# Renormalization Theory based on Flow Equations

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**Abstract**

I give an overview over some work on rigorous renormalization theory based on the differential flow equations of the Wilson-Wegner renormalization group. I first consider massive Euclidean $\varphi_4^4$-theory. The renormalization proofs are achieved through inductive bounds on regularized Schwinger functions. I present relatively crude bounds which are easily proven, and sharpened versions (which seem to be optimal as regards large momentum behaviour). Then renormalizability statements in Minkowski space are presented together with analyticity properties of the Schwinger functions. Finally I give a short description of further results.

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# 1 Introduction

In this lecture I would like to give a short overview over part of the work on renormalization theory based on flow equations I have been involved in since 1990. The differential flow equations of the Wilson renormalization group [1] appear for the first time in a paper of Wegner and Houghton [2] in 1972. In a seminal paper by Polchinski [3] in 1984 a renormalization proof for scalar $\varphi_4^4$-theory was performed. It is based on the observation that the flow equations give access to a tight inductive scheme wherefrom bounds on the regularized Schwinger functions implying renormalizability may be deduced. The Schwinger functions are regularized by an $UV$-cutoff $\Lambda_0$ and by an infrared cutoff $\Lambda$, the flow parameter. The bounds on these Schwinger functions obtained in [4] are *uniform in the $UV$ cutoff* and finite for $\Lambda \to 0$. This basically solves the renormalization problem. Later on renormalization theory with flow equations was extended in various directions, for a recent review covering some of the rigorous work see [5]. Among the issues treated are the renormalization of composite operators and the short distance expansion, sharp bounds on the Schwinger functions, massless theories, the transition to Minkowski space, the treatment of abelian and nonabelian gauge theories and field theories at finite temperatures.

I will adress subsequently massive Euclidean $\varphi_4^4$-theory. I first outline the simplest version of the renormalization proof based on relatively crude bounds on the regularized Schwinger functions. Then I present sharpened versions of these bounds in momentum and position space. Hereafter analyticity properties of the Schwinger functions and statements on renormalizability in Minkowski space are presented. These notes end with a short overview on further rigorous results obtained in the flow equation framework.

Colleagues I had the pleasure to work with on renormalization theory with flow equations are Frédéric Meunier, Walter Pedra, Thomas Reisz, Manfred Salmhofer, Clemens Schophaus and Vladimir Smirnov. In particular I am grateful for the fruitful long term collaborations with Georg Keller and Volkhard Müller.

# 2 RENORMALIZATION of $\varphi_4^4$-THEORY

Massive Euclidean $\varphi_4^4$-theory is the simplest theory on which the general issues of renormalization theory can be studied. When one analyzes the divergence structure of its bare Feynman amplitudes one faces the problem to be solved by general renormalization theory, namely the so-called overlapping divergences which had paved the way to a rigorous theory of renormalization with so many difficulties during the first few decades of existence of perturbative quantum field theory. The most astonishing message from the flow equation framework is that the focus on overlapping divergences and the heavy combinatorial and analytical machinery employed for its solution are not intrinsic to the problem[2]. In the flow equation approach overlapping divergences do not leave any trace.

## 2.1 The basic tools

The bare propagator of the theory is replaced by a regularized *flowing* propagator

$$C^{\Lambda,\Lambda_0}(p) \;=\; \frac{1}{p^2+m^2}\,\{e^{-\frac{p^2+m^2}{\Lambda_0^2}} - e^{-\frac{p^2+m^2}{\Lambda^2}}\}\;, \quad 0 \le \Lambda \le \Lambda_0 \le \infty\;.$$

The full propagator is recovered by taking the regulator $\Lambda_0$ to $\infty$ and the flow parameter $\Lambda$ to $0$. We calculate the derivative of $C^{\Lambda,\Lambda_0}$ as

$$\dot{C}^{\Lambda}(p) \;=\; \partial_\Lambda C^{\Lambda,\Lambda_0}(p) \;=\; -\,\frac{2}{\Lambda^3}\,e^{-\frac{p^2+m^2}{\Lambda^2}}\;.$$

[2] This is of course is of no consequence as for the merit attached to the hard and profound work done in early rigorous renormalization theory.

