Derivation of the Gross-Pitaevskii Hierarchy for the Dynamics of Bose-Einstein Condensate

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Abstract

Consider a system of N bosons on the three dimensional unit torus interacting via a pair potential $N^2V(N(x_i-x_j))$, where $\mathbf{x}=(x_1,\ldots,x_N)$ denotes the positions of the particles. Suppose that the initial data $\psi_{N,0}$ satisfies the condition

$$\langle \psi_{N,0}, H_N^2 \psi_{N,0}\rangle \leq CN^2$$

where H_N is the Hamiltonian of the Bose system. This condition is satisfied if $\psi_{N,0}=W_N\phi_{N,0}$ where W_N is an approximate ground state to H_N and $\phi_{N,0}$ is regular. Let $\psi_{N,t}$ denote the solution to the Schrödinger equation with Hamiltonian H_N. Gross and Pitaevskii proposed to model the dynamics of such system by a nonlinear Schrödinger equation, the Gross-Pitaevskii (GP) equation. The GP hierarchy is an infinite BBGKY hierarchy of equations so that if u_t solves the GP equation, then the family of k-particle density matrices $\{\otimes_k u_t, k\geq 1\}$ solves the GP hierarchy. We prove that as $N\to\infty$ the limit points of the k-particle density matrices of $\psi_{N,t}$ are solutions of the GP hierarchy. The uniqueness of the solutions to this hierarchy remains an open question. Our analysis requires that the N boson dynamics is described by a modified Hamiltonian which cuts off the pair interactions whenever at least three particles come into a region with diameter much smaller than the typical inter-particle distance. Our proof can be extended to a modified Hamiltonian which only forbids at least n particles from coming close together, for any fixed n.

1 Introduction

A very simple and useful way to understand Bose systems is to treat all bosons as independent particles. In particular, to prove the Bose-Einstein condensation is a simple exercise in the case of non-interacting bosons [12]. The true many-body problem with a pair interaction is a much harder problem. Gross [9, 10] and Pitaevskii [26] proposed to model the many-body effect by a nonlinear on-site self interaction of a complex order parameter (the "condensate wave function"). The strength

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of the nonlinear interaction in this model is given by the scattering length a_0 of the pair potential. The Gross-Pitaevskii (GP) equation is given by

$$i\partial_t u_t = -\Delta u_t + \sigma |u_t|^2 u_t = \frac{\delta \mathcal{E}(u,\bar{u})}{\delta \bar{u}}\Big|_{u_t}, \quad \mathcal{E}(u,\bar{u}) = \int \mathrm{d}x \Big[|\nabla u|^2 + \frac{\sigma}{2}|u|^4 \Big], \tag{1.1}$$

where $\mathcal{E}$ is the Gross-Pitaevskii energy functional and $\sigma = 8\pi a_0$. The Gross-Pitaevskii equation was considered as a mean-field model; some corrections were obtained in [14, 15] and more recently, e.g., in [28].

The many-body effects can be analyzed in much more details by the Bogoliubov transformation. The Bogoliubov's theory, it should be emphasized, postulates that the ratio between the non-condensate and the condensate is small. Apart from this assumption, the coupling constant σ derived from the Bogoliubov's theory is the semiclassical approximation to the scattering length. One can perform some higher order diagrammatic re-summation to recover the scattering length. In the case of hard core potential, the traditional theory relies on the non-rigorous pseudo-potential model [13].

The first rigorous result concerning the many-body effects of the Bose gas was Dyson's estimate of the ground state energy. Dyson [3] proved the correct leading upper bound to the energy and a lower bound off by a factor around 10. The matching lower bound was obtained by Lieb-Yngvason [24, 25] forty years later. The last result has inspired many subsequent works, including a proof [19, 24, 25] that the Gross-Pitaevskii energy functional correctly describes the ground state in a scaling limit to be specified later.

The experiments on the Bose-Einstein condensation were conducted by observing the dynamics of the condensate when the confining traps are switched off. It is remarkable that the Gross-Pitaevskii equation, despite being a mean-field equation, has provided a very good description for the dynamics of the condensate. The validity of the Gross-Pitaevskii equation asserts that the fundamental assumption of the Gross-Pitaevskii theory (i.e., that the many-body effects can be modelled by a nonlinear on-site self interaction of the order parameter) applies not only to the ground states, but to certain excited states and their subsequent time evolution. This remarkable and fundamental property of the Bose gas has mostly been taken for granted and has not been treated rigorously in the literature. In order to explain the issues involved, we now introduce some notations and set up the scaling for the Gross-Pitaevskii theory.

Let $V \geq 0$ be a fixed nonnegative, spherically symmetric, smooth potential with compact support. The zero energy scattering solution f satisfies the equation

$$\Big[-\Delta + \frac{1}{2} V(x) \Big] f(x) = 0\,. \tag{1.2}$$

If we fix the normalization $\lim_{|x|\to\infty} f(x) = 1$ and write $f(r) = q(r)/r$ (where $r = |x|$), the scattering length a_0 of V is defined by

$$a_0 = \lim_{r\to\infty} r - q(r) \tag{1.3}$$

and thus for x large,

$$f(x) \sim 1 - a_0/|x|\,. \tag{1.4}$$

From the zero energy equation, we also have the identity

$$\int \mathrm{d}x V(x) f(x) = 8\pi a_0\,. \tag{1.5}$$

Let Λ be the three-dimensional torus of unit side-length. Denote the position of N bosons in Λ by $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $x_j \in \Lambda$. Lieb, Seiringer and Yngvason [19] pointed out that, in order to obtain the GP functional, the length scale of the pair potential should be of order $1/N$. Notice that the density of the system is N and the typical inter-particle distance is $N^{-1/3}$, which is much bigger than the length scale of the potential. The system is really a dilute gas on the scale of the range of the interaction, but it is scaled in such a way that the size of the total system is of order one.

The Hamiltonian of the Bose system on the torus Λ is given by

$$H = -\sum_{j=1}^{N} \Delta_j + \sum_{j<k}^{N} N^2 V(N(x_j - x_k)) \,. \tag{1.6}$$

By scaling, the scattering length of the potential $N^2V(Nx)$ is $a := a_0/N$. We shall from now on use the notation

$$V_a(x) := N^2\, V\,(Nx)\ , \qquad a = a_0/N\,. \tag{1.7}$$

Notice that the pair potential in (1.6) is an approximation to the mean field Dirac delta interaction:

$$\frac{b_0}{N}\sum_{j<k}^{N} \delta(x_j - x_k), \qquad b_0 = \int \mathrm{d}x V(x)\,.$$

The Schrödinger equation is given by

$$i\partial_t \psi_t = H\psi_t, \quad \text{or} \quad i\partial_t \gamma_N = [H, \gamma_N]\ , \tag{1.8}$$

where γ_N is the N-particle density matrix. For a pure state, $\gamma_\psi := |\psi\rangle\langle\psi|$ is the orthogonal projection onto ψ.

Introduce the shorthand notation

$$\mathbf{x} := (x_1, x_2, \ldots, x_N), \quad \mathbf{x}_k := (x_1, \ldots, x_k), \quad \mathbf{x}_{N-k} := (x_{k+1}, \ldots, x_N)$$

and similarly for the primed variables, $\mathbf{x}'_k := (x'_1, \ldots, x'_k)$. The $k-$particle density matrix is given by

$$\gamma_N^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) := \int \mathrm{d}\mathbf{x}_{N-k} \gamma_N \left(\mathbf{x}_k, \mathbf{x}_{N-k}; \mathbf{x}'_k, \mathbf{x}_{N-k}\right)\ . \tag{1.9}$$

Our normalization implies that $\mathrm{Tr}\ \gamma_N^{(k)} = 1$ for all $k = 1, \ldots, N$.

The density matrix $\gamma_{N,t}^{(1)}$ satisfies the following equation

$$\begin{aligned} i\partial_t \gamma_{N,t}^{(1)}(x_1; x'_1) &= (-\Delta_{x_1} + \Delta_{x'_1})\gamma_{N,t}^{(1)}(x_1; x'_1) \\ &\quad + (N-1)\int \mathrm{d}x_2 (V_a(x_1 - x_2) - V_a(x'_1 - x_2))\gamma_{N,t}^{(2)}(x_1, x_2; x'_1, x_2), \end{aligned} \tag{1.10}$$

and similar equations hold for $\gamma_{N,t}^{(k)}$ for $k \geq 1$. To close this equation, one needs to assume some relation between $\gamma_{N,t}^{(2)}$ and $\gamma_{N,t}^{(1)}$. The simplest assumption would be the factorization property, i.e.,

$$\gamma_{N,t}^{(2)}(x_1, x_2; x'_1, x'_2) = \gamma_{N,t}^{(1)}(x_1; x'_1)\gamma_{N,t}^{(1)}(x_2; x'_2)\,. \tag{1.11}$$

This does not hold for finite N, but it may hold for a limit point $\gamma_t^{(k)}$ of $\gamma_{N,t}^{(k)}$ as $N \to \infty$, i.e.,

$$\gamma_t^{(2)}(x_1, x_2; x'_1, x'_2) = \gamma_t^{(1)}(x_1; x'_1)\gamma_t^{(1)}(x_2; x'_2)\,. \tag{1.12}$$

Under this assumption, $\gamma_t^{(1)}$ satisfies the limiting equation

$$i\partial_t \gamma_t^{(1)}(x_1;x_1') = \ (-\Delta_{x_1}+\Delta_{x_1'})\gamma_t^{(1)}(x_1;x_1') + (Q_t(x_1)-Q_t(x_1'))\gamma_t^{(1)}(x_1;x_1') \tag{1.13}$$

where

$$Q_t(x) = \lim_{N\to\infty} N\int \mathrm{d}y V_a(x-y)\rho_t(y), \quad \rho_t(x) = \gamma_t^{(1)}(x;x)\,. \tag{1.14}$$

If, instead of $V_a(x)$, we use the unscaled mean field potential $V(x)/N$, then Q_t is the convolution of V with the density $\rho_t(x)$. The equation (1.13) becomes the Hartree equation. For Bose systems with mean field interaction and product initial wave function, the factorization assumption (1.12) can be proved for a general class of potentials. See the work of Hepp [11] and Spohn [27] for bounded potentials and [2, 6] for potentials with Coulomb type singularity. For certain quasi-free initial data, Ginibre and Velo [8] can handle all integrable singularities. We note that in one dimension the convergence to the GP hierarchy (1.24) for the delta potential was established by Adami, Bardos, Golse and Teta in [1].

In our setting, $V_a(x)$ lives on scale $1/N$ and is much more singular than all the cases considered previously. If $\rho_t(x)$ is continuous, then Q_t is given by

$$Q_t(x) = b_0\rho_t(x).$$

Thus (1.13) gives the GP equation with the *incorrect* coupling constant $\sigma = b_0$ instead of $\sigma = 8\pi a_0$. It is known that $b_0/8\pi$ is the first Born approximation to the scattering length a_0 and the following inequality holds:

$$a_0 \le \frac{b_0}{8\pi} = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{2} V(x)\,dx\,. \tag{1.15}$$

To go beyond the Born approximation, we need to understand the short scale correlations of the ground state which we now review.

The ground state of a dilute Bose system with interaction potential V_a is believed to be very close to the form

$$W(\mathbf{x}) := \prod_{i<j} f(N(x_i - x_j)) \tag{1.16}$$

where f is the zero-energy solution (1.2). We remark that Dyson [3] took a different function which was not symmetric, but the short distance behavior was the same as in W. Since in the experiments the initial states were prepared with a trapping potential, living on a scale of order one, the ground state for such a trapped gas is of the form $\psi(\mathbf{x}) = W(\mathbf{x})\phi(\mathbf{x})$ [19, 20] where $\phi(\mathbf{x})$ is close to a product function. Thus we shall consider initial data of the form $W(\mathbf{x})\phi(\mathbf{x})$.

We assume for the moment that the ansatz, $\psi_t(\mathbf{x}) = W(\mathbf{x})\phi_t(\mathbf{x})$ with ϕ_t a product function, holds for all time. The reduced density matrices for $\psi_t(\mathbf{x})$ satisfy

$$\gamma_t^{(2)}(x_1,x_2;x_1',x_2') \sim f(N(x_1-x_2))f(N(x_1'-x_2'))\gamma_t^{(1)}(x_1;x_1')\gamma_t^{(1)}(x_2;x_2')\,. \tag{1.17}$$

Together with (1.5) and the assumption that ρ_t is smooth, we have

$$\lim_{N\to\infty} N\int \mathrm{d}x_2 V_a(x_1-x_2)\gamma_{N,t}^{(2)}(x_1,x_2;x_1',x_2) = 8\pi a_0 \gamma_t^{(1)}(x_1;x_1')\rho_t(x_1)\,. \tag{1.18}$$

We have used that $\lim_{|x|\to\infty} f(x) = 1$ and the last equation is valid for $|x_1 - x_1'| \gg 1/N$. This gives the GP equation with the correct dependence on the scattering length.

Notice that the relations (1.17) and (1.18) are very subtle. The correlation in $\gamma^{(2)}$ occurs at the scale $1/N$. Testing the relation (1.17) in a weak limit, all correlations at the scale $1/N$ disappear and the product relation (1.12) will hold. However, this short distance correlation shows up in the GP equation due to the singular potential $NV_a(x_1 - x_2)$. Therefore, a rigorous justification of the GP equation requires a proof that the relation (1.17) holds with such a precision that (1.18) is valid—a formidable task.

We would like to divide this task into two parts. The first question is whether the short scale structure of $\gamma^{(2)}_{N,t}$ is given by $f(N(x_1-x_2))f(N(x_1'-x_2'))$, i.e., whether, for large N,

$$\gamma^{(2)}_{N,t}(x_1, x_2; x_1', x_2') \sim f(N(x_1 - x_2))f(N(x_1' - x_2'))\gamma^{(2)}_t(x_1, x_2; x_1', x_2') \tag{1.19}$$

where $\gamma^{(2)}_t$ is a weak limit of $\gamma^{(2)}_{N,t}$ (the short distance correlations given by $f(N(x_1-x_2))f(N(x_1'-x_2'))$ vanish when the weak limit is taken) living on a scale of order one so that

$$\lim_{N\to\infty} \int \mathrm{d}x_2 V_a(x_1 - x_2)\gamma^{(2)}_{N,t}(x_1, x_2; x_1', x_2) = 8\pi a_0 \gamma^{(2)}_t(x_1, x_1; x_1', x_1)\,. \tag{1.20}$$

The second question is whether $\gamma^{(2)}_t$ factorizes, i.e., whether

$$\gamma^{(2)}_t(x_1, x_2; x_1', x_2') = \gamma^{(1)}_t(x_1; x_1')\gamma^{(1)}_t(x_2; x_2'). \tag{1.21}$$

If we replace the Hamiltonian (1.6) by a modified Hamiltonian $\widetilde{H}$, in which we remove the pair interaction when three or more particles come close together in a very short distance, then we can prove a certain version of (1.19) and (1.20). Let ℓ denote a distance much smaller than the typical inter-particle distance, say, $\ell = N^{-1/3-\delta}$ for some $\delta > 0$. The modified Hamiltonian is approximately of the form

$$\widetilde{H} \sim -\sum_{j=1}^{N} \Delta_j + \sum_{j<k}^{N} \Big[N^2 V(N(x_j - x_k)) \prod_{i\neq j,k} 1(|x_i - x_k| \geq \ell) \Big]\,. \tag{1.22}$$

The precise definition will be given in (2.9). This cutoff modification changes the Hamiltonian for events that are rare with respect to the expected typical particle distribution, therefore it should have little effect on the dynamics. Unfortunately we cannot control this effect rigorously. (In principle, the original unmodified dynamics may introduce local clustering of particles, despite that it is unfavorable for the local energy.) In the computation of the ground state energy by Lieb-Yngvason [24], no such modification was needed since the positivity of the contribution from these rare events could be exploited. Our method is based on the conservation of $\widetilde{H}^2$ along the dynamics and an inequality of the form:

$$\sum_{i<j} \int W^2 |\nabla_i \nabla_j \phi_t|^2 \leq C \int |(\widetilde{H} + N)\psi_t|^2 = C \int |(\widetilde{H} + N)\psi_0|^2 \leq CN^2, \tag{1.23}$$

with $\psi_t = W\phi_t$. The idea of using the conservation of higher power of the Hamiltonian was introduced in [6]. Clearly, the computation of $\widetilde{H}^2$ involves derivatives of the pair potential which have no definite sign. If we use the original Hamiltonian H instead of $\widetilde{H}$, these terms cannot be controlled by the kinetic energy operator in the rare situation when many particles come close together. The modified Hamiltonian (1.22) removes this technical obstacle.

Using (1.23) we will prove that weak limit points $\gamma^{(k)}_t$ of the k-particle density $\gamma^{(k)}_{N,t}$ (whose time evolution is generated by the modified Hamiltonian $\widetilde{H}$) satisfy the following infinite BBGKY hierarchy, which will be called GP-hierarchy:

$$\begin{aligned} i\partial_t \gamma_t^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) = &\sum_{j=1}^k (-\Delta_{x_j} + \Delta_{x'_j})\gamma_t^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) \\ &+ \sigma \sum_{j=1}^k \int \mathrm{d}x_{k+1}(\delta(x_j - x_{k+1}) - \delta(x'_j - x_{k+1}))\gamma_t^{(k+1)}(\mathbf{x}_k, x_{k+1};\mathbf{x}'_k, x_{k+1})\,. \end{aligned} \tag{1.24}$$

with the correct coupling constant $\sigma = 8\pi a_0$. Notice that if $u(t,x)$ is a solution of the GP equation (1.1), then

$$\gamma_t^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) = \prod_{j=1}^k u(t,x_j)\bar{u}(t,x'_j) \tag{1.25}$$

is a solution of the hierarchy (1.24). To conclude the factorization property (1.21), it remains to answer the following three open questions. The first one is to rigorously justify the cutoff modification of the Hamiltonian by controlling the clustering of particles. The second one is the uniqueness of the solution of the hierarchy in a certain space. The third one is to prove a-priori bounds on the density matrices of the Schrödinger equation so that their limits fall into the space needed in the uniqueness theorem. Recently the uniqueness problem was solved in [5]. Furthermore, the required a-priori estimates were established for certain mean field Hamiltonians without the cutoff modification (see Section 3.1 for more details). However, the a-priori estimates and the removal of the cutoff modification for the GP scaling remain interesting open problems.

2 Definition of the Model and the Main Result

We now define the modified Hamiltonian and state the main result. Throughout the paper we use the notation that $a \ll b$, if $a/b = O(N^{-\alpha})$ for some $\alpha > 0$. The notation $W^{k,p}$ will stand for the standard Sobolev spaces.

2.1 The Two-Body Problem

We consider the problem of a single particle in the field of the scaled potential

$$V_a(x)/2 = (1/2)N^2 V(Nx).$$

Here we assume $V(x)$ to be a smooth, spherically symmetric, and compactly supported potential. For $0 < \kappa \ll 1$ let e_κ and $(1 - w^\kappa)$ be the lowest Neumann eigenvalue and eigenfunction on the ball $\{y : |y| \le \kappa\}$,

$$(-\Delta + \tfrac{1}{2}V_a)(1 - w^\kappa) = e_\kappa(1 - w^\kappa) \tag{2.1}$$

with normalization $w^\kappa(|x| = \kappa) = 0$. We can extend w^κ to be identically zero for $|x| \ge \kappa$ so that w^κ satisfies

$$(-\Delta + \tfrac{1}{2}V_a)(1 - w^\kappa) = e_\kappa(1 - w^\kappa)\chi_\kappa(x), \tag{2.2}$$

where $\chi_\kappa(x) := \chi(|x| \le \kappa)$ is the characteristic function of the ball of radius κ. We will prove in Lemma A.1 that

$$e_\kappa = 3a\kappa^{-3}(1 + o(1)),$$

as long as $a \ll \kappa \ll 1$.

Let ℓ_1 be a scale to be fixed later on with $a \ll \ell_1 \ll 1$. Let $\mu(d\kappa) := g(\kappa)d\kappa$ be a probability measure supported on $[\ell_1/2, 3\ell_1/2]$ with smooth density g. Define w by

$$1 - w := \int (1 - w^\kappa)\mu(d\kappa) . \tag{2.3}$$

One can check that w satisfies the equation

$$(-\Delta + \tfrac{1}{2}V_a)(1 - w) = q(1 - w) \tag{2.4}$$

where

$$q(x) := \frac{\int e_\kappa 1(|x| \leq \kappa)(1 - w^\kappa(x))\mu(d\kappa)}{\int (1 - w^\kappa(x))\mu(d\kappa)} . \tag{2.5}$$

Some properties of the functions $w(x)$ and $q(x)$, which will be used in the rest of the paper, are collected in Appendix A (see, in particular, Lemma A.2).

2.2 Removal of triples

We introduce a new length-scale ℓ with $\ell_1 \ll \ell \ll 1$. The following procedure excludes configurations with more than three particles within a distance ℓ from each other.

Let h be the exponential cutoff at scale ℓ defined by

$$h(x) := e^{-\sqrt{x^2+\ell^2}/\ell} , \qquad x \in \Lambda .$$

For a configuration $\mathbf{x} = (x_1, \dots x_N)$ let $\mathcal{N}_{ij}(\mathbf{x})$ be the number of particles other than i and j that are within distance ℓ to either i or j:

$$\mathcal{N}_{ij}(\mathbf{x}) := \sum_{k \neq i,j} [h(x_k - x_i) + h(x_k - x_j)] , \qquad i \neq j .$$

Let $0 < \varepsilon < \frac{1}{10}$ be fixed. Define

$$F(u) := e^{-u/\ell^\varepsilon}, \quad u \geq 0$$

and

$$F_{ij}(\mathbf{x}) := F(\mathcal{N}_{ij}(\mathbf{x})) .$$

Thus $F_{ij}(\mathbf{x})$ is exponentially small if $|x_k - x_i| \lesssim \ell$ or $|x_k - x_j| \lesssim \ell$ for some $k \neq i, j$. The functions F_{kj}, modulo an exponentially small error, forbid particle triples within a distance ℓ and provide a strong non-overlap property. This fact and some other important properties of the functions F_{ij} will be presented in Appendix B.

Introduce the notations

$$\chi_{jk} = \chi_{jk}(\mathbf{x}) := 1(|x_j - x_k| \leq \ell_1) \qquad \text{if} \quad k \neq j ,$$
$$\widetilde{\chi}_{jk} = \widetilde{\chi}_{jk}(\mathbf{x}) := 1(|x_j - x_k| \leq \ell) \qquad \text{if} \quad k \neq j ,$$

and $\chi_{jk} \equiv \widetilde{\chi}_{jk} \equiv 0$ if $j = k$. Analogously, we will freely use the shorthand notation

$$V_{ij} = V_a(x_i - x_j), \quad w_{ij} = w(x_i - x_j), \quad h_{ij} = h(x_i - x_j) , \quad q_{ij} = q(x_i - x_j) ,$$

and similarly for their derivatives. In particular $w_{ii} = 0$, $(\nabla w)_{ii} = 0$.

We also fix a smooth function $\theta \in C_0^\infty(\mathbb{R})$ with $\theta(s) = 1$ for $s \leq 1$ and $\theta(s) = 0$ for $s > 2$. For some $K > 0$ we introduce the notation

$$\theta_{kj} := \theta\Big(\frac{|x_k - x_j|}{K\ell|\log \ell|}\Big) . \tag{2.6}$$

The constant K will be chosen sufficiently large but independent of N at the end of the proof. The K-dependence will be omitted from the notation.

2.3 The Modified Hamiltonian

Consider the function w defined in (2.3) for the length scale ℓ_1. We define

$$G_i(\mathbf{x}) := 1 - \sum_{j\neq i} w(x_i - x_j)F_{ij}(\mathbf{x}) , \quad \text{and} \quad W := \prod_{i=1}^{N} \sqrt{G_i} . \tag{2.7}$$

The function W is our approximation to the wave function of the ground state. Due to the cutoffs it differs slightly from the form $\prod_{i<j} f(x_i - x_j)$ mentioned in Section 1. It is also slightly different from Dyson's construction [3] of the ground state, which is non-symmetric. By Lemma A.2, $w \le d_0 < 1$ is separated away from 1. We will prove, in Section 8 (see also Lemma 4.1), that there exists a constant $c_1 > 0$ such that $c_1 \le G_i \le 1$.

Introduce the notation M_{kj} by

$$M_{kj} := \frac{F_{kj}}{2}(G_j^{-1} + G_k^{-1}) . \tag{2.8}$$

The modified Hamiltonian $\widetilde{H}$ is defined by

$$\widetilde{H} := H - \sum_{k\neq j}(\tfrac{1}{2}V_{kj} - q_{kj})[1 - (1 - w_{kj})M_{kj}] . \tag{2.9}$$

Since V and q have compact support of size at most of order ℓ_1 (Lemma A.2 in Appendix A), the factor $(1/2)V_{kj} - q_{kj}$ vanishes for any $k \neq j$ unless $|x_k - x_j| \le O(\ell_1)$. Suppose now $|x_k - x_j| \le O(\ell_1)$. If there is no third particle at x_m with $m \neq k, j$, such that $|x_k - x_m| \lesssim \ell/|\log \ell|$ then $G_j \approx G_k \approx 1 - w_{kj}$ and $F_{kj} \approx 1$ with exponential precision. Thus $[1 - (1 - w_{kj})M_{kj}]$ is exponentially small. This shows that the difference between H and $\widetilde{H}$ is exponential small unless three or more particles are closer than ℓ to each other. Since we will choose $\ell = N^{-2/5-\kappa}$, for some $\kappa > 0$, the probability for this to happen is, for large N, very small.

The reason for this modification will be clear in Section 6; without the subtraction of the three-body terms in (2.9) we are not able to prove the a priori estimate for H^2 (part ii) of Proposition 6.1.

Our methods can be easily generalized to remove only n-body collisions instead of removing triples collisions, for any $n \ge 3$. More precisely, let $f \ge 0$ be a smooth function such that $f(x) = 1$ for $x \le 0$, $f(x) \simeq e^{-x}$ for $x \gg 1$. We define

$$F^{(n)}(u) := f\left(\frac{u - (n-3)}{\ell^{\varepsilon}}\right)$$

and

$$F_{ij}^{(n)}(\mathbf{x}) := F^{(n)}(\mathcal{N}_{ij}(\mathbf{x})).$$

Then, apart from exponentially small errors, $F_{ij}^{(n)}(\mathbf{x}) = 1$, unless there are at least $n - 2$ other particles in the ℓ-vicinity of x_i and x_j (and $F_{ij}^{(n)} \simeq 0$ if this is the case). The modified Hamiltonian

$$\widetilde{H}^{(n)} = H - \sum_{k\neq j}(\tfrac{1}{2}V_{kj} - q_{kj})[1 - (1 - w_{kj})M_{kj}^{(n)}] \tag{2.10}$$

with $M_{kj}^{(n)}$ defined through (2.8), with F_{ij} replaced by $F_{ij}^{(n)}$. Note that $\widetilde{H}^{(n)}$ now differs from H by a term which is exponentially small unless there are n or more particles closer that $\ell = N^{-2/5-\kappa}$ to each other.

Since we are dealing with bosons, the Hamiltonian (2.9) acts on the Hilbert space $\mathcal{H}^{\otimes_s N} := L^2(\Lambda, \mathrm{d}x)^{\otimes_s N}$ of functions of $3N$ variables which are symmetric with respect to any permutation of the N particles, i.e., $\psi \in \mathcal{H}^{\otimes_s N}$ if and only if $\psi \in \mathcal{H}^{\otimes N}$ and $R_\pi \psi = \psi$ for all permutation $\pi \in S_N$, where

$$(R_\pi \psi)(x_1, \ldots, x_N) := \psi(x_{\pi 1}, \ldots, x_{\pi N}) \,. \tag{2.11}$$

The dynamics is given by the Schrödinger equation

$$i\partial_t \psi_t = \widetilde{H}\psi_t, \quad \text{or} \quad i\partial_t \gamma_N = [\widetilde{H}, \gamma_N] \,, \tag{2.12}$$

for a density matrix γ_N.

2.4 Weak* Topology for the Kernel of Density Matrices

Let $\mathcal{H} := L^2(\Lambda, \mathrm{d}x)$ be the one particle Hilbert space, and let $\mathcal{H}^{\otimes_s n}$ be the symmetric subspace of the n-fold tensor product $\mathcal{H}^{\otimes n}$. We denote by $\mathcal{L}^1_n := \mathcal{L}^1(\mathcal{H}^{\otimes n})$ the space of trace class operators on the Hilbert space $\mathcal{H}^{\otimes n}$. We will work with families of trace class operators $\Gamma = \{\gamma^{(n)}\}_{n\geq 1}$ with $\gamma^{(n)} \in \mathcal{L}^1_n$ for all $n \geq 1$.

In this section we consider the operator kernels $\gamma^{(n)}(\mathbf{x}_n; \mathbf{x}'_n)$ as elements of $L^2(\Lambda^n \times \Lambda^n)$, with the norm

$$\|\gamma^{(n)}\|_2 = \int \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{x}'_n \, |\gamma^{(n)}(\mathbf{x}_n; \mathbf{x}'_n)|^2 \,. \tag{2.13}$$

For $\Gamma = \{\gamma^{(n)}\}_{n\geq 1} \in \bigoplus_{n\geq 1} L^2(\Lambda^n \times \Lambda^n)$, and any fixed $\nu > 1$ we define the norm

$$\|\Gamma\|_{H_-^{(\nu)}} := \sum_{n=1}^{\infty} \nu^{-n} \|\gamma^{(n)}\|_2 \tag{2.14}$$

and we define the Banach space $H_-^{(\nu)} := \{\Gamma \in \bigoplus_{n\geq 1} L^2(\Lambda^n \times \Lambda^n) : \|\Gamma\|_{H_-^{(\nu)}} < \infty\}$. We also define $H_+^{(\nu)} := \{\Gamma \in \bigoplus_{n\geq 1} L^2(\Lambda^n \times \Lambda^n) : \lim_{n\to\infty} \nu^n \|\gamma^{(n)}\|_2 = 0\}$. We equip $H_+^{(\nu)}$ with the norm

$$\|\Gamma\|_{H_+^{(\nu)}} = \sup_{n\geq 1} \nu^n \|\gamma^{(n)}\|_2 \,.$$

Similarly to the standard proof of the duality $\ell^1 = c_0^*$ between scalar valued summable sequences and vanishing sequences, one readily checks the duality $H_-^{(\nu)} = \big(H_+^{(\nu)}\big)^*$. By the Banach-Alaoglu Theorem, the unit ball $B_-^{(\nu)}$ in $H_-^{(\nu)}$ is compact with respect to the weak$*$ topology.

We shall fix a time $T > 0$ for the rest of this paper. Denote by $C([0,T], H_-^{(\nu)})$ the space of weak* continuous functions from $t \in [0,T]$ to $H_-^{(\nu)}$. Since the space $H_+^{(\nu)}$ is separable, we can fix a dense countable subset in the unit ball of $H_+^{(\nu)}$, denoted by $\{J_i\}_{i\geq 1}$. Define the metric on $H_-^{(\nu)}$ by

$$\rho(\Gamma, \widetilde{\Gamma}) := \sum_{i=1}^{\infty} 2^{-i} \Big| \sum_{n=1}^{\infty} \int \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{x}'_n \overline{J_i^{(n)}(\mathbf{x}_n; \mathbf{x}'_k)} \big[\gamma^{(n)}(\mathbf{x}_n; \mathbf{x}'_n) - \widetilde{\gamma}^{(n)}(\mathbf{x}_n; \mathbf{x}'_n)\big] \Big| \,. \tag{2.15}$$

Note that the topology induced by $\rho(.,.)$ and the weak* topology are equivalent on the unit ball $B_-^{(\nu)}$. We equip $C([0,T], H_-^{(\nu)})$ with the metric

$$\widehat{\rho}(\Gamma, \widetilde{\Gamma}) := \sup_{0\leq t\leq T} \rho\left(\Gamma(t), \widetilde{\Gamma}(t)\right) \,. \tag{2.16}$$

2.5 The Main Result

Consider the rescaled potential $V_a(x) = (a_0/a)^2 V((a_0/a)x)$ (see (1.7)) whose scattering length a is chosen such that $Na = a_0$. Let ℓ, ℓ_1 be two lengthscales satisfying

$$a \ll \ell_1 \ll \ell \ll 1, \quad a\ell_1 \ll \ell^4, \quad a \ll \ell_1^{3/2}, \quad \ell_1 \ll \ell^{3/2}, \quad \ell_1^3 \ll a\ell^{9/4}, \quad \ell^5 \ll a^2 \tag{2.17}$$

(for example $\ell_1 = N^{-2/3+\kappa}$ and $\ell = N^{-2/5-\kappa}$ for $\kappa > 0$ small enough).

