QUANTITATIVE ESTIMATES ON THE ENHANCED BINDING FOR THE PAULI-FIERZ OPERATOR

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Abstract. For a quantum particle interacting with a short-range potential, we estimate from below the shift of its binding threshold, which is due to the particle interaction with a quantized radiation field.

1. Introduction

Recently, the question of enhanced binding in nonrelativistic QED has been extensively studied in several publications [10, 9, 6, 3, 2, 4]. Dressing a charged particle with photons increases the ability of a potential to confine it. For the Pauli-Fierz operator which describes a nonrelativistic particle interacting with a radiation field, this effect was proved for small values of the fine structure constant α, first under the simplifying assumption that the spin of the particle is absent [9], and later generalized to the case of a particle with spin [6, 3]. In [2], it was shown that the effect of the enhanced binding is asymptotically small in α in the sense that the binding threshold for the Pauli-Fierz operator tends to the binding threshold for the corresponding Schrödinger operator as α tends to zero. Some quantitative estimates on this effect were obtained in [4] where it was proved that the difference between the binding threshold for the Schrödinger operator and the corresponding Pauli-Fierz operator with spin zero is at least of the order α. In the work at hand, using a different method, we prove similar results for the more general case of a particle with spin zero or one half. Notice that studying the enhanced binding effect in the case of a particle with spin requires recovering one more term of the energy's expansion in powers of α than in the spinless case.

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The method of the proof is a further development of a method used in [9] and [6]. We prove that the Pauli-Fierz operator has a ground state even for some value of the potential coupling constant that is smaller than the binding threshold for the corresponding Schrödinger operator. To do so, we construct a trial function for which the quadratic form of the Pauli-Fierz operator with this coupling constant takes a value strictly less than the self-energy. Then we apply [8, Theorem 2.1] which tells us that this implies the existence of a ground state. The trial function we use is similar to the one in [6] with some modifications necessary to obtain quantitative estimates in the case with spin. It is constructed using the ground state of the self-energy operator with total momentum zero.

As in all previous papers [9, 6, 3, 2, 4], our method is asymptotic in α. Therefore, the problem of establishing the enhanced binding effect and estimating its strength for the physical value of $\alpha \approx 1/137$ still remains open.

2. Definitions and main result

The Pauli-Fierz Hamiltonian H for a charged particle with or without spin in an external electrostatic potential and coupled to the quantized electromagnetic radiation field is defined by

(1)

$$H = \big(-i\nabla_x \otimes I_f + \sqrt{\alpha} A(x)\big)^2 + g\sqrt{\alpha}\sigma \cdot B(x) + \lambda W(x) \otimes I_f + I_{el} \otimes H_f - c_{\text{n.o.}}\alpha.$$

The operator H acts on the Hilbert space $\mathfrak{H} := \mathfrak{H}^{el} \otimes \mathfrak{F}$. The Hilbert space $\mathfrak{H}^{el}$ of the nonrelativistic particle is $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ in the case $g = 1$ and $L^2(\mathbb{R}^3)$ in the case $g = 0$. Here $\mathbb{R}^3$ is the configuration space of a single particle, while $\mathbb{C}^2$ accommodates its spin in the case $g = 1$.

We will describe the quantized electromagnetic field by use of the Coulomb gauge condition. Accordingly, the one-photon Hilbert space is given by $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, where $\mathbb{R}^3$ denotes either the photon momentum or configuration space, and $\mathbb{C}^2$ accounts for the two independent transversal polarizations of the photon. The photon Fock space is then defined by

$$\mathfrak{F} = \bigoplus_{n=0}^{\infty} \mathfrak{F}_s^{(n)},$$

where the n-photons space $\mathfrak{F}_s^{(n)} = \bigotimes_s^n \left(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\right)$ is the symmetric tensor product of n copies of $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$.

We use units such that $\hbar = c = 1$, and where the mass of the particle equals $m = 1/2$. The particle charge is then given by $e = \sqrt{\alpha}$. As usual, we will consider α as a small parameter.

The operator that couples a particle to the quantized vector potential is given by

$$\begin{aligned} A(x) = & \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{2\pi|k|^{1/2}} \varepsilon_\lambda(k) \Big[e^{ikx} \otimes a_\lambda(k) + e^{-ikx} \otimes a_\lambda^*(k) \Big] \mathrm{d}k \\ =: & D(x) + D^*(x), \end{aligned}$$

where $\mathrm{div} A = 0$ by the Coulomb gauge condition. The operators a_λ, a_λ^* satisfy the usual commutation relations

$$[a_\nu(k), a_\lambda^*(k')] = \delta(k-k')\delta_{\lambda,\nu}, \quad [a_\nu(k), a_\lambda(k')] = 0.$$

The vectors $\varepsilon_\lambda(k) \in \mathbb{R}^3$ are the two orthonormal polarization vectors perpendicular to k,

$$\varepsilon_1(k) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \qquad \text{and} \qquad \varepsilon_2(k) = \frac{k}{|k|} \wedge \varepsilon_1(k). \tag{2}$$

The function $\zeta(|k|)$ describes the *ultraviolet cutoff* on the wavenumbers k. We assume ζ to be of class C^1 and to have compact support.

The constant $c_{\text{n.o.}}$ is

$$c_{\text{n.o.}} = [D, D^*] = \frac{2}{\pi} \int_0^\infty r|\zeta(r)|^2 \mathrm{d}r,$$

and subtraction of the constant $c_{\text{n.o.}}\alpha$ amounts to normal ordering of the operator A^2.