Subsequently we will denote by $d\mu_{\Lambda,\Lambda_0}$ the Gaussian measure with covariance $\hbar\, C^{\Lambda,\Lambda_0}$. The parameter $\hbar$ is introduced as usual to obtain a systematic expansion in the number of loops.
The theory we want to study is massive Euclidean $\varphi_4^4$-theory. This means that we start from the *bare action*

$$L_0(\phi) = \int d^4x \, \{ \frac{g}{4!}\, \varphi^4 \, + \, a_0\, \varphi^2 + b_0\, (\partial_\mu\, \varphi)^2 + c_0\, \varphi^4 \}$$

$$a_0\,,\; c_0 = O(\hbar)\,, \quad b_0 = O(\hbar^2)\ .$$

From the bare action and the flowing propagator we may define Wilson's *flowing effective action* $L^{\Lambda,\Lambda_0}$ by integrating out momenta in the region $\Lambda^2 \le p^2 \le \Lambda_0^2$. It is defined through

$$e^{-\frac{1}{\hbar} L^{\Lambda,\Lambda_0}(\varphi)} \; := \mathcal{N} \int d\mu_{\Lambda,\Lambda_0}(\phi)\; e^{-\frac{1}{\hbar} L_0(\phi\, +\, \varphi)}$$

and can be recognized to be the generating functional of the connected free propagator amputated Schwinger functions of the theory with propagator $C^{\Lambda,\Lambda_0}$ and bare action $L_0$. For the normalization factor $\mathcal{N}$ to be finite we have to restrict the theory to finite volume. All subsequent formulae are valid also in the thermodynamic limit since they do not involve any more the vacuum functional or partition function.

The fundamental tool for our study of the renormalization problem is then the functional *Flow Equation*

$$\partial_\Lambda\, L^{\Lambda,\Lambda_0} \; = \; \frac{\hbar}{2}\, \langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda\, \frac{\delta}{\delta\varphi} \rangle L^{\Lambda,\Lambda_0} \; - \; \frac{1}{2}\, \langle \frac{\delta L^{\Lambda,\Lambda_0}}{\delta\varphi}, \dot{C}^\Lambda\, \frac{\delta L^{\Lambda,\Lambda_0}}{\delta\varphi} \rangle\ .$$

It is obtained by deriving both sides of the previous equation w.r.t. $\Lambda$ and performing an integration by parts in the functional integral on the r.h.s. We then expand $L^{\Lambda,\Lambda_0}$ in moments w.r.t. $\varphi$

$$(2\pi)^{4(n-1)}\; \delta_{\varphi(p_1)} \dots \delta_{\varphi(p_n)} L^{\Lambda,\Lambda_0}|_{\varphi\equiv 0} \; = \; \delta^{(4)}(p_1 + \ldots + p_n)\, \mathcal{L}_n^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)$$

and also in a formal powers series w.r.t. $\hbar$ to select the loop order $l$

$$\mathcal{L}_n^{\Lambda,\Lambda_0} \; = \sum_{l=0}^{\infty} \hbar^l\, \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}\ .$$

From the functional flow equation we then obtain the perturbative flow equations for the (connected free propagator amputated) n-point functions by identifying coefficients

$$\partial_\Lambda \partial^w\, \mathcal{L}_{l,n}^{\Lambda,\Lambda_0} = \frac{1}{2} \int_k \partial^w\, \mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}(k, -k, \ldots)\; \dot{C}^\Lambda$$

$$- \sum_{l_i, n_i, w_i} c_{\{w_i\}} \left[ \partial^{w_1} \mathcal{L}_{l_1,n_1}^{\Lambda,\Lambda_0}\; (\partial^{w_3} \dot{C}^\Lambda)\; \partial^{w_2} \mathcal{L}_{l_2,n_2}^{\Lambda,\Lambda_0} \right]_{sym}\,,$$

$$l_1 + l_2 = l\,, \quad n_1 + n_2 = n + 2\,, \quad w_1 + w_2 + w_3 = w\ .$$

Here we wrote the equation directly in a form where a number $|w|$ of momentum derivatives, characterized by a multi-index $w$, act on both sides. Derived Schwinger functions are needed to make close the inductive scheme (see below). The $c_{\{w_i\}}$ are combinatoric constants The subscript *sym* indicates a symetrization procedure w.r.t. external momenta.

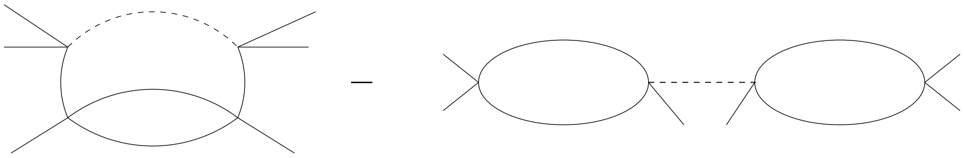

Fig. 1: A contribution to the r.h.s. of the flow equation for $l = 2,\, n = 6$. The dashed line represents the derived propagator $\dot{C}^{\Lambda}$.

