# On the numerical computation of Diophantine rotation numbers of analytic circle maps

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**Abstract**

In this paper we present a numerical method to compute Diophantine rotation numbers of circle maps with high accuracy. We mainly focus on analytic circle diffeomorphisms, but the method also works in the case of (enough) finite differentiability. The keystone of the method is that, under these conditions, the map is conjugate to a rigid rotation of the circle. Moreover, albeit it is not fully justified by our construction, the method turns to be quite efficient for computing rational rotation numbers. We discuss the method through several numerical examples.

# Contents

**1 Introduction** **3**

**2 Conjugacy to the rotation** **4**

**3 Numerical computation of the rotation number** **6**
3.1 The averaging procedure . . . . . 6
3.2 The extrapolation procedure . . . . . 8
3.3 Bounding the error of the method . . . . . 9

**4 Numerical results** **12**
4.1 The quadratic polynomial . . . . . 13
4.2 The Arnold family . . . . . 18
4.3 The Chirikov standard map . . . . . 20

**Acknowledgements** **25**

**References** **25**

# 1 Introduction

The main purpose of this work is to introduce a new numerical method to compute the rotation number of a circle map. This problem has been formerly considered by many other authors, and several algorithms have been developed. See for instance [31, 4, 20, 24, 23, 3, 11, 12, 7]. On the one side, the level of complexity of these algorithms ranges from the definition itself to sophisticated methods of frequency analysis. On the other side, some of them are efficient for the computation of rational rotation numbers and some others work better for irrational ones.

In this paper we are mainly concerned with analytic circle diffeomorphisms having Diophantine rotation number. So, we take strong advantage of the fact that the map is analytically conjugate to a rotation. The method we present is based on the computation of suitable averages of the iterates of the map, followed by Richardson's extrapolation. The keystone of this procedure is that we know *a-priori* which is the asymptotic behavior of these averages when the number of iterates goes to infinity. This algorithm provides numerical approximations to the rotation number, with very high accuracy in general.

To develop this method we strongly use the hypotheses on the map to be analytically conjugate to a rigid rotation and to have a (good) Diophantine rotation number. However, albeit we focus on the analytic case, the same procedure can be used for smooth circle diffeomorphisms, but we only expect the method to be efficient if the conjugation has enough regularity.

Of course, the set up of this method is restrictive and excludes a lot of cases. For instance, if we consider a (generic) one-parameter family of circle homeomorphisms, the set of parameters for which the rotation number is rational, and hence the map is not conjugate to a rotation (in general), is a dense set with interior. However, if these maps are smooth enough small perturbations of a rotation, then, under general hypotheses, the relative measure of the set of parameters for which the rotation number is Diophantine is big. On the other hand, if eventually the rotation number is rational, the method provides quite good results. We do not have a complete justification of this fact, but we refer to Remark 9 for a tentative explanation and to Section 4 for examples with rational rotation numbers.

From the practical point of view, the numerical method presented here is suitable if we are able to compute the iterates of the map with high precision, for instance if we can work with a computer arithmetic having a big number of decimal digits. In this case, we can try to use the method with high order extrapolation and, then, we can hope to obtain a good approximation for the rotation number from a moderate (big) number of iterates. If we have big round-off errors, it has no-sense to perform too much extrapolation.

One motivation for the method we present is the computation of Arnold tongues of two-parameter families of analytic circle diffeomorphisms, for instance the Arnold family (see Section 4.2). An Arnold tongue is defined as the set of parameters for which the corresponding map of the family has a prefixed rotation number. If we consider a (good) Diophantine rotation number, it is possible to compute, numerically but with high accuracy, its Arnold tongue for the Arnold family, which it is known to be an analytic curve. This accuracy is important if we need to compute a big number of iterates of a map of the family having parameters on this Arnold tongue, and hence, it is very convenient to know the values of such parameters with small error.

Our main reason for developing this method goes in this direction. More precisely, let us consider an Arnold tongue of the Arnold family having a Diophantine rotation number. We know that for any value of the parameters on this tongue the corresponding member of the Arnold family has an analytic conjugation to a rotation. This conjugation can be analytically continued to a maximal complex strip of width $\Delta(\varepsilon)$ (see (17)), where $\varepsilon$ is the perturbative parameter of the family. In a forthcoming paper [26] we are going to perform a numerical study of the asymptotic behavior of $\Delta(\varepsilon)$ when $\varepsilon$ goes to zero. See [10, 5] for rigorous results on this problem. To compute numerically $\Delta(\varepsilon)$ we will apply a result of Herman [17, 21], which requires to compute a big number of iterates of one critical point of the map. So, we need to know the parameters defining the map with very high accuracy.

There are other contexts in which the method presented in this paper can be useful. For instance, if we have an invariant curve of a map, of arbitrary dimension, and we can introduce an angular variable as a parameter on it, the dynamics on this curve induces a circle map. In the aim of KAM theory, we know that the hypothesis of Diophantine rotation number for the dynamics on the curve is consistent with its own existence. So, another application of our method is the computation of invariant curves with Diophantine rotation number. See Section 4.3.

Finally, we also observe that the method can be extended to higher dimensions, by considering maps on a $d$-dimensional torus whose dynamics is conjugate to a $d$-dimensional rotation, having a Diophantine rotation vector (see Remark 4). Moreover, one can also deal with continuous dynamical systems, by considering flows on a $d$-dimensional torus, whose dynamics is conjugate to a quasi-periodic linear flow having a Diophantine vector of basic frequencies (see Remark 5). Other extensions and generalizations of the method will be object of future research.

The contents of the paper are organized in the following form. In Section 2 we formalize the problem and we state some results giving theoretical support to the method. In Section 3 we properly develop the method for the computation of the rotation number. Moreover, in Section 3 we also give some rigorous bounds of the error. Section 4 is devoted to apply the method to different examples to check numerically its efficiency and accuracy.

## 2 Conjugacy to the rotation

In this section we introduce the basic definitions and properties of circle maps we need in the paper. We refer to [8] for details.

Let $f : \mathbb{T}^1 \to \mathbb{T}^1$ be an orientation-preserving homeomorphism of the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. If we denote by $\pi$ the projection $\pi : \mathbb{R} \to \mathbb{T}^1$, we can consider $\tilde{f}$, a lift of $f$ to $\mathbb{R}$, defined so that $f \circ \pi = \pi \circ \tilde{f}$. As we work with the lift rather than with the map itself, we skip the tilde from $\tilde{f}$ and we identify the circle map with its lift. Thus, the map on $\mathbb{T}^1$ is obtained from the lift simply by taking modulo one on the definition of $f$. To normalize the lift, we suppose that $f(0) \in [0, 1)$. To such a map one can assign its rotation number, defined as

$$\rho(f) = \lim_{n \to \infty} \frac{f^n(x_0) - x_0}{n}, \tag{1}$$

where $x_0 \in \mathbb{R}$. It is well known that $f$ being an orientation-preserving homeomorphism of $\mathbb{T}^1$ guarantees that this limit exists and is independent of the point $x_0$.

If $\theta = \rho(f)$ is an irrational number and $f$ is a $\mathcal{C}^2$-diffeomorphism of $\mathbb{T}^1$, Denjoy's theorem ensures that $f$ is topologically conjugate to the rigid rotation $\mathcal{T}_\theta(x) = x + \theta$. That is, there exists a homeomorphism $\eta : \mathbb{T}^1 \to \mathbb{T}^1$ such that $f \circ \eta = \eta \circ \mathcal{T}_\theta$, making the following diagram commute:

$$\begin{array}{ccc} \mathbb{T}^1 & \xrightarrow{\mathcal{T}_\theta} & \mathbb{T}^1 \\ \eta \big\downarrow & & \big\downarrow \eta \\ \mathbb{T}^1 & \xrightarrow{f} & \mathbb{T}^1 \end{array} \tag{2}$$

If we require $\eta(0) = x_0$, for a fixed $x_0 \in \mathbb{T}^1$, then the conjugacy $\eta$ is unique.

In this paper we are interested in the case when the conjugacy $\eta$ is a smooth function. More precisely, we are mainly concerned with the analytic case. To guarantee the regularity of the conjugation it is not enough to consider smooth diffeomorphisms of the circle, but we also need the rotation number $\theta$ to be "very irrational". For the theoretical discussion of the method, we suppose that $\theta$ is a Diophantine number.

**Definition 1.** *Given $\theta \in \mathbb{R}$, we say that $\theta$ is a Diophantine number if there exist constants $C' > 0$ and $\tau \geq 1$ such that $|k\theta - l|^{-1} \leq C'|k|^\tau$, for all $(k, l) \in \mathbb{Z}^2$ with $k \neq 0$, or, in equivalent form*

$$|1 - \mathrm{e}^{2\pi \mathrm{i} k\theta}|^{-1} \leq C|k|^\tau, \qquad \forall k \in \mathbb{Z} \setminus \{0\}, \tag{3}$$

*with $C > 0$. If we denote by $\mathcal{D}$ the set of Diophantine numbers, a remarkable property of $\mathcal{D}$ is that the Lebesgue measure of $\mathbb{R} \setminus \mathcal{D}$ is equal to zero.*

**Remark 2.** *From the numerical point of view, we need the constant $C$ to be not too small if we want the method of this paper to work efficiently. However, if $\theta$ is an arbitrary real number (even a rational one) but condition (3) is fulfilled for any $|k| \leq N$, for a big $N$ and for $C$ not too small, we expect the method to provide a good approximation for $\theta$ even if the map is not conjugate to a rotation. Of course, when working with a computer all the numbers are rational. See Figure 3 for a discussion of the method for "bad" Diophantine numbers.*

The theoretical support of the method is provided by the following result.

**Theorem 3 ([15, 33, 19, 28]).** *If $f$ is an orientation-preserving $\mathcal{C}^r$-diffeomorphism of $\mathbb{T}^1$ with Diophantine rotation number $\theta$ verifying (3), for certain $\tau \geq 1$ and $\tau + 1 < r \leq +\infty$, then $f$ is conjugate to $\mathcal{T}_\theta$ via a conjugacy $\eta$ which is a $\mathcal{C}^{r-\tau-\varepsilon}$-diffeomorphism, for any $\varepsilon > 0$. If $f$ is analytic and $\theta \in \mathcal{D}$, then the conjugacy $\eta$ is also analytic.*

We can write $\eta(x) = x + \xi(x)$, being $\xi$ a 1-periodic function normalized in such a way that $\xi(0) = x_0$. By using the fact that $\eta$ conjugates $f$ to a rigid rotation (see (2)), we have:

$$f^n(x_0) = f^n(\eta(0)) = \eta(n\theta) = n\theta + \sum_{k\in\mathbb{Z}} \xi_k \mathrm{e}^{2\pi \mathrm{i} kn\theta}, \qquad \forall n \in \mathbb{Z}, \tag{4}$$

where

$$\xi(x) = \sum_{k\in\mathbb{Z}} \xi_k \mathrm{e}^{2\pi \mathrm{i} kx}, \tag{5}$$

denotes the Fourier series of $\xi$. Clearly:

$$\frac{f^n(x_0) - x_0}{n} = \theta + \frac{1}{n} \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k (\mathrm{e}^{2\pi \mathrm{i} kn\theta} - 1).$$

Being $\xi$ a continuous function, the sum at the right-hand side is uniformly bounded for any $n \geq 1$, which makes clear the computation of the rotation number from definition (1). Unfortunately, the convergence speed of this limit is, roughly speaking, of $\mathcal{O}(1/n)$ when $n$ goes to infinity. This convergence is too slow if we want to obtain a good approximation for the rotation number from a moderate (big) number of iterates.

# 3 Numerical computation of the rotation number

From now on $f$ is a lift of an analytic circle diffeomorhism with Diophantine rotation number $\theta = \rho(f)$ and, hence, it is analytically conjugate to a rotation.