The operator that couples a particle to the magnetic field $B = \mathrm{curl} A$ is given by

$$\begin{aligned} B(x) = & \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{2\pi|k|^{1/2}} k \times i\varepsilon_\lambda(k) \Big[e^{ikx} \otimes a_\lambda(k) - e^{-ikx} \otimes a_\lambda^*(k) \Big] \mathrm{d}k \\ =: & K(x) + K^*(x). \end{aligned}$$

In Equation (1), $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the 3-component vector of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The photon field energy operator H_f is given by

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| a_\lambda^*(k) a_\lambda(k) \mathrm{d}k.$$

The multiplicative potential W is assumed to be short range and in $L^4_{\rm loc}(\mathbb{R}^3)$, and λ is a positive coupling constant. If the negative part of W is nontrivial, then there exists a critical value λ_0 such that the Schrödinger operator $-\Delta + \lambda W$ has discrete spectrum for all $\lambda > \lambda_0$, but does not have any discrete spectrum for $0 \leq \lambda < \lambda_0$. Analogously, the Pauli-Fierz operator also has a critical coupling constant λ_1, which depends on the fine structure constant α. It is known [2] that λ_1 converges to λ_0 from below as α goes to zero.

Before stating our main result, let us introduce some notations. For v a measurable function in $\mathbb{R}^3$, we define

$$d_v = \frac{1}{2\pi}\left(\int \frac{|v(x)||v(y|}{|x-y|^2}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}, \tag{3}$$

if v is not spherically symmetric and

$$d_v = \min\{\frac{1}{2\pi}\left(\int \frac{|v(x)||v(y|}{|x-y|^2}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}, \int_0^\infty t|v(t)|\mathrm{d}t\} \tag{4}$$

if v is spherically symmetric.

Our main result is thus

Theorem 2.1. *Assume that $W(x)$ satisfies the following conditions: $W \in L^4_{\rm loc}(\mathbb{R}^3)$ and there exists $a > 0$, $c > 0$ and $\delta > 0$ such that for all $|x| > a$, $|W(x)| \leq c(1+|x|)^{-2-\delta}$. Then*

$$\lambda_1 \leq \lambda_0(1 - \alpha\eta^2 + \mathcal{O}(\alpha^{\frac{5}{4}}))$$

with

$$\eta^2 = \frac{1}{6\pi^2}\int_{\mathbb{R}^3} \frac{\zeta(|k|)}{|k|(k^2+|k|+C_W)}\mathrm{d}k\,,$$

and

$$C_W = \lambda_0^2(1+\lambda_0 d_{W_+})d_{W^2} \quad \textit{and} \quad W_+ = (|W|+W)/2.$$

3. Proof of the main Theorem

In this section, we will prove the main theorem in the case of particle with spin $g = 1$. The proof for $g = 0$ can easily be deduced with several simplifications.

We start with establishing some useful preliminary estimates.

3.1. **Properties of the self-energy operator T(0) with zero total momentum.**

This section addresses the main properties of the self-energy operator $T(0)$. Let us consider the case of a free particle coupled to the quantized electromagnetic field. The self-energy operator T is given by

$$T = \big(-i\nabla_x \otimes I_f + \sqrt{\alpha}A(x)\big)^2 + g\sqrt{\alpha}\sigma \cdot B(x) + I_{el} \otimes H_f - c_{\text{n.o.}}\alpha.$$

We note that this system is translationally invariant, that is, T commutes with the operator of total momentum

$$P_{tot} = p_{el} \otimes I_f + I_{el} \otimes P_f,$$

where p_{el} and $P_f = \sum_{\lambda=1,2}\int k a_\lambda^*(k)a_\lambda(k)\mathrm{d}k$ denote the particle and the photon momentum operators.

Let $\mathfrak{H}_P \cong \mathbb{C}^2 \otimes \mathfrak{F}$ denotes the fibre Hilbert space corresponding to conserved total momentum P. For any fixed value P of the total momentum, the restriction of T to the fibre space $\mathfrak{H}_P$ is given by (see e.g. [5])

$$T(P) = (P - P_f + \sqrt{\alpha}A(0))^2 + g\sqrt{\alpha}\sigma \cdot B(0) + H_f - c_{\text{n.o.}}\alpha. \tag{5}$$

We denote $\Sigma_0 := \inf \sigma(T(0))$.

For the reader convenience, we first collect in the following theorem different known facts regarding the ground state of the operator $T(0)$, which will be used in the proof of the main theorem.

From now on, we will denote by Π_n the projection onto the subspace of $\mathbb{C}^2 \otimes \mathfrak{F}$ corresponding to vectors which have all components zero except the n-photon components. We also define $\Pi_n^{\geq} = 1 - \sum_{i=1}^{n-1}\Pi_n$.

For vectors in $\mathbb{C}^2 \otimes \mathfrak{F}$, the norm $\|.\|$ will refer to the standard norm in $\mathbb{C}^2 \otimes \mathfrak{F}$.

