ON ELLIPTIC PROBLEM WITH PERTURBED $p-$LAPLACE OPERATOR

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ABSTRACT. We consider elliptic boundary value problem with perturbed $p-$Laplace operator and prove weak existence theorem without monotonicity assumption.

1. INTRODUCTION

The $p-$Laplacian appears in the study of flow through porus media ($p = 3/2$), nonlinear elasticity ($p \geq 2$) and glaciology ($p \in (1, 4/3]$). We refer to [3] for more background material.

For $p-$Laplace equation with right hand side homogeneous in u, existence and nonexistence results were obtained by many authors, see for example [2], [6], [3], [4]. Variational methods were employed in [7] and others when trying to find positive solutions.

There are applications which provide elliptic problems with linear operators situated inside the $p-$Laplacian:

$$\Delta_p A u = f(x) \in H^{-1,p}(M), \quad u \mid_{\partial M} = 0, \tag{1.1}$$

here A is a certain bounded linear operator. If operator A is a small perturbation of the identity then, as it is not hard to show, the problem remains coercive, as for monotonicity, this property is much delicate to check, and possibly it is destroyed even under the small perturbation. In should be noted that in the linear case ($p = 2$) the monotonicity property is steady with respect to the small perturbation.

On the other hand, using standard variational technique we have to consider weak convergent sequences but the operator A has not necessarily to be continuous with respect to weak convergence.

Our version of variational method allows to prove the existence theorem avoiding of using the monotonicity or weak convergence.

2000 *Mathematics Subject Classification.* 35J60.
Key words and phrases. Boundary value problems, p-Laplacian, nonlocal problems.
Partially supported by grants RFBR 05-01-01119.

Linear case of equation (1.1) with nonlocal operator of the argument's dilation/compression

$$Au = \sum_{i=-k}^{k} a_i u(q^i x), \quad q > 1, \quad a_j \in \mathbb{R}$$

was studied in [9], [10]. In these articles the condition of closeness to identity for operator A is not assumed.

Boundary value problems for elliptic functional-differential equations have been studied in the articles [1], [8] and others.

Boundary value problems for elliptic equations with shifts in the space variables were considered in [5], [12].

A theory of boundary value problems for elliptic differential-difference equation in a bounded domain was constructed in [11].

2. Main theorem

Let M be a bounded domain in $\mathbb{R}^m$ with smooth boundary ∂M and $x = (x_1, \dots, x_m)$ be coordinates in $\mathbb{R}^m$. Denote by ∂_i the partial derivative in the variable x_i.

For any $v \in L^r(M), \quad r \geq 1$ and $w \in L^{r'}(M), \quad 1/r + 1/r' = 1$ we put

$$(v, w) = \int_M v(x) w(x)\, dx.$$

Let $G : H_0^{1,p}(M) \to H^{1,p}(M), \quad p \geq 2$ be a bounded linear operator. Suppose, there exists such a bounded operator $G^+ : L^p(M) \to L^p(M)$ that the following equality holds:

$$\partial_i G = G^+ \partial_i. \tag{2.1}$$

There is a conjugated operator

$$G^{+*} : L^{p'}(M) \to L^{p'}(M), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Similarly, assume that a bounded operator $G^{+*+} : H^{-1,p'}(M) \to H^{-1,p'}(M)$ satisfy the equation:

$$\partial_i G^{+*} = G^{+*+} \partial_i. \tag{2.2}$$

Introduce operator $A_\varepsilon = I + \varepsilon G$, here I is the identity operator.

Consider the p-Laplace operator:

$$\Delta_p v = \sum_{i=1}^{m} \partial_i (|\partial_i v|^{p-2} \partial_i v).$$

The main object of our study is the following elliptic problem:

$$\Delta_p A_\varepsilon u = f(x) \in L^{p'}(M), \quad u\,|_{\partial M} = 0. \tag{2.3}$$

Theorem 1. *If $G^{+*+}f \in L^{p'}(M)$ then there exists such a positive constant $\tilde{\varepsilon}$ that for any ε, $|\varepsilon| < \tilde{\varepsilon}$ problem (2.3) has a weak solution $u \in H_0^{1,p}(M)$ i.e. for any $h \in H_0^{1,p}(M)$ one has*

$$-\sum_{i=1}^{m}(|\partial_i A_\varepsilon u|^{p-2}\partial_i A_\varepsilon u, \partial_i h) = (f, h).$$