The purpose of this section is to introduce a numerical method to approximate $\theta$. From the formal point of view, this method allows to compute approximations of $\theta$ with very high precision. Concretely, in Section 3.3 we prove that the error can be controlled (roughly speaking and in the best case) by an expression of $\mathcal{O}(N^{-(\log_2 N)/2})$, where $N$ is the number of iterates.

This method also works if the conjugation $\eta$ is only $\mathcal{C}^r$, but in this case the number of steps of the extrapolation procedure of Section 3.2 is limited by the order of differentiability. Thus, we cannot expect to obtain approximations for the rotation number as good as in the analytic case. See Remark 7 for additional comments.

The data required are the usual one to approximate the rotation number. We take a fixed $x_0 \in \mathbb{R}$ and compute the iterates $\{f^n(x_0)\}_{n=1,\ldots,N}$ of the lift $f$, for a big $N$. The method is based on the computation of suitable averages of these iterates, that are defined from certain recurrent sums. These sums are introduced in Section 3.1, where their asymptotic behavior (when $N \to +\infty$) is also established. In Section 3.2 we use this asymptotic behavior to perform Richardson's extrapolation to approximate the rotation number. To carry out the extrapolation procedure, we have to compute such averaged sums for "different values" of $N$, in geometrical progression. To simplify the construction we suppose that $N$ is a power of two, $N = 2^q$. However, the only thing we need to use formula (15) for the $p$-order extrapolation is that $N = 2^p N_0$, for any $N_0 \in \mathbb{N}$. Furthermore, any general extrapolation method can be adapted to this context (see for instance [29]). The error committed when dealing with these averages in terms of its asymptotic approximation and the total error of the method is discussed in Section 3.3.

## 3.1 The averaging procedure

The main goal of this Section is to define the normalized sums $\tilde{S}_N^p$ (10) from the $p$-order sums $S_N^p$ (7) of the iterates of the lift.

Let us start by considering the sum of the first $N$ iterates of $f$. We define (see (4))

$$\begin{aligned} S_N &= \sum_{n=1}^{N} (f^n(x_0) - x_0) \\ &= \frac{N(N+1)}{2}\theta + \sum_{n=1}^{N} \sum_{k\in\mathbb{Z}\backslash\{0\}} \xi_k (\mathrm{e}^{2\pi \mathrm{i} k n\theta} - 1) \\ &= \frac{N(N+1)}{2}\theta - N \sum_{k\in\mathbb{Z}\backslash\{0\}} \xi_k + \sum_{k\in\mathbb{Z}\backslash\{0\}} \xi_k \frac{\mathrm{e}^{2\pi \mathrm{i} k\theta}(1-\mathrm{e}^{2\pi \mathrm{i} kN\theta})}{1-\mathrm{e}^{2\pi \mathrm{i} k\theta}}, \end{aligned}$$

and then:

$$\frac{2}{N(N+1)} S_N = \theta - \frac{2}{N+1} \sum_{k\in\mathbb{Z}\backslash\{0\}} \xi_k + \frac{2}{N(N+1)} \sum_{k\in\mathbb{Z}\backslash\{0\}} \xi_k \frac{\mathrm{e}^{2\pi \mathrm{i} k\theta}(1-\mathrm{e}^{2\pi \mathrm{i} kN\theta})}{1-\mathrm{e}^{2\pi \mathrm{i} k\theta}}.$$

This means that for a suitable constant $A_1 = -\sum_{k\in\mathbb{Z}\backslash\{0\}} \xi_k = -x_0 + \xi_0$, independent of $N$, we have

$$\frac{2}{N(N+1)} S_N = \theta + \frac{2}{N+1} A_1 + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{6}$$

where the term $\mathcal{O}(1/N^2)$ is uniformly bounded with respect to $N$ due to the analyticity of $\xi$ and the Diophantine character of $\theta$ (see Lemma 6). If we neglect the "error term" $\mathcal{O}(1/N^2)$ from (6), we can use $S_N$ and $S_{2N}$, for instance, to extrapolate a value of $\theta$ with error of $\mathcal{O}(1/N^2)$. However, faster speed of convergence can be obtained by considering "higher order sums". Hence, before formalizing the extrapolation process of Section 3.2, we generalize the definition of $S_N$ to introduce $p$-order sums of the iterates. We define

$$S_N^2 = \sum_{j=1}^{N} S_j,$$

and for this sum we obtain:

$$\begin{aligned} S_N^2 &= \frac{N(N+1)(N+2)}{6}\theta - \frac{N(N+1)}{2} \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k + N \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{\mathrm{e}^{2\pi \mathrm{i} k\theta}}{1-\mathrm{e}^{2\pi \mathrm{i} k\theta}} \\ &\quad - \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{\mathrm{e}^{4\pi \mathrm{i} k\theta}(1-\mathrm{e}^{2\pi \mathrm{i} kN\theta})}{(1-\mathrm{e}^{2\pi \mathrm{i} k\theta})^2}. \end{aligned}$$

By taking the same constant $A_1$, and $A_2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \mathrm{e}^{2\pi \mathrm{i} k\theta}/(1-\mathrm{e}^{2\pi \mathrm{i} k\theta})$, we have:

$$\frac{6}{N(N+1)(N+2)} S_N^2 = \theta + \frac{3}{N+2} A_1 + \frac{6}{(N+1)(N+2)} A_2 + \mathcal{O}\left(\frac{1}{N^3}\right).$$

Proceeding by induction, we define

$$S_N^1 = S_N, \qquad S_N^p = \sum_{j=1}^{N} S_j^{p-1}. \tag{7}$$

Thus, in the general case we obtain:

$$S_N^p = \binom{N+p}{p+1}\theta + \sum_{l=1}^{p} \binom{N+p-l}{p+1-l} A_l + (-1)^{p+1} \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{\mathrm{e}^{2p\pi \mathrm{i} k\theta}(1-\mathrm{e}^{2\pi \mathrm{i} kN\theta})}{(1-\mathrm{e}^{2\pi \mathrm{i} k\theta})^p}, \tag{8}$$

where the coefficients $A_l$ are independent of $p$ and $N$ and given by

$$A_l = (-1)^l \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{\mathrm{e}^{2(l-1)\pi \mathrm{i} k\theta}}{(1-\mathrm{e}^{2\pi \mathrm{i} k\theta})^{l-1}}. \tag{9}$$

Now, we define

$$\tilde{S}_N^p = \frac{(p+1)!}{N(N+1)\cdots(N+p)} S_N^p = \binom{N+p}{p+1}^{-1} S_N^p, \qquad \tilde{A}_l^{(p)} = (p-l+2)\cdots(p+1)A_l. \tag{10}$$

Then, for the normalized sum $\tilde{S}_N^p$ we have

$$\tilde{S}_N^p = \theta + \sum_{l=1}^{p} \frac{\tilde{A}_l^{(p)}}{(N+p-l+1)\cdots(N+p)} + E(p,N), \tag{11}$$

where

$$E(p,N) = (-1)^{p+1} \frac{(p+1)!}{N\cdots(N+p)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{\mathrm{e}^{2p\pi \mathrm{i} k\theta}(1-\mathrm{e}^{2\pi \mathrm{i} kN\theta})}{(1-\mathrm{e}^{2\pi \mathrm{i} k\theta})^p}, \tag{12}$$

can be bounded (for a fixed $p$) by an expression $\mathcal{O}(1/N^{p+1})$.

### 3.2 The extrapolation procedure

Let us explain the extrapolation procedure we carry out to obtain an approximation to the rotation number $\theta$. As we have mentioned before, we simplify the computations by assuming that $N = 2^q$. Then, we pick up a fixed $p$ (the extrapolation order), with $p \le q$, and we compute the normalized sums $\{\tilde{S}^p_{N_j}\}_{j=0,\ldots,p}$, with $N_j = 2^{q-p+j}$. These sums are related with $\theta$ through formula (11). Now, if we set to zero the error terms $E(p, N_j)$, for any $j = 0, \ldots, p$, we obtain a square system of linear equations for the unknowns $\theta$ and $\{\tilde{A}^{(p)}_l\}_{l=1,\ldots,p}$. By solving this system we compute the (extrapolated) value of $\theta$.

Unfortunately, and due to the denominator of $\tilde{A}^{(p)}_l$ in (11), the matrix of this linear system depends on $q$. This implies that if we fix the value of $p$ and consider different values of $q \ge p$, the systems to be solved have different matrices for different values of $q$. We can overcome this problem by considering the following alternative expression for (11):

$$\tilde{S}^p_N = \theta + \sum_{l=1}^{p} \frac{\hat{A}^{(p)}_l}{N^l} + \hat{E}(p, N), \tag{13}$$

for certain $\{\hat{A}^{(p)}_l\}_{l=1,\ldots,p}$, independent of $N$, where $\hat{E}(p, N)$ differs from $E(p, N)$ by an expression of $\mathcal{O}(1/N^{p+1})$. Hence, a similar error can be expected by neglecting $\hat{E}(p, N)$ in (13) instead of $E(p, N)$ in (11). The linear system thus obtained is

$$\begin{pmatrix} \tilde{S}^p_{2^{q-p}} \\ \tilde{S}^p_{2^{q-p+1}} \\ \tilde{S}^p_{2^{q-p+2}} \\ \cdots \\ \tilde{S}^p_{2^q} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{2^1} & \frac{1}{2^2} & \cdots & \frac{1}{2^p} \\ 1 & \frac{1}{2^2} & \frac{1}{2^4} & \cdots & \frac{1}{2^{2p}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \frac{1}{2^p} & \frac{1}{2^{2p}} & \cdots & \frac{1}{2^{p^2}} \end{pmatrix} \begin{pmatrix} \theta \\ \hat{A}^{(p)}_1/2^{1(q-p)} \\ \hat{A}^{(p)}_2/2^{2(q-p)} \\ \cdots \\ \hat{A}^{(p)}_p/2^{p(q-p)} \end{pmatrix}. \tag{14}$$

As the matrix of this system is independent of $q$, we obtain

$$\theta = \Theta(p, 2^q) + e(p, 2^q) = \sum_{l=0}^{p} c^{(p)}_l \tilde{S}^p_{2^{q-p+l}} + e(p, 2^q), \tag{15}$$

for certain coefficients $\{c^{(p)}_l\}_{l=0,\ldots,p}$, and where we expect $e(p, 2^q) = \mathcal{O}(1/2^{(p+1)q})$. Such coefficients are given by the first row of the inverse of the matrix of system (14). For instance, simple computations show that:

$$\begin{aligned}
\theta &= 2\tilde{S}^1_{2^q} - \tilde{S}^1_{2^{q-1}} + \mathcal{O}\left(\frac{1}{2^{2q}}\right), \\
\theta &= \frac{8}{3}\tilde{S}^2_{2^q} - 2\tilde{S}^2_{2^{q-1}} + \frac{1}{3}\tilde{S}^2_{2^{q-2}} + \mathcal{O}\left(\frac{1}{2^{3q}}\right), \\
\theta &= \frac{64}{21}\tilde{S}^2_{2^q} - \frac{8}{3}\tilde{S}^2_{2^{q-1}} + \frac{2}{3}\tilde{S}^2_{2^{q-2}} - \frac{1}{21}\tilde{S}^2_{2^{q-3}} + \mathcal{O}\left(\frac{1}{2^{4q}}\right), \\
\theta &= \frac{1024}{315}\tilde{S}^2_{2^q} - \frac{64}{21}\tilde{S}^2_{2^{q-1}} + \frac{8}{9}\tilde{S}^2_{2^{q-2}} - \frac{2}{21}\tilde{S}^2_{2^{q-3}} + \frac{1}{315}\tilde{S}^2_{2^{q-4}} + \mathcal{O}\left(\frac{1}{2^{5q}}\right), \\
\theta &= \frac{32768}{9765}\tilde{S}^2_{2^q} - \frac{1024}{315}\tilde{S}^2_{2^{q-1}} + \frac{64}{63}\tilde{S}^2_{2^{q-2}} - \frac{8}{63}\tilde{S}^2_{2^{q-3}} + \frac{2}{315}\tilde{S}^2_{2^{q-4}} - \frac{1}{9765}\tilde{S}^2_{2^{q-5}} + \mathcal{O}\left(\frac{1}{2^{6q}}\right).
\end{aligned}$$

In general, the coefficients $c_l^{(p)}$ of (15) are given by

$$c_l^{(p)} = (-1)^{p-l} \frac{2^{l(l+1)/2}}{\delta(l)\delta(p-l)}, \tag{16}$$

where we define $\delta(n) := (2^n - 1)(2^{n-1} - 1) \cdots (2^1 - 1)$ for $n \geq 1$ and $\delta(0) := 1$.