Let $\widetilde{H}$ be the modified Hamiltonian given in (2.9). Let $\psi_{N,t}$ be a normalized solution of the N-body Schrödinger equation

$$i\partial_t \psi_{N,t} = \widetilde{H} \psi_{N,t} \, . \tag{2.18}$$

and let $\gamma^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k)$ be its k-particle marginal densities defined in (1.9). These density matrices satisfy the normalization

$$\text{Tr } \gamma^{(k)}_{N,t} = 1 \tag{2.19}$$

for all N, k and time t. The main result of this paper is the following theorem.

Theorem 2.1. *Assume the unscaled pair potential V is positive, spherically symmetric, smooth and compactly supported and assume that $\varrho = (8\pi)^{-1}(\|V\|_1 + \|V\|_\infty)$ is sufficiently small (of order one). Let $T > 0$ be fixed. Suppose that the normalized initial wave function $\psi_{N,0}$ satisfies the energy bounds*

$$\langle \psi_{N,0}, \widetilde{H} \psi_{N,0} \rangle \leq C_1 N \quad \textit{and} \quad \langle \psi_{N,0}, \widetilde{H}^2 \psi_{N,0} \rangle \leq C_2 N^2. \tag{2.20}$$

Then the sequence $\gamma^{(k)}_{N,t}$ has at least one non-trivial limit point and any limit point satisfies the infinite Gross-Pitaevskii Hierarchy (1.24) in a weak sense with $\sigma = 8\pi a_0$. More precisely:

i) For sufficiently large ν, the sequence $\Gamma_{N,t} = \{\gamma^{(k)}_{N,t}\}_{k\geq 1} \subset C([0,T], B^{(\nu)}_-)$ is compact w.r.t. the topology induced by the $\widehat{\rho}$ metric.

ii) Let $\Gamma_{\infty,t} = \{\gamma^{(k)}_{\infty,t}\}_{k\geq 1} \in C([0,T], B^{(\nu)}_-)$ be any limit point of $\Gamma_{N,t}$ in the $\widehat{\rho}$ metric. Then $\Gamma_{\infty,t}$ satisfies

$$\mathit{Tr}\, (1-\Delta_i)(1-\Delta_j)\gamma^{(k)}_{\infty,t} < C^k \tag{2.21}$$

for all $i \neq j$, $i,j = 1, \dots, k$, for all $t \in [0,T]$, and all $k \geq 1$.

iii) Any limit point $\Gamma_{\infty,t}$ of the sequence $\Gamma_{N,t}$ is non-trivial. In particular

$$\mathit{Tr}\, \gamma^{(1)}_{\infty,t} = 1 \qquad \textit{and} \qquad \mathit{Tr}\, \gamma^{(2)}_{\infty,t} = 1\, . \tag{2.22}$$

iv) Let $\Gamma_{\infty,t}$ be a limit point of $\Gamma_{N,t}$ and assume $h_r(x) = (3/4\pi)\, r^{-3} h(x/r)$ for any $h \in C_0^\infty(\Lambda)$ with $\int_\Lambda h = 1$. Then, for any $k \geq 1$ and $t \in [0,T]$, the limit

$$\begin{aligned} \lim_{r,r'\to 0} \int \mathrm{d}x'_{k+1} \mathrm{d}x_{k+1}\, h_r(x'_{k+1} - x_{k+1}) h_{r'}(x_{k+1} - x_j) \gamma^{(k)}_{\infty,t}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x'_{k+1}) \\ =: \gamma^{(k)}_{\infty,t}(\mathbf{x}_k, x_j; \mathbf{x}'_k, x_j) \end{aligned} \tag{2.23}$$

exists in the weak $W^{-1,1}(\Lambda^k \times \Lambda^k)$-sense and defines $\gamma^{(k)}(\mathbf{x}_k, x_j; \mathbf{x}'_k, x_j)$ as a distribution of $2k$ variables.

v) For any $k \geq 1$ and any regular test function $J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)$ in the Sobolev space $W^{2,\infty}(\Lambda^k \times \Lambda^k)$, we have, for any $t \in [0,T]$,

$$\begin{aligned}
\int \mathrm{d}\mathbf{x_k}\mathrm{d}\mathbf{x}'_k J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\gamma^{(k)}_{\infty,t}(\mathbf{x}_k;\mathbf{x}'_k) = &\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\gamma^{(k)}_{\infty,0}(\mathbf{x}_k;\mathbf{x}'_k) \\
&-i\sum_{j=1}^{k}\int_0^t \mathrm{d}s \int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\,(-\Delta_j+\Delta'_j)\gamma^{(k)}_{\infty,s}(\mathbf{x}_k;\mathbf{x}'_k) \\
&-8\pi i a_0\sum_{j=1}^{k}\int_0^t \mathrm{d}s \int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}x_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) \\
&\qquad\times\left(\delta(x_j-x_{k+1})-\delta(x'_j-x_{k+1})\right)\gamma^{(k+1)}_{\infty,s}(\mathbf{x}_k,x_{k+1};\mathbf{x}'_k,x_{k+1})\,.
\end{aligned} \tag{2.24}$$

Remark 2.2. *The main assumption* (2.20) *is satisfied for $\psi_{N,0} = W\phi_N$ with ϕ_N sufficiently regular. See Lemma D.1 in the Appendix for a proof.*

For simplicity, we state the theorem for initial conditions that are pure states. The same result holds for initial conditions $\gamma_{N,0} \in \mathcal{L}^1(\mathcal{H}^{\otimes_s N})$, Tr $\gamma_{N,0} = 1$, with the bounds

$$\mathrm{Tr}\ \widetilde{H}\gamma_{N,0} \leq C_1 N \quad \text{and} \quad \mathrm{Tr}\ \widetilde{H}^2\gamma_{N,0} \leq C_2 N^2.$$

As pointed out in the introduction, the factorization of the limit point $\gamma^{(k)}_{\infty,t}$ depends on the uniqueness of the solution to the infinite GP hierarchy (2.24), which is an open problem.

A similar proof holds if we replace the Hamiltonian $\widetilde{H}$ (2.9) with $\widetilde{H}^{(n)}$ (2.10) for any fixed $n \geq 3$, if ϱ is sufficiently small, depending on n.

The structure of the rest of the paper is the following. In Section 3 we explain some of the main ideas behind the proof of the main theorem. In Section 4 we define an approximation $U_{N,t}$ of $\Gamma_{N,t}$. In Section 6 we prove energy estimates for $\widetilde{H}$ and $\widetilde{H}^2$, which rely on some Hardy type and Sobolev type inequalities, which are stated and proved in Section 5. The energy estimates are turned into a-priori estimates for the approximation $U_{N,t}$ and for any limit point $U_{\infty,t}$ of $U_{N,t}$ in Section 7. In Section 8 we show how these a-priori estimates can be used to regularize the delta-function appearing in the limiting GP-Hierarchy. Finally, in Section 9, we prove Theorem 2.1. Some technical estimates are collected in Section 10 and in the Appendices.

3 Idea of the Proof

We write the solution of the Schrödinger equation in the form $\psi_{N,t} = W\phi_{N,t}$, where W, defined by (2.7), is an approximation of the wave function of the ground state of the N boson system. We shall implement a general idea that consists of two steps. A similar idea in a quite different context lies behind the relative entropy method [29].

(i) Construct an approximate solution (in our case, W) to the dynamics and factor out this approximation from the full solution ψ.

(ii) Derive an effective inequality governing the remaining part ϕ. In general, one can prove a stronger estimate on ϕ than on ψ. This estimate often involves Dirichlet form with respect to the approximate solution.

In our case, we obtain the following key estimate on $\phi_{N,t}$ (Corollary 6.2):

$$\int W^2 |\nabla_i \nabla_j \phi_{N,t}|^2 \leq C \tag{3.1}$$

for any fixed $i \neq j$. We define $U_{N,t}^{(k)}$ to be roughly the k-particle density matrix corresponding to the wave function $\phi_{N,t}$ (the precise definition of $U_{N,t}^{(k)}$ will be given in Section 4). We are going to use $U_{N,t}^{(k)}$ as an approximation of the k-particle density $\gamma_{N,t}^{(k)}$ (they will coincide in the weak $N \to \infty$ limit). Then (3.1) implies that

$$\mathrm{Tr}\,(1-\Delta_i)(1-\Delta_j) U_{N,t}^{(k)} \leq C^k\,. \tag{3.2}$$

Let now $U_{\infty,t}^{(k)}$ be a weak limit point of $U_{N,t}^{(k)}$ which also satisfies (3.2). Then, using this bound, we can show, by Proposition 8.1, that $U_{\infty,t}^{(k)}(\mathbf{x}_k, x_j; \mathbf{x}'_k, x_j)$ is well-defined. Moreover we will show (Lemma 9.5) that the limit points of $\gamma_{N,t}^{(k)}$ and of $U_{N,t}^{(k)}$ coincide, i.e. that $\gamma_{\infty,t}^{(k)} = U_{\infty,t}^{(k)}$. Therefore (2.23) holds true, and the right hand side of (2.24) is well-defined. Note that (2.23) and (2.24) cannot be proven directly, since we do not have estimates for $\mathrm{Tr}\,(1-\Delta_i)(1-\Delta_j)\gamma_{N,t}^{(k)}$; this is why we need to introduce the auxiliary densities $U_{N,t}^{(k)}$.

A more refined calculation is needed to obtain $\sigma = 8\pi a_0$. The idea of this calculation was explained in the introduction; it can be made rigorous (Section 9) via the key estimate (3.1) and certain generalizations of the Hardy and Poincaré inequalities (Section 5). To explain (3.1) in more details, we first review the related mean field models.

3.1 The mean-field model

The mean field Hamiltonian is defined by

$$H_N = -\sum_{j=1}^N \Delta_j + N^{-1} \sum_{j<k}^N V^{(\tau)}(x_j - x_k) \tag{3.3}$$

where we have introduced a scale parameter τ and $V^{(\tau)}(x) = \tau^{-3} V(x/\tau)$. If we fix τ and take the limit $N \to \infty$ (the mean field limit), then with a product initial wave function Hepp [11] and Spohn [27] proved that the one particle density matrix of the solution of the Schrödinger equation (1.8) converges to the Hartree equation

$$i\partial_t u_t = -\Delta u_t + (V^{(\tau)} * |u_t|^2) u_t\,. \tag{3.4}$$

If we take $\tau \to 0$ in this equation, we recover the GP equation (1.1) but with the so-called mean-field interaction constant $\sigma = b_0 = \int V$ instead of $\sigma = 8\pi a_0$ involving a_0, the scattering length of V. To investigate the simultaneous limit $\tau \to 0$, $N \to \infty$ in the many body problem, we set $\tau = N^{-\beta}$. If $0 < \beta < 1$, then the scattering length of the interaction potential, $N^{-1}V^{(\tau)}$, is much smaller than the range of the potential. So we are in the mean-field regime and the one particle density matrix of the solution of the Schrödinger equation (1.8) is expected to converge to the GP equation with $\sigma = b_0$. If $\beta = 1$, then the Hamiltonian in (3.3) recovers the Hamiltonian in the GP scaling (1.6) and the limit is expected to be the GP equation with $\sigma = 8\pi a_0$.

In a joint work with A. Elgart we prove that, *for product initial data and* $0 < \beta < 2/3$, *the density matrices of the Schrödinger equation* (1.8) *converge to a solution of the GP hierarchy* (1.24) *with* $\sigma = b_0$. In [5], we also prove that the GP hierarchy has a unique solution in an appropriate Sobolev space. Moreover, we show in [5] that for $0 < \beta < 1/2$ any limit of the solution to the Schrödinger equation with product initial data satisfies the a-priori Sobolev bound. Hence, in this case, we prove the uniqueness as well, and therefore we complete the derivation of the GP equation with $\sigma = b_0$ in a mean-field scaling. The same result is expected to hold also for $1/2 \le \beta < 1$, but the a-priori bound in this regime is open.

The outline of the proof in [4] is similar to this paper's. The first step is to prove the estimate

$$\sum_{i,j} \int |\nabla_i \nabla_j \psi_{N,t}|^2 \le CN^2 . \tag{3.5}$$

The proof of the convergence to the hierarchy makes use of arguments similar to the ones used in Section 8 and Section 9. Since the proof of the estimate (3.5) is instructive, we reproduce it here.

Proposition 3.1. *Suppose* $\psi(\mathbf{x})$ *is a smooth normalized function. Then the following inequality holds*

$$\begin{aligned}(\psi, H_N^2 \psi) \ge C_1 \left(1 - \frac{1}{N\tau^{3/2}}\right) \sum_{j,\ell=1}^N \int \mathrm{d}\mathbf{x} |\nabla_j \nabla_\ell \psi(\mathbf{x})|^2 \\ - \frac{C_2 N}{N\tau^{3/2}} \sum_{j=1}^N \int \mathrm{d}\mathbf{x} |\nabla_j \psi(\mathbf{x})|^2 - \frac{C_3 N^2}{N\tau^{3/2}} \int \mathrm{d}\mathbf{x} |\psi(\mathbf{x})|^2 .\end{aligned} \tag{3.6}$$

In particular, for $\tau = N^{-\beta}$ *with* $0 \le \beta < 2/3$, *we have*

$$\sum_{j,\ell=1}^N \int \mathrm{d}\mathbf{x} |\nabla_j \nabla_\ell \psi(\mathbf{x})|^2 \le C(\psi, H_N^2 \psi) + CN^2 . \tag{3.7}$$

Proof. By expanding the square of the energy operator and dropping some positive diagonal terms, we have

$$\begin{aligned}\|H_N \psi\|^2 \ge & \sum_{j,\ell} \int \mathrm{d}\mathbf{x} \Delta_j \bar{\psi}(\mathbf{x}) \Delta_\ell \psi(\mathbf{x}) \\ & + \frac{1}{N} \sum_{j=1}^N \sum_{\ell \ne m} \int \mathrm{d}\mathbf{x} (-\Delta_j \bar{\psi}(\mathbf{x})) V^{(\tau)}(x_\ell - x_m) \psi(\mathbf{x}) \\ & + \frac{1}{N} \sum_{j=1}^N \sum_{\ell \ne m} \int \mathrm{d}\mathbf{x} (-\Delta_j \psi(\mathbf{x})) V^{(\tau)}(x_\ell - x_m) \bar{\psi}(\mathbf{x}) .\end{aligned} \tag{3.8}$$

From integration by parts, the second term on the right hand side of the previous equation equals

$$\begin{aligned}\frac{1}{N} \sum_{j=1}^N \sum_{\ell \ne m} \int \mathrm{d}\mathbf{x} (-\Delta_j \bar{\psi}(\mathbf{x})) V^{(\tau)}(x_\ell - x_m) \psi(\mathbf{x}) = & \frac{1}{N} \sum_{j=1}^N \sum_{\ell \ne m} \int \mathrm{d}\mathbf{x} |\nabla_j \psi(\mathbf{x}))|^2 V^{(\tau)}(x_\ell - x_m) \\ & + \frac{2}{N} \sum_{j \ne \ell}^N \int \mathrm{d}\mathbf{x} \big(\nabla_j \bar{\psi}\big)(\mathbf{x}) \psi(\mathbf{x}) \nabla_j V^{(\tau)}(x_j - x_\ell) .\end{aligned}$$

From the Schwarz inequality, the last term is bounded by

$$\begin{aligned}\int \mathrm{d}\mathbf{x} \nabla_j \bar{\psi}(\mathbf{x}) \psi(\mathbf{x}) \nabla_j V^{(\tau)}(x_j - x_\ell) \geq & -\frac{1}{\tau^{3/2}} \int \mathrm{d}\mathbf{x}\, |\nabla_j \psi(\mathbf{x})|^2 \frac{1}{\tau^2} \Big| \nabla V\left(\tfrac{x_j - x_\ell}{\tau}\right) \Big| \\ & - \frac{1}{\tau^{3/2}} \int \mathrm{d}\mathbf{x} |\psi(\mathbf{x})|^2 \, \frac{1}{\tau^3} \Big| \nabla V\left(\tfrac{x_j - x_\ell}{\tau}\right) \Big| \, .\end{aligned}$$

By Lemma 5.2, the last two terms are bounded below by

$$-\frac{C_1}{\tau^{3/2}} \sum_{j,\ell=1}^{N} \int \mathrm{d}\mathbf{x} \left(|\nabla_\ell \nabla_j \psi(\mathbf{x})|^2 + |\nabla_j \psi(\mathbf{x})|^2 + |\psi(\mathbf{x})|^2 \right) \, .$$

The last term in (3.8) can be bounded in a similar way. Combining these equations, we have proved Proposition 3.1. □

To understand the restriction $\tau \gg N^{-2/3}$, consider a single particle in the potential $N^{-1} V^{(\tau)}$. Define the operator $\mathfrak{h}$ in $\mathbb{R}^3$ by

$$\mathfrak{h} := -\Delta_x + N^{-1} V^{(\tau)}(x) \, .$$

Clearly, $\langle f, \mathfrak{h}^2 f \rangle$ is given by

$$\langle f, \mathfrak{h}^2 f \rangle = \Big\langle \, f, \, \Big\{ \Delta\Delta - \Delta N^{-1} V^{(\tau)} - N^{-1} V^{(\tau)} \Delta + N^{-2} (V^{(\tau)})^2 \Big\} \, f \, \Big\rangle . \tag{3.9}$$

Notice that for a general smooth function f living on scale one, the last term $\langle f, N^{-2}(V^{(\tau)})^2 f \rangle$ is of order $N^{-2}\tau^{-3}$. This term diverges in the large N limit if $\tau \ll N^{-2/3}$. The middle terms, on the other hand, are bounded by CN^{-1}. Hence for smooth functions living on scale one, $\langle f, \mathfrak{h}^2 f \rangle$ typically diverges. Therefore, functions with $\langle f, \mathfrak{h}^2 f \rangle$ finite (in particular, the eigenstates of $\mathfrak{h}$) should have short scale structure. For such functions, the divergent parts of $N^{-2}(V^{(\tau)})^2$ (and that of $\Delta\Delta$) are cancelled by the middle terms $\Delta N^{-1} V^{(\tau)} + N^{-1} V^{(\tau)} \Delta$. This cancellation can be understood using the following idea of "resolution of singularities" with the help of an approximate solution W. We explain the method in the setting of our modified Hamiltonian $\widetilde{H}$.

3.2 Resolution of singularities

In this section we explain the idea behind the estimate (see (1.23))

$$\sum_{i,j} \int W^2 |\nabla_i \nabla_j \phi|^2 \leq C \langle W\phi, (\widetilde{H}^2 + N^2) W\phi \rangle = C \langle \psi, (\widetilde{H}^2 + N^2) \psi \rangle \, . \tag{3.10}$$

Since the right side is a constant of motion, this implies the a-priori bound (3.1). Recall that $\widetilde{H}$ contains the singular potential $N^2 V(Nx)$. The difficulty of the singular potential was resolved in the work of Lieb-Yngvason [24] for the ground state energy problem. The basic idea was to replace the singular potential by some flattened out potential and to use the variational principle and the positivity of the potential. Our aim, instead, is to estimate the Sobolev norm of ϕ in terms of $\widetilde{H}^2$. Therefore, there is no variational principle and the usage of positivity of the potential is limited. Our strategy is to factor out the approximate ground state W. The resulting effective Hamiltonian contains a flattened interaction potential, similar to the one in [24]. In this way, we also maintain the almost perfect cancellation between the potential and kinetic energy operators, a critical property explained in the previous section.

The action of the Hamiltonian $\widetilde{H}$ on a product function $\psi = W\phi$ can be written in a more convenient form. Let B be the function

$$B := W^{-1}\widetilde{H}W\,. \tag{3.11}$$

Define the operator

$$L := \sum_k \Big(-\Delta_k - 2\nabla_k(\log W)\nabla_k\Big)\,. \tag{3.12}$$

The operator L is self-adjoint with respect to W^2, since

$$\int W^2\bar{\phi}_1(L\phi_2) = \sum_{k=1}^N \int W^2(\nabla_k\bar{\phi}_1)(\nabla_k\phi_2) = \int W^2(L\bar{\phi}_1)\phi_2\,.$$

Then we have

$$W^{-1}\widetilde{H}\,(W\phi) = L\phi + B\phi, \qquad \int \bar{\psi}\widetilde{H}\psi = \sum_k \int W^2|\nabla_k\phi|^2 + \int W^2 B|\phi|^2\,,$$

and the following identity

$$\int |\widetilde{H}W\phi|^2 = \int W^2|L\phi|^2 + \int W^2 B\bar{\phi}L\phi + \int W^2 B\phi L\bar{\phi} + \int |\phi|^2(\widetilde{H}W)^2\,.$$

Last equation can be rewritten in the following more convenient form:

$$\begin{aligned}\int |\widetilde{H}W\phi|^2 = &\sum_{i,j}\int W^2|\nabla_i\nabla_j\phi|^2 - 2\sum_{i,j}\int W^2(\nabla_i\nabla_j\log W)\,\nabla_i\bar{\phi}\nabla_j\phi \\ &+2\sum_m \int W^2 B|\nabla_m\phi|^2 + \int W^2(\nabla_m B)\left(\bar{\phi}\nabla_m\phi + \phi\nabla_m\bar{\phi}\right) + \int B^2|\phi|^2W^2.\end{aligned} \tag{3.13}$$

For more details see (6.15)-(6.17). If W is the true ground state of $\widetilde{H}$, then B is a positive constant and the middle term in the second line of the last equation vanishes. From the Schwarz inequality, we have

$$\int |\widetilde{H}W\phi|^2 \geq \sum_{i,j}\int W^2|\nabla_i\nabla_j\phi|^2 - 2\sum_{i,j}\int W^2|\nabla_i\nabla_j\log W|\,|\nabla_i\phi|^2. \tag{3.14}$$

The singularities of the ground state wave function are of the form $1-Ca/|x_i-x_j|$ for some constant C and for $r \leq |x_i - x_j| \ll 1$, where r is the radius of the support of V_a (r is of order a). The second derivative of the ground state wave function is bounded by $Ca/|x_i - x_j|^3 \leq C\varrho/|x_i - x_j|^2$ in this regime, since $a \leq C\int V_a \leq Ca^{-2}\|V\|_\infty r^3 \leq Cr\varrho$. Recall that $\varrho = (8\pi)^{-1}(\|V\|_1 + \|V\|_\infty)$. The same estimate also holds for $|x_i - x_j| \leq r$ (see Appendix A). Thus $|\nabla_i\nabla_j\log W| \leq C\varrho/|x_i-x_j|^2$. Applying the Hardy inequality, the last term can be bounded by

$$\int W^2|\nabla_i\nabla_j\log W|\,|\nabla_i\phi|^2 \leq C\varrho\sum_{i,j}\int W^2|\nabla_i\nabla_j\phi|^2\,.$$

Assuming that ϱ is small enough (of order one), together with (3.14) we get the key estimate (3.10).

In practice, we do not know the exact ground state function and its approximation will be used. When x_i is near x_j, the approximate ground state behaves like the ground state of the Neumann boundary problem given in (2.4). Thus the singularity of B for $|x_i - x_j| \ll 1$ behaves like $q(x_i - x_j)$ where q, defined in (2.5), lives on scale $\ell_1 \gg 1/N$. This procedure, roughly speaking, replaces the singular potential V_a by an effective potential living on a bigger scale ℓ_1. The price to pay is the two middle terms in (3.13). Since the singularities of W and q are milder than that of V_a, this procedure in a sense resolves the singularity of V_a. This is the key idea behind the proof of the estimate (3.1).

4 Construction of the approximate solution

Recall the definition of the functions G_i and W,

$$G_i(x) := 1 - \sum_{j \neq i} w(x_i - x_j) F_{ij}(x) \; , \quad \text{and} \quad W := \prod_{i=1}^{N} \sqrt{G_i} \; . \tag{4.1}$$

In the next lemma we prove that G_i is separated away from zero.

Lemma 4.1. *There is a positive constant c_1 depending only on V such that for sufficiently large N*

$$c_1 \leq G_i \leq 1 \, . \tag{4.2}$$

Proof. This can easily be seen because, for fixed i, the sum $\sum_{j \neq i} w_{ij} F_{ij}$ equals zero, if $|x_i - x_j| > \ell$ for all $j \neq i$, while, if $|x_i - x_m| \leq \ell$ for some $m \neq i$, it is bounded by

$$\chi(|x_i - x_m| \leq \ell) \sum_{j \neq i} w_{ij} F_{ij} \leq w_{im} F_{im} + N e^{-c\ell^{-\varepsilon}} \leq c_1 + N e^{-c\ell^{-\varepsilon}}$$

where we applied Lemma B.1. So if N is large enough we obtain (4.2) with some $c_1 \in (c_0, 1)$. $\square$

We will also need versions of G_i and W which are independent of some of the variables $x_1, \dots x_N$. We define, for $i, m \neq k_1, \dots, k_\alpha$ fixed, a version of F_{im} independent of $x_{k_1}, \cdots, x_{k_\alpha}$:

$$F_{im}^{(k_1 \dots k_\alpha)} := F(\mathcal{N}_{im}^{(k_1 \dots k_\alpha)}), \qquad \mathcal{N}_{im}^{(k_1 \dots k_\alpha)} := \sum_{u \neq i, m, k_1, \dots, k_\alpha} [h_{iu} + h_{mu}] \, .$$

Moreover, for $i \neq k_1, \dots, k_\alpha$, we define

$$G_i^{(k_1 \dots k_\alpha)} := \left(1 - \sum_{m \neq i, k_1, \dots, k_\alpha} w_{im} F_{im}^{(k_1 \dots k_\alpha)} \right)$$

and

$$W^{(k_1 \dots k_\alpha)} := \prod_{i \neq k_1, \dots, k_\alpha} \sqrt{G_i^{(k_1 \dots k_\alpha)}} \, . \tag{4.3}$$

Thus $W^{(k_1 \dots k_\alpha)}$ is independent of $x_{k_1}, \dots, x_{k_\alpha}$. When $\alpha = 1$, we shall drop the label $\alpha = 1$ and denote the function by $W^{(k)}$. Notice that we have

$$G_i^{(k)} = G_i + w_{ik} F_{ik} - \sum_{m \neq k, i} w_{im} (F_{im}^{(k)} - F_{im}) \, .$$

We shall also use the notations $W^{[k]} := W^{(1 \dots k)}$, $F_{im}^{[k]} := F_{im}^{(1 \dots k)}$ for any $i, m > k$, and $G_i^{[k]} := G_i^{(1 \dots k)}$, for $i > k$. In Appendix C we shall prove the existence of a constant C, such that, for any choice of the indices $k_1, ..k_\alpha$,

$$C^{-\alpha} \leq W^{(k_1 .. k_\alpha)} / W \leq C^{\alpha} \, .$$

Since $W \neq 0$ (by Lemma 4.1), we can write the solution of the Schrödinger equation (2.18) as

$$\psi_{N,t} = W \phi_{N,t}. \tag{4.4}$$

We define a modified k-particle marginal distributions by

$$U^{(k)}_{N,t}(x_1,\dots,x_k;x'_1,\dots,x'_k) := \int \mathrm{d}x_{k+1}\dots\mathrm{d}x_N (W^{[k]}(x_{k+1},\dots,x_N))^2 \\ \times\phi_{N,t}(x_1,,\dots,x_k,x_{k+1},\dots,x_N)\overline{\phi}_{N,t}(x'_1,\dots,x'_k,x_{k+1},\dots,x_N)\ . \tag{4.5}$$

These density matrices will satisfy the requirements of the Main Theorem 2.1.

Clearly $U^{(k)}_{N,t}\geq 0$ as an operator on $L^2(\Lambda,\mathrm{d}x)^{\otimes_s k}$. Furthermore, by (C.2), we have

$$\begin{aligned}\mathrm{Tr}\, U^{(k)}_{N,t} &= \int \mathrm{d}x_1\dots\mathrm{d}x_N(W^{[k]}(x_{k+1},\dots,x_N))^2|\phi_{N,t}(x_1,\dots x_k,x_{k+1}..x_N)|^2\\ &\leq C_0^k\int \mathrm{d}x_1\dots\mathrm{d}x_N\, W(x)^2|\phi(x)|^2\leq C^k\int \mathrm{d}x|\psi_{N,t}(x)|^2 = C_0^k\ .\end{aligned}$$

It is instructive to compare the approximate densities $U^{(k)}_{N,t}$ with the true marginal densities $\gamma^{(k)}_{N,t}$, which are given by

$$\gamma^{(k)}_{N,t}(x_1,\dots,x_k;x'_1,\dots,x'_k) := \int \mathrm{d}x_{k+1}\dots\mathrm{d}x_N\ W(x_1,\dots,x_N)W(x'_1,\dots,x'_N) \\ \times\ \phi_{N,t}(x_1,,\dots,x_k,x_{k+1},\dots,x_N)\overline{\phi}_{N,t}(x'_1,\dots,x'_k,x_{k+1},\dots,x_N)\ .$$

A simple computation shows that $\gamma^{(2)}_{N,t}$ and $U^{(2)}_{N,t}$ satisfy the relation (1.19).

5 Hardy type and Sobolev type inequalities

We need the following finite volume version of the usual Hardy inequality

$$\int_{\mathbb{R}^3}\frac{|f|^2}{|x|^2}\leq C\int_{\mathbb{R}^3}|\nabla f|^2\ . \tag{5.1}$$

Lemma 5.1 (Hardy Type Inequality). *Let $0\leq U(x)\leq c|x|^{-2}$ on $B:=\{|x|\leq \ell\}\subset\mathbb{R}^3$, and let*

$$\langle f\rangle := |B|^{-1}\int_B f$$

be the average of f on B ($|B|=(4/3)\pi\ell^3$ is the volume of B). Then

$$\langle U|f|^2\rangle\leq C\langle|\nabla f|^2\rangle + C\langle U\rangle\langle|f|^2\rangle\ . \tag{5.2}$$

Proof. Let $\bar f := \langle f\rangle$, then

$$\langle U\ |f|^2\rangle \leq 2\langle U\ |f-\bar f|^2\rangle + 2\bar f^2\langle U\rangle\ .$$

By the usual Hardy inequality

$$\langle U|f-\bar f|^2\rangle\leq C\langle|\nabla f|^2\rangle + C\ell^{-2}\langle|f-\bar f|^2\rangle\ .$$

By the Neumann spectral gap we have

$$\ell^{-2}\langle|f-\bar f|^2\rangle\leq C\langle|\nabla f|^2\rangle\,.$$

The lemma now follows from

$$|\bar f|^2 = |\langle f\rangle|^2\leq\langle|f|^2\rangle.$$

□

We also need some well-known inequalities, collected in the following lemma.

Lemma 5.2 (Sobolev Type Inequalities). *The following two inequalities are standard.*

i) Suppose $U \in L^{3/2}(\Lambda)$, $U \geq 0$, and $\psi \in W^{1,2}(\Lambda)$. Then

$$\int_\Lambda |\psi|^2 U \leq C\|U\|_{L^{3/2}(\Lambda)} \int_\Lambda \Big[|\nabla\psi|^2 + |\psi|^2\Big] . \tag{5.3}$$

ii) Suppose $U \in L^1(\Lambda)$. Then, considering $U(x-y)$ as an operator on the Hilbert space $L^2(\Lambda, \mathrm{d}x) \otimes L^2(\Lambda, \mathrm{d}y)$ we have the operator inequality

$$U(x-y) \leq C\|U\|_{L^1} (1-\Delta_x)(1-\Delta_y) . \tag{5.4}$$

Proof. i) Applying the Hölder inequality we find

$$\int U\,|\psi|^2 \leq \Big(\int_\Lambda U^{3/2}\Big)^{2/3} \Big(\int_\Lambda |\psi|^6\Big)^{1/3} .$$

By the Sobolev inequality we have

$$\Big(\int_\Lambda |\psi|^6\Big)^{1/3} \leq C\int_\Lambda \Big[|\nabla\psi|^2 + |\psi|^2\Big] ,$$

which implies part i). The proof of part ii) can be found in [6]. □

Let

$$\lambda(x) := \frac{\chi\{|x| \leq 3\ell_1/2\}}{|x|^2} \quad \text{and} \quad \sigma(x) := \frac{\chi\{|x| \leq 3\ell_1/2\}}{|x|^3 + a^3} \qquad x \in \Lambda , \tag{5.5}$$

and, as usual, $\lambda_{ij} := \lambda(x_i - x_j)$, $\sigma_{ij} := \sigma(x_i - x_j)$. We note the following inequalities, whose proof is given in Appendix A

$$\begin{gathered} w \leq \widetilde{\chi}, \qquad w \leq Ca\ell_1\lambda, \qquad |\nabla w| \leq Ca^{-1}, \qquad |\nabla w| \leq Ca\lambda, \\ |\nabla^2 w| \leq C\varrho\lambda, \qquad |\nabla^2 w| \leq Ca\sigma, \qquad |\nabla w|^2 \leq Ca\sigma, \qquad w^2 \leq Ca^2\lambda, \end{gathered} \tag{5.6}$$

where $\varrho = (8\pi)^{-1}(\|V\|_\infty + \|V\|_1)$.

