(r) = \frac{e^{\beta\mathcal{L}^u_k(r,\vec{\rho}_{D^c})}}{\sum_l e^{\beta\mathcal{L}^u_l(r,\vec{\rho}_{D^c})}} \tag{9.38}$$

We need to evaluate the difference:

$$\begin{aligned} &F^0_{\gamma,D,u}(\vec{\rho}_D|\vec{\rho}_{D^c}) - F^0_{\gamma,D,u}(\vec{\rho}^{\,*}_D|\vec{\rho}_{D^c}) = \\ &\qquad\int_D dr'\left(-\vec{\rho}_D(r')\cdot\vec{\mathcal{L}}^u(r,\vec{\rho}_{D^c}) + \frac{1}{\beta}\vec{\rho}_D(r')\cdot\ln\vec{\rho}_D(r')\right) - \\ &\qquad\qquad\left(-\vec{\rho}^{\,*}_D(r')\cdot\vec{\mathcal{L}}^u(r,\vec{\rho}_{D^c}) + \frac{1}{\beta}\vec{\rho}^{\,*}_D(r')\cdot\ln\vec{\rho}^{\,*}_D(r')\right)\end{aligned}$$

Using (9.38), we write $\vec{\mathcal{L}}^u(r,\vec{\rho}_{D^c})$ in terms of $\vec{\rho}^{\,*}$ and get:

$$F^0_{\gamma,D,u}(\vec{\rho}_D|\vec{\rho}_{D^c}) - F^0_{\gamma,D,u}(\vec{\rho}^{\,*}_D|\vec{\rho}_{D^c}) = \frac{1}{\beta}\int_D dr'\sum_{k=1}^{Q}\rho_k(r')\ln\frac{\rho_k(r')}{\rho_k^*(r')} \tag{9.39}$$

Thus by the Kullback-Leibler inequality, one gets

$$F^0_{\gamma,D,u}(\vec{\rho}_D|\vec{\rho}_{D^c}) - F^0_{\gamma,D,u}(\vec{\rho}^{\,*}_D|\vec{\rho}_{D^c}) \quad\ge\quad \frac{1}{2\beta}\int_D dr'|\vec{\rho}_D(r') - \vec{\rho}^{\,*}_D(r')|^2$$

We claim that there is $\epsilon>0$ such that, for any $\vec{\rho}_{D^c}$ and for $\gamma>0$ small enough,

$$\|\vec{\rho}^{\,*} - \vec{\rho}^{\pm}\|_\star \le (1-\epsilon)\gamma^a \tag{9.40}$$

Using Cauchy-Schwartz inequality, we thus get (taking $\zeta'<\epsilon$):

$$F^0_{\gamma,D,u}(\vec{\rho}_D|\vec{\rho}_{D^c}) - F^0_{\gamma,D,u}(\vec{\rho}^{\,*}_D|\vec{\rho}_{D^c}) \ge \frac{1}{2\beta}|D|(\epsilon-\zeta')^2\gamma^{2a} \tag{9.41}$$

Postponing the proof of (9.40), we get the following bound by using proposition 9.6 together with (9.37) and (9.41):

$$
\begin{aligned}
\sup_{\vec{\xi}^*\in\mathcal{X}^{(\pm)}} \ln\ \mu_{\mathrm{abs},D,u}(\mathcal{A}_x^{\pm}|\vec{\xi}^{\,*}) &\leq \exp\left\{-\left(\frac{1}{2\beta}|D|(\epsilon-\zeta')^2\gamma^{2a}-c'\gamma^d|D|^2\right)+c\gamma^{1/2}\ell_{-,d}^d\right\}\\
&= \exp\left\{-\left(\frac{1}{2\beta}(\epsilon-\zeta')^2\gamma^{2a}-c'\gamma^{d\alpha}+c\gamma^{1/2}\right)\ell_{-,d}^d\right\}
\end{aligned}
$$

Assumption (9.33) is then proven. We now turn to the proof of (9.40):

In order to prove (9.40), we show that there is $b\in(0,1)$:

$$
|\rho_k^*(r)-\rho_k^{\pm}|\leq b|\mathcal{L}_k^u(r)-\rho_k^{\pm}| \tag{9.42}
$$

while $\|\vec{\mathcal{L}}^u(r)-\vec{\rho}^{\pm}\|_\star$ is small and close enough to γ^a.

We first prove that $|\mathcal{L}_k^u(r)-\rho_k^{\pm}|$ is small enough. We define

$$
J_\gamma^{(\ell_-,\gamma)}(r,\langle r'\rangle):=\frac{1}{|C_{r'}^{\ell_-,\gamma}|}\int J_\gamma(r,r'')dr''
$$

For γ small enough, by hypothesis on $\mathcal{J}$ (see (2.1))

$$
\left|J_\gamma(r,r')-J_\gamma^{(\ell_-,\gamma)}(r,\langle r'\rangle)\right|\leq\sqrt{d}\|\nabla\mathcal{J}\|_\infty\gamma^{d+1}\ell_{-,\gamma}\mathbf{1}_{\{|r-r'|\leq 2\gamma^{-1}\}} \tag{9.43}
$$

Then, for any $\vec{\rho}\in L^\infty(\mathbb{R}_+^d,S_Q)$, there is a constant $c_d=2^d\sqrt{d}\|\nabla\mathcal{J}\|_\infty$

$$
\left\|J_\gamma*\vec{\rho}-J_\gamma^{(\ell_-,\gamma)}*\vec{\rho}\right\| \ \leq\ c_d\ \gamma\ell_{-,\gamma}=c_d\gamma^\alpha
$$

Using the fact that $\vec{\rho}_{D^c}\in\mathcal{X}_{D^c}^{\pm}$, we write:

$$
\begin{aligned}
\left\|\vec{\mathcal{L}}^u(r)-\vec{\rho}^{\pm}\right\| &= u\left\|\int_{D^c}J_\gamma(r,r')\vec{\rho}_{D^c}(r')dr'-\vec{\rho}^{\pm}\right\|\\
&\leq \left\|\int_{D^c}\left(J_\gamma(r,r')-J_\gamma^{(\ell_-,\gamma)}(r,r')\right)\vec{\rho}_{D^c}(r')dr'\right\|\\
&\qquad+\left\|\int_{D^c}J_\gamma^{(\ell_-,\gamma)}(r,r')\left(\vec{\rho}_{D^c}(r')-\vec{\rho}^{\pm}\right)dr'\right\|+\left\|\int_D J_\gamma^{(\ell_-,\gamma)}(r,r')\vec{\rho}^{\pm}dr'\right\|\\
&\leq c_d\gamma^\alpha+\gamma^a+c'_d\gamma^{\alpha d}
\end{aligned}
$$

with $c'_d=\|\mathcal{J}\|_\infty$. Recalling (9.38), we have

$$
\left\|\vec{\rho}_D^{\,*}(r)-\vec{\rho}^{\pm}\right\|=\left\|\vec{g}(\vec{\mathcal{L}}^u(r))-\vec{g}(\vec{\rho}^{\pm})\right\| \tag{9.44}
$$

Applying Theorem A.3, we get

$$
\begin{aligned}
\left\|\vec{\rho}_D^{\,*}(r)-\vec{\rho}^{\pm}\right\| &\leq (1-\frac{1}{2Q})\left(\gamma^a+c_d\gamma^\alpha+c'_d\gamma^{\alpha d}\right)\\
&\leq (1-\frac{1}{4Q})\gamma^a-\left(\frac{1}{4Q}\gamma^a-(1-\frac{1}{2Q})(c_d\gamma^\alpha+c'_d\gamma^{\alpha d})\right)\\
&\leq \gamma^a(1-\epsilon)
\end{aligned}
$$

for γ small enough and $a<\alpha$, taking $\epsilon=1-\frac{1}{4Q}$.

9.1.3. *Third Assumption:*
Let

$$\mathcal{B}(n) := \{m \in (\ell_{-,\gamma}\mathbb{Z})^d : \mathrm{dist}(C_n^{\ell_-,\gamma}, C_m^{\ell_-,\gamma}) \le \gamma^{-1}\}$$

there is $\tilde{r}_{nm} : (\ell_{-,\gamma}\mathbb{Z})^d \times (\ell_{-,\gamma}\mathbb{Z})^d \to \mathbb{R}^+$ s.t. for any $\vec{\xi}^1, \vec{\xi}^2 : \vec{\xi}^1(j) = \vec{\xi}^2(j)\ \forall j \in [\mathbb{Z}^d \sqcap \mathcal{B}(n)] \setminus i$, it holds that:

$$R_{C_n}(\mu_C(\cdot|\vec{\xi}^{(1)}_{C_n^c}), \mu_{C_n}(\cdot|\vec{\xi}^{(2)}_{C_n^c})) \le \sum_{m \in (\ell_{-,\gamma}\mathbb{Z})^d \setminus \mathcal{B}(n)} \tilde{r}_{nm} \mathrm{dist}(\vec{\xi}^{(1)}_{C_m}, \vec{\xi}^{(2)}_{C_m}) \tag{9.45}$$

$$\sup_n \sum_{m \notin \mathcal{B}(n)} \tilde{r}_{nm} < 1 \tag{9.46}$$

This is a condition on the tail of the interaction, which in our case is satisfied because of the exponential decay of the interaction $h_u^{\pm}$ due to the Peierls estimates.

9.1.4. *Fourth Assumption:*
Let $r(n,m) : (\ell_{-,\gamma}\mathbb{Z})^d \times (\ell_{-,\gamma}\mathbb{Z})^d \to \mathbb{R}^+$ defined as follows:

$$r(n,m) := \begin{cases} 0 & \text{if } m = n \\ \tilde{r}_{nm} & \text{if } m \notin \mathcal{B}(n) \\ \tilde{r}^*_{nm} & \text{if } m \in \mathcal{B}(n) \setminus n \end{cases}$$

where:

$$\tilde{r}^*_{nm} := \sup_{j \in C_m^{\ell_-,\gamma}} \sum_{i \in C_n^{\ell_-,\gamma}} \sum_{k>0} b^k_{C_n^{\ell_-,\gamma}}(i,j) + 2e^{-c\gamma^{2a}\ell^d_{-,\gamma}} \left(\ell^d_{-,\gamma} + \sum_{i,i' \in C_n^{\ell_-,\gamma}} \sum_{k>0} b^k_{C_n^{\ell_-,\gamma}}(i,i') \right)$$

with

$$b_{C_n^{\ell_-,\gamma}}(i,j) := b(i,j)\mathbf{1}_{i \in C_n}$$

and $b^k_{C_n^{\ell_-,\gamma}}(i,j)$ is the k-th convolution of $b_{C_n^{\ell_-,\gamma}}(i,j)$. Assumption 4 then states:

$$\sup_n \sum_m r(n,m) \le \delta \qquad\qquad 0 < \delta < 1 \tag{9.47}$$

9.1.5. *Fifth Assumption:*
There is a metric $\psi(n,m)$ on $(\ell_{-,\gamma}\mathbb{Z})^d$ s.t. $\psi(n,m) \ge d_0 > 0$ for all $n \ne m \in (\ell_{-,\gamma}\mathbb{Z})^d$, and

$$\sup_n \left[\delta e^{\sup_{j \in \mathcal{B}(n)} \psi(n,m)} + \sum_{m \notin \mathcal{B}(n)} r(n,m) e^{\psi(n,m)} \right] < 1 \tag{9.48}$$

for δ as in (9.47).

9.1.6. *Conclusion of subsection* (9.1).

We have proved that the assumptions 1, 2, 3, 4 hold for the two abstract models, which imply uniqueness of the measure. Moreover assumption 5 holds and implies exponential decay of the correlation of the measures $\tilde{\mu}^{\pm}_{\mathrm{abs},u}$. see [**3**].

Hence the validity of assumptions 1, 2, 3, 4 5 imply Theorem 9.3 and Corollary 9.4 □

9.2. Small deviation estimates.

In this subsection we prove the estimate (9.18)

$$\int_0^1 du \sum_{i\in A}\{|\langle g_{\gamma,i}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs};u}}|+|\langle g_{\gamma,i,\Lambda}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs},\Lambda;u}}|\}\le c\gamma^{1/8}|\delta^{\ell_+,\gamma}_{\mathrm{in}}[\Lambda]|$$

Let $A:=\delta^{\ell_+,\gamma}_{\mathrm{in}}[\Lambda]$ and define the set $S^{\pm}(\vec{\xi})$ as:

$$S^{\pm}(\vec{\xi}):=\{i\in\check{A}:\|\vec{\xi}^{(\ell_0)}(i)-\vec{\rho}^{\pm}\|_{\star}\ge\gamma^{1/8}\} \tag{9.49}$$

where

$$\check{A}:=A\sqcup\delta^{\gamma^{-1}}_{\mathrm{out}}[A] \tag{9.50}$$

We first prove the following bound:

$$\begin{aligned}&\int_0^1 du \sum_{i\in A}\{|\langle g_{\gamma,i}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs};u}}|+|\langle g_{\gamma,i,\Lambda}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs},\Lambda;u}}|\} \\ &\qquad\le\mathfrak{I}_\gamma\int_0^1 du\left[\langle\mathbf{1}_{|S^{\pm}|\ge\gamma^{1/8}|A|}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs};u}}+\langle\mathbf{1}_{|S^{\pm}|\ge\gamma^{1/8}|A|}\rangle_{\tilde{\mu}^{\pm}_{\mathrm{abs},\Lambda;u}}\right]+c(\gamma^{1/2}|A|+\gamma^{1/8}|A|)\end{aligned} \tag{9.51}$$

PROOF OF (9.51). We recall the definition of $g_{\gamma,i,\Lambda}(\vec{\xi})$ (9.13) and $g_{\gamma,i}(\vec{\xi})$ (9.14):

$$\begin{aligned}g_{\gamma,i,\Lambda}(\vec{\xi}) &:= -\frac{1}{2}\delta\vec{\xi}_i\sum_{\substack{j\in\Lambda\\ j\ne i}}J_\gamma(i,j)\delta\vec{\xi}_j+\sum_{\substack{\Delta\ni i\\ \Delta\sqsubset\Lambda}}\frac{1}{|\Delta|}U^{\pm}_{\Delta}(\vec{\xi}) \\ g_{\gamma,i}(\vec{\xi}) &:= \lim_{\Lambda\nearrow\mathbb{R}^d}g_{\gamma,i,\Lambda}(\vec{\xi})=-\frac{1}{2}\delta\vec{\xi}_i\sum_{j\ne i}J_\gamma(i,j)\delta\vec{\xi}_j+\sum_{\Delta\ni i}\frac{1}{|\Delta|}U^{\pm}_{\Delta}(\vec{\xi})\end{aligned}$$

By (7.41),

$$\sum_{\Delta\ni i}\frac{1}{|\Delta|}U^{\pm}_{\Delta}(\vec{\xi})\le e^{-(\frac{\mathfrak{K}_\gamma}{2}-b)}$$

Hence, we have:

$$\begin{aligned}\sum_{i\in A}\left|g_{\gamma,i}(\vec{\xi})\right| &\le \frac{1}{2}\sum_{i\in A}\left|\delta\vec{\xi}_i\sum_{j\neq i}J_\gamma^{(\ell_0)}(i,j)\delta\vec{\xi}_j\right| + c\gamma\ell_0|A|\\ &\le \sum_{i\in A}\sum_{j\in\check{A}}J_\gamma^{(\ell_0)}(i,j)\|\delta\vec{\xi}_j^{(\ell_0)}\|_\star + c\gamma^{1/2}|A|\\ &\le \mathfrak{J}_\gamma\sum_{j\in\check{A}}\|\delta\vec{\xi}_j^{(\ell_0)}\|_\star + c\gamma^{1/2}|A|\\ &\le \mathfrak{J}_\gamma|S^\pm(\vec{\xi})| + c(\gamma^{1/2}|A| + \gamma^{1/8}|A|)\end{aligned}$$

where we have defined

$$J_\gamma^{(\ell_0)}(i,\langle j\rangle) := \frac{1}{|C^{\ell_0}|}\sum_{j'\in C_j^{\ell_0}}J_\gamma(i,j)$$

and used that by definition (9.49), $\|\delta\vec{\xi}_j\|_\star \le \gamma^{1/8}$ for all $j\notin S$. A similar estimate for $g_{\gamma,i,\Lambda}$ holds and (9.51) follows. □

Since the two terms in the integral in the right hand side of (9.51) are very similar, we give the derivation of an estimate for the first term only. We prove the following

$$\langle \mathbf{1}_{|S^\pm|\ge\gamma^{1/8}|A|}\rangle_{\tilde{\mu}^\pm_{\mathrm{abs};u}} \le e^{-c\gamma^{3/8}|A|} \tag{9.52}$$

PROOF OF (9.52).