**Remark 4.** *We point out that everything is analogous if we consider a map $f : \mathbb{T}^d \to \mathbb{T}^d$, where $\mathbb{T}^d$ is the d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, and we assume that it admits an analytic (or smooth enough) conjugation to a rotation, with rotation vector $\omega \in \mathbb{R}^d$. This means that there is an analytic (or smooth) diffeomorphism $\eta : \mathbb{T}^d \to \mathbb{T}^d$, such that $f \circ \eta = \eta \circ \mathcal{T}_\omega$, where $\mathcal{T}_\omega(x) = x + \omega$ is defined analogously to the one-dimensional case (see [32] for a tutorial on toral maps and flows). We observe that the Diophantine condition on $\omega$ is now*

$$|\mathrm{e}^{2\pi \mathrm{i} \langle k, \omega \rangle} - 1|^{-1} \leq C(|k_1| + \cdots + |k_d|)^\tau, \qquad \forall k \in \mathbb{Z}^d \setminus \{0\},$$

*for certain $C > 0$ and $\tau \geq d$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^d$. In this case, the normalized sums $\tilde{S}_N^p$ belong to $\mathbb{R}^d$ ($f$ plays the rôle of a lift of the map to the universal covering $\mathbb{R}^d$), but the formulas for $\omega$ are still given by (15), with the same coefficients (16).*

**Remark 5.** *Let $\varphi_t$ be a flow on $\mathbb{T}^d$. If we assume that this flow is conjugate to a linear quasi-periodic flow, with vector of basic frequencies $\omega \in \mathbb{R}^d$, then we can also extend the method to the numerical computation of $\omega$. As $\varphi_t$ takes the following form (in the covering space)*

$$\varphi_t(x) = x + \omega t + \sum_{k \in \mathbb{Z}^d} \xi_k \mathrm{e}^{2\pi \mathrm{i} \langle k, \omega \rangle t},$$

*there are two ways to deal with this case. The first one is to consider a Poincaré section of the flow so that we work with a map on $\mathbb{T}^{d-1}$. The second one is to compute the values of $\varphi_t(x_0)$, with a fixed $x_0 \in \mathbb{R}^d$, for a sequence of equi-spaced values of $t$. If this constant time step is the unity, then everything is identical than for a map on $\mathbb{T}^d$.*

It is clear that the numerical implementation of this method in a computer presents several problems. The most evident arises from the fact that when computing $S_N^p$, for high values of $p$ and $N$, one obtains very big numbers (of order $N^{p+1}$) which can give rise to an important lose of precision. Another source of problems is the computation of the iterates itself. If we require a great number of them, the accuracy of $f^n(x_0)$ decreases with $n$ due to the accumulation of round-off errors. If the iterates have big error, it makes no-sense to use high extrapolation orders. The most natural way to overcome these problems is to do the computations by using a representation of real numbers with a computer arithmetic having a great number of decimal digits (better multiple precision), and to be very careful with the manipulation of big numbers, to prevent the lose of significative digits (for instance, by storing separately its integer and decimal part) and beware not to "saturate" them.

### 3.3 Bounding the error of the method

Once we have introduced the extrapolation procedure, in this section we are going to discuss how the error $e(p, 2^q)$ in the extrapolation process (see (15)) behaves as function of $p$ and $N = 2^q$.

It is clear that, for a fixed $p$, the expressions $E(p, N)$ in (12), $\hat{E}(p, N)$ in (13) and $e(p, N)$ in (15) are of $\mathcal{O}(1/N^{p+1})$. However, the coefficients given this order depend on $p$, and thus, a natural question is how to select $p$ as function of $N$ so that the error on the approximation of $\theta$ becomes as smaller as possible.

For this purpose, let us start with the following (standard) bound on small divisors.

**Lemma 6.** *Let $\xi(x)$ be a real analytic function in the complex strip of width $\Delta > 0$,*

$$\mathcal{A}_\Delta = \{x \in \mathbb{C} : |\mathrm{Im}(x)| < \Delta\}, \tag{17}$$

*with $|\xi(x)| \le M$ up to the boundary of the strip and 1-periodic in $x$. If we expand $\xi$ in Fourier series (5) and we consider a Diophantine number $\theta$ verifying (3), we have*

$$\left| \sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k \frac{\mathrm{e}^{2p\pi \mathrm{i}k\theta}(1-\mathrm{e}^{2\pi \mathrm{i}kN\theta})}{(1-\mathrm{e}^{2\pi \mathrm{i}k\theta})^p} \right| \le \frac{\mathrm{e}^{-\pi\Delta}}{1-\mathrm{e}^{-\pi\Delta}} 4MC^p \left(\frac{\tau p}{\pi\Delta \mathrm{e}}\right)^{\tau p}. \tag{18}$$

**Proof :** By using standard estimates on the Fourier coefficients of a real analytic function, we have that $|\xi_k| \le M\mathrm{e}^{-2\pi\Delta|k|}$ and from (3) we have that $|1-\mathrm{e}^{2\pi \mathrm{i}k\theta}|^{-p} \le C^p|k|^{\tau p}$. Moreover, we observe that $\sup_{x\ge 0}\{\mathrm{e}^{-\pi\Delta x}x^{\tau p}\} = (\tau p/(\pi\Delta\mathrm{e}))^{\tau p}$. Thus, the bound is obtained from the sum of a geometric progression of ratio $\mathrm{e}^{-\pi\Delta}$.

□

The estimate given by Lemma 6 is not optimal, but is good enough for our purposes and simplifies the computations.

**Remark 7.** *It is clear that the expression on the left hand side of (18), and thus the error $E(p,N)$ of (12), is still convergent if the conjugacy $\eta(x) = x + \xi(x)$ is only a smooth function with enough regularity, and $p$ is not too big. More precisely, it is known that if $\xi \in \mathcal{C}^r$ then $|\xi_k| \sim \mathcal{O}(|k|^{-r})$. Thus, if $r > p\tau + 1$, the expression on the left hand side of (18) is of order $\mathcal{O}(C^p/(r-\tau p-1))$.*

We apply Lemma 6 to $E(p,N)$ in (12) to obtain

$$|E(p,N)| \le \frac{(p+1)!}{N(N+1)\cdots(N+p)} \frac{\mathrm{e}^{-\pi\Delta}}{1-\mathrm{e}^{-\pi\Delta}} 4MC^p \left(\frac{\tau p}{\pi\Delta\mathrm{e}}\right)^{\tau p}. \tag{19}$$

By applying Stirling's formula to (19), $j! = \sqrt{2\pi} j^{j+1/2}\mathrm{e}^{-j+\chi_j/(12j)}$ with $0 < \chi_j < 1$, we have

$$|E(p,N)| \le \frac{(p+1)!}{N^{p+1}} \frac{\mathrm{e}^{-\pi\Delta}}{1-\mathrm{e}^{-\pi\Delta}} 4MC^p \left(\frac{\tau p}{\pi\Delta\mathrm{e}}\right)^{\tau p} \le ab^p \frac{p^{p(\tau+1)}}{N^{p+1}}, \tag{20}$$

for certain constants $a$ and $b$, independent of $p$ and $N$.

However, if we use the alternative expression (13), we can ensure that the new error $\hat{E}(p,N)$ is bounded by an analogous estimate to (20), with different constants $a, b$. The reason of this fact is that when changing (11) by (13), the error $\hat{E}(p,N)$ is given by

$$\hat{E}(p,N) = E(p,N) + \left( \sum_{l=1}^{p} \frac{\tilde{A}_l^{(p)}}{(N+p-l+1)\cdots(N+p)} - \sum_{l=1}^{p} \frac{\hat{A}_l^{(p)}}{N^l} \right),$$

where the new coefficients $\hat{A}_l^{(p)}$ are defined from the old ones $\tilde{A}_l^{(p)}$ so that the error $\hat{E}(p,N)$ is still of order $\mathcal{O}(1/N^{p+1})$. This implies that $\hat{A}_l^{(p)}$, for any $l = 1, \ldots, p$, is a linear combination of $\{\tilde{A}_j^{(p)}\}_{j=1,\ldots,l}$ whose coefficients are polynomials in p. The other important thing here is that formulas (9) and (10) for $\tilde{A}_l^{(p)}$ show that the contribution of the small divisors $1-\mathrm{e}^{2\pi \mathrm{i}k\theta}$ to $\tilde{A}_l^{(p)}$ comes with a smaller power than for $E(p,N)$ in (12). As a summary, one can check that the final bound for $\hat{E}(p,N)$ is of the form

$$|\hat{E}(p,N)| \leq \hat{a}\hat{b}^p \frac{p^{p(\tau+1)}}{N^{p+1}}, \tag{21}$$

for some constants $\hat{a}$ and $\hat{b}$ independent of $p$ and $N$.

Now, let us resume the extrapolation method. We pick up a fixed $p$, compute $N = 2^q$ iterates of the map and the averaged sums $\tilde{S}^p_{N_j}$, with $N_j = 2^{q-p+j}$, for $j = 0,\ldots,p$. By using formula (15) to compute $\theta$, we obtain that the error of the extrapolation is given by

$$e(p,2^q) = -\sum_{l=0}^{p} c_l^{(p)} \hat{E}(p, 2^{q-p+l}).$$

To bound this error we need some idea about how behave the coefficients $c_l^{(p)}$ given in (16). From the following lower bound for $\delta(n)$:

$$\delta(n) = 2^{n(n+1)/2}\prod_{j=1}^{n}(1-2^{-j}) \geq 2^{n(n+1)/2}K,$$

where $K := \prod_{j\geq 1}(1-2^{-j})$, we have

$$|c_l^{(p)}| \leq \frac{1}{K^2} 2^{-(p-l)(p-l+1)/2}.$$

In this way, using (21), we obtain:

$$\begin{aligned} |e(p,2^q)| &\leq \sum_{l=0}^{p} |c_l^{(p)}||\hat{E}(p,2^{q-p+l})| \leq \frac{\hat{a}}{K^2}\hat{b}^p p^{p(\tau+1)} \frac{1}{2^{q(p+1)}} \sum_{l=0}^{p} 2^{(p-l)(p+l+1)/2} \\ &\leq \tilde{a}\tilde{b}^p p^{p(\tau+1)} 2^{(p/2-q)(p+1)}, \end{aligned} \tag{22}$$

for some constants $\tilde{a}$, $\tilde{b}$ independent of $p$ and $q$ (having into account that the biggest term in the last sum corresponds to $l = 0$).