In the sequel we will use the convention that integrals without specified measure and domain will always refer to Lebesgue integration on Λ^N:

$$\int \Big(\dots\Big) := \int_{\Lambda^N} \Big(\dots\Big)\mathrm{d}\mathbf{x} .$$

We will use the Hardy-type and the Sobolev type inequalities in the following form:

Lemma 5.3. *Let $\phi(\mathbf{x}) \in L^2(\Lambda^N)$. For any $q > 0$ and for fixed k, j indices and fixed $x_1, .., \widehat{x}_k, .., x_N$ we have*

$$\int \mathrm{d}x_k W^2 F_{kj}^q \lambda_{kj}|\phi|^2 \leq C_q \int \mathrm{d}x_k W^2 F_{kj}^{q/4}\widetilde{\chi}_{kj}|\nabla_k\phi|^2 + C_q\ell_1\ell^{-3} \int \mathrm{d}x_k W^2 F_{kj}^{q/4}\widetilde{\chi}_{kj}|\phi|^2 \tag{5.7}$$

for sufficiently large N. In particular

$$\sum_{kj} \int W^2 F_{kj}^q \lambda_{kj}|\phi|^2 \leq C_q \int W^2 \Big[\sum_k |\nabla_k\phi|^2 + N\ell_1\ell^{-3} \int |\phi|^2\Big] \tag{5.8}$$

by (B.2). Moreover we have

$$\sum_{kj}\int W^2\sigma_{kj}|\phi|^2 \le C(\log N)\int W^2\left(\sum_{jk}|\nabla_j\nabla_k\phi|^2 + N\sum_j|\nabla_j\phi|^2 + N^2|\phi|^2\right). \tag{5.9}$$

The same estimates hold if we set $W\equiv 1$ in all integrals.

Proof. We define

$$F_j^{(k)} := e^{-\ell^{-\varepsilon}\sum_{m\neq k,j}h(x_j-x_m)}.$$

Note that $F_j^{(k)}$ is independent of the variable x_k, and that, trivially, $F_{kj}\le F_j^{(k)}$. Moreover we have

$$\begin{aligned}\widetilde{\chi}_{kj}\,F_{kj}^q &= \widetilde{\chi}_{kj}\,e^{-q\ell^{-\varepsilon}\sum_{m\neq k,j}(h(x_j-x_m)+h(x_k-x_m))}\\ &\ge \widetilde{\chi}_{kj}\,e^{-q\ell^{-\varepsilon}\sum_{m\neq k,j}h(x_j-x_m)\left(1+e^{|x_j-x_k|/\ell}\right)}\\ &\ge \widetilde{\chi}_{kj}\,[F_j^{(k)}]^{4q},\end{aligned}\tag{5.10}$$

because $1+e\le 4$. Here we used that $((x_k-x_m)^2+\ell^2)^{1/2}\le((x_j-x_m)^2+\ell^2)^{1/2}+|x_j-x_k|$. Using (C.1) to replace W^2 by $[W^{(k)}]^2$, which is independent of x_k, applying (5.2) for x_k, and finally changing the measure back, we find:

$$\begin{aligned}\int W^2F_{kj}^q\lambda_{kj}|\phi|^2 &\le C_0^2\int[W^{(k)}]^2(F_j^{(k)})^q\lambda_{kj}|\phi|^2\\ &\le C\int \mathrm{d}x_1\ldots\widehat{\mathrm{d}x_k}\ldots\mathrm{d}x_N\,[W^{(k)}]^2(F_j^{(k)})^q\int\mathrm{d}x_k\Big[|\nabla_k\phi|^2+\ell_1\ell^{-3}|\phi|^2\Big]\widetilde{\chi}_{kj}\\ &\le C\int W^2(F_j^{(k)})^q\,\widetilde{\chi}_{kj}\Big[|\nabla_k\phi|^2+\ell_1\ell^{-3}|\phi|^2\Big].\end{aligned}$$

The estimate (5.7) follows now from (5.10). Similarly the estimate (5.9) follows from (5.4), after replacing W by $W^{(k,j)}$. It is clear that setting $W\equiv 1$ would not alter the validity of the proof. □

6 Energy Estimate

We begin by computing the action of the Hamiltonian H defined in (1.6) on the wave function W, defined in Section 2.3. For any fixed index k we have

$$W^{-1}(-\Delta_k)W = \frac{-\Delta_k G_j}{2G_j} - 1(i\neq j)\frac{1}{4}\frac{\nabla_k G_i}{G_i}\frac{\nabla_k G_j}{G_j} + \frac{1}{4}\left(\frac{\nabla_k G_j}{G_j}\right)^2.$$

Here, and in the rest of the paper, we use the summation convention: all unspecified indices are summed up; in this case the j and i indices on the right hand side.

Direct computation shows that for any fixed k

$$-\Delta_k G_k = (\Delta w)_{kj}F_{kj} + \Omega_k\,,$$

where

$$\Omega_k := 2(\nabla w)_{kj}\nabla_k F_{kj} + w_{kj}\Delta_k F_{kj} \tag{6.1}$$

and, if $k\neq j$ are fixed, then

$$-\Delta_k G_j = (\Delta w)_{kj}F_{kj} + \Omega_{kj}$$

with

$$\Omega_{kj} := 2(\nabla w)_{kj}\nabla_k F_{kj} + w_{ij}\Delta_k F_{ij} .$$

From (2.4) it follows that

$$\Delta w = -(1/2)V_a(1-w) + (1-w)q .$$

Then we have

$$W^{-1}HW = ((1/2)V_{kj} - q_{kj})[1-(1-w_{kj})M_{kj}] + q_{kj} + \Omega \tag{6.2}$$

with

$$\Omega := \frac{\Omega_k}{2G_k} + \frac{\Omega_{kj}}{2G_j} - \frac{1}{4}\frac{\nabla_k G_i}{G_i}\frac{\nabla_k G_j}{G_j}1(i\neq j) + \frac{1}{4}\left(\frac{\nabla_k G_i}{G_i}\right)^2 \tag{6.3}$$

and, for any k,j (see 2.8)

$$M_{kj} := \frac{F_{kj}}{2}(G_j^{-1} + G_k^{-1}) . \tag{6.4}$$

The action of the modified Hamiltonian (2.9) on the wave function W is then simply given by

$$W^{-1}\widetilde{H}W = q_{kj} + \Omega .$$

It follows that the action of $\widetilde{H}$ on the product $W\phi$ is given by

$$W^{-1}\widetilde{H}W\phi = L\phi + B\phi \tag{6.5}$$

where we defined the operator

$$L := \sum_k \Big(-\Delta_k - 2\nabla_k(\log W)\nabla_k \Big) , \tag{6.6}$$

and the function

$$B := W^{-1}\widetilde{H}W = q_{kj} + \Omega . \tag{6.7}$$

Note that the operator L has the self-adjointness property that (with the summation convention)

$$\int W^2\bar{\psi}(L\phi) = \int W^2(\nabla_k\bar{\psi})(\nabla_k\phi) = \int W^2(L\bar{\psi})\phi . \tag{6.8}$$

Next we compute the last two terms in Ω (see (6.3)). For fixed i,k we have

$$\nabla_k G_i = (\nabla w)_{ik}F_{ik} - (\nabla w)_{k\ell}F_{k\ell}\delta_{ki} - w_{i\ell}\nabla_k F_{i\ell} .$$

Hence we obtain

$$\begin{aligned}(\nabla_k G_i)^2 = (\nabla w)_{ik}^2 F_{ik}^2 + (\nabla w)_{k\ell}(\nabla w)_{ks}F_{k\ell}F_{ks}\delta_{ki} - 2(\nabla w)_{ik}w_{i\ell}F_{ik}\nabla_k F_{i\ell} \\ + 2(\nabla w)_{k\ell}w_{ks}F_{k\ell}\nabla_k F_{ks}\delta_{ki} + w_{i\ell}w_{is}\nabla_k F_{i\ell}\nabla_k F_{is} .\end{aligned}$$

Using Lemma B.1 we note that the second term on the r.h.s. is exponentially small unless $\ell = s$. Analogously the third term is exponentially small unless $k=\ell$, while the fourth and fifth terms are exponentially small unless $\ell = s$. Thus summing over i,k on the left and the right side of the equation, we obtain (changing the name of the indices in the second and fourth term)

$$\frac{(\nabla_k G_i)^2}{4G_i^2} = \frac{(\nabla w)_{ik}^2 F_{ik}^2}{2(1-w_{ik}F_{ik})^2} - \frac{(\nabla w)_{ik}w_{ik}F_{ik}(\nabla_k F_{ik} - \nabla_i F_{ik})}{2G_i^2} + \frac{w_{i\ell}^2(\nabla_k F_{i\ell})^2}{4G_i^2} + O\left(e^{-c\ell^{-\varepsilon}}\right) . \tag{6.9}$$

Here we also used that, on the support of $(\nabla w)_{ik}$, we have $G_i = 1 - w_{ik}F_{ik} + O(e^{-c/\ell^\varepsilon})$. Analogously, for $i \neq j$ we have

$$\frac{\nabla_k G_i \nabla_k G_j}{4G_i G_j} = \frac{(\nabla w)^2_{ik} F^2_{ik}}{2(1-w_{ik}F_{ik})^2} - \frac{(\nabla w)_{ik} w_{j\ell} F_{ik} (\nabla_k F_{j\ell} - \nabla_i F_{j\ell})}{2G_i G_j} + \frac{w_{i\ell} w_{js} \nabla_k F_{i\ell} \nabla_k F_{js}}{4G_i G_j} + O\left(e^{-c\ell^{-\varepsilon}}\right) \tag{6.10}$$

where we are summing over all k and all $i \neq j$ on the left and the right side of the equation. We see that there are some cancellations between (6.9) and (6.10). It follows that

$$\Omega = \widetilde{\Omega} + O\left(e^{-c\ell^{-\varepsilon}}\right) \,, \tag{6.11}$$

where

$$\widetilde{\Omega} = \frac{\Omega_k}{2G_k} + \frac{\Omega_{kj}}{2G_j} + \Gamma \tag{6.12}$$

and

$$\Gamma = s_{ij} \left(\frac{(\nabla w)_{ik} w_{j\ell} F_{ik} (\nabla_k F_{j\ell} - \nabla_i F_{j\ell})}{2G_i G_j} - \frac{w_{i\ell} w_{js} \nabla_k F_{i\ell} \nabla_k F_{js}}{4G_i G_j} \right) .$$

Here $s_{ij} = -1$ if $i = j$ and $s_{ij} = 1$ if $i \neq j$ (and sum over all indices is understood). A more careful analysis also shows that, analogously to (6.11), for any fixed m,

$$\nabla_m \Omega = \nabla_m \widetilde{\Omega} + O\left(e^{-c\ell^{-\varepsilon}}\right) \,. \tag{6.13}$$

This fact will be used in the proof of the next proposition, which is the key energy estimate.

Proposition 6.1. *Assume $a = a_0 N^{-1}$, $a \ll \ell_1 \ll \ell \ll 1$, $a\ell_1 \ll \ell^4$, $a \ll \ell_1^{3/2}$ and $a \ll \ell^2$ (for example $\ell_1 = N^{-2/3+\kappa}$ and $\ell = N^{-2/5-\kappa}$, for $\kappa > 0$ sufficiently small). Put $\varrho = (8\pi)^{-1}(\|V\|_\infty + \|V\|_1)$.*

i) Then we have

$$(W\phi, \widetilde{H} W\phi) = \int W\bar{\phi}(\widetilde{H} W\phi) \geq (1 - o(1)) \int W^2 \sum_k |\nabla_k \phi|^2 - o(N) \int W^2 |\phi|^2 \,,$$

for $N \to \infty$.

ii) Moreover, if ϱ is sufficiently small, there is $C > 0$ such that

$$\int |\tilde{H} W\phi|^2 \geq (C - o(1)) \int W^2 \sum_{ij} |\nabla_i \nabla_j \phi|^2 - o(1) \left(N \int W^2 \sum_i |\nabla_i \phi|^2 + N^2 \int W^2 |\phi|^2 \right)$$

as $N \to \infty$.

This Proposition and the conservation of $\widetilde{H}$ and $\widetilde{H}^2$ along the solution of the Schrödinger equation (2.18) immediately implies the following

Corollary 6.2. *Let $\psi_{N,t}(\mathbf{x}) = W(\mathbf{x})\phi_{N,t}(\mathbf{x})$ be a symmetric (with respect to permutations of the particles) wave function solving the Schrödinger equation (2.18), with initial data $\psi_{N,0}$ satisfying the bounds*

$$(\psi_{N,0}, \widetilde{H}\psi_{N,0}) \leq C_1 N \,, \quad \textit{and} \quad (\psi_{N,0}, \widetilde{H}^2 \psi_{N,0}) \leq C_2 N^2 \,.$$

Suppose moreover that the assumptions of Proposition 6.1 are satisfied. Then for any time t and for any fixed indices $i \neq j$, we have

$$\int W^2 |\nabla_i \phi_{N,t}|^2 \leq C, \qquad \int W^2 |\nabla_i \nabla_j \phi_{N,t}|^2 \leq C \tag{6.14}$$

for some constant C, independent of t and of the indices i, j.

Proof of Proposition 6.1. From (6.5), (6.8) we have (recall the summation convention)

$$\int W\bar{\phi}(\widetilde{H}W\phi) = \int W^2|\nabla_k\phi|^2 + \int W^2 B|\phi|^2 \geq \int W^2|\nabla_k\phi|^2 - \int W^2|\Omega||\phi|^2 .$$

Here we used that $B = q_{kj} + \Omega \geq \Omega$, because $q_{kj} \geq 0$. Part i) thus follows immediately from (6.11) and Lemma 6.3 below.

As for part ii) we note that, from (6.5),

$$\int |\widetilde{H}W\phi|^2 = \int W^2|L\phi|^2 + \int W^2 B\bar{\phi}L\phi + \int W^2 B\phi L\bar{\phi} + \int |\phi|^2(\widetilde{H}W)^2 . \tag{6.15}$$

Using (6.8), we can rewrite the second and third term in the last equation as

$$\begin{aligned}\int W^2 B\bar{\phi}L\phi + \int W^2 B\phi L\bar{\phi} &= 2\int W^2 B|\nabla_m\phi|^2 + \int W^2(\nabla_m B)\left(\bar{\phi}\nabla_m\phi + \phi\nabla_m\bar{\phi}\right) \\ &\geq 2\int W^2\Omega|\nabla_m\phi|^2 - 2\int W^2|\nabla_m B||\phi||\nabla_m\phi|.\end{aligned} \tag{6.16}$$

Here we used again the positivity of q. From Eqs. (6.11) and (6.13), we find

$$\begin{aligned}&\int W^2 B\bar{\phi}L\phi + \int W^2 B\phi L\bar{\phi} \\ &\quad \geq -2\int W^2|\widetilde{\Omega}||\nabla_m\phi|^2 - 2\int W^2|\nabla_m q_{kj}||\phi||\nabla_m\phi| \\ &\qquad -2\int W^2|\nabla_m\widetilde{\Omega}||\phi||\nabla_m\phi| + O\left(e^{-c\ell^{-\varepsilon}}\right)\left(\int W^2|\nabla_m\phi|^2 + N\int W^2|\phi|^2\right) .\end{aligned}$$

Analogously, the first term on the r.h.s. of (6.15) can be written as

$$\begin{aligned}\int W^2|L\phi|^2 &= \int W^2\nabla_j(L\bar{\phi})\nabla_j\phi \\ &= \int W^2(L\nabla_j\bar{\phi})\nabla_j\phi + \int W^2\Big([\nabla_j, L]\bar{\phi}\Big)\nabla_j\phi \\ &= \int W^2\nabla_i\nabla_j\bar{\phi}\nabla_i\nabla_j\phi - 2\int W^2\nabla_j\Big(\frac{\nabla_i W}{W}\Big)\nabla_i\bar{\phi}\nabla_j\phi \\ &= \int W^2|\nabla_i\nabla_j\phi|^2 - 2\int W^2(\nabla_i\nabla_j\log W)\,\nabla_i\bar{\phi}\nabla_j\phi .\end{aligned} \tag{6.17}$$

Next we note that, for any fixed $i.j$,

$$\nabla_i\nabla_j\log W = \frac{1}{2}\sum_\ell \nabla_i\nabla_j\log G_\ell = \frac{1}{2}\sum_\ell \frac{\nabla_i\nabla_j G_\ell}{G_\ell} - \frac{\nabla_i G_\ell\nabla_j G_\ell}{G_\ell^2} .$$

The second term gives a positive contribution. Hence we have

$$\int W^2|L\phi|^2 \geq \int W^2|\nabla_i\nabla_j\phi|^2 - \int W^2\frac{\nabla_i\nabla_j G_\ell}{G_\ell}\nabla_i\bar{\phi}\nabla_j\phi.$$

Thus, using $G_\ell \geq c_1 > 0$, we obtain

$$\begin{aligned}\int |\widetilde{H}W\phi|^2 \geq &\int W^2|\nabla_i\nabla_j\phi|^2 - (1/c_1)\int W^2|\nabla_i\nabla_j G_\ell|\,|\nabla_i\phi||\nabla_j\phi| \\ &-2\int W^2|\widetilde{\Omega}||\nabla_m\phi|^2 - 2\int W^2|\nabla_m q_{kj}||\phi||\nabla_m\phi| \\ &-2\int W^2|\nabla_m\widetilde{\Omega}||\phi||\nabla_m\phi| + O\left(e^{-c\ell^{-\varepsilon}}\right)\left(\int W^2|\nabla_m\phi|^2 + N\int W^2|\phi|^2\right) .\end{aligned} \tag{6.18}$$

The first term on the r.h.s. is the positive contribution we are interested in, all the other terms are errors terms which we estimate separately in Lemmas 6.3 (note the remark after Lemma 6.3), 6.4, 6.5 and 6.6. The proposition follows then from last equation and from the results of these lemmas. □

In the rest of this section we use the following notations for brevity

$$\mathcal{E}(\phi) := \int W^2 \sum_k |\nabla_k \phi|^2 + N \int W^2 |\phi|^2$$

$$\mathcal{F}(\phi) := \int W^2 \sum_{kj} |\nabla_k \nabla_j \phi|^2 + N \int W^2 \sum_k |\nabla_k \phi|^2 + N^2 \int W^2 |\phi|^2 \ ,$$

where the dependence on N is omitted.

Lemma 6.3. *Assume $a \ll \ell_1 \ll \ell \ll 1$ and $a\ell_1 \ll \ell^4$. Then*

$$\int W^2 |\widetilde{\Omega}| |\phi|^2 \leq o(1) \mathcal{E}(\phi)$$

as $N \to \infty$.

Remark. Replacing ϕ by $\nabla_m \phi$, and summing over m, it also follows from Lemma 6.3 that

$$\int W^2 |\widetilde{\Omega}| \sum_m |\nabla_m \phi|^2 \leq o(1) \mathcal{F}(\phi) \ .$$

Proof. Using $G_j \geq c_1 > 0$, we have from (6.12) (using the summation convention)

$$\begin{aligned}
\int W^2 |\widetilde{\Omega}| |\phi|^2 \leq C & \int W^2 \Big[|(\nabla w)_{kj}| |\nabla_k F_{kj}| + w_{ij} |\Delta_k F_{ij}| \Big] |\phi|^2 \\
& + C \int W^2 \Big[|(\nabla w)_{ik}| w_{jr} F_{ik} |\nabla_k F_{jr}| + w_{im} w_{jr} |\nabla_k F_{im}| |\nabla_k F_{jr}| \Big] |\phi|^2 \\
& \leq C \int W^2 (a\ell^{-1} + a\ell_1 \ell^{-2}) \lambda_{kj} F_{kj}^{1/2} |\phi|^2 \ ,
\end{aligned}$$

where we used the estimates $w_{jr} \leq \widetilde{\chi}_{jr}$, $w_{ij} \leq Ca\ell_1 \lambda_{ij}$ and $|\nabla w_{ij}| \leq Ca\lambda_{ij}$ from (5.6), and we also used the formulas (B.3), (B.6) to sum up indices. Using $\ell_1 \leq \ell$ and (5.8), we obtain

$$\int W^2 |\widetilde{\Omega}| |\phi|^2 \leq Ca\ell^{-1} \int W^2 \Big[|\nabla_k \phi|^2 + N \ell_1 \ell^{-3} |\phi|^2 \Big] = o(1) \mathcal{E}(\phi) \ .$$

□

Lemma 6.4. *Assume $a \ll \ell_1 \ll \ell$ and $a\ell_1 \ll \ell^4$. Then there is a constant C such that*

$$\int W^2 \sum_{i,j,m} |\nabla_i \nabla_j G_m| |\nabla_i \phi| |\nabla_j \phi| \leq \ C\varrho \int W^2 \sum_{i,j} |\nabla_i \nabla_j \phi|^2 + o(1) \mathcal{E}(\phi)$$

for $N \to \infty$. Recall that $\varrho = (8\pi)^{-1} (\|V\|_\infty + \|V\|_1)$.

Proof. A direct computation of $\nabla_i\nabla_j G_m$ and an application of Schwarz inequality (to separate $|\nabla_j\phi|$ and $|\nabla_i\phi|$) show that (with the summation convention)

$$\begin{aligned}\int & W^2|\nabla_i\nabla_j G_m||\nabla_i\phi||\nabla_j\phi| \\ &\le 4\int W^2|(\nabla^2 w)_{ij}|F_{ij}|\nabla_i\phi|^2 + C\int W^2|(\nabla w)_{kj}||\nabla_i F_{kj}||\nabla_i\phi|^2 \\ &\quad + C\int W^2|(\nabla w)_{kj}||\nabla_i F_{kj}||\nabla_j\phi|^2 + \int W^2 w_{km}|\nabla_i\nabla_j F_{km}||\nabla_i\phi|^2 \,.\end{aligned} \tag{6.19}$$

Applying $|\nabla w|\le Ca\lambda$ (see (5.6)), (B.3), and (5.8), the second term is bounded by

$$\int W^2|(\nabla w)_{kj}||\nabla_i F_{kj}||\nabla_i\phi|^2 \le Ca\ell^{-1}\int W^2\lambda_{kj}F_{kj}^{1/2}|\nabla_i\phi|^2 \le o(1)\mathcal{F}(\phi) \tag{6.20}$$

if $a\ell_1 \ll \ell^4$, and $a\ll\ell$. The third term on the r.h.s. of (6.19) can be bounded analogously. As for the fourth term in (6.19), we note that, using (B.3) and $w\le Ca\ell_1\lambda$ (by (5.6)), it is bounded by

$$\int W^2 w_{km}|\nabla_i\nabla_j F_{km}||\nabla_i\phi|^2 \le Ca\ell_1\ell^{-2}\int W^2\lambda_{km}F_{km}^{1/2}|\nabla_i\phi|^2$$

and the same estimate holds as in (6.20) since $a\ell_1\ell^{-2}\le a\ell^{-1}$.

Finally we consider the first term on the r.h.s. of (6.19). Using $|\nabla^2\omega|\le C\varrho\lambda$ from (5.6) and the estimate (5.7) we obtain

$$\begin{aligned}\int W^2|(\nabla^2 w)_{ij}|F_{ij}|\nabla_i\phi|^2 &\le C\varrho\int W^2\lambda_{ij}F_{ij}|\nabla_i\phi|^2 \\ &\le C\varrho\int W^2\widetilde{\chi}_{ij}F_{ij}^{1/4}|\nabla_i\nabla_j\phi|^2 + C\varrho\ell_1\ell^{-3}\int W^2\widetilde{\chi}_{ij}F_{ij}^{1/4}|\nabla_i\phi|^2 \\ &\le C\varrho\int W^2|\nabla_i\nabla_j\phi|^2 + C\frac{\ell_1}{N\ell^3}N\int W^2|\nabla_i\phi|^2 \,,\end{aligned}$$

where we also used (B.2). This completes the proof because $\ell_1\ell^{-3}N^{-1}=o(1)$, as $N\to\infty$. □

Lemma 6.5. *Assume $a\ll\ell_1\ll\ell$ and $a\ll\ell_1^{3/2}$. Then*

$$\int W^2\sum_{m,k,j}|\nabla_m q_{kj}||\phi||\nabla_m\phi| \le o(1)\mathcal{F}(\phi) \,,$$

for $N\to\infty$.

Proof. By part iii) of Lemma A.2 we have (with the summation convention)

$$\int W^2|\nabla_m q_{kj}||\phi||\nabla_m\phi| \le Ca\ell_1^{-3}\int W^2\,\frac{\chi(|x_m-x_j|\le\frac{3}{2}\ell_1)}{|x_m-x_j|}|\phi||\nabla_m\phi| \,.$$

Next we apply a weighted Schwarz inequality:

$$\begin{aligned}Ca\ell_1^{-3}\int & W^2\,\frac{\chi(|x_m-x_j|\le\frac{3}{2}\ell_1)}{|x_m-x_j|}|\phi||\nabla_m\phi| \\ &\le C\alpha a\ell_1^{-3}\int W^2\,\frac{\chi(|x_m-x_j|\le\frac{3}{2}\ell_1)}{|x_m-x_j|}|\phi|^2 \\ &\quad + C\alpha^{-1}a\ell_1^{-3}\int W^2\,\frac{\chi(|x_m-x_j|\le\frac{3}{2}\ell_1)}{|x_m-x_j|}|\nabla_m\phi|^2 \,.\end{aligned}$$

In the first term we use (5.4) from Lemma 5.2, in the second one we estimate $\chi(|x_m - x_j| \le \frac{3}{2}\ell_1) \le C\ell_1|x_m - x_j|^{-1}$ and apply the usual Hardy inequality (5.1). In both cases we first have to change W to $W^{(j)}$ by (C.1) then back again to W to make the weight function W independent of the x_j variable. The result is

$$Ca\ell_1^{-3}\int W^2\chi(|x_m - x_j| \le \tfrac{3}{2}\ell_1)|\phi||\nabla_m\phi| \le C\left(\frac{\alpha a\ell_1^2}{\ell_1^3} + \frac{a\ell_1}{\ell_1^3\alpha}\right)\mathcal{F}(\phi)\ .$$

We choose $\alpha = \ell_1^{-1/2}$. Then

$$Ca\ell_1^{-4}\int W^2\chi(|x_m - x_j| \le \tfrac{3}{2}\ell_1)|\phi||\nabla_m\phi| \le o(1)\mathcal{F}(\phi)$$

provided $a \ll \ell_1^{3/2}$. □

Lemma 6.6. *Assume $a \ll \ell_1 \ll \ell \ll 1$, $a \ll \ell^2$ and $a\ell_1 \ll \ell^4$. Then*

$$\int W^2\sum_m|\nabla_m\widetilde{\Omega}||\phi||\nabla_m\phi| \le o(1)\mathcal{F}(\phi)$$

as $N \to \infty$.

In order to prove this lemma we need the following estimates (we recall the definition of θ_{km} from (2.6)).

Lemma 6.7. *Let $a \ll \ell_1 \ll \ell \ll 1$ and $a \ll \ell^2$, then for any $q > 0$*

$$\sum_{m,k}\int W^2\sigma_{km}\,F_{km}^q|\phi||\nabla_m\phi| \le o(N\ell)\mathcal{F}(\phi)\ , \tag{6.21}$$

$$\sum_{m,k,j}\int W^2\lambda_{kj}F_{kj}^q\theta_{km}|\phi||\nabla_m\phi| \le o(1)\mathcal{F}(\phi)\ . \tag{6.22}$$

Proof. With a Schwarz inequality we obtain (using the summation convention)

$$\begin{aligned}\int W^2\sigma_{km}F_{km}^q|\phi||\nabla_m\phi| &\le \int W^2\sigma_{km}F_{km}^q\left(\alpha|\phi|^2 + \alpha^{-1}|\nabla_m\phi|^2\right)\\ &\le \alpha\int W^2\sigma_{km}|\phi|^2 + \alpha^{-1}a^{-1}\int W^2\lambda_{km}F_{km}^q|\nabla_m\phi|^2\end{aligned}$$

where we used that $\sigma_{km} \le a^{-1}\lambda_{km}$. Now we apply (5.9) and (5.7) from Lemma 5.3 to the r.h.s. of the last equation. We find

$$\int W^2\sigma_{km}F_{km}^q|\phi||\nabla_m\phi| \le C_q\left(\alpha\log N + \alpha^{-1}a^{-1} + \alpha^{-1}\ell_1\ell^{-3}\right)\mathcal{F}(\phi)\ .$$

Eq. (6.21) follows choosing $\alpha = \ell^{-1-\delta}$ for sufficiently small $\delta > 0$.