Let

$$\hat{A} := A\sqcup\delta_{\mathrm{out}}^{\ell_+,\gamma}[A]$$

$$\langle \mathbf{1}_{|S^\pm|\ge\gamma^{1/8}|A|}\rangle_{\tilde{\mu}^\pm_{\mathrm{abs};u}} = \langle\frac{\hat{\mu}^\pm_{\mathrm{abs};\hat{A},u}(\mathbf{1}_{|S^\pm|\ge\gamma^{1/8}|A|}e^{-\beta\sum_{\Delta:\Delta\cap\hat{A}\neq\varnothing}U_\Delta^\pm(\vec{\xi})}|\vec{\xi}_{\hat{A}^c})}{\hat{\mu}^\pm_{\mathrm{abs};\hat{A},u}(e^{-\beta\sum_{\Delta:\Delta\cap\hat{A}\neq\varnothing}U_\Delta^\pm(\vec{\xi})}|\vec{\xi}_{\hat{A}^c})}\rangle_{\tilde{\mu}^\pm_{\mathrm{abs};u}} \tag{9.53}$$

where $\hat{\mu}^\pm_{\mathrm{abs};\hat{A},u}$ is the measure on $\mathcal{X}^\pm_{\hat{A}}$ associated to the finite range interpolating Hamiltonian $\hat{h}_u^\pm(\vec{\xi}_{\hat{A}}|\vec{\xi}_{\hat{A}^c})$

$$\hat{h}_u^\pm(\vec{\xi}_{\hat{A}}|\vec{\xi}_{\hat{A}^c}) := uH_{\gamma,\hat{A}}(\vec{\xi}_{\hat{A}}|\vec{\xi}_{\hat{A}^c}) + (1-u)\mathfrak{H}^\pm_{\gamma,\hat{A}}(\vec{\xi})$$

Recalling (7.41), we get:

$$\langle \mathbf{1}_{|S^\pm|\ge\gamma^{1/8}|A|}\rangle_{\tilde{\mu}^\pm_{\mathrm{abs};u}} \le \langle\hat{\mu}^\pm_{\mathrm{abs};\hat{A},u}(\mathbf{1}_{|S^\pm|\ge\gamma^{1/8}|A|}|\vec{\xi}_{\hat{A}^c})\rangle_{\tilde{\mu}^\pm_{\mathrm{abs};u}} \times e^{\beta|\hat{A}|e^{-(\mathfrak{K}_\gamma-4b)}} \tag{9.54}$$

we write

$$\hat{\mu}^\pm_{\mathrm{abs};\hat{A},u}(\mathbf{1}_{|S^\pm|\ge\gamma^{1/8}|A|}|\vec{\xi}_{\hat{A}^c}) = \frac{\hat{Z}^\pm_{\mathrm{abs},\beta,\hat{A};u}(S^\pm|\vec{\xi}_{\hat{A}^c})}{\hat{Z}^\pm_{\mathrm{abs},\beta,\hat{A};u}(\vec{\xi}_{\hat{A}^c})} \tag{9.55}$$

where

$$\hat{Z}^{\pm}_{\mathrm{abs},\beta,\hat{A};u}(S^{\pm}|\vec{\xi}_{\hat{A}^c}) := \sum_{\vec{\xi}_{\hat{A}}\in\mathcal{X}^{\pm}_{\hat{A}}} \mathbf{1}_{|S^{\pm}|\geq\gamma^{1/8}|A|} e^{-\beta\hat{h}^{\pm}_u(\vec{\xi}_{\hat{A}}|\vec{\xi}_{\hat{A}^c})}$$

$$\hat{Z}^{\pm}_{\mathrm{abs},\beta,\hat{A};u}(\vec{\xi}_{\hat{A}^c}) := \sum_{\vec{\xi}_{\hat{A}}\in\mathcal{X}^{\pm}_{\hat{A}}} e^{-\beta\hat{h}^{\pm}_u(\vec{\xi}_{\hat{A}}|\vec{\xi}_{\hat{A}^c})}$$

Now the partition function $\hat{Z}^{\pm}_{\mathrm{abs},\beta,\hat{A};u}(S^{\pm}|\vec{\xi}_{\hat{A}^c})$ can be estimated using an approximation to the continuum. Defining

$$\begin{aligned} F_{\gamma,\hat{A},u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) := &-\frac{1}{2}\int_{\hat{A}\times\hat{A}} J_\gamma(r,r')\big(u\vec{\xi}_{\hat{A}}(r)\cdot\vec{\xi}_{\hat{A}}(r') + (1-u)\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm}\big)drdr' \\ &-\int_{\hat{A}\times\hat{A}^c} J_\gamma(r,r')\big(u\vec{\xi}_{\hat{A}}(r)\cdot\vec{\xi}^{(\ell_0)}_{\hat{A}^c}(r') + (1-u)\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm}\big)drdr' \\ &-(1-u)\int_{\hat{A}}\big(\vec{\xi}_{\hat{A}}(r)\cdot\vec{\rho}^{\pm}\big)dr + \frac{1}{\beta}\int_{\hat{A}}\vec{\xi}_{\hat{A}}(r)\cdot\ln\vec{\xi}_{\hat{A}}(r)dr \end{aligned}$$

and

$$\mathfrak{Z}^{\pm} := \left\{\vec{\xi}_{\hat{A}}\in\Omega^{\ell_0}_{\Lambda} : \vec{\xi}_{\hat{A}}(r) = \vec{\xi}^{(\ell_0)}(r)\ ;\ \eta(\vec{\xi};r) = a_{\pm}\ r\in\hat{A}\ ;\ \int_{\breve{A}}\mathbf{1}_{\|\vec{\xi}_{\hat{A}}-\vec{\rho}^{\pm}\|_\star\geq\gamma^{1/8}} \geq \gamma^{1/8}|A|\right\}$$

A result similar to theorem 5.1 holds for the above functional and leads to

$$\ln\hat{Z}^{\pm}_{\mathrm{abs},\beta,\hat{A};u}(S^{\pm}|\vec{\xi}_{\hat{A}^c}) \leq -\beta\inf_{\vec{\xi}_{\hat{A}}\in\mathfrak{Z}^{\pm}} F_{\gamma,\hat{A},u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) + c\gamma^{1/2}|\hat{A}| \tag{9.56}$$

where c is a constant independent on u.

We are then reduced to study the variational problem in (9.56) for $F_{\gamma,\hat{A},u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c})$ on $\mathfrak{Z}^{\pm}$. We define the "excess free energy functional":

$$\mathcal{F}^{\mathrm{eff}}_{\gamma,\hat{A},u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) := F_{\gamma,\hat{A},u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) + \mathfrak{A} \tag{9.57}$$

$$\mathfrak{A} := -\inf_{\vec{v}:\|\vec{v}\|=1}\phi^{\mathrm{mf}}_u(\vec{v})|\hat{A}| + \frac{u}{2}\int_{\hat{A}\times\hat{A}^c} J_\gamma(r,r')\big(\vec{\xi}^{(\ell_0)}_{\hat{A}^c}(r')\cdot\vec{\xi}^{(\ell_0)}_{\hat{A}^c}(r')\big)\ dr\ dr' \tag{9.58}$$

$$\phi^{\mathrm{mf},\pm}_u(\vec{v}) := -u\frac{(\vec{v}\cdot\vec{v})}{2} - (1-u)(\vec{v}\cdot\rho^{\pm}) + \frac{1}{\beta}(\vec{v}\cdot\ln\vec{v}) \tag{9.59}$$

$\mathcal{F}^{\mathrm{eff}}_{\gamma,\hat{A},u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c})$ is positive and differs from $F_{\gamma,\hat{A},u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c})$ by a constant (hence has the same minimizers and its minimum is finite)

Denoting by:

$$\Phi^{\mathrm{eff},\pm}_u(\vec{\xi}_{\hat{A}}) := \phi^{\mathrm{mf},\pm}_u(\vec{\xi}_{\hat{A}}) - \inf_{\vec{v}:\|\vec{v}\|=1}\phi^{\mathrm{mf},\pm}_u(\vec{v}) \tag{9.60}$$

$\mathcal{F}^{\mathrm{eff}}_{\gamma,u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c})$ can be rewritten then as:

$$\begin{aligned} \mathcal{F}^{\mathrm{eff}}_{\gamma,\hat{A},u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) = &\int_{\hat{A}}\Phi^{\mathrm{eff},\pm}_u(\vec{\xi}_{\hat{A}}) + \frac{u}{4}\int_{\hat{A}\times\mathbb{B}} J_\gamma(r,r')[\vec{\xi}_{\hat{A}}(r)-\vec{\xi}_{\hat{A}}(r')]^2 \\ &+\frac{u}{2}\int_{\hat{A}\times\hat{A}^c} J_\gamma(r,r')[\vec{\xi}_{\hat{A}}(r)-\vec{\xi}^{(\ell_0)}_{\hat{A}^c}(r')]^2 \end{aligned}$$

The analysis in appendix B, see corollary B.5, proves that there are positive constants ω, c_ω, so that for any $\vec{\xi} \in \mathfrak{Z}^\pm$ there is $\vec{\psi}_{\hat{A}}$:

$$\mathcal{F}^{\rm eff}_{\gamma,\hat{A},u}(\vec{\xi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) \geq \mathcal{F}^{\rm eff}_{\gamma,\hat{A},u}(\vec{\psi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) - c_\omega e^{-\omega\gamma\ell_{+,\gamma/4}}|\hat{A}| \tag{9.61}$$

where $\vec{\Psi}_{\hat{A}}$ has the following properties:

$$\begin{aligned}
&\eta(\vec{\psi}_{\hat{A}};r) = a_\pm \ \forall r \in \hat{A}\\
&\vec{\psi}_{\hat{A}}(r) = \vec{\rho}^{\pm} \ \forall r \in \Sigma := \big(\hat{A}\setminus A\big) \setminus \delta_{\rm in}^{\ell_{+,\gamma/4}}\big(\hat{A}\setminus A\big)\\
&\vec{\psi}_{\hat{A}}(r) = \vec{\xi}(r) \ \forall r \in \check{A}\\
&\sup_q |\psi_{\hat{A},q}(r) - \rho^\pm_q| \leq (1-\kappa_0)\gamma^a \ \forall r \in \delta_{\rm in}^{\ell_{+,\gamma/4}}[\hat{A}] \ \kappa_0 > 0
\end{aligned} \tag{9.62}$$

We can write:

$$\ln \hat{Z}^\pm_{\beta,\hat{A};u}(S^\pm|\vec{\xi}_{\hat{A}^c}) \leq -\beta \inf_{\vec{\psi}_{\hat{A}}\in\mathcal{B}^0} \mathcal{F}^{\rm eff}_{\gamma,\hat{A},u}(\vec{\psi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) + c\gamma\ell_0|\hat{A}| + \beta c_\omega e^{-\omega\gamma\ell_{+,\gamma/4}}|\hat{A}| + \mathfrak{A} \tag{9.63}$$

where

$$\begin{aligned}
\mathcal{B}^0 := \{\vec{\psi}_{\hat{A}} : \sup_q |\psi_{\hat{A},q} - \rho^\pm_q| \leq (1-\kappa_0)\gamma^a,\ r \in \delta_{\rm in}^{\ell_{+,\gamma/4}}[\hat{A}];\ \eta(\vec{\psi}_{\hat{A}};r) = a_\pm,\ r \in \hat{A};\\
\vec{\psi}_{\hat{A}} = \vec{\rho}^{\pm}\ r \in \Sigma;\ \vec{\psi}_{\hat{A}}(r) = \vec{\psi}^{(\ell_0)}_{\hat{A}}(r),\ r \in \check{A};\ \int_{\check{A}} \mathbf{1}_{\|\vec{\psi}_{\hat{A},q} - \vec{\rho}^{\pm}\|_\star \geq \gamma^{1/8}} > \gamma^{1/8}|A|\}
\end{aligned} \tag{9.64}$$

In appendix E it is proved that for any $\psi_{\hat{A}} \in \mathcal{B}^0$ there is $\vec{\psi}^*_{\hat{A}}$:

$$\vec{\psi}^*_{\hat{A}} = \begin{cases} \vec{\psi}_{\hat{A}} & \text{on } \delta_{\rm in}^{\ell_{+,\gamma/4}}[\hat{A}],\\ \vec{\rho}^{\pm} & \text{elsewhere.}\end{cases} \tag{9.65}$$

so that:

$$\mathcal{F}^{\rm eff}_{\gamma,\hat{A},u}(\vec{\psi}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) \geq \mathcal{F}^{\rm eff}_{\gamma,\hat{A},u}(\vec{\psi}^*_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) + c\gamma^{1/4}(\gamma^{1/8}|A|) \tag{9.66}$$

Let $[\psi^*]^{\ell_0}$ defined as in (5.15). By theorem 5.1 we then have:

$$-\beta F_{\gamma,\hat{A},u}(\vec{\psi}^{\,*}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) \leq \ln \hat{Z}_{\beta,\hat{A};u}\big(\{\vec{\xi}^{\ell_0}_{\hat{A}} = [\vec{\psi}^{\,*}_{\hat{A}}]^{\ell_0}\}\big|\vec{\xi}_{\hat{A}^c}\big) + \gamma\ell_0|\hat{A}|$$

and since by definition :

$$\|[\vec{\psi}^{\,*}_{\hat{A}}]^{\ell_0} - \vec{\psi}^{\,*}_{\hat{A}}\|_\star < \frac{1}{2}\ell_0^{-d}$$

we have:

$$\|[\vec{\psi}^{\,*}_{\hat{A}}]^{\ell_0}_q - \vec{\rho}^{\,\pm}\|_\star < \frac{1}{2}\ell_0^{-d} + (1-\kappa_0)\gamma^a$$

so that the set $\{\vec{\xi}^{\ell_0}_{\hat{A}} = [\vec{\psi}^{\,*}_{\hat{A}}]^{\ell_0}\} \in \mathcal{X}^\pm$ and

$$-\beta F_{\gamma,\hat{A},u}(\vec{\psi}^{\,*}_{\hat{A}}|\vec{\xi}^{(\ell_0)}_{\hat{A}^c}) \leq \ln \hat{Z}^\pm_{{\rm abs},\beta,\hat{A};u}\big(\vec{\xi}_{\hat{A}^c}\big) + \gamma\ell_0|\hat{A}|$$

By (9.57) and (9.63)

$$\ln \hat{Z}^{\pm}_{\text{abs},\beta,\hat{A};u}(S^{\pm}|\vec{\xi}_{\hat{A}^c}) \le \ln \hat{Z}^{\pm}_{\text{abs},\beta,\hat{A};u}\big(\vec{\xi}_{\hat{A}^c}\big) + \beta c_\omega e^{-\omega\gamma\ell_{+,\gamma}/4}|\hat{A}| - c\gamma^{1/4}(\gamma^{1/8}|A|) + c'\gamma\ell_0|\hat{A}|$$

Inserting this inequality in (9.53), we get (9.52) for γ small enough.

$\square$

10. Large deviation estimates

Let Γ a $\hat{p}$-contour. Recalling the definition (7.15) of $G^{\hat{p}}$,

$$G^{\hat{p}} := \mathrm{sp}(\Gamma) \bigsqcup_{\hat{q}\neq\hat{p}} A^{\hat{q}}$$

we define $\mathcal{R}(\Gamma)$ and $\mathcal{R}(\not{\Gamma})$ as the two subsets of $L^\infty(G^{\hat{p}}, S_Q)$ such that

$$\mathcal{R}(\Gamma) := \{\vec{\rho} \in L^\infty(G^{\hat{p}}, S_Q) : \eta_x(\vec{\rho}) = \eta_\Gamma(x)\ \forall x \in \mathrm{sp}(\Gamma); \quad \vec{\rho}(x) = \vec{\rho}^{\hat{q}}\ \forall x \in A^{\hat{q}}\ \forall \hat{q} \neq \hat{p}\}$$

$$\mathcal{R}(\not{\Gamma}) := \{\vec{\rho} \in L^\infty(G^{\hat{p}}, S_Q) : \eta(\vec{\rho}, x) = a_{\hat{p}}\ \forall x \in \Gamma; \quad \rho(x) = \vec{\rho}^{\hat{p}}\ \forall x \in \bigsqcup_{\hat{q}\neq\hat{p}} A^{\hat{q}}\}$$

We prove the following:

Theorem 10.1. *There exists $\bar{\gamma} > 0$ and a constant $c_f > 0$ such that for all $\gamma \le \bar{\gamma}$ and all β such that $|\beta - \beta_c^{\mathrm{mf}}(Q)| \le c_b\gamma^{\frac{1}{2}}$, the following large deviation estimate holds:*

$$\inf_{\vec{\rho}\in\mathcal{R}(\Gamma)} F_{\gamma,\beta,G^{\hat{p}}}(\vec{\rho}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) - \inf_{\vec{\rho}\in\mathcal{R}(\not{\Gamma})} F_{\gamma,\beta,G^{\hat{p}}}(\vec{\rho}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) \ge c_f|\Gamma|\gamma^{2a}\Big(\frac{\ell_{-,\gamma}}{\ell_{+,\gamma}}\Big)^d + \sum_{\hat{q}\neq\hat{p}} \mathcal{I}_{A^{\hat{q}}}(\vec{\rho}^{\,2}_{\hat{q}} - \vec{\rho}^{\,2}_{\hat{p}})$$

where

$$\mathcal{I}_{A^{\hat{q}}} = \frac{1}{2}\int_{A^{\hat{q}}}\int_{(\mathrm{Int}^{\hat{q}}(\Gamma)\setminus A^{\hat{q}})} J_\gamma(x,y)\,dx\,dy$$

Proof. For each $\mathcal{D}^{\ell_{+,\gamma}}$-measurable cube C in $\mathrm{sp}(\Gamma)$, at least one of the two following events occurs (recall definition (6.3)):

(a) there is x in Γ such that $C_x^{\ell_{+,\gamma}} \sim C$ and $\eta_x^{\ell_{-,\gamma}} = 0$.

