The spectrum minimum for random Schrödinger operators with indefinite sign potentials

24th October 2005

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Abstract

This paper sets out to study the spectral minimum for operator belonging to the family of random Schrödinger operators of the form $H_{\lambda,\omega} = -\Delta + W_{\text{per}} + \lambda V_\omega$, where we suppose that V_ω is of Anderson type and the single site is assumed to be with an indefinite sign. Under some assumptions we prove that there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$, the minimum of the spectrum of $H_{\lambda,\omega}$ is obtained by a given realization of the random variables.

2000 Mathematics Subject Classification:81Q10, 35P05, 37A30, 47F05.
Keywords and phrases: spectral theory, random Schrödinger operators.

1 Introduction

Among the most investigated and dealt with operators in the field of mathematical physics problems are random Schrödinger operators of the form

$$H_\omega = -\Delta + W_{\text{per}} + V_\omega = H_0 + V_\omega, \tag{1.1}$$

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Researches partially supported by CMCU N 04/S1404 and Research Unity 01/UR/ 15-01 projects.

where W_{per} is a $\mathbb{Z}^d$-periodic function and V_ω is a random potential having the Anderson form, i.e $V_\omega(\cdot) = \sum_{n\in\mathbb{Z}^d}\omega_\gamma f(\cdot - \gamma)$. See [2, 12], for the physical motivations.
The study of the spectral theory of operators of the form (1.5) have drawn the attention of many researchers for the importance of the related results. In fact, it is linked to the systems evolutions for which the Hamiltonian is described by (1.5). The goal of this paper is to discuss one of the problems that remain unsolved: the spectrum location of H_ω, precisely the spectrum infimum. This will be carried out in the case when the single site f does not have a definite sign.
As the main object is to study the location of the spectrum, let us recall the following basic results already known on this subject and stated by Kirsch and Martinelli [2, 3]:

Theorem 1.1

$$\Sigma(H_{,\omega}) = \overline{\bigcup_{\omega_\gamma\in\mathcal{P}} \Sigma\Big(-\Delta + W_{per} + \sum_{\gamma\in\mathbb{Z}^d}\omega_\gamma f(x-\gamma)\Big)}. \tag{1.2}$$

Here $\mathcal{P}$ is the set of all periodic sequences $\{\omega_n\}_{n\in\mathbb{Z}^d}$, with an arbitrary period such that ω_n is in the support of μ for all n and $\Sigma(H)$ is the spectrum of H.

As has been said above the proof of Theorem 1.1 exists in [2, 3] and is based on Weyl sequences and probabilistic arguments. Notice that this theorem reduces the determination of the spectra of random Schrödinger operators for the case of periodic Schrödinger operators. As it is well known [9] that the spectrum of periodic operators have a band structure, this will be the case for $\Sigma(H_\omega)$ with the possibility to close gaps.
Under additional assumptions on f more is known:

Theorem 1.2 *If f has a fixed sign, i.e $f \le 0$ or $f \ge 0$ and if μ is supported $[\omega^-, \omega^+]$, then*

$$\Sigma(H_{,\omega}) = \bigcup_{\omega\in[\omega^-,\omega^+]} \Sigma\Big(-\Delta + W_{per} + \omega\sum_{\gamma\in\mathbb{Z}^d} f(x-\gamma)\Big). \tag{1.3}$$

In particular,

$$\inf\Sigma(H_{,\omega}) = \begin{cases} \inf\Sigma\Big(-\Delta + W_{per} + \omega^+\sum_{\gamma\in\mathbb{Z}^d} f(x-\gamma)\Big) & if\ \ f\le 0,\\ \inf\Sigma\Big(-\Delta + W_{per} + \omega^-\sum_{\gamma\in\mathbb{Z}^d} f(x-\gamma)\Big) & if\ \ f\ge 0.\end{cases} \tag{1.4}$$