Theorem 3.1. [7, 5, 6, 1] *For α sufficiently small we have:*

- *Σ_0 is an eigenvalue bordering to continuous spectrum of $T(0)$ and $\Sigma_0 = \inf \sigma(T)$.*
- *For any $\Omega_0 \in \mathrm{Ker}(T(0) - \Sigma_0)$, its projection $\Pi_0\Omega_0$ onto the zero-photon sector of $\mathbb{C}^2 \otimes \mathfrak{F}$ fulfils $\|\Pi_0\Omega_0\| \neq 0$. If Ω_0 is normalized by $\|\Pi_0\Omega_0\| = 1$, then the following inequalities are satisfied: $\|\Omega_0\| = 1 + \mathcal{O}(\alpha^{1/2})$, $\|D(0)\Omega_0\| = \mathcal{O}(\alpha^{1/2})$, and $\|H_f^{1/2}\Omega_0\| = \mathcal{O}(\alpha^{1/2})$.*
- *For the photon number operator $N_f := \sum_{\lambda=1,2}\int a_\lambda^*(k)a_\lambda(k)\mathrm{d}k$, we have $\|N_f^{1/2}\Omega_0\| = \mathcal{O}(\alpha^{1/2})$.*

Corollary 3.2. *For any vector $\Omega_0 \in \mathrm{Ker}(T(0) - \Sigma_0)$ normalized by $\|\Pi_0\Omega_0\| = 1$, we have $\|\Omega_0\| = 1 + \mathcal{O}(\alpha)$, $\|D^*(0)\Pi_1^{\geq}\Omega_0\| = \mathcal{O}(\alpha^{1/2})$ and $\|\sigma \cdot K^*(0)\Pi_1^{\geq}\Omega_0\| = \mathcal{O}(\alpha^{1/2})$.*

In the following, we consider two 4-vectors in $\mathbb{C}^2 \otimes (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)$, of the form $(\xi(\uparrow, k, \lambda_1), \xi(\uparrow, k, \lambda_2), \xi(\downarrow, k, \lambda_1), \xi(\downarrow, k, \lambda_2))$, where $\uparrow$ and $\downarrow$

refer to the spin up and spin down of the particle, and λ_1, λ_2 refer to the two polarizations of the transverse photons.

$$\Gamma_{a,b} := \begin{pmatrix} \Gamma(\uparrow, k, \lambda_1) \\ \Gamma(\uparrow, k, \lambda_2) \\ \Gamma(\downarrow, k, \lambda_1) \\ \Gamma(\downarrow, k, \lambda_2) \end{pmatrix} := \begin{pmatrix} \frac{\zeta_\Lambda(k)}{|k|^{\frac12}}(-a\sqrt{k_1^2+k_2^2} + b\,\frac{(k_1-ik_2)k_3}{\sqrt{k_1^2+k_2^2}}) \\ b\zeta_\Lambda(k)\frac{-k_2-ik_1}{\sqrt{k_1^2+k_2^2}}|k|^{\frac12} \\ \frac{\zeta_\Lambda(k)}{|k|^{\frac12}}(b\sqrt{k_1^2+k_2^2} + a\frac{(k_1+ik_2)k_3}{\sqrt{k_1^2+k_2^2}}) \\ a\,\zeta_\Lambda(k)\frac{-k_2+ik_1}{\sqrt{k_1^2+k_2^2}}|k|^{\frac12} \end{pmatrix}.$$

Let

$$\varphi_{a,b} = \sqrt{\alpha}\frac{i}{2\pi|k|(1+|k|)}\Gamma_{a,b}. \tag{6}$$

Proposition 3.3 (Approximate ground state of the Pauli-Fierz operator)**.** *For a and b in $\mathbb{C}$ such that $|a|^2+|b|^2=1$, we consider the family of real-valued functionals $L_{a,b}$ defined on $\mathbb{C}^2\otimes L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ by*

$$L_{a,b}(\xi) = \langle (k^2+|k|)\xi, \xi\rangle + 2\sqrt{\alpha}\mathcal{R}e\langle \xi, \Pi_1\sigma\cdot K^*(0)(\begin{pmatrix} a \\ b\end{pmatrix} 0, 0, \cdots)\rangle,$$

where as before $B(0)=K(0)+K^*(0)$. Then we have

i) The vector $\varphi_{a,b}$ defined by (6) is the unique minimizer of $L_{a,b}$.

ii) $|\Sigma_0 - \inf L_{a,b}(\xi)| = \mathcal{O}(\alpha^{3/2})$.

iii) Let $\Omega_0\in \mathrm{Ker}(T(0)-\Sigma_0)$ be normalized by $\|\Pi_0\Omega_0\|=1$. Let us denote by $(a,b):=\Pi_0\Omega_0$. We define the scalar product $\langle .,.\rangle_1$ onto the one-photon sector $\Pi_1(\mathbb{C}^2\otimes\mathfrak{F}) = \mathbb{C}^2\otimes L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ by $\langle f,g\rangle_1 = \langle (k^2+|k|)f, g\rangle_{\Pi_1(\mathbb{C}^2\otimes\mathfrak{F})}$. Then for $\gamma\in\mathbb{R}$ and $R\in\mathbb{C}^2\otimes L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ such that

$$\Pi_1\Omega_0 = \gamma\varphi_{a,b} + R$$

and $\langle\varphi_{a,b}, R\rangle_1 = 0$, we have

$$\langle R,\, R\rangle_1 = \mathcal{O}(\alpha^{3/2}) \text{ and } |\gamma - 1| = \mathcal{O}(\alpha^{3/4}) \tag{7}$$

Remark 3.4. *In the above proposition, and in the sequel, we use the same notation for $\Pi_1\Omega_0$ as a vector in $\mathbb{C}^2\otimes\mathfrak{F}$ which has all components zero except its one-photon component $(\Pi_1\Omega_0)^{(1)}$, as well as for the vector $(\Pi_1\Omega_0)^{(1)}$ in $\mathbb{C}^2\otimes L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$.*