(b) there are x_1, x_2 in Γ such that $C_{x_1}^{\ell_{+,\gamma}} \sim C$, $C_{x_2}^{\ell_{+,\gamma}} \sim C$, $C_{x_1}^{\ell_{-,\gamma}} \sim C_{x_2}^{\ell_{-,\gamma}}$ and $\eta_{x_1} \neq 0$,$\eta_{x_2} \neq 0$,$\eta_{x_1} \neq \eta_{x_2}$.

Let N_Γ the number of cubes of $\mathcal{D}^{\ell_{+,\gamma}}$ contained in $\mathrm{sp}(\Gamma)$. Since a single event can be associated to at most 3^d cubes, there are at least $3^{-d}N_\Gamma$ distinct and nonintersecting events. Let consider a maximal family of nonintersecting events and denote by $(x_0^i)_{i\in I_a}$, respectively $(x_1^j, x_2^j)_{j\in I_b}$ a set of points (respectively a set of pairs of points) characterizing the events of type a (respectively of type b). We also define

$$B_a = \bigsqcup_{i\in I_a} C^{\ell_-,\gamma}_{x_0^i} \tag{10.1}$$

$$B_b = \bigsqcup_{j\in I_b} (C^{\ell_-,\gamma}_{x_1^j} \sqcup C^{\ell_-,\gamma}_{x_2^j}) \tag{10.2}$$

We have

$$|I_a| + |I_b| \geq \frac{1}{3^d} N_\Gamma \tag{10.3}$$

$$|B_a| + |B_b| \geq \frac{1}{3^d} \Big(\frac{\ell_{-,\gamma}}{\ell_{+,\gamma}}\Big)^d |\Gamma| \tag{10.4}$$

We separate $G^{\hat p}$ in two components $G^{\hat p} = G_1 \sqcup G_2$,

$$G_1 = \delta^{\ell_{+,\gamma}}_{\text{out}}(A^{\hat p}) \sqcap \text{sp}(\Gamma) \tag{10.5}$$

$$G_2 = G^{\hat p} \setminus G_1 \tag{10.6}$$

and write for all $\vec\rho$ in $\mathcal{R}(\Gamma)$

$$F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{(\ell_0)}_{A^{\hat p}}) = F_{\gamma,\beta,G_1}(\vec\rho_{G_1}|\vec\xi^{(\ell_0)}_{A^{\hat p}}) + F_{\gamma,\beta,G_2}(\vec\rho_{G_2}|\vec\rho_{G_1}) \tag{10.7}$$

We first apply corollary B.5 with $\Lambda = G_1 \setminus \delta^{\gamma^{-1}}_{\text{in}}(G_1)$, $\rho^* = \rho$ and $u = 0$. Thus there are positive constants ω and c_ω and there exists $\psi \in \mathcal{R}(\Gamma)$ such that

$$F_{\gamma,\beta,G_1}(\vec\rho_{G_1}|\vec\xi^{(\ell_0)}_{A^{\hat p}}) \geq F_{\gamma,\beta,G_1}(\vec\psi_{G_1}|\vec\xi^{(\ell_0)}_{A^{\hat p}}) - c_\omega |G_1| e^{-\omega\ell_{+,\gamma}/4} \tag{10.8}$$

with

$$\vec\psi_{G_1} = \begin{cases} \vec\rho_{G_1} & \text{on } \delta^{\gamma^{-1}}_{\text{in}}(G_1), \\ \vec\rho^{\hat p} & \text{on } G_1 \setminus \delta^{\ell_{+,\gamma}/4}_{\text{in}}(G_1) \end{cases} \tag{10.9}$$

and

$$F_{\gamma,\beta,G_2}(\vec\rho_{G_2}|\vec\rho_{G_1}) = F_{\gamma,\beta,G_2}(\vec\rho_{G_2}|\vec\psi_{G_1}) \tag{10.10}$$

Thus we have

$$F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{(\ell_0)}_{A^{\hat p}}) \geq F_{\gamma,\beta,G^{\hat p}}(\vec\psi|\vec\xi^{(\ell_0)}_{A^{\hat p}}) - c_\omega |G_1| e^{-\omega\ell_{+,\gamma}/4} \tag{10.11}$$

For $i \in I_a$, we write $C_i = C^{\ell_-,\gamma}_{x_i}$ as a shorthand notation, and define the function $\vec\psi^0$ on $G^{\hat p}$ as

$$\vec\psi^0(r) = \begin{cases} \dfrac{1}{|C_i|} \displaystyle\int_{C_r^{\ell_-,\gamma}} \vec\psi(r')dr' & \text{if } r \in B_a, \\ \psi(r) & \text{otherwise.} \end{cases} \tag{10.12}$$

From the definition of a contour, it follows that $\mathrm{dist}(B_a, \mathrm{sp}(\Gamma)^c) \geq \gamma^{-1}$. Thus the energy terms in the free energies of $\vec{\psi}$ and $\vec{\psi}^0$ differ by the quantity

$$
\begin{aligned}
&|U_{\gamma,G^{\hat{p}}}(\vec{\psi}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) - U_{\gamma,G^{\hat{p}}}(\vec{\psi}^0|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}})| \\
&= \frac{1}{2}\Big|\int_{\mathrm{sp}(\Gamma)\times\mathrm{sp}(\Gamma)} J_\gamma(x,y)\vec{\psi}(x)\cdot\vec{\psi}(y)dxdy - \int_{\mathrm{sp}(\Gamma)\times\mathrm{sp}(\Gamma)} J_\gamma(x,y)\vec{\psi}^0(x)\cdot\vec{\psi}^0(y)dxdy\Big| \\
&= \frac{1}{2}\Big|\int_{\mathrm{sp}(\Gamma)\times\mathrm{sp}(\Gamma)} J_\gamma(x,y)(\vec{\psi}(x)+\vec{\psi}^0(x))\cdot(\vec{\psi}(y)-\vec{\psi}^0(y))dxdy\Big| \\
&= \frac{1}{2}\sum_{i\in I_a}\Big|\frac{1}{|C_i|}\int_{\mathrm{sp}(\Gamma)\times C_i} dxdyJ_\gamma(x,y)(\vec{\psi}(x)+\vec{\psi}^0(x))\cdot\int_{C_i} dy'(\vec{\psi}(y)-\vec{\psi}(y'))\Big| \\
&= \frac{1}{2}\sum_{i\in I_a}\Big|\frac{1}{|C_i|}\int_{\mathrm{sp}(\Gamma)\times C_i} dxdy(\vec{\psi}(x)+\vec{\psi}^0(x))\cdot\vec{\psi}(y)\int_{C_i} dy'(J_\gamma(x,y)-J_\gamma(x,y'))\Big| \\
&\leq \frac{1}{2}\sum_{i\in I_a}\frac{1}{|C_i|}\int_{\mathrm{sp}(\Gamma)\times C_i} dxdy|(\vec{\psi}(x)+\vec{\psi}^0(x))\cdot\vec{\psi}(y)|\int_{C_i} dy'|J_\gamma(x,y)-J_\gamma(x,y')| \\
&\leq 2^d\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_a|
\end{aligned}
$$

where in the last inequality we used

$$
|(\vec{\psi}(x)+\vec{\psi}^0(x))\cdot\vec{\psi}(y)| \leq 2 \tag{10.13}
$$

and

$$
|J_\gamma(x,y)-J_\gamma(x,y')| \leq \sqrt{d}\ell_{-,\gamma}\gamma^{d+1}\|\nabla\mathcal{J}\|_\infty\mathbf{1}_{\{|x-y|\leq 2\gamma^{-1}\}} \tag{10.14}
$$

that for all y, y' in C_i. By concavity of the entropy, $I(\vec{\psi}) \leq I(\vec{\psi}^0)$ and we get

$$
F_{\gamma,\beta,G^{\hat{p}}}(\vec{\psi}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) \geq F_{\gamma,\beta,G^{\hat{p}}}(\vec{\psi}^0|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) - 2^d\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_a| \tag{10.15}
$$

We look for a lower bound of the free energy of $\vec{\psi}^0$. We again divide $G^{\hat{p}}$ in two parts $G^{\hat{p}} = G'_1 \sqcup G'_2$ where

$$
G'_1 = \delta^{\ell_{+,\gamma}/2}_{\mathrm{out}}(A^{\hat{p}}) \sqcap \mathrm{sp}(\Gamma) \tag{10.16}
$$
$$
G'_2 = G \setminus G'_1 \tag{10.17}
$$

We have

$$
F_{\gamma,\beta,G^{\hat{p}}}(\vec{\psi}^0|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) = F_{\gamma,\beta,G'_1}(\vec{\psi}^0_{G'_1}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) + F_{\gamma,\beta,G'_2}(\vec{\psi}^0_{G'_2}|\vec{\psi}^0_{G'_1}) \tag{10.18}
$$

We write the second term as:

$$
\begin{aligned}
F_{\gamma,\beta,G'_2}(\vec{\psi}^0_{G'_2}|\vec{\psi}^0_{G'_1}) &= \int_{G'_2} dx\phi^{mf}_\beta(\vec{\psi}^0(x)) + \frac{1}{4}\int_{G'_2\times G'_2} dxdyJ_\gamma(x,y)|\vec{\psi}^0(x)-\vec{\psi}^0(y)|^2 \\
&- \int_{G'_2\times G'_1} dxdyJ_\gamma(x,y)(\vec{\psi}^0(x)\cdot\vec{\psi}^0(y)) + \frac{1}{2}\int_{G'_2\times G'^c_2} dxdyJ_\gamma(x,y)|\vec{\psi}^0(x)|^2
\end{aligned} \tag{10.19}
$$

The last two terms can be calculated since, by (10.9), $\vec{\psi}^0$ is constant and equal to $\vec{\rho}^{\hat{p}}$ on both $\delta_{\rm out}^{\gamma^{-1}}(G'_1) \sqcap G'_2$ and $G'_1 \sqcap \delta_{\rm out}^{\gamma^{-1}}(G'_2)$, and equal to $\vec{\rho}^{\hat{q}}$ on $A^{\hat{q}}$, $\hat{q} \neq \hat{p}$. We get:

$$F_{\gamma,\beta,G'_2}(\vec{\psi}^0_{G'_2}|\vec{\psi}^0_{G'_1}) = \int_{G'_2} dx \phi_\beta^{\rm mf}(\vec{\psi}^0(x)) + \frac{1}{4}\int_{G'_2\times G'_2} dxdy J_\gamma(x,y)|\vec{\psi}^0(x) - \vec{\psi}^0(y)|^2$$
$$+\frac{1}{2}\sum_{\hat{q}\neq\hat{p}}\int_{G'_2\times({\rm Int}^{\hat{q}}(\Gamma)\setminus A^{\hat{q}})} dxdy J_\gamma(x,y)|\vec{\rho}^{\hat{q}}|^2 - \frac{1}{2}\int_{G'_2\times G'_1} dxdy J_\gamma(x,y)|\vec{\rho}^{\hat{p}}|^2$$

Since $B_b \sqsubset G'_2$, the second term can be bounded by the contributions of the b-events. We have the following estimate:

$$\frac{1}{4}\int_{G'_2\times G'_2} dxdy J_\gamma(x,y)|\vec{\psi}^0(x) - \vec{\psi}^0(y)|^2$$
$$\geq \frac{1}{4}\sum_{j\in I_b}\Big(\int_{G'_2\times C_j} dxdy J_\gamma(x,y)|\vec{\psi}^0(x) - \vec{\psi}^0(y)|^2 + \int_{G'_2\times C'_j} dxdy' J_\gamma(x,y')|\vec{\psi}^0(x) - \vec{\psi}^0(y')|^2\Big)$$
$$= \frac{1}{4}\sum_{j\in I_b}\frac{1}{|C_j|}\int_{G'_2} dx \int_{C_j\times C'_j} dydy'\big(J_\gamma(x,y)|\vec{\psi}^0(x) - \vec{\psi}^0(y)|^2 + J_\gamma(x,y')|\vec{\psi}^0(x) - \vec{\psi}^0(y')|^2\big)$$

where we used the notations $C_j = C^{\ell_-,\gamma}_{x_1^j}$, $C'_j = C^{\ell_-,\gamma}_{x_2^j}$ for all $j \in I_b$. Using now the fact that for all (y, y') in $C_j \times C'_j$,

$$\big|J_\gamma(x,y) - J_\gamma(x,y')\big| \leq 2\sqrt{d}\ell_{-,\gamma}\gamma^{d+1}\|\nabla\mathcal{J}\|_\infty \mathbf{1}_{\{|x-y|\leq 2\gamma^{-1}\}} \tag{10.20}$$

we get

$$\frac{1}{4}\int_{G'_2\times G'_2} dxdy J_\gamma(x,y)|\vec{\psi}^0(x) - \vec{\psi}^0(y)|^2 \tag{10.21}$$
$$\geq \frac{1}{4}\sum_{j\in I_b}\int_{G'_2} dx \frac{1}{2|C_j|}\cdot \tag{10.22}$$
$$\int_{C_j\times C'_j} dydy'(J_\gamma(x,y) + J_\gamma(x,y'))\big(|\vec{\psi}^0(x) - \vec{\psi}^0(y)|^2 + |\vec{\psi}^0(x) - \vec{\psi}^0(y')|^2\big)$$
$$-2^{d-1}\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_b|$$

Now using successively the inequality

$$|a-b|^2 + |a-c|^2 \geq \frac{1}{2}|b-c|^2 \tag{10.23}$$

and $\mathrm{dist}(B_b, G'^c_2) \geq \gamma^{-1}$, we can sum over the x variable to obtain

$$\frac{1}{4}\int_{G'_2\times G'_2} dxdy J_\gamma(x,y)|\vec{\psi}^0(x) - \vec{\psi}^0(y)|^2 \tag{10.24}$$
$$\geq \frac{1}{8}\sum_{j\in I_b}\int_{G'_2} dx \frac{1}{2|C_j|}\int_{C_j\times C'_j} dydy'(J_\gamma(x,y) + J_\gamma(x,y'))|\vec{\psi}^0(y) - \vec{\psi}^0(y')|^2$$
$$-2^{d-1}\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_b|$$
$$\geq \frac{1}{8}\sum_{j\in I_b}\frac{1}{|C_j|}\int_{C_j\times C'_j} dydy'|\vec{\psi}^0(y) - \vec{\psi}^0(y')|^2 - 2^{d-1}\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_b|$$

We finally get

$$\begin{aligned}&\frac{1}{4}\int_{G_2'\times G_2'} dxdy J_\gamma(x,y)|\vec{\psi}^0(x)-\vec{\psi}^0(y)|^2\\&\geq \frac{1}{8}\sum_{j\in I_b}\frac{1}{|C_j|}\Big|\int_{C_j} dy\vec{\psi}^0(y)-\int_{C_j'} dy'\vec{\psi}^0(y')\Big|^2 - 2^{d-1}\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_b|\end{aligned}$$