## 2.2 Renormalizability

Before bounding the solutions of the system of flow equations we first have to specify the boundary conditions : At $\Lambda = \Lambda_0$ we find as a consequence of our choice of the bare action $L_0 = \ L^{\Lambda_0,\Lambda_0}$

$$\partial^w \mathcal{L}_{l,n}^{\Lambda_0,\Lambda_0} \equiv 0 \quad \text{for} \quad n + |w| \geq 5 \ .$$

The so-called relevant parameters of the theory or renormalization constants are explicitly fixed by renormalization conditions imposed for the fully integrated theory at $\Lambda = 0$:

$$\mathcal{L}_4^{0,\Lambda_0}(0) = \ g \, , \quad \mathcal{L}_2^{0,\Lambda_0}(0) = \ 0 \, , \quad \partial_{p^2} \mathcal{L}_2^{0,\Lambda_0}(0) = \ 0$$

(where we chose for simlicity BPHZ renormalization conditions).
Once the boundary conditions are specified the renormalization problem can be solved *inductively* as follows :
The *inductive scheme* may for example be chosen as in the subequent figure, namely by ascending in $n + 2l$ and for fixed $n + 2l$ ascending in $l$. For this scheme to work it is important to note that by definition there is no 0-loop two-point function in $L^{\Lambda,\Lambda_0}$ .

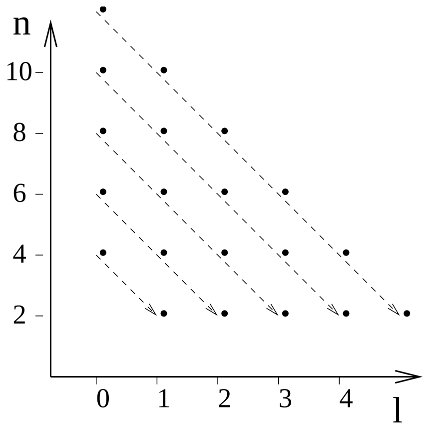


Fig. 2: The inductive scheme starts from the point (l=0, n=4) and follows the arrows, along the lines of fixed value of $n + 2l$ in increasing order.

We then may verify the follwing
*Induction hypothesis :*

$$|\partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\vec{p})| \leq (\Lambda_m)^{4-n-|w|} \, \mathcal{P}_1(\ln \frac{\Lambda_m}{m}) \, \mathcal{P}_2(\frac{|\vec{p}|}{\Lambda_m}) \ .$$

Here we note $\vec{p} = (p_1, \ldots, p_n)\,,\ \ |\vec{p}| = \sup\{|p_1|, \ldots, |p_n|\}\,,\ \ \Lambda_m \ = \ \sup(\Lambda, m)$, and the $\mathcal{P}_i$ are (each time they appear possibly new) suitable polynomials with *coefficients*, which *do not depend on* $\Lambda_0$. The important point to note on this induction hypothesis is that it is *independent of (and thus uniform in) the UV cutoff* $\Lambda_0$. This means that its verification implies essentially the solution of the renormalization problem. We shortly describe the

*Method of proof :*
The induction hypothesis is proven by first verifying it for the boundary conditions specified previously and then by inserting it on the right hand side of the flow equation bounding the integrals in an elementary way. On has to distinguish *irrelevant terms with* $n+|w|\geq 5$ *which are integrated downwards from* $\Lambda_0$ *to* $\Lambda$ *and relevant terms which are integrated upwards from* $0$ *to* $\Lambda$. We note that the relevant terms are imposed by the renormalization conditions at *fixed* external momentum. To move away from the renormalization point one uses the Schlömilch interpolation formula, e.g. for the two-point function

$$\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(p) = \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(0) + \sum_\mu p_\mu \int_0^1 d\lambda\, (\frac{\partial}{\partial p_\mu}\mathcal{L}_{l,2}^{\Lambda,\Lambda_0})(\lambda p)\ ,$$

and similar formulas where $\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(p)$ is replaced by $\frac{\partial}{\partial p_\mu}\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(p)$ or by $\frac{\partial^2}{\partial p_\mu \partial p_\nu}\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(p)$. When applying three derivatives to the two-point function or one derivative to the four-point function the contribution becomes irrelevant and is integrated downwards from $\Lambda_0$. In this way the analogue of the previous equation written for $\frac{\partial^2}{\partial p_\mu \partial p_\nu}\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(p)$ gives full control of the twice derived two-point function. Then the equation for $\frac{\partial}{\partial p_\mu}\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(p)$ gives control of the once derived two-point function, and finally the one for $\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(p)$ gives control of the two-point function itself. By Euclidean invariance one realizes that no renormalization conditions are needed for terms which are not scalars w.r.t. this symmetry. For the four-point function only one step is required.
As an example we now bound by induction the first term on the r.h.s. of the flow equation in the irrelevant case, i.e. for $n+|w|\geq 5$