Once we have a bound of the error of the method, it is natural to guess which is the optimal value of $p$ to use for the extrapolation. This is a very realistic setting: we compute $N = 2^q$ iterates and we want to select $p$ so that the error $|e(p,2^q)|$ becomes as smaller as possible. To this end, we define (for a fixed $q$) the function

$$g(p) = \log_2 \tilde{a} + p\log_2 \tilde{b} - (q-p/2)(p+1) + (\tau+1)p\log_2 p,$$

obtained by taking binary logarithm of the right hand side of formula (22), and we try to minimize this function. Thus, we consider the equation $g'(p) = 0$,

$$p - q + 1/2 + \log_2 \tilde{b} + (\tau+1)\log_2 p + (\tau+1)\log_2(\mathrm{e}) = 0,$$

from which we can compute a zero $p^* = p^*(N)$ (not an integer in general) that behaves (for big values of $q$) as $p^* \simeq q - (\tau+1)\log_2 q = \log_2 N - (\tau+1)\log_2(\log_2 N)$. By using this value of $p^*$

(in fact, one has to pick up the most closed integer), we optimize the bound (22) of the error obtaining

$$|e(p^*, 2^q)| \leq \frac{1}{N^{\frac{1}{2}\log_2 N - (\tau+1)\log_2(\log_2 N) + \mathcal{O}(1)}}.$$

So, if we compute $N = 2^q$ iterates and use $p^* = p^*(N)$ as extrapolation order, we obtain an asymptotic expression for the error smaller than any power of $1/N$. In Section 4.1 we perform some numerical comparisons between the real error and the bound (22) for different values of $p$ (see Figure 2).

**Remark 8.** *From the practical (numerical) point of view it is difficult to take advantage of this theoretical discussion in order to optimize the error of the method. Let us introduce the strategy we use in Section 4 to estimate the error of the method.*

*If we fix the extrapolation order $p$ and compute $\Theta(p, 2^q)$, we know that*

$$|e(p, 2^q)| = |\Theta(p, 2^q) - \theta| \leq c/2^{q(p+1)}, \tag{23}$$

*for certain (unknown) constant $c$, independent of $q$ (see (22)). If we want to control the size of $|e(p, 2^q)|$ we need to estimate $c$. To do that, we suppose also known $\Theta(p, 2^{q-1})$ and we consider (23) for $|e(p, 2^{q-1})|$. Then, we replace in this inequality the exact value of $\theta$ by $\Theta(p, 2^q)$, as we expect $\Theta(p, 2^q)$ to be closer to $\theta$ than $\Theta(p, 2^{q-1})$. After that, we estimate $c$ by*

$$c \sim 2^{(q-1)(p+1)}|\Theta(p, 2^{q-1}) - \Theta(p, 2^q)|.$$

*Now, we replace $c$ in (23) by this value and we estimate the error of $\Theta(p, 2^q)$ by*

$$|e(p, 2^q)| \leq 2^{-(p+1)}\nu|\Theta(p, 2^{q-1}) - \Theta(p, 2^q)|, \tag{24}$$

*where $\nu$ is a "safety parameter", to prevent from the fact that the true value of $c$ can oscillate as function of $q$. In the numerical computations of Section 4 we take $\nu = 10$.*

**Remark 9.** *All the discussions during this section are only valid when the rotation number $\theta$ is Diophantine. If $\theta$ is a rational number, the sums $S_N^p$ in (7) can be computed from the iterates of the map, but formula (8) has no sense because the map is not conjugate to a rotation. Nevertheless, the numerical results of Section 4 show that, even in the rational case, the method works as well as in the Diophantine one.*

*We do not have a complete justification for the efficiency of the method in the rational case, but we know that for any circle homeomorphism having a rational rotation number, every orbit is either periodic or its iterates converge to a periodic orbit (see [8]). Then, at the limit, the iterates of the map behave as periodic points. For a periodic point one can see that the normalized sums $\tilde{S}_N^p$ in (10) behave also as in (13), with $\hat{E}(p, N) = \mathcal{O}(1/N^{p+1})$, which is the only thing we need for the extrapolation to work.*

*In fact, what we observe, numerically, is that the worst case for this method is when $\theta$ is an irrational number too close to the rational ones (i.e., it is very close to resonance but it is not exactly resonant). See Figure 3 (bottom) and Figure 4 (bottom).*

## 4 Numerical results

In this section we consider some numerical applications of the method introduced in Section 3. The computations presented here have been done by using the *quad-double and double-double* *computation package* (see [18]), which provides with a *double-double* data type of approximately 32 decimal digits and a *quad-double* data type of approximately 64 decimal digits for a C++ compilator. The reason for which we use these extended arithmetics, and not for instance the usual *double* data type of a PC, with approximately 16 decimal digits, is because by working with the *double* data type the method "saturates" all the significative digits faster (the better error we can expect is $10^{-16}$), and hence, we cannot appreciate the features of the method for "big values" of $p$ and $q$.

We consider three different contexts. The first one, which is done in Section 4.1, is the Siegel disk of the quadratic polynomial $F(z) = \lambda(z - \frac{1}{2}z^2)$. We use this example, where the rotation number is known *a-priori*, as a test of the method. Section 4.2 is devoted to compute some of the most irrational Arnold tongues of the Arnold family (27). Moreover, and mainly to test the method for the case of rational rotation numbers, we also compute the Devil's Staircase of (27) for a fixed value of $\varepsilon$. Finally, in Section 4.3 we consider the two-dimensional Chirikov standard map (28). First, we perform a "frequency analysis" of the map for some values of $\varepsilon$. Next to that, we use the method as a tool to compute the invariant curve of rotation number the Golden Mean, for increasing values of $\varepsilon$. We compare the critical value of $\varepsilon$, up to which we can compute numerically this invariant curve, with the one obtained by using the classical Greene's criterion.

## 4.1 The quadratic polynomial

The first numerical application of the method is a test of the method itself and, in particular, of the behavior of the error discussed in Section 3.3. For this purpose, it is better to use examples for which the rotation number is known *a-priori*, and hence, the error of the method can be computed exactly.

The most simple context is to consider a Siegel disk in the complex plane. Let $F : \mathcal{U} \to \mathbb{C}$ be an analytic map, where $\mathcal{U} \subset \mathbb{C}$ is an open set, such that $F(0) = 0$ and $F'(0) = \mathrm{e}^{\mathrm{i}\omega}$, with $\omega = 2\pi\theta$. It is well-known that if $\theta$ is a (irrational) Brjuno number (in particular if it is Diophantine), then the map $F$ is analytically linearizable around 0 (see [2, 35]). This means that there is a unique $R > 0$ (maximal for this property) and a unique conformal isomorphism

$$\varphi : \mathbb{D}_R \to U, \qquad \varphi(0) = 0, \qquad \varphi'(0) = 1,$$

where $\mathbb{D}_R$ is the open disk of center 0 and radius $R$, such that $\varphi$ conjugates $F$ to the rotation of angle $\omega$ around the origin. That is, $F \circ \varphi = \varphi \circ \mathcal{R}_\omega$ in $\mathbb{D}_R$, where $\mathcal{R}_\omega(z) = \mathrm{e}^{\mathrm{i}\omega} z$. The (topological) rotation disk $U$ is called a Siegel disk of $F$.

It is clear that $U$ is foliated by invariant curves under the action of $F$, any of them defined as $\varphi(\mathbf{C}_s)$, with $0 < s < R$, where $\mathbf{C}_s$ is the circle of radius $s$ around the origin. The dynamics on any of these curves is analytically conjugate to a rotation on $\mathbb{T}^1$, with rotation number $\theta$. Let us suppose that, for a given $s$, the curve $\varphi(\mathbf{C}_s)$ can be analytically parameterized by $\arg(z)/2\pi$ (defined as a map from $\mathbb{C}\setminus\{0\}$ to $\mathbb{T}^1$). This holds, for instance, if $s$ is small enough, because then $\varphi(\mathbf{C}_s)$ is close to $\mathbf{C}_s$. Under this assumption, we can consider the circle map $\hat{f}_s$ defined as follows (see Figure 1). Given $x_0 \in \mathbb{T}^1$, let $z_0 \in \varphi(\mathbf{C}_s)$ be the unique point such that $x_0 = \arg(z_0)/2\pi$. Then

$$\begin{array}{rccl} \hat{f}_s : & \mathbb{T}^1 & \to & \mathbb{T}^1 \\ & x_0 = \arg(z_0)/2\pi & \mapsto & x_1 = \arg(F(z_0))/2\pi \end{array} \qquad (25)$$

is an orientation-preserving analytic circle diffeomorphism, with rotation number $\theta$. To define the lift of $\hat{f}_s$ to $\mathbb{R}$, for which we keep the same name, we only have to select the suitable determination of $\arg(\cdot)$ in any case.

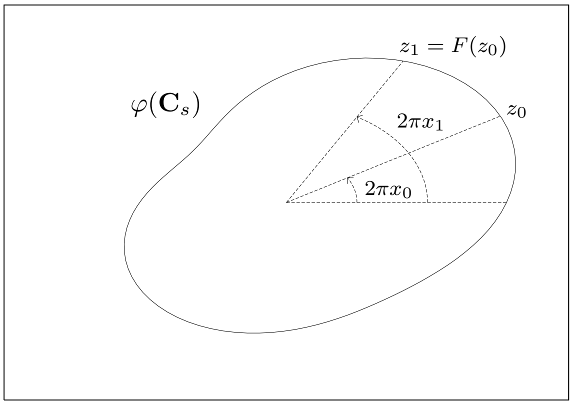


Figure 1: The circle map $\hat{f}_s$ induced by the invariant curve $\varphi(\mathbf{C}_s)$ of $F$.

The most simple case of a (non-trivial) Siegel disk is when $F$ is a quadratic polynomial. Thus, in this section we present several numerical examples working with the widely studied map $F(z) = \lambda(z - \frac{1}{2}z^2)$, where $\lambda = \mathrm{e}^{2\pi\mathrm{i}\theta}$ (see for instance [35]). We observe that $F$ has a critical point at $z = 1$ which cannot belong to the Siegel disk $U$. A remarkable property of $F$ is that if $\theta$ is a Diophantine number then this critical point belongs to the boundary of $U$ (see [16]). In particular, this implies that the invariant curves of the Siegel disk can be parameterized by its cut with the positive real axis in the interval $(0, 1)$. Thus, given any $r \in (0, 1)$, we define $f_r : \mathbb{T}^1 \to \mathbb{T}^1$ as the map $\hat{f}_s$ introduced in (25) with $s = s(r)$ so that the invariant curve $\varphi(\mathbf{C}_s)$ contains the point $z_0 = r$.

We do not have an explicit formula for this map, but to apply the method of Section 3 it is enough to know the iterates of $z_0 = r$, whose argument is $x_0 = 0$. Hence,

$$f_r^n(0) = \arg(F^n(z_0))/2\pi.$$

Moreover, if we pick up the critical point $z_0 = 1$, it is known that the closure of the set defined by its iterates gives the boundary of the Siegel disk (the limit invariant curve). This boundary is known to be a quasi-circle but is no-longer an analytic curve. This means that the width of the strip of analyticity (see (17)) around the real axis of the map $f_r$ , decreases from $+\infty$ to 0 when $r$ increases from 0 to 1. At the limit $r = 1$, the function is no-longer differentiable, but only Hölder continuous (see [14, 30]).

Let us describe now the numerical examples we consider for the quadratic polynomial $F$. For the rotation number we mainly take the Golden Mean, $\theta = (\sqrt{5} - 1)/2$, because it is known to be the "best choice" in terms of the Diophantine condition (3). In particular, as $\theta$ is a quadratic irrational, we can take $\tau = 1$. Two characterizations of the Golden Mean are that it is a zero of the equation $\theta^2 + \theta - 1 = 0$ and that its continuous fraction expansion is of constant type, $\theta = [1, 1, \ldots]$. We use these properties as a motivation to introduce the other rotation numbers we consider, which are also quadratic irrationals. We define $\theta_s$ from the (constant) continuous fraction expansion given by $\theta_s = [s, s, \ldots]$, which is a zero of $\theta^2 + s\theta - 1 = 0$. It is clear that $\theta_s$ is a Diophantine number for any $s \geq 1$, with $\tau = 1$ but with a bigger constant $C$ when $s$ increases. Roughly speaking, for a big $s$ then $\theta_s = 1/s - 6/s^3 + \mathcal{O}(1/s^5)$ is "very close" to the rational numbers.

As we know *a-priori* the rotation number of the map, we can compute (numerically) the exact error $e(p, 2^q)$, introduced in (15), of the numerical approximation $\Theta(p, 2^q)$ obtained by solving the system (14), i.e., $2^q$ is the number of iterates computed and $p$ is the extrapolation order. We expect for $e(p, 2^q)$ a similar behavior as for its bound (22).