Now collecting all estimates (10.11),(10.15) and (10.25), we get for all $\vec{\rho}$ in $\mathcal{R}(\Gamma)$:

$$\begin{aligned}&F_{\gamma,\beta,G^{\hat{p}}}(\vec{\rho}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) \geq F_{\gamma,\beta,G_1'}(\vec{\psi}^0_{G_1'}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) + \sum_{\hat{q}\neq\hat{p}}\mathcal{I}_{A^{\hat{q}}}|\vec{\rho}^q|^2 - \frac{1}{2}\int_{G_2'\times G_1'} dxdy J_\gamma(x,y)|\vec{\rho}^{\hat{p}}|^2\\&+\int_{G_2'} dx\phi^{\mathrm{mf}}_\beta(\vec{\psi}^0(x)) + \frac{1}{8}\sum_{j\in I_b}\frac{1}{|C_j|}\Big|\int_{C_j} dy'\vec{\psi}^0(y')-\int_{C_j'} dy''\vec{\psi}^0(y'')\Big|^2\\&-c_\omega|G_1|e^{-\omega\ell_{+,\gamma}/4} - 2^d\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_a| - 2^{d-1}\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_b|\end{aligned}$$

Using the definitions of B_a and B_b and the results of appendix A, we have for all β such that $\beta-\beta_c^{\mathrm{mf}}(Q)\leq c_b\gamma^{\frac{1}{2}}$,

$$\phi^{mf}_\beta(\vec{\psi}^0(x)) \geq \phi^{mf}_\beta(\vec{\rho}^{\hat{p}}) + (\frac{Q}{\beta}-1)\gamma^{2a} - c\gamma^{\frac{1}{2}} \tag{10.25}$$

for all x in B_a and

$$\frac{1}{|C_j|^2}\Big|\int_{C_j} dy\vec{\psi}^0(y)-\int_{C_j'} dy'\vec{\psi}^0(y')\Big|^2 \geq (\frac{Q-1}{Q}-2\gamma^a)^2 \tag{10.26}$$

on all j in I_b Hence

$$\begin{aligned}&F_{\gamma,\beta,G^{\hat{p}}}(\vec{\rho}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) \geq F_{\gamma,\beta,G_1'}(\vec{\psi}|\vec{\xi}^{(\ell_0)}_{A^{\hat{p}}}) + |G_2'|\phi^{\mathrm{mf}}_\beta(\vec{\rho}^{\hat{p}}) + \sum_{\hat{q}\neq\hat{p}}\mathcal{I}_{A^{\hat{q}}}|\vec{\rho}^q|^2 - \frac{1}{2}\int_{G_2'\times G_1'} dxdy J_\gamma(x,y)|\vec{\rho}^{\hat{p}}|^2\\&+(\frac{Q}{\beta}-1)\gamma^{2a}|B_a| + \frac{1}{4}(\frac{Q-1}{2Q}-\gamma^a)^2|B_b|\\&-c_\omega|G_1|e^{-\omega\ell_{+,\gamma}/4} - 2^d\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_a| - c\gamma^{\frac{1}{2}} - 2^{d-1}\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty|B_b|\end{aligned}$$

Now for $a<\min(\frac{1}{4},\frac{\alpha}{2})$, we may choose γ so that the last three error terms are a fraction of the respective gain terms. In addition, since we have both $|B_a|+|B_b|\geq\frac{1}{3^d}\big(\frac{\ell_{-,\gamma}}{\ell_{+,\gamma}}\big)^d|\Gamma|$ and $G_1\sqsubset\Gamma$, the remaining error term can be also compensated for γ small enough. We define $\bar{\gamma}$ as the largest value of γ such that the following inequalities hold simultaneously

$$\begin{aligned}&2^{d-1}\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty \leq \frac{1}{3}(\frac{Q-1}{2Q}-\gamma^a)^2\\&2^d\sqrt{d}\ell_{-,\gamma}\gamma\|\nabla\mathcal{J}\|_\infty \leq \frac{1}{3}(\frac{Q}{\beta}-1)\gamma^{2a}\\&c_\omega e^{-\omega\ell_{+,\gamma}/4} \leq \frac{1}{3}(\frac{Q}{\beta}-1)\gamma^{2a}\\&\frac{1}{4}(\frac{Q-1}{2Q}-\gamma^a)^2 \geq \frac{1}{3}(\frac{Q}{\beta}-1)\gamma^{2a}\end{aligned}$$

For all $\gamma \le \bar{\gamma}$, we have

$$F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{(\ell_0)}_{A^{\hat p}}) \ge F_{\gamma,\beta,G'_1}(\vec\psi|\vec\xi^{(\ell_0)}_{A^{\hat p}}) + |G'_2|\phi^{mf}_\beta(\vec\rho^{\hat p}) - \frac{1}{2}\int_{G'_2\times G'_1} dxdy J_\gamma(x,y)|\vec\rho^{\hat p}|^2$$
$$+\sum_{\hat q\ne\hat p}\mathcal{I}_{A^{\hat q}}|\vec\rho^{q}|^2 + \frac{1}{3^{d+1}}(\frac{Q}{\beta}-1)\gamma^{2a}(\frac{\ell_{-,\gamma}}{\ell_{+,\gamma}})^d|\Gamma| \tag{10.27}$$

Now consider the function $\vec\varphi$ in $L^\infty(G^{\hat p}, S_Q)$ defined as

$$\vec\varphi(r) = \begin{cases} \vec\psi(r) & r\in G'_1, \\ \vec\rho^{\hat p} & r\in G'_2. \end{cases} \tag{10.28}$$

$\vec\varphi(r)$ belongs clearly to $\mathcal{R}(\not\Gamma)$, and since $\vec\varphi(r) = \vec\rho^{\hat p}$ on $G'_1 \sqcap \delta^{\gamma^{-1}}_{\text{out}}(G'_2)$, its free energy reads

$$F_{\gamma,\beta,G^{\hat p}}(\vec\varphi|\vec\xi^{(\ell_0)}_{A^{\hat p}}) = F_{\gamma,\beta,G'_1}(\vec\varphi|\vec\xi^{(\ell_0)}_{A^{\hat p}})$$
$$+|G'_2|\phi^{mf}_\beta(\vec\rho^{\hat p}) - \frac{1}{2}\int_{G'_2\times G'_1} dxdy J_\gamma(x,y)|\vec\rho^{\hat p}|^2 + \sum_{\hat q\ne\hat p}\mathcal{I}_{A^{\hat q}}|\vec\rho^{p}|^2$$

Hence we have for all $\vec\rho$ in $\mathcal{R}(\Gamma)$,

$$F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{(\ell_0)}_{A^{\hat p}}) \ge F_{\gamma,\beta,G^{\hat p}}(\vec\varphi|\vec\xi^{(\ell_0)}_{A^{\hat p}})$$
$$+\sum_{\hat q\ne\hat p}\mathcal{I}_{A^{\hat q}}(|\vec\rho^{\hat q}|^2 - |\vec\rho^{\hat p}|^2) + \frac{1}{3^{d+1}}(\frac{Q}{\beta}-1)\gamma^{2a}(\frac{\ell_{-,\gamma}}{\ell_{+,\gamma}})^d|\Gamma| \tag{10.29}$$

and since $\vec\varphi$ is in $\mathcal{R}(\not\Gamma)$,

$$F_{\gamma,\beta,G^{\hat p}}(\vec\varphi|\vec\xi^{(\ell_0)}_{A^{\hat p}}) \ge \inf_{\rho\in\mathcal{R}(\not\Gamma)} F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{(\ell_0)}_{A^{\hat p}}) \tag{10.30}$$

Taking now the infimum over $\mathcal{R}(\Gamma)$ in (10.29), we get for all $\gamma\le\bar\gamma$,

$$\inf_{\vec\rho\in\mathcal{R}(\Gamma)} F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{(\ell_0)}_{A^{\hat p}}) \ge \inf_{\rho\in\mathcal{R}(\not\Gamma)} F_{\gamma,\beta,G^{\hat p}}(\vec\rho|\vec\xi^{(\ell_0)}_{A^{\hat p}})$$
$$+\sum_{\hat q\ne\hat p}\mathcal{I}_{A^{\hat q}}(|\vec\rho^{\hat q}|^2 - |\vec\rho^{\hat p}|^2) + c_f\gamma^{2a}(\frac{\ell_{-,\gamma}}{\ell_{+,\gamma}})^d|\Gamma| \tag{10.31}$$

with

$$c_f = \frac{1}{3^{d+1}}(\frac{Q}{\beta}-1) \tag{10.32}$$

□

Acknowledgments

We thank Marzio Cassandro, Roberto Fernandez, Joel Lebowitz and Errico Presutti for many helpful discussions. T.G. thanks the warm hospitality of the Mathematics departments of Universities of Rome 2 and l'Aquila where most of the work has been done.

Appendix A. Mean Field Model

In this appendix, we review briefly the mean field theory of the Potts model [**45**], and derive the various quantities needed in the rest of the paper.

We consider a Q-state Potts model defined on a complete graph with N sites and derive its behavior in the large N limit. A variable σ_i, $\sigma_i \in [1, \cdots, Q]$, is attached to each site i of the graph so that the space of configurations is $\Omega_N = [1, \cdots, Q]^N$. The mean field Hamiltonian on Ω_N is:

$$H^{\mathrm{mf}}(\sigma) := -\frac{1}{2N} \sum_{i \neq j} \mathbf{1}_{\{\sigma_i = \sigma_j\}}$$

H^{mf} is invariant under any permutation of sites so that its value on a given configuration depends only on the number of sites with color q, says N_q, $q \in [1, \cdots, Q]$. The partition function of the model is thus:

$$Z_{N,\beta} = \sum_{\{N_q\}: \sum N_q = N} \frac{N!}{\prod_q N_q!} e^{\frac{\beta}{2N} \sum_q N_q (N_q - 1)} \tag{A.1}$$

For N large, $Z_{N,\beta}$ is dominated by the configurations which realize the minimum of the free energy density ϕ_β^{mf} defined on $S_Q = \{\vec{\rho} \in \mathbb{R}_+^Q, \sum_q \rho_q = 1\}$ as

$$\phi_\beta^{\mathrm{mf}}(\vec{\rho}) = -\frac{1}{2} \sum_q \rho_q^2 + \frac{1}{\beta} \sum_q \rho_q \ln(\rho_q) \tag{A.2}$$

with the correspondence $\rho_q = \frac{N_q}{N}$. In fact, for our present purpose, we are also interested in all local minimizers of ϕ_β^{mf}, which appear to be of two kinds: one "disordered" (or uniform) state in which all colors have the same density and Q degenerated "ordered" (or colored) states in which one color dominates. As can be expected, the first one exists for small values of β, while the other Q exist only for β large enough. In addition, there is a critical value of β which determines which kind of local minimizer is the actual absolute minimizer for ϕ_β^{mf}. We make these statements precise in the two following theorems. We first characterize all local minimizers:

Theorem A.1. *For all $Q > 2$, there exists $\beta_0 < Q$ such that the mean field free energy ϕ_β^{mf} (A.2) on S_Q has exactly:*

- *1 local minimizer for $\beta \leq \beta_0$,*
- *$Q + 1$ local minimizers for $\beta_0 < \beta < Q$, and*
- *Q local minimizers for $\beta \geq Q$.*

These minimizers are of two kinds and characterized as follows:

- *For all $\beta < Q$, the uniform state $\vec{\rho}^{(-1)}$, with components*

$$\rho_q^{(-1)} = \frac{1}{Q} \quad \textit{for all } q \in \{1, \cdots, Q\} \tag{A.3}$$

and free energy

$$\phi_\beta^{\mathrm{mf}}(\vec{\rho}^{(-1)}) = \frac{-1}{2Q} - \frac{1}{\beta}\log(Q) \tag{A.4}$$

- *For all $\beta > \beta_0$, Q colored states $\vec{\rho}^{(p)} \equiv \vec{\rho}_\beta^{(p)}$, $p \in \{1, \cdots, Q\}$, with components*

$$\rho_q^{(p)} = \begin{cases} \rho_A & \text{if } q = p, \\ \rho_B & \text{otherwise.} \end{cases} \tag{A.5}$$

where (ρ_A, ρ_B), $\rho_A > \rho_B$, is the solution of the set of equations

$$\frac{\log(\rho_A) - \log(\rho_B)}{\rho_A - \rho_B} = \beta \tag{A.6}$$

$$\rho_A + (Q-1)\rho_B = 1 \tag{A.7}$$

which verify

$$Q\beta\rho_A\rho_B < 1 \tag{A.8}$$

These Q states are degenerate and have free energy

$$\phi_\beta^{\mathrm{mf}}(\vec{\rho}^{(p)}) = \frac{-1}{2}Q\rho_A\rho_B + \frac{1}{2\beta}\log(\rho_A\rho_B) \tag{A.9}$$

The mean field first order transition is described in the following

Theorem A.2. *For all $Q > 2$, there exists a critical value of β, in (β_0, Q),*

$$\beta_c^{\mathrm{mf}}(Q) \equiv \frac{2(Q-1)}{Q-2}\log(Q-1) \tag{A.10}$$

such that ϕ_β^{mf} has:

- *1 minimizer $\vec{\rho}^{(-1)}$ for all $\beta < \beta_c^{\mathrm{mf}}(Q)$;*
- *$Q+1$ minimizers $\vec{\rho}^{(\hat{p})}$, $\hat{p} \in \{-1, 1, \cdots, Q\}$ for $\beta = \beta_c^{\mathrm{mf}}(Q)$;*
- *Q minimizers $\vec{\rho}^{(p)}$, $p \in \{1, \cdots, Q\}$ for $\beta > \beta_c^{\mathrm{mf}}(Q)$.*

Finally, the relevance of local minimizers in our problem arises from their local stability, which is stated in the following theorem:

Theorem A.3. *For all $Q > 2$ and all β in (β_0, Q), the map*

$$\vec{\rho} \to \vec{g}(\vec{\rho}) \tag{A.11}$$

where

$$g_q(\vec{\rho}) = \frac{\exp(\beta\rho_q)}{\sum_{p=1}^{Q}\exp(\beta\rho_p)} \tag{A.12}$$

is a contraction around the $Q+1$ local minimizers $\vec{\rho}^{(\hat{p})}$ of the mean field free energy ϕ_β^{mf}. In particular,

$$\sup_{q,q'} \left|\frac{\partial g_q(\vec{\rho})}{\partial \rho_{q'}}\right| \leq 1 - \frac{1}{2Q} \tag{A.13}$$

for all $\vec{\rho}$ such that $\sup_q |\rho_q - \rho_q^{\hat{p}}| \leq \frac{1}{4\beta^2 Q^2}$ for some $\hat{p}$.

PROOF. **of Theorem A.1**:
We consider the variational problem for $\phi_\beta^{\mathrm{mf}}(\vec\rho)$.

$$\inf_{\vec\rho:\sum_{q=1}^{Q}\rho_q=1}\phi_\beta^{\mathrm{mf}}(\vec\rho)=\min_{\vec\rho,\lambda}{}^{\mathrm{loc}}\left[\phi_\beta^{\mathrm{mf}}(\vec\rho)+\lambda\left(\sum_{j=q}^{Q}\rho_q-1\right)\right]$$

where λ is a Lagrange parameter associated to the constraint $\sum_q\rho_q=1$. Since the gradient of the free energy points inward the simplex S_Q, the (local) minima cannot stay on the boundary of S_Q and are thus solutions of the set of equations:

$$\frac{\partial\phi_\beta^{\mathrm{mf}}(\vec\rho)}{\partial\rho_q}+\lambda=0,\qquad q=1,\cdots,Q \tag{A.14}$$

together with the condition

$$\sum_{q1,q2=1}^{Q}\frac{\partial^2\phi_\beta^{\mathrm{mf}}(\vec\rho)}{\partial\rho_{q_1}\rho_{q_2}}x_{q_1}x_{q_2}\geq 0 \tag{A.15}$$

for all $\vec x\in\mathbb{R}^Q$ such that $\sum_{q=1}^{Q}x_q=0$.

Explicitly the first derivatives of the free energy read

$$\frac{\partial\phi_\beta^{\mathrm{mf}}(\vec\rho)}{\partial\rho_q}\equiv 1-\rho_q+\frac{1}{\beta}(\ln\rho_q+1) \tag{A.16}$$

while the Hessian matrix of $\phi_\beta^{\mathrm{mf}}(\vec\rho)$ is diagonal and

$$\frac{\partial^2\phi_\beta^{\mathrm{mf}}(\vec\rho)}{\partial\rho_{q_1}\rho_{q_2}}=\delta_{q_1,q_2}\big(-1+\frac{1}{\beta\rho_{q_1}}\big) \tag{A.17}$$

As a function of ρ_q alone, $\frac{\partial\phi_\beta^{\mathrm{mf}}(\vec\rho)}{\partial\rho_q}$ is a strictly concave C^∞ function and hence cannot take the same value more than twice. Thus there are two kind of solutions for (A.14), depending on whether ρ_q takes one or two values.