Applications. Define an operator $\sigma : L^r(M) \to L^r(\mathbb{R}^m)$, $r \geq 1$ as follows:

$$\sigma v = \begin{cases} v(x) & \text{if} \quad x \in M, \\ 0 & \text{if} \quad x \in \mathbb{R}^m \backslash \overline{M}. \end{cases}$$

Use this operator to construct $R_q v = (\sigma v)(qx)$, $q > 0$. This operator dilates/compresses the graph of the function $v(x)$ in q times, here q is a constant.

One can show that the operator R_q is bounded as an operator of the space $H_0^{1,r}(M)$, $r \geq 1$ to $H^{1,r}(M)$ and as an operator of $L^r(M)$ to itself.

The operator R_q commutes with the partial derivatives in the following fashion:

$$\partial_i R_q = R_q^+ \partial_i, \quad R_q^+ = qR_q.$$

The operator R_q^* can also be written in the explicit form: letting $R_q^* = q^{-m}R_{q^{-1}}$ and by the changing of variable $x \mapsto q^{-1}x$ in the integral we make sure that $(R_q v, w) = (v, R_q^* w)$. So, one has: $R_q^{+*} = q^{1-m}R_{q^{-1}}$ and $R_q^{+*+} = q^{-m}R_{q^{-1}} = R_q^*$. To define this operator on $H^{-1,p'}(M)$ for any $g \in H^{-1,p}(M)$ we put:

$$(R_q^{+*+}g, h) = (g, R_q h), \quad h \in H_0^{1,p'}(M).$$

Now it is easy to see that formulas (2.1) and (2.2) will be fulfilled if as an operator G we take the following linear combination:

$$G = \sum_{k=1}^{l} a_k R_{q_k} \quad a_j \in \mathbb{R}, \quad q_j > 0.$$

In the conclusion of this section, note that there are another sources of proper operators G. They are linear combinations of shift operators of the form $\sum_{k=1}^{l} a_k v(x + \tau_k)$, $v(x) \in L^r(M)$, $\tau_k \in \mathbb{R}^m$. Taking into account the arguments above, one sees that the operator G can also be a linear combination of dilation/compressions and shifts operators.

3. Proof of Theorem 1

Consider the operators $A_\varepsilon^\iota = I + \varepsilon G^\iota$, $\iota \in \{+, +*, +*+\}$. Here I is the identity operator of the corresponding space.

Define a liner function $J_\varepsilon : H_0^{1,p}(M) \to \mathbb{R}$ by the formula $J_\varepsilon(v) = (A_\varepsilon^{+*+}f, v)$.

We are going to minimize this function on the level set of a function

$$F_\varepsilon : H_0^{1,p}(M) \to \mathbb{R}, \quad F_\varepsilon(v) = \int_M \sum_{i=1}^m |\partial_i A_\varepsilon v|^p \, dx.$$

Lemma 1. *Take a positive constant ε_0 and a constant $\gamma \geq 1$. Then for any $\varepsilon, \quad |\varepsilon| < \varepsilon_0$ and for any $a, b \in \mathbb{R}$ one has*

$$|a + \varepsilon b|^\gamma = |a|^\gamma + \varepsilon \psi(a, b, \varepsilon),$$

here the function ψ satisfies the inequality:

$$|\psi(a, b, \varepsilon)| \leq c_1(|a|^\gamma + |b|^\gamma).$$

Positive constant c_1 depends only on ε_0 and γ.

Proof. If $a = 0$ or $\varepsilon = 0$ then the assertion is trivial, otherwise letting $\xi = b/a$ we have:

$$\frac{|\psi(a, b, \varepsilon)|}{|a|^\gamma + |b|^\gamma} = \frac{||a + \varepsilon b|^\gamma - |a|^\gamma|}{|\varepsilon|(|a|^\gamma + |b|^\gamma)} = \frac{|\xi|}{1 + |\xi|^\gamma} \cdot \frac{||1 + \varepsilon\xi|^\gamma - 1|}{|\varepsilon\xi|} \leq c_1.$$

□

The following Lemma states the coercivity of the operator $\Delta_p A_\varepsilon$ when the parameter ε is small.

