Completeness of Averaged Scattering Solutions and Inverse Scattering at a Fixed Energy *†

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Abstract

We prove that the averaged scattering solutions to the Schrödinger equation with short-range electromagnetic potentials (V, A) where $V(x) = O(|x|^{-\rho}), A(x) = O(|x|^{-\rho}), |x| \to \infty, \rho > 1$, are dense in the set of all solutions to the Schrödinger equation that are in $L^2(K)$ where K is any connected bounded open set in $\mathbb{R}^n, n \geq 2$, with smooth boundary.
We use this result to prove that if two short-range electromagnetic potentials (V_1, A_1) and (V_2, A_2) in $\mathbb{R}^n, n \geq 3$, have the same scattering matrix at a fixed positive energy and if the electric potentials V_j and the magnetic fields $F_j := \mathrm{curl} A_j, j = 1, 2$, coincide outside of some ball they necessarily coincide everywhere.
In a previous paper of Weder and Yafaev the case of electric potentials and magnetic fields that are asymptotic sums of homogeneous terms at infinity was studied. It was proven that all these terms can be uniquely reconstructed from the singularities in the forward direction of the scattering amplitude at a fixed positive energy.
The combination of the new uniqueness result of this paper and the result of Weder and Yafaev implies that the scattering matrix at a fixed positive energy uniquely determines electric potentials and magnetic fields that are a finite sum of homogeneous terms at infinity, or more generally, that are asymptotic sums of homogeneous terms that actually converge, respectively, to the electric potential and to the magnetic field.

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1 Introduction

Let us first consider the stationary Schrödinger equation

$$-\Delta\phi + V(x)\phi = E\phi, E = k^2, k > 0, \tag{1.1}$$

on $\mathbb{R}^n, n \geq 2$, where the real-valued potential V satisfies,

$$V(x) = O(|x|^{-\rho}), |x| \to \infty, \rho > 1. \tag{1.2}$$

Later we give precise conditions on V and we introduce magnetic potentials.

If $\rho > n$, for any $E > 0$ and any unit vector $\omega \in \mathbb{S}^{n-1}$ (1.1) has a unique solution $\phi_+(x, \omega; E)$ with the asymptotics as $|x| \to \infty$

$$\phi_+(x, \omega; E) = e^{ikx\cdot\omega} + f\frac{e^{ik|x|}}{|x|^{(n-1)/2}} + o(|x|^{-(n-1)/2}). \tag{1.3}$$

If we only have that $\rho > (n+1)/2$ the unique solution with the asymptotics (1.3) exists but now the $o(|x|^{(n-1)/2})$ has to be interpreted in an appropriate averaged sense. For a discussion of this issue see, for example, [35]. The unique solution with the asymptotics (1.3) is called the scattering solution. The coefficient $f = f(\nu, \omega; E)$ depends upon the incident direction ω of the incoming plane wave $e^{ikx\cdot\omega}$ its energy, $E = k^2$, and the direction $\nu := x/|x|$ of observation of the outgoing spherical wave $e^{ik|x|}/|x|^{(n-1)/2}$. The function $f(\nu, \omega; E)$ is known as the scattering amplitude. In potential scattering in quantum mechanics the plane wave describes a beam of particles incident on a scattering center described by V, and the outgoing spherical wave corresponds to the scattered particles.

The unitary operator that corresponds to this scattering process is the scattering matrix, that is the unitary operator in $L^2(\mathbb{S}^{n-1})$ that is defined in terms of the scattering amplitude by the formula,

$$(S(E)u)(\omega) = u(\omega) + ie^{i\pi(n-3)/4}E^{(n-1)/4}\,(2\pi)^{-(n-1)/2}\int_{\mathbb{S}^{n-1}} f(\nu, \omega; \lambda)u(\omega)\,d\omega. \tag{1.4}$$

In the case of general short-range potentials that satisfy (1.2) with $\rho > 1$ the scattering amplitude can be defined by (1.4) where the scattering matrix $S(E)$ is defined via the time-dependent wave operators. See Section 2.

An important property of the scattering solutions is that for any connected bounded open set $K \subset \mathbb{R}^n, n \geq 2$, with smooth boundary, the set

$$\{\phi_+(x, \omega; E)\}_{\omega\,\in\,\mathbb{S}^{n-1}} \tag{1.5}$$

is strongly dense in the set of all solutions to (1.1) that are in $L^2(K)$. This was proven by D. Eidus in [5] for bounded potentials that satisfy (1.2) with $\rho > n$ and, independently, [26] proved a similar result. The proof in [26] applies to potentials that satisfy (1.2) with $\rho > (n+1)/2$. The density of the scattering solutions was applied in [5] to prove that (1.1) has the Runge property and in [26] to prove that if two electric potentials in $\mathbb{R}^n, n \geq 3$, have the same scattering matrix at a fixed positive energy and if they coincide outside of some ball they necessarily coincide everywhere. For further references on the completeness of the scattering solutions and for the application to inverse boundary value and inverse scattering problems see [24], [18], [10] and the references quoted there.

In this paper we wish to generalize the completeness of the scattering solutions to the case of potentials that satisfy (1.2) with $\rho > 1$. The first problem that we have to address is that in general (1.1) has no solutions with the asymptotics (1.3) if (1.2) only holds for some $\rho > 1$. For a discussion of this issue see [22] and [35]. To see what would be an appropriate generalization we observe that the completeness of the scattering solutions is equivalent to the completeness of the set of solutions

$$\phi_{+,f}(x;E) := \int_{\mathbb{S}^{n-1}} \phi_+(x,\omega;E)\, f(\omega)\, d\omega, f \in L^2(\mathbb{S}^{n-1}) \tag{1.6}$$

that are obtained by taking the average on the angular variables of the scattering solutions with arbitrary functions in $L^2(\mathbb{S}^{n-1})$. The point is that in the general case where (1.2) only holds for some $\rho > 1$ it is possible to define solutions to (1.1) $\phi_{+,f}(x;E)$ for all $f \in L^2(\mathbb{S}^{n-1})$ that for regular f are asymptotic to a linear combination of incoming and outgoing spherical waves and furthermore, the action of the scattering matrix $S(E)$ can be determined in terms of these solutions. Moreover, if (1.2) holds with $\rho > (n+1)/2$ the solutions $\phi_{+,f}$ are given by the right-hand side of (1.6). For these results see [33] -where also a generalized eigenfunctions expansion theorem in terms of these solutions is proven- and [2].