$$\int_\Lambda^{\Lambda_0} d\lambda \int_k \frac{2}{\lambda^3}\, e^{-\frac{k^2+m^2}{\lambda^2}}\ \lambda_m^{4-(n+2)-|w|}\ \mathcal{P}_1(\ln\frac{\lambda_m}{m})\,\mathcal{P}_2(\frac{|(\vec{p},k,-k)|}{\lambda_m})$$

$$\leq \int_\Lambda^{\Lambda_0} d\lambda\ \lambda_m^{4+1-(n+2)-|w|}\ \mathcal{P}_1(\ln\frac{\lambda_m}{m})\,\tilde{\mathcal{P}}_2(\frac{|\vec{p}|}{\lambda_m})\ \leq\ (\Lambda_m)^{4-n-|w|}\ \tilde{\mathcal{P}}_1(\ln\frac{\Lambda_m}{m})\ \tilde{\mathcal{P}}_2(\frac{|\vec{p}|}{\Lambda_m})$$

which verifies the induction hypothesis.
To complete the proof of renormalizability one also proves

$$|\partial_{\Lambda_0}\partial^w\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\vec{p})| \leq \frac{1}{\Lambda_0^2}(\Lambda_m)^{5-|n|-|w|}\,\mathcal{P}_1(\ln(\Lambda_0/m))\,\mathcal{P}_2(\frac{|\vec{p}|}{\Lambda_m})\ .$$

This bound is obtained applying the same inductive scheme on the flow equation derived once w.r.t. $\Lambda_0$ and using the previous bounds on $\partial^w\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\vec{p})$. It implies convergence of the $\partial^w\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\vec{p})$ for $\Lambda_0\to\infty$. whereas the previous induction hypothesis still allows for solutions of the flow equations which are uniformly bounded but oscillating in terms of $\Lambda_0$.

## 2.3 Generalizations and Improvements

The previous statement of renormalizability may be improved and generalized in various ways. A first short remark is that we may generalize the boundary conditions by enlarging the class of bare actions : we also admit *nonvanishing irrelevant terms in the bare action* as long as they satisfy the bound

$$|\partial^w\mathcal{L}_{l,n}^{\Lambda_0,\Lambda_0}(\vec{p})|\ \leq \Lambda_0^{4-n-|w|}\,\mathcal{P}_1(\ln\frac{\Lambda_0}{m})\,\mathcal{P}_2(\frac{|\vec{p}|}{\Lambda_0})\ \text{ for } n+|w|\geq 5\ .$$

Remembering the previous method of proof we only have to note that these boundary conditions still satisfy the induction hypothesis (for $\Lambda=\Lambda_0$) so that the proof goes through without change. One might note that

this generalization is of practical importance on one hand, since such irrelevant terms typically will appear, e.g. in effective actions used to describe critical behaviour in statistical mechanics [6]. On can calculate subdominant corrections to scaling near second order phase transitions due to such terms.
On the other hand these terms pose considerable problems in the BPHZ type renormalization proofs of which we are not sure whether they have been solved in full rigour up the present day [7].

It is also possible to give much *sharper bounds* on the Schwinger functions with the aid of the flow equations which do not only imply renormalizability but also restrict the high momentum or large distance behaviour of the Schwinger functions in a basically optimal way (in the sense that they are saturated by individual Feynman amplutudes). To phrase those bounds we need the following concept of

*Weighted trees*:
We regard trees with $n$ external lines and $\mathcal{V}$ vertices of coordination number 3 or 4. To the $n$ external lines of a tree $T$ we associate $n$ external incoming momenta $\vec{p} = (p_1, \ldots, p_n)$ respectively positions $\vec{x} = (x_1, \ldots, x_n)$. We then define weight factors in momentum and position space.

