SOLVABILITY OF PERTURBED $p-$LAPLACE EQUATION

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ABSTRACT. We consider elliptic boundary value problem with perturbed $p-$Laplace operator, $p>1$ and prove weak existence theorem.

1. INTRODUCTION

The $p-$Laplacian appears in the study of flow through porus media ($p=3/2$), nonlinear elasticity ($p\geq 2$) and glaciology ($p\in(1,4/3]$). We refer to [4] for more background material.

For $p-$Laplace equation with right hand side homogeneous in u, existence and nonexistence results were obtained by many authors, see for example [3], [7], [4], [5]. Variational methods were employed in [8] and others when trying to find positive solutions.

There are applications which provide elliptic problems with nonlocal linear operators situated inside the $p-$Laplacian:

$$\Delta_p Au=f(x)\in H^{-1,p'}(M),\quad u\mid_{\partial M}=0,\quad \frac{1}{p}+\frac{1}{p'}=1, \tag{1.1}$$

here A is a certain bounded linear operator. If operator A is a small perturbation of the identity then, as it is not hard to show, the problem remains coercive, as for monotonicity, this property is much delicate to check, and possibly it is destroyed even under the small perturbation. It should be noted that in the linear case ($p=2$) the monotonicity property is steady under the small perturbation.

On the other hand, using standard variational technique we have to consider weak convergent sequences but the operator A has not necessarily to be continuous with respect to weak convergence.

Our version of variational method allows to prove the existence theorem avoiding of using the monotonicity or weak convergence.

2000 *Mathematics Subject Classification.* 35J60.

Key words and phrases. Boundary value problems, p-Laplacian, nonlocal problems.

Partially supported by grants RFBR 05-01-01119.

Linear case of equation (1.1) with nonlocal operator of the argument's dilation/compression

$$Au=\sum_{i=-k}^{k}a_iu(q^ix),\quad q>1,\quad a_j\in\mathbb{R}$$

was studied in [10], [11]. In these articles the condition of closeness to identity for operator A is not assumed. The case of coefficients a_k dependent on x was considered in [12] in the linear setup.

Boundary value problems for elliptic functional-differential equations have been studied in the articles [2], [9] and others.

Boundary value problems for elliptic equations with shifts in the space variables were considered in [6], [14].

A theory of boundary value problems for elliptic differential-difference equation in a bounded domain was constructed in [13].

2. Main theorem

Let M be a bounded domain in $\mathbb{R}^m$ with smooth boundary ∂M and $x=(x_1,\ldots,x_m)$ be coordinates in $\mathbb{R}^m$. Denote by ∂_i the partial derivative in the variable x_i.

For any $v\in L^r(M),\quad r\ge 1$ and $w\in L^{r'}(M),\quad 1/r+1/r'=1$ we put

$$(v,w)=\int_M v(x)w(x)\,dx.$$

Denote by $\mathcal{W}$ a space of bounded linear operators $G:H^{1,p}_0(M)\to H^{1,p}(M),\quad p>1$ with the following properties.

For any operator $G\in\mathcal{W}$ there exists a bounded operator $G^+:L^p(M)\to L^p(M)$ such that:

$$\partial_iG=G^+\partial_i. \tag{2.1}$$

Let

$$G^{+*}:L^{p'}(M)\to L^{p'}(M),\quad \frac{1}{p}+\frac{1}{p'}=1$$

be conjugated operator, then there is a bounded operator $G^{+*+}:H^{-1,p'}(M)\to H^{-1,p'}(M)$ satisfying the equation:

$$\partial_iG^{+*}=G^{+*+}\partial_i. \tag{2.2}$$

Denote by $\mathcal{L}(H^{1,p}_0(M))$ the space of bounded linear operators of $H^{1,p}_0(M)$ to itself.

Let $\varepsilon\in\mathbb{R}$ be a real parameter, then introduce an operator $A_\varepsilon:H^{1,p}_0(M)\to H^{1,p}(M)$ by the formula:

$$A_\varepsilon=\mathrm{id}_{H^{1,p}_0(M)}+\varepsilon\sum_{j=1}^{l}G_jT_j,\quad G_j\in\mathcal{W},\quad T_j\in\mathcal{L}(H^{1,p}_0(M)). \tag{2.3}$$

Consider the p-Laplace operator:

$$\Delta_p v = \sum_{i=1}^{m} \partial_i(|\partial_i v|^{p-2}\partial_i v).$$

The main object of our study is the following elliptic problem:

$$\Delta_p A_\varepsilon u = f(x) \in L^{p'}(M), \quad u\,|_{\partial M} = 0. \tag{2.4}$$

Theorem 1. *If $T_j^* G_j^{+*+} f \in L^{p'}(M), \quad j = 1, \dots, l$ then there exists such a positive constant $\tilde{\varepsilon}$ that for any $\varepsilon, \quad |\varepsilon| < \tilde{\varepsilon}$ problem (2.4) has a weak solution $u \in H_0^{1,p}(M)$ i.e. for any $h \in H_0^{1,p}(M)$ one has*

$$-\sum_{i=1}^{m}(|\partial_i A_\varepsilon u|^{p-2}\partial_i A_\varepsilon u, \partial_i h) = (f, h).$$

3. Applications: Dilation/Compression Operators

In this section we regard the domain M as star-shaped with respect to the origin.