Related problems appear in different settings. See [27] for the case of acoustic and electromagnetic waves in perturbed stratified media.

We prove in Theorem 3.1 the completeness of the averaged scattering solutions $\phi_{+,f}$ in the case of the stationary Schrödinger equation with electric potential V and magnetic potential A,

$$(i\nabla + A)^2\, \phi + V\phi = E\phi, E = k^2, k > 0 \tag{1.7}$$

where V satisfies (1.2) and

$$A(x) = O(|x|^{-\rho}), \partial_j A^{(l)} = O(|x|^{-\rho}), |x| \to \infty, 1 \leq j,l \leq n, \tag{1.8}$$

for some $\rho > 1$. For precise conditions see Theorem 3.1.

With the help of our result on the completeness of the averaged scattering solutions we prove in Theorems 4.2 and 4.3 that if two electromagnetic potentials (V_1, A_1) and (V_2, A_2) in $\mathbb{R}^n, n \geq 3$, that satisfy (1.2) and (1.8) with $\rho > 1$ have the same scattering matrix at a fixed positive energy and if the electric potentials V_j and the the magnetic fields

$F_j := \mathrm{curl} A_j, j = 1, 2$, are equal outside of some ball, then $V_1 = V_2$ and $F_1 = F_2$ everywhere. This generalizes our result of [26] in two directions. First, we allow now for general short-range electric potentials that satisfy (1.2) with $\rho > 1$ and second, we consider now the case where there is also a short-range magnetic potential that satisfies (1.8) with $\rho > 1$. This decay is optimal for short-range potentials. Note that as the scattering matrix is invariant under short-range gauge transformations uniqueness of A does not hold

We recall that in Theorem 4.2 of [31] the case of electric potentials and magnetic fields that are asymptotic sums of homogeneous terms at infinity was studied. It was proven that all these terms can be uniquely reconstructed from the singularities in the forward direction of the scattering amplitude at a fixed positive energy.

By combining our new uniqueness result in Theorem 4.3 with Theorem 4.2 of [31] we prove in Theorem 4.4 that the scattering matrix at a fixed positive energy uniquely determines electric potentials and magnetic fields in $\mathbb{R}^n, n \geq 3$, that are finite sums of homogeneous terms at infinity, or more generally, that are asymptotic sums of homogeneous terms at infinity that actually converge, respectively, to the electric potential and to the magnetic field. This result generalizes Theorem 4.6 of [31] to the case where there is also a magnetic potential and where both the electric potential and the magnetic field satisfy the optimal short-range decay condition.

It is known since quite some time that the scattering matrix at a fixed positive energy uniquely determines electric potentials and magnetic fields if strong restrictions on the decay at infinity are imposed. The paper [16] considers potentials of compact support, and [17, 6, 11, 25] potentials decaying exponentially at infinity. On the contrary, for general short-range potentials the scattering matrix at a fixed positive energy does not determine uniquely the potential. Indeed, in [4] examples -in three dimensions- are given of non-trivial radial oscillating potentials with decay as $|x|^{-3/2}$ at infinity such that the corresponding scattering amplitude is identically zero at some positive energy. Moreover, in dimension two there are examples [7] of potentials with a regular decay as $|x|^{-2}$ at infinity that have zero scattering amplitude at some positive energy. Nevertheless, as we discussed above if two general short-range electric potentials and magnetic fields coincide outside of some ball and if they have the same scattering matrix at some positive energy they are equal everywhere.

Actually, the same problem appears in different settings. Thus, it is proven in [12, 28, 8] that the scattering matrix at a fixed positive energy uniquely determines an exponentially decreasing perturbation of a stratified media. As another example, we mention that the scattering matrix at a fixed quasi-energy uniquely determines time-periodic potentials that decay exponentially at spatial infinity [30].

Theorem 4.6 of [31] and its generalization in Theorem 4.4 below show a new aspect of the inverse scattering problem at a fixed energy. Namely, that uniqueness holds for general short-range electric potentials and magnetic fields without strongly restricting the decay at infinity, provided that the electric potential and the magnetic field have a regular behaviour at infinity. Of course, this eliminates the oscillations and hence there is no contradiction with the examples of [4]. Furthermore, as we consider three or more dimensions there is no contradiction with the two dimensional examples of [8].

The paper is organized as follows. In Section 2 we discuss some basic results on the limiting absorption principle and on stationary scattering theory, we consider the averaged scattering solutions and we give a representation of the scattering matrix in terms of these solutions. In Section 3 we prove Theorem 3.1 on the completeness of the averaged scattering solutions by generalizing the proof given in [26]. In Section 4 we prove Theorems 4.2 and 4.3 extending to this case the proof of Theorem 1 of [26] and, finally, we prove Theorem 4.4.

2 Basic Results

In this section we recall some well known results on the stationary scattering theory of the Schrödinger operator in $\mathbb{R}^n, n \geq 2$, with short-range electromagnetic potentials [13, 2, 20, 14, 34]. For any $\alpha \in \mathbb{R}$ let us denote by $\mathbb{H}^\alpha = \mathbb{H}^{\alpha,2}$ the L^2-based Sobolev space. See for example [21].

We consider the Schrödinger operator,

$$H := (i\nabla + A)^2 + V = H_0 + Q, \tag{2.1}$$

where the free Hamiltonian, $H_0 := -\Delta$ is a self-adjoint operator with domain the Sobolev space $\mathbb{H}^2$ and

$$Q := 2iA \cdot \nabla + i\mathrm{Div}A + A^2 + V \tag{2.2}$$

is the perturbation.