*Weight factor in momentum space :*
To each internal line $I \in \mathcal{I}$ of the tree is attached a weight $\mu(I) \in \{1, 2\}$ such that $\sum_{I \in \mathcal{I}} \mu(I) = n - 4$. Let $p(I)$ be the momentum flowing through the internal line $I \in \mathcal{I}$. For given $\Lambda$ and tree $T$ we define

$$g^\Lambda(T) = \prod_{I \in \mathcal{I}} \frac{1}{(\sup(\Lambda_m, |p(I)|)^{\mu(I)}} \ .$$

*Weight factor in position space :*
For $\alpha = 1 - \varepsilon$ fixed and letting $|I|$ be the distance of the points joined by $I$ in position space we define

$$\mathcal{F}_\Lambda(T) \ := \Lambda_m^{4-n} \prod_{I \in \mathcal{I}} e^{-[\Lambda_m \, |I|]^\alpha} \ .$$

In terms of these weight factors we may now state
*Sharp bounds in momentum space :* [8]

$$|\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\vec{p})| \le \sup_T g^\Lambda(T) \ \mathcal{P}_l\big(\ln \sup(\frac{|\vec{p}|}{\Lambda_m}, \frac{\Lambda_m}{m})\big) \ , \ n \ge 4 \, ,$$

$$|\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(p)| \le \sup(|p|, \Lambda_m)^2 \ \mathcal{P}_{l-1}\big(\ln \sup(\frac{|p|}{\Lambda_m}, \frac{\Lambda_m}{m})\big) \ ,$$

$$\deg \mathcal{P}_l \le \ l \ .$$

The most important ingredient to obtain such sharpened bounds is to implement in the inductive procedure the fact that momentum derivatives improve the large momentum behaviour. These derivatives necessarily appear due to the use of the Schlömilch interpolation formula. Here the main problem to solve is to find optimal paths in the interpolation formula for the four point function so as to avoid that derivatives only lead to a net gain of a small external momentum only whereas subsequent application of the interpolation formula then may lead to multiplication by a large one. Thus the problem is related to the exceptional momentum problem.

*Sharp bounds in position space :*
On smearing out the positions of $n-1$ external vertices with smooth bounded test functions $\varphi \ := \varphi_2 \cdot \ldots \cdot \varphi_n$, $\varphi_i$ being supported in a square of side length $1/\Lambda_m$ around $x_i$, we have for suitable (maximizing) choice of the positions of the internal vertices of the trees for $n \ge 4$

$$|\, \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(\varphi, x_1)| \ \le \ \sup_T \mathcal{F}_\Lambda(T) \ \mathcal{P}_l(\ln(\varphi, \Lambda_m)) \, ||\varphi|| \ ,$$

$$|\mathcal{L}^{\Lambda,\Lambda_0}_{l,2}(\varphi, x_1)| \leq \Big(||\varphi|| + \Lambda_m^{-2}\,||\varphi^{(2)}|| + \Lambda_m^{-3}\,||\varphi^{(3)}||\Big)\; e^{-[\Lambda_m\, dist(x_1,\, supp\varphi)]^\alpha}\; \mathcal{P}_{l-1}(\ln(\varphi, \Lambda_m)) \; .$$

Here the definiton of $\mathcal{P}_{l-1}(\ln(\varphi, \Lambda_m))$ is given through

$$\mathcal{P}_l(\ln(\varphi, \Lambda_m)) := \mathcal{P}_l(\ln \sup(\frac{||\varphi'||}{||\varphi||\Lambda_m}, \frac{\Lambda_m}{m})) \; ,$$

with the definition $||\varphi^{(k)}|| := \sup_{w,|w|\leq k} ||\varphi^{(w)}||_\infty$ . This means that the polynomial is in logarithms of the maximal momentum content of the test functions if the latter exceeds $\Lambda_m/m$ .
The proof of these bounds has not been published so far.

*Sharp combinatoric bounds :* [9]
Finally the flow equation method also permits to recover the bounds on large orders of perturbation theory which were established by de Calan and Rivasseau using BPHZ type methods [10]. They can be stated now in a form somewhat more compact than the one given in the original paper [9]:

$$|\mathcal{L}^{\Lambda,\Lambda_0}_{l,n}(\vec{p})| \leq \Lambda_m^{4-n}\; K^{\frac{n-3}{2}+2l}\; (\frac{n}{2})!\; \sum_{s=0}^{l} (\frac{n}{2}+s-2)!\; \ln^{l-s}(\frac{\Lambda_m}{m})$$

for $|\vec{p}| \leq \sup(2\Lambda_m, k)$ with $k$ fixed, and for $K$ sufficiently large.

