Spectral properties of gauge-periodic Schrödinger operators on graphs decorated by Riemannian manifolds

Jochen Brüning [1], Vladimir Geyler [2,*], Igor Lobanov [1,2].

[1] Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin, Germany

[2] Mathematical Faculty, Mordovian State University, 430000 Saransk, Russia

[*] Corresponding author; e-mail addresses: geyler@mrsu.ru, geyler@mail.ru

Abstract. Gauge-periodic Schrödinger operators on periodic graphs decorated by compact Riemannian manifolds are considered. Results concerning the dependence of the spectral structure of such operators on properties of the periodicity group are announced.

Spectral properties of the Laplace–Beltrami operators on periodic Riemannian manifolds attract noticeable interest recently (see, e.g. [1] and the references therein). In particular, this interest is motivated by transport properties of the carbon nanostructures composed from fullerenes and nanotubes [2], [3]; the geometry of such structures is simulated adequately by means of so-called hybrid manifolds, i.e. by means of the topological spaces glued from manifolds of distinct dimensions. We consider here a particular case of hybrid manifolds, so-called periodic decorated graphs, and announce some results concerning the spectrum of the periodic Schrödinger operators on such spaces. Note that the class of periodic decorated graphs includes periodic metric graphs (quantum networks) [4] as a very special particular case.

Below Γ denotes an oriented graph with vertex set V and edge set E. For every $v \in V$ we fix a compact Riemannian manifold M_v with dimension less or equal to 3, and for every $e \in E$ we fix a closed interval $M_e = [a_1(e), a_2(e)] \subset \mathbb{R},\ a_1(e) \leq a_2(e)$. By definition, a *decorated graph* is a topological space X obtained from the disjoint union Y of the manifolds M_v and the intervals M_e by gluing each of the point $a_1(e)$ (respectively, $a_2(e)$) to the manifold M_v where v is the initial (respectively, terminal) vertex of the edge e. Note that if all the manifolds M_v ($v \in V$) are zero-dimensional and all the intervals M_e ($e \in E$) are nondegenerate, then X is just a *metric graph*.

There is a natural measure on X such that $L^2(X)$ is isomorphic to the direct sum $\bigoplus_{\xi\in\Xi} L^2(M_\xi)$ where Ξ denotes the disjoint union of the sets V and E. For every $\xi \in \Xi$ we fix a Schrödinger operator H_ξ on $L^2(M_\xi)$ defined by a second order differential expression with sufficiently smooth coefficients on M_ξ; if $\xi \in E$ we impose some self-adjoint boundary conditions on the boundary of the interval M_ξ. Then we can define a Schrödinger operator on X in the following way. Let S_ξ be the restriction of H_ξ to the set of all the functions from the domain $\mathcal{D}(H_\xi)$ vanishing at the gluing points, and let S be the direct sum of all the operators S_ξ, $\xi \in \Xi$. Every self-adjoint extension of S is called a *Schrödinger operator on* X. It is convenient to describe all the Schrödniger operators generated by S with the help of the Krein resolvent formula. For this purpose we denote by H^0 the direct sum of H_ξ ($\xi \in \Xi$) and introduce for every ξ an auxiliary Hilbert space $\mathcal{G}_\xi$ with the dimension equal to the deficiency index of the operator S_ξ. Let γ_ξ and Q_ξ be the Γ-field and Q-function respectively for the pair (H_ξ, S_ξ) [5]; in particular, $\gamma_\xi(z)$ (respectively $Q_\xi(z)$) is a bounded linear operator from $\mathcal{G}_\xi$ into $L^2(M_\xi)$ (respectively, from $\mathcal{G}_\xi$ into $\mathcal{G}_\xi$) for every $z \in \mathbb{C} \setminus \operatorname{spec}(H^0)$. Let now

$\mathcal{G} = \bigoplus_{\xi\in\Xi} \mathcal{G}_\xi$, $\gamma = \bigoplus_{\xi\in\Xi} \gamma_\xi$, $Q = \bigoplus_{\xi\in\Xi} Q_\xi$. Then the following formula gives a bijection between the resolvents of all the self-adjoint extensions H of the operator S and self-adjoint (generally speaking, multi-valued) operators Λ in $\mathcal{G}$:

$$(H-z)^{-1} = (H^0-z)^{-1} - \gamma(z)[Q(z)-\Lambda]^{-1}\gamma^*(\bar{z}).$$

We will denote by H_Λ the extension H corresponding to Λ. The extensions H_Λ and H^0 of S are *disjoint* (i.e., $\mathcal{D}(H_\Lambda) \cap \mathcal{D}(H^0) = \mathcal{D}(S)$) if and only if the operator Λ is single-valued. To simplify the statements below, we suppose further that H_Λ and H^0 are disjoint. In essence, this supposition does not restrict generality since the set of all the Lagrangian subspaces in the space $\mathcal{G}\times\mathcal{G}$ equipped with the standard skew-Hermitian product coincides with the set of the graphs of all Λ [6]; it is clear that the graphs of all single-valued Λ are points of general position in this set.