In this section we always assume that for some $\epsilon > 0$ the operator $(1+|x|)^{1+\epsilon} Q$ is compact from $\mathbb{H}^2$ into L^2. Sufficient conditions that assure that this is true are well known. See for example [21]. In particular, it follows from Theorem 5.2 of [21] that this is the case if the following is true. If $n = 2, 3, V, \mathrm{DivA}, A^2 \in L^2_{\mathrm{loc}}$, if $n = 4, V, \mathrm{Div}A, A^2 \in L^{2+\delta}_{\mathrm{loc}}$ for some $\delta > 0$ and if $n \geq 5, V, A, \mathrm{DivA}, A^2 \in L^{n/2}_{\mathrm{loc}}$ and if moreover, for some constants $C, R > 0$,

$$|V(x)| + |A(x)| + |\partial_j A^{(l)}(x)| \leq C(1 + |x|)^{-\rho},\ \rho > 1, 1 \leq j, l \leq n, \text{ for } |x| \geq R > 0. \tag{2.3}$$

Under these conditions the Schrödinger operator H is self-adjoint and bounded below with domain $\mathbb{H}^2$. It has no singular continuous spectrum and its absolutely-continuous spectrum is $[0, \infty)$. By unique continuation [9], [32], and Theorem 1.2 of [3] H has no positive eigenvalues. The negative spectrum consists of eigenvalues with finite multiplicity and they can only accumulate at zero.

To state the limiting absorption principle we introduce weighted L^2 spaces for $s \in \mathbb{R}$.

$$L^2_s := \left\{ f : (1+|x|^2)^{s/2} f(x) \in L^2 \right\}, \|f\|_{L^2_s} := \|(1+|x|^2)^{s/2} f(x)\|_{L^2},$$

and for any $\alpha, s \in \mathbb{R}$,

$$\mathbb{H}^{\alpha,s} := \left\{ f(x) : (1+|x|^2)^{s/2} f(x) \in \mathbb{H}^\alpha \right\}, \|f\|_{\mathbb{H}^{\alpha,s}} := \|(1+|x|^2)^{s/2} f(x)\|_{\mathbb{H}^\alpha}.$$

$\mathbb{C}^\pm$ denotes, respectively, the upper, lower, complex half-plane.

The limiting absorption principle is the following statement. For z in the resolvent set of H let $R(z) := (H-z)^{-1}$ be the resolvent. Then, for every $E \in (0,\infty)$ the following limits,

$$R(E \pm i0) := \lim_{\epsilon \downarrow 0} R(E \pm i\epsilon),$$

exist in the uniform operator topology in $\mathcal{B}\left(L^2_s, \mathbb{H}^{\alpha,-s}\right), s > 1/2, \alpha \leq 2$, where for any pair of Banach spaces X, Y, $\mathcal{B}(X,Y)$ denotes the Banach space of all bounded operators from X into Y. The functions,

$$R_\pm(E) := \begin{cases} R(E), & \operatorname{Im} E \neq 0, \\ R(E \pm i0) & , E \in (0,\infty), \end{cases}$$

defined for $E \in \mathbb{C}^\pm \cup (0,\infty)$ with values in $\mathcal{B}\left(L^2_s, \mathbb{H}^{\alpha,-s}\right)$ are analytic for $\operatorname{Im} E \neq 0$ and locally Hölder continuous for $E \in (0,\infty)$ with exponent γ satisfying $\gamma < 1, \gamma < s - 1/2$.

The wave operators,

$$W_\pm := \mathrm{s} - \lim_{t \to \pm\infty} e^{itH}\, e^{-itH_0}$$

exist as strong limits and are complete, i.e., Range $W_\pm = \mathcal{H}_{ac}$ where $\mathcal{H}_{ac}$ denotes the subspace of absolute continuity of H. Moreover, they have the intertwining property, $HW_\pm = W_\pm H_0$. The scattering operator,

$$\mathbf{S} := W_+^* W_-$$

is unitary.

Let us denote by $T_0(E)$ the following trace operator,

$$(T_0(E)\phi)(\omega) := 2^{-1/2}\, E^{(n-2)/4} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iE^{1/2} x \cdot \omega}\, \phi(x)\, dx, \tag{2.4}$$

that is bounded from $L^2_s, s > 1/2$, into $L^2(\mathbb{S}^{n-1})$, and furthermore, the operator valued function $E \to T_0(E)$ from $(0,\infty)$ into $\mathcal{B}(L^2_s, L^2(\mathbb{S}^{n-1}))$ is locally Hölder continuous with exponent $\gamma < 1, \gamma < s - 1/2$. Moreover, the operator,

$$\left(\mathcal{F}_0\phi\right)(E,\omega) := \left(T_0(E)\phi\right)(\omega), \tag{2.5}$$

extends to a unitary operator from L^2 onto $\hat{\mathcal{H}} := L^2((0,\infty); L^2(\mathbb{S}^{n-1}))$ that gives a spectral representation for H_0, i.e.,

$$\mathcal{F}_0 H_0 \mathcal{F}_0^* = E, \tag{2.6}$$

the operator of multiplication by E in $\hat{\mathcal{H}}$.

The perturbed trace operators are defined as follows,

$$\left(T_\pm(E)\phi\right)(\omega) := T_0(E)(I - QR_\pm(E))\phi, \tag{2.7}$$

for $E \in (0,\infty)$. They are bounded from $L^2_s, s > 1/2$, into $L^2(\mathbb{S}^{n-1})$, and furthermore, the operator valued functions $E \to T_\pm(E)$ from $(0,\infty)$ into $\mathcal{B}(L^2_s, L^2(\mathbb{S}^{n-1}))$ are locally Hölder continuous with exponent $\gamma < 1, \gamma < s - 1/2$. The operators,

$$\left(\mathcal{F}_\pm\phi\right)(E,\omega) := \left(T_\pm(E)\phi\right)(\omega), \tag{2.8}$$

extend to unitary operators from $\mathcal{H}_{ac}$ onto $\hat{\mathcal{H}}$ and they give spectral representations for the restriction of H to $\mathcal{H}_{ac}$,

$$\mathcal{F}_\pm H \mathcal{F}_\pm^* = E \tag{2.9}$$

the operator of multiplication by E in $\hat{\mathcal{H}}$. Furthermore, the stationary formulae for the wave operators hold,

$$W_\pm = \mathcal{F}_\pm^* \mathcal{F}_0.$$

As $\mathbf{S}$ commutes with H_0 we have that,

$$\left(\mathcal{F}_0 S \mathcal{F}_0^* \phi\right)(E,\omega) = S(E)\phi,$$

where $S(E), E > 0$, is unitary on $L^2(\mathbb{S}^{n-1})$. The operator $S(E)$ is the scattering matrix. This time-dependent definition of the scattering matrix generalizes to general short-range potentials the definition given in Section 1.