Another point we consider for this numerical test is the comparison with another method to compute the rotation number. The alternative method we use is based on the idea that the rotation number is the constant rotation that better fits with the map if we compare it with a rotation. However, instead of computing the rotation average of the iterates as in definition (1), we look for a rational approximation for the rotation number by selecting the iterates that are closest to be periodic points ("closest returns"). Let us compute the iterates $\{f^n(0)\}_{n=1,\ldots,N}$ of a lift $f$ of a circle map, and let $P_N, Q_N \in \mathbb{N}$ be such that $|f^{Q_N}(0) - P_N| = \min_{1 \leq n \leq N} \operatorname{dist}(f^n(0), \mathbb{Z})$. Then, we take the rational number $P_N/Q_N$ as an approximation to the rotation number. This method converges to the rotation number $\theta = \rho(f)$ with an error

$$\tilde{e}(N) = P_N/Q_N - \theta \tag{26}$$

that behaves, roughly speaking, as $\mathcal{O}(1/N^2)$ (see [22]). Hence, it can be considered as "equivalent" to use extrapolation order $p = 1$ for the method of Section 3. The advantage of this alternative method is that it works independently of the arithmetic properties of the rotation number and of the smooth or analytic character of the map. Thus, it is worth to compare this method with the one we present in this paper, specially in the "critical cases", that is, for "bad" Diophantine numbers or for non-smooth maps.

For what refers to the iterates of the map, we compute them up to $2^{23} = 8388608$ at most, by using the *quad-double* data type. For this number of iterates we have not found extremely bad effects due to round-off errors.

The numerical results obtained are displayed in Figure 2 and Figure 3. To understand the meaning of the axis in the different plots, we can take into account the next general rules. The vertical axis is always a quantity related with the error of the method. In the top-left plot of Figure 2 and in Figure 3 it is always $\log_{10}$ of the error, which gives (minus) the number of correct decimal digits. For the concrete meaning of the vertical axis in the remaining three plots of Figure 2, see the explanations below. The horizontal axis means, depending on the plot, the extrapolation order $p$ or $q = \log_2 N$, where $N = 2^q$ is the number of iterates used. Finally, as all these error graphs are discontinuous, to plot them we join consecutive points by lines. The detailed explanation of these plots is given as follows.

The four plots displayed in Figure 2 correspond to the same example. We take the invariant curve of $F(z)$, with rotation number the Golden Mean, having as initial condition $z_0 = \frac{1}{2}$. For this initial condition we compute up to $2^{23}$ iterates of the map. Then, our purpose in this figure is to illustrate the results obtained by using different values of $q$ and different extrapolation orders $p$, and to compare the exact errors thus obtained with the "asymptotic behavior" (22) of the error.

**Top-left:** The dashed curve is the graph of the map $q \in \{10, \ldots, 23\} \mapsto \log_{10} |\tilde{e}(2^q)|$ (see (26)). For the rest of the (continuous) curves in the plot we consider different values of the extrapolation order $p \in \{0, 1, 2, 4, 6, 10\}$ (recall $p = 0$ means definition (1)). For any of these values of $p$ we plot the graph of the function $q \in \{10, \ldots, 23\} \mapsto \log_{10} |e(p, 2^q)|$. The error-curves thus obtained appear ordered in decreasing order with respect to $p$ by its value at $q = 23$. As expected, the bigger is the $p$ the smaller is the error for $N = 2^{23}$. However, we observe that the fact that some of them have self-intersections makes clear that, for a given $q$, to choose the greatest value of $p$

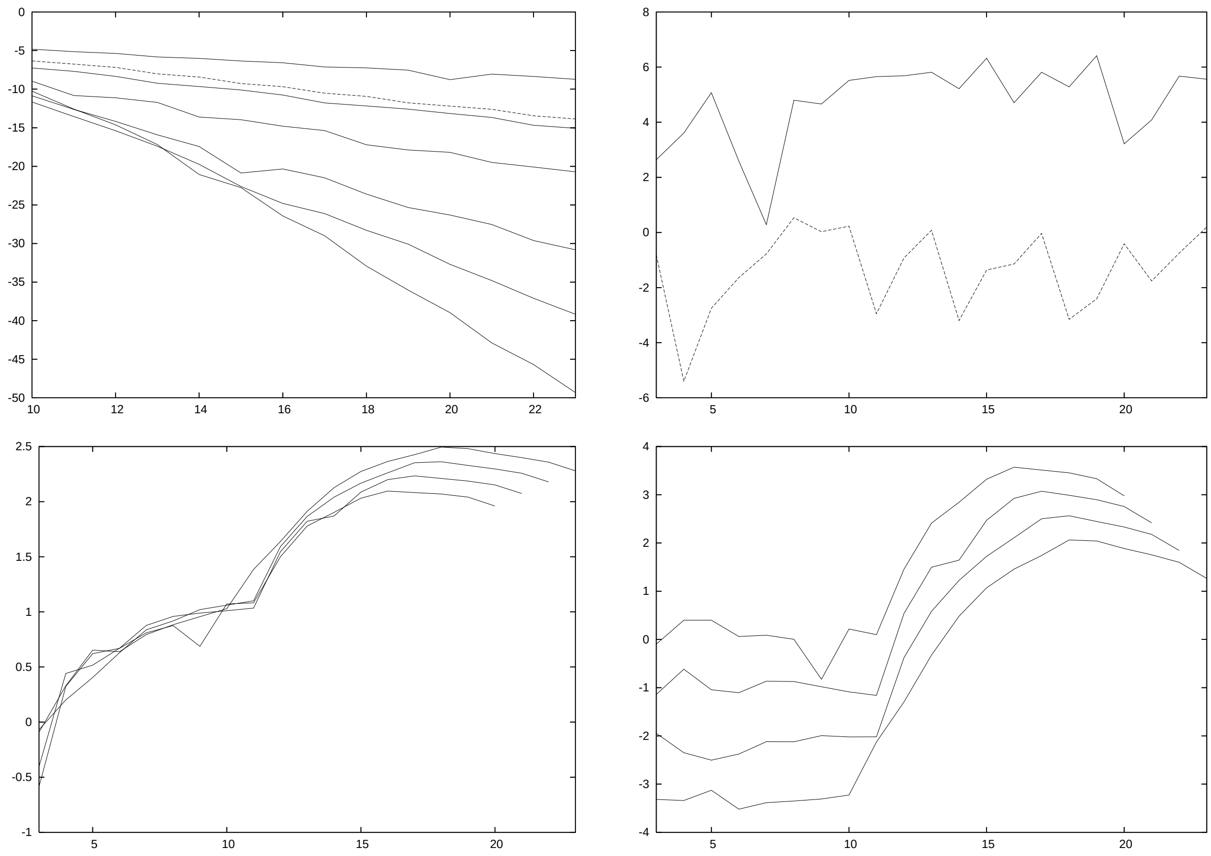


Figure 2: Numerical tests of the error of the method applied to the computation of the rotation number of the invariant curve of Siegel disk of the quadratic polynomial $F(z) = \lambda(z - \frac{1}{2}z^2)$, with rotation number the Golden Mean and initial condition $z_0 = \frac{1}{2}$. See the text for full details.

is not always the best choice (to have the smaller error). See also Figure 3. We note that the dashed curve is very close to the one corresponding to $p = 1$.

**Top-right:** We plot the graph of $q \in \{3, \ldots, 23\} \mapsto \log_2 |e(p, 2^q)| + q(p+1)$ for $p = 1$ (continuous curve) and $p = 2$ (dashed curve). From the bound on the error $e(p, 2^q)$ given in (22), we expect these curves to remain bounded when $q$ goes to infinity.

**Bottom-left:** For any $q \in \{20, 21, 22, 23\}$ we compute the error $e(p, 2^q)$ for all the possible values of $p$, and we plot the graph of

$$p \in \{3, \ldots, q\} \mapsto \frac{\log_2 |e(p, 2^q)| + (q - p/2)(p+1)}{p \log_2 p}.$$

We expect these curves to be close to $\tau + 1 = 2$ at the "limit" $p = q$ (see (22)), which fits quite nicely.

**Bottom-right:** The same as in the previous plot, but now for the graph of the function

$$p \in \{3, \ldots, q\} \mapsto \frac{\log_2 |e(p, 2^q)| + (q - p/2)(p+1) - 2p \log_2 p}{p}.$$

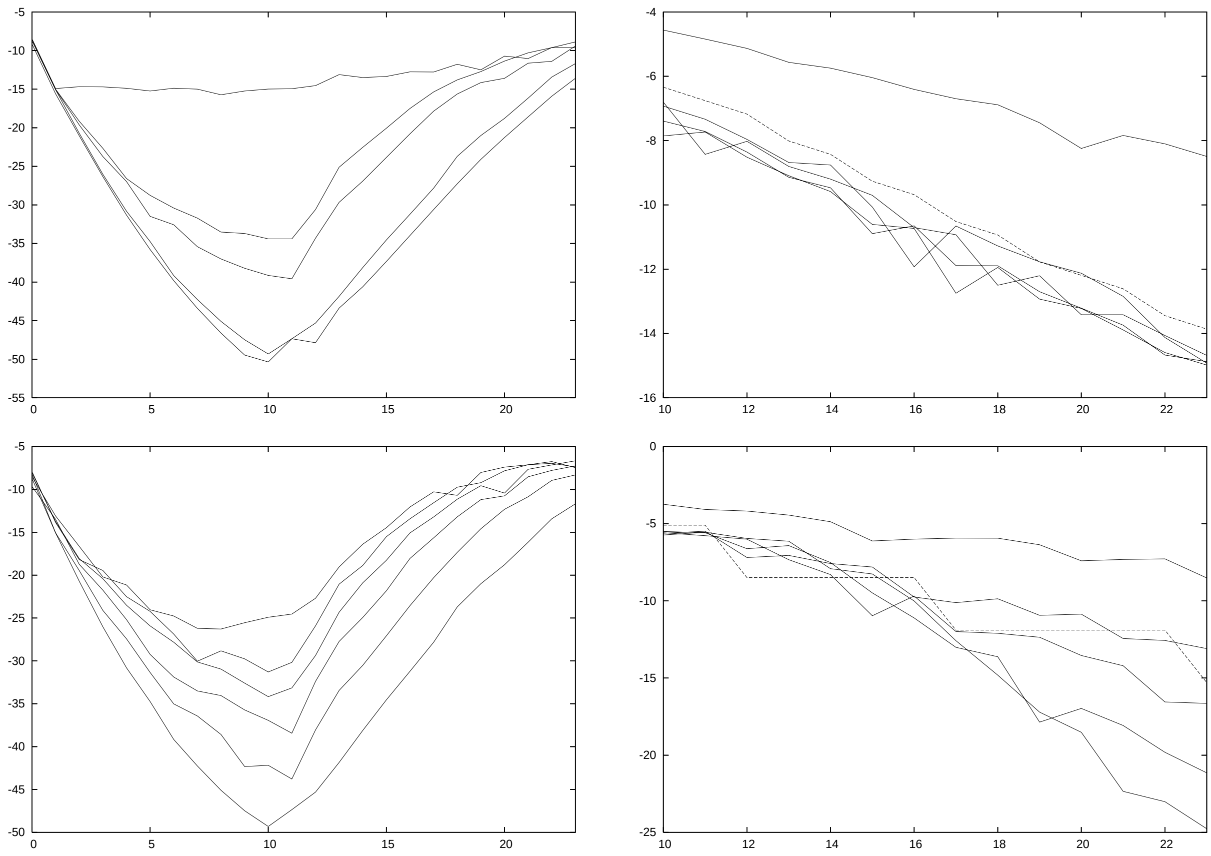


Figure 3: More numerical tests of the error of the method for the quadratic polynomial $F(z)$. See the text for full details.