The first case correspond to a "disordered" solution $\vec\rho^{(-1)}$ in which each color has the same density:

$$\rho_q^{(-1)} = \frac{1}{Q}\qquad\text{for all } q=1,\cdots,Q \tag{A.18}$$

$$\phi_\beta^{\mathrm{mf}}(\vec\rho^{(-1)}) = \frac{-1}{2Q}-\frac{1}{\beta}\ln(Q) \tag{A.19}$$

Using (A.17), $\vec\rho^{(-1)}$ is a local minimum of $\phi_\beta^{\mathrm{mf}}(\vec\rho)$ if and only if $\beta<Q$

In the second case, let $\vec\rho$ a vector in S_Q which components takes two values, says ρ_A and ρ_B with $\rho_A>\rho_B$, and let n, $0<n<Q$, the number of components equal to ρ_A (the remaining $Q-n$ are thus equal to ρ_B). According to (A.14), $\vec\rho$ is a critical point for $\phi_\beta^{\mathrm{mf}}(\vec\rho)$ in S_Q whenever the constraint is satisfied and both ρ_A and ρ_B are associated to the same

value of the Lagrange parameter:

$$n\rho_A + (Q-n)\rho_B = 1 \tag{A.20}$$

$$\lambda = \rho_A - 1 - \frac{1}{\beta}(\ln\rho_A + 1) = \rho_B - 1 - \frac{1}{\beta}(\ln\rho_B + 1) \tag{A.21}$$

However, all such points are not local minima: equation (A.21) implies the relation (A.6) between β, ρ_A and ρ_B,

$$\beta = \frac{\ln\rho_A - \ln\rho_B}{\rho_A - \rho_B} \tag{A.22}$$

and by concavity of the logarithmic function, we have necessarily

$$\frac{1}{\rho_A} < \beta < \frac{1}{\rho_B} \tag{A.23}$$

Thus the second derivative of $\phi_\beta^{\mathrm{mf}}(\vec{\rho})$ is negative in each direction in which $\rho_q = \rho_A$. This is obviously not compatible with the condition (A.15) for $n \geq 2$. In the case $n = 1$, the constraint (A.21) reduces to (A.7):

$$\rho_A + (Q-1)\rho_B = 1 \tag{A.24}$$

while the condition (A.15) can be made explicit using (A.17):

$$\big(-1 + \frac{1}{\beta\rho_A}\big) + \frac{1}{Q-1}\big(-1 + \frac{1}{\beta\rho_B}\big) \geq 0$$

or equivalently using the constraint (A.24), one gets (A.8):

$$\beta Q\rho_A\rho_B \leq 1 \tag{A.25}$$

We need to find out all triples (β, ρ_A, ρ_B) which solve simultaneously (A.22), (A.24) and (A.25). For ρ_A in $(\frac{1}{Q}, 1)$, we consider the function $\tilde{\beta}(\rho_A)$ as

$$\tilde{\beta}(\rho_A) = \frac{\ln\rho_A - \ln\rho_B}{\rho_A - \rho_B} \tag{A.26}$$

where ρ_B is taken implicitly as a function of ρ_A, through (A.24) Its first derivative reads

$$\frac{\partial\tilde{\beta}(\rho_A)}{\partial\rho_A} = \frac{1}{(Q-1)(\rho_A - \rho_B)}\big[\frac{1}{\rho_A\rho_B} - \beta Q\big] \tag{A.27}$$

while the second derivative can be cast in the form

$$\frac{\partial^2\tilde{\beta}(\rho_A)}{\partial\rho_A^2} = \frac{\varphi(\rho_A) - \varphi(\rho_B)}{(Q-1)^2(\rho_A - \rho_B)^3} \tag{A.28}$$

where

$$\varphi(\rho) = 2Q^2\log(\rho) + \frac{4Q}{\rho} - \frac{1}{\rho^2} \tag{A.29}$$

One can check that the function $\varphi(\rho)$ is strictly increasing and thus the second derivative of $\tilde{\beta}(\rho_A)$ is always positive. It follows that there exists a unique value ρ_0 in $(\frac{1}{Q}, 1)$ so that the first derivative is zero in ρ_0, strictly negative for $\rho < \rho_0$ and strictly positive for $\rho > \rho_0$. We define $\beta_0 \equiv \tilde{\beta}(\rho_0)$. Since equation (A.22) is equivalent to $\beta = \tilde{\beta}(\rho_A)$, solutions to (A.22) will exist only for β in the image of $\tilde{\beta}(\cdot)$, and thus for $\beta \geq \beta_0$. Now the condition for a local minimum (A.25) is equivalent to $\frac{\partial\tilde{\beta}(\rho_A)}{\partial\rho_A} \geq 0$ and thus to $\rho_A \geq \rho_0$. Furthermore the function $\tilde{\beta}(\cdot)$ is invertible from $(\rho_0, 1)$ onto $(\beta_0, +\infty)$ and therefore, for all $\beta > \beta_0$ there is a unique couple (ρ_A, ρ_B) for which the vectors (A.5) are Q local minima, and there is no "colored" solutions for $\beta < \beta_0$. Whenever they exist, those minima are degenerate and their mean field free energy is given by (A.9). We postpone the proof that $\beta_0 < Q$ at the end of the proof of the next theorem. □

Proof. **of Theorem A.2** For β in (β_0, Q), we consider the difference of free energy between ordered and disordered local minima

$$\Delta(\phi_\beta^{\mathrm{mf}}) \equiv \phi_\beta^{\mathrm{mf}}(\vec{\rho}^{(p)}) - \phi_\beta^{\mathrm{mf}}(\vec{\rho}^{(-1)}) = \frac{1 - Q^2\rho_A\rho_B}{2Q} + \frac{1}{2\beta}\log(Q^2\rho_A\rho_B) \tag{A.30}$$

We first note that $\beta\Delta(\phi_\beta^{\mathrm{mf}})$ is a strictly decreasing function of β: using (A.26), we have for β in (β_0, Q),

$$\frac{\partial}{\partial\beta}\big(\beta\Delta(\phi_\beta^{\mathrm{mf}})\big) = \Big(\frac{\partial\tilde{\beta}}{\partial\rho_A}\Big)^{-1}\frac{\partial\tilde{\beta}\Delta(\phi_{\tilde{\beta}}^{\mathrm{mf}})}{\partial\rho_A} = -\frac{(1-Q\rho_A)^2}{2Q(Q-1)} < 0 \tag{A.31}$$

Furthermore, $\Delta(\phi_\beta^{\mathrm{mf}})$ has one zero for $Q^2\rho_A\rho_B = 1$ or equivalently

$$\rho_A = \frac{Q-1}{Q} \tag{A.32}$$

This zero is thus necessarily unique and defines a critical value of β

$$\beta_c^{\mathrm{mf}}(Q) = \tilde{\beta}\Big(\frac{Q-1}{Q}\Big) = \frac{2(Q-1)}{Q-2}\log(Q-1) \tag{A.33}$$

We complete the proof by showing that (A.25) holds at $\beta = \beta_c^{\mathrm{mf}}(Q)$. We have for all $Q > 2$:

$$Q\beta_c^{\mathrm{mf}}(Q)\rho_A\rho_B = \frac{2(Q-1)}{Q(Q-2)}\log(Q-1) = \frac{\log(Q-1)}{\sinh(\log(Q-1))} < 1 \tag{A.34}$$

Thus $\beta_c^{\mathrm{mf}}(Q) > \beta_0$. On the other hand, since $\rho_A\rho_B = Q^{-2}$ at $\beta = \beta_c^{\mathrm{mf}}(Q)$, (A.34) proves also that $\beta_c^{\mathrm{mf}}(Q) < Q$, and thus $\beta_0 < \beta_c^{\mathrm{mf}}(Q) < Q$. This also complete the proof of theorem A.1.

□

Proof. **of Theorem A.3**

Let β be in the interval (β_0, Q) and consider the map $\vec{\rho} \to \vec{g}(\vec{\rho})$ defined in (A.12). We have

$$\frac{\partial g_q(\vec{\rho})}{\partial\rho_{q'}} = \beta g_q(\vec{\rho})\big(\delta_{q,q'} - g_{q'}(\vec{\rho})\big) \tag{A.35}$$

and since $0 \le g_q(\vec{\rho}) \le 1$, we have a first bound uniform in $\vec{\rho}$:

$$\sup_{q,q'}\Big|\frac{\partial g_q}{\partial\rho_{q'}}(\vec{\rho})\Big| \le \beta \tag{A.36}$$

On the other hand, from (A.35), one can also write

$$\sup_{q,q'}\Big|\frac{\partial g_q}{\partial\rho_{q'}}(\vec{\rho})\Big| \le \sup_{q,q'}\Big|\frac{\partial g_q}{\partial\rho_{q'}}(\vec{\rho}^{\,p})\Big| + \sup_{q,q'}\Big|\frac{\partial g_q}{\partial\rho_{q'}}(\vec{\rho}) - \frac{\partial g_q}{\partial\rho_{q'}}(\vec{\rho}^{\,p})\Big| \tag{A.37}$$

The first term can be bounded by

$$\sup_{q,q'} |\frac{\partial g_q}{\partial \rho_{q'}}(\vec{\rho}^{\hat{p}})| = \beta \sup_{q,q'} |\rho_q^{\hat{p}}(\delta_{q,q'} - \rho_{q'}^{\hat{p}})| \leq 1 - \frac{1}{Q} \tag{A.38}$$

where the inequality follows from (A.3) and $\beta \leq Q$ for $\hat{p} = -1$, and from (A.8) and (A.7) for $\hat{p} > 0$. The second term in (A.37) can be bounded using (A.36) as

$$\sup_{q,q'} |\frac{\partial g_q}{\partial \rho_{q'}}(\vec{\rho}) - \frac{\partial g_q}{\partial \rho_{q'}}(\vec{\rho}^{\hat{p}})| \leq 2\beta \sup_q |g_q(\vec{\rho}) - g_q(\vec{\rho}^{\hat{p}})| \leq 2Q\beta^2 \sup_q |\rho_q - \rho_q^{\hat{p}}| \tag{A.39}$$

Thus for all ρ such that $\sup_q |\rho_q - \rho_q^{\hat{p}}| \leq \frac{1}{4\beta^2 Q^2}$ for some $\hat{p}$ one gets

$$\sup_{q,q'} |\frac{\partial g_q}{\partial \rho_{q'}}(\vec{\rho})| \leq 1 - \frac{1}{2Q} \tag{A.40}$$

□

We conclude this appendix by a proof of (8.4):
From equation (A.31) and the definition of $\beta_c^{\mathrm{mf}}(Q)$ (A.10), one gets explicitly:

$$\begin{aligned} \frac{d}{d\beta}\left[P^-_{\mathrm{mf},\beta} - P^+_{\mathrm{mf},\beta}\right]\Big|_{\beta=\beta_c^{\mathrm{mf}}(Q)} &= \frac{1}{\beta_c^{\mathrm{mf}}(Q)}\frac{\partial}{\partial\beta}\left(\beta\Delta(\phi_\beta^{\mathrm{mf}})\right)|_{\beta=\beta_c^{\mathrm{mf}}(Q)} \\ &= -\frac{(Q-2)^2}{2Q(Q-1)\beta_c^{\mathrm{mf}}(Q)} < 0 \end{aligned} \tag{A.41}$$

Appendix B. Local equilibrium

The main result of this appendix is the proof that, for suitable values of the temperature, if a density profile is in a neighborhood of an equilibrium value in a region $\Lambda \sqcup \delta_{\mathrm{out}}^{\gamma^{-1}}[\Lambda]$, then it can be made closer to equilibrium inside Λ at an exponential rate from its boundary, decreasing the free energy. This result is essentially due to the stability properties of the free energy functionals discussed in section 3, namely the contraction property of the map $\vec{\mathfrak{M}}$ around its constant fixed point. The precise result is stated in the Theorem B.1 below. The proof follows the lines developed in [**40**],(see also [**2**]) for Ising model and continuum particle models, and we will stress here only the points specific for our model while we will only sketch the points that are quite analogous to the other cases.

Without lost of generality we study the local equilibrium around the phase " $\hat{p}$ ", $\hat{p} \in \{-1, 1, \dots Q\}$. Let Λ a bounded $\mathcal{D}^{\ell_+,\gamma}$-measurable region, and $\eta_x^\zeta(\vec{\rho}) \in L^\infty(\mathbb{R}^d, S_Q)$ defined analogously as $\eta_x(\vec{\rho})$ in (6.1), but with an accuracy parameter, denoted by ζ, that here we leave free

$$\eta_x^\zeta(\vec{\rho}) = \begin{cases} a_{\hat{p}} & \text{if } \|\vec{\rho}^{\ell_-,\gamma}(x) - \vec{\rho}_\beta^{(\hat{p})}\|_\star < \zeta \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{M}^{\hat{p}}_{\Lambda,\zeta} := \{\vec{\rho} \in L^\infty(R^d, S_Q) : \eta^\zeta_x(\vec{\rho}) = a_{\hat{p}} \ \forall x \in \Lambda \sqcup \delta^{\gamma^{-1}}_{\text{out}}[\Lambda]\}$$

and for any $\vec{\rho}^{\,*} \in \mathcal{M}^{\hat{p}}_{\Lambda}$, we define

$$\mathcal{X}^{\hat{p}}_{\Lambda,\vec{\rho}^{\,*}} := \{\vec{\rho} \in M^{\hat{p}}_{\Lambda,\zeta} : \vec{\rho}_{\Lambda^c}(x) \equiv \vec{\rho}^{\,*}_{\Lambda^c}(x)\}$$

Theorem B.1. *There are positive constant ζ_0, ω, c_ω so that for any $\zeta < \zeta_0$, $\gamma^\alpha < \kappa_0\zeta$ and any $\vec{\rho}^{\,*} \in M^{\hat{p}}_{\Lambda,\zeta}$, s.t.*

- *there is a unique $\vec{\psi} \in \mathcal{X}_{\Lambda,\vec{\rho}^{\,*}}$ s.t.:*

$$\inf_{\vec{\rho} \in \mathcal{X}^{\hat{p}}_{\Lambda,\vec{\rho}^{\,*}}} \mathcal{F}_{\gamma,u,\Lambda}(\vec{\rho}|\vec{\rho}^{\,*}_{\Lambda^c}) = \mathcal{F}_{\gamma,u,\Lambda}(\vec{\psi}|\vec{\rho}^{\,*}_{\Lambda^c}) \tag{B.1}$$

- *$\vec{\psi}$ is the unique solution of the mean field equation and has the following properties:*
  - $\vec{\psi}_\Lambda \in C^\infty(\Lambda, M^{\hat{p}}_{\Lambda,(1-\kappa_0)\zeta})$, $\quad \sup_{r\in\Lambda} \|\nabla\psi_\Lambda(r)\|_\star \le \beta\|\nabla J_\gamma\|_1$
  - $$\|\vec{\psi}_\Lambda(r) - \vec{\rho}^{\hat{p}}\| \le c_\omega e^{-\omega\ \text{dist}(r,\Lambda^c_{\neq})}$$

    where $\Lambda^c_{\neq} := \{r \in \Lambda^c : \text{dist}(r,\Lambda) \le \gamma^{-1}; \vec{\rho}^{\,*}(r) \neq \vec{\rho}^{\hat{p}}\}$

- *If $\vec{\psi}, \vec{\phi}$ are minimizers resp. in $\mathcal{X}^{\hat{p}}_{\Lambda,\vec{\rho}_1}, \mathcal{X}^{\hat{p}}_{\Lambda,\vec{\rho}_2}$ then:*

$$\|\vec{\psi}_\Lambda(r) - \vec{\phi}_\Lambda(r)\| \le c_\omega e^{-\omega \text{dist}(r,\Lambda^c_{1,2,\neq})}$$

where $\Lambda^c_{1,2,\neq} := \{r \in \Lambda^c : \text{dist}(r,\Lambda) \le \gamma^{-1}; \vec{\rho}_1(r) \neq \vec{\rho}_2(r)\}$

The proof of Theorem B.1 is obtained by defining a dynamic $\vec{T}^{u,\Lambda,\hat{p}}_t$ on $L^\infty(\mathbb{R}^d, S_Q)$, and proving that this dynamic maps $M^{\hat{p}}_{\Lambda,\zeta}$ into itself and that it is dissipative for the free energy $\mathcal{F}_{\gamma,u,\Lambda}$. The minimizer $\vec{\psi}$ is then obtained as the limit point of the orbit $\vec{T}^{u,\Lambda,\hat{p}}_t$ as $t \to \infty$.