The scattering matrix has the following stationary representation,

$$S(E) = I - 2\pi i \mathcal{F}_0\, Q \left[I - R_+(E)\, Q\right] \mathcal{F}_0^*, E \in (0,\infty). \tag{2.10}$$

The scattering matrix can be represented in terms of averaged scattering solutions as follows (see [27] for a similar representation in the case of acoustic and electromagnetic waves in perturbed stratified media).

For any $f \in L^2(\mathbb{S}^{n-1})$ let us define the unperturbed averaged scattering solutions as follows,

$$\phi_{0,f}(x;E) := \int_{\mathbb{S}^{n-1}} e^{iE^{1/2} x\cdot\omega}\, f(\omega)\, d\omega. \tag{2.11}$$

Observe that $\phi_{0,f} \in L^2_{-s}, s > 1/2$, and that $H_0\phi_{0,f} = E\phi_{0,f}$. The perturbed averaged scattering solutions are defined as,

$$\phi_{+,f}(x;E) := [I - R_+(E)Q]\phi_{0,f},\ E \in (0,\infty),\ f \in L^2(\mathbb{S}^{n-1}). \tag{2.12}$$

Then, $\phi_{+,f} \in L^2_{-s}, s > 1/2$, and $H\phi_{+,f} = E\phi_{+,f}$.

By (2.10) for $f, g \in L^2(\mathbb{S}^{n-1})$,

$$(S(E)f,g)_{L^2(\mathbb{S}^{n-1})} = (f,g)_{L^2(\mathbb{S}^{n-1})} - i\frac{E^{(n-2)/2}}{2(2\pi)^{n-1}}\left(Q\,\phi_{+,f}, \phi_{0,g}\right)_{L^2}. \tag{2.13}$$

If $\rho > (n+1)/2$ in (2.3), $V, A, A^2, \mathrm{Div}A \in L^2_s, s > 1/2$, and we can define the scattering solution,

$$\phi_+(x,\omega;E) := e^{iE^{1/2}x\cdot\omega} - R_+(E)\left(Qe^{iE^{1/2}x\cdot\omega}\right).$$

In this case

$$\phi_{+,f}(x;E) = \int_{\mathbb{S}^{n-1}} \phi_+(x,\omega;E)\, f(\omega)\, d\omega,$$

what justifies the name averaged scattering solutions. See [2] for further discussions on this point.

3 Completeness of Solutions

In this section we prove our result on the completeness of the averaged scattering solutions.

THEOREM 3.1. *Suppose that if $n = 2,3, V, \mathrm{DivA}, A^2 \in L^2_{\mathrm{loc}}$, if $n = 4, V, \mathrm{Div}A, A^2 \in L^{2+\delta}_{\mathrm{loc}}$ for some $\delta > 0$ and if $n \geq 5, V, A, \mathrm{DivA}, A^2 \in L^{n/2}_{\mathrm{loc}}$ and that*

$$|V(x)| + |A(x)| + |\partial_j A^{(l)}(x)| \leq C(1+|x|)^{-\rho},\ \rho > 1, 1 \leq j,l \leq n, \text{ for } |x| \geq R > 0. \tag{3.1}$$

Let K be a connected open bounded set with smooth boundary. Then, the set of averaged scattering solutions, $\phi_{+,f}, f \in L^2(\mathbb{S}^{n-1})$, is strongly dense on the set of all solutions to (1.7) in $L^2(K)$.

Proof: We follow the proof given in [26]. Suppose that $\phi \in L^2(K)$ is a solution to (1.7) that is orthogonal to all the averaged scattering solutions, i.e.,

$$(\varphi, \phi_{+,f})_{L^2(K)} = 0, f \in L^2(\mathbb{S}^{n-1}), \tag{3.2}$$

and define,

$$\psi := R_+(E)\varphi, \tag{3.3}$$

where we have extended φ by zero to $\mathbb{R}^n \setminus K$.

By (2.9), and as the trace operator $T_-(E)$ is locally Hölder continuous, it follows from Privalov's theorem that,

$$\psi = \psi_1 + \psi_2, \tag{3.4}$$

where,

$$\psi_1 := \text{P.V.} \int_I d\lambda \, \frac{1}{\lambda - E} \, T_-^*(\lambda) T_-(\lambda)\varphi + i\pi T_-{}^*(E) T_-(E)\varphi, \tag{3.5}$$

$$\psi_2 = R(E)\, \mathcal{E}(\tilde{I})\varphi, \tag{3.6}$$

where $I := [a,b], 0 < a < E - \delta, b > E + \delta$ for some $\delta > 0, \tilde{I} := \mathbb{R} \setminus I$, and $\mathcal{E}(\cdot)$ is the spectral family of H.

Clearly,

$$\psi_2 \in L^2. \tag{3.7}$$

Moreover, by (2.7) and (2.12) equation (3.2) implies that,

$$T_-(E)\varphi = 0, \tag{3.8}$$

and as T_- is Hölder continuous we can eliminate the P.V. in the integral in the right-hand side of (3.5) and then,

$$\psi_1 = \int_I d\lambda \, \frac{1}{\lambda - E} \, T_-^*(\lambda) T_-(\lambda)\varphi. \tag{3.9}$$

Let us denote by J_I the operator,

$$J_I := \mathcal{E}(I)\, \mathcal{F}_-^* = \mathcal{F}_-^* \, \chi_I(E), \tag{3.10}$$

where χ_I is the characteristic function of I. Since $\mathcal{F}_-$ is unitary from $\mathcal{H}_{ac}$ onto $\hat{\mathcal{H}}$ and (2.9) holds, it follows that J_I is unitary from $L^2(I, L^2(\mathbb{S}^{n-1}))$ onto $\mathcal{E}(I)\mathcal{H}_{ac}$, and in consequence it is bounded from $L^2(I, L^2(\mathbb{S}^{n-1}))$ into L^2. Moreover, by (2.8)

$$J_I\varphi = \int_I T_-^*(\lambda)\, \varphi(\lambda)\, d\lambda, \tag{3.11}$$

and it follows that J_I is bounded from $L^1(I, L^2(\mathbb{S}^{n-1}))$ into L^2_{-s}, for any $s > 1/2$. Then, by interpolation [19] J_I is bounded from $L^p(I, L^2(\mathbb{S}^{n-1}))$ into $L^2_{-\epsilon_p s}, \epsilon_p := \frac{2}{p} - 1, 1 \leq p \leq 2$. Observe that by (3.8), (3.9)

$$\psi_1 = J_I \frac{1}{\lambda - E} \left(T_-(\lambda) - T_-(E)\right) \varphi. \tag{3.12}$$