We consider periodic decorated graphs and gauge-periodic operators H_Λ on the graphs. With this in mind we assume that the graph Γ is equipped with a free action of a discrete group G, in particular, G acts on Ξ. We will suppose that $M_{g\xi} = M_\xi$ ($g \in G$, $\xi \in \Xi$) and that the gluing points are invariant with respect to action of G on Y; as a result we obtain a natural action of G on X. Let now σ be a 2- cocycle on G with values in $\mathbf{U}(1)$. We say that a closed linear operator A in $L^2(X)$ is *gauge-periodic*, if the resolvent of A is $(G,\bar{\sigma})$-invariant. If, in addition, $\sigma = 1$, then A is called *periodic*. We assume further that $H_{g\xi} = H_\xi$ $\forall g \in G$. Under this assumption, it is easy to prove that *a Schrödinger operator H_Λ is gauge-periodic if and only if Λ is $(G,\bar{\sigma})$-invariant*. Below we consider only $(G,\bar{\sigma})$-invariant operators Λ.

Further we suppose that the action of G on Γ has finite number of orbits; denote by F a fundamental set of the action of G on Ξ, i.e. F is a full system of distinct representatives for the orbits of this action. Denote by $C_r^*(G,\sigma)$ the reduced twisted group C^*-algebra for G with the multiplier σ [7]. We denote by $\mathcal{H}_f$ (respectively, by $\mathcal{G}_f$) the space $\bigoplus_{\xi\in F} L^2(M_\xi)$ (respectively, the space $\bigoplus_{\xi\in F} \mathcal{G}_\xi$). We denote by $\mathcal{K}(\mathcal{H})$ the space of all compact linear operators in a Hilbert space $\mathcal{H}$, the standard trace in $C_r^*(G,\sigma)\otimes\mathcal{K}(\mathcal{H})$ will be denoted by Tr. Since $\mathcal{G}_f$ is finite-dimensional, the following theorem considerably simplifies the study of the spectrum of a gauge-periodic operator H_Λ; in this theorem and further, we identify the space $L^2(X)$ (respectively, $\mathcal{G}$) with $l^2(G,\mathcal{H}_f)$ (respectively $l^2(G,\mathcal{G}_f)$).

Theorem 1. *If Λ belongs to the C^*-algebra $C_r^*(G,\sigma)\otimes\mathcal{K}(\mathcal{G}_f)$, then the resolvent of the operator H_Λ belongs to $C_r^*(G,\sigma)\otimes\mathcal{K}(\mathcal{H}_f)$.*

The condition of Theorem 1 is satisfied in the following very important particular case: Λ is a bounded linear operator in $\mathcal{G}$ and the standard matrix representation of Λ has only finite numbers non-vanishing diagonals. Moreover, let G be equipped with a left-invariant metric ρ. Then using some methods from [8] we can prove the following theorem.

2. *Let Λ be a bounded operator such that for some constants $C,\delta>0$ all the matrix elements $\Lambda(\alpha,\beta)$ satisfy the condition*

$$||\Lambda(\alpha,\beta)||\leq C\eta(\rho(\alpha,\beta))^{-1-\delta},$$

where $\eta(R)$ is the numbers of points in any ball of radius R. Then $\Lambda\in C_r^*(G,\sigma)\otimes\mathcal{K}(\mathcal{G}_f)$.

Further we assume that H_Λ is semi-bounded below. Fix a real number $E'<$

$\inf \operatorname{spec}(H^\Lambda)$ and define for $E \notin \operatorname{spec}(H^\Lambda)$

$$N(E) := \begin{cases} \operatorname{Tr} P_{[E',E]}, & E \geq E'; \\ 0, & E < E'; \end{cases}$$

where P is the spectral projection for H_Λ.

Corollary 1. *Under the conditions of Theorem 1, all the values of $N(E)$ lie in a countable subset of $\operatorname{Tr}(K_0 C_r^*(G,\sigma))$, where K_0 is the corresponding component of the K-functor on the category of C^*-algebras.*

By definition, the pair (G,σ) has *the Kadison property* if the traces of all non-zero self-adjoint projectors from $C_r^*(G,\sigma)\otimes\mathcal{K}(l^2)$ are bounded away from zero. In particular, each pair of the form $(\mathbb{Z}^m,1)$, $m\geq 1$ has the Kadison property; further examples can be found in [9].