Theorem 1.2 is a simple consequence of Theorem 1.1. Indeed, using (1.2) and the fact that constant sequences $\{\omega_\gamma = \omega \in \ supp\ \mu\} \in \mathcal{P}$, we get that the r.h.s of (1.3) is naturally contained in $\Sigma(H_{,\omega})$. For the inverse inclusion, let $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d} \in \mathcal{P}$ be k-periodic and let $[a, b]$ be the n-th band of the k periodic operator $H_{\omega,k} = -\Delta + W_{\text{per}} + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma f(x-\gamma)$. Let $[a_1, b_2]$ and $[a_2, b_2]$ be the n-th bands of respectively $-\Delta + W_{\text{per}} + \omega^- \sum_{\gamma\in\mathbb{Z}^d} f(x-\gamma)$ and $-\Delta + W_{\text{per}} + \omega^+ \sum_{\gamma\in\mathbb{Z}^d} f(x-\gamma)$ (both seen as k-periodic operators). By the min-max principle we have $a_1 \le a \le a_2$ and $b_1 \le b \le b_2$. As the bands of $-\Delta + W_{\text{per}} + \omega \sum_{\gamma\in\mathbb{Z}^d} f(x-\gamma)$ depend continuously on ω, we deduce that $\Sigma\Big(-\Delta + W_{\text{per}} + \sum_{\gamma\in\mathbb{Z}^d} \omega f(x-\gamma)\Big)$ is contained in the r.h.s of (1.3). The proof of (1.3) is ended by taking into account the fact that these sets are closed.
For (1.4) it is a simple consequence of monotonicity of the model, it is increasing when $f \ge 0$ and decreasing when $f \le 0$. indeed if $f \le 0$, and $\widetilde{\omega}_\gamma \le \hat{\omega}_\gamma$ for any $\gamma \in \mathbb{Z}^d$ then in the sense form we have

$$-\Delta + W_{\text{per}} + \sum_{\gamma\in\mathbb{Z}^d} \hat{\omega}_\gamma f(x-\gamma) \le -\Delta + W_{\text{per}} + \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma f(x-\gamma).$$

The situation is more complicated and different when the single site f changes the sign, as the monotonicity property is not true in this case. We notice that recently Lott and Stolz have conjectured [11] that in dimension one, the spectral minimum of random displacement models is realized through the pair formation of the single site.

1.1 The Model

Our basic object of study is the so-called *Anderson model*, a random Schrödinger operator of the form

$$H_{\lambda,\omega} = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + W_{\text{per}} + \lambda \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma f(x-\gamma), \tag{1.5}$$

where,

- W_{per} is a $\mathbb{Z}^d$-periodic and bounded function.

- λ is a positive parameter
- $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a family of independent, identically-distributed random variables taking values in $[\omega^-, \omega^+]$. We denote by μ the probability distribution.
- For $C_0 = [-\frac{1}{2}, \frac{1}{2}]$, the single site potential $f \in C_0^\infty(C_0)$ such that $f \in l^1(L^p(\mathbb{R}^d))$, with $p = 2$ if $d \leq 3$, $p > 2$ if $d = 4$ and $p = d/2$ if $d \geq 5$ and $f = f^+ + f^-$, with $f^+ \geq 0,\ f^- \leq 0$ and $f^+ \cdot\ f^- = 0$.

By [2, 8], we know that $H_{\lambda,\omega}$ is an essentially self-adjoint operator on $L^2(\mathbb{R}^d)$ with the domain $C_0^\infty(\mathbb{R}^d)$, we denote by $H_{\lambda,\omega}$ its self adjoint extension.
It is an ergodic operator so, according to [2, 8], we know that there exist $\Sigma_\lambda, \Sigma_{\lambda,pp}, \Sigma_{\lambda,ac}$ and $\Sigma_{\lambda,sc}$ closed and non-random sets of $\mathbb{R}$ such that Σ_λ is the spectrum of $H_{\lambda,\omega}$ with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of $H_{\lambda,\omega}$, then $\Sigma_{pp} = \sigma_{pp}, \Sigma_{\lambda,ac} = \sigma_{\lambda,ac}$ and $\Sigma_{\lambda,sc} = \sigma_{\lambda,sc}$ with probability one.

1.2 The result

As we will see (subsection2.1) $H_{\lambda,\omega}$ can be considered as a perturbation of some periodic operator H_{λ,ω^-}. Let $\varphi_{\lambda,1}(x, \theta(\lambda))$ be the Floquet eigenfunction associated to the first Floquet eigenvalue $E_1(\lambda, \theta)$ of H_{λ,ω^-}. Let $(\theta_k(\lambda))_{1\leq k\leq m}$ be the points where $E_1(\lambda, \theta)$ attains its minimum. We set

$$A(0) = \Big(\langle f\varphi_{0,1}(\cdot, \theta_k(0)), \varphi_{0,1}(\cdot, \theta_{k'}(0))\rangle_{L^2(C_0)} \Big)_{1\leq k,k'\leq m}.$$