Proof. In this proof, for the sake of simplicity of notations, we will drop the argument 0 in the operators $A(0)$, $B(0)$, $D(0)$, $K(0)$ and their adjoint. We first prove i). Denoting

$$g_{a,b} := \frac{1}{(k^2+|k|)}\Pi_1\sigma\cdot K^*((\begin{matrix} a \\ b \end{matrix}), 0, 0, \cdots),$$

we have

$$L_{a,b}(\xi) = \langle \xi, \xi\rangle_1 + 2\sqrt{\alpha}\mathcal{R}e\langle \xi, g_{a,b}\rangle_1 = \|\xi + \sqrt{\alpha} g_{a,b}\|_1^2 - \|\sqrt{\alpha} g_{a,b}\|_1^2, \tag{8}$$

where $\|.\|_1$ is the norm associated to the scalar product $\langle ., .\rangle_1$. Therefore, the minimizer of $L_{a,b}$ is $-\sqrt{\alpha} g_{a,b}$. A straightforward computation shows that $-\sqrt{\alpha} g_{a,b} = \varphi_{a,b}$. This implies that

$$\inf L_{a,b} = L_{a,b}(\varphi_{a,b}) = -\|\varphi_{a,b}\|_1^2. \tag{9}$$

We now prove ii). We have

$$\begin{aligned} \langle T(0)\Omega_0, \Omega_0\rangle =& \langle P_f^2 \Omega_0, \Omega_0\rangle - \sqrt{\alpha} 2\mathcal{R}e\langle P_f\Omega_0, A\Omega_0\rangle + \alpha\langle A^2\Omega_0, \Omega_0\rangle \\ &+ \sqrt{\alpha}\langle \sigma\cdot B\Omega_0, \Omega_0\rangle + \langle H_f\Omega_0, \Omega_0\rangle - c_{\text{n.o.}}\alpha \end{aligned} \tag{10}$$

Let us estimate the terms in the above equality in order to identify those who are of order $\alpha^{3/2}$ and higher.

$$\begin{aligned} \langle P_f^2\Omega_0, \Omega_0\rangle =& \langle P_f^2 \Pi_0\Omega_0, \Pi_0\Omega_0\rangle + \langle P_f^2\Pi_1\Omega_0, \Pi_1\Omega_0\rangle + \langle P_f^2 \Pi_2^{\geq}\Omega_0, \Pi_2^{\geq}\Omega_0\rangle \\ =& \langle P_f^2\Pi_1\Omega_0, \Pi_1\Omega_0\rangle + \langle P_f^2\Pi_2^{\geq}\Omega_0, \Pi_2^{\geq}\Omega_0\rangle. \end{aligned} \tag{11}$$

$$\begin{aligned} \langle H_f\Omega_0, \Omega_0\rangle =& \langle H_f \Pi_0\Omega_0, \Pi_0\Omega_0\rangle + \langle H_f\Pi_1\Omega_0, \Pi_1\Omega_0\rangle + \langle H_f \Pi_2^{\geq}\Omega_0, \Pi_2^{\geq}\Omega_0\rangle \\ =& \langle H_f\Pi_1\Omega_0, \Pi_1\Omega_0\rangle + \langle H_f\Pi_2^{\geq}\Omega_0, \Pi_2^{\geq}\Omega_0\rangle. \end{aligned} \tag{12}$$

Now, using the fact that n-photon sectors are invariant under P_f, $P_f\Pi_0\Omega_0 = 0$, and $\langle P_f\Omega_0, A\Omega_0\rangle = \langle AP_f\Pi_1^{\geq}\Omega_0, \Omega_0\rangle = \langle A\Pi_1^{\geq}\Omega_0, P_f\Omega_0\rangle = \langle A\Pi_1^{\geq}\Omega_0, P_f\Pi_1^{\geq}\Omega_0\rangle$, we get

$$\begin{aligned} |\langle P_f\Omega_0, A\Omega_0\rangle| &= |\langle P_f\Pi_1^{\geq}\Omega_0, A\Pi_1^{\geq}\Omega_0\rangle| \\ &\leq |\langle P_f\Pi_1^{\geq}\Omega_0, D\Pi_2^{\geq}\Omega_0\rangle| + |\langle P_f\Pi_2^{\geq}\Omega_0, D^*\Pi_1^{\geq}\Omega_0\rangle| \\ &\leq |\langle P_f\Pi_1\Omega_0, D\Pi_2\Omega_0\rangle| + |\langle P_f\Pi_2^{\geq}\Omega_0, D\Pi_3^{\geq}\Omega_0\rangle| \\ &\quad + |\langle P_f\Pi_2^{\geq}\Omega_0, D^*\Pi_1^{\geq}\Omega_0\rangle| \\ &\leq \|P_f\Pi_1\Omega_0\|\,\|D\Pi_2\Omega_0\| + \frac{1}{2}\|P_f\Pi_2^{\geq}\Omega_0\|^2 + \\ &\quad 2\|D\Pi_3^{\geq}\Omega_0\|^2 + 2\|D^*\Pi_1^{\geq}\Omega_0\|^2. \end{aligned}$$

Using Theorem 3.1 and Corollary 3.2 and the fact that $\|P_f\Pi_1\Omega_0\| \leq c(\Lambda)\|\Pi_1\Omega_0\| = \mathcal{O}(\alpha^{1/2})$, where $c(\Lambda)$ depends only on the ultraviolet cutoff, yields

$$|\langle P_f\Omega_0, A\Omega_0\rangle| \leq \frac{1}{2}\|P_f\Pi_2^{\geq}\Omega_0\|^2 + \mathcal{O}(\alpha). \tag{13}$$