We expect these curves to be bounded at the "limit" $p = q$, by a constant independent of $q$. From the results displayed in this plot, it is clear that the error predicted by formula (22) is quite correct from the asymptotic point of view. However, we cannot say the same about the "transitory regime", because it seems that for values of the extrapolation order $p$ not "too big" with respect to $q$, the error $e(p, 2^q)$ is quite smaller than its bound (22). Of course, this fact is not a bad new, but from the practical point of view it makes more difficult to select the optimal value $p^* = p^*(q)$. See Figure 3 for a more clear view of the behavior of $p^*$.

Our purpose in Figure 3 is to show how the error $e(p, 2^q)$ is affected by the two different aspects we have considered in the theoretical analysis: the width of the strip of analyticity of the conjugation (or its lack of smoothness) and the good or bad arithmetic properties of the rotation number. We consider again invariant curves of the quadratic polynomial $F(z)$, but now we apply the method to different initial conditions and different values of the rotation number $\theta$. In any case, the number of iterates we compute is up to $2^{23}$.

**Top-left:** In this plot we show the effect of the width of analyticity of the conjugation $\eta$ (see (2)) on the numerical precision of the rotation number. To do that, we take $F(z)$ with rotation number the Golden Mean and consider different initial conditions $z_0 = r$, with $r \in \{0.2, 0.5, 0.9, 0.95, 1.\}$. Then, for any initial condition, we plot the graph of $p \in \{0, \ldots, 23\} \mapsto$

$\log_{10}|e(p,2^{23})|$. We obtain five different curves that are ordered from down to up as $r$ increases (and so the width of analyticity decreases). We recall that for the limit case $z_0 = 1$ the invariant curve is only continuous.

**Top-right:** Here we discuss more precisely the effect of the non-smoothness of the conjugation on the precision of the method, by taking the limit case $z_0 = 1$ of the previous plot. Then, we compute $2^{23}$ iterates of the critical point, consider five different extrapolation orders, $p \in \{0, 1, 2, 6, 10\}$, and for any of them we plot the error curve $q \in \{10, \ldots, 23\} \mapsto \log_{10}|e(p, 2^q)|$. The upper one corresponds to $p = 0$. The dashed curve is the graph of $q \in \{10, \ldots, 23\} \mapsto \log_{10}|\tilde{e}(2^q)|$ (see (26)). What we observe is that all the error curves of the plot seem to have the same behavior for $p \geq 1$, with an error of $\mathcal{O}((1/2^q)^2)$, and hence the method is useless for $p > 1$. But albeit the map at the boundary is only Hölder continuous but not differentiable, and thus there is no justification for the extrapolation, the method for $p \geq 1$ is not worst than to compute the closest returns.

**Bottom-left:** Now we discuss the effect of the Diophantine properties of the rotation number on the precision of the method. We consider $F(z)$ for six different rotation numbers of constant type, $\theta \in \{\theta_1, \theta_{10}, \theta_{20}, \theta_{30}, \theta_{40}, \theta_{50}\}$, and we compute $2^{23}$ iterates of the initial condition $z_0 = \frac{1}{2}$. We plot the error curves $p \in \{0, \ldots, 23\} \mapsto \log_{10}|e(p, 2^{23})|$, which appear ordered from down to up as a function of the subscript $j$ of $\theta_j$. As expected, the method works better for "good" Diophantine numbers.

**Bottom-right:** Among the Diophantine numbers of the previous plot, here we focus on the worst case, $\theta = \theta_{50}$. We consider the invariant curve with initial condition $z_0 = \frac{1}{2}$ and we plot the error curves $q \in \{10, \ldots, 23\} \mapsto \log_{10}|e(p, 2^q)|$ for five different extrapolation orders, $p \in \{0, 1, 2, 6, 10\}$. The dashed curve is the error $q \in \{10, \ldots, 23\} \mapsto \log_{10}|\tilde{e}(2^q)|$. We observe that, even though for moderate values of $q$ the error $\tilde{e}(2^q)$ is the smallest one, when $q$ increases the extrapolation effects arise giving better results if $p \geq 1$.

## 4.2 The Arnold family

Let us consider the Arnold family of circle maps,

$$\begin{array}{rccl} f_{\alpha,\varepsilon}: & \mathbb{T}^1 & \longrightarrow & \mathbb{T}^1 \\ & x & \longrightarrow & x + \frac{\alpha}{2\pi} + \frac{\varepsilon}{2\pi}\sin(2\pi x) \end{array} \tag{27}$$

where $(\alpha, \varepsilon)$ are real parameters, $\alpha \in [0, 2\pi)$, $\varepsilon \in [0, 1)$. For any pair of values of the parameters, the map $f_{\alpha,\varepsilon}$ is an orientation-preserving analytic circle diffeomorphism, so that we can define its rotation number as a function of $(\alpha, \varepsilon)$, namely $\rho(\alpha, \varepsilon)$. Given an arbitrary $\theta \in [0, 1)$, the set $T_\theta = \{(\alpha, \varepsilon) : \rho(\alpha, \varepsilon) = \theta\}$ is called the Arnold tongue of rotation number $\theta$. If $\theta$ is a rational number, then $T_\theta$ is a set with interior. If $\theta$ is irrational, then $T_\theta$ is a continuous curve connecting $\varepsilon = 0$ with $\varepsilon = 1$, which is the graph of a function $\varepsilon \mapsto \alpha(\varepsilon)$, with $\alpha(0) = 2\pi\theta$. In the Diophantine case this curve is known to be analytic for any $\varepsilon \in [0, 1)$ (see [25, 9]).

The first application of the method is the numerical computation of some irrational Arnold tongues for this family. To do that, we fix a Diophantine number $\theta$ and solve the equation $g(\alpha, \varepsilon) := \rho(\alpha, \varepsilon) - \theta = 0$. As we know the solution of this equation for $\varepsilon = 0$, we use numerical continuation with respect to $\varepsilon$ to obtain the curve $\alpha(\varepsilon)$. To be more precise, we pick up a finite sequence of values of $\varepsilon$, $\{\varepsilon_j\}_{j=0,\ldots,K}$, with $\varepsilon_0 = 0$ and $\varepsilon_K = 1$ (for instance $\varepsilon_j = j/K$) and compute a numerical approximation $\alpha_j^*$ of $\alpha(\varepsilon_j)$. We obtain $\alpha_j^*$ by solving the equation $g(\alpha, \varepsilon_j) = 0$ by means of the secant method. To start up the secant method we need two initial

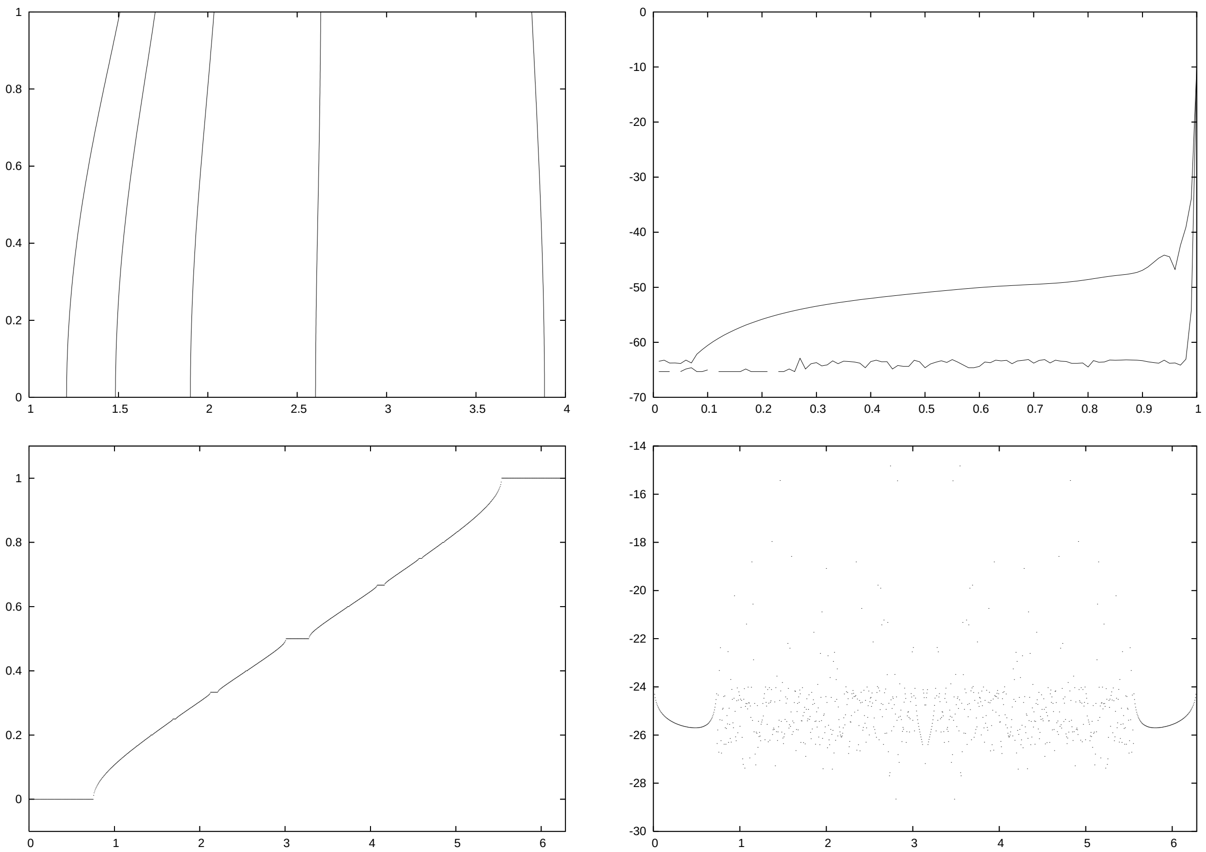


Figure 4: Some Arnold tongues of the Arnold family, the Devil's Staircase for $\varepsilon = 0.75$ and two plots showing the errors in the computation of the figures. See the text for full details.

approximations of $\alpha_j^*$. In the general case $j = 2, \ldots, K$, these two approximations are $\alpha_{j-1}^*$ and the value obtained by linear interpolation between $(\varepsilon_{j-2}, \alpha_{j-2}^*)$ and $(\varepsilon_{j-1}, \alpha_{j-1}^*)$.

To evaluate $\rho(\alpha, \varepsilon)$ we use the method of Section 3. Of course, for a given pair $(\alpha, \varepsilon)$ we cannot ensure $\rho(\alpha, \varepsilon)$ to be Diophantine. However, if $(\alpha, \varepsilon)$ is close to a very irrational Arnold tongue $T_\theta$, we expect the method to work quite well (see Remark 2).

The second application is the numerical computation of the Devil's Staircase for a given $\varepsilon \in (0, 1)$. Thus, we set $\varepsilon$ fixed in (27) and consider the one parameter family of circle maps $\{f_{\alpha,\varepsilon}\}_{\alpha \in [0,2\pi)}$. The (continuous) graph of the function $\alpha \mapsto \rho(\alpha, \varepsilon)$ is called a Devil's Staircase (see [8]). We observe that if $\rho(\alpha^*, \varepsilon) \in \mathbb{Q}$, for certain $\alpha^*$, then this function is constant in a neighborhood of $\alpha^*$. If $\rho(\alpha^*, \varepsilon) \notin \mathbb{Q}$, then $\alpha \mapsto \rho(\alpha, \varepsilon)$ is strictly increasing at $\alpha = \alpha^*$. As the values of $\alpha$ for which $\rho(\alpha, \varepsilon) \in \mathbb{Q}$ are dense in $[0, 2\pi)$ (the complementary is a cantor set), there are an infinite number of intervals in which the function is locally constant, given rise to a staircase with a dense number of stairs.

The results obtained for the Arnold family are displayed in Figure 4. Here we give a detailed explanation of them.