Following [**40**] we define an opportune dynamic $\vec{T}^{u,\Lambda,\hat{p}}_t$ (suitable for our model) that has the properties that allow to conclude as in reference [**40**].

The essential point in the proof of the Theorem B.1 is the contraction property of the map $\vec{\mathfrak{M}}(\cdot)$ discussed in section 3, that extends to the map $\vec{\mathfrak{M}}^{(u,\hat{p})}(\cdot)$ parameterized by u, $u \in [0,1]$, defined as follows:

$$\mathfrak{M}^{(u,\hat{p})}_q(\vec{\rho})(r) := \frac{e^{\beta\mathcal{L}^{u,\hat{p}}_q(\vec{\rho};r)}}{\sum_{q'} e^{\beta\mathcal{L}_{q'}(\vec{\rho};r)}}$$
$$\mathcal{L}^{u,\hat{p}}_q(\vec{\rho};r) := u\int dr'\ J_\gamma(r,r')\rho_q(r') + (1-u)\ \rho^{\hat{p}}_q$$

We state here a lemma which proof is postponed at the end of this appendix :

Lemma B.2. *There are ζ_0' and κ_0 positive, so that for any $\zeta < \zeta_0'$ and $\gamma^\alpha < \kappa_0 \zeta$, any bounded $\mathcal{D}^{\ell_-,\gamma}$-measurable region Λ, $r \in \Lambda$, $\vec\rho \in M^{\hat p}_{\Lambda,\zeta}$*

$$\sup_{r\in\Lambda} \|\vec{\mathcal{L}}^{u,\hat p}(\vec\rho)(r) - \vec\rho^{\,\hat p}(r)\|_\star \le u(1 + c_d\kappa_0)\zeta < 2\zeta \tag{B.2}$$

$$\sup_{r\in\Lambda} \|\vec{\mathfrak{M}}^{(u,\hat p)}(\vec\rho)(r) - \vec\rho^{\,\hat p}(r)\|_\star \le u(1 - \kappa_0)\zeta \tag{B.3}$$

We then define a dynamic given by the semigroup $\vec T^{u,\Lambda,\hat p}_t(\cdot)$, $t \ge 0$ on $L^\infty(\Lambda, S_Q)$,

$$\vec T^{u,\Lambda,\hat p}_t(\vec\rho) := \big(T^{u,\Lambda,\hat p}_{1,t}(\rho_1), \dots, T^{u,\Lambda,\hat p}_{Q,t}(\rho_Q)) \qquad \text{for any } \vec\rho \in L^\infty(\mathbb{R}^d, S_Q) \tag{B.4}$$

where $T^{u,\Lambda,\hat p}_{q,t}(\rho_i^*)$ are solutions of the Cauchy problem:

$$\begin{cases} \dfrac{d\rho_q^\Lambda(r,t)}{dt} = -\rho_q^\Lambda(r,t) + \mathfrak{M}_q^{(u,\hat p)}(\vec\rho^{\,\Lambda}) & \text{for any } q = 1,\dots,Q \\ \vec\rho^{\,\Lambda}(r,t) = \vec\rho^{\,*}(r,t) & (r,t) \in [\Lambda^c \times \{t \ge 0\}] \sqcup [\mathbb{R}^d \times \{t = 0\}] \end{cases} \tag{B.5}$$

Existence, uniqueness and continuity w.r.t. the initial datum of the solution follows by the continuity and the Lipschitz property of the r.h.s. of equation (B.5)

Notice that $\vec T^{u,\Lambda}_t(\cdot)$ maps $L^\infty(\mathbb{R}^d, S_Q)$ in itself, and has as a fixed point $\vec\rho^{\hat p}$. We next prove the following properties:

(1) $\vec T^{u,\Lambda,\hat p}_t(M^{\hat p}_{\Lambda,\zeta}) \sqsubset M^{\hat p}_{\Lambda,\zeta}$ for any $t \ge 0$

(2) For any $\vec\rho_0 \in L^\infty(\mathbb{R}^d, S_Q)$ and Λ a Borel set, $T^{u,\Lambda,\hat p}_{q,t}(\vec\rho_0)$, $q = 1,\dots,Q$, converges by subsequences as $t \to \infty$ to functions v_q that are bounded in Λ and with ∇v_q bounded in Λ. The limit points are solutions of (B.6) below.

(3) $\mathcal{F}_{\gamma,u,\Lambda}(\vec T^{u,\Lambda,\hat p}_t(\vec\rho_0))$ decrease with t, strictly unless $\vec\rho_0$ is stationary, in which case satisfies:

$$\rho_{0,q} = \frac{e^{\beta\mathcal{L}^u_q(\vec\rho_0(r,t))}}{\sum_{q'=1}^{Q} e^{\beta\mathcal{L}^u_{q'}(\vec\rho_0(r,t))}} \qquad \forall q = 1,\dots Q\ \forall r \in \Lambda \tag{B.6}$$

(4) As a consequence of the property (3), for any $\vec\rho^{\,*}(r) \in M^{\hat p}_{\Lambda,\zeta}$, the minimizers of $\mathcal{F}_{\gamma,u,\Lambda}(\cdot)$ in $\mathcal{X}_{\Lambda,\vec\rho^{\,*}} := \{\vec\rho \in M^{\hat p}_{\Lambda,\zeta} : \vec\rho_{\Lambda^c}(r) \equiv \vec\rho^{\,*}_{\Lambda^c}(r)\}$ are solutions of (B.6): $\vec\rho = \vec{\mathfrak{M}}^{u,\hat p}(\vec\rho)$. By the contraction property of the map $\vec{\mathfrak{M}}^{u,\hat p}(\cdot)$ we get uniqueness of the minimizer.

Proof of the properties 1,2,3

(1) Clearly we have $\vec{T}_t^{u,\Lambda,\hat p}(S_Q) \sqsubset S_Q$. To prove the first point, let then $\tau > 0$, $\vec\rho_0 \in M^{\hat p}_{\Lambda,\zeta}$ and

$$\mathcal{X}_{\tau,\vec\rho_0} := \{\vec\rho \in L^\infty(\mathbb{R}^d \times [0,\tau]; S_Q) : \vec\rho(r,t) = \vec\rho_0(r)\ (r,t) \in \big[\mathbb{R}^d \times 0\big] \sqcup \big[\Lambda^c \times [0,\tau]\big]\}$$

Let $\hat\Omega^{\hat p}(\cdot)$ the map from $\mathcal{X}_{\tau,\vec\rho_0}$ into itself defined for any $t \in [0,\tau]$, $r \in \Lambda$ as

$$\vec\Omega^{\hat p}(\vec\rho)(r,t) = e^{-t}\vec\rho_0(r) + \int_0^t e^{-(t-s)}\vec{\mathfrak{M}}^{(u,\hat p)}(\vec\rho)\ ds$$

if τ is small enough $\vec\Omega^{\hat p}$ is a contraction and its fixed point is the solution of (B.5), $T_t^{u,\Lambda,\hat p}(\vec\rho_0)$, $t \in [0,\tau]$. By (B.3) the set:

$$\{\vec v \in \mathcal{X}_{\tau,\vec\rho_0} : \vec v(\cdot,t) \in M^{\hat p}_{\Lambda,\zeta}\ \forall t \in [0,\tau]\}$$

is invariant under the map $\hat\Omega$, and since it is closed, it contains the fixed point of $\hat\Omega$. By induction the statement can be extended fo any t: $T_t^{u,\Lambda,\hat p}(\vec\rho_0) \sqsubset M^{\hat p}_{\Lambda,\zeta}$, $t \geq 0$.

(2) Convergence on subsequences follows by Ascoli-Arzelà theorem, after having written the integral expression of the evolution (B.5) and observed $T_{q,t}^{u,\Lambda,\hat p}(\vec\rho) - e^{-t}\rho_q$ is bounded with bounded gradient.

(3) The decreasing of the free energy functional, follows by observing that:

$$\frac{d}{dt}\mathcal{F}_{\gamma,u,\Lambda}([\vec{T}_t^{u,\Lambda,\hat p}(\vec\rho)] \mid \vec\rho_{\Lambda^c}) < 0$$

an explicit calculation gives in fact :

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_{\gamma,u,\Lambda}([\vec{T}_t^{u,\Lambda,\hat p}(\vec\rho)] \mid \vec\rho_{\Lambda^c} &= \sum_q \int_\Lambda dr\ \Big(-\mathcal{L}_q^u(\vec{T}_t^{u,\Lambda,\hat p}(\vec\rho)) + \frac{1}{\beta}(1+\ln T_{q,t}^{u,\Lambda,\hat p}(\vec\rho))\Big) \\
&\qquad \cdot\Big(- T_{q,t}^{u,\Lambda,\hat p}(\vec\rho) + \mathfrak{M}_q^{(u,\hat p)}(\vec{T}_t^{u,\Lambda,\hat p}(\vec\rho))\Big) \\
&= \sum_q \int_\Lambda dr\ \frac{1}{\beta}\Big(\ln \frac{T_{q,t}^{u,\Lambda,\hat p}(\vec\rho)}{\mathfrak{M}_q^{(u,\hat p)}(\vec{T}_t^{u,\Lambda,\hat p}(\vec\rho))} + 1 - \ln \sum_{q'} e^{\beta\mathcal{L}^u_{q'}(\vec{T}_t^{u,\Lambda,\hat p}(\vec\rho))}\Big) \\
&\qquad \cdot\Big(- T_{q,t}^{u,\Lambda,\hat p}(\vec\rho) + \mathfrak{M}_q^{(u,\hat p)}(\vec{T}_t^{u,\Lambda,\hat p}(\vec\rho))\Big) \\
&= \frac{1}{\beta}\sum_q \int_\Lambda dr\ \Big(\ln \frac{T_{q,t}^{u,\Lambda,\hat p}(\vec\rho)}{\mathfrak{M}_q^{(u,\hat p)}(\vec{T}_t^{u,\Lambda}(\vec\rho))}\Big)\Big(- T_{q,t}^{u,\Lambda,\hat p}(\vec\rho) + \mathfrak{M}_q^{(u,\hat p)}(\vec{T}_t^{u,\Lambda,\hat p}(\vec\rho))\Big) \\
&\qquad + \frac{K}{\beta}\sum_q \int_\Lambda dr\ \Big(- T_{q,t}^{u,\Lambda,\hat p}(\vec\rho) + \mathfrak{M}_q^{(u,\hat p)}(\vec{T}_t^{u,\Lambda,\hat p}(\vec\rho))\Big)
\end{aligned}$$

with $K = (1 - \ln \sum_{q'} e^{\beta\mathcal{L}^u_{q'}(\vec\rho)})$. By normalization condition, last term is null, while the first one is negative (in fact if the first factor inside the integral is positive, the last one is negative and viceversa). We denote by:

$$\mathcal{D}_{\Lambda}^{u,\hat{p}}(\vec{\rho}) := \frac{1}{\beta}\sum_i \left(\ln \frac{\rho_i}{\mathfrak{M}_i^{(u,\hat{p})}(\vec{\rho})}\right)\left(-\rho_i + \mathfrak{M}_i^{(u,\hat{p})}(\vec{\rho})\right)$$

$\mathcal{D}_{\Lambda}^{u,\hat{p}}(\vec{\rho}) = 0$ if and only if $\vec{\rho}$ satisfies the equation:

$$\rho_q = \mathfrak{M}_q^{(u,\hat{p})}(\vec{\rho}) \qquad\qquad \forall q = 1,\dots,Q$$

To conclude we need a lower bound on $\vec{T}_t^{u,\Lambda,\hat{p}}(\vec{\rho})$ that assure that the dynamic is always well defined for any time $t > 0$. (notice that $\mathcal{D}_{\Lambda}^{u,\hat{p}}(\vec{\rho})$ diverges if one of the coordinates $\rho_q(t)$ becomes null.)

Lemma B.3. *Let $\vec{\rho} \in L^{\infty}(\mathbb{R}^d, S_Q)$, Λ a Borel set in $\mathbb{R}^d$, $\vec{\rho}^{\Lambda}(\cdot,t) = \vec{T}_t^{u,\Lambda,\hat{p}}(m)$ and $v_q^0 := \inf_{r'\in\Lambda} \rho_q^{\Lambda}(r')$. Then for all $r \in \Lambda$ and $t \geq 0$:*

$$\rho_q(r,t) \geq \left[v_q^0 - c_q(u,\beta)\right]e^{-t} + c_q(u,\beta) \tag{B.7}$$

where $c_q(u,\beta) = \frac{1}{e^{\beta u}} \frac{e^{(1-u)\beta\rho_q^{\hat{p}}}}{\sum_{q'} e^{(1-u)\beta\rho_{q'}^{\hat{p}}}} \geq \frac{1}{Qe^{\beta[u+(1-u)]}} \geq \frac{1}{Qe^{\beta}}$

Proof. For any $q = 1,\dots,Q$, let $v_q(t)$ the solutions of the Cauchy problems:

$$\begin{cases} \dfrac{dv_q(t)}{dt} = -v_q(t) + \dfrac{1}{e^{\beta u}} c_q(u,\beta) \\ \vec{v}(t) = \vec{v}^0 \end{cases} \tag{B.8}$$

Let $w_q(r,t) := v_q(t)\mathbf{1}_{r\in\Lambda} + \vec{\rho}^*(r,t)\mathbf{1}_{r\in\Lambda^c}$, Since $-w_q(r,t) + \frac{1}{e^{\beta u}}c_q(u,\beta) \leq -w_q(r,t) + \mathfrak{M}_q^{(u,\hat{p})}(\vec{w})$ uniformly in r and $w_q \in [0,1]$.By Gronwall Lemma , $\rho_q(r,t) \geq w_q(r,t)$ for any $r \in \mathbb{R}^d, t > 0$, and it is strictly positive for any $t > s > 0$.

□

By Lemma B.3 for any $s > 0$, the functions $[T_{q,t}^{u,\Lambda,\hat{p}}(\vec{\rho})]_{\Lambda}$, $t \geq s$, for any $q \in [1,\dots,Q]$ are bounded away from 0, so that:

$$\mathcal{F}_{\gamma,u,\Lambda}([\vec{T}_t^{u,\Lambda,\hat{p}}(\vec{\rho})]_{\Lambda}|\vec{\rho}^{\,*}) - \mathcal{F}_{\gamma,u,\Lambda}([\vec{T}_s^{u,\Lambda,\hat{p}}(\vec{\rho})]_{\Lambda}|\vec{\rho}^{\,*}) = \int_s^t \mathcal{D}_{\Lambda}^{u,\hat{p}}(\vec{T}_{t'}^{u,\Lambda,\hat{p}}(\vec{\rho}))dt'$$

Since $\mathcal{D}_{\Lambda}^{u,\hat{p}}(\vec{T}_s^{u,\Lambda,\hat{p}}(\vec{\rho}))$ is monotone in s, by the Lebesgue dominated convergence theorem, the limit $s \to 0$ exists, and we get:

$$\mathcal{F}_{\gamma,u,\Lambda}([\vec{T}_t^{u,\Lambda,\hat{p}}(\vec{\rho})]_{\Lambda}|\vec{\rho}^{\,*}) - \mathcal{F}_{\gamma,u,\Lambda}(\vec{\rho}_{\Lambda}|\vec{\rho}^{\,*}) = \int_0^t \mathcal{D}_{\Lambda}^{u,\hat{p}}(\vec{T}_{t'}^{u,\Lambda,\hat{p}}(\vec{\rho}))dt'$$

We omit the proof of the following theorem that follows by previous analysis:

Theorem B.4. *Let $\vec{\rho} \in L^\infty(\mathbb{R}^d, S_Q)$ and Λ a bounded, Borel set. Then, any limit point of $\vec{T}_t^\Lambda(\vec{\rho})$ satisfies* (B.6) *and for any $\vec{\rho}^{\,*} \in L^\infty(\Lambda^c, S_Q)$ there is $\vec{v} \in C^1(\Lambda, S_Q)$, with ∇v_x bounded for any x in Λ s.t.*

$$\mathcal{F}_{\gamma,u,\Lambda}(\vec{v}|\vec{\rho}^{\,*}_{\Lambda^c}) \le \mathcal{F}_{\gamma,u,\Lambda}(\vec{\rho}^{\,*}_{\Lambda}|\vec{\rho}^{\,*}_{\Lambda^c})$$

As a corollary of the Theorem B.1 we have the following result used in Subsection 9.2, and 10