## 3 Relativistic Theory

We want to indicate how the flow equation method, which at first sight is restricted to euclidean theories, can be employed to prove renormalizability in Minkowski space [11]. In this respect we emphasize the beautiful framework for renormalization theory developed by Epstein and Glaser, which takes into account the structural locality and causality properties of relativistic theories from the beginning, and which was presented at the Saclay conference in the contributions of Dorothea Bahns and Klaus Fredenhagen. In our framework we address the relativistic renormalization problem in momentum space, and we want to recover the results from BPHZ renormalization theory saying that the perturbative relativistic Green functions are distributions in momentum space, and at the same time analytic functions in those domains where the external incoming energy stays below the physical thresholds. For a fully satisfactory renormalization theory in Minkowski space one would also like to have continuity properties of the Green functions above threshold, sufficient at least to define a physical renormalized coupling for physical values of the external momenta which necessarily lie above threshold. Satisfactory results in this respect do not seem to exist, and we hope to come back on this issue in the future.

### 3.1 The Flow Equation for one particle irreducible Schwinger functions

To discuss analyticity properties it is preferable to work with one particle irreducible (1PI) Schwinger functions, the generating functional of which is obtained from the one for connected Schwinger functions by a Legendre transform. Using this relation one can deduce the (still Euclidean) flow equations for the perturbative 1PI Schwinger functions $\Gamma^{\Lambda,\Lambda_0}_{l,n}$ which take the following form

$$\partial_\Lambda\, \Gamma^{\Lambda,\Lambda_0}_{l,n} = \frac{1}{2}\int_k \hat{\Gamma}^{\Lambda,\Lambda_0}_{l-1,n+2}(k,-k,\ldots)\; \dot{C}^{\Lambda,\Lambda_0}(k)$$

$$\hat{\Gamma}^{\Lambda,\Lambda_0}_{l,n} = \sum_{i\geq 1} (-1)^{i+1} \sum_{l_i,n_i} \Big[\Big(\prod_{j=1}^{i-1} C^{\Lambda,\Lambda_0}(k_j)\; \Gamma^{\Lambda,\Lambda_0}_{l_j,n_j+2}\Big)\; \Gamma^{\Lambda,\Lambda_0}_{l_i,n_i+2}\Big]_{sym} \; ,$$

$$\sum_i l_i = l \; , \quad \sum_i n_i = n - 2\,(1-\delta_{i,1}) \; .$$

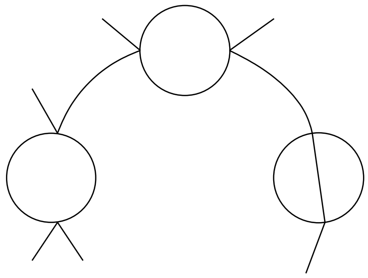

Fig. 3: A contribution to $\hat{\Gamma}_{l,n}$ for $l = 4,\ n = 6$ .

The momentum arguments $k_j$ are determined by momentum conservation. The are a sum of the loop momentum $k$ and a subsum of incoming momenta $p_i$ One immediately realizes that the inductive scheme used for the renormalization proof of connected Schwinger functions is also viable in the 1PI case, and the same inductive bounds may also be proven for the euclidean 1PI functions $\Gamma^{\Lambda}_{l,n}(\vec{p})$ .

## 3.2 Analyticity of the Euclidean theory

Using the 1PI flow equation it possible to extend the inductive renormalizability proof to the *complex domain*

$$\mathcal{D} = \Big\{ (p_{01}, \underline{p}_1, \ldots, p_{0n-1}, \underline{p}_{n-1}) \mid \underline{p}_i \in \mathbf{R}^3\ ,\ p_{0i} \in \mathbf{C}\ , \quad |Im \sum_{i \in J} p_{0i}| < 2m - \eta \quad \forall\, J \subset \{1, \ldots, n\} \Big\}$$

(with $\eta > 0$ arbitrarily small)

that is to say :
*The 1PI Schwinger functions* $\Gamma^{\Lambda,\Lambda_0}_{l,n}(\vec{p})$ *are analytic in* $\mathcal{D}$ *with respect to the arguments* $p_{01}, \ldots, p_{0n}$ and still uniformly bounded with respect to $\Lambda_0$ by bounds analogous to those given for real external momenta (for $\eta > 0$ fixed).
The proof of this statement obviously uses the fact that we have chosen an analytic regulator in momentum space. It relies on the observation that the external momentum sets of the terms $\Gamma^{\Lambda,\Lambda_0}_{l_i,n_i+2}$ appearing on the r.h.s. of the 1PI flow equation also fulfill the conditions appearing in the definition of $\mathcal{D}$, if the full external momentum set does. Thus the inductive procedure is still valid. Then by displacing the integration contour of the first component of the integration variable $k$ in the flow equation by an amount of modulus smaller than $m$ in the imaginary direction - in fact one has to move towards the centre of the interval of width $2m - \eta$ appearing in the definition of $\mathcal{D}$ - one can show that all propagators $C^{\Lambda,\Lambda_0}(k_i)$ in the 1PI flow equation have still $|Im\, k_{0i}| < m - \eta$ so that the $k$-integral is still controlled by the exponential fall-off $\exp(-\frac{k_i^2+m^2}{\Lambda^2})$ of the regulating factors. We note again that the bounds thus obtained are (obviously) not uniform in $\eta$.