Lemma 2. *There is a positive constants ε_1, c_2 such that for any $\varepsilon, \quad |\varepsilon| < \varepsilon_1$ and for any $v \in H_0^{1,p}(M)$ one has*

$$F_\varepsilon(v) \geq c_2 \|v\|^p_{H_0^{1,p}(M)}.$$

Proof. According to Lemma 1 write

$$\begin{aligned} F_\varepsilon(v) &= \int_M \sum_{i=1}^m |\partial_i v + \varepsilon \partial_i G v|^p \, dx = \int_M \sum_{i=1}^m |\partial_i v|^p + \varepsilon \psi(\partial_i v, \partial_i G v, \varepsilon) \, dx \\ &= \|v\|^p_{H_0^{1,p}(M)} + \varepsilon \int_M \sum_{i=1}^m \psi(\partial_i v, \partial_i G v, \varepsilon) \, dx. \end{aligned} \tag{3.1}$$

Now the proof follows from the estimate of the last term in (3.1):

$$\Big| \int_M \sum_{i=1}^m \psi(\partial_i v, \partial_i G v, \varepsilon) \, dx \Big| \leq c_1 \int_M \sum_{i=1}^m |\partial_i v|^p + |\partial_i G v|^p \, dx \leq c_3 \|v\|^p_{H_0^{1,p}(M)},$$

here c_3 is a positive constant independent on v and ε. □

Lemma 3. *Consider a set*

$$B_\varepsilon = \{v \in H_0^{1,p}(M) \mid F_\varepsilon(v) = 1\}.$$

For any $\varepsilon, \quad |\varepsilon| < \varepsilon_1$ the function $J_\varepsilon |_{B_\varepsilon}$ attains its minimum, say at $\hat{v}_\varepsilon$.

Proof. By the conditions of the Theorem, the function J_ε can be extended continuously to the space $L^p(M)$, we do not introduce another notation for this extension.

As it follows from Lemma 2 the set B_ε is bounded in $H_0^{1,p}(M)$, but the space $H_0^{1,p}(M)$ is compactly embedded in $L^p(M)$. Consequently, B_ε is a compact subset of $L^p(M)$. Thus the function J_ε attains its minimum $\hat{v}_\varepsilon$. $\square$

Consider weak derivatives:

$$J'_\varepsilon h = \frac{d}{ds}\Big|_{s=0} J_\varepsilon(\hat{v}_\varepsilon + sh) = (A_\varepsilon^{+*+} f, h) = J_\varepsilon(h), \quad h \in H_0^{1,p}(M), \tag{3.2}$$

$$F'_\varepsilon(\hat{v}_\varepsilon)h = \frac{d}{ds}\Big|_{s=0} F_\varepsilon(\hat{v}_\varepsilon + sh) = p\sum_{i=1}^{m}(|\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2}\partial_i A_\varepsilon \hat{v}_\varepsilon, \partial_i A_\varepsilon h). \tag{3.3}$$

There is standard relation between these derivatives. This relation is described as follows.

Lemma 4. *For any $\varepsilon, \quad |\varepsilon| < \varepsilon_1$ the following inclusion holds true:*

$$\ker F'_\varepsilon(\hat{v}_\varepsilon) \subseteq \ker J'_\varepsilon. \tag{3.4}$$

Proof. Fix $\varepsilon, \quad |\varepsilon| < \varepsilon_1$ and let

$$h \in \ker F'_\varepsilon(\hat{v}_\varepsilon). \tag{3.5}$$

Define a function of two real arguments by the formula:

$$\varphi(y,t) = F_\varepsilon(y\hat{v}_\varepsilon + th).$$

We want to show that for some positive t_0 there exists such a function $y(t) \in C^1(-t_0, t_0), \quad y(0) = 1$ that satisfies an equation

$$\varphi(y(t), t) = 1. \tag{3.6}$$

If it would be true, then the set $\{y(t)\hat{v}_\varepsilon + th \mid |t| < t_0\}$ is a curve on the manifold B_ε. This curve passes across the point $\hat{v}_\varepsilon$ and has the tangent vector h at this point.