We also have

$$\begin{aligned}
&\langle A^2\Omega_0, \Omega_0\rangle \\
&= \langle (D+D^*)^2\Omega_0, \Omega_0\rangle \\
&= 2\mathcal{R}e\langle DD\Omega_0, \Omega_0\rangle + 2\|D\Omega_0\|^2 + \|[D,D^*]\|\|\Omega_0\|^2 \\
&= \mathcal{O}(\alpha^{1/2}) + \mathcal{O}(\alpha) + \|[D,D^*]\|(1+\mathcal{O}(\alpha))\,,
\end{aligned}$$

where we used from Theorem 3.1 that $\|\Omega_0\| = 1+\mathcal{O}(\alpha)$ and $\|D\Omega_0\| = \mathcal{O}(\alpha^{1/2})$. Since the commutator $[D, D^*]$ equals $c_{\text{n.o.}}$ we arrive at

$$\langle A^2\Omega_0, \Omega_0\rangle = c_{\text{n.o.}} + \mathcal{O}(\alpha^{1/2}). \tag{14}$$

Finally we have, writing $B = K + K^*$

$$\begin{aligned}
\langle \sigma\cdot B\Omega_0, \Omega_0\rangle &= \langle \sigma\cdot K\Pi_1\Omega_0, \Pi_0\Omega_0\rangle + \langle \sigma\cdot K\Pi_2^{\geq}\Omega_0, \Pi_1^{\geq}\Omega_0\rangle \\
&\quad + \langle \sigma\cdot K^*\Pi_0\Omega_0, \Pi_1\Omega_0\rangle + \langle \sigma\cdot K^*\Pi_1^{\geq}\Omega_0, \Pi_2^{\geq}\Omega_0\rangle \\
&= 2\mathcal{R}e\langle \sigma\cdot K\Pi_1\Omega_0, \Pi_0\Omega_0\rangle + 2\mathcal{R}e\langle \sigma\cdot K^*\Pi_1^{\geq}\Omega_0, \Pi_2^{\geq}\Omega_0\rangle.
\end{aligned}$$

using Theorem 3.1 and Corollary 3.2 we obtain

$$\langle \sigma\cdot B\Omega_0, \Omega_0\rangle = 2\mathcal{R}e\langle \sigma\cdot K\Pi_1\Omega_0, \Pi_0\Omega_0\rangle + \mathcal{O}(\alpha). \tag{15}$$

Collecting (10)-(15) and using $\langle H_f\Pi_2^{\geq}\Omega_0, \Pi_2^{\geq}\Omega_0\rangle \geq 0$ we obtain

$$\begin{aligned}
\langle T(0)\Omega_0, \Omega_0\rangle &\geq \langle P_f^2\Pi_1\Omega_0, \Pi_1\Omega_0\rangle + \langle H_f\Pi_1\Omega_0, \Pi_1\Omega_0\rangle \\
&\quad + 2\sqrt{\alpha}\mathcal{R}e\langle \sigma\cdot K\Pi_1\Omega_0, \Pi_0\Omega_0\rangle + \mathcal{O}(\alpha^{\frac{3}{2}}).
\end{aligned} \tag{16}$$

Since on the one-photon sector the operator P_f^2 reduces to multiplication by k^2, and the operator H_f reduced to multiplication by $|k|$, we obtain

$$\Sigma_0 = \frac{\langle T(0)\Omega_0, \Omega_0\rangle}{||\Omega_0||^2} \geq L_{a,b}(\Pi_1\Omega_0) + \mathcal{O}(\alpha^{3/2}) \geq \inf L_{a,b} + \mathcal{O}(\alpha^{3/2}). \tag{17}$$

On the other hand, using i), and for $\psi_{a,b} = (\begin{pmatrix} a \\ b \end{pmatrix}, \varphi_{a,b}, 0, 0, \cdots)$ we have

$$\begin{aligned}
\inf L_{a,b} &= L_{a,b}(\varphi_{a,b}) = \langle T(0)\psi_{a,b}, \psi_{a,b}\rangle \geq \Sigma_0\|\psi_{a,b}\|^2 \\
&= \Sigma_0(1+\mathcal{O}(\alpha)) \geq \Sigma_0 + \mathcal{O}(\alpha^2)
\end{aligned} \tag{18}$$

Inequalities (17) and (18) conclude the proof of ii).

Eventually, we prove iii). Due to the Inequalities (17) and (18), we have $\inf L_{a,b} + \mathcal{O}(\alpha^{3/2}) = L_{a,b}(\Pi_1\Omega_0)$. Using (8), the fact that $-\sqrt{\alpha}g_{a,b} = \varphi_{a,b}$, and (9), we thus get

$$\begin{aligned}
\inf L_{a,b} + \mathcal{O}(\alpha^{3/2}) &= L_{a,b}(\Pi_1\Omega_0) = \|\gamma\varphi_{a,b} + R - \varphi_{a,b}\|_1^2 - \|\varphi_{a,b}\|_1^2 \\
&= (\gamma-1)^2\|\varphi_{a,b}\|_1^2 + \|R\|_1^2 + \inf L_{a,b}\,,
\end{aligned} \tag{19}$$

which proves (7). □

3.2. **Proof of Theorem 2.1.** As it was mentioned in the introduction, we prove the theorem by constructing a trial function Ψ for which the quadratic form of H takes a value strictly smaller than $\Sigma_0\|\Psi\|^2$.

Let us start by proving an auxiliary result. For $\gamma \in (0,1)$, we define $f_\gamma \in L^2(\mathbb{R}^3)$ to be a normalized real valued eigenfunction, with associated eigenvalue e_γ, of the Schrödinger operator

$$h_\gamma := -(1-\gamma)\Delta + \lambda_0 W(x).$$

Here λ_0 is the critical coupling constant defined in Section 2.

Lemma 3.5. *Then for $\lambda \le \lambda_0$, we have*

$$\sum_i \langle(-\Delta + \lambda W)\frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i}\rangle \le C_W \|\nabla f_\gamma\|^2 + o_\gamma(1)\|\nabla f_\gamma\|^2,$$

with $C_W := \lambda_0^2(1+\lambda_0 d_{W_+})d_{W^2}$, where $W_+ = (W+|W|)/2$ and d_{W^2} and d_{W_+} are defined by (3)-(4).