Let us take $s_0 := 1/2 + \gamma$, where $\gamma < \min[1/2, \rho/2]$. Since T_- is locally Hölder continuous from $L^2_{s_0}$ into $L^2(\mathbb{S}^{n-1})$ with exponent γ, and as $\varphi \in L^2_{s_0}$ it follows that,

$$\frac{1}{\lambda - E} \left(T_-(\lambda) - T_-(E)\right) \varphi \in L^p(I, L^2(\mathbb{S}^{n-1})), p < 1/(1-\gamma)$$

and by taking s close enough to $1/2$ we conclude that $\psi_1 \in L^2_{-\beta}$ for some $0 < \beta < 1/2$. Hence, by (3.4) and (3.7) $\psi \in L^2_{-\beta}$. Furthermore,

$$H\psi = E\psi + \varphi,$$

and as $\varphi(x) = 0$, for $x \in \mathbb{R}^n \setminus K$ it follows from Theorem 1.2 of [3] that $\psi(x)$ is identically zero in the complement of a large enough ball and then, by unique continuation [9], [32] it is identically zero on $\mathbb{R}^n \setminus K$. In particular, $\psi(x) = \nabla\psi(x) = 0$ on ∂K in trace sense. Finally, approximating ψ in the norm of $\mathcal{H}^2(K)$ by functions in $C_0^\infty(K)$ (it is here that the smoothness of ∂K is used) we prove that,

$$\|\varphi\|^2_{L^2(K)} = ((H-E)\psi, \varphi)_{L^2(K)} = (\psi, (H-E)\varphi)_{L^2(K)} = 0,$$

and it follows that $\varphi = 0$.

4 Inverse Problem

Note that it follows from the definition of the wave operators that the scattering operator $\mathbf{S}$ and the scattering matrix $S(E)$ are invariant under the gauge transformation, $A \to A + \nabla\psi$, where $|\psi(x)| \leq C(1+|x|)^{-\mu}, |\nabla\psi(x)| \leq C(1+|x|)^{-1-\mu}, \mu > 0$. This invariance suggest that we should associate the scattering operator and the scattering matrix to the magnetic field $F = \text{curl}A$. The problem is that in the Schrödinger equation (1.7) as well as in the definition of the Hamiltonian (2.1), the magnetic potential appears explicitly and, in general, it is not possible to express $\mathbf{S}$ and $S(E)$ only in terms of F. A striking manifestation of this fact is the famous Aharonov-Bohm effect [1]. For a study of inverse scattering in the context of the Aharonov-Bohm effect see [29].

In our case we can proceed as follows. We consider $\mathbb{R}^n, n \geq 3$, and we assume that A satisfies,

$$|\partial^\alpha A(x)| \leq C_\alpha (1+|x|)^{-\rho-|\alpha|}, \quad \rho > 1, 0 \leq |\alpha| \leq 1. \tag{4.1}$$

We use a three-dimensional notation for the curl and the divergence keeping in mind that in the general case A is a 1-form and F is a 2-form. By definition, $F(x) = \text{curl}\, A(x)$, and in terms of components it is given by,

$$F^{(ij)}(x) = \partial_i A^{(j)}(x) - \partial_j A^{(i)}(x). \tag{4.2}$$

Note that, $\text{div}F = 0$. Clearly, from a magnetic field $F(x)$ such that $\text{div}\, F(x) = 0$ we can only reconstruct the magnetic potential up to arbitrary gauge transformations.

We find it convenient use the procedure given in in [36], [31] to construct a short-range magnetic potential for an arbitrary magnetic field satisfying $\mathrm{div}F = 0$ and the estimate,

$$|\partial^\alpha F(x)| \le C(1+|x|)^{-1-\rho-|\alpha|}, \quad \rho > 1, 0 \le |\alpha| \le 1. \tag{4.3}$$

Let us define the auxiliary potentials

$$A^{(i)}_{reg}(x) = \int_1^\infty s \sum_{j=1}^d F^{(ij)}(sx) x_j \, ds, \quad A^{(i)}_\infty(x) = -\int_0^\infty s \sum_{j=1}^d F^{(ij)}(sx) x_j \, ds. \tag{4.4}$$

Observe that A_∞ is a homogeneous function of order -1, and that $\mathrm{curl}\, A_\infty(x) = 0$ for $x \neq 0$. Let us now define the function $U(x)$ for $x \neq 0$ as a curvilinear integral

$$U(x) = \int_{\Gamma_{x_0,x}} A_\infty(y) \cdot dy \tag{4.5}$$

taken between some fixed point $x_0 \neq 0$ and a variable point x. We require that $0 \notin \Gamma_{x_0,x}$. Then, by Stokes theorem, the function $U(x)$ does not depend on the choice of the contour $\Gamma_{x_0,x}$ and $\nabla U(x) = A_\infty(x)$.

Finally, we choose an arbitrary function $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta(x) = 0$ in a neighbourhood of zero, $\eta(x) = 1$ for $|x| \ge R$, for some $R > 0$, and define,

$$A(x) := A_{reg}(x) + (1 - \eta(x)) A_\infty(x) - U(x) \nabla \eta(x). \tag{4.6}$$

Then, $\mathrm{curl}\, A(x) = F(x)$, A satisfies (4.1) and $A(x) = A_{reg}(x)$ for $|x| \ge R$.