To solve equation (3.6) we use the implicit function theorem. Observing that $\varphi(1,0) = F_\varepsilon(\hat{v}_\varepsilon) = 1$ and by formula

$$\begin{aligned}\varphi_y(y,t)\mid_{(y,t)=(1,0)} &= F'_\varepsilon(y\hat{v}_\varepsilon + th)\hat{v}_\varepsilon \mid_{(y,t)=(1,0)} \\ &= F'_\varepsilon(\hat{v}_\varepsilon)\hat{v}_\varepsilon = pF_\varepsilon(\hat{v}_\varepsilon) = p \neq 0,\end{aligned} \tag{3.7}$$

we obtain desired function $y(t)$.

Note that the derivative of the function $y(t)$ vanishes at zero:

$$y_t(0) = 0. \tag{3.8}$$

Indeed, since $F_\varepsilon(y(t)\hat{v}_\varepsilon + th) = 1$ for $|t| < t_0$, we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} F_\varepsilon(y(t)\hat{v}_\varepsilon + th) &= F'_\varepsilon(y(0)\hat{v}_\varepsilon)(y_t(0)\hat{v}_\varepsilon + h) \\ &= y_t(0)F'_\varepsilon(\hat{v}_\varepsilon)\hat{v}_\varepsilon + F'_\varepsilon(\hat{v}_\varepsilon)h = 0.\end{aligned} \tag{3.9}$$

Combining formula (3.9) with (3.5) and (3.7) we obtain (3.8).

Since $y(t)\hat{v}_\varepsilon + th \in B_\varepsilon$ for $|t| < t_0$, and by Lemma 3 a function $\rho(t) = J_\varepsilon(y(t)\hat{v}_\varepsilon + th)$ attains its local minimum at $t = 0$ so we have:

$$\rho_t(0) = J'_\varepsilon(y_t(0)\hat{v}_\varepsilon + h) = 0.$$

This formula and formula (3.8) imply: $J'_\varepsilon h = 0$. □

According to the Lagrange multipliers theorem, formula (3.4) implies that there exists such a constant λ_ε that $\lambda_\varepsilon F'_\varepsilon(\hat{v}_\varepsilon) = J'_\varepsilon$. In other words by formulas (3.2), (3.3) for any $h \in H^{1,p}_0(M)$ it follows that:

$$\lambda_\varepsilon p \sum_{i=1}^m (|\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon, \partial_i A_\varepsilon h) = (A^{+*+}_\varepsilon f, h). \tag{3.10}$$

Using formulas (2.1), (2.2) write:

$$\begin{aligned}\sum_{i=1}^m (|\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon, \partial_i A_\varepsilon h) &= \sum_{i=1}^m (|\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon, A^+_\varepsilon \partial_i h) \\ = \sum_{i=1}^m (A^{+*}_\varepsilon |\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon, \partial_i h) &= -\sum_{i=1}^m (\partial_i A^{+*}_\varepsilon |\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon, h) \\ &= -\sum_{i=1}^m (A^{+*+}_\varepsilon \partial_i (|\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon), h).\end{aligned}$$

This formula and formula (3.10) imply:

$$-\lambda_\varepsilon p \sum_{i=1}^m A^{+*+}_\varepsilon \partial_i (|\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon) = A^{+*+}_\varepsilon f. \tag{3.11}$$

It is easy to see that the operator $A^{+*+}_\varepsilon : H^{-1,p}(M) \to H^{-1,p}(M)$ is invertible, provided ε is small enough, so that equation (3.11) is equivalent to the following one:

$$-\lambda_\varepsilon p \sum_{i=1}^m \partial_i (|\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon) = f. \tag{3.12}$$

Take a constant τ_ε so that the equation $-\lambda_\varepsilon p |\tau_\varepsilon|^{p-2} \tau_\varepsilon = 1$ is satisfied, and substituting $\hat{v}_\varepsilon = \tau_\varepsilon u$ in (3.12) we see that the function u solves (2.3).

The Theorem is proved.

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