We prove that

Theorem 1.3 *Let $H_{\lambda,\omega}$ be the operator defined by (1.5).*
If the matrix $A(0)$ is positive-definite. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$ we have:

$$\inf(\Sigma_\lambda) = \inf(\Sigma_{\lambda,\omega^-}).$$

If the matrix $A(0)$ is negative-definite. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$ we have:

$$\inf(\Sigma_\lambda) = \inf(\Sigma_{\lambda,\omega^+}).$$

Remark 1.4 *Theorem 1.3 is stated for the infimum of the spectrum. Under some additional assumptions the same result is still true for the internal edges of the spectrum.*
The analogous problem for the random magnetic Schrödinger operator is considered and studied in [1].
Theorem 1.3 can be considered as a first step toward the physically-motivated applications. One of them is the study of the so-called Lifshitz tails of the integrated density of states. This could be done under some additional assumptions on the behavior of the random variables in the vicinity of ω^- or ω^+, [4, 6, 7]. Another one is the spectral localization [12].

The proof of Theorem 1.3, is given in section 3. It is based on the reduction procedure. This powerful technique was predicted by Klopp [4] and used in several works, [1, 6, 7].
As stated the proof of Theorem 1.3 can be divided naturally divides into two parts, we shall discuss them separately.
Indeed, if $A(0)$ is positive-definite we will conjugate H_ω with $\Pi_{\lambda,0}$, the spectral projection for H_{λ,ω^-} on the first band. Then we prove that

$$\Pi_{\lambda,0} H_{\lambda,\omega} \Pi_{\lambda,0} \geq E_{\lambda,\omega^-} \Pi_{\lambda,0}.$$

Here E_{λ,ω^-} is the bottom of the spectrum of the periodic operator H_{λ,ω^-}.
If $A(0)$ is negative-definite we will conjugate H_ω with $\Pi_{\lambda,0}$, the spectral projection for H_{λ,ω^+} on the first band. Then we prove

$$\Pi_{\lambda,0} H_{\lambda,\omega} \Pi_{\lambda,0} \geq E_{\lambda,\omega^+} \Pi_{\lambda,0}.$$

Here E_{λ,ω^+} is the bottom of the spectrum of the periodic operator H_{λ,ω^+}.

2 Preliminary

Let us consider the following periodic operator

$$H_{\lambda,\omega^-} = -\Delta + W_{\text{per}} + \lambda \sum_{\gamma \in \mathbb{Z}^d} \omega^- f(\cdot - \gamma). \tag{2.6}$$

For this, it is convenient to consider $H_{\lambda,\omega}$ as a perturbation of H_{λ,ω^-}. Indeed, we have:

$$H_{\lambda,\omega} = H_{\lambda,\omega^-} + \lambda \sum_{\gamma \in \mathbb{Z}^d} (\omega_\gamma - \omega^-) f(\cdot - \gamma).$$

For this for $\gamma \in \mathbb{Z}^d$, we set $\widetilde{\omega}_\gamma = \omega_\gamma - \omega^-$ and $V_{\widetilde{\omega}}(\cdot) = \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma f(\cdot - \gamma)$. We notice that according to the definition of $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ we get that, $(\widetilde{\omega}_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a family of random positive and bounded variables.

2.1 Some Floquet Theory

For $\gamma \in \mathbb{Z}^d$, we denote by τ_γ the translation by γ operator i.e $(\tau_\gamma\varphi)(x) = \varphi(x-\gamma)$. We have, for any $\gamma \in \mathbb{Z}^d$

$$\tau_\gamma H_{\lambda,\omega^-}\tau_\gamma^* = \tau_\gamma^* H_{\lambda,\omega^-}\tau_\gamma = H_{\lambda,\omega^-}.$$

Then the so called, Floquet Theory, can be used to study H_{λ,ω^-}. For this, we review some standard facts from the Floquet theory for periodic operators. Basic references for this material are [5, 9, 10]. Let $\mathbb{T}^* = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$. We define $\mathcal{H}$ by

$$\mathcal{H} = \{u(x,\theta) \in L^2_{loc}(\mathbb{R}^d)\otimes L^2(\mathbb{T}^*); \forall (x,\theta,\gamma) \in \mathbb{R}^d\times\mathbb{T}^*\times\mathbb{Z}^d;\ u(x+\gamma,\theta) = e^{i\gamma\theta}u(x,\theta)\}.$$

$\mathcal{H}$ is equipped with the norm

$$\frac{1}{\mathrm{vol}(\mathbb{T}^*)}\int_{\mathbb{T}^*} \|u(x,\theta)\|^2_{L^2(C_0)}d\theta.$$