**Top-left:** We plot the Arnold tongues $T_\theta$ for the quadratic irrationals $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ (recall that $\theta_s = [s, s, \cdots])$. The horizontal axis means the value of $\alpha \in [0, 2\pi)$ and the vertical one the value of $\varepsilon \in [0,1]$. The computation of these Arnold tongues has been done by using the *quad-double* data type and a fixed extrapolation order $p = 9$. The continuation step with respect to $\varepsilon$ is $10^{-2}$, so we plot 100 points for any tongue $T_\theta$. The errors we allow for the numerical continuation are, at most, $10^{-32}$ for the evaluation of the rotation number (by using the estimate (24) with $\nu = 10$) and $10^{-30}$ for the secant method (distance between two consecutive iterates). This means that to evaluate the rotation number we compute iterates of the map up to $2^{23}$ at most, and we stop when the estimated error (24) is smaller than $10^{-32}$. The required number of iterates increases from $2^{18}$ to $2^{23}$ as $\varepsilon$ approaches to 1. The number of iterates of the secant method is not limited, but typically we need four iterates to determine the points of $T_{\theta_1}$ and $T_{\theta_2}$ and five iterates for the remaining three tongues.

We remark that for $\varepsilon = 1$ the map (27) is an analytic orientation-preserving homeomorphism, but not a diffeomorphism. Nevertheless, Yoccoz [34] proved that such a map is still conjugate to a rotation if the rotation number is irrational. What we observe for $\varepsilon = 1$ is that the numerical computation of the rotation number works quite well. However, the secant method only has linear speed of convergence and a big number of iterates (from 18 to 24 depending on the tongue) is needed.

**Top-right:** In this plot we illustrate the typical behavior of the error when computing the Arnold tongues. The horizontal axis is $\varepsilon \in [0,1]$ and the vertical one $\log_{10}$ of the errors. The two error curves we plot correspond to the computation of the Arnold tongue $T_{\theta_1}$. The upper one shows the error of the secant method after five iterates. The lower one is the exact error $e(9, 2^{20})$, see (15), for the point $(\alpha_j^*, \varepsilon_j)$ obtained after five iterates of the secant method. The gaps of the lower curve correspond to values of $\varepsilon_j$ for which the numerical error is zero. For most of the values of $\varepsilon$ the errors obtained are smaller than the ones required to compute $T_{\theta_1}$ in the previous plot.

**Bottom-left:** We plot the Devil's Staircase for $\varepsilon = 0.75$. In the horizontal axis we plot $\alpha \in [0, 2\pi]$, with a step $2\pi \times 10^{-3}$, and in the vertical one the corresponding rotation number $\rho(\alpha, 0.75)$. The computations have been done by using the *double-double* data type, a fixed extrapolation order $p = 7$ and up to $2^{20}$ iterates of the map, at most. We estimate the error of the rotation number by using (24) with $\nu = 10$, and we validate the rotation number when this error is smaller than $10^{-24}$.

**Bottom-right:** We plot the error (24) for the points displayed in the previous figure. The horizontal axis is $\alpha \in [0, 2\pi]$ and the vertical one is $\log_{10}$ of this error. For 91% of the points this error is smaller than $10^{-24}$ after at most $2^{20}$ iterates (for 60% we need at most $2^{18}$ iterates). For the remaining 9%, the estimate on the error does not achieve this critical tolerance after $2^{20}$ iterates, but it is always smaller than $10^{-19}$ except for six points. As we pointed out in Remark 9, the rotation numbers of these six points seem to be irrational numbers very close to resonance (thus having a big constant $C$ in (3) in the Diophantine case). For instance, for the point $\alpha = 872 \times 10^{-3}\pi$ we have computed the corresponding rotation number $\theta = \rho(\alpha, 0.75)$ with an error of $1.5 \times 10^{-15}$. The computed $\theta$ verifies $|73 \times \theta - 15| \sim 1.3 \times 10^{-5}$, which means that it is very close to the rational 15/73.

### 4.3 The Chirikov standard map

We consider the following family of exact symplectic analytic diffeomorphisms of the cylinder,

$$SM_\varepsilon : (x, y) \in \mathbb{T}^1 \times \mathbb{R} \mapsto (x + y + \varepsilon \sin(2\pi x), y + \varepsilon \sin(2\pi x)) \in \mathbb{T}^1 \times \mathbb{R} \tag{28}$$

where $\varepsilon \geq 0$ is a parameter. The map $SM_\varepsilon$ is usually referred as the Chirikov standard map [6].

For $\varepsilon = 0$ the cylinder is filled by invariant curves given by $\mathbb{T}^1 \times \{y_0\}$. The dynamics of the variable $x$ on any of these invariant circles is a rotation of rotation number $y_0$. As the map (28) is a perturbation of an integrable twist map, we can apply Moser's Twist Theorem to it (see [27]). Then, if we consider a fixed Diophantine rotation number $\theta \in [0, 1)$, there exists $\varepsilon_C(\theta)$ such that, for any $0 \le \varepsilon < \varepsilon_C(\theta)$, the map $SM_\varepsilon$ has an analytic invariant curve whose dynamics is analytically conjugate to a rigid rotation of rotation number $\theta$. This curve is a small perturbation of the circle $\mathbb{T}^1 \times \{\theta\}$. Moreover, from the twist character of the map $SM_\varepsilon$, we can also apply a result due to Birkhoff (see [1]) which ensures that all these curves can be written as graphs of the variable $y$ over the variable $x$. In this way, the dynamics on any of these curves induces a map on $\mathbb{T}^1$ simply by projecting the iterates of $SM_\varepsilon$ on $\mathbb{T}^1$.

Let us introduce this circle map more precisely. We take $(x_0, y_0) \in \mathbb{T}^1 \times \mathbb{R}$, belonging to one of these invariant curves, and compute $(x_n, y_n) = (SM_\varepsilon)^n(x_0, y_0)$, for $n \ge 0$. If we call $f$ the circle map induced by this curve, we have $f^n(x_0) = x_n$. Consequently, we can apply the method of Section 3 to this sequence to compute (with high precision) the rotation number of this curve.

In this section we use this method from two different points of view. First, we perform a "frequency analysis" of $SM_\varepsilon$ for some given values of $\varepsilon$, and we detect which initial conditions on the "vertical line" $x = 0$ give rise to an invariant curve simply by computing (if possible) its (irrational) rotation number. See [20] for a similar set up.

Second, we use this method to follow the evolution, when $\varepsilon$ increases, of the invariant curve of $SM_\varepsilon$ having a prefixed rotation number $\theta$, up to its critical value $\varepsilon_C(\theta)$ for which the curve is destroyed. We denote by $\mathcal{Y}(\theta, \varepsilon)$ the function given the $y$-coordinate of the cut of this invariant curve with $x = 0$. For a given $\theta$, the function $\mathcal{Y}(\theta, \cdot)$ is defined for any $0 \le \varepsilon < \varepsilon_C(\theta)$ and verifies $\mathcal{Y}(\theta, 0) = \theta$.

The method we use to obtain the function $\varepsilon \mapsto \mathcal{Y}(\theta, \varepsilon)$ is completely analogous to the one used in the computation of the Arnold tongues in Section 4.2. We fix $\theta$ and consider the equation $g(y, \varepsilon) := \rho(y, \varepsilon) - \theta = 0$, where $\rho(y, \varepsilon)$ is the rotation number associated to the initial condition $(0, y)$ for the map $SM_\varepsilon$ (if the point $(0, y)$ belongs to an invariant curve of $SM_\varepsilon$). The solution with respect to $y$ of this equation is $y = \mathcal{Y}(\theta, \varepsilon)$. The function $\rho(y, \varepsilon)$ is not properly defined for any couple $(y, \varepsilon)$. However, if $y$ is close to $\mathcal{Y}(\theta, \varepsilon)$ then, in the Lebesgue measure sense, mostly of the points of the form $(y, 0)$ belong to an invariant curve of $SM_\varepsilon$, and the function $\rho(y, \varepsilon)$ is well-defined. From the numerical point of view, what we observe is that the method works quite well for computing $\rho(y, \varepsilon)$ for values of $(y, \varepsilon)$ close to this invariant curve.

To solve the equation $g(y, \varepsilon) = 0$ we use numerical continuation with respect to $\varepsilon$. We construct a finite and increasing sequence of values of $\varepsilon$, $\{\varepsilon_j\}_{j=0,\ldots,K}$, with $\varepsilon_0 = 0$ and variable step-size. For any $j = 0, \ldots, K$, we compute a numerical approximation $\mathcal{Y}_j^*$ of $\mathcal{Y}(\theta, \varepsilon_j)$, beginning with $\mathcal{Y}_0^* = \theta$. To obtain $\mathcal{Y}_j^*$ we solve numerically the equation $g(y, \varepsilon_j) = 0$ by using the secant method. If the secant method does not converge, this means that either we are working with a value of $\varepsilon$ bigger than $\varepsilon_C(\theta)$ or that the continuation step-size is too big. In any of these cases we are forced to go back to $\varepsilon_{j-1}$ and to reduce the step-size.

Since there is strong (numerical) evidence that the "most robust" invariant curve is the one having rotation number $\theta = (\sqrt{5} - 1)/2$ the Golden Mean, we apply the continuation method to this value of $\theta$. Our purpose is to compare the numerical approximation thus obtained for $\varepsilon_C(\theta)$ with the value $\varepsilon_G(\theta) = 0.971635/2\pi \approx 0.1546405$ obtained by applying Greene's criterion to the same problem (see [13]).

The numerical results related with Chirikov standard map are displayed in Figure 5 and Figure 6. A detailed explanation of these figures is given as follows.

In Figure 5 we consider the frequency analysis of $SM_\varepsilon$ for three different values of $\varepsilon$, con-

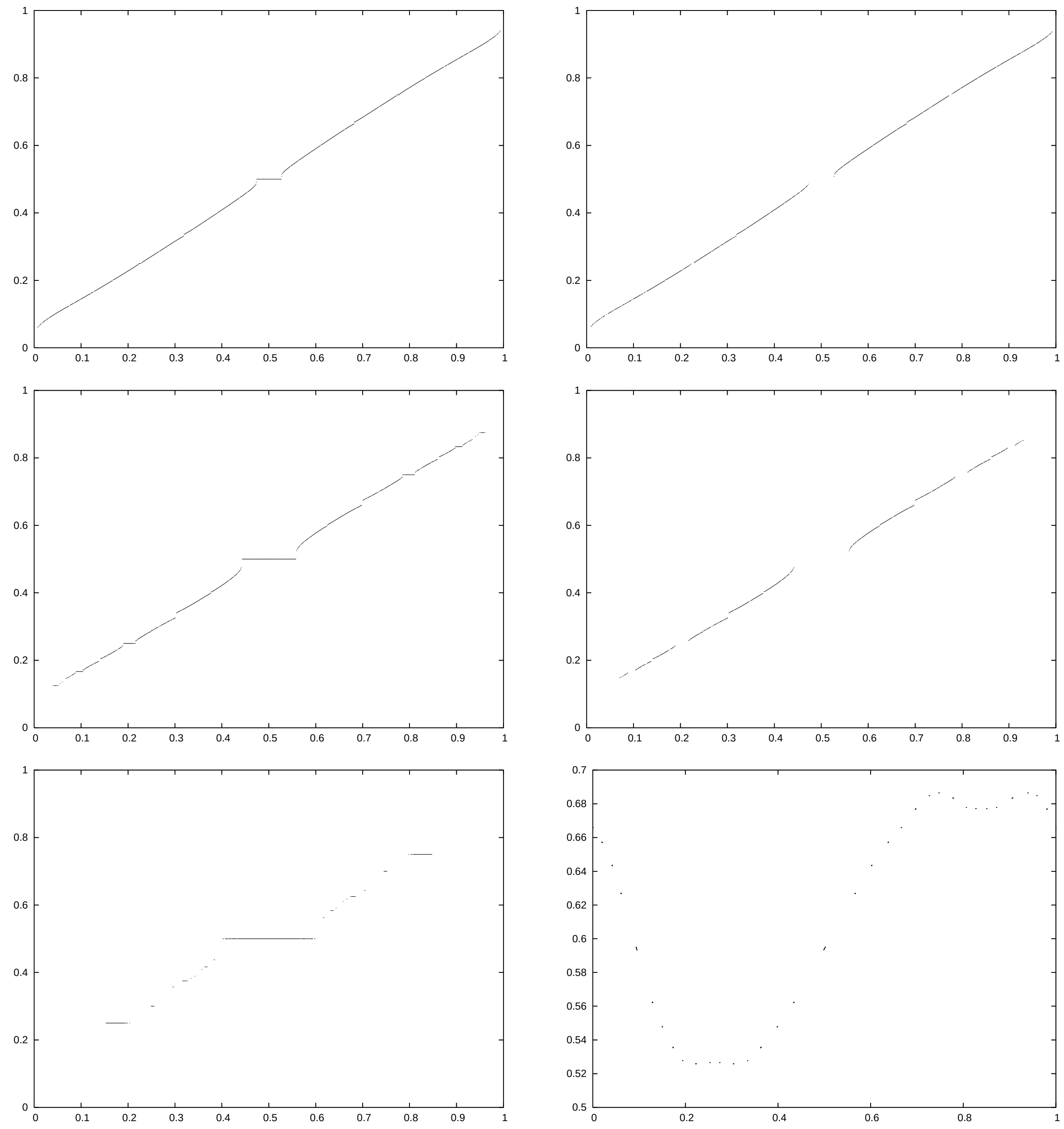


Figure 5: Frequency analysis of the Chirikov standard map for $\varepsilon \in \{0.05, 0.1, 0.16\}$. See the text for full details.