Define an operator $\sigma : L^r(M) \to L^r(\mathbb{R}^m), \quad r \geq 1$ as follows:

$$\sigma v = \begin{cases} v(x) & \text{if} \quad x \in M, \\ 0 & \text{if} \quad x \in \mathbb{R}^m \backslash \overline{M}. \end{cases}$$

Use this operator to construct $R_q v = (\sigma v)(qx), \quad q > 0$. This operator dilates/compresses the graph of the function $v(x)$ in q times, here q is a constant.

One can show that the operator R_q is bounded as an operator of the space $H_0^{1,r}(M), \quad r \geq 1$ to $H^{1,r}(M)$ and as an operator of $L^r(M)$ to itself.

The operator R_q commutes with the partial derivatives in the following fashion:

$$\partial_i R_q = R_q^+ \partial_i, \quad R_q^+ = q R_q.$$

The operator R_q^* can also be written in the explicit form: letting $R_q^* = q^{-m} R_{q^{-1}}$ and by the change of variable $x \mapsto q^{-1}x$ in the integral we make sure that $(R_q v, w) = (v, R_q^* w)$. So, one has: $R_q^{+*} = q^{1-m} R_{q^{-1}}$ and $R_q^{+*+} = q^{-m} R_{q^{-1}} = R_q^*$.

To define operator R_q^{+*+} on $H^{-1,p'}(M)$ note that any element $g \in H^{-1,p'}(M)$ presents as follows [1]:

$$g = g_0 + \sum_{i=1}^{m} \partial_i g_i, \quad g_j \in L^{p'}(M), \quad j = 0, \dots, m.$$

Now we put:

$$(R_q^{+*+} g, h) = (g_0, R_q h) - \sum_{i=1}^{m}(g_i, \partial_i R_q h), \quad h \in H_0^{1,p}(M).$$

It is easy to see that formulas (2.1) and (2.2) are fulfilled and $R_q \in \mathcal{W}$.

Show that an operator

$$Q_\varepsilon = \mathrm{id}_{H_0^{1,p}(M)} + \varepsilon \sum_{j=1}^{l} a_j(x) R_{q_j}, \quad a_j \in C^1(\overline{M}) \tag{3.1}$$

can be presented in the form (2.3). Consider the cases $q_j \geq 1$ and $q_j < 1$ separately.

Introduce an operator of multiplication by a by the formula: $U_a v = av(x)$. If $a(x) \in C^1(\overline{M})$ then $U_{a(x)} \in \mathcal{L}(H_0^{1,p}(M))$.

Assume $q_j \geq 1$. Then obviously we have: $a_j(x) R_{q_j} = R_{q_j} U_{a_j(q_j^{-1}x)}$.

If $q_j < 1$ take a function $\widetilde{a}_j(x) \in C^1(\overline{M})$ such that $\widetilde{a}_j(x) = a_j(q_j^{-1}x)$ for $x \in q_j\overline{M}$, then we have $a_j(x) R_{q_j} = R_{q_j} U_{\widetilde{a}_j(x)}$.

4. Proof of Theorem 1

Define operator $B_\varepsilon : H^{-1,p'}(M) \to H^{-1,p'}(M)$ by the formula:

$$B_\varepsilon = \mathrm{id}_{H^{-1,p'}(M)} + \varepsilon \sum_{j=1}^{l} T_j^* G_j^{+*+}.$$

Consider a linear function $J_\varepsilon(v) = (B_\varepsilon f, v)$. We are going to minimize this function on the level set of a function

$$F_\varepsilon : H_0^{1,p}(M) \to \mathbb{R}, \quad F_\varepsilon(v) = \int_M \sum_{i=1}^{m} |\partial_i A_\varepsilon v|^p \, dx.$$

Such a variational scheme is not quite standard: problems with homogeneous right hand side are usually treated in the opposite way: it is the function $F_\varepsilon(v)$ whose conditional extremum is looked for.

Though our case differs from that one, the proposed technique provides much more direct and simple way to obtain existence results even for the homogeneous problems (in the subcritical case of course !).

Since this technique gives possibility to avoid from dealing with weak convergent sequences one can apply it to a problem which contains nonlocal linear operators in the right hand side.