As for (6.22) we have, from (5.7),

$$\begin{aligned}\int W^2\lambda_{kj}F_{kj}^q\theta_{km}|\phi||\nabla_m\phi| &\le \int W^2\lambda_{kj}F_{kj}^q\theta_{km}(\alpha^{-1}|\nabla_m\phi|^2 + \alpha|\phi|^2)\\ &\le \int W^2\widetilde{\chi}_{kj}F_{kj}^{q/4}\theta_{km}\\ &\quad\times\left(\alpha^{-1}|\nabla_j\nabla_m\phi|^2 + \alpha^{-1}\ell_1\ell^{-3}|\nabla_m\phi|^2 + \alpha|\nabla_j\phi|^2 + \alpha\ell_1\ell^{-3}|\phi|^2\right)\ .\end{aligned}$$

Applying (5.3) for the second and third terms, and (5.4) for the last term, we get

$$\int W^2 \lambda_{kj} F^q_{kj} \theta_{km} |\phi||\nabla_m \phi| \\ \leq C\left(\alpha^{-1} + \alpha^{-1}\ell_1 \ell^{-3}\ell^2 |\log \ell|^2 + \alpha \ell^2 |\log \ell|^2 + \alpha \ell_1 \ell^{-3} \ell^3 |\log \ell|^3\right) \mathcal{F}(\phi) .$$

Choosing for example $\alpha = \ell^{-1}$ we find (6.22). □

Proof of Lemma 6.6. By the definition of $\widetilde{\Omega}$ we find (using the summation convention)

$$\int W^2 |\nabla_m \widetilde{\Omega}||\phi||\nabla_m \phi| \leq \int W^2 \Big|\nabla_m \frac{\Omega_k}{2G_k}\Big||\phi||\nabla_m \phi| \\ + \int W^2 \Big|\nabla_m \frac{\Omega_{jk}}{2G_j}\Big||\phi||\nabla_m \phi| + \int W^2 \Big|\nabla_m \Gamma\Big||\phi||\nabla_m \phi| . \tag{6.23}$$

We begin considering the first term on the r.h.s. of (6.23).

$$\int W^2 \Big|\nabla_m \frac{\Omega_k}{2G_k}\Big||\phi||\nabla_m \phi| \leq C \int W^2 \left(|\nabla_m \Omega_k| + |\Omega_k||\nabla_m G_k|\right) |\phi||\nabla_m \phi| . \tag{6.24}$$

From the definition of Ω_k (see (6.1)), using Lemma B.2 to bound the derivatives of F_{mj}, we find

$$\int W^2 |\nabla_m \Omega_k||\phi||\nabla_m \phi| \leq C\ell^{-1} \int W^2 |(\nabla^2 w)_{mj}| F^{1/2}_{mj} |\phi||\nabla_m \phi| \\ + C\ell^{-2} \int W^2 |(\nabla w)_{kj}| F^{1/2}_{kj} (\theta_{mk} + \theta_{mj}) |\phi||\nabla_m \phi| \\ + C\ell^{-3} \int W^2 w_{kj} F^{1/2}_{kj} (\theta_{mk} + \theta_{mj}) |\phi||\nabla_m \phi| + O(\ell^{K-1-\varepsilon}) \mathcal{F}(\phi) . \tag{6.25}$$

We use that $|(\nabla^2 w)_{mj}| \leq Ca\sigma_{mj}$, $|(\nabla w)_{kj}| \leq Ca\lambda_{kj}$ and $w_{kj} \leq Ca\ell_1 \lambda_{kj}$ (see (5.6)), and applying Lemma 6.7 we obtain

$$\int W^2 |\nabla_m \Omega_k||\phi||\nabla_m \phi| = o(1)\mathcal{F}(\phi) . \tag{6.26}$$

For the other term in (6.24), using the definition of G_k (2.7), we have

$$\int W^2 |\Omega_k|\, |\nabla_m G_k|\, |\phi|\, |\nabla_m \phi| \\ \leq C \int W^2 \Bigg(|(\nabla w)_{mj}|\, |(\nabla w)_{mn}|\, F_{mn}\, |\nabla_m F_{mj}| + |(\nabla w)_{kj}|\, |(\nabla w)_{km}|\, F_{km}\, |\nabla_k F_{kj}| \\ + |(\nabla w)_{mj}|\, w_{mj} F_{mn}\, |\Delta_m F_{mj}| + |(\nabla w)_{km}|\, w_{kj} F_{km}\, |\Delta_k F_{kj}| \\ + |(\nabla w)_{kj}|\, w_{kn}\, |\nabla_m F_{kn}|\, |\nabla_k F_{kj}| + w_{kj} w_{kn}\, |\Delta_k F_{kj}|\, |\nabla_m F_{kn}\, | \Bigg) |\phi||\nabla_m \phi| .$$

Taking as example the first term on the r.h.s. of the last equation, we note that, unless $j = n$, the term is actually exponentially small because, by Lemma B.1, $\|\widetilde{\chi}_{mj}\widetilde{\chi}_{mn}F_{mn}\|_\infty \leq \exp(-c\ell^{-\varepsilon})$ if $j \neq n$. Applying similar arguments to all other terms as well, and applying Lemma B.2 to bound the derivatives of F, we obtain

$$\begin{aligned}
&\int W^2|\Omega_k||\nabla_m G_k||\phi||\nabla_m\phi| \\
&\qquad \le C\int W^2|\phi||\nabla_m\phi|\Big(\ell^{-1}\,|(\nabla w)_{mn}|^2 F_{mn} + \big(\ell^{-2}\,|(\nabla w)_{kj}|F_{kj} + \ell^{-3}w_{kj}^2 F_{kj}\big)\,(\theta_{mk}+\theta_{mj})\Big) \\
&\qquad\quad + \Big(O\left(e^{-c\ell^{-\varepsilon}}\right) + O(\ell^{K-2-\varepsilon})\Big)\,\mathcal{F}(\phi) \\
&\qquad = o(1)\mathcal{F}(\phi)
\end{aligned}$$

following the estimate of the term (6.25) and using the bounds $(\nabla w)^2 \le Ca\sigma$ and $w^2 \le w$ from (5.6).

So, together with (6.26) we find the following estimate for the first term of (6.23)

$$\int W^2\Big|\nabla_m\frac{\Omega_k}{2G_k}\Big||\phi||\nabla_m\phi| \le o(1)\mathcal{F}(\phi)\ . \tag{6.27}$$

We consider next the second term on the r.h.s. of (6.23). Clearly we have

$$\int W^2\Big|\nabla_m\frac{\Omega_{jk}}{2G_j}\Big||\phi||\nabla_m\phi| \le C\int W^2\left(|\nabla_m\Omega_{kj}| + |\Omega_{kj}||\nabla_m G_j|\right)|\phi||\nabla_m\phi|\ .$$

Comparing Ω_{kj} with Ω_k, it is clear that it only remains to control the two terms

$$\int W^2 w_{ij}|\Delta_k F_{ij}||\nabla_m G_j||\phi||\nabla_m\phi| \tag{6.28}$$

and

$$\int W^2|\nabla_m(w_{ij}\Delta_k F_{ij})||\phi||\nabla_m\phi|\ . \tag{6.29}$$

We begin with (6.28). The summation over k is performed by (B.3). We have

$$\begin{aligned}
\int W^2 w_{ij}|\Delta_k F_{ij}||\nabla_m G_j||\phi||\nabla_m\phi| \le\ & C\ell^{-2}\int W^2\Big(w_{ij}F_{ij}^{1/2}|(\nabla w)_{jm}|F_{jm} \\
&+ w_{im}F_{im}^{1/2}|(\nabla w)_{mn}|F_{mn} + w_{ij}F_{ij}^{1/2}w_{jn}|\nabla_m F_{jn}|\Big)\,|\phi||\nabla_m\phi|\ .
\end{aligned}$$

By Lemma B.1 we know that, for example the first term is exponentially small unless $i = m$. Similarly, apart from exponentially small contributions, in the other two terms we can consider only the case $i = n$. Hence we find

$$\begin{aligned}
\int W^2 w_{ij}|\Delta_k F_{ij}||\nabla_m G_j||\phi||\nabla_m\phi| \le\ & C\ell^{-2}\int W^2 w_{mj}F_{mj}^{1/2}|(\nabla w)_{mj}|\ F_{mj}\ |\phi|\ |\nabla_m\phi| \\
&+ C\ell^{-2}\int W^2 w_{jn}^2 F_{jn}^{1/2}|\nabla_m F_{jn}|\ |\phi|\ |\nabla_m\phi| \\
&+ O\left(\exp(-c\ell^{-\varepsilon})\right)\mathcal{E}(\phi)\ .
\end{aligned} \tag{6.30}$$

On the right hand side we estimate $|\nabla w| \le Ca\lambda$ by (5.6). In the first term, using $F \le 1$, we have

$$C\ell^{-2}\int W^2 w_{mj}F_{mj}^{1/2}|(\nabla w)_{mj}|\ F_{mj}\ |\phi|\ |\nabla_m\phi| \le Ca\ell^{-2}\int W^2\lambda_{mj}\ F_{mj}\ |\phi|\ |\nabla_m\phi| \tag{6.31}$$

that can be easily estimated by $o(1)\mathcal{F}(\phi)$, using the assumption $a \ll \ell^2$ and $a \ll \ell_1 \ll \ell$.

In the second term of (6.30) we estimate $w^2 \le Ca^2\lambda$. By Lemma B.2 we find

$$C\ell^{-2}\int W^2 w_{jn}^2 F_{jn}^{1/2}\,|\nabla_m F_{jn}|\,|\phi|\,|\nabla_m\phi| \le Ca^2\ell^{-2}\int W^2\lambda_{jn}F_{jn}^{1/2}(\theta_{mj}+\theta_{mn})|\phi|\,|\nabla_m\phi| = o(1)\mathcal{F}(\phi)\ ,$$

where we also applied Lemma 6.7.

We consider next (6.29). We have

$$\begin{aligned}\int W^2|\nabla_m(w_{ij}\Delta_k F_{ij})||\phi||\nabla_m\phi| \le &\int W^2|(\nabla w)_{mj}||\Delta_k F_{mj}||\phi||\nabla_m\phi| \\ &+\int W^2 w_{ij}|\nabla_m\Delta_k F_{ij}||\phi||\nabla_m\phi|\ .\end{aligned} \tag{6.32}$$

After summation over k using (B.3), the first term can be treated as in (6.31).

As for the second term in (6.32), we use that $w \le Ca\ell_1\lambda$ and Lemma B.2. We find

$$\int W^2 w_{ij}|\nabla_m\Delta_k F_{ij}||\phi||\nabla_m\phi| \le a\ell_1\ell^{-3}\int W^2\lambda_{ij}F_{ij}^{1/2}(\theta_{mi}+\theta_{mj})|\phi||\nabla_m\phi|$$

which can be estimated by $o(1)\mathcal{F}(\phi)$ by Lemma 6.7.

Finally we control the third term on the r.h.s. of (6.23). After applying Lemma B.1 we find, for any fixed m,

$$\begin{aligned}\Big|\nabla_m\Gamma\Big| \le C\Bigg[&|(\nabla^2 w)_{mk}|\ w_{jn}F_{mk}(\nabla_k F_{jn}+\nabla_m F_{jn}) + |(\nabla w)_{ik}|\ |(\nabla w)_{jm}|\ F_{ik}\ |\nabla_k F_{jm}| \\ &+|(\nabla w)_{ik}|\ w_{jn}\ |\nabla_m F_{ik}|\ |\nabla_k F_{jn}| + |(\nabla w)_{ik}|\ w_{jn}\ |\nabla_m\nabla_k F_{jn}|\ F_{ik} \\ &+|(\nabla w)^2_{im}|\ w_{jn}F^2_{im}\ |\nabla_m F_{jn}| + |(\nabla w)^2_{km}|\ w_{jn}F^2_{km}\ |\nabla_k F_{jn}| \\ &+|(\nabla w)_{ik}|\ w_{ik}w_{jn}F_{ik}\ |\nabla_k F_{jn}|\ |\nabla_m F_{ik}| + |(\nabla w)_{ik}|\ w^2_{jn}F_{ik}\ |\nabla_k F_{jn}|\ |\nabla_m F_{jn}| \\ &+|(\nabla w)_{ik}|\ |(\nabla w)_{jm}|\ w_{jm}F_{ik}F_{jm}\ |\nabla_k F_{jm}| + |(\nabla w)_{mi}|\ w_{jn}\ |\nabla_k F_{mi}|\ |\nabla_k F_{jn}| \\ &+|(\nabla w)_{mi}|\ w_{mi}w_{jn}F_{mi}\ |\nabla_k F_{mi}||\nabla_k F_{jn}| + w_{in}w_{js}|\nabla_k\nabla_m F_{in}|\ |\nabla_k F_{js}| \\ &+w^2_{in}w_{js}|\nabla_k F_{in}|\ |\nabla_m F_{in}|\ |\nabla_k F_{js}|\Bigg] + O\left(e^{-c\ell^{-\varepsilon}}\right)\ .\end{aligned} \tag{6.33}$$

Using the estimates (B.3), (B.6) and (B.7) from Lemma B.2 and the bounds (5.6), we obtain

$$\begin{aligned}\Big|\nabla_m\Gamma\Big| \le C\Bigg[&a\ell^{-1}\sigma_{mk}F_{mk}^{1/2} + (a\ell^{-2}+a\ell_1\ell^{-3}+a^2\ell^{-3})\lambda_{ik}F_{ik}^{1/2}(\theta_{im}+\theta_{km}) \\ &+|(\nabla w)_{ik}|\ |(\nabla w)_{jm}|\ F_{ik}\ |\nabla_k F_{jm}| + |(\nabla w)_{ik}|\ w^2_{jn}F_{ik}\ |\nabla_m F_{jn}|\ |\nabla_k F_{jn}|\Bigg]\ .\end{aligned} \tag{6.34}$$

The terms on the first line of the last equation can be bounded using directly Lemma 6.7. Consider next the first term on the second line. Its contribution, after two Schwarz inequalities is given by

$$\begin{aligned}
&\int W^2|(\nabla w)_{ik}||(\nabla w)_{jm}|F_{ik}|\nabla_k F_{jm}||\phi||\nabla_m\phi| \\
&\quad\le \int W^2\alpha|(\nabla w)_{ik}|^2\Big[\,\beta|\phi|^2+\beta^{-1}|\nabla_m\phi|^2\,\Big]\chi_{ik}F_{ik}\chi_{jm}|\nabla_k F_{jm}| \\
&\qquad+\int W^2\alpha^{-1}|(\nabla w)_{jm}|^2\Big[\,\gamma|\phi|^2+\gamma^{-1}|\nabla_m\phi|^2\,\Big]\chi_{ik}F_{ik}\chi_{jm}|\nabla_k F_{jm}| \\
&\quad\le C\alpha\beta a\ell^{-1}\int W^2\sigma_{ik}F_{ik}|\phi|^2+C\alpha\beta^{-1}\ell^{-1}\int W^2\lambda_{ik}F_{ik}|\nabla_m\phi|^2 \\
&\qquad+C\alpha^{-1}\gamma a\ell^{-1}\int W^2\sigma_{jm}F_{jm}^{1/2}|\phi|^2+C\alpha^{-1}\gamma^{-1}\ell^{-1}\int W^2\lambda_{jm}F_{jm}^{1/2}|\nabla_m\phi|^2 \\
&\quad\le C\Big(\alpha\beta a\ell^{-1}\log N+\alpha\beta^{-1}\ell^{-1}(1+\ell_1\ell^{-3})+\alpha^{-1}\gamma a\ell^{-1}\log N \\
&\qquad+\alpha^{-1}\gamma^{-1}(1+a\ell_1\ell^{-3})\Big)\mathcal{F}(\phi)\,.
\end{aligned}$$

Optimizing the choice of α, β and γ, we find that this term is of order $o(1)\mathcal{F}(\varphi)$ (using that $a \ll \ell^2$ and $a\ell_1 \ll \ell^4$).

As for the last term on the r.h.s. of (6.34) we use that $w_{jn}^2 \le \chi_{jn}$. By Lemma B.2 and Lemma 5.3 we get

$$\begin{aligned}
&\int W^2|(\nabla w)_{ik}|\; w_{jn}^2F_{ik}\;|\nabla_m F_{jn}|\;|\nabla_k F_{jn}|\;|\phi|\;|\nabla_m\phi| \\
&\qquad\qquad\le Ca\ell^{-2}\int W^2\lambda_{ik}F_{ik}(\alpha|\phi|^2+\alpha^{-1}|\nabla_m\phi|^2) \\
&\qquad\qquad\le C\left(\alpha N^{-1}(a\ell^{-2}+a\ell_1\ell^{-5})+\alpha^{-1}(a\ell^{-2}+a\ell_1\ell^{-5})\right)\mathcal{F}(\phi) \\
&\qquad\qquad\le C(a^{3/2}\ell^{-2}+a^{3/2}\ell_1\ell^{-})\mathcal{F}(\phi)=o(1)\mathcal{F}(\phi).
\end{aligned}$$

From (6.34), we find

$$\int W^2\Big|\nabla_m\Gamma\Big|\,|\phi|\,|\nabla_m\phi| = o(1)\mathcal{F}(\phi)\,.$$

This completes the proof of Lemma 6.6. □

7 A-priori Estimates on $U_{N,t}$

Recall that $\mathcal{H} = L^2(\Lambda, \mathrm{d}x)$ is the one particle Hilbert space, and that $\mathcal{H}^{\otimes n}$ denotes the n-fold tensor product $\mathcal{H}^{\otimes n}$. We denote by $\mathcal{L}^1_n := \mathcal{L}^1(\mathcal{H}^{\otimes n})$ and by $\mathcal{K}_n := \mathcal{K}(\mathcal{H}^{\otimes n})$ the space of trace class operators and the space of compact operators on the Hilbert space $\mathcal{H}^{\otimes n}$ respectively. We equip these spaces with the trace norm, $\|\cdot\|_1 := \mathrm{Tr}|\cdot|$, and with the operator norm, $\|\cdot\|$, respectively. It is a well-known fact that $\mathcal{K}_n^* = \mathcal{L}^1_n$, that is $\mathcal{L}^1_n$ is the dual Banach space of $\mathcal{K}_n$.

We define Sobolev-type norms on trace class operators. For $\gamma^{(n)} \in \mathcal{L}^1(\mathcal{H}^{\otimes n})$, we define the norm

$$\|\gamma^{(n)}\|_{\mathcal{W}_n} := \begin{cases} \mathrm{Tr}|S_1\gamma^{(1)}S_1| & \text{if } n=1 \\ \mathrm{Tr}|S_1S_2\gamma^{(n)}S_1S_2| & \text{if } n\ge 2 \end{cases}$$

where $S_j = (1-\Delta_j)^{1/2}$, and the space $\mathcal{W}_n = \{\gamma^{(n)} \in \mathcal{L}^1(\mathcal{H}^{\otimes n}) : \|\gamma^{(n)}\|_{\mathcal{W}_n} < \infty\}$.

Note that $\mathcal{W}_n$ is the dual of the space

$$\mathcal{A}_n := \begin{cases} \{S_1 K^{(1)} S_1 : K^{(1)} \in \mathcal{K}_1\} & \text{if } n = 1 \\ \{S_1 S_2 K^{(n)} S_2 S_1 : K^{(n)} \in \mathcal{K}_n\} & \text{if } n \geq 2 \end{cases}$$

equipped with the norms

$$\|T^{(1)}\|_{\mathcal{A}_1} = \|S_1^{-1} T^{(1)} S_1^{-1}\| \quad \text{and} \quad \|T^{(n)}\|_{\mathcal{A}_n} := \|S_1^{-1} S_2^{-1} T^{(n)} S_2^{-1} S_1^{-1}\| \quad \text{if } n \geq 2.$$

Lemma 7.1. *Suppose $U_{N,t} = \{U_{N,t}^{(k)}\}_{k=1}^N$ is defined as in (4.5), and that the assumptions of Theorem 2.1 are satisfied. Then there is $\nu > 1$ large enough such that $\|U_{N,t}\|_{H_-^{(\nu)}} \leq 1$. Moreover there exists a constant $C > 0$ such that*

$$\|U_{N,t}^{(k)}\|_{\mathcal{W}_k} \leq C^k \tag{7.1}$$

for any $t \in [0,T]$ and $k \leq N$.

Let $U_{\infty,t}$ be a limit point of $U_{N,t}$ in $C([0,T], H_-^{(\nu)})$ with respect to the metric $\widehat{\rho}$ (defined in (2.16)). Then we also have $\|U_{\infty,t}\|_{H_-^{(\nu)}} \leq 1$ and there is a version of $U_{\infty,t}$ such that

$$Tr(1-\Delta_1) U_{\infty,t}^{(1)} \leq C \qquad and \qquad Tr(1-\Delta_i)(1-\Delta_j) U_{\infty,t}^{(k)} \leq C^k \quad if \quad k \geq 2 \tag{7.2}$$

for every k, every $t \in [0,T]$, and every $i,j = 1,\dots,k$ with $i \neq j$.

Remark 7.2. *This lemma does not yet prove the compactness of the sequence $U_{N,t} \in C([0,T], H_-^{(\nu)})$: to this end we still need the equicontinuity of $U_{N,t}$, which will be proven in Section 9.2. Here we prove that, if a limit $U_{\infty,t} \in C([0,T], H_-^{(\nu)})$ exists, then it satisfies (7.2).*

Proof. We have

$$\|U_{N,t}^{(k)}\|^2 = \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}_k' |U_{N,t}^{(k)}(\mathbf{x}_k; \mathbf{x}_k')|^2 \leq C_0^k \operatorname{Tr} U_{N,t}^{(k)} \leq C_0^{2k}.$$

Here we used that $U_{N,t}^{(k)}$ is the kernel of a positive operator with trace bounded by C_0^k: this implies that also the operator norm of $U_{N,t}^{(k)}$ is bounded by C_0^k. Choosing $\nu > 2C_0$ we immediately see that $\|U_{N,t}\|_{H_-^{(\nu)}} \leq 1$. The bound $\|U_{\infty,t}\|_{H_-^{(\nu)}} \leq 1$ follows because the norm can only drop when a weak limit is taken.

From Corollary 6.2 and (C.2) we have

$$\operatorname{Tr}|S_1 U_{N,t}^{(1)} S_1| = \int |W^{(1)}|^2 \left(|\nabla_1 \phi_{N,t}|^2 + |\phi_{N,t}|^2 \right) \leq C \int W^2 \left(|\nabla_1 \phi_{N,t}|^2 + |\phi_{N,t}|^2 \right) \leq C$$

and, for any $k \geq 2$,

$$\begin{aligned} \operatorname{Tr}|S_1 S_2 U_{N,t}^{(k)} S_1 S_2| &= \int |W^{[k]}|^2 \Big[|\nabla_1 \nabla_2 \phi_{N,t}|^2 + |\nabla_1 \phi_{N,t}|^2 + |\nabla_2 \phi_{N,t}|^2 + |\phi_{N,t}|^2 \Big] \\ &\leq C_0^k \int |W|^2 \Big[|\nabla_1 \nabla_2 \phi_{N,t}|^2 + |\nabla_1 \phi_{N,t}|^2 + |\nabla_2 \phi_{N,t}|^2 + |\phi_{N,t}|^2 \Big] \\ &\leq C^k \end{aligned} \tag{7.3}$$

for some constant C independent of k. This proves (7.1).

For each fixed t, we can assume that $U_{N,t}$ converges to $U_{\infty,t}$ in the weak$*$ topology of $H_-^{(\nu)}$ (otherwise we choose an appropriate subsequence). This follows because $U_{N,t} \in B_-^{(\nu)}$ (the unit ball of $H_-^{(\nu)}$) and because, on $B_-^{(\nu)}$, the topology induced by the metric ρ is equivalent to the weak$*$ topology. This in particular implies that, for every fixed k, $U_{N,t}^{(k)}$ converges to $U_{\infty,t}^{(k)}$ in the weak topology of $L^2(\Lambda^k \times \Lambda^k)$. Note that $U_{\infty,t}^{(k)}$ is symmetric w.r.t. permutations of the k particles, because $U_{N,t}^{(k)}$ is symmetric, and because the symmetry is clearly preserved in the limiting process. By passing to a subsequence and using the Alaoglu-Banach Theorem, we can assume that $U_{N,t}^{(k)}$ also converges in the weak$*$ topology of $\mathcal{W}_k$, and let $\widetilde{U}_{\infty,t}$ be the limit. Testing these two limits against operators with smooth kernels, one easily sees that $\widetilde{U}_{\infty,t}^{(k)} = U_{\infty,t}^{(k)}$ as elements of $L^2(\Lambda^k \times \Lambda^k)$, hence $U_{N,t}^{(k)}$ converges to $U_{\infty,t}^{(k)}$ in both topologies. Then the estimate in (7.2) for the version of $U_{\infty,t}^{(k)}$ given by $\widetilde{U}_{\infty,t}^{(k)}$ follow from (7.1), from the permutation symmetry of $U_{\infty,t}^{(k)}$, and from the fact that the norm does not increase under weak$*$ limit. $\square$

The next lemma will be used in Section 9 to prove part iii) of Theorem 2.1.

Lemma 7.3. *Assume that $a \ll \ell_1 \ll \ell \ll 1$. Then, for any fixed k and t,*

i) We have

$$\mathrm{Tr}\left(U_{N,t}^{(k)} - \gamma_{N,t}^{(k)}\right) \to 0$$

as $N \to \infty$.

ii) Assume that $U_{\infty,t} = \{U_{\infty,t}^{(k)}\}_{k\geq 1} \in C([0,T], H_-^{(\nu)})$ is a limit point of $U_{N,t} = \{U_{N,t}^{(k)}\}_{k=1}^N$ with respect to the metric $\widehat{\rho}$. Then

$$\mathrm{Tr}\, U_{\infty,t}^{(1)} = 1 \qquad \text{and} \qquad \mathrm{Tr}\, U_{\infty,t}^{(2)} = 1\,. \tag{7.4}$$

Proof. i) We have (with $\phi = \phi_{N,t}(\mathbf{x})$),

$$\begin{aligned}\mathrm{Tr}\left(U_{N,t}^{(k)} - \gamma_{N,t}^{(k)}\right) &= \int (W^2 - (W^{[k]})^2)|\phi|^2 \\ &= \int (1 - \prod_{j=1}^k G_j) \prod_{j=k+1}^N G_j |\phi|^2 + \sum_{m=k+1}^N \int \prod_{j=k+1}^{m-1} G_j^{[k]}(G_m^{[k]} - G_m) \prod_{j=m+1}^N G_j |\phi|^2\end{aligned}$$

and hence, using (C.6),

$$\begin{aligned}\left|\mathrm{Tr}\left(U_{N,t}^{(k)} - \gamma_{N,t}^{(k)}\right)\right| &\leq C^k \int (1 - \prod_{j=1}^k G_j) W^2 |\phi|^2 + C^k \sum_{m=k+1}^N \int |G_m^{[k]} - G_m| W^2 |\phi|^2 \\ &\leq C^k \sum_{j\leq k} \sum_{m=1}^N \int W^2 w_{jm} F_{jm} |\phi|^2 \\ &\quad + C^k \sum_{m,j>k} \sum_{r\leq k} \int W^2 w_{mj} F_{mj}^{[k]} \theta_{mr} |\phi|^2 + O(\ell^{K-\varepsilon}).\end{aligned}$$

So applying Lemma 5.3 for the first term and part (ii) of Lemma 5.2 for the second term (with

$U = \theta_{mr}$), we find

$$
\begin{aligned}
\left|\mathrm{Tr}\Big(U_{N,t}^{(k)} - \gamma_{N,t}^{(k)}\Big)\right| \le {} & C_k \left(a\ell_1 \sum_{j\le k} \int W^2 |\nabla_j \phi|^2 + a\ell_1 \ell^{-3} k \int W^2 |\phi|^2 \right) + O(\ell^{K-\varepsilon}) \\
& + C_k \Bigg(a\ell_1 (\ell |\log \ell|)^2 \sum_{j>k} \sum_{r<k} \int W^2 |\nabla_r \nabla_j \phi|^2 \\
& + a\ell_1^2 \ell^{-3} (\ell |\log \ell|)^3 \sum_{m>k} \sum_{r\le k} \int W^2 (|\nabla_r \nabla_m \phi|^2 + |\nabla_r \phi|^2 + |\nabla_m \phi|^2 + |\phi|^2) \Bigg) \\
\le {} & C_k (a\ell_1 + a\ell_1^2 \ell^{-3} + \ell_1 \ell^2 |\log \ell|^2 + \ell_1^2 |\log \ell|^3) + O(\ell^{K-\varepsilon}) = o(1)
\end{aligned}
$$

for $N \to \infty$ using (6.14).

ii) From part i) we have, for any fixed k and t, Tr $U_{N,t}^{(k)} \to 1$ as $N \to \infty$ using the normalization (2.19). By Lemma 7.1, the sequence $U_{N,t}^{(1)}$ in $\mathcal{W}_1$ is compact w.r.t. the weak$*$ topology of $\mathcal{W}_1$. By passing to a subsequence we can assume $U_{N,t}^{(1)}$ converges to $U_{\infty,t}^{(1)}$ in the weak$*$ topology of $\mathcal{W}_1$. In particular,

$$
\mathrm{Tr}\ U_{\infty,t}^{(1)} = \mathrm{Tr}\ S_1^{-2} S_1 U_{\infty,t}^{(1)} S_1 = \lim_{N\to\infty} \mathrm{Tr}\ S_1^{-2} S_1 U_{N,t}^{(1)} S_1 = \lim_{N\to\infty} \mathrm{Tr}\ U_{N,t}^{(1)} = 1
$$

since the operator $S_1^{-2} = (1 - \Delta_1)^{-1}$ is compact on the finite periodic box Λ. The proof of (7.4) for $U_{\infty,t}^{(2)}$ is similar. □

8 Approximation of the Delta Function

We consider the sequence of density matrices $U_{N,t} = \{U_{N,t}^{(k)}\}_{k\ge 1}$. By viewing their kernels as distributions, we will prove in Section 9.2 that $U_{N,t}$ is compact in $C([0,T], H_-^{(\nu)})$. Assuming this property for the moment, we denote a limit point by $U_{\infty,t}$. Since for each fixed t the kernel $U_{\infty,t}$ is defined only as a distribution, the restriction of $U_{\infty,t}^{(k+1)}$ on the diagonal $x_{k+1} = x'_{k+1}$, i.e.,

$$
U_{\infty,t}^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) \tag{8.1}
$$

has no a-priori meaning. On the other hand, based upon Lemma 7.1 we can view $U_{N,t}$ and $U_{\infty,t}$ (or, more precisely, the families of operators with kernels given by $U_{N,t}^{(k)}$ and $U_{\infty,t}^{(k)}$) as elements of $\mathcal{W}^{(\nu)}$, for every $t \in [0,T]$.

We shall show in the next proposition that since $U_{\infty,t} \in \mathcal{W}^{(\nu)}$, the diagonal element (8.1) is well-defined. For the rest of this section, we shall assume that $U^{(k+1)}$ is the kernel of a density matrix with Tr $U^{(k+1)} < \infty$.

Proposition 8.1. *Suppose $\delta_\beta(x)$ is a radially symmetric function, with $0 \le \delta_\beta(x) \le C\beta^{-3}\chi(|x| \le \beta)$ and $\int \delta_\beta(x) \mathrm{d}x = 1$ (for example $\delta_\beta(x) = \beta^{-3} h(x/\beta)$, for a radially symmetric probability density $h(x)$ supported in $\{x : |x| \le 1\}$). Then, for any $J^{(k)} \in W^{1,\infty}(\Lambda^k \times \Lambda^k)$ and for any smooth function*

$U^{(k+1)}(\mathbf{x}_{k+1};\mathbf{x}'_{k+1})$ *corresponding to a $(k+1)$-particle density matrix, we have, for any fixed $j \leq k$,*

$$\begin{aligned}\Big|\int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}x_{k+1}\mathrm{d}x'_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\, U^{(k+1)}(\mathbf{x}_k, x_{k+1};\mathbf{x}'_k, x'_{k+1})\\ \times \left(\delta_{\beta_1}(x'_{k+1}-x_{k+1})\delta_{\beta_2}(x_j - x_{k+1}) - \delta(x'_{k+1}-x_{k+1})\delta(x_j-x_{k+1})\right)\Big| \\ \leq C[\,\|J\|_\infty + \|\nabla_j J\|_\infty\,]\,(\beta_1 + \sqrt{\beta_2})\ Tr|S_j S_{k+1} U^{(k+1)} S_j S_{k+1}|\,.\end{aligned} \tag{8.2}$$

Remark 8.2. *From (8.2), using a standard approximation argument, it also follows that*

$$\begin{aligned}\Big|\int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}x_{k+1}\mathrm{d}x'_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\, U^{(k+1)}(\mathbf{x}_k, x_{k+1};\mathbf{x}'_k, x'_{k+1})\\ \times\left(\delta_{\beta_1}(x'_{k+1}-x_{k+1})\delta_{\beta_2}(x_j-x_{k+1}) - \delta_{\beta'_1}(x'_{k+1}-x_{k+1})\delta_{\beta'_2}(x_j-x_{k+1})\right)\Big|\\ \leq C[\,\|J\|_\infty + \|\nabla_j J\|_\infty\,]\,\left(\beta_1+\beta'_1+\sqrt{\beta_2}+\sqrt{\beta'_2}\right)\ Tr|S_j S_{k+1} U^{(k+1)} S_j S_{k+1}|\end{aligned}$$

for any $U^{(k+1)}$ for which the r.h.s. is finite.