To obtain the statement on the existence of the relativistic Green functions as distributions we need more explicit information on the dependence of the 1PI Schwinger functions on the external momenta. This information is obtained from the following $\alpha$-parametric integral representation for these Schwinger functions

$$\Gamma^{\Lambda,\Lambda_0}_{l,n}(\vec{p}) = \int_{1/\Lambda_0^2}^{1/\Lambda^2} d\alpha_1 \ldots d\alpha_s\ G^{\Lambda,\Lambda_0}_{l,n}(\vec{\alpha}, \vec{p})$$

with

$$G^{\Lambda,\Lambda_0}_{l,n}(\vec{\alpha}, \vec{p})\ =\ \sum_j V_j(\vec{\alpha})\, P_j(\vec{p})\, Q_j(\vec{\alpha})\, e^{-(\vec{p}, A_j\, \vec{p})_{eu} - m^2 \sum_{k=1}^{s} \alpha_k}\ .$$

By induction one verifies the following statements :
(i) $P_j$ is a monomial in the $O(4)$-invariant scalar products of the external momenta.

(ii) $Q_j$ is a rational function homogeneous of degree $d_j > -s$ in the $\alpha_i$. Here $s = 2l-2+n/2$ is the number of internal lines of $\Gamma_{l,n}$.
(iii) $A_j$ is a positive-semidefinite symmetric $(n-1 \times n-1)$-matrix, homogeneous of degree 1 in $\vec{\alpha}$.
(iv) $V_j$ is a product of $\theta$-functions restricting the $\alpha$-integration domain. Its origin traces back to previous integrations over the flow parameter, which either start from $\Lambda_0$ for the irrelevant terms or from 0 (corresponding to $\alpha \to \infty$) for the relevant terms.
On changing variables

$$\alpha_k \, = \, \tau \beta_k \, , \quad d\vec{\alpha} \, = \, \tau^{s-1} \delta(1 - \sum \beta_k) d\vec{\beta} \, d\tau$$

and *using homogeneity of* $A_j$ *und* $Q_j$ we may write

$$\Gamma^{\Lambda,\Lambda_0}_{l,n}(\vec{p}) \, = \, \int_0^1 d\vec{\beta} \, \delta(1 - \sum \beta_k) \sum_j V_j(\vec{\beta}) \; P_j(\vec{p}) \; Q_j(\vec{\beta}) \; \frac{1}{[(\vec{p}, A_j \, \vec{p})_{eu} + m^2]^{d_j+s}} \, .$$

The integrand is absolutely integrable with respect to $\vec{\beta}$ for $\vec{p} \in \mathcal{D}$. This statement follows from the previous one on the degree of $Q_j$.
The previous integral representation is obtained quite naturally from the starting observation that

$$C^{\Lambda,\Lambda_0}(k) = \int_{1/\Lambda_0^2}^{1/\Lambda^2} e^{-\alpha(k^2+m^2)} \; d\alpha$$

and on using recursively (inductively) the flow equation where each successive $\Lambda$-integration is written as a new $\alpha$-integration.

## 3.3 Transition to Minkowski space

We use the relativistic Feynman propagator

$$C_{rel}(p) \, = \, \frac{i}{p_{rel}^2 - m^2 + i\varepsilon(p_{eu}^2 + m^2)}$$

$$C_{rel}^{\Lambda,\Lambda_0}(p) \, = \, \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} e^{i\alpha[p_{rel}^2 - m^2 + i\varepsilon(p_{eu}^2 + m^2)]} \, d\alpha \, ,$$

which for finite $\varepsilon$ can be bounded in terms of the Euclidean one and $1/\varepsilon$ to make all integrals well-defined. Then following the steps which led to the integral representation for the Euclidean theory one obtains the same representation for the relativistic theory, replacing Euclidean by Minkowski scalar products and

$$\frac{1}{(\vec{p}, A_j \, \vec{p})_{eu} + m^2}$$

by

$$\frac{1}{(\vec{p}, A_j \, \vec{p})_{rel} - m^2 + i\varepsilon((\vec{p}, A_j \, \vec{p})_{eu} + m^2)} \, .$$