Proof. For a potential V such that $V \in L^2_{\mathrm{loc}}$ and short range we have

$$|\langle V\psi, \psi\rangle| \le d_V \|\nabla\psi\|^2 .$$

Moreover, we know that f_γ is an eigenfunction of $-(1-\gamma)\Delta + \lambda_0 W$ and that the associated eigenvalue e_γ tends to zero as γ tends to zero, since λ_0 is the critical coupling constant. Therefore we obtain the following sequence of inequalities

(20)
$$\begin{aligned}
\sum_i \langle(-\Delta + \lambda W)\frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i}\rangle &\le \sum_i \langle(-\Delta + \lambda W_+)\frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i}\rangle \\
\le \sum_i \langle(-\Delta + \lambda_0 W_+)\frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i}\rangle &\le \sum_i \langle -\Delta\frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i}\rangle(1 + d_{\lambda_0 W_+}) \\
= (1 + d_{\lambda_0 W_+})\langle -\Delta f_\gamma, -\Delta f_\gamma\rangle &= (1 + d_{\lambda_0 W_+})\langle \frac{e_\gamma - \lambda_0 W}{1-\gamma} f_\gamma, -\Delta f_\gamma\rangle \\
= \frac{e_\gamma(1 + d_{\lambda_0 W_+})}{1-\gamma}\langle f_\gamma, -\Delta f_\gamma\rangle &- \frac{\lambda_0(1 + d_{\lambda_0 W_+})}{1-\gamma}\langle W f_\gamma, -\Delta f_\gamma\rangle
\end{aligned}$$

We estimate the last term in the right hand side by

$$\begin{aligned}
-\langle W f_\gamma, -\Delta f_\gamma\rangle &= \langle W f_\gamma, \frac{\lambda_0 W - e_\gamma}{1-\gamma} f_\gamma\rangle \\
&\le -\frac{e_\gamma d_{W_+}}{1-\gamma}\|\nabla f_\gamma\|^2 + \frac{\lambda_0}{1-\gamma} d_{W^2}\|\nabla f_\gamma\|^2
\end{aligned} \tag{21}$$

The Inequalities (20) and (21) imply for $\lambda \le \lambda_0$

(22)
$$\sum_i \langle (-\Delta + \lambda W)\frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i}\rangle \le \left(\lambda_0^2(1+\lambda_0 d_{W_+})d_{W^2} + o_\gamma(1)\right)\|\nabla f_\gamma\|^2$$

□

In the rest of this section, we will mainly work in the space representation for both particle and photons. Following [6], let us introduce, for given $x \in \mathbb{R}^3$, the shift operator on the photon space variables $\tau_x : \mathbb{C}^2 \otimes \mathfrak{F} \to \mathbb{C}^2 \otimes \mathfrak{F}$. For $\phi = (\phi_0, \phi_1, \ldots, \phi_n, \ldots) \in \mathbb{C}^2 \otimes \mathcal{F}$, we have, writing by abuse of notation $\tau_x\phi = (\tau_x\phi_0, \tau_x\phi_1, \ldots)$,

$$\tau_x\phi_n(s; y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_n) = \phi_n(s; y_1 - x, \ldots, y_n - x; \lambda_1, \ldots, \lambda_n),$$

where s is the spin of the particle and takes value in $\{\uparrow, \downarrow\}$.

We denote by Ω_0^x the ground state Ω_0 written in space representation and shifted by x, i.e.

$$\Omega_0^x := \tau_x\mathcal{F}^{-1}\Omega_0,$$

where $\mathcal{F}$ stands for the Fourier transform.

Recall that $D^*(0)$ is an operator valued vector with 3 components which we denote by $D^*(0)_i$ $(i = 1, 2, 3)$. Then we consider the functions

$$\theta_i = (0, \theta_i^{(1)}, 0, \ldots) \in \mathbb{C}^2 \otimes \mathfrak{F} \tag{23}$$

with

$$\theta_i^{(1)} = (k^2 + |k| + C_W)^{-1}\Pi_1 D^*(0)_i\left(\left(\begin{array}{c} a \\ b \end{array}\right), 0, \ldots\right) \tag{24}$$

and

$$\theta_i^x = \tau_x\mathcal{F}^{-1}\theta_i. \tag{25}$$

We first state some properties of θ_i.

Lemma 3.6.
i) For $i \ne j$ we have

$$\langle \theta_i, \theta_j\rangle = 0 \quad \textit{and} \quad \langle \theta_i, \theta_j\rangle_1 = 0.$$

ii) For $i = 1, 2, 3$ holds

$$\|\theta_i\sqrt{k^2 + |k| + C_W}\|^2 = \frac{1}{6\pi^2}\int_{\mathbb{R}^3}\frac{\zeta(|k|)}{|k|(k^2 + |k| + C_W)}\mathrm{d}k,$$

iii) For $i = 1, 2, 3$, $\langle k_i\varphi_{a,b}, \Pi_1\theta_i\rangle = 0$.