Proof. It is enough to prove (8.2) for $U^{(k+1)}$ of the form $U^{(k+1)}(\mathbf{x}_{k+1},\mathbf{x}'_{k+1}) = f(\mathbf{x}_{k+1})\overline{f}(\mathbf{x}'_{k+1})$ (in this section we use the notation $\mathbf{x}_{k+1} = (x_1,\dots x_{k+1})$ and $\mathbf{x}'_{k+1} = (x'_1,\dots,x'_{k+1})$). Then, for a general density matrix $U^{(k+1)}$, the proposition follows by considering the spectral decomposition:

$$U^{(k+1)}(\mathbf{x}_{k+1};\mathbf{x}'_{k+1}) = \sum_n c_n f_n(\mathbf{x}_{k+1})\overline{f_n}(\mathbf{x}'_{k+1})$$

with $c_n > 0$ for all n and $\|f_n\|_{L^2} = 1$, and using the fact that Tr $U^{(k+1)} = \sum c_n < \infty$. If $U^{(k+1)}(\mathbf{x}_{k+1};\mathbf{x}'_{k+1}) = f(\mathbf{x}_{k+1})\overline{f}(\mathbf{x}'_{k+1})$ we can bound the l.h.s. of (8.2) by the sum

$$\begin{aligned}\Big|\int \mathrm{d}\mathbf{x}_{k+1}\mathrm{d}\mathbf{x}'_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\,\left(\delta_{\beta_1}(x'_{k+1}-x_{k+1}) - \delta(x'_{k+1}-x_{k+1})\right)\delta_{\beta_2}(x_j-x_{k+1})\, f(\mathbf{x}_{k+1})\overline{f}(\mathbf{x}'_{k+1})\Big|\\ + \Big|\int \mathrm{d}\mathbf{x}_{k+1}\mathrm{d}\mathbf{x}'_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\,(\delta_{\beta_2}(x_j-x_{k+1}) - \delta(x_j-x_{k+1}))\,\delta(x'_{k+1}-x_{k+1}) f(\mathbf{x}_{k+1})\overline{f}(\mathbf{x}'_{k+1})\Big|.\end{aligned} \tag{8.3}$$

We next bound the first term by

$$\begin{aligned}\Big|\int \mathrm{d}\mathbf{x}_{k+1}\mathrm{d}\mathbf{x}'_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\delta_{\beta_2}(x_j-x_{k+1})\left(\delta_{\beta_1}(x'_{k+1}-x_{k+1}) - \delta(x'_{k+1}-x_{k+1})\right) f(\mathbf{x}_{k+1})\overline{f}(\mathbf{x}'_{k+1})\Big|\\ \leq\ \|J^{(k)}\|_\infty \int \mathrm{d}\mathbf{x}_{k+1}\mathrm{d}\mathbf{x}'_k\, \delta_{\beta_2}(x_j-x_{k+1})\,|f(\mathbf{x}_{k+1})|\\ \times\Big|f(\mathbf{x}'_k, x_{k+1}) - \int \mathrm{d}x'_{k+1}\,\delta_{\beta_1}(x_{k+1}-x'_{k+1}) f(\mathbf{x}'_k, x'_{k+1})\Big|.\end{aligned} \tag{8.4}$$

A slight generalization of a standard Poincaré-type inequality (see, e.g. Lemma 7.16 in [7]) yields that

$$\Big|f(\mathbf{x}'_k, x_{k+1}) - \int \mathrm{d}x'_{k+1}\,\delta_\beta(x_{k+1}-x'_{k+1})\, f(\mathbf{x}'_k, x'_{k+1})\Big| \leq C\int_{|y|\leq\beta_1} \frac{|\nabla_{k+1} f(\mathbf{x}'_k, x_{k+1}+y)|}{|y|^2}\,\mathrm{d}y \tag{8.5}$$

holds for any $\mathbf{x}'_k$ and x_{k+1}. Inserting this inequality on the r.h.s. of (8.4) and applying a Schwarz inequality we get

$$\begin{aligned}
&\Big|\int \mathrm{d}\mathbf{x}_{k+1}\mathrm{d}\mathbf{x}'_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\, \left(\delta_{\beta_1}(x'_{k+1}-x_{k+1})-\delta(x'_{k+1}-x_{k+1})\right)\delta_{\beta_2}(x_j-x_{k+1})\, f(\mathbf{x}_{k+1})\overline{f}(\mathbf{x}'_{k+1})\Big| \\
&\quad\leq \|J^{(k)}\|_\infty \int \mathrm{d}\mathbf{x}_{k+1}\mathrm{d}\mathbf{x}'_k\mathrm{d}y\, \frac{\chi(|y|\leq\beta_1)}{|y|^2}\,\delta_{\beta_2}(x_j-x_{k+1})\left(|f(\mathbf{x}_k,x_{k+1})|^2+|\nabla_{k+1}f(\mathbf{x}'_k,x_{k+1}+y)|^2\right) \\
&\quad\leq C\|J^{(k)}\|_\infty\beta_1\int \mathrm{d}\mathbf{x}_{k+1}\,\delta_{\beta_2}(x_j-x_{k+1})|f(\mathbf{x}_k,x_{k+1})|^2 \\
&\qquad+\|J^{(k)}\|_\infty\int \mathrm{d}\mathbf{x}'_k\mathrm{d}x_{k+1}\mathrm{d}y\,\frac{\chi(|y|\leq\beta_1)}{|y|^2}\,|\nabla_{k+1}f(\mathbf{x}'_k,x_{k+1}+y)|^2
\end{aligned}$$

where in the first term we integrated over $\mathrm{d}\mathbf{x}'_k$ and over $\mathrm{d}y$ (the integration over $\mathrm{d}y$ gives the factor β_1), while in the second term we integrated over $\mathrm{d}\mathbf{x}_k$ (in particular over $\mathrm{d}x_j$). To control the first term on the r.h.s. of the last equation we apply (5.4). In the second term we shift the x_{k+1} variable and then we integrate over $\mathrm{d}y$. We find, using that $U^{(k+1)}(\mathbf{x}_{k+1},\mathbf{x}'_{k+1})=f(\mathbf{x}_{k+1})\overline{f}(\mathbf{x}'_{k+1})$,

$$\begin{aligned}
&\Big|\int \mathrm{d}\mathbf{x}_{k+1}\mathrm{d}\mathbf{x}'_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\, \left(\delta_{\beta_1}(x'_{k+1}-x_{k+1})-\delta(x'_{k+1}-x_{k+1})\right)\delta_{\beta_2}(x_j-x_{k+1})\, f(\mathbf{x}_{k+1})\overline{f}(\mathbf{x}'_{k+1})\Big| \\
&\qquad\qquad\leq C\|J^{(k)}\|_\infty\,\beta_1\,\mathrm{Tr}\,(1-\Delta_{k+1})(1-\Delta_j)U^{(k+1)}
\end{aligned} \tag{8.6}$$

uniformly in β_2. In order to control the second term on the r.h.s. of (8.3), we use that

$$\begin{aligned}
&\int \mathrm{d}\mathbf{x}_{k+1}\mathrm{d}\mathbf{x}'_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\delta(x'_{k+1}-x_{k+1})\left(\delta_{\beta_2}(x_j-x_{k+1})-\delta(x_j-x_{k+1})\right)\, f(\mathbf{x}_{k+1})\overline{f}(\mathbf{x}'_{k+1}) \\
&\qquad=\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}x_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\left(\delta_{\beta_2}(x_j-x_{k+1})-\delta(x_j-x_{k+1})\right)\, f(\mathbf{x}_k,x_{k+1})\overline{f}(\mathbf{x}'_k,x_{k+1})
\end{aligned}$$

and then we apply Lemma 8.3 below, keeping all variables fixed, apart from x_j and x_{k+1}. This completes the proof of the proposition. □

Lemma 8.3. *Assume δ_β is defined as in Proposition 8.1. Then we have*

$$\begin{aligned}
&\Big|\int \mathrm{d}x\mathrm{d}z\mathrm{d}x'\, J(x,x')(\delta_\beta(x-z)-\delta(x-z))f(x,z)\overline{f}(x',z)\Big| \\
&\qquad\leq C\sqrt{\beta}\left[\,\|J\|_\infty+\|\nabla_x J\|_\infty\,\right]\Big|\int \mathrm{d}x\mathrm{d}z\overline{f}(x,z)(1-\Delta_x)(1-\Delta_z)f(x,z)\Big|\,.
\end{aligned} \tag{8.7}$$

Proof. From the version (8.5) of the Poincaré inequality, we can bound the left side of (8.7) by

$$\begin{aligned}
&C\int \mathrm{d}z\,\mathrm{d}x'\int \mathrm{d}x\; 1(|x-z|\leq 2\beta)\frac{|\nabla_x[J(x,x')f(x,z)]|}{|x-z|^2}|f(x',z)| \\
&\quad\leq C\int \mathrm{d}z\,\mathrm{d}x'\int \mathrm{d}x\; 1(|x-z|\leq 2\beta)\Big[\kappa\frac{|\nabla_x[J(x,x')f(x,z)]|^2}{|x-z|^2}+\kappa^{-1}\frac{|f(x',z)|^2}{|x-z|^2}\Big]\,.
\end{aligned} \tag{8.8}$$

In the first term we drop the restriction $1(|x-z|\leq 2\beta)$ and apply the Hardy-type inequality (5.2) to the z variable on the domain Λ. In the second term we perform the $\mathrm{d}x$ integration. Therefore the l.h.s. of (8.7) is controlled by

$$C\int \mathrm{d}z\,\mathrm{d}x'\int \mathrm{d}x\quad\Big[\kappa|\nabla_x[J(x,x')\nabla_z f(x,z)]|^2\,+\,\kappa|\nabla_x[J(x,x')f(x,z)]|^2\,+\,\kappa^{-1}\beta|f(x',z)|^2\Big]\quad. \tag{8.9}$$

The first two terms of the last expression are bounded by

$$C\kappa\left[\|J\|_\infty + \|\nabla_x J\|_\infty\right]^2\left|\int \overline{f}(x,z)(1-\Delta_x)(1-\Delta_z)f(x,z)\mathrm{d}x\mathrm{d}z\right|,$$

while the last term is bounded by $C\beta\kappa^{-1}\|f\|_2^2$. Optimizing the choice of κ, we obtain (8.7). $\square$

As a corollary of the proof of this lemma, we can prove that for all $0<\beta<1$,

$$\int \mathrm{d}x\mathrm{d}z\,\delta_\beta(x-z)|f(x,z)|^2 \le C\int \mathrm{d}x\mathrm{d}z\,\overline{f}(x,z)(1-\Delta_x)(1-\Delta_z)f(x,z)$$

with the constant C independent of β. This reproves the second part of Lemma 5.2 (originally proven in Lemma 5.3 of [6]) for the case of radially symmetric potential U and integration over a finite volume Λ.

9 Proof of the Main Theorem

The strategy for the proof of Theorem 2.1 is as follows. First, in Section 9.1, we derive a hierarchy of equation for the time evolution of the densities $U_{N,t}^{(k)}$, for large, but finite N.

Using this Hierarchy we prove in Section 9.2 that the sequence $U_{N,t}=\{U_{N,t}^{(k)}\}_{k=1}^N$ is equicontinuous in t with respect to the weak$*$ topology of $H_-^{(\nu)}$. This result will be used to prove that $U_{N,t}$ is compact in the space $C([0,T],H_-^{(\nu)})$ with respect to the topology induced by the metric $\widehat{\rho}$ (defined in (2.16)).

In Section 9.3 we prove then that any limit point of $U_{N,t}$ in $C([0,T],H_-^{(\nu)})$ (w.r.t. the metric $\widehat{\rho}$) is also a limit point of the sequence $\Gamma_{N,t}=\{\gamma_{N,t}^{(k)}\}_{k=1}^N$ (and that any limit point of $\Gamma_{N,t}$ is also a limit point of $U_{N,t}$). This result implies the compactness of $\Gamma_{N,t}$. Also part ii) and iii) follow immediately from the fact that the limit points of $U_{N,t}$ and $\Gamma_{N,t}$ coincide, and from the results of Lemma 7.1 (in particular, the bound (7.2)) and Lemma 7.3. Having part ii), part iv) of Theorem 2.1 follows easily from Proposition 8.1.

Finally, in Section 9.4, we complete the proof of Theorem 2.1, by proving part v): here we will start from the hierarchy we are going to derive in Section 9.1, and we will take the limit $N\to\infty$, using again Proposition 8.1.

9.1 Convergence to a Regularized Gross-Pitaevskii Hierarchy

In this section we begin our analysis of the hierarchy of equations governing the time evolution of the densities $U_{N,t}^{(k)}$, for large N. Here and henceforth we use the pairing

$$\langle J^{(k)},U^{(k)}\rangle = \int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\,J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)U^{(k)}(\mathbf{x}_k;\mathbf{x}'_k).$$

The following lemma will be used in Sections 9.2 and 9.4, in order to prove the equicontinuity of $U_{N,t}^{(k)}$ with respect to the weak$*$ topology of $H_-^{(\nu)}$ and to prove that any limit point of $\Gamma_{N,t}$ satisfies the Gross-Pitaevskii Hierarchy (2.24).

Proposition 9.1. *Suppose $J^{(k)}\in W^{1,\infty}(\Lambda^k\times\Lambda^k)$. For $0<\beta\le 1$ we choose a radially symmetric function $\delta_\beta\in C_0^\infty(\mathbb{R}^3)$ such that $0\le\delta_\beta(y)\le C\beta^{-3}\chi(|y|\le\beta)$ and $\int\mathrm{d}y\,\delta_\beta(y)=1$. Then, for any*

$t > 0$,

$$
\begin{aligned}
\langle J^{(k)}, U^{(k)}_{N,t}\rangle = \langle J^{(k)}, U^{(k)}_{N,0}\rangle - i \sum_{j=1}^{k} \int_0^t \mathrm{d}s \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \, J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) \, (-\Delta_j + \Delta'_j) U^{(k)}_{N,s}(\mathbf{x}_k; \mathbf{x}'_k) \\
- 8\pi i a_0 \sum_{j=1}^{k} \int_0^t \mathrm{d}s \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) \\
\times \int \mathrm{d}x_{k+1} (\delta_\beta(x_j - x_{k+1}) - \delta_\beta(x'_j - x_{k+1})) U^{(k+1)}_{N,s}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) \\
+ t\, O(\beta^{1/2}) + t\, o(1)
\end{aligned}
\tag{9.1}
$$

as $N \to \infty$.

Proof. Put $\phi_t = \phi_{N,t}(\mathbf{x})$. From

$$
i\partial_t \phi_t = L\phi_t + B\phi_t
$$

(see (6.6) and (6.7)) it follows that

$$
i\partial_t U^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k) = \int \mathrm{d}\mathbf{x}_{N-k} (W^{[k]})^2 \left[\left((L\phi_t)\overline{\phi'_t} - (\overline{L'\phi'_t})\phi_t \right) + (B - B')\phi_t \overline{\phi'_t} \right], \tag{9.2}
$$

where the superscript $'$ means that the coordinates $x_1, .., x_k$ are replaced by $x'_1, .., x'_k$. Similarly, we will use $\nabla'_m := \nabla_{x'_m}$. In most cases we will not write out the arguments of all functions fully, but it is understood that functions ϕ_t, B, Ω, etc. with unspecified arguments depend on the variables $x_1, x_2, \ldots, x_N$ and their primed versions, ϕ'_t, B', Ω', etc. depend on $x'_1, x'_2, \ldots, x'_k, x_{k+1}, \ldots, x_N$.

Next we note that for any fixed k

$$
\begin{aligned}
L = & - \sum_{m=1}^{N} \left(\Delta_m + 2\nabla_m (\log W) \nabla_m \right) \\
= & - \sum_{m=1}^{k} \Delta_m - 2 \sum_{m=1}^{N} \left(\nabla_m \log \frac{W}{W^{[k]}} \right) \nabla_m - \sum_{m=k+1}^{N} \left(\Delta_m + 2\nabla_m (\log W^{[k]}) \nabla_m \right)
\end{aligned}
$$

since $W^{[k]}$ is independent of the first k variables. The contribution of the last term $\sum_m \Delta_m + 2\nabla_m(\log W^{[k]})\nabla_m$ cancels the analogous contribution from L' in (9.2) by self adjointness on the space of the last $N - k$ variables. Moreover we use the estimate (see (6.11))

$$
B - B' = (q_{jm} - q'_{jm}) + (\widetilde{\Omega} - \widetilde{\Omega}') + O(\exp(-c\ell^{-\varepsilon}))
$$

(with the summation convention). Thus, from (9.2) we get

$$
\begin{aligned}
& i\partial_t U^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k) \\
= & \sum_{j=1}^{k} (-\Delta_j + \Delta'_j) U^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k) + \sum_{m,j=1}^{N} \int \mathrm{d}\mathbf{x}_{N-k} (W^{[k]})^2 (q_{jm} - q'_{jm}) \phi_t \overline{\phi'_t} \\
& - 2 \sum_{m=1}^{N} \int \mathrm{d}\mathbf{x}_{N-k} (W^{[k]})^2 \left[\left(\nabla_m \log \frac{W}{W^{[k]}} \right) (\nabla_m \phi_t) \overline{\phi'_t} - \left(\nabla'_m \log \frac{W'}{W^{[k]}} \right) (\nabla'_m \overline{\phi'_t}) \phi_t \right] \\
& + \int \mathrm{d}\mathbf{x}_{N-k} (W^{[k]})^2 (\widetilde{\Omega} - \widetilde{\Omega}') \phi_t \overline{\phi'_t} + O(\exp(-c\ell^{-\varepsilon})) \, .
\end{aligned}
\tag{9.3}
$$

We can rewrite this equation in integral form

$$
\begin{aligned}
U^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k) = & \, U^{(k)}_{N,0}(\mathbf{x}_k;\mathbf{x}'_k) - i\sum_{j=1}^{k}\int_0^t \mathrm{d}s\,(-\Delta_j+\Delta'_j)U^{(k)}_{N,s}(\mathbf{x}_k;\mathbf{x}'_k) \\
& - i\sum_{m,j=1}^{N}\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2(q_{jm}-q'_{jm})\phi_s\overline{\phi'_s} \\
& + 2i\sum_{m=1}^{N}\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2 \\
& \quad \times\left[\left(\nabla_m \log\frac{W}{W^{[k]}}\right)(\nabla_m\phi_s)\overline{\phi'_s} - \left(\nabla'_m \log\frac{W'}{W^{[k]}}\right)(\nabla'_m\overline{\phi'_s})\phi_s\right] \\
& - i\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2(\widetilde{\Omega}-\widetilde{\Omega}')\phi_s\overline{\phi'_s} + O(t\exp(-c\ell^{-\varepsilon}))\,.
\end{aligned}
\tag{9.4}
$$

Only the terms in the first three lines of last equation survive in the limit $N\to\infty$. The other terms vanish as $N\to\infty$, after they are tested against a function $J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\in W^{1,\infty}(\Lambda^k\times\Lambda^k)$: this is proven in Lemma 10.1 and Lemma 10.2 in Section 10. Thus we are left with

$$
\begin{aligned}
\langle J^{(k)}, U^{(k)}_{N,t}\rangle = & \, \langle J^{(k)}, U^{(k)}_{N,0}\rangle - i\sum_{j=1}^{k}\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\, J^{(k)}(-\Delta_j+\Delta'_j)U^{(k)}_{N,s} \\
& - i\sum_{j,m=1}^{N}\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}\mathbf{x}_{N-k}\, J^{(k)}(W^{[k]})^2(q_{jm}-q'_{jm})\phi_s\overline{\phi_s}' + t\,o(1)
\end{aligned}
\tag{9.5}
$$

as $N\to\infty$.

Next we note that $q_{jm}-q'_{jm}=0$ if $j,m>k$. On the other hand if $j,m\le k$, then we first use a Schwarz inequality to separate ϕ and ϕ', then we use (5.3) with $U=\chi(|x_m-x_j|\le 2\ell_1)$ and we get

$$
\begin{aligned}
& \sum_{j,m\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}\mathbf{x}_{N-k}|J^{(k)}|(W^{[k]})^2 q_{mj}|\phi_s|\,|\overline{\phi}'_s| \\
& \quad \le a\ell_1^{-3}\|J^{(k)}\|_\infty \sum_{j,m\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2\chi(|x_m-x_j|\le 2\ell_1)\left(|\phi_s|^2+|\overline{\phi}'_s|^2\right) \\
& \quad \le Cka\ell_1^{-3}\ell_1^2\|J^{(k)}\|_\infty \sum_{j=1}^{k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2\left(|\nabla_j\phi_s|^2+|\phi_s|^2+|\overline{\phi}'_s|^2\right) \\
& \quad \le C^k ka\ell_1^{-1}\|J^{(k)}\|_\infty \sum_{j=1}^{k}\left[\int \mathrm{d}\mathbf{x}\,W^2(|\nabla_j\phi_s|^2+|\phi_s|^2)+\int \mathrm{d}\mathbf{x}'\,(W')^2|\phi'_s|^2\right].
\end{aligned}
\tag{9.6}
$$

In the last line we used that the volume is finite and that $W^{[k]}\le C^k W'$ (by (C.1)). Since $a\ll\ell_1$ by assumption, the r.h.s. of (9.6) vanishes in the limit $N\to\infty$.

The argument above implies that the factor $q_{jm}-q'_{jm}$ only gives an important contribution to

(9.5) if $j \le k < m$ or $m \le k < j$. Using the permutation symmetry of the wave function we get

$$\begin{aligned} \langle J^{(k)}, U^{(k)}_{N,t}\rangle = \langle J^{(k)}, U^{(k)}_{N,0}\rangle - i \sum_{j=1}^{k} \int_0^t \mathrm{d}s \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \, J^{(k)}(-\Delta_j + \Delta'_j) U^{(k)}_{N,s} \\ - 2i(N-k) \sum_{j=1}^{k} \int_0^t \mathrm{d}s \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k} \, J^{(k)} (W^{[k]})^2 \\ \times (q(x_j - x_{k+1}) - q(x'_j - x_{k+1})) \phi_s(\mathbf{x}_k, \mathbf{x}_{N-k}) \overline{\phi_s}(\mathbf{x}'_k, \mathbf{x}_{N-k}) + t\, o(1) \end{aligned} \tag{9.7}$$

for $N \to \infty$.

In the next step we replace $W^{[k]}$ by $W^{[k+1]}$. To this end we use that

$$\begin{aligned} (W^{[k]})^2 - (W^{[k+1]})^2 = (G^{[k]}_{k+1} - 1) \prod_{m=k+2}^{N} G^{[k]}_m \\ + \sum_{i=k+2}^{N} (G^{[k]}_i - G^{[k+1]}_i) \prod_{m=k+2}^{i-1} G^{[k+1]}_m \prod_{m=i+1}^{N} G^{[k]}_m \end{aligned} \tag{9.8}$$

and estimate the effect of this difference in (9.7). For the first term in (9.8) we use

$$G^{[k]}_{k+1} = 1 - \sum_{r>k+1} w_{k+1,r} F^{[k]}_{k+1,r}$$

and its contribution to (9.7) for any fixed $j \le k$ can be bounded by

$$\begin{aligned} (N-k)\|J^{(k)}\|_\infty \sum_{r>k+1} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k} \, w_{k+1,r} F^{[k]}_{k+1,r} \left(\prod_{m=k+2}^{N} G^{[k]}_m \right) |q_{j,k+1}| |\phi_s| \, |\overline{\phi}'_s| \\ \le C a \ell_1^{-3} (N-k) \|J^{(k)}\|_\infty \sum_{r>k+1} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k} \, w_{k+1,r} F^{[k]}_{k+1,r} \left(\prod_{m=k+2}^{N} G^{[k]}_m \right) \\ \times \chi(|x_j - x_{k+1}| \le 2\ell_1) \left(|\phi_s|^2 + |\overline{\phi}'_s|^2 \right) \\ \le C a \ell_1^{-1} (N-k) \|J^{(k)}\|_\infty \sum_{r>k+1} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k} w_{k+1,r} F^{[k]}_{k+1,r} \left(\prod_{m=k+2}^{N} G^{[k]}_m \right) \\ \times \left(|\nabla_j \phi_s|^2 + |\phi_s|^2 + |\overline{\phi}'_s|^2 \right) \ , \end{aligned}$$

where we used $\chi_{j,k+1} \le C\ell_1^2 \lambda_{j,k+1}$ and we applied the usual Hardy inequality (5.1) for the variable x_j. Next we use $w \le Ca\ell_1\lambda$, $F^{[k]}_{k+1,r} \le CF_{k+1,r}$ and we estimate $\prod_m G^{[k]}_m \le C^{k+1} W^2$ (following from Lemma C.1), so that we can use (5.7) from Lemma 5.3 with respect to the variable x_{k+1}. We find, for any fixed j,

$$\begin{aligned} (N-k) \sum_{r>k+1} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}_{N-k} \, w_{k+1,r} F^{[k]}_{k+1,r} \left(\prod_{m=k+2}^{N} G^{[k]}_m \right) |q_{j,k+1}| |\phi_s| \, |\overline{\phi}'_s| \\ \le C a \ell_1^{-1} (N-k) \Big\{ a\ell_1 \int \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{x}'_k \mathrm{d}\, W^2 \Big[|\nabla_{k+1} \nabla_j \phi_s|^2 + |\nabla_{k+1} \phi_s|^2 + |\nabla_{k+1} \overline{\phi}'_s|^2 \Big] \\ + a\ell_1^2 \ell^{-3} \int \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{x}'_k \, W^2 \Big[|\nabla_j \phi_s|^2 + |\phi_s|^2 + |\overline{\phi}'_s|^2 \Big] \Big\} \ . \end{aligned} \tag{9.9}$$

Finally we apply the estimate $W \le C^k W'$ for the terms with ϕ' to obtain the energy norm. Using Corollary 6.2 and the finiteness of the volume we find (with $a\ell_1 \ll \ell^3$)

$$(N-k)\sum_{j<k}\sum_{r>k+1}\int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\, w_{k+1,r} F^{[k]}_{k+1,r}\left(\prod_{m=k+2}^{N} G^{[k]}_m\right)|q_{j,k+1}||\phi_s|\,|\overline{\phi}'_s|$$
$$\le C_k(a + a\ell_1\ell^{-3}) \to 0 \qquad \text{as } N\to\infty\,.$$

The other contributions coming from the second term of (9.8) can be bounded analogously. From (9.7), and since

$$\int \mathrm{d}x_{k+2}\dots\mathrm{d}x_N\,(W^{[k+1]})^2\phi_s(\mathbf{x}_k,\mathbf{x}_{N-k})\overline{\phi_s}(\mathbf{x}'_k,\mathbf{x}_{N-k}) = U^{(k+1)}_{N,s}(\mathbf{x}_k,x_{k+1};\mathbf{x}'_k,x_{k+1})$$

it follows that

$$\begin{aligned}\langle J^{(k)},U^{(k)}_{N,t}\rangle = \langle J^{(k)},U^{(k)}_{N,0}\rangle &- i\sum_{j=1}^{k}\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\,(-\Delta_j+\Delta'_j)U^{(k)}_{N,s}(\mathbf{x}_k;\mathbf{x}'_k)\\ &- 2i(N-k)\sum_{j=1}^{k}\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}x_{k+1}\,J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\\ &\qquad\times(q(x_j-x_{k+1})-q(x'_j-x_{k+1}))U^{(k+1)}_{N,s}(\mathbf{x}_k,x_{k+1};\mathbf{x}'_k,x_{k+1})\\ &+ t\,o(1)\,.\end{aligned}\tag{9.10}$$

Note that the function q depends on N, and that $Nq(x)$ approaches $4\pi a_0$ times a Dirac delta function as $N\to\infty$ (Lemma A.2). Using Proposition 8.1 (with $\beta_1 = 0$, that is $\delta_{\beta_1}(x) = \delta(x)$, and $\beta_2 = 3/2\ell_1$) we can replace $Nq(x)$ by $4\pi a_0$ times a δ-function. We can also replace $Nq(x)$ by $4\pi a_0$ times a smoothed version of the δ-function at some fixed length scale β. More precisely, we choose a radially symmetric function $\delta_\beta \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \delta_\beta(x) \le C\beta^{-3}\chi(|x|\le\beta)$ and $\int \mathrm{d}x\,\delta_\beta(x) = 1$. By the assumption $J^{(k)} \in W^{1,\infty}(\Lambda^k\times\Lambda^k)$ and the a-priori bound from Lemma 7.1, we have

$$\begin{aligned}\Big|\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}x_{k+1}\,J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\,&[(N-k)q(x_j-x_{k+1}) - 4\pi a_0\delta_\beta(x_j-x_{k+1})]\\ &\times U^{(k+1)}_{N,s}(\mathbf{x}_k,x_{k+1};\mathbf{x}'_k,x_{k+1})\Big| \le Ct(\ell_1^{1/2}+\beta^{1/2})\,,\end{aligned}$$

for some constant C depending on k and $J^{(k)}$, but independent of N and β. Here we used Lemma 8.3 twice, once for the function q with lengthscale ℓ_1 (Lemma A.2) and once for the function δ_β. From (9.10) we find

$$\begin{aligned}\langle J^{(k)},U^{(k)}_{N,t}\rangle = \langle J^{(k)},U^{(k)}_{N,0}\rangle &- i\sum_{j=1}^{k}\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\,(-\Delta_j+\Delta'_j)U^{(k)}_{N,s}(\mathbf{x}_k;\mathbf{x}'_k)\\ &- 8\pi i a_0\sum_{j=1}^{k}\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\\ &\times\int \mathrm{d}x_{k+1}(\delta_\beta(x_j-x_{k+1})-\delta_\beta(x'_j-x_{k+1}))U^{(k+1)}_{N,s}(\mathbf{x}_k,x_{k+1};\mathbf{x}'_k,x_{k+1})\\ &+ tO(\beta^{1/2}) + t\,o(1)\end{aligned}$$

as $N\to\infty$. □

9.2 Compactness of $U_{N,t}$

Using Proposition 9.1 we can now prove the equicontinuity of $U_{N,t}^{(k)}$ in t with respect to the metric $\widehat{\rho}$ on $C([0,T],H_-^{(\nu)})$, and thus the compactness of $U_{N,t}$.

We recall the bound

$$\|U_N\|_{C([0,T],H_-^{(\nu)})} = \sup_{t\in[0,T]} \sum_{k\geq 1} \nu^{-k} \|U_{N,t}^{(k)}\|_2 \leq 1 \tag{9.11}$$

for some sufficiently large $\nu > 1$ (Lemma 7.1) and we recall the definition of the metric $\widehat{\rho}$ from (2.16). In order to prove the compactness of the sequence $U_{N,t}$ with respect to the topology induced on $C([0,T],H_-^{(\nu)})$ by the metric $\widehat{\rho}$, it is enough, by the Arzela-Ascoli Theorem, to prove the equicontinuity of the sequence $U_{N,t}$.

Lemma 9.2. *The sequence of families of density matrices $U_{N,t} = \{U_{N,t}^{(k)}\}_{k=1}^N$, $N = 1,2,\dots$ satisfying (9.11) is equicontinuous on $H_-^{(\nu)}$ with respect to the metric ρ if and only if for every fixed $k \geq 1$, for arbitrary $J^{(k)} \in W^{1,\infty}(\Lambda^k \times \Lambda^k)$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left| \langle J^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)} \rangle \right| \leq \varepsilon \tag{9.12}$$

whenever $|t-s| \leq \delta$.

Proof. Equicontinuity in the metric ρ means that, for any $\varepsilon > 0$ there exists $\delta > 0$ (independent of N), such that

$$\rho(U_{N,t}, U_{N,s}) = \sum_{j=1}^{\infty} 2^{-j} \Big| \sum_{k\geq 1} \langle J_j^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)} \rangle \Big| \leq \varepsilon \tag{9.13}$$

whenever $|t-s| \leq \delta$. Recall that $J_j = \{J_j^{(k)}\}_{k\geq 1}$, for $j \geq 1$ was chosen as a dense countable subset of the unit ball of $H_+^{(\nu)}$. Using the uniform bound (9.11), one can approximate any given $J = (J^{(1)}, J^{(2)}, \dots) \in H_+^{(\nu)}$ by an appropriate finite linear combinations of J_j and one can easily prove that (9.13) implies (9.12).