This representation together with results from distribution theory [12] implies the following
*Results* :
1) The relativistic 1PI Green functions are *Lorentz-invariant tempered distributions.*
2) For external momenta $(p_{01}, \underline{p}_1, \ldots, p_{0n}, \underline{p}_n)$ with $|\sum_{i \in J} p_{0i}| < 2m$ $\forall J \subset \{1, \ldots, n\}$ they agree with the Euclidean ones for $(ip_{01}, \underline{p}_1, \ldots, ip_{0n}, \underline{p}_n)$ and are thus *smooth functions* in the (image of the) corresponding domain under the Lorentz group.

# 4 A short look on further results

Among the results of rigorous renormalization theory with flow equations not covered here we mention the following :

1. One of the earliest applications of Polchinski's method was by Mitter and Ramadas [13], who gave a renormalization proof for the two-dimesnional nonlinear $\sigma$-model on the ultraviolet side.

2. Massless theories [14]
Massles theories or partially massless theories have been treated in momentum space. In this case the inductive bounds of course have to allow for the singularities present in the connected Schwinger functions at exceptional external momentum, i.e. when subsums of external momenta vanish. One then has to find and prove bounds which characterize these singularities in a basically optimal way so that the successive integration procedures of the flow equation reveal how the possible singularities disappear when passing from $\mathcal{L}_{l-1,n+2}$ to $\mathcal{L}_{l,n}$ . We think that a better and more transparent way of treating this problem than that of [14] is by the more recent method of [8]. There we give sharp large momentum bounds for the massive theory. It should be possible to translate them to the massless theory, taking care of restrictions on the admissible renormalization conditions in this case.

2. Composite Operators, Zimmermann identities [15] and the Wilson short distance expansion [16]
The renormalization of Schwinger functions with composite operator insertions is an indispensable tool in many applications of renormalization theory. In particular in gauge theories these composite operator insertions appear as quantities which characterize the violation of the Ward identities due to the presence of cutoffs. Besides, inserted Green functions are of physical interest in their own rigth, e.g. current-current-correlators in relativistic field theory, or the $< \varphi^2 \varphi^2 >$-correlator from which one reads the behaviour of the specific heat when applying Euclidean field theory to critical phenomena. The Zimmermann identities relate these composite operator insertions in different renormalization schemes with each other. They are obtained in a straightforward way from the linear structure of flow equations for inserted Green functions. Having at hand the general composite operator renormalziation techniques one can then prove asymptotic expansions for perturbative Green functions of the Wilson type. This has been done in Euclidean space only.

3. Gauge theories [17, 18]
We mentioned that the Ward identities are violated in momentum space regulated gauge theories. In the flow equation framework the renormalization of gauge theories then requires to prove that the Ward identities can be restored in the limit where the cutoffs are taken away. To do so one disposes of the freedom of imposing renormalization conditions on all relevant terms of the theory. This programme has been achieved explicitly for QED and for massive SU(2) Yang-Mills-Higgs theory.

4. Finite temperature theory [19]
Using flow equations it is possible to study explicitly the difference between the (massive Euclidean scalar) finite temperature and zero temperature field theories and to show that this difference theory is completely irrelevant in the sense of the renormalization group. That is to say, both theories ($T > 0$ and $T = 0$) are renormalzied with identical counter terms. This result is new (though expected by experts) and its proof is simple : our framework is particularly suited for this kind analysis where one has to lay hand on the bare and renormalized actions at the same time, since both appear automatically as opposite boundary values of the flow to be controlled.

Two further topics of related nature have been treated within the flow equation framework. First the Symanzik programme of improved improved actions [20] has been carried out: it is possible to modify the bare action through irrelevant terms such that the convergence of the regularized towards the renormalized theory is accelerated, in terms of inverse powers of $1/\Lambda_0$ . Secondly decoupling theorems [21] for heavy particles have been proven.

In conclusion we may say that

1. The Wilson-Wegner flow equation allows for a simple transparent rigorous solution of the perturbative renormalization problem, without introducing Feynman diagrams.
2. The method gives new results and perspectives on various aspects of the problem.
3. It places the problem in a more general, more physical and less technical context, relating perturbative and nonperturbative aspects.

We should also mention that flow equations in different forms and in various approximation schemes are presently applied to many problems in high energy and solid state physics, a topic not covered in this lecture. For a relatively recent review see [22].

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