In this section we always associate to a magnetic field F satisfying $\mathrm{div}F = 0$ and (4.3) the magnetic potential A given by formulae (4.4) – (4.6) and then construct the scattering operator $\mathbf{S}$ and the scattering matrix $S(E)$ in terms of the Schrödinger operator (2.1) with this potential. If another short-range potential $\tilde{A}$ satisfies (4.1) and moreover, $\mathrm{curl}\, \tilde{A}(x) = F(x)$, then necessarily A and $\tilde{A}$ are related by a gauge transformation and the scattering operators and scattering matrices corresponding to A and $\tilde{A}$ coincide. It is in this sense that we speak about the scattering operator $\mathbf{S}$ and the scattering matrix $S(E)$ corresponding to the magnetic field F. The key issue that makes the magnetic potential (4.4-4.6) important for us is that if two magnetic fields that satisfy (4.3) coincide outside of a ball of radius bigger or equal to R, then, the corresponding magnetic potentials given by (4.4-4.6) also coincide outside of the same ball.

LEMMA 4.1. *Let $V_j, j = 1,2$ be electric potentials in $\mathbb{R}^n, n \ge 3$, such that, if $n = 3, V_j \in L^2_{\mathrm{loc}}$, if $n = 4, V_j \in L^{2+\delta}_{\mathrm{loc}}$ for some $\delta > 0$ and if $n \ge 5, V_j \in L^{n/2}_{\mathrm{loc}}$ and for some $R > 0$,*

$$|V_j(x)| \le C(1+|x|)^{-\rho}, \quad \rho > 1, \, |x| \ge R > 0, j = 1,2. \tag{4.7}$$

Furthermore, let $F_j, j = 1,2,$ be magnetic potentials that satisfy (4.3). Let $S_j(E)$ be the scattering matrices corresponding, respectively, to $(V_j, F_j), j = 1,2$. Suppose that for some $E > 0$, $S_1(E) = S_2(E)$ and that for some $R_1 > 0, V_1(x) = V_2(x), F_1(x) = F_2(x)$ for $|x| \ge R_1 > 0$. Then, the averaged scattering solutions $\phi^{(j)}_{+,f}(x;E), j = 1,2,$ coincide for $|x| \ge R_1$, i.e.,

$$\phi^{(1)}_{+,f}(x;E) = \phi^{(2)}_{+,f}(x;E), \text{ for } |x| \geq R_1, f \in L^2(\mathbb{S}^{n-1}). \tag{4.8}$$

Proof: We follow the proof of [26]. Let A_j be the magnetic potentials (4.4-4.6) corresponding to F_j, and let Q_j be defined as in (2.2) with A_j instead of A, for $j = 1, 2$. Denote,

$$\psi := \phi^{(2)}_{+,f} - \phi^{(1)}_{+,f}, \tag{4.9}$$

$$\varphi := Q_1\phi^{(1)}_{+,f} - Q_2\phi^{(2)}_{+,f}. \tag{4.10}$$

Then,

$$(H_0 - E)\psi = \varphi \tag{4.11}$$

and furthermore, $\psi \in \mathcal{H}^2_{-s}, \varphi \in L^2_s, 0 < s < (1+\rho)/2$.

As $S_1(E) = S_2(E)$ it follows from (2.11) and (2.13) that,

$$\int_{\mathbb{R}^n} e^{-iE^{1/2}x\cdot\omega}\,\varphi(x)\,dx = 0 \tag{4.12}$$

in trace sense. Let us denote by $\hat{\varphi}(\xi)$ the Fourier transform of φ. Then, $\hat{\varphi} \in \mathcal{H}^s$ and $\hat{\varphi}(\xi) = 0$ in trace sense on the sphere $|\xi| = E^{1/2}$. Furthermore, denoting by $\hat{\psi}$ the Fourier transform of ψ, it follows from (4.11) that,

$$(\xi^2 - E)\hat{\psi} = \hat{\varphi}(\xi).$$

Then, by Theorem 3.5 of [2], $\hat{\psi} \in \mathcal{H}^{s-1}$, what implies that $\psi \in L^2_{s-1}$.

Moreover, as $V_1 = V_2, A_1 = A_2$ for $|x| \geq R_1$,

$$(H_0 + Q_1)\psi = E\psi, \text{ for } |x| \geq R_1,$$

and since $s - 1 > -1/2$, it follows from Theorem 1.2 of [3] that $\psi(x) = 0$ for $|x|$ large enough, and then, by unique continuation [9], [32], $\psi(x) = 0$, for $|x| \geq R_1$, and (4.8) holds.

□

In our first uniqueness result we consider the case where the magnetic field is identically zero.

THEOREM 4.2. *Suppose that $F = 0$ and let $V_j, j = 1, 2$ be electric potentials in $\mathbb{R}^n, n \geq 3$, that satisfy, $V_j \in L^n_{\text{loc}}$ and such that for some $R > 0$,*

$$|V_j(x)| \leq C(1 + |x|)^{-\rho}, \quad \rho > 1, \ |x| \geq R > 0, j = 1, 2. \tag{4.13}$$

Let $S_j(E)$ be the scattering matrices corresponding, respectively, to $V_j, j = 1, 2$. Then, if for some $E > 0$, $S_1(E) = S_2(E)$ and $V_1(x) = V_2(x)$ for $|x| \geq R > 0$, the electric potentials coincide everywhere, i.e. $V_1(x) = V_2(x), x \in \mathbb{R}^n$.

Proof: Let B_R denote the open the ball of center zero and radius R. Let $\varphi_j \in \mathcal{H}^2(B_R), j=1,2$, be any solutions to

$$(H_0 + V_j - E)\,\varphi_j = 0, j=1,2. \tag{4.14}$$

Then, multiplying the equation for $j=1$ by $\overline{\varphi_2}$ and the complex conjugate of the equation for $j=2$ by φ_1 integrating over B_R substracting the resulting equations and using Green's formula we obtain the following identity,

$$\int_{B_R} (V_1 - V_2)\,\varphi_1\,\overline{\varphi_2}\,dx = \int_{\partial B_R} \left(\overline{\varphi_2}\partial_\nu \varphi_1 - \varphi_1 \overline{\partial_\nu \varphi_2}\right)\,dS, \tag{4.15}$$

where ν is the exterior unit normal to ∂B_R.