On the other hand it is clear, by a standard approximation argument (and because $W^{1,\infty}(\Lambda^k \times \Lambda^k)$ is dense in $L^2(\Lambda^k \times \Lambda^k)$), that (9.12) for all $J^{(k)} \in W^{1,\infty}(\Lambda^k \times \Lambda^k)$ implies the same bound for all $J^{(k)} \in L^2(\Lambda^k \times \Lambda^k)$. To prove that (9.12) for all $J^{(k)} \in L^2(\Lambda^k \times \Lambda^k)$ implies (9.13) one can proceed as follow. Given $\varepsilon > 0$, we note that

$$\begin{aligned}\sum_{j>m} 2^{-j} \Big| \sum_{k\geq 1} \langle J_j^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)} \rangle \Big| &\leq \sum_{j>m} 2^{-j} \|J_j\|_{H_+^{(\nu)}} \left(\|U_{N,t}\|_{H_-^{(\nu)}} + \|U_{N,s}\|_{H_-^{(\nu)}} \right) \\ &\leq 2 \sum_{j>m} 2^{-j} \leq \varepsilon/3\end{aligned}$$

if we choose m sufficiently large. Hence

$$\sum_{j\geq 1} 2^{-j} \Big| \sum_{k\geq 1} \langle J_j^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)} \rangle \Big| \leq \varepsilon/3 + \sum_{j\leq m} 2^{-j} \Big| \sum_{k\geq 1} \langle J_j^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)} \rangle \Big|. \tag{9.14}$$

Moreover we note that

$$\begin{aligned}\sum_{j\leq m} 2^{-j} \Big| \sum_{k>p} \langle J_j^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)} \rangle \Big| &\leq 2 \sup_{t\in[0,T]} \|U_N(t)\|_{H_-^{(\nu)}} \sum_{j\leq m} 2^{-j} \left(\sup_{k>p} \nu^k \|J_j^{(k)}\|_{k,+} \right) \\ &\leq \varepsilon/3\end{aligned}$$

if we choose p large enough, depending on m, because $\lim_{k\to\infty}\nu^k \|J_j^{(k)}\|_{k,+} = 0$ for every j. From (9.14) we therefore have

$$\sum_{j\geq 1} 2^{-j}\Big|\sum_{k\geq 1}\langle J_j^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)}\rangle\Big| \leq 2\varepsilon/3 + \sum_{j\leq m} 2^{-j}\Big|\sum_{k\leq p}\langle J_j^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)}\rangle\Big|. \tag{9.15}$$

Now, for every $j \leq m$ and $k \leq p$, we can find $\delta_{jk} > 0$ such that

$$\Big|\langle J_j^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)}\rangle\Big| \leq 2^{-k}\varepsilon/3$$

if $|t-s| \leq \delta_{jk}$. Thus, for $|t-s| \leq \delta := \min\{\delta_{jk} \,:\, j \leq m,\ k \leq p\}$, we have

$$\sum_{j\geq 1} 2^{-j}\Big|\sum_{k\geq 1}\langle J_j^{(k)}, U_{N,t}^{(k)} - U_{N,s}^{(k)}\rangle\Big| \leq \varepsilon.$$

This proves that (9.12) implies (9.13). □

Lemma 9.3. *The sequence $U_{N,t} = \{U_{N,t}^{(k)}\}_{k=1}^N \in C([0,T], H_-^{(\nu)})$ is equicontinuous in t with respect to the metric ρ (defined in (2.15)). In particular, by the Arzela-Ascoli Theorem, the sequence $U_{N,t}$ is compact in $C([0,T], H_-^{(\nu)})$ with respect to $\widehat{\rho}$.*

Proof. We prove (9.12) for all $J^{(k)} \in W^{1,\infty}(\Lambda^k \times \Lambda^k)$. For such $J^{(k)}$ we can apply Proposition 9.1 and we find

$$\begin{aligned}
&\Big|\langle J^{(k)}, U_{N,t_1}^{(k)}\rangle - \langle J^{(k)}, U_{N,t_2}^{(k)}\rangle\Big| \\
&\quad \leq \sum_{j=1}^k \int_{t_1}^{t_2} \mathrm{d}s \,\Big|\langle J^{(k)}, (-\Delta_j + \Delta_j')U_{N,s}^{(k)}\rangle\Big| \\
&\quad\quad + 2a_0 \sum_{j=1}^k \int_{t_1}^{t_2} \mathrm{d}s \Big|\int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}_k' J^{(k)}(\mathbf{x}_k;\mathbf{x}_k') \\
&\quad\quad \times \int \mathrm{d}x_{k+1}(\delta_\beta(x_j - x_{k+1}) - \delta_\beta(x_j' - x_{k+1}))U_{N,s}^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}_k', x_{k+1})\Big| \\
&\quad\quad + |t_1 - t_2|\, O(\beta^{1/2}) + o(|t_1 - t_2|)\,.
\end{aligned} \tag{9.16}$$

Next we bound

$$\Big|\langle J^{(k)}, (-\Delta_j + \Delta_j')U_{N,s}^{(k)}\rangle\Big| \leq \Big|\mathrm{Tr} J^{(k)}\Delta_j U_{N,s}^{(k)}\Big| + \Big|\mathrm{Tr}\, J^{(k)} U_{N,s}^{(k)}\Delta_j\Big|,$$

and use

$$\Big|\mathrm{Tr}\, J^{(k)}\Delta_j U_{N,s}^{(k)}\Big| \leq \|S_j^{-1}J^{(k)}S_j\|\|S_j^{-1}\Delta_j S_j^{-1}\|\mathrm{Tr}\Big|S_j U_{N,s}^{(k)} S_j\Big|\,. \tag{9.17}$$

Using that $J^{(k)} \in W^{1,\infty}(\Lambda^k \times \Lambda^k)$ and the finiteness of the volume of Λ we obtain

$$\int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}_k'\, |\nabla_j J(\mathbf{x}_k;\mathbf{x}_k')|^2 < \infty.$$

It follows that $S_j J^{(k)}$ is a Hilbert-Schmidt operator, and thus compact. This implies in particular that $S_j^{-1} J^{(k)} S_j$ is a bounded operator, and thus, by Lemma 7.1, the r.h.s. of (9.17) is bounded. Analogously, it also follows that $|\mathrm{Tr}\, J^{(k)} U_{N,s}^{(k)}\Delta_j|$ is bounded, uniformly in N and s.

As for the second term on the r.h.s. of (9.16), we use that for fixed β, δ_β is bounded, $J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)$ is bounded and $\mathrm{Tr}\, U^{(k+1)}_{N,s} \leq 1$: we obtain

$$\left| \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}x_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)(\delta_\beta(x_j - x_{k+1}) - \delta_\beta(x'_j - x_{k+1}))U^{(k+1)}_{N,s}(\mathbf{x}_k, x_{k+1};\mathbf{x}'_k, x_{k+1})\right| \leq C \tag{9.18}$$

for a constant C, independent of N and of s. It follows from (9.16) that, for any fixed $J^{(k)} \in W^{1,\infty}(\Lambda^k \times \Lambda^k)$,

$$\left|\langle J^{(k)}, U^{(k)}_{N,t_1}\rangle - \langle J^{(k)}, U^{(k)}_{N,t_2}\rangle\right| \leq C|t_1 - t_2|$$

for a constant C which depends on k and on $J^{(k)}$, but is independent of N and of t_1 and t_2. This implies, by Lemma 9.2, that $U_N(t)$ is compact w.r.t. the topology induced by the metric $\widehat{\rho}$ on $C([0,T], H^{(\nu)}_-)$. □

9.3 Compactness of $\Gamma_{N,t}$

The aim of this section is to prove part i) of Theorem 2.1, stating the compactness of the sequence $\Gamma_{N,t} = \{\gamma^{(k)}_{N,t}\}_{k=1}^N$ in $C([0,T], B^{(\nu)}_-)$ with respect to the metric $\widehat{\rho}$.

First of all, we note that, for any $\nu > 2$,

$$\|\Gamma_{N,t}\|_{H^{(\nu)}_-} = \sum_{k\geq 1} \nu^{-k} \|\gamma^{(k)}_{N,t}\|_2 \leq 1$$

because $\|\gamma^{(k)}_{N,t}\|_2 \leq 1$ for every k. Thus $\Gamma_{N,t} \in B^{(\nu)}_-$.

In order to prove the compactness of $\Gamma_{N,t}$ we use that, by Lemma 9.3, the sequence $U_{N,t}$ is compact. It only remains to prove that limit points of $U_{N,t}$ are also limit points of $\Gamma_{N,t}$. This is the aim of the next two lemmas.

Recall that

$$\gamma^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k) = \int \mathrm{d}\mathbf{x}_{N-k}\, W(\mathbf{x}_k, \mathbf{x}_{N-k})W(\mathbf{x}'_k, \mathbf{x}_{N-k})\phi_{N,t}(\mathbf{x}_k, \mathbf{x}_{N-k})\overline{\phi}_{N,t}(\mathbf{x}'_k, \mathbf{x}_{N-k})$$

are the marginal densities corresponding to $\psi_{N,t}(x) = W(x)\phi_{N,t}(x)$, while

$$U^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k) = \int \mathrm{d}\mathbf{x}_{N-k}\, W^{[k]}(\mathbf{x}_{N-k})^2 \phi_{N,t}(\mathbf{x}_k, \mathbf{x}_{N-k})\overline{\phi}_{N,t}(\mathbf{x}'_k, \mathbf{x}_{N-k}).$$

Recall that $W^{[k]} = W^{(1\ldots k)}$ denotes the wave function W after removing its dependence on $\mathbf{x}_k = (x_1, \ldots, x_k)$ (see (4.3)).

Lemma 9.4. *Assume that $\Gamma_{N,t} = \{\gamma^{(k)}_{N,t}\}_{k=1}^N$ and $U_{N,t} = \{U^{(k)}_{N,t}\}_{k=1}^N$ are defined as above and that the assumptions of Theorem 2.1 are satisfied. Then we have, for every fixed $k \geq 1$ and $t \in [0,T]$,*

$$\int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \left|\gamma^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k) - U^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k)\right| \to 0 \qquad \text{as } N \to \infty.$$

Proof. In this proof k is considered fixed and all constants may depend on it. We have

$$\begin{aligned}\int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \left|\gamma^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k) - U^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k)\right| \leq & \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\, |\phi_{N,t}(\mathbf{x}_k, \mathbf{x}_{N-k})||\phi_{N,t}(\mathbf{x}'_k, \mathbf{x}_{N-k})| \\ & \times \left|W(\mathbf{x}_k, \mathbf{x}_{N-k})W(\mathbf{x}'_k, \mathbf{x}_{N-k}) - W^{[k]}(\mathbf{x}_{N-k})^2\right|.\end{aligned}$$

Using the shorthand notation $\phi = \phi_{N,t}(\mathbf{x}_k, \mathbf{x}_{N-k})$, $\phi' = \phi_{N,t}(\mathbf{x}'_k, \mathbf{x}_{N-k})$, and $W = W(\mathbf{x}_k, \mathbf{x}_{N-k})$, $W' = W(\mathbf{x}'_k, \mathbf{x}_{N-k})$, and $W^{[k]} = W^{[k]}(\mathbf{x}_{N-k})$, we need to bound

$$\int |WW' - (W^{[k]})^2||\phi||\phi'| \leq \int |W - W^{[k]}|W'|\phi||\phi'| + \int |W' - W^{[k]}|W^{[k]}|\phi||\phi'| \,. \tag{9.19}$$

Using $W = \prod_{j=1}^N G_j^{1/2}$ and $W^{[k]} = \prod_{j=k+1}^N (G_j^{[k]})^{1/2}$, we get

$$\begin{aligned}|W - W^{[k]}| \leq &\left|1 - \prod_{j=1}^k G_j^{1/2}\right| \prod_{j=k+1}^N G_j^{1/2} \\ &+ \sum_{m=k+1}^N \left(\prod_{j=k+1}^{m-1} G_j^{1/2}\right) \left|G_m^{1/2} - (G_m^{[k]})^{1/2}\right| \left(\prod_{j=m+1}^N (G_j^{[k]})^{1/2}\right) \,.\end{aligned}$$

Thus, the first term on the r.h.s. of (9.19) can be bounded by

$$\begin{aligned}\int |W - W^{[k]}|W'|\phi||\phi'| \leq &\int \left|1 - \prod_{j=1}^k G_j^{1/2}\right| \left(\prod_{j=k+1}^N G_j^{1/2}\right) W'|\phi||\phi'| \\ &+ \sum_{m=k+1}^N \int \left(\prod_{j=k+1}^{m-1} G_j^{1/2}\right) \left|G_m^{1/2} - (G_m^{[k]})^{1/2}\right| \left(\prod_{j=m+1}^N (G_j^{[k]})^{1/2}\right) W'|\phi||\phi'| \,.\end{aligned}$$

Applying Schwarz inequality, with some $\alpha > 0$ which will be specified later on, we find

$$\begin{aligned}\int \left|W - W^{[k]}\right| W'|\phi||\phi'| \leq &\,\alpha \int \left|1 - \prod_{j=1}^k G_j^{1/2}\right|^2 \left(\prod_{j=k+1}^N G_j\right) |\phi|^2 \\ &+ \alpha \sum_{m=k+1}^N \int \left(\prod_{j=k+1}^{m-1} G_j\right) \left|G_m^{1/2} - (G_m^{[k]})^{1/2}\right|^2 \left(\prod_{j=m+1}^N G_j^{[k]}\right) |\phi|^2 \\ &+ CN\alpha^{-1} \int (W')^2|\phi'|^2 \\ \leq &\, C\alpha \int \left|1 - \prod_{j=1}^k G_j^{1/2}\right|^2 W^2|\phi|^2 \\ &+ C\alpha \sum_{m=k+1}^N \int \left|G_m^{1/2} - (G_m^{[k]})^{1/2}\right|^2 W^2|\phi|^2 + CN\alpha^{-1} \int (W')^2|\phi'|^2 \,.\end{aligned} \tag{9.20}$$

Next we note that, since $0 < G_j \leq 1$, we have

$$\left|1 - \prod_{j=1}^k G_j^{1/2}\right| \leq \sum_{j \leq k} \sum_m w_{jm} F_{jm} \,.$$

Using (5.7) summed up for all $1 \leq j \leq k$ with the help of (B.2), the first term on the r.h.s. of (9.20) can be bounded by

$$\int \Big|1-\prod_{j=1}^{k} G_j^{1/2}\Big|^2 W^2|\phi|^2 \le \sum_{j\le k}\sum_{m}\int w_{jm}F_{jm}W^2|\phi|^2$$
$$\le a\ell_1 \sum_{j\le k}\int W^2|\nabla_j\phi|^2 + a\ell_1^2\ell^{-3}k\int W^2|\phi|^2$$
$$\le C_k(a\ell_1 + a\ell_1^2\ell^{-3}) = o(N^{-1}\ell^{\delta})$$

for $N\to\infty$ and for some $\delta>0$ (because $\ell_1 \ll \ell^{3/2}$). As for the second term on the r.h.s. of (9.20) we note that, since $G_j \ge c_1 > 0$ (pointwise, for N large enough), we have for each fixed m

$$|G_m^{1/2} - (G_m^{[k]})^{1/2}| \le C|G_m - G_m^{[k]}| \le \sum_{n\le k} w_{mn}F_{mn} + \sum_{n>k} w_{mn}|F_{mn} - F_{mn}^{[k]}|$$
$$\le \sum_{n\le k} w_{mn}F_{mn} + \ell^{-\varepsilon}\sum_{n>k}\sum_{r\le k} w_{mn}\theta_{mr}F_{mn}^{[k]} + O(\ell^{K-\varepsilon})$$

using (C.4). Thus, using Lemma B.1 and the fact that $w \le \widetilde{\chi}$, we find,

$$\sum_{m>k}\int W^2\Big|G_m^{1/2} - (G_m^{[k]})^{1/2}\Big|^2|\phi|^2 \le C\sum_{m>k}\sum_{n_1,n_2\le k}\int W^2 w_{mn_1}w_{mn_2}F_{mn_1}F_{mn_2}|\phi|^2 + O(\ell^{K-\varepsilon})$$
$$+ C\ell^{-2\varepsilon}\sum_{m,n_1,n_2>k}\sum_{r_1,r_2\le k}\int W^2 w_{mn_1}w_{mn_2}F_{mn_1}^{[k]}F_{mn_2}^{[k]}\theta_{mr_1}\theta_{mr_2}|\phi|^2$$
$$\le C\sum_{m>k}\sum_{n\le k}\int W^2 w_{mn}^2 F_{mn}|\phi|^2$$
$$+ C\ell^{-2\varepsilon}\sum_{m,n>k}\sum_{r\le k}\int W^2 w_{mn}^2 F_{mn}^{[k]}\theta_{mr}|\phi|^2 + O(\ell^{K-\varepsilon})\,.$$

Using the estimates (5.6), Lemma 5.3 and (5.3) with $U = \theta_{mr}$, we obtain

$$\sum_{m=k+1}^{N}\int W^2\Big|G_m^{1/2} - (G_m^{[k]})^{1/2}\Big|^2|\phi|^2$$
$$\le Ca^2\sum_{n\le k}\int W^2|\nabla_n\phi|^2 + Ca^2\ell_1\ell^{-3}k\int W^2|\phi|^2$$
$$+ C\ell^{-2\varepsilon}(\ell|\log\ell|)^2a^2\sum_{m>k}\sum_{r\le k}\int W^2(|\nabla_m\nabla_r\phi|^2 + k|\nabla_m\phi|^2)$$
$$+ C\ell^{-2\varepsilon}(\ell|\log\ell|)^2a^2\ell_1\ell^{-3}N\sum_{r\le k}\int W^2(|\nabla_r\phi|^2 + |\phi|^2) + O(\ell^{K-\varepsilon})$$
$$\le C\Big(a^2 + a^2\ell_1\ell^{-3} + a\ell^{-2\varepsilon}(\ell|\log\ell|)^2 + a\ell_1\ell^{-1-2\varepsilon}|\log\ell|^2\Big) + O(\ell^{K-\varepsilon})$$
$$= o(N^{-1}\ell^{\delta})$$

as $N\to\infty$, for some sufficiently small $\delta>0$ (here we use (6.14), that $\ell_1 \ll \ell^{3/2}$, $a\ell_1 \ll \ell^4$ and that $\varepsilon < 1/10$). So choosing $\alpha = N\ell^{-\delta}$ (for some δ small enough), we get, from (9.20),

$$\int |W - W^{[k]}|W'|\phi||\phi'| \le o(1)$$

for $N \to \infty$. Analogously we can bound the second term on the r.h.s. of (9.19). $\square$

Lemma 9.5. *For any increasing subsequence N_j, the subsequence $U_{N_j,t}$ converges if and only if $\Gamma_{N_j,t}$ converges (the convergence is in $C([0,T], H_-^{(\nu)})$ with respect to the metric $\widehat{\rho}$). Moreover the limits coincide.*

Proof. Suppose that, for a given subsequence N_j, $U_{N_j,t} \to U_{\infty,t} = \{U_{\infty,t}^{(k)}\}_{k\geq 1}$ as $j \to \infty$, with respect to the metric $\widehat{\rho}$. Then we prove that $\Gamma_{N_j,t} \to U_{\infty,t}$ w.r.t. $\widehat{\rho}$ for $j \to \infty$. Since $\Gamma_{N_j,t} \in B_-^{(\nu)}$ (the unit ball of $H_-^{(\nu)}$) it is enough to prove that for every fixed $k \geq 1$ and $t \in [0,T]$, and for all $J^{(k)}$ from any dense subset of $L^2(\Lambda^k \times \Lambda^k)$,

$$\int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) \left(\gamma_{N_j,t}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) - U_{\infty,t}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\right) \to 0$$

as $j \to \infty$. Assume now that $J^{(k)} \in W^{0,\infty}(\Lambda^k \times \Lambda^k)$ (which is a dense subset of $L^2(\Lambda^k \times \Lambda^k)$). Then we have

$$\begin{aligned} \Big| \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)&(\gamma_{N_j,t}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) - U_{\infty,t}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k))\Big| \\ &\leq \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \left|\gamma_{N_j,t}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) - U_{N_j,t}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\right| \\ &\quad + \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) \left(U_{N_j,t}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) - U_{\infty,t}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\right) . \end{aligned}$$

The second term converges to zero, as $j \to \infty$, because we assumed that $U_{N_j,t} \to U_{\infty,t}$ w.r.t. the metric $\widehat{\rho}$ as $j \to \infty$ (and because $U_{N_j,t} \in B_-^{(\nu)}$). The first term converges to zero as $j \to \infty$, by Lemma 9.4. This proves that $\Gamma_{N_j,t} \to U_{\infty,t}$ w.r.t. $\widehat{\rho}$ as $j \to \infty$. Analogously one can prove that, if $\Gamma_{N_j,t} \to \Gamma_{\infty,t}$, then also $U_{N_j,t} \to \Gamma_{\infty,t}$. $\square$

The last lemma, together with Lemma 9.3, implies that the sequence $\Gamma_{N,t}$ is compact, and completes the proof of part i) of Theorem 2.1.

Part ii) of Theorem 2.1 follows from Lemma 9.5 and from (7.2).

Part iii) of Theorem 2.1 follows on the other hand by Lemmas 9.5 and 7.3.

Part iv) follows by the remark after Proposition 8.1, using that $\Gamma_{\infty,t}$ satisfies the bound from part ii) of Theorem 2.1, $\mathrm{Tr}\,(1-\Delta_i)(1-\Delta_j)\gamma_{\infty,t}^{(k)} \leq C^k$, for all $i \neq j$.

9.4 Convergence to the Gross-Pitaevskii Hierarchy

In this section we prove that the limit point $U_{\infty,t}$ satisfies the Gross-Pitaevskii Hierarchy, in the sense of (2.24).

Proof of part v) of Theorem 2.1. Using the assumption $J^{(k)} \in W^{2,\infty}(\Lambda^k \times \Lambda^k)$ we can apply Proposition 9.1. We find

$$\begin{aligned} \langle J^{(k)}, U_{N,t}^{(k)}\rangle = \langle J^{(k)}, U_{N,0}^{(k)}\rangle - i\sum_{j=1}^k \int_0^t \mathrm{d}s \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\,(-\Delta_j + \Delta'_j) U_{N,s}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) \\ - 8\pi i a_0 \sum_{j=1}^k \int_0^t \mathrm{d}s \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) \\ \times \int \mathrm{d}x_{k+1} (\delta_\beta(x_j - x_{k+1}) - \delta_\beta(x'_j - x_{k+1})) U_{N,s}^{(k+1)}(\mathbf{x}_k, x_{k+1};\mathbf{x}'_k, x_{k+1}) \\ + tO(\beta^{1/2}) + to(1) \end{aligned} \tag{9.21}$$

as $N \to \infty$.

By passing to a subsequence and by Lemma 9.5, we can assume here that $U_{N,t} \to \Gamma_{\infty,t} = \{\gamma^{(k)}_{\infty,t}\}_{k\geq 1}$ with respect to the topology induced by the metric $\widehat{\rho}$ on $C([0,T], H^{(\nu)}_{-})$. This in particular implies that $U^{(k)}_{N,t} \to \gamma^{(k)}_{\infty,t}$ for every $k \geq 1$ and for every $t \in [0,T]$ w.r.t. the weak topology of $L^2(\Lambda^k \times \Lambda^k)$. Since $J^{(k)} \in W^{2,\infty}(\Lambda^k \times \Lambda^k)$ and $|\Lambda| < \infty$, we have $J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) \in L^2(\Lambda^k \times \Lambda^k)$. This means that

$$\langle J^{(k)}, U^{(k)}_{N,t} - \gamma^{(k)}_{\infty,t}\rangle \to 0 \quad \text{and} \quad \langle J^{(k)}, U^{(k)}_{N,0} - \gamma^{(k)}_{\infty,0}\rangle \to 0$$

as $N \to \infty$.

Moreover, since $J^{(k)} \in W^{2,\infty}(\Lambda^k \times \Lambda^k)$, it also follows that $\Delta_{x_j} J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k)$ and $\Delta_{x'_j} J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k)$ are elements of $L^2(\Lambda^k \times \Lambda^k)$. This and the fact that $U_{N,t} \to \Gamma_{\infty,t}$ w.r.t. the topology induced by the metric $\widehat{\rho}$, imply that

$$\sup_{s\in[0,T]} \sum_{j=1}^{k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \, \Delta_j J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) \left(U^{(k)}_{N,s}(\mathbf{x}_k; \mathbf{x}'_k) - \gamma^{(k)}_{\infty,s}(\mathbf{x}_k; \mathbf{x}'_k)\right) \to 0$$

as $N \to \infty$.

Finally we consider the limit $N \to \infty$ of the last term on the r.h.s. of (9.21). From the proof of Proposition 8.1 (see in particular (8.6)), we have

$$\begin{aligned} \Big| \int & \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}x_{k+1} \mathrm{d}x'_{k+1} J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) \delta_\beta(x_j - x_{k+1}) \\ & \times \left(\delta_\eta(x_{k+1} - x'_{k+1}) - \delta(x_{k+1} - x'_{k+1})\right) U^{(k+1)}_{N,s}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x'_{k+1}) \Big| \\ & \leq C\, \eta\, \|J^{(k)}\|_\infty \mathrm{Tr}\; S_{k+1} S_j U^{(k+1)}_{N,s} S_j S_{k+1} \end{aligned}$$

for some finite constant C independent of $s \in [0,T]$, of N, and of β. In particular,

$$\begin{aligned} \int & \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}x_{k+1} \, J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) \left(\delta_\beta(x_j - x_{k+1}) - \delta_\beta(x'_j - x_{k+1})\right) U^{(k+1)}_{N,s}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) \\ & = \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}x_{k+1} \mathrm{d}x'_{k+1} \, J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) \left(\delta_\beta(x_j - x_{k+1}) - \delta_\beta(x'_j - x_{k+1})\right) \\ & \qquad \times \delta_\eta(x_{k+1} - x'_{k+1}) U^{(k+1)}_{N,s}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x'_{k+1}) + O(\eta)\,. \end{aligned}$$

On the r.h.s. of the last equation we can now let $N \to \infty$ keeping β and η fixed. By the assumptions on $J^{(k)}$ and by the choice of the functions δ_β and δ_η, it is easy to see that $J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k)\delta_\beta(x_j - x_{k+1})\delta_\eta(x_{k+1} - x'_{k+1})$ is an element of $L^2(\Lambda^{k+1} \times \Lambda^{k+1})$ for any fixed β and η. Hence

$$\begin{aligned} \int & \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}x_{k+1} \mathrm{d}x'_{k+1} \, J^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) \left(\delta_\beta(x_j - x_{k+1}) - \delta_\beta(x'_j - x_{k+1})\right) \delta_\eta(x_{k+1} - x'_{k+1}) \\ & \times \left(U^{(k+1)}_{N,s}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x'_{k+1}) - \gamma^{(k+1)}_{\infty,s}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x'_{k+1})\right) \to 0 \end{aligned}$$

for $N\to\infty$, uniformly in s. So, after taking the limit $N\to\infty$, (9.21) becomes

$$
\begin{aligned}
\langle J^{(k)},\gamma^{(k)}_{\infty,t}\rangle = \langle J^{(k)},\gamma^{(k)}_{\infty,0}\rangle &- i\sum_{j=1}^k\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\,(-\Delta_j+\Delta'_j)\gamma^{(k)}_{\infty,s}(\mathbf{x}_k;\mathbf{x}'_k)\\
&- 8\pi i a_0\sum_{j=1}^k\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}x_{k+1}\mathrm{d}x'_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\delta_\eta(x_{k+1}-x'_{k+1})\\
&\qquad\times(\delta_\beta(x_j-x_{k+1})-\delta_\beta(x'_j-x_{k+1}))\gamma^{(k+1)}_{\infty,s}(\mathbf{x}_k,x_{k+1};\mathbf{x}'_k,x'_{k+1})\\
&+O(\beta^{1/2})+O(\eta)
\end{aligned}
$$

for any fixed t and k. Next we apply Proposition 8.1 to replace $\delta_\eta(x_{k+1}-x'_{k+1})$ by $\delta(x_{k+1}-x'_{k+1})$ and $\delta_\beta(x_j-x_{k+1})$ (respectively, $\delta_\beta(x'_j-x_{k+1})$) by $\delta(x_j-x_{k+1})$ (respectively, by $\delta(x'_j-x_{k+1})$). The error here is of order $\beta^{1/2}+\eta$. Hence, letting $\eta\to 0$ and $\beta\to 0$ we find

$$
\begin{aligned}
\langle J^{(k)},\gamma^{(k)}_{\infty,t}\rangle = \langle J^{(k)},\gamma^{(k)}_{\infty,0}\rangle &- i\sum_{j=1}^k\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\,(-\Delta_j+\Delta'_j)\gamma^{(k)}_{\infty,s}(\mathbf{x}_k;\mathbf{x}'_k)\\
&- 8\pi i a_0\sum_{j=1}^k\int_0^t \mathrm{d}s\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}x_{k+1}\, J^{(k)}(\mathbf{x}_k;\mathbf{x}'_k)\\
&\times(\delta(x_j-x_{k+1})-\delta(x'_j-x_{k+1}))\gamma^{(k+1)}_{\infty,s}(x_1,\dots,x_{k+1};x'_1,\dots,x'_k,x_{k+1})\,.
\end{aligned}
$$

□

10 Control of Some Error Terms

In this section we use the notation

$$\mathcal{D}(\phi)=\int \mathrm{d}x\, W^2\,\left(|\nabla_1\nabla_2\phi|^2+N^{-1}|\nabla_1^2\phi|^2+|\nabla_1\phi|^2+|\phi|^2\right).$$

By Corollary 6.2, $\mathcal{D}(\phi)\le C$ if $(W\phi,\widetilde{H}W\phi)\le C_1N$ and $(W\phi,\widetilde{H}^2W\phi)\le C_2N^2$.

Lemma 10.1. *Assume $a\ll\ell_1\ll\ell\ll 1$, $a\ell_1\ll\ell^4$, $\ell_1\ll\ell^{3/2}$. Then for any fixed k and any $J^{(k)}\in W^{0,\infty}(\Lambda^k\times\Lambda^k)$, and for every wave function ϕ symmetric w.r.t. permutation, we have*

$$\left|\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}\mathbf{x}_{N-k}\, J^{(k)}(\mathbf{x}_k,\mathbf{x}'_k)(W^{[k]})^2(\widetilde{\Omega}-\widetilde{\Omega}')\phi(\mathbf{x}_k,\mathbf{x}_{N-k})\,\overline{\phi}(\mathbf{x}'_k,\mathbf{x}_{N-k})\right|\le o(1)\mathcal{D}(\phi)\qquad(10.1)$$

as $N\to\infty$

Proof. We recall that

$$\widetilde{\Omega}=\frac{\Omega_p}{2G_p}+\frac{\Omega_{pj}}{2G_j}+\Gamma.$$

Using the definitions of Ω_p, Ω_{pj} and Γ from the beginning of Section 6 and the estimates from the end of Appendix C, we find

$$
\begin{aligned}
|\widetilde{\Omega}-\widetilde{\Omega}'| \leq C\ell^{-1}\sum_i\sum_{j\leq k}&\Big(|(\nabla w)_{ij}|F_{ij}^{1/2}+|(\nabla w')_{ij}|(F')_{ij}^{1/2}\Big)\\
+C\ell^{-2}\sum_i\sum_{j\leq k}&\Big(w_{ij}F_{ij}^{1/2}+w'_{ij}(F'_{ij})^{1/2}\Big)\\
+C\ell^{-1-\varepsilon}\sum_{i,j>k}\sum_{r,r'=1}^{k}&|(\nabla w)_{ij}|(F_{ij}^{[k]})^{1/2}(\theta_{ri}+\theta_{r'i})\\
+C\ell^{-2-\varepsilon}\sum_{i,j>k}\sum_{r,r'=1}^{k}&w_{ij}(\theta_{ri}+\theta_{r'i})(F_{ij}^{[k]})^{1/2}\\
+O(N^4\ell^{K-3-\varepsilon})&\,.
\end{aligned}\tag{10.2}
$$

This inequality relies on the fact that if both indices $i,j>k$ then there is a cancellation between F_{ij} and F'_{ij}.

Using $|w|\leq Ca\ell_1\lambda \ll Ca\ell\lambda$ and $|\nabla w|\leq Ca\lambda$, it is sufficient to control the first and the third terms on the r.h.s of (10.2) with $|(\nabla w)_{ij}|$ replaced with $Ca\lambda_{ij}$.

Inserting the first term on the r.h.s. of the last equation into (10.1), taking the absolute value, and estimating $J^{(k)}$ by its sup-norm, we find that this contribution is bounded by

$$
\begin{aligned}
Ca\ell^{-1}\sum_i\sum_{j\leq k}&\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2\lambda_{ij}F_{ij}^{1/2}\,|\phi(\mathbf{x}'_k,\mathbf{x}_{N-k})|\,|\phi(\mathbf{x}_k,\mathbf{x}_{N-k})|\\
&\leq Ca\ell^{-1}\sum_i\sum_{j\leq k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2\lambda_{ij}F_{ij}^{1/2}\left(|\phi|^2+|\phi'|^2\right)\\
&\leq C\sum_i\sum_{j\leq k}\Big[a\ell^{-1}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2\widetilde{\chi}_{ij}F_{ij}^{1/8}\,|\nabla_j\phi|^2\\
&\qquad+a\ell_1\ell^{-4}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}'_k\,\mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\widetilde{\chi}_{ij}F_{ij}^{1/8}\left(|\phi|^2+|\phi'|^2\right)\Big]\\
&\leq C^k\sum_{j\leq k}a\ell^{-1}\int \mathrm{d}\mathbf{x}\,W^2\,|\nabla_j\phi|^2\\
&\qquad+C^kka\ell_1\ell^{-4}\left(\int \mathrm{d}\mathbf{x}\,W^2\,|\phi|^2+\int \mathrm{d}\mathbf{x}'\,(W')^2\,|\phi'|^2\right)\\
&\leq C_k(a\ell^{-1}+a\ell_1\ell^{-4})\mathcal{D}(\phi)\\
&=o(1)\mathcal{D}(\phi)
\end{aligned}
$$

because $a\ell_1\ell^{-4}\ll 1$. Here $\mathbf{x}'=(\mathbf{x}'_k,\mathbf{x}_{N-k})$.