Proof. To prove this Lemma we remind that θ_i has only a non zero component $\Pi_1\theta_i$ in the one photon sector, and

$$\Pi_1\theta_i = \begin{pmatrix} a\frac{\varepsilon_{1,i}(k)\zeta(|k|)}{|k|^{\frac{1}{2}}(k^2+|k|+C_W)} \\ a\frac{\varepsilon_{2,i}(k)\zeta(|k|)}{|k|^{\frac{1}{2}}(k^2+|k|+C_W)} \\ b\frac{\varepsilon_{1,i}(k)\zeta(|k|)}{|k|^{\frac{1}{2}}(k^2+|k|+C_W)} \\ b\frac{\varepsilon_{2,i}(k)\zeta(|k|)}{|k|^{\frac{1}{2}}(k^2+|k|+C_W)} \end{pmatrix},$$

where the two polarization vectors $\varepsilon_1(k)$ and $\varepsilon_2(k)$ are defined in (2). The properties stated in the Lemma follow straightforwardly from computations of the corresponding integrals. □

We consider the trial function $\Psi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes \mathfrak{F}$:

$$\Psi := \Psi_1 + \Psi_2 := f_\gamma(x)\Omega_0^x + i\sqrt{\alpha}\sum_{i=1}^{3}\theta_i^x \frac{\partial f_\gamma(x)}{\partial x_i}. \tag{26}$$

Now we compute the expectation value of H in the state Ψ. We have

$$\langle H\Psi, \Psi\rangle = \langle H\Psi_1, \Psi_1\rangle + \langle H\Psi_2, \Psi_2\rangle + 2\mathcal{R}e\langle H\Psi_1, \Psi_2\rangle.$$

As usual [6], due to the orthogonality $\langle f, \partial f/\partial x_i\rangle = 0$, we have

$$\langle H\Psi_1, \Psi_1\rangle = \Sigma_0\|\Psi_1\|^2 + \langle(-\Delta + \lambda W(x))f_\gamma, f_\gamma\rangle\|\Omega_0\|^2 \tag{27}$$

Since Ψ_2 has only a non zero component in the one photon sector, in the quadratic form $\langle H\Psi_2, \Psi_2\rangle$, all the terms involving $A(0)$ or $B(0)$ vanish. Moreover, using Lemma 3.6 and the orthogonalities $\langle \frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_j}\rangle = 0$ and $\langle \frac{\partial f_\gamma}{\partial x_i}, \frac{\partial^2 f_\gamma}{\partial x_i \partial x_j}\rangle = 0$, for $i \neq j$, we arrive at

$$\begin{aligned}\langle H\Psi_2, \Psi_2\rangle =& \alpha\sum_l \|\theta_l^x\|^2\langle(-\Delta+\lambda W)\frac{\partial f_\gamma}{\partial x_l}, \frac{\partial f_\gamma}{\partial x_l}\rangle + \mathcal{O}(\alpha^2)\|\nabla f_\gamma\|^2 \\ &+ \alpha\sum_l \|\frac{\partial f_\gamma}{\partial x_l}\|^2\langle(|k|+k^2)\Pi_1\theta_l^x, \Pi_1\theta_l^x\rangle.\end{aligned} \tag{28}$$

To compute the last term $\langle H\Psi_1, \Psi_2\rangle$, we first note that

$$\begin{aligned}\langle(-\Delta+\lambda W)f_\gamma, \frac{\partial f_\gamma}{\partial x_i}\rangle =& \langle(-(1-\gamma)\Delta + \lambda W)f_\gamma, \frac{\partial f_\gamma}{\partial x_i}\rangle - \gamma\langle \Delta f_\gamma, \frac{\partial f_\gamma}{\partial x_i}\rangle \\ =& e_\gamma\langle f_\gamma, \frac{\partial f_\gamma}{\partial x_i}\rangle + \gamma\sum_j\langle\frac{\partial^2 f_\gamma}{\partial x_j^2}, \frac{\partial f_\gamma}{\partial x_i}\rangle = 0.\end{aligned}$$

The last equality holds since f_γ is a real function vanishing at infinity. Moreover, all other terms in the quadratic form $\langle H\Psi_1, \Psi_2\rangle$ which contain $\langle f_\gamma, \frac{\partial f_\gamma}{\partial x_i}\rangle$ vanish also. So we arrive at

$$\begin{aligned} 2\mathcal{R}e\langle H\Psi_1, \Psi_2\rangle =& -2\mathcal{R}e\langle P\cdot(P_f - \sqrt{\alpha}A(0))\Psi_1, \Psi_2\rangle \\ =& 2\sqrt{\alpha}\sum_i \|\frac{\partial f_\gamma}{\partial x_i}\|^2 \mathcal{R}e\langle (P_f - \sqrt{\alpha}A(0))_i \Omega_0, \theta_i\rangle \end{aligned} \tag{29}$$

The term with P_f on the right hand side is estimated as follows

$$\begin{aligned} \mathcal{R}e\langle (P_f)_i\Omega_0, \theta_i\rangle_{\mathbb{C}^2\otimes\mathfrak{F}} =& \mathcal{R}e\langle \Pi_1(P_f)_i(\gamma\varphi_{a,b} + R), \Pi_1\theta_i\rangle_{\mathbb{C}^2\otimes L^2(\mathbb{R}^3)\otimes\mathbb{C}^2} \\ =& \mathcal{R}e\langle k_i(\gamma\varphi_{a,b} + R), \Pi_1\theta_i\rangle_{\mathbb{C}^2\otimes L^2(\mathbb{R}^3)\otimes\mathbb{C}^2} \\ \le& \|R\|_1 \||k|^{\frac{1}{2}}\Pi_1\theta_i\| + \gamma\mathcal{R}e\langle k_i\varphi_{a,b}, \Pi_1\theta_i\rangle_{\mathbb{C}^2\otimes L^2(\mathbb{R}^3)\otimes\mathbb{C}^2} \end{aligned} \tag{30}$$