Corollary 2. *If the pair (G,σ) from Theorem 1 has the Kadison property, then the spectrum of H_Λ has the band structure.*

By definition, a C^*-algebra A has *the property RRI_0*, if A has a faithful state F such that every self-adjoint element from A can be approximated by elements $a\in A$ with finite spectra and with arbitrary small values $F(a)$ [10]. The proof of the following corollary use some arguments from [11].

Corollary 3. *Let the condition of Theorem 1 be satisfied and let C^*-algebra $C_r^*(G,\sigma)\otimes\mathcal{K}(\mathcal{H}_f)$ have the property RRI_0. Then the operator H_Λ is approximated in the uniform resolvent sense by operators from $C_r^*(G,\sigma)\otimes\mathcal{K}(\mathcal{H}_f)$ that have Cantor spectra.*

Suppose now that G is a torsion-free Abelian group; then $G\simeq\mathbb{Z}^d$, and in G we fix the standard metric ρ of the group $\mathbb{Z}^d$.

Theorem 3. *Let $\Lambda(\alpha,\beta)\leq c_1\exp(-c_2\rho(\alpha,\beta))$ with some constants $c_1,c_2>0$.*

Then the spectrum of H_Λ has a band structure and has no singular continuous component.

Remark. Simple examples show that under the condition of Theorem 3, the operator H_Λ can have eigenvalues in the spectrum.

There are examples of periodic Schrödinger operators H on periodic decorated graphs X with $\dim X \geq 2$ such that H have infinite number of gaps in their spectrum (see [12]); hence, the Bethe–Sommerfeld conjecture [13] is not valid for such operators. The next theorem shows that really this conjecture is not valid for a wide class of periodic decorated graphs.

Theorem 4. *Let the action of G on the set of vertices V be transitive. Suppose that every operator H_v (in the considered case, H_v is independent of v) has infinite number of eigenvalues with multiplicity greater than the degree of v (this degree is also independent of v). Suppose in addition that all the intervals degenerate into a point. Then under generic position of the gluing points (both in the sense of the Riemann–Lebesgue measure and the Baire category) H_Λ has infinite number of gaps in its spectrum.*

Acknowledgements. The paper is partially supported by grants of RFBR (#02-01-00804), DFG, and INTAS.

References

[1] O. Post. Periodic manifolds with spectral gaps. *J. Diff. Eqs.* **187** (2003), 23–45.

[2] H. Aoki, M. Koshino, D. Takeda, H. Morise, K. Kuroki. Electronic structure of periodic curved surfaces: Topological band structure. *Phys. Rev. B.* **65** (2002), 035102.1–035102.8.

[3] A. R. Sabirov, I. V. Stankevich, L. A. Chernozatonskii. Hybrids of Carbyne and Fullerene (in Russian). *Pis'ma ZhETF*. **79** (2004), 153–157; English transl.: *JETP Letters*. **79** (2004), 121–125.

[4] P. Kuchment. Quantum graphs: I. Some basic structures. *Waves in Random Media.* **14** (2004), 107–128.

[5] M.G. Krein, H. K. Langer. On defect subspaces and generalized resolvents of a Hermitian operator in the space Π_κ (in Russian). *Funk. Analiz i Prilozh.* **5**, No. 2 (1971), 59–71; English transl.:*Funct. Anal. Appl.* **5** (1971), 136–146.

[6] B. S.Pavlov. The theory of extensions and explicitly solvable models (in Russian). *Uspekhi Mat. Nauk*. **42**, No. 6 (1987), 99–131; English transl.: *Russ. Math. Surv.* **42**, No. 6 (1987), 127–168.

[7] L. Auslander, C. C. Moore. Unitary representations of solvable Lie group. *Mem. Amer. Math. Soc.* No. **62** (1966), 1–199.

[8] J. Brüning, T. Sunada. On the spectrum of gauge-periodic elliptic operator. *Astérisque*. **210** (1992), 65–74.

[9] V. Mathai. On positivity of the Kadison constant and noncommutative Bloch theory. *Tôhoku Math. Publ.* **20** (2001), 107–124.

[10] M. Choi, G. Elliott. Density of the self-adjoint elements with finite spectrum in an irrational rotation C^*-algebra. *Math. Scand.* **67** (1990), 73–86.

[11] M. J. Gruber. Noncommutative Bloch theory. *J. Math. Phys.* **42** (2001), 2438–2465.

[12] J. Brüning, P. Exner, V. Geyler. Large gaps in point-coupled periodic systems of manifolds. *J. Phys. A: Math. Gen.* bf 36 (2003), 4875–4890.

[13] M. M. Skriganov. Geometric and arithmetical methods in the spectral theory of multidimensional periodic operators (in Russian). *Trudy MIAN* **171** (1985), 1–124; English transl.: *Proc. Steklov Inst. Math.* **171** (1987), 1–121.