It follows from Lemma 4.1 that

$$\int_{B_R} (V_2 - V_1)\,\phi^{(1)}_{+,f}\,\overline{\phi^{(2)}_{+,g}}\,dx = 0,\ f, g \in L^2(\mathbb{S}^{n-1}). \tag{4.16}$$

Then, by Theorem 3.1,

$$\int_{B_R} (V_2 - V_1)\,\varphi_1\,\overline{\varphi_2}\,dx = 0 \tag{4.17}$$

for every $\varphi_j \in \mathcal{H}^2(B_R)$ that are solutions to (4.14), j=1,2.

Now as in [26] for any $p \in \mathbb{C}^n, p^2 = E, \mathrm{I}mp \neq 0$, and $|p|$ large enough we construct solutions $\varphi_j(x,p) \in \mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}^n)$ to the equations

$$(H_0 + \chi_{B_R}(x) V_j)\varphi_j(x,p) = E\varphi_j(x,p), j=1,2,$$

where χ_{B_R} is the characteristic function of B_R, such that,

$$\varphi_j(x,p) = e^{ip\cdot x}(1 + \psi_j(x,p)),$$

where $\|\psi_j(x,p)\|_{\mathbb{H}^1_{-s}}\| \leq C_s, s > 1/2$, and

$$\mathrm{s}-\lim_{|p|\to\infty} \|\psi_j(x,p)\|_{L^2_{-s}} = 0.$$

For this purpose, note that as $V \in L^n_{\mathrm{loc}}$ and (4.7) holds it follows from Theorem 5.2 of [21] that $(1+|x|)^\rho V_j, j=1,2$ are bounded operators from $\mathcal{H}^1$ into L^2 and that their norm can be bounded by a constant times $1 + \|V_j\|_{L^n(B_R)}$.

Given any $\xi \in \mathbb{R}^n$ take a sequence $p^{(j)}_l$ satisfying $\left(p^{(j)}_l\right)^2 = E, \mathrm{Re}\,p^{(1)}_l - \mathrm{Re}\,p^{(2)}_l = \xi, \mathrm{Im}\,p^{(1)}_l = -\mathrm{Im}\,p^{(2)}_l \neq 0, \lim_{l\to\infty} |p^{(j)}|_l = \infty$. This is possible as $n \geq 3$. Since $\varphi\left(x, p^{(j)}_l\right)$ are solutions to (4.14) in B_R we have that,

$$\int_{B_R} (V_2(x) - V_1(x))\,\varphi_1(x, p^{(1)}_l)\,\overline{\varphi_2(x, p^{(2)}_l)}\,dx = 0.$$

But then,

$$\int_{B_R} e^{i\xi\cdot x}\,(V_2(x)-V_1(x))\,dx = \lim_{l\to\infty}\int_{B_R}(V_2(x)-V_1(x))\,\varphi_1(x,p_l^{(1)})\,\overline{\varphi_2(x,p_l^{(2)})}\,dx = 0,$$

and it follows that $V_1(x)=V_2(x), x\in B_R$.

□

We now consider the case where there is also a magnetic potential.

THEOREM 4.3. *Suppose that $V_j, F_j \in C^\infty(\mathbb{R}^n), n\geq 3$, that V_j satisfies (4.7) and F_j satisfies (4.3), j=1,2. Let $S_j(E)$ be the scattering matrices corresponding, respectively, to $(V_j,F_j), j=1,2$. Then, if some $E>0$, $S_1(E)=S_2(E)$ and $V_1(x)=V_2(x), F_1(x)=F_2(x)$ for $|x|\geq R>0$, we have that the electric potentials and the magnetic fields coincide everywhere, i.e. $V_1(x)=V_2(x), F_1(x)=F_2(x), x\in\mathbb{R}^n$.*

Proof: Let us consider the following Dirichlet problems on B_R,

$$(H_0+Q_j-E)\varphi = 0,\ \varphi|_{\partial B_R} = f, \tag{4.18}$$

where $Q_j, j=1,2$, are defined as in (2.2) with V_j, A_j instead of V, A and A_j given by (4.4)-(4.6) with the corresponding F_j. We can always take a R such that zero is not an eigenvalue of H_0+Q_j for both, $j=1$ and $j=2$ [15]. Then, for every $f\in\mathcal{H}^{1/2}(\partial\Omega)$ there is a unique $\varphi_j\in\mathcal{H}^1(B_R)$ that solves (4.18) for $j=1,2$. The Dirichlet to Neumann maps, Λ_j, are the operators mapping $\mathcal{H}^{1/2}(\partial\Omega)$ into $\mathcal{H}^{-1/2}(\partial\Omega)$ defined as,

$$\Lambda_j f = \frac{\partial}{\partial\nu}\varphi_j|_{\partial B_R} - iA_j\cdot\nu f, j=1,2, \tag{4.19}$$

where ν is the exterior unit normal to ∂B_R and with $\frac{\partial}{\partial\nu}\varphi_j|_{\partial B_R}$ defined in trace sense.

The following identity is proven by integration by parts [23],

$$\begin{aligned} &i\int_{B_R}(A_2-A_1)\cdot(\varphi_1\,\nabla\overline{\varphi_2}-\overline{\varphi_2}\,\nabla\varphi_1)\;dx+\int_{B_R}(A_1^2-A_2^2+V_1-V_2)\varphi_1\,\overline{\varphi_2}\,dx\\ &= -\int_{\partial B_R}\overline{\varphi_2}\,[\Lambda_1-\Lambda_2]\varphi_1\,dS, \end{aligned} \tag{4.20}$$

where $\varphi_j\in\mathcal{H}^1(B_R)$ are any pair of solutions to $(H_0+Q_j-E)\varphi_j=0, j=1,2$.