Lemma 1. *Take a positive constant ε_0 and a constant $\gamma \geq 1$. Then for any ε, $|\varepsilon| < \varepsilon_0$ and for any $a, b \in \mathbb{R}$ one has*

$$|a + \varepsilon b|^\gamma = |a|^\gamma + \varepsilon \psi(a, b, \varepsilon),$$

here the function ψ satisfies the inequality:

$$|\psi(a, b, \varepsilon)| \leq c_1(|a|^\gamma + |b|^\gamma).$$

Positive constant c_1 depends only on ε_0 and γ.

Proof. If $a = 0$ or $\varepsilon = 0$ then the assertion is trivial, otherwise letting $\xi = b/a$ we have:

$$\frac{|\psi(a,b,\varepsilon)|}{|a|^\gamma+|b|^\gamma} = \frac{||a+\varepsilon b|^\gamma - |a|^\gamma|}{|\varepsilon|(|a|^\gamma+|b|^\gamma)} = \frac{|\xi|}{1+|\xi|^\gamma}\cdot\frac{||1+\varepsilon\xi|^\gamma-1|}{|\varepsilon\xi|} \le c_1.$$

□

Lemma 2. *There are positive constants ε_1, c_2 such that for any ε, $|\varepsilon| < \varepsilon_1$ and for any $v \in H_0^{1,p}(M)$ one has*

$$F_\varepsilon(v) \ge c_2\|v\|^p_{H_0^{1,p}(M)}.$$

Proof. According to Lemma 1 write

$$F_\varepsilon(v) = \int_M \sum_{i=1}^m |\partial_i v + \varepsilon\partial_i Gv|^p\,dx = \int_M \sum_{i=1}^m |\partial_i v|^p + \varepsilon\psi(\partial_i v, \partial_i Gv, \varepsilon)\,dx$$

$$= \|v\|^p_{H_0^{1,p}(M)} + \varepsilon\int_M \sum_{i=1}^m \psi(\partial_i v, \partial_i Gv, \varepsilon)\,dx. \tag{4.1}$$

Now the proof follows from the estimate of the last term in (4.1):

$$\Big|\int_M \sum_{i=1}^m \psi(\partial_i v, \partial_i Gv, \varepsilon)\,dx\Big| \le c_1\int_M \sum_{i=1}^m |\partial_i v|^p + |\partial_i Gv|^p\,dx \le c_3\|v\|^p_{H_0^{1,p}(M)},$$

here c_3 is a positive constant independent on v and ε. □

Lemma 3. *Consider a set*

$$S_\varepsilon = \{v \in H_0^{1,p}(M) \mid F_\varepsilon(v) = 1\}.$$

For any ε, $|\varepsilon| < \varepsilon_1$ the function $J_\varepsilon |_{S_\varepsilon}$ attains its minimum, say at $\hat v_\varepsilon$.

Proof. By the conditions of the Theorem, the function J_ε can be extended continuously to the space $L^p(M)$, we do not introduce another notation for this extension.

As it follows from Lemma 2 the set S_ε is bounded in $H_0^{1,p}(M)$, but the space $H_0^{1,p}(M)$ is compactly embedded in $L^p(M)$. Consequently, S_ε is a compact subset of $L^p(M)$. Thus the function J_ε attains its minimum $\hat v_\varepsilon$. □

Consider weak derivatives:

$$J'_\varepsilon h = \frac{d}{ds}\Big|_{s=0} J_\varepsilon(\hat v_\varepsilon + sh) = (B_\varepsilon f, h) = J_\varepsilon(h), \quad h \in H_0^{1,p}(M), \tag{4.2}$$

$$F'_\varepsilon(\hat v_\varepsilon)h = \frac{d}{ds}\Big|_{s=0} F_\varepsilon(\hat v_\varepsilon + sh) = p\sum_{i=1}^m (|\partial_i A_\varepsilon \hat v_\varepsilon|^{p-2}\partial_i A_\varepsilon \hat v_\varepsilon, \partial_i A_\varepsilon h). \tag{4.3}$$

There is standard relation between these derivatives. This relation is described as follows.

Lemma 4. *For any ε, $|\varepsilon| < \varepsilon_1$ the following inclusion holds true:*

$$\ker F'_\varepsilon(\hat v_\varepsilon) \subseteq \ker J'_\varepsilon. \tag{4.4}$$

Proof. Fix $\varepsilon,\quad |\varepsilon|<\varepsilon_1$ and let

$$h\in\ker F'_\varepsilon(\hat v_\varepsilon). \tag{4.5}$$

Define a function of two real arguments by the formula:

$$\varphi(y,t)=F_\varepsilon(y\hat v_\varepsilon+th).$$

We want to show that for some positive t_0 there exists such a function $y(t)\in C^1(-t_0,t_0),\quad y(0)=1$ that satisfies an equation

$$\varphi(y(t),t)=1. \tag{4.6}$$

If it would be true, then the set $\{y(t)\hat v_\varepsilon+th\mid |t|<t_0\}$ is a curve on the manifold S_ε. This curve passes across the point $\hat v_\varepsilon$ and has h as a tangent vector at this point.