The contribution from the third term on the r.h.s. of (10.2) can be bounded as follows. Also here we estimate $J^{(k)}$ by its sup-norm.

$$
\begin{aligned}
&Ca\ell^{-1-\varepsilon}\sum_{i,j>k}\sum_{r\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2\lambda_{ij}F_{ij}^{[k]}\theta_{ri}\,|\phi'|\,|\phi| \\
&\quad\le Ca\ell^{-1-\varepsilon}\sum_{i,j>k}\sum_{r\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\lambda_{ij}F_{ij}^{[k]}\theta_{ri}\left(|\phi|^2+|\phi'|^2\right) \\
&\quad\le Ca\ell^{-1-\varepsilon}\sum_{i,j>k}\sum_{r\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2\widetilde{\chi}_{ij}(F_{ij}^{[k]})^{1/4}\theta_{ir}\left(|\nabla_j\phi|^2+|\nabla_j\phi'|^2\right) \\
&\qquad+Ca\ell_1\ell^{-4-\varepsilon}\sum_{i,j>k}\sum_{r\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\widetilde{\chi}_{ij}(F_{ij}^{[k]})^{1/4}\theta_{ir}\left(|\phi|^2+|\phi'|^2\right)\,.
\end{aligned}
$$

Next we apply Lemma 5.2. We find

$$
\begin{aligned}
&C\ell^{-1-\varepsilon}\sum_{i,j>k}\sum_{r\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2\lambda_{ij}F_{ij}^{[k]}\theta_{ri}|\phi'||\phi| \\
&\quad\le C\sum_{i,j>k}\sum_{r\le k}\Big\{a\ell^{-1-\varepsilon}\ell^2|\log\ell|^2\int \mathrm{d}\mathbf{x}(W^{[k]})^2\widetilde{\chi}_{ij}\,(F_{ij}^{[k]})^{1/4}\left(|\nabla_r\nabla_j\phi|^2+|\nabla_j\phi|^2\right) \\
&\qquad+a\ell^{-4-\varepsilon}\ell_1(\ell|\log\ell|)^3 \\
&\qquad\qquad\times\int \mathrm{d}\mathbf{x}\,(W^{[k]})^2\,\widetilde{\chi}_{ij}\,(F_{ij}^{[k]})^{1/4}\,\left(|\nabla_i\nabla_r\phi|^2+|\nabla_r\phi|^2+|\nabla_i\phi|^2+|\phi|^2\right)\Big\} \\
&\qquad+Ck\sum_{i,j>k}\Big\{a\ell^{-1-\varepsilon}\ell^2|\log\ell|^2\int \mathrm{d}\mathbf{x}'(W^{[k]})^2\widetilde{\chi}_{ij}\,(F_{ij}^{[k]})^{1/4}\,|\nabla_j\phi'|^2 \\
&\qquad+a\ell^{-4-\varepsilon}\ell_1(\ell|\log\ell|)^3\int \mathrm{d}\mathbf{x}'\,(W^{[k]})^2\,\widetilde{\chi}_{ij}\,(F_{ij}^{[k]})^{1/4}\,\left(|\nabla_i\phi'|^2+|\phi'|^2\right)\Big\} \\
&\quad\le C_k(\ell^{1-\varepsilon}|\log\ell|^2+\ell_1\ell^{-1-\varepsilon}|\log\ell|^3)\mathcal{D}(\phi)=o(1)\mathcal{D}(\phi)\,.
\end{aligned}
$$

We used the assumption $\ell_1\ll\ell^{3/2}$ (and $0<\varepsilon<1/4$). □

Lemma 10.2. *Assume $a\ll\ell_1\ll\ell$, $\ell_1\ll\ell^{3/2}$, $N\ell_1^3\ll\ell^{9/4}$, and $N^2\ell^5\ll1$. Then for any fixed k, any $J^{(k)}\in W^{0,\infty}(\Lambda^k\times\Lambda^k)$, and any wave function ϕ symmetric w.r.t. permutations, we have*

$$
\Big|\sum_{m=1}^N\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}J^{(k)}(\mathbf{x}_k,\mathbf{x}_k')\,(W^{[k]})^2\left(\nabla_m\log\frac{W}{W^{[k]}}\right)(\nabla_m\phi)\,\overline{\phi}'\Big|\le o(1)\mathcal{D}(\phi) \tag{10.3}
$$

as $N\to\infty$.

Proof. We have

$$
\nabla_m\log\frac{W}{W^{[k]}}=\frac{1}{2}\sum_{j\le k}\frac{\nabla_mG_j}{G_j}+\frac{1}{2}\sum_{j>k}\left(\frac{\nabla_mG_j}{G_j}-\frac{\nabla_mG_j^{[k]}}{G_j^{[k]}}\right).
$$

Using that for each fixed j

$$
G_j=1-w_{jn}F_{jn}\qquad\text{and}\qquad G_j^{[k]}=1-w_{jn}F_{jn}^{[k]}\chi(n>k)\qquad\text{for }\ j>k
$$

we have, for each fixed $j>k$ and m,

$$
|G_j-G_j^{[k]}|\le\sum_{n\le k}w_{jn}F_{jn}+\sum_{n>k}w_{jn}|F_{jn}-F_{jn}^{[k]}|
$$

and

$$|\nabla_m(G_j - G_j^{[k]})| \leq |(\nabla w)_{jm}|\,|F_{jm} - F_{jm}^{[k]}| + w_{jn}|\nabla_m(F_{jn} - F_{jn}^{[k]})| \\ + |(\nabla w)_{jm}|F_{jm}\chi(m \leq k)\ .$$

We also use that w_{jn} is supported on $|x_j - x_n| \leq 2\ell_1$, and

$$\chi(|x_j - x_n| \leq 2\ell_1)\, F_{jn}\, G_j = \chi(|x_j - x_n| \leq 2\ell_1)\, F_{jn}\,(1 - w_{jn}F_{jn}) + O(\exp(-c\ell^{-\varepsilon})).$$

Estimating $J^{(k)}$ by its sup-norm, we find

$$\begin{aligned}
&\sum_m \Big| \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k} J^{(k)}(\mathbf{x}_k, \mathbf{x}'_k)\,(W^{[k]})^2 \left(\nabla_m \log \frac{W}{W^{[k]}}\right) (\nabla_m \phi)\,\overline{\phi}' \Big| \\
&\quad \leq \sum_m \sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\,(|(\nabla w)_{jm}|F_{jm} + w_{jn}|\nabla_m F_{jn}|)\,|\nabla_m\phi|\,|\phi'| \\
&\quad + \sum_m \sum_{n,j>k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2 w_{jn}|\nabla_m(F_{jn} - F_{jn}^{[k]})||\nabla_m\phi|\,|\phi'| \\
&\quad + \sum_m \sum_{j>k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2|(\nabla w)_{jm}||(F_{jm} - F_{jm}^{[k]})||\nabla_m\phi|\,|\phi'| \\
&\quad + O(\exp(-c\ell^{-\varepsilon}))\mathcal{D}(\phi)\ .
\end{aligned} \tag{10.4}$$

Applying a Schwarz inequality we can bound the first term on the r.h.s of the last equation by (using the summation convention for m)

$$\begin{aligned}
&\sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\,|(\nabla w)_{jm}|F_{jm}|\nabla_m\phi||\phi'| \\
&\qquad \leq a\alpha \sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\,\lambda_{jm}F_{jm}|\nabla_m\phi|^2 \\
&\qquad + a\alpha^{-1} \sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\,\lambda_{jm}F_{jm}|\phi'|^2.
\end{aligned}$$

To bound the first term on the r.h.s. of the last equation we use Lemma 5.3; to bound the second one, we estimate $F_{jm} \leq 1$, and, for each fixed m, we integrate over the variable x_j using that ϕ' doesn't depend on it. We get

$$\begin{aligned}
&\sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\,|(\nabla w)_{jm}|F_{jm}|\nabla_m\phi||\phi'| \\
&\qquad \leq Ca\alpha \sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\widetilde{\chi}_{jm}F_{jm}^{1/4}|\nabla_j\nabla_m\phi|^2 \\
&\qquad + Ca\alpha\ell_1\ell^{-3} \sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}(W^{[k]})^2\widetilde{\chi}_{jm}F_{jm}^{1/4}|\nabla_m\phi|^2 \\
&\qquad + Ca\alpha^{-1}\ell_1 N \int \mathrm{d}\mathbf{x}'\,(W^{[k]})^2|\phi'|^2 \\
&\qquad \leq C_k\left(\alpha a N + \alpha a\ell_1\ell^{-3}N + \alpha^{-1}a\ell_1 N\right)\mathcal{D}(\phi) = o(1)\mathcal{D}(\phi)
\end{aligned}$$

where we used $\alpha = \ell^{3/2}$ and that $\ell_1 \ll \ell^{3/2}$. Next we consider the second term on the r.h.s. of (10.4). We have, applying first Lemma 5.3, and then Eq. (5.3) from Lemma 5.2, (using the summation convention for the indices m, n)

$$
\begin{aligned}
&\sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\, (W^{[k]})^2 w_{jn} |\nabla_m F_{jn}|\, |\nabla_m \phi||\phi'| \\
&\qquad \leq C a \ell_1 \sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k} (W^{[k]})^2 \lambda_{jn} |\nabla_m F_{jn}| \left(|\nabla_m \phi|^2 + |\phi'|^2 \right) \\
&\qquad \leq C a \ell_1 \ell^{-1} \sum_{j\leq k} \int \mathrm{d}\mathbf{x} (W^{[k]})^2 F_{jn}^{1/2} \widetilde{\chi}_{jn} |\nabla_j \nabla_m \phi|^2 \\
&\qquad\quad + C a \ell_1^2 \ell^{-4} \sum_{j\leq k} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k} (W^{[k]})^2 \theta_{jm} F_{jn}^{1/2} \widetilde{\chi}_{jn} \left(|\nabla_m \phi|^2 + |\phi'|^2 \right) \\
&\qquad \leq C_k (\ell_1 \ell^{-1} + \ell_1^2 \ell^{-4} \ell^2 |\log \ell|^2) \mathcal{D}(\phi) = o(1) \mathcal{D}(\phi)
\end{aligned}
$$

for $N \to \infty$, because $\ell_1 \ll \ell$.

We consider now the second term on the r.h.s. of (10.4). The bound (C.5) implies that (here we are summing over $n, j > k$ and $r \leq k$)

$$
\begin{aligned}
&\int \mathrm{d}\mathbf{x}_k \mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\, (W^{[k]})^2 w_{jn} |\nabla_m (F_{jn} - F_{jn}^{[k]})||\nabla_m \phi||\phi'| \\
&\qquad \leq C \ell^{-1-\varepsilon} \int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}'_k \mathrm{d}\mathbf{x}_{N-k}\, (W^{[k]})^2 w_{jn} F_{jn}^{1/2} \theta_{jr} \left(\alpha |\nabla_m \phi|^2 + \alpha^{-1} |\phi'|^2 \right) + O(\ell^{K-1-\varepsilon}) \\
&\qquad \leq C k \alpha a \ell_1 \ell^{-1-\varepsilon} \int \mathrm{d}\mathbf{x}\, (W^{[k]})^2 \widetilde{\chi}_{jn} F_{jn}^{1/8} |\nabla_j \nabla_m \phi|^2 \\
&\qquad\quad + C \alpha a \ell_1^2 \ell^{-4-\varepsilon} \int \mathrm{d}\mathbf{x}\, (W^{[k]})^2 \widetilde{\chi}_{jn} F_{jn}^{1/8} \theta_{jr} |\nabla_m \phi|^2 \\
&\qquad\quad + C \alpha^{-1} a \ell_1 \ell^{-1-\varepsilon} \int \mathrm{d}\mathbf{x}' \mathrm{d}\mathbf{x}_k\, (W^{[k]})^2 \widetilde{\chi}_{jn} F_{jn}^{1/8} \theta_{jr} |\nabla_j \phi'|^2 \\
&\qquad\quad + C \alpha^{-1} a \ell_1^2 \ell^{-4-\varepsilon} \int \mathrm{d}\mathbf{x}' \mathrm{d}\mathbf{x}_k\, (W^{[k]})^2 \widetilde{\chi}_{jn} F_{jn}^{1/8} \theta_{jr} |\phi'\Big|^2 + O(\ell^{K-1-\varepsilon}) \mathcal{D}(\phi)\ .
\end{aligned}
$$

Applying Lemma 5.2, we find

$$
\begin{aligned}
&\int \mathrm{d}\mathbf{x}_k \mathrm{d}\mathbf{x}_{N-k}\, (W^{[k]})^2 w_{jn} |\nabla_m (F_{jn} - F_{jn}^{[k]})||\nabla_m \phi||\phi'| \\
&\qquad \leq C \alpha a \ell_1 \ell^{-1-\varepsilon} k \int \mathrm{d}\mathbf{x}\, (W^{[k]})^2 |\nabla_j \nabla_m \phi|^2 \\
&\qquad\quad + C \alpha a \ell_1^2 \ell^{-4-\varepsilon} \ell^2 |\log \ell|^2 k \int \mathrm{d}\mathbf{x}\, (W^{[k]})^2 \left(|\nabla_j \nabla_m \phi|^2 + N |\nabla_m \phi|^2 \right) \\
&\qquad\quad + C k \alpha^{-1} a \ell_1 \ell^{-1-\varepsilon} \ell^3 |\log \ell|^3 \int \mathrm{d}\mathbf{x}' (W^{[k]})^2 |\nabla_j \phi'|^2 \\
&\qquad\quad + C k \alpha^{-1} a \ell_1^2 \ell^{-4-\varepsilon} \ell^3 |\log \ell|^3 \int \mathrm{d}\mathbf{x}\, (W^{[k]})^2 \left(|\nabla_j \phi'|^2 + N |\phi'|^2 \right) + O(\ell^{K-1-\varepsilon}) \\
&\qquad \leq C_k |\log \ell|^3 (\alpha a \ell_1 \ell^{-1-\varepsilon} N^2 + \alpha a \ell_1^2 \ell^{-2-\varepsilon} N^2 + \alpha^{-1} a \ell_1 \ell^{2-\varepsilon} N \\
&\qquad\quad + \alpha^{-1} a \ell_1^2 \ell^{-1-\varepsilon} N + \ell^{K-1-\varepsilon}) \mathcal{D}(\phi) \\
&\qquad \leq C_k (\alpha N \ell_1 \ell^{-1-\varepsilon} + \alpha^{-1} \ell_1^2 \ell^{-1-\varepsilon}) \mathcal{D}(\phi) \leq C_k \ell_1^{3/2} N^{1/2} \ell^{-1-\varepsilon} \mathcal{D}(\phi) = o(1) \mathcal{D}(\phi)
\end{aligned}
$$

for $N \to \infty$. Here we first used that $\ell_1\ell^2 \ll \ell_1^2\ell^{-1}$ (as follows from $N^2\ell^5 \ll 1$ and $a \ll \ell_1 \ll \ell \ll 1$), then we optimized the choice of α and we used that $N\ell_1^3 \ll \ell^{9/4}$ (and that $\varepsilon < 1/8$).

Finally we consider the third term on the r.h.s. of (10.4). Using (C.4) we find (using the summation convention for m)

$$\sum_{j>k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2|(\nabla w)_{jm}||(F_{jm}-F_{jm}^{[k]})||\nabla_m\phi|\,|\phi'|$$

$$\le a\sum_{j>k}\sum_{r\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\lambda_{jm}\theta_{mr}(F_{jm}^{[k]})^{1/2}|\nabla_m\phi|\,|\phi'|.$$

Next we apply a weighted Schwarz inequality. We find

$$\sum_{j>k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2|(\nabla w)_{jm}||(F_{jm}-F_{jm}^{[k]})||\nabla_m\phi|\,|\phi'|$$

$$\le \alpha\sum_{j>k}\sum_{r\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\,\theta_{mr}\widetilde{\chi}_{jm}(F_{jm}^{[k]})^{1/2}|\nabla_m\phi|^2$$

$$+\alpha^{-1}a\sum_{j>k}\sum_{r\le k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2\sigma_{mj}\theta_{mr}(F_{jm}^{[k]})^{1/2}|\phi'|^2$$

where we used that $a\lambda_{jm}^{1/2} \le \widetilde{\chi}_{jm}$ and that $\lambda_{jm}^{3/2} \le \sigma_{mj}$ (recall the definition of σ_{jm} from (5.5)). In the first term on the r.h.s. of the last equation we can sum over the index j (using Lemma B.1) and then we can apply Lemma 5.2 for the integration over the variable x_r. As for the second term on the r.h.s. of the last equation we can first integrate over the variable x_r (using that ϕ' is independent of x_r) and then we can apply (5.9). We find (using the summation convention for the index m),

$$\sum_{j>k}\int \mathrm{d}\mathbf{x}_k\mathrm{d}\mathbf{x}_k'\mathrm{d}\mathbf{x}_{N-k}\,(W^{[k]})^2|(\nabla w)_{jm}||(F_{jm}-F_{jm}^{[k]})||\nabla_m\phi|\,|\phi'|$$

$$\le C\alpha\ell^2|\log\ell|^2\sum_{r\le k}\int \mathrm{d}\mathbf{x}\,(W^{[k]})^2|\nabla_m\nabla_r\phi|^2$$

$$+C\alpha^{-1}a\ell^3(\log N)\sum_{j>k}\int \mathrm{d}\mathbf{x}'\,(W^{[k]})^2|\nabla_j\nabla_m\phi'|^2$$

$$\le C_k(\log N)(\alpha N\ell^2+\alpha^{-1}\ell^3N)\mathcal{D}(\phi) = o(1)\mathcal{D}(\phi)$$

where we chose $\alpha = \ell^{1/2}$ and we used that $N^2\ell^5 \ll 1$. This completes the proof of the lemma. $\square$

A Properties of the Two Body Problem

We set $r = |x|$ for $x \in \mathbb{R}^3$ and for a radial function $g(x)$ on $\mathbb{R}^3$ we use the notation $g(r)$ for the function $r \mapsto g(x)|_{|x|=r}$. Let $g'(r) := \partial_r g(r)$ be the derivative of this radial function g w.r.t. r.

Consider the (unique) solution to the zero energy problem

$$\mathfrak{h}\omega = 0, \quad \mathfrak{h} := -\Delta + (1/2)V\ ,$$

on $\mathbb{R}^3$, where ω is radially symmetric and satisfies the condition $\lim_{r\to\infty}\omega(r) = 1$. It is known that ω is always non-negative and by Harnack inequality actually $\omega > 0$ since V is regular. By the maximum principle, the C^2 norm of ω is also bounded. By writing $\omega(r) = f(r)/r$, the scattering length, a, of a potential V is defined by

$$a := \lim_{r\to\infty} r - f(r)\,. \tag{A.1}$$

When V has a compact support, that is, $V(x) = 0$ for $|x| \geq R$, then $f(r) = r - a$ for $r \geq R$. It is known that a is bounded by

$$a \leq \frac{1}{8\pi}\int V \mathrm{d}x \tag{A.2}$$

and also $a < R$ since $f > 0$.

Lemma A.1. *Let V be a smooth, positive, spherical symmetric function such that* $\operatorname{supp} V \subset \{x \in \mathbb{R}^3 : |x| \leq R\}$*, for some $R > 0$. Let $\varrho := (8\pi)^{-1}(\|V\|_\infty + \|V\|_{L_1})$ and denote the scattering length of V by a. Let φ be the ground state of the Neumann problem*

$$(-\Delta + \tfrac{1}{2}V)\varphi = E\varphi \tag{A.3}$$

on the sphere of radius L, with the boundary condition

$$\varphi(L) = 1, \quad \partial_r\varphi(L) = 0\,.$$

Then, if L is sufficiently large, we have

i)

$$E = 3aL^{-3}(1 + O(1/L)) \quad \textit{as} \quad L \to \infty\,. \tag{A.4}$$

ii) There is a constant $0 < c_0 < 1$ such that for all $|x| \leq L$ we have

$$c_0 \leq \varphi(x) \leq 1, \quad 1 - \varphi(x) \leq \frac{C}{|x|} \tag{A.5}$$

where C is a constant, depending only on the potential.

iii) For all $|x| \leq L$ we also have the following bounds:

$$|\nabla\varphi(x)| \leq \frac{C\varrho}{|x|^2 + R^2}, \quad \textit{and} \quad |\nabla^2\varphi(x)| \leq \frac{C\varrho}{|x|^3 + R^3}\,. \tag{A.6}$$

Note that, in ii) and iii), the constant C is independent of L, if L is large enough.

Proof. i) We first prove an upper bound for the energy E. We write the zero energy state $\omega(r)$ as $\omega(r) = f(r)/r$ and let

$$\psi(r) := \sin(kf(r))/r. \tag{A.7}$$

Note that

$$\partial_r\psi(r) = \frac{kf'(r)r\cos(kf(r)) - \sin(kf(r))}{r^2}\,.$$

Therefore, assuming $L \geq R$, ψ satisfies Neumann boundary conditions at $r = L$ if and only if

$$kL = \tan(k(L - a)) \tag{A.8}$$

where a is the scattering length defined in (A.1). We define k to be the smallest positive real number satisfying equation (A.8), in particular $\psi > 0$. It is easy to check that there are constants C_1 and C_2 such that

$$\frac{3a}{L^3}\Big(1 - \frac{C_1R}{L}\Big) \leq k^2 \leq \frac{3a}{L^3}\Big(1 + \frac{C_1R}{L}\Big)\,. \tag{A.9}$$

Let $\widetilde{\mathfrak{h}} = -\partial_r^2 + (1/2)V$, then $\widetilde{\mathfrak{h}} f = r\mathfrak{h}\omega = 0$, i.e.

$$-f''(r) + \frac{1}{2}V(r)f(r) = 0 \; . \tag{A.10}$$

In particular, f is linear for $r > R$ and by the normalization $\lim_{r\to\infty}\omega(r) = 1$, we have $f'(r) = 1$ for $r > R$. Moreover, f satisfies uniform bounds $f(r) \le C(1+r)$, $\|f'\|_\infty, \|f''\|_\infty \le C$ by the boundedness of ω and f does not vanish.

With the help of the identity

$$-\big[\sin(kf(r))\big]'' = k^2 f'(r)^2 \sin(kf(r)) - kf''(r)\cos(kf(r)) \; , \tag{A.11}$$

we compute

$$\psi\mathfrak{h}\psi = k^2\psi^2 + k^2(f'-1)\Big(\frac{\sin kf}{r}\Big)^2 + \frac{1}{r^2}\Big[-kf''(\sin kf)(\cos kf) + \frac{1}{2}V(\sin kf)^2\Big] \; .$$

The last two terms are supported on $|x| \le R$. Using the equation (A.10) and the boundedness properties of f, we see that the $O(k^2)$ terms cancel in the square bracket:

$$\psi\mathfrak{h}\psi = k^2\psi^2 + r^{-2}\chi(r \le R)O(k^3) \tag{A.12}$$

and therefore

$$\langle\psi, \mathfrak{h}\psi\rangle = k^2\langle\psi,\psi\rangle + O(k^3) \; ,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L^2(|x| \le L)$.

Using the estimate

$$\sin kf(r) \ge Ckf(r) \ge Ckr \tag{A.13}$$

$r \ge 2R$, we also get the lower bound

$$\langle\psi,\psi\rangle \ge 4\pi\int_0^L (\sin kf(r))^2 \mathrm{d}r \ge C\int_{2R}^L k^2r^2\mathrm{d}r = Ck^2L^3 \; .$$

So we can use ψ as a trial function for the upper bound on the energy

$$E \le \frac{\langle\psi,\mathfrak{h}\psi\rangle}{\langle\psi,\psi\rangle} \le k^2 + O(kL^{-3}) = \frac{3a}{L^3}(1 + O(L^{-1})) \; .$$

This proves the upper bound in (A.4). Before proving the lower bound in (A.4), we prove parts ii) and iii) of the lemma.

ii), iii) We now prove (A.5) and (A.6). We set $m(r) := r\varphi(r)$ and we rewrite the eigenfunction equation (A.3) as

$$\widetilde{\mathfrak{h}}m = (-\partial_r^2 + \tfrac{1}{2}V)m = Em \; . \tag{A.14}$$

Since V has compact support, we can explicitly write $m(r)$ for $r \in (R, L]$. Let $\lambda := \sqrt{E}$. From the boundary conditions on φ, we have

$$m(r) = \lambda^{-1}\sin(\lambda(r-L)) + L\cos(\lambda(r-L)), \quad R < r \le L \; . \tag{A.15}$$

From i) we have $\lambda \le CL^{-3/2}$. This allows us to expand $m(r)$ up to $O(\lambda^4)$, we find

$$m(r) = r - a + O(1/L) \tag{A.16}$$

uniformly in r, for $r \in (R, L]$. Using that $a < R$, this proves that $\varphi(r) = m(r)/r$ satisfies the inequalities in (A.5) for $r \in [R, L]$, if L is large enough. The properties (A.6) on the interval $[R, L]$ can be easily proved using the explicit formula (A.15) (and using that $a < R$).

We now consider the region $r \leq R$. From the Harnack inequality, the ratio between the supremum and the infimum of φ in a given ball is bounded. Since φ satisfies (A.5) for $r \leq R$, it follows that $\varphi(r)$ is bounded away from zero for $r \leq R$. This proves the lower bound of $\varphi(r)$ in (A.5).

To prove the bound $\varphi(x) \leq 1$ for $r \leq R$, consider the ball B_R of radius R about the origin. On the boundary of this domain $\varphi(x) \leq 1 - \delta$ with some $\delta > 0$, uniformly for all sufficiently large L from (A.16). Inside this ball,

$$-\Delta\varphi \leq -\Delta\varphi + \frac{1}{2}V\varphi = E\varphi \leq CL^{-3}$$

since φ is bounded by Harnack principle, i.e. $-\Delta(\varphi + CL^{-3}x^2) \leq 0$. By maximum principle, $\varphi(x) \leq 1 - \delta + CL^{-3}R^2 < 1$ for large enough L for all $x \in B_R$.

Finally we have to prove (A.6) for $r < R$. Since $\varphi(0)$ is bounded, we have $m(0) = 0$. From the equation of m inside $r \leq R$, $|m''(r)| \leq C\varrho$. By integrating m' from 0 to r we obtain $m'(r) = m'(0) + O(\varrho r)$. Integrating once more yields

$$m(r) = m'(0)r + O(\varrho r^2) ,$$

hence the bound $|\varphi'| = O(\varrho)$ follows. Differentiating the equation (A.3), we obtain $|m'''| \leq C\varrho$ for $r \in [0, R]$ and integrating three times we obtain

$$\begin{aligned} m''(r) &= m''(0) + O(\varrho r), \qquad m'(r) = m'(0) + m''(0)r + O(\varrho r^2), \\ m(r) &= m'(0)r + \frac{m''(0)r^2}{2} + O(\varrho r^3) , \end{aligned}$$

and the bound on φ'' follows.

Next we prove the lower bound in (A.4). Given any wave function ϕ satisfying the Neumann boundary condition at $|x| = L$, we can write it as $\phi(x) = g(x)\psi(x)$, where ψ is given in (A.7), and $g > 0$ satisfies Neumann boundary condition at $|x| = L$ as well. From the identity $\mathfrak{h}\phi = (\mathfrak{h}\psi)g - (\Delta g)\psi - 2\nabla g \nabla\psi$, we have

$$\int_{|x|\leq L} \mathrm{d}x\, \overline{\phi}\mathfrak{h}\phi = \int_{|x|\leq L} \mathrm{d}x |\nabla g|^2\psi^2 + \int_{|x|\leq L} \mathrm{d}x\, |g|^2\psi\mathfrak{h}\psi$$

and from (A.12)

$$\int_{|x|\leq L} \mathrm{d}x\, \overline{\phi}\mathfrak{h}\phi \geq\; k^2\|\phi\|^2 + \int_{|x|\leq L} \mathrm{d}x |\nabla g|^2\psi^2 - Ck^3 \int_{|x|\leq R} \mathrm{d}x\, \frac{|g(x)|^2}{|x|^2} .$$

Using (A.13) and that f does not vanish, we have $\psi(r) \geq ck$ with some positive constant c, thus

$$\int_{|x|\leq L} \mathrm{d}x\, \overline{\phi}\mathfrak{h}\phi \geq\; k^2\|\phi\|^2 + ck^2 \int_{|x|\leq L} \mathrm{d}x |\nabla g|^2 - Ck^3 \int_{|x|\leq R} \mathrm{d}x\, \frac{g^2(x)}{|x|^2} .$$

Using Hardy's inequality (Lemma 5.1)

$$k^3 \int_{|x|\leq R} \frac{g^2(x)}{|x|^2}\, \mathrm{d}x \leq k^3 \int_{|x|\leq L} \mathrm{d}x\, |\nabla g|^2 + \frac{k^3 R}{L^3} \int_{|x|\leq L} \mathrm{d}x\, |g|^2$$

and by $\psi(r) \geq ck$ the last term can be controlled by

$$\frac{k^3 R}{L^3} \int_{|x|\leq L} \mathrm{d}x\, |g|^2 \leq \frac{kR}{c^2 L^3} \int_{|x|\leq L} \mathrm{d}x\, |g|^2 \psi^2 \leq C k^2 L^{-3/2} \|\phi\|^2 \,.$$

This implies the lower bound

$$\int_{|y|\leq L} \mathrm{d}y\, \overline{\phi} \mathfrak{h} \phi \geq \; k^2(1 + O(L^{-3/2})) \|\phi\|^2 \geq \frac{3a}{L^3}(1 + O(L^{-1})) \|\phi\|^2$$

in (A.4). □

Next we apply last lemma to prove some important properties of the function $w(y)$ and $q(y)$, defined in Section 2.1.

Lemma A.2. *Assume $V_a(x) = (a_0/a)^2 V((a_0/a)x) = N^2 V(Nx)$ (because $a = a_0/N$), where $V \geq 0$ is a smooth, spherical symmetric potential with scattering length a_0 and with $\operatorname{supp} V \subset \{x \in \mathbb{R}^3 : |x| \leq R_0\}$. Let $\varrho = (8\pi)^{-1} (\|V\|_1 + \|V\|_\infty)$. Suppose that the functions $w(x)$ and $q(x)$ are defined as in (2.3) and (2.5), with $a \ll \ell_1 \ll 1$.*

i) There is a constant $c_0 > 0$ such that

$$c_0 \leq 1 - w(x) \leq 1 \tag{A.17}$$

for all $x \in \mathbb{R}^3$. Moreover

$$w(x) \leq Ca \frac{\chi(|x| \leq 3\ell_1/2)}{|x|}, \tag{A.18}$$

for some constant C independent of N.

ii) For $x \in \mathbb{R}^3$ we have the following bounds on the derivatives of $w(x)$:

$$\begin{aligned} |\nabla w(x)| &\leq Ca \frac{\chi(|x| \leq 3\ell_1/2)}{|x|^2 + a^2}, \\ |\nabla^2 w(x)| &\leq C\varrho\, a \frac{\chi(|x| \leq 3\ell_1/2)}{|x|^3 + a^3} \leq C\varrho \frac{\chi(|x| \leq 3\ell_1/2)}{|x|^2} \,. \end{aligned} \tag{A.19}$$

iii) We have $\operatorname{supp} q \subset \{x \in \mathbb{R}^3 : |x| \leq 3\ell_1/2\}$. Moreover

$$0 \leq q(x) \leq C a \ell_1^{-3} \chi(|x| \leq 3\ell_1/2)$$

and

$$|\nabla q(x)| \leq C \frac{a}{|x| \ell_1^3} \chi(|x| \leq 3\ell_1/2)$$

for all $x \in \mathbb{R}^3$.

iv) The L^1 norm of $q(x)$ is given by

$$\int_{\mathbb{R}^3} q(x) \mathrm{d}x = 4\pi a (1 + o(1)) \,, \qquad a \to 0 \,.$$

The constant C in i),ii) and iii) is independent of N, if N is large enough.