Corollary B.5. *There are positive constants ω and c'_ω so that for any $u \in [0,1]$ and any $\vec{\rho}^{\,*} \in M^{\hat{p}}_{\Lambda,\zeta,\ell}$ there is $\vec{\psi}^{(u)} \in \mathcal{X}_{\Lambda,\vec{\rho}^{\,*}}$ with the following properties:*

$$\begin{aligned}
&\vec{\psi}^{(u)}(r) = \vec{\rho}^{\,*}(r) && r \in \Lambda^c \sqcup \delta_{\text{in}}^{\gamma^{-1}}[\Lambda] \\
&\vec{\psi}^{(u)}(r) = \vec{\rho}^{\hat{p}}(r) && r \in \Lambda \setminus \delta_{\text{in}}^{\ell_{+,\gamma}/4}[\Lambda] \\
&\mathcal{F}_{\gamma,u,\lambda}(\vec{\psi}^{(u)}) \le \mathcal{F}_{\gamma,u,\lambda}(\vec{\rho}^{\,*}) + c_\omega |\Lambda| e^{-\omega \ell_{+,\gamma}/4}
\end{aligned} \tag{B.9}$$

We conclude this appendix by giving the proof of the Lemma B.2

Proof of Lemma B.2: In order to prove the first statement, we define a $\mathcal{D}^{(\ell_{-,\gamma})}$- measurable approximation of the interaction kernel as:

$$J_\gamma^{(\ell_{-,\gamma})}(x, \langle y \rangle) = \frac{1}{|C^{(\ell_{-,\gamma})}|} \int_{y' \in C_y^{(\ell_{-,\gamma})}} J_\gamma(x, y') dy' \tag{B.10}$$

We have for $\ell_{-,\gamma} = \gamma^{-1+\alpha} \ll \gamma^{-1}$,

$$\begin{aligned}
|J_\gamma^{(\ell_{-,\gamma})}(x, \langle y \rangle) - J_\gamma(x,y)| \quad &\le \sup_{y' \in C_y^{(\ell_{-,\gamma})}} |J_\gamma^{(\ell_{-,\gamma})}(x, \langle y \rangle) - J_\gamma(x, y')| && \text{(B.11)} \\
&\le \sqrt{d}\ \ell_{-,\gamma} \gamma^{d+1} \|\nabla J\|_\infty \mathbf{1}_{\{|x-y|\} \le 2\gamma^{-1}} && \text{(B.12)}
\end{aligned}$$

Using this result, we can write for all r and all q,

$$\begin{aligned}
&|\mathcal{L}_q^u(\vec{\rho})(r) - \rho_q^{\hat{p}}| = u \left| \int dr'\ J_\gamma(r,r')(\rho_q(r') - \rho_q^{\hat{p}}) \right| \\
&\le u \left| \int dr'\ J_\gamma^{(\ell_{-,\gamma})}(r, \langle r' \rangle)(\rho_q(r') - \rho_q^{\hat{p}}) \right| + u \left| \int dr'\ (J_\gamma(r,r') - J_\gamma^{(\ell_{-,\gamma})}(r, \langle r' \rangle))(\rho_q(r') - \rho_q^{\hat{p}}) \right| \\
&\le u \Big(\sum_{j \in (\ell_{-,\gamma}\mathbb{Z})^d} J_\gamma^{(\ell_{-,\gamma})}(r, \langle x \rangle) \int_{r' \in C_j^{(\ell_{-,\gamma})}} |\rho_q(r') - \rho_q^{\hat{p}}| dr' + 2^d \sqrt{d} \|\nabla J\|_\infty \gamma \ell_{-,\gamma} \Big) \\
&\le u(1 + c_d \kappa_0)\zeta
\end{aligned}$$

for $\gamma^\alpha \le \kappa_0 \zeta$ and $c_d = 2^d \sqrt{d} \|\nabla J\|_\infty$. Hence

$$\sup_r \|\vec{\mathcal{L}}^u(\vec{\rho}) - \vec{\rho}^{\,\hat{p}}\|_\star \le u(1 + c_d\kappa_0)\zeta \le 2\zeta \tag{B.13}$$

for κ_0 small enough.

In order to prove (B.3), we take ζ_0' small enough so that theorem A.3 holds (for example $\zeta_0' < \left(\frac{1}{2\beta Q}\right)^2$) for all $\zeta < \zeta_0'$. We get for any $r \in \Lambda$

$$\begin{aligned} \|\vec{\mathfrak{M}}^{(u,\hat{p})}(\vec{\rho})(r) - \vec{\rho}^{\,\hat{p}}(r)\|_\star &= \|\vec{g}(\vec{\mathcal{L}}^u(\vec{\rho}))(r) - \vec{g}(\vec{\rho}^{\,\hat{p}})(r)\|_\star \\ &\le (1 - \frac{1}{2Q})\|\vec{\mathcal{L}}^u(\vec{\rho})(r) - \vec{\rho}^{\,\hat{p}}(r)\|_\star \\ &\le (1 - \frac{1}{2Q})u(1 + c_d\kappa_0)\zeta \le u(1-\kappa_0)\zeta \end{aligned} \tag{B.14}$$

having chosen κ_0 so small that

$$\kappa_0 \le \frac{1}{2Q(1+c_d)} \tag{B.15}$$

and (B.3) holds. □

Appendix C. Existence of the pressures $P^{\pm}_{\mathrm{abs},\gamma,\beta}$ of the abstract models

Let $\{\Lambda_n\}$ a sequence of sets of side $2^n\ell_{+,\gamma}$ and

$$D_{\gamma,\beta}(n) := \frac{\ln Z^{\pm}_{\mathrm{abs},\Lambda_n\gamma,\beta}(\vec{\rho}^{\,\pm}_\beta)}{\beta|\Lambda_n|} - \frac{\ln Z^{\pm}_{\mathrm{abs},\Lambda_{n-1},\gamma,\beta}(\vec{\rho}^{\,\pm}_\beta)}{\beta|\Lambda_{n-1}|}$$

The proof of existence and continuity in β o the abstract pressures follows by the continuity in β of $D_{\gamma,\beta}(n)$ and by proving that there is a constant κ_7:

$$|D_{\gamma,\beta}(n)| \le \kappa_7 2^{-n} \tag{C.1}$$

Proof. Decomposing Λ_n into cubes $\Lambda_{n-1}(k)$, $k = 1, \dots, 2^d$, since the interaction energy is bounded uniformly in $\vec{\xi}$ and recalling (7.41), we have:

$$\begin{aligned} &\ln Z^{\pm}_{\mathrm{abs},\Lambda_n\gamma,\beta}(\vec{\rho}^{\,\pm}_\beta) \ge \\ &\quad 2^d \ln Z^{\pm}_{\mathrm{abs},\Lambda_{n-1}\gamma,\beta}(\vec{\rho}^{\,\pm}_\beta) - c\gamma^{-1}(2^n\ell_{+,\gamma})^{d-1} - 2d(2^{n-1})^{d-1}e^{-(\mathfrak{K}_\gamma - 2b)} \end{aligned} \tag{C.2}$$

where, denoting by $[\delta\Lambda_n^0] := \sqcup_{k=1}^{2^d}\delta_{in}^{\ell_{+,\gamma}}[\Lambda_{n-1}(k)]$, we have used the estimate

$$|\sum_{\Delta\sqsubset\Lambda_n} U_\Delta(\vec{\xi}) - \sum_k \sum_{\Delta\sqsubset\Lambda_{n-1}(k)} U_\Delta(\vec{\xi})| = \sum_{\Delta:\Delta\sqcap[\delta\Lambda_n^0]\neq\emptyset} \|U_\Delta(\vec{\xi})\|_\infty \le 2d(2^{n-1})^{d-1}e^{-(\mathfrak{K}_\gamma - 2b)}$$

(C.2) gives:

$$D_{\gamma,\beta}(n) \ge -c2^{-n}$$

the same arguments give also the upper bound:

$$D_{\gamma,\beta}(n) \le -c2^{-n}$$

$\square$

Appendix D. Existence of the pressure $P^{\pm}_{\text{abs},\gamma;0}$ and surface correction.

In this appendix we prove the following Lemma:

Lemma D.1. *There exists a constant* c_d *such that for* γ *small enough*

$$|P^{\pm}_{\text{abs},\gamma;0} + \phi^{\text{mf}}_{\beta}(\rho^{\pm})| \le c_d\gamma \tag{D.1}$$

where $P^{\pm}_{\text{abs},\gamma;0}$ *is defined in* (9.5) *and* (9.9), *and*

$$R^{\pm}_{\text{abs},\Lambda;0} = R^{\text{mf}}_{\gamma,\Lambda} := \frac{\beta}{2}\sum_{\substack{x\in\Lambda\\ y\in\Lambda^c}} J_\gamma(x,y)(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm}) \tag{D.2}$$

Proof.

We denote by $C_0 \equiv C_0^{\ell_+,\gamma}$, the cube of the partition $\mathcal{D}^{\ell_+,\gamma}$ that contains the point 0,

$$\begin{aligned}
\ln Z^{\pm}_{\text{abs},\beta,\Lambda;0} &= \ln\{\sum_{\vec{\xi}_\Lambda}\prod_{x\in\Lambda} e^{\beta\left((\vec{\xi}_x-\rho^{\pm})\cdot\vec{\mathcal{L}}(\rho^{\pm})\right)}\mathbf{1}_{\vec{\xi}\in\mathcal{X}^{\pm}_{C_0}}\} - \beta H_{\gamma,\Lambda}(\vec{\rho}^{\pm}_{\Lambda}|\vec{\rho}^{\pm}_{\Lambda^c})\\
&= \ln\{\sum_{\vec{\xi}_\Lambda}\prod_{x\in\Lambda} e^{\beta\left(\vec{\xi}_x\cdot\vec{\mathcal{L}}(\rho^{\pm})\right)}\mathbf{1}_{\vec{\xi}\in\mathcal{X}^{\pm}_{C_0}}\} - \beta|\Lambda|\mathfrak{J}_\gamma(\rho^{\pm}\cdot\vec{\rho}^{\pm}) - \beta H_{\gamma,\Lambda}(\vec{\rho}^{\pm}_{\Lambda}|\vec{\rho}^{\pm}_{\Lambda^c})\\
&= \frac{|\Lambda|}{|C_0|}\ln\{\sum_{\vec{\xi}_{C_0}}\prod_{x\in C_0} e^{\beta\left(\vec{\xi}_x\cdot\vec{\mathcal{L}}(\rho^{\pm})\right)}\mathbf{1}_{\vec{\xi}\in\mathcal{X}^{\pm}_{C_0}}\} - \frac{\beta|\Lambda|}{2}\mathfrak{J}_\gamma(\rho^{\pm}\cdot\vec{\rho}^{\pm}) \\
&\qquad + \frac{\beta|}{2}\sum_{\substack{x\in\Lambda\\ y\in\Lambda^c}} J_\gamma(x,y)(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm})
\end{aligned} \tag{D.3}$$

where we recall $H_{\gamma,\Lambda}(\vec{\rho}^{\pm}_{\Lambda}|\vec{\rho}^{\pm}_{\Lambda^c}) = -\frac{1}{2}\sum_{\substack{x\in\Lambda\\ y\in\Lambda}} J_\gamma(x,y)(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm}) - \sum_{\substack{x\in\Lambda\\ y\in\Lambda^c}} J_\gamma(x,y)(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm}) = -\frac{1}{2}\sum_{\substack{x\in\Lambda\\ y\in\mathbb{Z}^d}} J_\gamma(x,y)(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm}) - \frac{1}{2}\sum_{\substack{x\in\Lambda\\ y\in\Lambda^c}} J_\gamma(x,y)(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm})$. Then we get:

$$\begin{aligned}
P^{\pm}_{\text{abs},\gamma,0} &:= \lim_{\Lambda\nearrow\infty}\frac{1}{\beta|\Lambda|}\ln Z^{\pm}_{\text{abs},\beta,\Lambda;0} = \lim_{\Lambda\nearrow\infty}\frac{1}{\beta|\Lambda|}\ln\{\sum_{\vec{\xi}_{C_0}}\prod_{x\in C_0} e^{-\beta\mathfrak{H}^{\pm}_{\gamma,\Lambda}(\vec{\xi})}\mathbf{1}_{\vec{\xi}\in\mathcal{X}^{\pm}_{C_0}}\}\\
&\equiv \frac{1}{\beta|C_0|}\ln\{\sum_{\vec{\xi}_{C_0}}\prod_{x\in C_0} e^{\beta\left(\vec{\xi}_x\cdot\vec{\mathcal{L}}(\rho^{\pm})\right)}\mathbf{1}_{\vec{\xi}\in\mathcal{X}^{\pm}_{C_0}}\} - \frac{1}{2}\mathfrak{J}_\gamma(\vec{\rho}^{\pm}\cdot\vec{\rho}^{\pm})
\end{aligned} \tag{D.4}$$

A comparison of (D.3) and (D.4) gives directly (D.2).

We now prove (D.1)

Let consider the first term of (D.4):

$$\frac{1}{\beta|C_0|}\ln\{\sum_{\vec{\xi}_{C_0}}\prod_{x\in C_0}e^{\beta\left(\vec{\xi}_x\cdot\vec{\mathcal{L}}(\rho^\pm)\right)}\mathbf{1}_{\vec{\xi}\in\mathcal{X}^\pm_{C_0}}\}$$

$$=\frac{1}{\beta|C_0|}\ln G_{\pm,\gamma;0}[\mathbf{1}_{\vec{\xi}\in\mathcal{X}^\pm_{C_0}}]+\frac{1}{\beta|C_0|}\ln\sum_{\vec{\xi}_{C_0}}\prod_{x\in C_0}e^{\beta\left(\vec{\xi}_x\cdot\vec{\mathcal{L}}(\rho^\pm)\right)}$$

$$=\frac{1}{\beta|C_0|}\ln G_{\pm,\gamma;0}[\mathbf{1}_{\vec{\xi}\in\mathcal{X}^\pm_{C_0}}]+\frac{1}{\beta}\ln\sum_{i=1,Q}e^{\beta\mathfrak{J}_\gamma\rho_i^\pm}$$

where $G_{\pm,\gamma;0}[\mathbf{1}_{\vec{\xi}\in\mathcal{X}^\pm_{C_0}}]$ is the probability of the event $\mathbf{1}_{\vec{\xi}\in\mathcal{X}^\pm_{C_0}}$ w.r.t. the Gibbs measure specified by:

$$\mu^\pm_{\gamma;0}:=\frac{e^{\beta\sum_x\left(\vec{\xi}_x\cdot\vec{\mathcal{L}}(\rho^\pm)\right)}}{Z^\pm_{\gamma;0}}$$

Postponing the proof that the first term is negligible, as

$$G_{\pm,\gamma;0}[\mathbf{1}_{\vec{\xi}\in\mathcal{X}^\pm_{C_0}}]>1-Ce^{-c\gamma^{2a}|C^{\ell_-,\gamma}|}\tag{D.5}$$

we consider the second term. We have

$$|\frac{1}{\beta}\ln\sum_{i=1,Q}e^{\beta\mathfrak{J}_\gamma\rho_i^\pm}-\frac{1}{\beta}\ln\sum_{i=1,Q}e^{\beta\rho_i^\pm}|\le 2^d\sqrt{d}\|\nabla\mathcal{J}\|_\infty\gamma$$

and

$$\begin{aligned}\frac{1}{\beta}\ln\sum_{i=1,Q}e^{\beta\rho_i^\pm} &=\frac{1}{\beta}\sum_j\rho_j^\pm\ln\sum_{i=1,Q}e^{\beta\rho_i^\pm}\\ &=\frac{1}{\beta}\sum_j\rho_j^\pm\left[-\ln\rho_j^\pm+\beta\rho_j^\pm\right]=-\phi_\beta^{\mathrm{mf}}(\vec{\rho}^\pm)\end{aligned}$$

Going back to (D.4), and using (D.5), we get

$$|P^\pm_{\mathrm{abs},\gamma,0}+\phi_\beta^{\mathrm{mf}}(\vec{\rho}^\pm)|\le c_d\gamma+Ce^{-c\gamma^{2a}|C^{\ell_-,\gamma}|}\le c'_d\gamma$$

for γ small enough.