For $\theta \in \mathbb{R}^d$ and $u \in \mathcal{S}(\mathbb{R}^d)$; the Schwartz space of rapidly decreasing functions we define

$$(Uu)(x,\theta) = \sum_{\gamma\in\mathbb{Z}^d} e^{i\gamma\cdot\theta}u(x-\gamma).$$

U can be extended as a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H}$. Its inverse is given by the formula,

$$\text{for } u \in \mathcal{H},\quad (U^*u)(x) = \frac{1}{\mathrm{vol}(\mathbb{T}^*)}\int_{\mathbb{T}^*} u(x,\theta)d\theta.$$

U is a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H}$ and $H_{\lambda,\omega}$ admits the Floquet decomposition [5, 10]

$$UH_{\lambda,\omega^-}U^* = \frac{1}{\mathrm{vol}(\mathbb{T}^*)}\int_{\mathbb{T}^*}^{\oplus} H_{\lambda,\omega-}(\theta)d\theta.$$

Here $H_{\lambda,\omega^-}(\theta)$ is the operator H_{λ,ω^-} acting on $\mathcal{H}_\theta$, defined by

$$\mathcal{H}_\theta = \{u \in L^2_{loc}(\mathbb{R}^d); \forall\gamma \in \mathbb{Z}^d, u(x+\gamma) = e^{i\gamma\theta}u(x)\}.$$

As H_{λ,ω^-} is elliptic, we know that, $H_{\lambda,\omega^-}(\theta)$ has a compact resolvent; hence its spectrum is discrete [9]. We denote its eigenvalues, called Floquet eigenvalues of H_{λ,ω^-}, by

$$E_1(\lambda,\theta) \leq E_2(\lambda,\theta) \leq \cdots \leq E_n(\lambda,\theta) \leq \cdots.$$

The corresponding Floquet eigenfunctions are denoted by $(\varphi_{\lambda,j}(x,\cdot))_{j\in\mathbb{N}^*}$. The functions $(\theta \to E_n(\lambda,\theta))_{n\in\mathbb{N}^*}$ are Lipshitz-continuous, and we have

$$E_n(\lambda,\theta) \to +\infty \quad \text{as } n \to +\infty \quad \text{uniformly in} \quad \theta.$$

The spectrum $\Sigma_{\lambda,\omega^-}$ of H_{λ,ω^-} is made of bands (i.e $\Sigma_{\lambda,\omega^-} = \cup_{n\in\mathbb{N}^*} E_n(\lambda,\mathbb{T}^*)$.) Let us note by E_{λ,ω^-} the bottom of the spectrum of $\Sigma_{\lambda,\omega^-}$, i.e $E_{\lambda,\omega^-} = \inf_{\theta\in\mathbb{T}^d} E_1(\lambda,\theta)$.
It is a well-known fact that, in any dimension the bottom (the first band) of the spectrum of a periodic Schrödinger operators is given by a simple Floquet eigenvalue and that the minimum of this Floquet eigenvalue is non-degenerate and quadratic. More precisely let $\theta(\lambda)$ be an element of

$$Z_\lambda = \{\theta \in \mathbb{T}^*;\ E_1(\lambda,\theta) = E_{\lambda,\omega^-}\}.$$

Then there exist $C > 0$ and $\delta > 0$ such that

$$\forall\ |\theta - \theta(\lambda)| \leq \delta, \qquad \frac{1}{C}|\theta-\theta(\lambda)|^2 \leq E_1(\lambda,\theta) - E_{\lambda,\omega^-} \leq C|\theta-\theta(\lambda)|^2.$$

Hence, the points where $E_1(\lambda,\theta)$ reaches E_{λ,ω^-} are isolated and as $\mathbb{T}^*$ is compact, one concludes that Z_λ contains only finitely many of elements. Let m be the cardinal of Z_λ and let us denote them by $(\theta_k(\lambda))_{1\leq k\leq m}$. One can check that $\theta_k(\lambda)$ depends continuously on λ. For the sake of brevity, we use the notation $\theta_k = \theta_k(\lambda)$.
For $1 \leq k \leq m$ and $\theta \in \mathbb{T}^*$, we set

$$\zeta_{k,\lambda}(\theta) = \sum_{1\leq i\leq d} (\theta_i - \theta_{k,i})^2. \tag{2.7}$$

We notice that for any $1 \leq k \leq m$,

$$\theta \mapsto \varphi_{\lambda,1}(x,\theta),$$

is analytic in a neighborhood of θ_k.