Since by Lemma 4.1, $\phi^{(1)}_{+,f}=\phi^{(2)}_{+,f}$, $\frac{\partial}{\partial\nu}\phi^{(1)}_{+,f}=\frac{\partial}{\partial\nu}\phi^{(2)}_{+,f}$ and $A_1=A_2$ on ∂B_R we have that,

$$\Lambda_2\phi^{(1)}_{+,f}|_{\partial B_R}=\Lambda_2\phi^{(2)}_{+,f}|_{\partial B_R}=\frac{\partial}{\partial\nu}\phi^{(2)}_{+,f}|_{\partial B_R}-iA_2\phi^{(2)}_{+,f}|_{\partial B_R}=\Lambda_1\phi^{(1)}_{+,f}|_{\partial B_R},$$

and taking $\varphi_1=\phi^{(1)}_{+,f}$ in (4.20) we obtain that,

$$i\int_{B_R}(A_2-A_1)\cdot\left(\phi^{(1)}_{+,f}\,\nabla\overline{\varphi_2}-\overline{\varphi_2}\,\nabla\phi^{(1)}_{+,f}\right)\,dx+\int_{B_R}(A_1^2-A_2^2+V_1-V_2)\,\phi^{(1)}_{+,f}\,\overline{\varphi_2}\,dx=0. \tag{4.21}$$

Let $\phi^{(1)}_{+,f_n}$ be a sequence that converges to φ_1 in $L^2(B_R)$. Then,

$$\begin{aligned}&\lim_{n\to\infty}\int_{B_R}(A_2-A_1)\cdot\overline{\varphi_2}\,\nabla\phi^{(1)}_{+,f_n}\,dx=-\lim_{n\to\infty}\int_{B_R}\left[\nabla((A_2-A_1)\cdot\overline{\varphi_2})\right]\phi^{(1)}_{+,f_n}\,dx=\\&-\int_{B_R}\left[\nabla((A_2-A_1)\cdot\overline{\varphi_2})\right]\varphi_1\,dx=\int_{B_R}(A_2-A_1)\cdot\overline{\varphi_2}\,\nabla\varphi_1\,dx.\end{aligned}\tag{4.22}$$

Replacing $\phi^{(1)}_{+,f}$ by $\phi^{(1)}_{+,f_n}$ in (4.21), taking the limit when $n\to\infty$ and using (4.22) we obtain that

$$i\int_{B_R}(A_2-A_1)\cdot(\varphi_1\nabla\overline{\varphi_2}-\overline{\varphi_2}\,\nabla\varphi_1)\ dx+\int_{B_R}(A_1^2-A_2^2+V_1-V_2)\varphi_1\,\overline{\varphi_2}\,dx=0.$$

But as this holds for any $\varphi_j\in\mathcal{H}^{(1)}(B_R)$ with $(H_0+Q_j-E)\varphi_j=0, j=1,2$, it follows from (4.20) that $\Lambda_1=\Lambda_2$. Then, by Theorem (B) of [16], $V_1=V_2, F_1=F_2$ on B_R and the theorem follows.

□

In order to state our uniqueness result for electric potentials and magnetic fields with a regular behaviour at infinity we introduce some notation [31].

Let us denote by $\dot{\mathcal{S}}^{-\rho}$ the set of $C^\infty(\mathbb{R}^n\setminus\{0\})$-functions $f(x)$ such that $\partial^\alpha f(x)=O(|x|^{-\rho-|\alpha|})$ as $|x|\to\infty$ for all α. An important example of functions from the class $\dot{\mathcal{S}}^{-\rho}$ are homogeneous functions $f\in C^\infty(\mathbb{R}^n\setminus\{0\})$ of order $-\rho$ such that $f(\lambda x)=\lambda^{-\rho}f(x)$ for all $x\in\mathbb{R}^n, x\neq 0$, and $\lambda>0$.

Let the functions $f_l\in\dot{\mathcal{S}}^{-\rho_l}$ where $\rho_l\to\infty$ (but the condition $\rho_l<\rho_{l+1}$ is not required). The notation

$$f(x)\simeq\sum_{l=1}^{\infty}f_l(x)\tag{4.23}$$

means that, for any N, the remainder

$$f-\sum_{l=1}^{N}f_j\in\dot{\mathcal{S}}^{-\rho}\quad\text{where}\quad\rho=\min_{l\geq N+1}\rho_l.\tag{4.24}$$

In particular, if the sum (4.23) consists of a finite number N of terms, then the inclusion (4.24) should be satisfied for all ρ. A function $f\in C^\infty$ is determined by its asymptotic expansion (4.23) up to a term from the Schwarz class $\mathcal{S}=\mathcal{S}^{-\infty}$.

THEOREM 4.4. *Let the electric potentials V_j and the magnetic fields F_j be $C^\infty(\mathbb{R}^n)$ functions, $n\geq 3, j=1,2,$ and assume that they satisfy,*

$$|\partial^\alpha V_j(x)|\leq C(1+|x|)^{-\rho-|\alpha|},|\partial^\alpha F_j(x)|\leq C(1+|x|)^{-1-\rho-|\alpha|},\quad\rho>1,\tag{4.25}$$

for all α. Moreover, suppose that they admit the asymptotic expansions

$$V_j(x)\simeq\sum_{l=1}^{\infty}V_{j,l}(x),\quad F_j(x)\simeq\sum_{l=1}^{\infty}F_{j,l}(x),j=1,2,\tag{4.26}$$

where $V_{j,l}$ and $F_{j,l}$ are homogeneous functions of orders, respectively, $-\rho_{j,l}$ and $-r_{j,l}$, with, $1 < \rho_{j,1} < \rho_{j,2} < \cdots$, and $2 < r_{j,1} < r_{j,2} < \cdots, j = 1, 2$. Assume, moreover, that the asymptotic expansions (4.26) actually converge, respectively, to V_j and $F_j, j = 1, 2$, in pointwise sense, for $|x|$ large enough, or just that the sums in (4.26) are finite. Let $S_j(E)$ be, respectively, the scattering matrices corresponding to $(V_j, F_j), j = 1, 2$. Then, if for some $E > 0$, $S_1(E) = S_2(E)$, we have that $V_1(x) = V_2(x)$ and $F_1(x) = F_2(x), x \in \mathbb{R}^n$.

Proof: By Theorem 4.2 of [31] $V_{1,l} = V_{2,l}, F_{1,l} = F_{2,l}, l = 1, 2, \cdots$. Moreover, since the asymptotic expansions in (4.26) actually converge, respectively, to V_j, F_j for $|x|$ large enough, we have that $V_1(x) = V_2(x), F_1(x) = F_2(x)$ for $|x|$ large enough, and then by Theorem 4.3 $V_1(x) = V_2(x), F_1(x) = F_2(x), x \in \mathbb{R}^n$. The same argument applies if the sums in (4.26) are finite.

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