To conclude we are then left with the proof of (D.5). Since the derivation is similar (and simpler) to what is done in section 10 but with a different free energy functional, we just sketch the proof here. We consider the one body functional $\mathcal{F}^{(1)}_{\beta,C_0}$ on C_0,

$$\mathcal{F}^{(1)}_{\beta,C_0}(\vec{\rho}_{C_0})=-\frac{1}{2}\int_{C_0}(\vec{\rho}^\pm\cdot\vec{\rho}_{C_0}(x))dx-\frac{1}{\beta}\int_{C_0}I(\vec{\rho}_{C_0}(x))dx\tag{D.6}$$

Using a result similar to 5.1 leads to an estimate valid for γ small enough of $G_{\pm,\gamma;0}[\mathbf{1}_{\vec\xi\notin\mathcal{X}^{\pm}_{C_0}}]$ in terms of the functional $\mathcal{F}^{(1)}_{\beta,C_0}$, as

$$\begin{aligned} G_{\pm,\gamma;0}[\mathbf{1}_{\vec\xi\notin\mathcal{X}^{\pm}_{C_0}}] &= \frac{\sum_{\vec\xi_{C_0}}\prod_{x\in C_0} e^{\beta\left(\vec\xi_x\cdot\vec{\mathcal{L}}(\rho^{\pm})\right)}\mathbf{1}_{\vec\xi\notin\mathcal{X}^{\pm}_{C_0^+}}}{\sum_{\vec\xi_{C_0}}\prod_{x\in C_0} e^{\beta\left(\vec\xi_x\cdot\vec{\mathcal{L}}(\rho^{\pm})\right)}} \\ &= \frac{\sum_{\vec\xi_{C_0}}\prod_{x\in C_0} e^{\beta\mathfrak{I}_\gamma\left(\vec\xi_x\cdot\vec\rho^{\pm}\right)}\mathbf{1}_{\vec\xi\notin\mathcal{X}^{\pm}_{C_0}}}{\sum_{\vec\xi_{C_0}}\prod_{x\in C_0} e^{\beta\mathfrak{I}_\gamma\left(\vec\xi_x\cdot\vec\rho^{\pm}\right)}} \\ &\approx \exp\left\{-\beta\left(\inf_{\vec\rho_{C_0}\notin\mathcal{X}^{\pm}_{C_0}} \mathcal{F}^{(1)}_{\beta,C_0}(\vec\rho_{C_0}) - \inf_{\vec\rho_{C_0}}\mathcal{F}^{(1)}_{\beta,C_0}(\vec\rho_{C_0})\right)\right\} \end{aligned}$$

Following the analysis of section 10, one gets a similar estimate for the large deviation cost as

$$\inf_{\vec\rho_{C_0}\notin\mathcal{X}^{\pm}_{C_0}} \mathcal{F}^{(1)}_{\beta,C_0}(\vec\rho_{C_0}) - \inf_{\vec\rho_{C_0}}\mathcal{F}^{(1)}_{\beta,C_0}(\vec\rho_{C_0}) \geq c\gamma^{2a}|C^{\ell_{\gamma,-}}| \tag{D.7}$$

for some constant c and γ small enough. The estimate (D.5) then follows. $\square$

Appendix E. Proof of (9.65)-(9.66)

In this appendix we prove that for any $\psi_B\in\mathcal{B}^0$ there is $\vec\psi^*_B$:

$$\vec\psi^*_B = \begin{cases} \vec\psi_B & \text{on } \delta^{\ell_{+,\gamma/4}}_{\text{in}}[B], \\ \vec\rho^{\pm} & \text{elsewhere.} \end{cases} \tag{E.1}$$

so that:

$$\mathcal{F}^{\text{eff}}_{\gamma,B,u}(\vec\psi_B|\vec\xi^{(\ell_0)}_{B^c}) \geq \mathcal{F}^{\text{eff}}_{\gamma,B,u}(\vec\psi^*_B|\vec\xi^{(\ell_0)}_{B^c}) + c\gamma^{1/4}(\gamma^{1/8}|A|) \tag{E.2}$$

The proof is analogous to the case of the Ising model widely analyzed in [**40**] to which we refer for details. A sketchy version is reported here for completeness.

Let Σ as in (9.62), and

$$\Delta = A\sqcup\delta^{\ell_{+,\gamma/4}}_{\text{out}}[A]$$

Then, recalling that the interaction term appearing in the excess of free energy is always positive, we get a lower bound by neglecting the interaction between Δ and $B\setminus\Delta$:

$$\mathcal{F}^{\text{eff}}_{\gamma,B,u}(\vec\psi_B|\vec\xi^{(\ell_0)}_{B^c}) \geq \mathcal{F}^{\text{eff}}_{\gamma,B\setminus\Delta,u}(\vec\psi_{B\setminus\Delta}|\vec\xi^{(\ell_0)}_{B^c}) + \mathcal{F}^{\text{eff}}_{\gamma,\Delta,u}(\vec\psi_\Delta)$$

where, for any sets $D,F\sqsubset\mathbb{R}^d$:

$$\mathcal{F}^{\text{eff}}_{\gamma,D,u}(\vec{\psi}_D) := \int_D \Phi^{\text{eff},\pm}_u(\vec{\psi}_D) + \frac{u}{4}\int_{D\times D} J_\gamma(r,r')[\vec{\psi}_D(r)-\vec{\psi}_D(r')]^2$$
$$\mathcal{F}^{\text{eff}}_{\gamma,D,u}(\vec{\psi}_D|\vec{\xi}_F) := \int_D \Phi^{\text{eff},\pm}_u(\vec{\psi}_D) + \frac{u}{4}\int_{D\times D} J_\gamma(r,r')[\vec{\psi}_D(r)-\vec{\psi}_D(r')]^2 + \frac{u}{2}\int_{D\times F} J_\gamma(r,r')[\vec{\psi}_D(r)-\vec{\xi}_F(r')]^2$$

For any $\psi_B \in \mathcal{B}^0$ ((9.64)), since $\vec{\psi}^*_B(r) \equiv \vec{\psi}_B(r)$ on $r \in \delta^{\ell_+,\gamma/4}_{\text{in}}[B] \sqcup \Sigma$, and $\vec{\psi}_B(r) = \vec{\rho}^{\pm}$ on $r \in \Sigma$, we have that:

$$\mathcal{F}^{\text{eff}}_{\gamma,B\setminus\Delta,u}(\vec{\psi}_{B\setminus\Delta}|\vec{\xi}^{(\ell_0)}_{B^c}) = \mathcal{F}^{\text{eff}}_{\gamma,B,u}(\vec{\psi}^*_B|\vec{\xi}^{(\ell_0)}_{B^c})$$

in fact the distance between the sets $\delta^{\ell_+,\gamma/4}_{\text{in}}[B]$ and Δ is larger than γ^{-1} and $\mathcal{F}_{\gamma,\Delta,u}(\vec{\rho}^{\pm}) \equiv 0$:

Hence we need to prove that for any $\vec{\psi}_B \in \mathcal{B}^0$:

$$\mathcal{F}^{\text{eff}}_{\gamma,\Delta,u}(\vec{\psi}_\Delta) \geq c\gamma^{1/4}\left(\gamma^{1/8}|A|\right) \tag{E.3}$$

It is convenient here to fix a specific color $\hat{p}$ instead of distinguish only disordered and ordered configurations.

We then denote by

$$\mathcal{S}^{\hat{p}} := \{r \in \hat{A} : \|\vec{\psi}_B(r) - \vec{\rho}^{\hat{p}}\|_\star \geq \gamma^{1/8}\}$$

that can be written as the sum of two sets $\mathcal{S}_0$, $\mathcal{S}_1$:

$$\begin{aligned}
\mathcal{S}_0 &:= \{r \in \mathcal{S} : \|\vec{\psi}_B(r) - \vec{\rho}^{\hat{p}}\|_\star \geq \gamma^{1/8}\ \forall \hat{q} = -1, 1, \dots, Q\} \\
\mathcal{S}_1 &:= \{r \in \mathcal{S} : \exists \hat{q} \neq \hat{p} : \|\vec{\psi}_B(r) - \vec{\rho}^{\hat{q}}\|_\star \leq \gamma^{1/8}\}
\end{aligned}$$

Recalling the definition of $\Phi^{\text{eff},\hat{p}}_u(\vec{v})$ in (9.60), and (9.57)-(9.59), we will prove that there are positive constants, c_0, c_1, c_2, so that:

$$\int_{\mathcal{S}_0} \Phi^{\text{eff},\hat{p}}_u(\vec{\psi}(r)) \geq c_0\gamma^{1/4}|\mathcal{S}_0| \tag{E.4}$$

$$\int_{\mathcal{S}_1} \Phi^{\text{eff},\hat{p}}_u(\vec{\psi}(r)) \geq (c_1 u + c_2(1-u))|\mathcal{S}_1| \tag{E.5}$$

(E.4) and (E.5) prove (E.3)

PROOF OF (E.4). (E.4) follows from the bound immediately obtained by the explicit expression of $\Phi_u^{\text{eff},\hat{p}}(\vec{v})$:

$$\inf_{u\in(0,1)} \inf_{\{\vec{v}:\|\vec{v}-\rho^{\hat{q}}\|_\star\ge\gamma^{1/8}\forall\hat{q}\}} \Phi_u^{\text{eff},\hat{p}}(\vec{v}) \ge c_0\gamma^{1/4}$$

c_0 a suitable constant □

PROOF OF (E.5). Suppose $\|\vec{v}-\vec{\rho}^{\hat{q}}\|_\star \le \gamma^{1/8}$ for some $\hat{q}\neq\hat{p}$. We will prove separately two bounds:

$$\int_{\mathcal{S}_1} \Phi_u^{\text{eff},\hat{p}}(\vec{\psi}(r)) \ge 2c_2(1-u)|\mathcal{S}_1| \tag{E.6}$$

$$\int_{\mathcal{S}_1} \Phi_u^{\text{eff},\hat{p}}(\vec{\psi}(r)) \ge 2c_1 u|\mathcal{S}_1| \tag{E.7}$$

that together give (E.5)

Proof of (E.6)

$$\Phi_u^{\text{eff},\hat{p}}(\vec{v}) \ge (1-u)\Big[-\vec{\rho}^{\hat{p}}\cdot(\vec{v}-\vec{\rho}^{\hat{p}}) + \frac{1}{\beta}\Big[\vec{v}\ln\vec{v} - \vec{\rho}^{\hat{p}}\ln\vec{\rho}^{\hat{p}}\Big]\Big] \ge 0 \tag{E.8}$$

Since $\vec{\rho}^{\hat{p}}$ is a solution of the mean field equations $\vec{\rho}^{\hat{p}} = \frac{e^{\beta\vec{\rho}^{\hat{p}}}}{\sum_i e^{\beta\rho_i^{\hat{p}}}}$, it satisfies:

$$\beta\rho_i^{\hat{p}} = \ln\rho_i^{\hat{p}} - \ln C$$

with $C=\sum_i e^{\beta\rho_i^{\hat{p}}}$ and the square parenthesis in r.h.s. of (E.8), can be rewritten as:

$$\begin{aligned}
&\Big[-\vec{\rho}^{\hat{p}}\cdot(\vec{v}-\vec{\rho}^{\hat{p}}) + \frac{1}{\beta}\Big[\vec{v}\ln\vec{v} - \vec{\rho}^{\hat{p}}\ln\vec{\rho}^{\hat{p}}\Big]\Big]\\
&\qquad = -\frac{1}{\beta}\sum_i(v_i-\rho_i^{\hat{p}})\ln\rho_i^{p} + \frac{1}{\beta}\sum_i(v_i-\rho_i^{\hat{p}})\ln C^{-1} + \frac{1}{\beta}\Big[\vec{v}\ln\vec{v} - \vec{\rho}^{\hat{p}}\ln\vec{\rho}^{\hat{p}}\Big]\\
&\qquad = -\frac{1}{\beta}\sum_i(v_i-\rho_i^{\hat{p}})\ln\rho_i^{p} + \frac{1}{\beta}\Big[\vec{v}\ln\vec{v} - \vec{\rho}^{\hat{p}}\ln\vec{\rho}^{\hat{p}}\Big]
\end{aligned}$$

where in the last equality we used the fact that $\sum_i v_i = \sum_i \rho_i^{\hat{p}} = 1$. We then have:

$$\Phi_u^{\text{eff},\hat{p}}(\vec{v}) \ge \frac{(1-u)}{\beta}\,\vec{v}\ln\frac{\vec{v}}{\vec{\rho}^{\hat{p}}}$$

and by Kullback-Leibler inequality:

$$\begin{aligned}
\Phi_u^{\text{eff},\hat{p}}(\vec{v}) &\ge \frac{(1-u)}{2\beta}(\vec{v}-\vec{\rho}^{\hat{p}})^2\\
&\ge \frac{(1-u)}{2\beta}\Big[(\vec{\rho}^{\hat{q}}-\vec{\rho}^{\hat{p}})^2 - \gamma^{1/4}\Big]
\end{aligned}$$

We consider separately the case when $\hat{q}=-1$, $\hat{p}>0$ (or viceversa) and the case where both $\hat{q},\hat{p}$ are positive.

In the first case:

$$\begin{aligned}\Phi_u^{\mathrm{eff},\hat{p}}(\vec{v}) &\geq \frac{(1-u)}{2\beta}\left[\left(\rho_A-\frac{1}{Q}\right)^2+\left(\rho_B-\frac{1}{Q}\right)^2(Q-1)-\gamma^{1/4}\right]\\ &\geq \frac{(1-u)}{2\beta}\left[\frac{Q(1-2/Q)^2}{(Q-1)}-\gamma^{1/4}\right]\end{aligned}$$

If both $\hat{q},\hat{p}$ are positive

$$\begin{aligned}\Phi_u^{\mathrm{eff},\hat{p}}(\vec{v}) &\geq \frac{(1-u)}{2\beta}\left[2\left(\rho_A-\rho_B\right)^2-\gamma^{1/4}\right]\\ &\geq \frac{(1-u)}{2\beta}\left[\left(\frac{Q(1-2/Q)}{(Q-1)}\right)^2-\gamma^{1/4}\right]\end{aligned}$$

Finally, for γ small enough:

$$\Phi_u^{\mathrm{eff},\hat{p}}(\vec{v}) \geq \frac{(1-u)}{20\beta}$$

We now prove (E.7). Let $r\in\mathcal{S}_1$

$$\int dr' J_\gamma(r,r')\left(\vec{\psi}_B(r)-\vec{\psi}_B(r')\right)^2 \geq \int dr' J_\gamma^{(\ell_{-,\gamma})}(r,\langle r'\rangle)\left(\vec{\psi}_B(r)-\vec{\psi}_B(r')\right)^2 - c\gamma\ell_{-,\gamma}$$

where

$$J_\gamma^{(\ell_{-,\gamma})}(r,\langle r'\rangle) := \frac{1}{|C^{\ell_{-,\gamma}}|}\int_{C_{r'}^{\ell_{-,\gamma}}} J_\gamma(r,r'')dr''$$

and it is constant on the cubes of the partition $\mathcal{D}^{\ell_{-,\gamma}}$. By Cauchy-Schwartz inequality:

$$\begin{aligned}&\frac{1}{|C^{\ell_{-,\gamma}}|}\int_{C_{r'}^{\ell_{-,\gamma}}} dr' J_\gamma^{(\ell_{-,\gamma})}(r,\langle r'\rangle)\left(\vec{\psi}_B(r)-\vec{\psi}_B(r')\right)^2\\ &\geq J_\gamma^{(\ell_{-,\gamma})}(r,\langle r'\rangle)\left(\vec{\psi}_B(r)-\frac{1}{|C^{\ell_{-,\gamma}}|}\int_{C_{r'}^{\ell_{-,\gamma}}}\vec{\psi}_B(r'')\,dr''\right)^2\\ &\geq J_\gamma^{(\ell_{-,\gamma})}(r,\langle r'\rangle)\left(\left(\vec{\rho}^{\hat{q}}-\vec{\rho}^{\hat{p}}\right)^2-\gamma^{1/4}-\gamma^{2a}\right)\end{aligned}$$

if both $\hat{q},\hat{p}>0$, $\left(\vec{\rho}^{\hat{q}}-\vec{\rho}^{\hat{p}}\right)^2=2(\rho_A-\rho_B)^2=2\left(\frac{Q(1-2/Q)}{Q-1}\right)^2$. While if $\hat{q}$ or $\hat{p}$ is equal to -1: $\left(\vec{\rho}^{\hat{q}}-\vec{\rho}^{\hat{p}}\right)^2=2(1-2/Q)^2$ Then:

$$\int dr' J_\gamma(r,r')\left(\vec{\psi}_B(r)-\vec{\psi}_B(r')\right)^2 \geq \frac{1}{5}-c\gamma\ell_{-,\gamma}-\gamma^{1/4}-\gamma^{2a}$$

and

$$\frac{u}{4}\int_{\mathcal{S}_1} dr\, dr' J_\gamma(r,r')\left(\vec{\psi}_B(r)-\vec{\psi}_B(r')\right)^2 \geq \left(\frac{1}{30}\right)u|\mathcal{S}_1|$$

$\square$

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