Remark A.3. *Using the functions $\lambda(x)$ and $\sigma(x)$ introduced in (5.5), this lemma in particular proves all the bounds in (5.6). The bound $|\nabla w|^2 \leq Ca\sigma$ follows because $a(|x|^2+a^2)^2 \leq C(|x|^3+a^3)^{-1}$, for all $x \in \mathbb{R}^3$.*

Proof. Let e_κ and $\psi_\kappa(x) = 1 - w_\kappa(x)$ be the lowest Neumann eigenvalue and eigenfunction on the ball $\{|x| \leq \kappa\}$, that is

$$(-\Delta + \frac{1}{2}V_a(x))\psi_\kappa(x) = e_\kappa \psi_\kappa(x)$$

with the condition that $\psi_\kappa(x) = 1$ if $|x| = \kappa$. $\psi_\kappa(x)$ is then extended to be one, for $|x| \geq \kappa$. We define $\phi_\kappa(x) := \psi_\kappa((a/a_0)x)$. Then we have

$$(-\Delta + \frac{1}{2}V(y))\phi_\kappa(y) = (a/a_0)^2 e_\kappa \phi_\kappa(y)$$

for $|y| \leq (a_0/a)\kappa =: L$, and with $\phi_\kappa(L) = 1$. By Lemma A.1, part i), we have

$$(a/a_0)^2 e_\kappa = 3a_0 L^{-3}(1+o(1))$$

and thus

$$e_\kappa = 3a\kappa^{-3}(1+o(1)). \tag{A.20}$$

From Lemma A.1, part ii) we get immediately

$$c_0 \leq \psi_\kappa(x) \leq 1 \quad \text{and} \quad w_\kappa(x) \leq Ca\frac{\chi(|x| \leq \kappa)}{|x|} \tag{A.21}$$

where the constants c_0 and C are independent of κ and a (they depend only on the properties of the unscaled potential $V(x)$). Moreover from Lemma A.1, part iii) we find

$$|\nabla w_\kappa(x)| \leq C\varrho\, a\frac{\chi(|x| \leq \kappa)}{|x|^2+a^2} \quad \text{and} \quad |\nabla^2 w_\kappa(x)| \leq C\varrho\, a\frac{\chi(|x| \leq \kappa)}{|x|^3+a^3} \tag{A.22}$$

where C is independent of κ and a.

From (A.21), taking the average over κ w.r.t. the probability measure μ, part i) follows trivially. As for part ii), from (A.22) we have

$$\begin{aligned} |\nabla w(x)| &\leq \int |\nabla w_\kappa(x)|\mu(\mathrm{d}\kappa) \leq \frac{C\varrho\, a}{|x|^2+a^2}\int \chi(|x| \leq \kappa)\mu(\mathrm{d}\kappa) \\ &\leq C\varrho\, a\frac{\chi(|x| \leq 3\ell_1/2)}{|x|^2+a^2} \end{aligned} \tag{A.23}$$

because the measure μ is supported on $[\ell_1/2, 3\ell_1/2]$. The bound for $|\nabla^2 w(x)|$ in (A.19) can be proven analogously, using (A.22).

In order to prove iii) and iv) recall that $q(x)$ was defined by

$$q(x) = \frac{\int e_\kappa \chi(|x| \leq \kappa)\psi_\kappa(x)\mu(\mathrm{d}\kappa)}{\int \psi_\kappa(x)\mu(\mathrm{d}\kappa)} = \frac{\int e_\kappa \chi(|x| \leq \kappa)\psi_\kappa(x)\mu(\mathrm{d}\kappa)}{1-w(x)}.$$

From (A.20) and since the measure μ is supported on $[\ell_1/2, 3\ell_1/2]$, we get $q(x) \leq Ca\ell_1^{-3}$. The gradient of q can be estimated by

$$\begin{aligned} |\nabla q(x)| \leq\; &\frac{\int e_\kappa \kappa^{-1}\delta(|x|-\kappa)\mu(\mathrm{d}\kappa)}{1-w(x)} + \frac{\int e_\kappa \chi(|x| \leq \kappa)|\nabla_x w_\kappa(x)|\mu(\mathrm{d}\kappa)}{1-w(x)} \\ &+ \frac{|\nabla w(x)|}{1-w(x)}q(x)\,. \end{aligned} \tag{A.24}$$

The first term on the r.h.s. of the last equation (where we already used the condition $\psi_\kappa(x) = 1$ if $|x| = \kappa$) is different from zero only if $|x| \in [\ell_1/2, 3\ell_1/2]$. Since the delta function forces $\kappa = |x|$ we find, because of (A.20) and because $1 - w(x) \geq c_0$,

$$\frac{\int e_\kappa \delta(|x| - \kappa)\mu(\mathrm{d}\kappa)}{1 - w(x)} \leq C\frac{a}{\ell_1^3}\,\chi(|x| \leq 3\ell_1/2)\,.$$

The second term on the r.h.s. of (A.24) can be controlled using (A.20) and (A.22). The third term on the r.h.s. of (A.24) can be estimated using the bounds (A.23) and $q(x) \leq Ca\ell_1^{-3}$. This completes the proof of iii).

Finally, to prove the estimate for the L^1 norm of $q(x)$, we note that

$$\int \mathrm{d}x\, q(x) = \int \mathrm{d}x\, \frac{\int e_\kappa \chi(|x| \leq \kappa)\psi_\kappa(x)\mu(\mathrm{d}\kappa)}{\int \psi_\kappa(x)\mu(\mathrm{d}\kappa)} = \int \mathrm{d}x\, \frac{\int e_\kappa \chi(|x| \leq \kappa)\mu(\mathrm{d}\kappa)}{\int \psi_\kappa(x)\mu(\mathrm{d}\kappa)} + a\,o(1) \tag{A.25}$$

for $a \to 0$, because $\psi_\kappa(x) = 1 - w_\kappa(x)$ and

$$\begin{aligned}\Big|\int \mathrm{d}x \int e_\kappa \chi(|x| \leq \kappa) w_\kappa(x)\mu(\mathrm{d}\kappa)\Big| &\leq Ca\int e_\kappa \left(\int \mathrm{d}x \frac{\chi(|x| \leq \kappa)}{|x|}\right)\mu(\mathrm{d}\kappa)\\ &\leq Ca^2 \int \kappa^{-1}\mu(\mathrm{d}\kappa) \leq Ca^2\ell_1^{-1} = aO(a/\ell_1)\,.\end{aligned}$$

Analogously we can estimate the contribution of $w_\kappa(x)$ in the denominator on the r.h.s. of (A.25). We get

$$\begin{aligned}\int \mathrm{d}x\, q(x) &= \int \mathrm{d}x \int e_\kappa \chi(|x| \leq \kappa)\mu(\mathrm{d}\kappa) + a\,o(1) = \int e_\kappa \left(\int \mathrm{d}x\, \chi(|x| \leq \kappa)\right)\mu(\mathrm{d}\kappa) + a\,o(1)\\ &= \frac{4\pi}{3}\int e_\kappa \kappa^3 \mu(\mathrm{d}\kappa) + a\,o(1)\end{aligned}$$

and thus, with (A.20),

$$\int \mathrm{d}x\, q(x) = 4\pi a(1 + o(1))$$

for $a \to 0$, which proves iv). □

B Properties of the Triple Cutoff Function F_{ij}

In this appendix we collect some properties of the function F_{ij}, defined in Section 2.2.

Lemma B.1 (No overlap). *For any exponent $q > 0$ and any fixed i and $j \neq j'$*

$$\|\widetilde{\chi}_{ij}\widetilde{\chi}_{ij'}F_{ij}^q\|_\infty \leq e^{-cq/\ell^\varepsilon}\,. \tag{B.1}$$

Moreover, for any fixed i,

$$\sum_j \widetilde{\chi}_{ij}F_{ij}^q \leq c_q\,. \tag{B.2}$$

Proof. Clearly $|F| \leq 1$. Suppose there is an overlap between two functions $\widetilde{\chi}_{ij}$, $\widetilde{\chi}_{ij'}$, i.e. for some $j \neq j'$ we have $\widetilde{\chi}_{ij} \neq 0$ and $\widetilde{\chi}_{ij'} \neq 0$. Then $|x_i - x_j|, |x_i - x_{j'}| \leq \ell$, so

$$F_{ij} \leq e^{-\ell^{-\varepsilon}h(x_i - x_{j'})} \leq e^{-c\ell^{-\varepsilon}}$$

so any overlap forces F_{ij} to be exponentially small. This proves (B.1) and also (B.2). □

We have analogous controls on the derivatives of F_{ij} as well. We define the sum norm of a vector $\mathbf{x} = (x_1, \dots x_N) \in \mathbb{R}^{3N}$ as

$$\| \mathbf{x} \| := \sum_{k=1}^{N} |x_k|$$

where $|x|$ is the Euclidean length in $\mathbb{R}^3$. Similarly, if $\mathbf{A} = (A_1, \dots, A_N)$, where A_j is an n-tensor on $\mathbb{R}^3$ for each j, then

$$\| \mathbf{A} \| := \sum_{k=1}^{N} |A_k|_{(\mathbb{R}^3)^{\otimes n}} \ ,$$

where $|A_k| = |A_k|_{(\mathbb{R}^3)^{\otimes n}}$ denotes the tensor norm on $(\mathbb{R}^3)^{\otimes n}$ derived from the Euclidean norm. In particular, if $\boldsymbol{\nabla} = (\nabla_1, \dots, \nabla_N)$ denotes the $3N$ dimensional derivative, and $\alpha \in \mathbb{N}$, we have

$$\| \boldsymbol{\nabla}^\alpha f \| := \sum_{k_1} \sum_{k_2} \dots \sum_{k_\alpha} |\nabla_{k_1} \dots \nabla_{k_\alpha} f|_{(\mathbb{R}^3)^{\otimes \alpha}}$$

where

$$|\nabla_{k_1} \dots \nabla_{k_\alpha} f|_{(\mathbb{R}^3)^{\otimes \alpha}} = \sup \Big\{ \Big| (\nabla_{k_1} \dots \nabla_{k_\alpha} f)(v_1 \otimes v_2 \otimes \dots \otimes v_\alpha) \Big| \ : \ v_j \in \mathbb{R}^3 \ , \ |v_j| = 1 \Big\}$$

with

$$(\nabla_{k_1} \dots \nabla_{k_\alpha} f)(v_1 \otimes v_2 \otimes \dots \otimes v_\alpha) = \sum_{\beta_1, \dots \beta_\alpha = 1}^{3} \Big(\prod_{j=1}^{\alpha} v_{j\beta_j} \Big) \Big(\prod_{j=1}^{\alpha} \frac{\partial}{\partial x_{j\beta_j}} \Big) f$$

with $v_j = (v_{j1}, v_{j2}, v_{j3})$.

Lemma B.2 (Control of the derivatives of F_{ij}). *Let $\ell \ll 1$, i.e. $\ell \leq N^{-\kappa}$ for some $\kappa > 0$. For sufficiently large N the following pointwise bounds hold:*

i) Let $\alpha \in \mathbb{N}$ and $q > 0$. Then for any fixed i, j

$$\| \boldsymbol{\nabla}^\alpha F_{ij}^q \| \leq c_\alpha \ell^{-\alpha} F_{ij}^{q/2} \ . \tag{B.3}$$

ii) Let $K > 0$ and recall the definition of θ_{kj} from (2.6). For any $\alpha \in \mathbb{N}$ and for $k \neq i, j$, and arbitrary m

$$|\nabla_k^\alpha F_{ij}^q| \leq c\ell^{-\alpha} (\theta_{ik} + \theta_{jk}) F_{ij}^{q/2} + O(\ell^{K-\alpha-\varepsilon}), \tag{B.4}$$

$$|\nabla_k \nabla_m F_{ij}^q| \leq c\ell^{-2} (\theta_{ik} + \theta_{jk}) F_{ij}^{q/2} + O(\ell^{K-2-\varepsilon}) \ . \tag{B.5}$$

iii) For any fixed $\alpha \in \mathbb{N}$ and index k

$$\sum_{ij} |\nabla_k^\alpha F_{ij}^q| \widetilde{\chi}_{ij} \leq c_{q,\alpha} \ell^{-\alpha} \ . \tag{B.6}$$

iv) For any fixed $\alpha, \beta \in \mathbb{N}$ and index k

$$\sum_{ijm} |\nabla_k^\alpha \nabla_m^\beta F_{ij}^q| \widetilde{\chi}_{ij} \leq c_{q,\alpha,\beta} \ell^{-\alpha-\beta} \ . \tag{B.7}$$

Proof. For i) note that

$$|\nabla^\alpha h(x)| \le c_\alpha \ell^{-\alpha} h(x)$$

for $\alpha \in \mathbb{N}$. Then

$$\| \boldsymbol{\nabla} F_{ij}^q \| = |\nabla_i F_{ij}^q| + |\nabla_j F_{ij}^q| + \sum_{k\neq i,j} |\nabla_k F_{ij}^q| \le cq\ell^{-1}\ell^{-\varepsilon} \sum_{k\neq i,j} [h_{ki} + h_{kj}] F_{ij}^q \le c\ell^{-1} F_{ij}^{q/2}$$

using that $ze^{-z} \le ce^{-z/2}$ with $z = cqN\ell^{-\varepsilon}$. For higher derivatives the proof is similar:

$$\| \boldsymbol{\nabla}^\alpha F_{ij}^q \| \le c\ell^{-\alpha} \Big[\ell^{-\varepsilon} \sum_{k\neq i,j} [h_{ki} + h_{kj}] \Big]^\alpha F_{ij}^q \le c_\alpha \ell^{-\alpha} F_{ij}^{q/2}$$

using that $z^\alpha e^{-z} \le c_\alpha e^{-z/2}$ with $z = \mathcal{N}\ell^{-\varepsilon}$.

For the proof of ii) we note that for $k \neq i, j$ the derivative $\nabla_k F_{ij}$ is smaller than $\ell^{K-\varepsilon}$ unless $|x_i - x_k|$ or $|x_j - x_k|$ is smaller than $K\ell|\log \ell|$, therefore

$$\begin{aligned} |\nabla_k F_{ij}^q| &\le c\ell^{-1-\varepsilon} \Big[\theta_{ki} + \theta_{kj} \Big] \, (h_{kj} + h_{ki}) \, F_{ij}^q + \ell^{K-1-\varepsilon} \\ &\le c\ell^{-1} \Big[\theta_{ki} + \theta_{kj} \Big] F_{ij}^{q/2} + \ell^{K-1-\varepsilon} \qquad k \neq i, j \; . \end{aligned}$$

The proof for higher derivatives is similar.

As for iii), we use the previous estimate when $i, j \neq k$

$$\sum_{i,j\neq k} |\nabla_k^\alpha F_{ij}^q| \widetilde{\chi}_{ij} \le c\ell^{-\alpha} \sum_{i,j\neq k} \theta_{kj} F_{ij}^{q/2} \widetilde{\chi}_{ij} + O(N^2 \ell^{K-\alpha-\varepsilon}) \; .$$

We consider the set $S_k = \{j \ : \ |x_j - x_k| \le K\ell|\log \ell|\}$. If $j, j' \in S_k$, and $|x_j - x_{j'}| \le \frac{\varepsilon}{2}\ell|\log \ell|$, then $F_{ij} \le \exp(-\ell^{-\varepsilon/2})$. Therefore, apart from exponentially small error, the cardinality of S_k is $(2K/\varepsilon)^3$, since if there are more j's in the set S_k, then $|x_j - x_{j'}| \le (\varepsilon/2)\ell|\log \ell|$, at least for two indices $j, j' \in S_k$. The same argument holds for the i indices, showing that the i summation is only over a finite set. This implies that

$$\sum_{i,j\neq k} \theta_{kj} F_{ij}^{q/2} \widetilde{\chi}_{ij} \le \sum_{j\in S_k, j\neq k} \; \sum_{i \in S_k i\neq k} F_{ij}^{q/2} \widetilde{\chi}_{ij} \le c_q (K/\varepsilon)^6 \; .$$

Choosing K sufficiently large we obtain the bound (B.6) for $i, j \neq k$. For the terms with $k = i$ or $k = j$ we use

$$|\nabla_k^\alpha F_{kj}^q| \le |\boldsymbol{\nabla}^\alpha F_{kj}^q| \le c\ell^{-\alpha} F_{kj}^{q/2} \tag{B.8}$$

and the j summation is finite by (B.2).

Finally, the proof of iv) is similar. If $k = i$, then we have

$$\sum_{jm} |\nabla_k^\alpha \nabla_m^\beta F_{kj}^q| \widetilde{\chi}_{kj} \le c_{\alpha,\beta} \ell^{-\alpha-\beta} \sum_j F_{kj}^{q/2} \widetilde{\chi}_{kj} = c_{q,\alpha,\beta} \ell^{-\alpha-\beta}$$

using (B.3) and (B.2) and the same estimate holds if $k = j$. If $i, j \neq k$, then again the cardinality of the index set S_k for i and j is bounded, apart from an exponentially small error. Therefore

$$\sum_{\substack{ijm \\ i,j\neq k}} |\nabla_k^\alpha \nabla_m^\beta F_{ij}^q| \widetilde{\chi}_{ij} \le \sum_{\substack{i,j\in S_k \\ i,j\neq k}} \sum_m |\nabla_k^\alpha \nabla_m^\beta F_{ij}^q| \le c_{\alpha,\beta} \ell^{-\alpha-\beta} (K/\varepsilon)^6 + O(N^3 \ell^{K-\alpha-\beta-2\varepsilon})$$

using (B.3). ∎

C Local Structure after Particle Removal

Lemma C.1. *There exists a constant $C_0 > 0$ such that for any k*

$$C_0^{-1} \leq \frac{W^{(k)}}{W} \leq C_0 \tag{C.1}$$

pointwise for sufficiently large N. Moreover, for any $\alpha \in \mathbb{N}$ and sufficiently large N we have the pointwise estimate

$$C_0^{-\alpha} \leq \frac{W^{(k_1 \dots k_\alpha)}}{W} \leq C_0^{\alpha} \tag{C.2}$$

uniformly for any family of indices $k_1, \dots, k_\alpha$.

Proof. Similarly to the proof of (4.2), we see that for any fixed k and sufficiently big $N \geq N(k)$, all $G_i^{(k)}$ are separated away from zero uniformly in i. Therefore it is sufficient to show that

$$\sum_{i \neq k} |G_i^{(k)} - G_i| \leq \sum_{i \neq k} \Big[w_{ik} F_{ik} + \sum_{m \neq i,k} w_{im} |F_{im}^{(k)} - F_{im}| \Big] \tag{C.3}$$

is uniformly bounded for any fixed k. The boundedness of the first term follows from (B.2). As for the second term, note that

$$|F_{im}^{(k)} - F_{im}| \leq \Big| 1 - e^{-\ell^{-\varepsilon}[h_{ik} + h_{mk}]} \Big| F_{im}^{(k)} \leq \ell^{-\varepsilon} [h_{ik} + h_{mk}] F_{im}^{(k)} ,$$

so

$$\sum_{i \neq k} \sum_{m \neq i,k} w_{im} |F_{im}^{(k)} - F_{im}| \leq 2\ell^{-\varepsilon} \sum_{i,m \neq k} \chi_{im} h_{ik} F_{im}^{(k)}$$

since $w_{im} \leq \chi_{im}$ by the support of w. If $|x_i - x_k| \geq K\ell |\log \ell|$ for some i, then $h_{ik} \leq \ell^K$ and and if K is a large constant, then $\ell^{K-\varepsilon} N^2 \to 0$, so this term is negligible even after the summation. Now we look at the set

$$S_k := \{ i \; : \; i \neq k, |x_k - x_i| \leq K\ell |\log \ell| \} .$$

If $i, i' \in S_k$, $i \neq i'$ and $|x_i - x_{i'}| \leq \frac{\varepsilon}{2} \ell |\log \ell|$ then $F_{im}^{(k)} \leq \exp(-\ell^{-\varepsilon} h_{ii'}) \leq \exp(-\ell^{-\varepsilon/2})$ that is exponentially small. Therefore, modulo exponentially small errors, $|S_k| \leq (K/(\varepsilon/2))^3$, which guarantees that the summation over i in (C.3) is finite. For each fixed i the summation over m is finite by (B.2). The second bound (C.2) follows easily by induction. $\square$

The same proof immediately gives the following two bounds that are used in the proof. Here k is fixed and the constants may depend on k.

$$|F_{im}^{[k]} - F_{im}| \leq C\ell^{-\varepsilon} \sum_{r \leq k} (\theta_{ir} + \theta_{mr}) F_{im}^{[k]} + O(\ell^{K-\varepsilon}) \tag{C.4}$$

and

$$\sum_j |\nabla_j^{\alpha} (F_{im}^{[k]} - F_{im})| \leq C\ell^{-\alpha-\varepsilon} \sum_{r \leq k} (\theta_{ir} + \theta_{mr}) (F_{im}^{[k]})^{1/2} + O(\ell^{K-\alpha-\varepsilon}) . \tag{C.5}$$

We will often multiply these inequalities by w_{im}, then on the support of w_{im} we can use that θ_{ir} and θ_{mr} are comparable. In particular, we also obtain for each m

$$|G_m^{[k]} - G_m| \leq C\ell^{-\varepsilon} \sum_{j > k} \sum_{r \leq k} w_{mj} F_{mj}^{[k]} \theta_{mr} + O(\ell^{K-\varepsilon}) . \tag{C.6}$$

D Upper bound for $\widetilde{H}^2$

The aim of this appendix is to prove that the assumptions of Theorem 2.1 about the energy distribution of the initial data $\psi_{N,0} = W\phi_{N,0}$ are satisfied for a large class of $\phi_{N,0}$. In particular, in the next lemma we prove that $(W\phi_{N,0}, \widetilde{H}^2 W\phi_{N,0}) \leq CN^2$ is satisfied, if the function $\phi_{N,0}$ is sufficiently smooth. The proof of the inequality $(W\phi_{N,0}, \widetilde{H} W\phi_{N,0}) \leq CN$ for sufficiently smooth $\phi_{N,0}$ is similar (but much easier) and therefore omitted.

Lemma D.1. *Assume $a \ll \ell_1 \ll \ell \ll 1$, $a\ell_1 \ll \ell^4$ and $\ell_1^3 \ll a\ell^{5/2}$. Suppose moreover that ϕ satisfies*

$$\sum_{i,j,k,m} \int \overline{\phi}(1-\Delta_i)(1-\Delta_j)(1-\Delta_k)(1-\Delta_m)\phi \leq DN^4 \tag{D.1}$$

for some $D > 0$. Then there is a constant $C > 0$ such that

$$(W\phi, \widetilde{H}^2\, W\phi) \leq CN^2\,.$$

Proof. Following the steps from (6.15) to (6.18) (but this time without neglecting the positive contributions), we find (using the summation convention)

$$\begin{aligned}(W\phi, \widetilde{H}^2\, W\phi) = & \int |\widetilde{H} W\phi|^2 \\ \leq & \int W^2 |\nabla_i \nabla_j \phi|^2 + C\int W^2 \left(|\nabla_i\nabla_j G_\ell| + |\nabla_i G_\ell||\nabla_j G_\ell|\right) |\nabla_i \phi||\nabla_j\phi| \\ & + 2\int W^2 |B||\nabla_m \phi|^2 + 2\int W^2 |\nabla_m B||\phi||\nabla_m\phi| + \int W^2 B^2 |\phi|^2,\end{aligned}$$

with $B = q_{kj} + \Omega = q_{kj} + \widetilde{\Omega} + O\left(e^{-C\ell^{-\varepsilon}}\right)$. Using Lemmas 6.3 - 6.6, we get

$$\begin{aligned}\int |\widetilde{H} W\phi|^2 \leq C\int W^2 \left(|\nabla_i\nabla_j\phi|^2 + N|\nabla_i\phi|^2 + N^2|\phi|^2\right) + C\int W^2 q_{ij}|\nabla_m\phi|^2 \\ + C\int W^2 |\nabla_i G_\ell||\nabla_j G_\ell||\nabla_i\phi||\nabla_j\phi| + \int W^2 B^2|\phi|^2\,.\end{aligned}$$

Since $W \leq 1$, and by (D.1), we find

$$\begin{aligned}\int |\widetilde{H} W\phi|^2 \leq CN^2 + C\int |\nabla_i G_\ell||\nabla_j G_\ell||\nabla_i\phi|^2 + C\int q_{ij}|\nabla_m\phi|^2 \\ + C\int q_{ij}q_{km}|\phi|^2 + C\int \widetilde{\Omega}^2|\phi|^2\,.\end{aligned} \tag{D.2}$$

We start by considering the second term on the r.h.s. of the last equation. Using Lemma B.1, we find

$$\begin{aligned}\int |\nabla_i G_\ell||\nabla_j G_\ell||\nabla_i\phi|^2 \leq C\int \Bigg(& |(\nabla w)_{mi}|^2 F_{mi} + |(\nabla w)_{im}| w_{im} F_{im} |\nabla_j F_{im}| \\ & + |(\nabla w)_{jm}| w_{jm} F_{jm}|\nabla_i F_{jm}| + w_{jm}^2 |\nabla_i F_{jm}||\nabla_n F_{jm}|\Bigg)|\nabla_i\phi|^2 \\ & + O(e^{-c\ell^{-\varepsilon}})\,.\end{aligned}$$

Using $|\nabla w|^2 \le C\lambda$, $|\nabla w| \le Ca\lambda$, Lemma B.2, Lemma 5.3 (with $W \equiv 1$) and (5.6) we find

$$\begin{aligned}
\int |\nabla_i G_\ell||\nabla_j G_\ell||\nabla_i \phi|^2 &\le C \int \lambda_{mi} F_{mi}^{1/2} |\nabla_i \phi|^2 + Ca\ell^{-1} \int \lambda_{mj} F_{mj} |\nabla_i \phi|^2 \\
&\quad + C\ell^{-2} \int |\nabla_i \phi|^2 + O(e^{-c\ell^{-\varepsilon}}) \\
&\le C \int \widetilde{\chi}_{mi} F_{mi}^{1/2} |\nabla_m \nabla_i \phi|^2 + \ell_1 \ell^{-3} \int \widetilde{\chi}_{mi} F_{mi}^{1/2} |\nabla_i \phi|^2 \\
&\quad + Ca\ell^{-1} \int \widetilde{\chi}_{mj} F_{mj}^{1/2} |\nabla_j \nabla_i \phi|^2 \\
&\quad + Ca\ell_1 \ell^{-4} \int \widetilde{\chi}_{mj} F_{mj}^{1/2} |\nabla_i \phi|^2 + O(e^{-c\ell^{-\varepsilon}}) \\
&\le C(1 + a\ell^{-1}) \int |\nabla_m \nabla_i \phi|^2 + \ell_1 \ell^{-4} \int |\nabla_i \phi|^2 \\
&\quad + O(e^{-c\ell^{-\varepsilon}}) ,
\end{aligned}$$

which is bounded by CN^2, because $a\ell_1\ell^{-4} \ll 1$, and because of (D.1).

Next we consider the third term in (D.2); we obtain

$$\begin{aligned}
\int q_{ij} |\nabla_m \phi|^2 &\le Ca\ell_1^{-3} \int \chi_{ij} |\nabla_m \phi|^2 \\
&\le Ca \int \left(|\nabla_i \nabla_j \nabla_m \phi|^2 + N |\nabla_i \nabla_m \phi|^2 + N^2 |\nabla_m \phi|^2 \right) \\
&\le CN^2
\end{aligned}$$

using (5.4) and (D.1). As for the fourth term on the r.h.s. of (D.2), we have

$$\begin{aligned}
\int q_{ij} q_{km} |\phi|^2 &\le Ca^2 \ell_1^{-6} \Big(\sum_{(i,j) \neq (k,m)} \int \chi_{ij} \chi_{km} |\phi|^2 + \sum_{i,j} \int \chi_{ij} |\phi|^2 \Big) \\
&\le Ca^2 \sum_{(i,j) \neq (k,m)} \int \overline{\phi} (1 - \Delta_i)(1 - \Delta_j)(1 - \Delta_k)(1 - \Delta_m) \phi \\
&\quad + Ca^2 \ell_1^{-3} \int \overline{\phi} (1 - \Delta_i)(1 - \Delta_j) \phi \\
&\le Ca^2 N^4 + a^2 \ell_1^{-3} N^2 \le CN^2
\end{aligned}$$

because of $a \ll \ell_1^{3/2}$ and of (D.1).

Finally we consider the last term on the r.h.s. of (D.2). We use that, by (6.12), $\widetilde{\Omega}^2 \le C\Omega_{kj}\Omega_{im} + C\Gamma^2$. By Lemma B.2 and by the estimate $w \le a\ell\lambda$, $|\nabla w| \le a\lambda$, we obtain

$$\begin{aligned}
\int \Omega_{kj} \Omega_{im} |\phi|^2 &\le C \int \Bigg(w_{sj} w_{rm} |\Delta_k F_{sj}||\Delta_i F_{rm}| \\
&\quad + |(\nabla w)_{kj}||(\nabla w)_{im}||\nabla_k F_{kj}||\nabla_i F_{im}| \Bigg) |\phi|^2 \\
&\le Ca^2 \ell^{-2} \int \lambda_{kj} \lambda_{im} F_{kj}^{1/2} F_{im}^{1/2} |\phi|^2 .
\end{aligned} \tag{D.3}$$

We note that, if, for example $k = i$, then j is forced to be equal to m, up to exponentially small errors, due to the strong non-overlapping properties of the functions F_{kj}. Hence, up to errors which are exponentially small in N and using Lemma 5.3, we can estimate the second term on the r.h.s of (D.3) by

$$\begin{aligned}
\int \Omega_{kj}\Omega_{im}|\phi|^2 &\leq Ca^2\ell^{-2}\sum_{kj}\int \lambda_{kj}^2 F_{kj}|\phi|^2 + Ca^2\ell^{-2}\sum_{(k,j)\neq(i,m)}\int \lambda_{kj}\lambda_{im}F_{kj}^{1/2}F_{im}^{1/2}|\phi|^2 \\
&\leq C\ell^{-2}a\int \sigma_{kj}|\phi|^2 + C\ell^{-2}a^2\sum_{(k,j)\neq(i,m)}\int \lambda_{kj}\lambda_{im}F_{kj}^{1/2}F_{im}^{1/2}|\phi|^2 \\
&\leq Ca\ell^{-2}(\log N)\int \overline{\phi}(1-\Delta_k)(1-\Delta_j)\phi \\
&\quad + Ca^2\ell^{-2}\sum_{(k,j)\neq(i,m)}\left(\int \lambda_{im}F_{im}^{1/2}|\nabla_j\phi|^2 + \ell_1\ell^{-3}N\int \lambda_{im}F_{im}^{1/2}|\phi|^2\right) \\
&\leq o(N^2) + Ca^2\ell^{-2}\int |\nabla_i\nabla_j\phi|^2 + Ca^2\ell^{-5}\ell_1 N\int |\nabla_j\phi|^2 \\
&\quad + Ca^2\ell^{-8}\ell_1^2N^2\int |\phi|^2\,,
\end{aligned}$$

which is $o(N^2)$ because $a\ell_1 \ll \ell^4$. We still have to control the contribution from Γ^2. Using Lemma B.2 we have

$$\begin{aligned}
\int \Gamma^2|\phi|^2 \leq C\int &\Bigg(|(\nabla w)_{ik}||(\nabla w)_{mr}|w_{jp}w_{ns}F_{ik}F_{mr}|\nabla_k F_{jp}||\nabla_r F_{ns}| \\
&\quad + w_{ik}w_{mr}w_{jp}w_{ns}|\nabla_q F_{ik}||\nabla_q F_{mr}||\nabla_d F_{jp}||\nabla_d F_{ns}|\Bigg)|\phi|^2 \\
\leq C\int &\left(\ell^{-2}|(\nabla w)_{ik}||(\nabla w)_{mr}|F_{ik}F_{mr} + \ell^{-4}w_{ik}w_{mr}F_{ik}F_{mr}\right)|\phi|^2 \\
\leq Ca^2&\ell^{-2}\int \lambda_{ik}\lambda_{mr}F_{ik}F_{mr}|\phi|^2
\end{aligned} \tag{D.4}$$

which can be bounded by CN^2 in the same way we bounded the r.h.s. of (D.3). This completes the proof of the lemma. □

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