To solve equation (4.6) we use the implicit function theorem. Observing that $\varphi(1,0)=F_\varepsilon(\hat v_\varepsilon)=1$ and by formula

$$\begin{aligned}\varphi_y(y,t)\mid_{(y,t)=(1,0)}&=F'_\varepsilon(y\hat v_\varepsilon+th)\hat v_\varepsilon\mid_{(y,t)=(1,0)}\\&=F'_\varepsilon(\hat v_\varepsilon)\hat v_\varepsilon=pF_\varepsilon(\hat v_\varepsilon)=p\neq0,\end{aligned} \tag{4.7}$$

we obtain desired function $y(t)$.

Note that the derivative of the function $y(t)$ vanishes at zero:

$$y_t(0)=0. \tag{4.8}$$

Indeed, since $F_\varepsilon(y(t)\hat v_\varepsilon+th)=1$ for $|t|<t_0$, we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}F_\varepsilon(y(t)\hat v_\varepsilon+th)&=F'_\varepsilon(y(0)\hat v_\varepsilon)(y_t(0)\hat v_\varepsilon+h)\\&=y_t(0)F'_\varepsilon(\hat v_\varepsilon)\hat v_\varepsilon+F'_\varepsilon(\hat v_\varepsilon)h=0.\end{aligned} \tag{4.9}$$

Combining formula (4.9) with (4.5) and (4.7) we obtain (4.8).

Since $y(t)\hat v_\varepsilon+th\in S_\varepsilon$ for $|t|<t_0$, and by Lemma 3 a function $\rho(t)=J_\varepsilon(y(t)\hat v_\varepsilon+th)$ attains its local minimum at $t=0$ so we have:

$$\rho_t(0)=J'_\varepsilon(y_t(0)\hat v_\varepsilon+h)=0.$$

This formula and formula (4.8) imply: $J'_\varepsilon h=0$. □

According to the Lagrange multipliers theorem, formula (4.4) implies that there exists such a constant λ_ε that $\lambda_\varepsilon F'_\varepsilon(\hat v_\varepsilon)=J'_\varepsilon$. In other words for any $h\in H^{1,p}_0(M)$ formulas (4.2), (4.3) give:

$$\lambda_\varepsilon p\sum_{i=1}^m(|\partial_iA_\varepsilon\hat v_\varepsilon|^{p-2}\partial_iA_\varepsilon\hat v_\varepsilon,\partial_iA_\varepsilon h)=(B_\varepsilon f,h). \tag{4.10}$$

Let us transform the left side of (4.10). Put $w_i=|\partial_iA_\varepsilon\hat v_\varepsilon|^{p-2}\partial_iA_\varepsilon\hat v_\varepsilon$, and write

$$(w_i,\partial_iA_\varepsilon h)=-(\partial_iw_i,h)+\varepsilon\sum_{j=1}^l(w_i,\partial_iG_jT_jh). \tag{4.11}$$

Using formulas (2.1), (2.2) we calculate:

$$(w_i, \partial_i G_j T_j h) = (w_i, G_j^{+} \partial_i T_j h) = (G_j^{+*} w_i, \partial_i T_j h)$$
$$= -(\partial_i G_j^{+*} w_i, T_j h) = -(G_j^{+*+} \partial_i w_i, T_j h) = -(T_j^* G_j^{+*+} \partial_i w_i, h).$$

By this formula and (4.11) it follows that

$$(w_i, \partial_i A_\varepsilon h) = -(B_\varepsilon \partial_i w_i, h). \tag{4.12}$$

Substituting (4.12) to (4.10) one has:

$$-\lambda_\varepsilon p \sum_{i=1}^{m} B_\varepsilon \partial_i (|\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon) = B_\varepsilon f. \tag{4.13}$$

It is easy to see that the operator B_ε is invertible, provided ε is small enough, so that equation (4.13) is equivalent to the following one:

$$-\lambda_\varepsilon p \sum_{i=1}^{m} \partial_i (|\partial_i A_\varepsilon \hat{v}_\varepsilon|^{p-2} \partial_i A_\varepsilon \hat{v}_\varepsilon) = f. \tag{4.14}$$

Let a constant τ_ε be a root of the equation $-\lambda_\varepsilon p |\tau_\varepsilon|^{p-2} \tau_\varepsilon = 1$, then substituting in (4.14) the expression $\hat{v}_\varepsilon = \tau_\varepsilon u$ we see that the function u solves (2.4).

The Theorem is proved.

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