cretely $\varepsilon \in \{0.05, 0.1, 0.16\}$. We point out that $\varepsilon = 0.16$ is (slightly) bigger than the critical value corresponding to the breakdown of the invariant curve of rotation number the Golden Mean. This implies that there are no invariant curves of the map for such value of $\varepsilon$, but only "islands" around periodic points.

**Top-left:** This plot corresponds to the frequency analysis of $SM_\varepsilon$ for $\varepsilon = 0.05$. We consider points of the form $(0, y_j)$, with $y_j = j \times 10^{-3}$ and $j = 0, \ldots, 999$. Given the initial condition $(0, y_j)$, we compute (if possible) the rotation number $\rho(y_j, 0.05)$ of this point by assuming that it belongs to an invariant curve of $SM_{0.05}$. The computations have been done by using the *double-double* data type, a fixed extrapolation order $p = 7$ and up to $2^{20}$ iterates of the map, at most. We estimate the error of the rotation number by using (24) with $\nu = 10$, and we validate the rotation number when this error is smaller than $10^{-24}$. What we plot in this figure is the graph of the function $y_j \mapsto \rho(y_j, 0.05)$ (when defined).

As the selected value of $\varepsilon$ is "small", most of the points, in the Lebesgue measure sense, belong to an invariant curve. Then, we have been able to validate the rotation number for more than the 93% of them (98% if we decrease the tolerance of the error of the rotation number to $10^{-19}$). Nevertheless, some of the rotation numbers thus obtained are rational numbers, computed with high precision (the plot resembles a Devil's Staircase). Of course, the points to which we assign a rational rotation number cannot belong to an invariant curve. This phenomena can be understood by remembering that the resonant invariant curves of $\varepsilon = 0$ give rise, for $\varepsilon > 0$, to isolated periodic orbits. Some of these periodic orbits are linearly stable and most of the initial conditions around them fall into "secondary invariant curves" or "islands", which are invariant curves of a suitable power of $SM_\varepsilon$ (depending on the period of the orbit). Thus, for a point on these islands, what we obtain is the "rotation number" of the periodic orbit in the middle of the island.

**Top-right:** The same as in the previous plot, but we have skipped the points $y_j$ for which the rotation number $\theta = \rho(y_j, 0, 05)$ is a rational number. We consider that $\theta$ is rational if the difference between $\theta$ and its truncated continuous fraction expansion $[a_1, a_2, a_3, a_4, a_5]$ is smaller than $10^{-8}$. In this way, we expect that the points on this plot correspond to initial conditions of invariant curves of $SM_{0.05}$. The surviving points are 84.7%.

**Center-left:** The same as in the top-left plot but for $\varepsilon = 0.1$. The intervals with (constant) rational rotation number are now more evident. We can validate the rotation number with error $10^{-24}$ for 78% of the points (87% if the error is $10^{-19}$).

**Center-right:** The same as in the top-right plot but for $\varepsilon = 0.1$. Of course the number of invariant curves decreases when $\varepsilon$ increases, and we obtain 67.3% points.

**Bottom-left:** The same as in the top-left and center-left plots but now for $\varepsilon = 0.16$. Only rational rotation numbers are plotted. However, we can validate the "rotation number" with error $10^{-24}$ for 32% of the points (35.5% if the error is $10^{-19}$).

**Bottom-right:** For $\varepsilon = 0.16$, only two points $y_j$ pass the test we use to detect "irrational numbers". They are $\tilde{y} = 3.33 \times 10^{-1}$ and $\bar{y} = 6.66 \times 10^{-1}$. Let us consider, for instance, $\theta = \rho(\bar{y}, 0.16)$. Then, we observe that $\theta$ is a rational number close to the Golden Mean $\theta_1$, given by the continuous fraction expansion $[1, 1, 1, 1, 1, 2]$. Similarly, $\rho(\tilde{y}, 0.16) \simeq 1 - \theta_1$. Here we plot 1000 iterates of the orbit having initial condition $(\bar{y}, 0)$ for the map $SM_{0.16}$. If we perform a zoom around the "big points" appearing in the plot, we obtain very narrow islands (the biggest one has a diameter of approximately $4 \times 10^{-3}$ and width of $8 \times 10^{-6}$). This orbit seems to follow "the path" where we would expect to found the invariant curve of rotation number $\theta_1$ ("the last" invariant curve of the map), that is destroyed at a value of $\varepsilon$ slightly smaller than 0.16 (see Figure 6, top-right).

In Figure 6 we show some results for the continuation, with respect to $\varepsilon$, of the invariant curve of rotation number the Golden Mean.

**Top-left:** Here we plot the graph $\varepsilon_j \mapsto \mathcal{Y}_j^*$ of the initial condition $(0, \mathcal{Y}_j^*)$ of the invariant curve having rotation number $\theta$ the Golden Mean. The computations have been done by using the *double-double* data type, a fixed extrapolation order $p = 9$ and up to $2^{23}$ iterates of the map, at

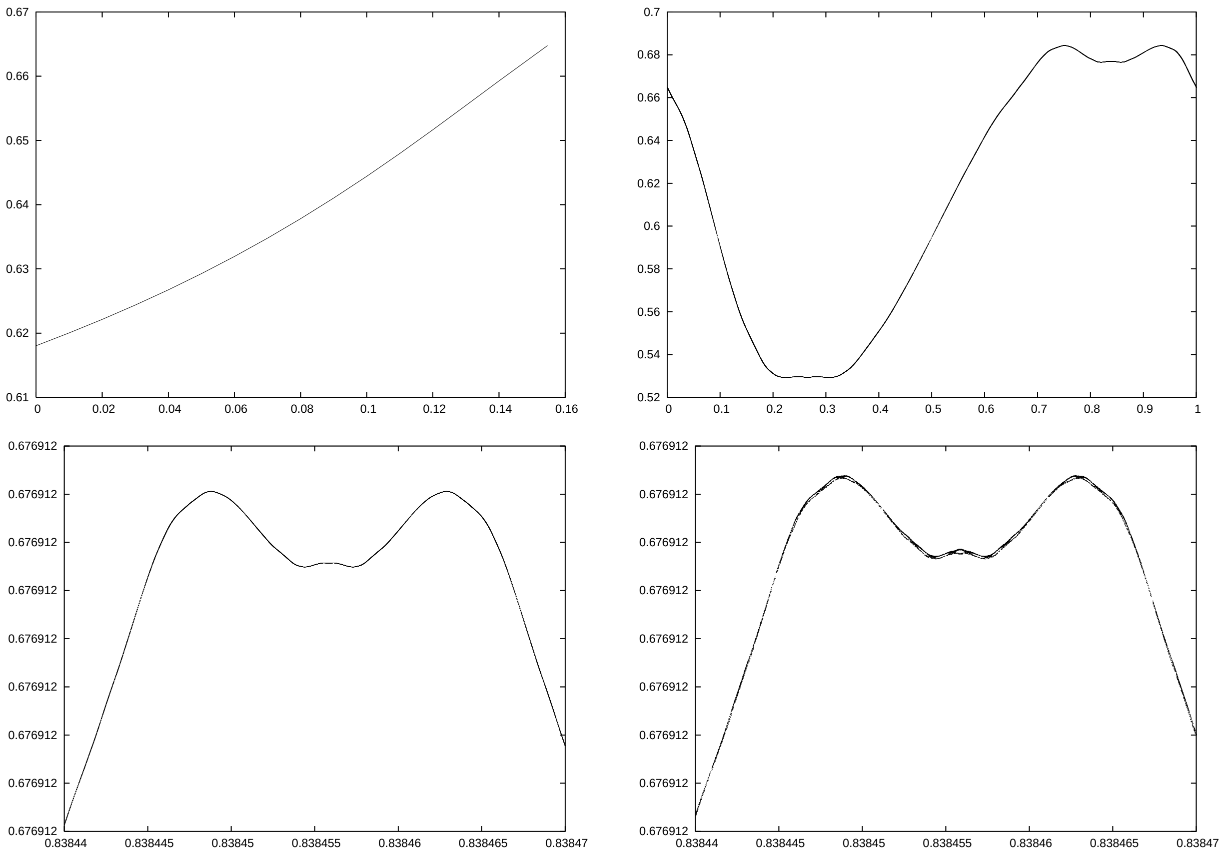


Figure 6: Numerical continuation with respect to $\varepsilon$ of the invariant curve of rotation number the Golden Mean for the Chirikov standard map. See the text for full details.

most. We estimate the error of the rotation number by using (24) with $\nu = 10$, and we validate the rotation number when this error is smaller than $10^{-30}$ and $10^{-25}$ for the secant method.

The critical value we obtain for $\varepsilon$ is $\varepsilon_C = 0.154640922$. We also notice that if we increase the tolerance for the rotation number to $10^{-20}$ and of the secant method to $10^{-16}$, we are able to continue the invariant curve up to $\varepsilon_C = 0.154643$. Any of these values for $\varepsilon_C$ is bigger than the critical one known from Greene's criterion, $\varepsilon_G \approx 0.1546405$. However, the question is up to which value of $\varepsilon$ we can ensure that the initial condition computed corresponds to a true invariant curve of $SM_\varepsilon$.

**Top-right:** We plot 10000 iterates of the map $SM_\varepsilon$, for $\varepsilon = 0.154640922$, of the initial condition displayed in the previous graph. The horizontal axis is the variable $x \in [0, 1]$ and the vertical one the variable $y$. It looks like an invariant curve. But, as we discuss in the next two plots, we are not completely sure about this.

**Bottom-left:** We compute $10^8$ iterates of $SM_\varepsilon$ for the initial condition displayed in the top-left plot for $\varepsilon = 0.1546405$, and we perform a zoom on it. What we see looks like what we expect for an invariant curve.

**Bottom-right:** The same zoom as in the previous plot but now for $\varepsilon = 0.1546407$. In this case we cannot ensure that it corresponds to a true invariant curve. Of course, the "islands" on the figure can be originated by the error in the determination of the initial condition or by its numerical propagation along $10^8$ iterates. Nevertheless, we point out that the numerical errors of this initial condition are comparable to the ones we made in the previous figure (concretely, $10^{-31}$ for the secant method and $10^{-42}$ for the rotation number).

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