Using Proposition 3.3 yields the following bound for the first term in the right hand side of (30)

$$\|R\|_1 \||k|^{\frac{1}{2}}\Pi_1\theta_i\| = \mathcal{O}(\alpha^{\frac{3}{4}}) \tag{31}$$

According to Lemma 3.6 iii), the second term in the right hand side of (30) equals zero. Therefore, collecting (29)-(31), we arrive at

$$2\mathcal{R}e\langle H\Psi_1, \Psi_2\rangle = -2\alpha\sum_i \|\frac{\partial f_\gamma}{\partial x_i}\|^2 \mathcal{R}e\langle A(0)_i\Omega_0, \theta_i\rangle + \mathcal{O}(\alpha^{\frac{5}{4}})\|\nabla f_\gamma\|^2 \tag{32}$$

Now we have, using Theorem 3.1 and the fact that $D(0)$ restricted to the 2-photon sector is a bounded operator

$$\begin{aligned} \mathcal{R}e\langle A(0)_i\Omega_0, \theta_i\rangle =& \mathcal{R}e\langle D(0)_i\Pi_2\Omega_0, \theta_i\rangle + \mathcal{R}e\langle D^*(0)_i\Pi_0\Omega_0, \theta_i\rangle \\ =& \mathcal{O}(\alpha^{\frac{1}{2}}) + \mathcal{R}e\langle D^*(0)_i\Pi_0\Omega_0, \theta_i\rangle. \end{aligned} \tag{33}$$

Due to the definition (24) of θ_i, the second term on the right hand side of (33) is $\|\theta_i\sqrt{k^2 + |k| + C_W}\|^2$. Therefore, collecting the Equalities (27), (28), (32) and (33) we obtain

(34)
$$\begin{aligned}\langle H\Psi,\Psi\rangle =& \Sigma_0\|\Psi_1\|^2+\|\Omega_0\|^2\langle(-\Delta+\lambda W)f_\gamma,f_\gamma\rangle\\ &+\alpha\sum_l\|\theta_l\|^2\langle(-\Delta+\lambda W)\frac{\partial f_\gamma}{\partial x_l},\frac{\partial f_\gamma}{\partial x_l}\rangle\\ &-2\alpha\sum_l\|\frac{\partial f_\gamma}{\partial x_l}\|^2\|\theta_i\sqrt{k^2+|k|+C_W}\|^2+\mathcal{O}(\alpha^{\frac{5}{4}})\|\nabla f_\gamma\|^2\\ &+\alpha\sum_l\|\frac{\partial f_\gamma}{\partial x_l}\|^2\|\theta_l\|_1^2.\end{aligned}$$

From Lemma 3.6, we know that $\|\theta_l\sqrt{k^2+|k|+C_W}\|^2$ is independent of l. We denote this constant by η^2. With Lemma 3.5 we thus arrive at

$$\begin{aligned}\langle H\Psi,\Psi\rangle \le{}& \Sigma_0\|\Psi\|^2-\Sigma_0\|\Psi_2\|^2+\|\Omega_0\|^2\left(\|\nabla f_\gamma\|^2+\langle\lambda W f_\gamma,f_\gamma\rangle\right)\\ &-\alpha\sum_l\|\frac{\partial f_\gamma}{\partial x_l}\|^2\eta^2+\alpha o_\gamma(1)\|\nabla f_\gamma\|^2+\mathcal{O}(\alpha^{\frac{5}{4}})\|\nabla f_\gamma\|^2\,.\end{aligned}\tag{35}$$

Note that $\Sigma_0\|\psi_2\|^2=\mathcal{O}(\alpha^2)\|\nabla f_\gamma\|^2$. We thus obtain

(36)
$$\begin{aligned}&\langle H\Psi,\Psi\rangle-\Sigma_0\|\Psi\|^2\le\\ &\|\Omega_0\|^2\Big((1-\frac{\alpha}{\|\Omega_0\|^2}\eta^2+\frac{\alpha}{\|\Omega_0\|^2}(o_\gamma(1)+\mathcal{O}(\alpha^{\frac{1}{4}})))\|\nabla f_\gamma\|^2+\langle\lambda W f_\gamma,f_\gamma\rangle\Big)\end{aligned}$$

Using from Corollary 3.2 that $\|\Omega_0\|^2=1+\mathcal{O}(\alpha)$, we obtain

$$\begin{aligned}&\langle H\Psi,\Psi\rangle-\Sigma_0\|\Psi\|^2\le\\ &\|\Omega_0\|^2\Big((1-\alpha\eta^2+\alpha o_\gamma(1)+\mathcal{O}(\alpha^{\frac{5}{4}}))\|\nabla f_\gamma\|^2+\langle\lambda W f_\gamma,f_\gamma\rangle\Big)\,.\end{aligned}\tag{37}$$

Therefore

$$\begin{aligned}\langle H\Psi,\Psi\rangle-\Sigma_0\|\Psi\|^2\le{}&\|\Omega_0\|^2(1-\alpha\eta^2+\alpha o_\gamma(1)+\mathcal{O}(\alpha^{\frac{5}{4}}))\\ &\times\Big(\|\nabla f_\gamma\|^2+(1+\alpha\eta^2+\alpha o_\gamma(1)+\mathcal{O}(\alpha^{\frac{5}{4}}))^{-1}\langle\lambda W f_\gamma,f_\gamma\rangle\Big)\,.\end{aligned}\tag{38}$$

If $\lambda>\lambda_0(1-\alpha\eta^2+\mathcal{O}(\alpha^{\frac{5}{4}}))$, choosing γ (depending on α) small enough, we arrive at $\langle H\Psi,\Psi\rangle-\Sigma_0\|\Psi\|^2<0$. Due to Griesemer et al. [8], this implies the existence of a ground state for H.

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