2.2 Wannier basis

We recall concepts used in [4]. Let $\mathcal{E} \subset L^2(\mathbb{R}^d)$ be a closed subspace invariant by the $\mathbb{Z}^d$-translations, i.e. such that $\Pi^{\mathcal{E}}$, the orthogonal projection on $\mathcal{E}$, satisfies

$$\forall \gamma \in \mathbb{Z}^d,\ \Pi^{\mathcal{E}} = \tau_{-\gamma} \Pi^{\mathcal{E}} \tau_\gamma.$$

Following the computations done in section 1.2 of [4], we see that there exists an orthonormal system of vectors $(\widetilde{\varphi}_{j,0})_{j \in N}$ such that for $\widetilde{\varphi}_{j,\gamma} = \tau_\gamma(\widetilde{\varphi}_{j,0})$; $(\widetilde{\varphi}_{j,\gamma})_{(j \in N; \gamma \in \mathbb{Z}^d)}$ is an orthonormal basis of $\mathcal{E}$. Such system is called **Wannier basis** of $\mathcal{E}$. The vectors $(\widetilde{\varphi}_{n,0})_{n \in N}$ are called **Wannier generators** of $\mathcal{E}$.
Let $\mathcal{E} \subset L^2(\mathbb{R}^d)$ be a space which is translation-invariant. $\mathcal{E}$ is said to be of finite energy for $H_{\lambda,\omega}$ if $\Pi^{\mathcal{E}} H_{\lambda,\omega} \Pi^{\mathcal{E}}$ is a bounded operator. In this case, $\mathcal{E}$ admits a finite set of Wannier generators.
Let $\Pi_{\lambda,0}(\theta)$ (respectively $\Pi_{\lambda,+}(\theta)$) be the orthogonal projection in $\mathcal{H}_\theta$ on the vector space generated by $\varphi_{\lambda,1}(\cdot,\theta)$ (respectively by $(\varphi_{\lambda,j}(\cdot,\theta))_{j \geq 2}$). These projections are two-by-two orthogonal and their sum is the identity for all $\theta \in \mathbb{T}^*$. One defines

$$\Pi_{\lambda,\alpha} = U^{-1}\Big(\int_{\mathbb{T}^*} \Pi_{\lambda,\alpha}(\theta) d\theta\Big) U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),$$

where $\alpha \in \{0,+\}$. $\Pi_{\lambda,\alpha}$ is an orthogonal projection on $L^2(\mathbb{R}^d)$ and for all $\gamma \in \mathbb{Z}^d$, we have $\tau_\gamma^* \Pi_{\lambda,\alpha} \tau_\gamma = \Pi_{\lambda,\alpha}$.
For $\alpha \in \{0,+\}$, we set $\mathcal{E}_{\lambda,\alpha} = \Pi_{\lambda,\alpha}(L^2(\mathbb{R}^d))$. These spaces are translation-invariant. Moreover $\mathcal{E}_{\lambda,0}$ is of finished energies for H_{λ,ω^-}. The reduction procedure consists in decomposing the operator H_{λ,ω^-} according to various translation-invariants subspaces. The random operators thus obtained are reference operators.

3 The proof of Theorem 1.3

As we have indicated, our aim in this section is to prove Theorem 1.3, but first, let us introduce some notations and useful lemma.
For $u \in L^2(\mathbb{T}^*)$, let

$$\mathcal{T}_{\varphi_{\lambda,1}}(u) = U^*(u\varphi_{\lambda,1}(x,\theta)) = \int_{\mathbb{T}^*} u(\theta)\varphi_{\lambda,1}(x,\theta) d\theta. \tag{3.8}$$

So, $\mathcal{T}_{\varphi_{\lambda,1}}$ define a unitary transformation from $L^2(\mathbb{T}^*)$ to $\mathcal{E}_{\lambda,0}$ and for $v \in \mathcal{E}_{\lambda,0}$ we have

$$\mathcal{T}^*_{\varphi_{\lambda,1}}(v) = \langle (Uv)(\cdot,\theta), \varphi_{\lambda,1}(\cdot,\theta)\rangle.$$

For $1 \leq k \leq m$ and $(x,\theta) \in \mathbb{R}^d \times \mathbb{T}^*$, let

$$\widetilde{\varphi}_{\lambda,1,k}(x,\theta) = \varphi_{\lambda,1}(\theta_k(\lambda), x)e^{(\theta-\theta_k)\cdot x}.$$

We set,

$$\delta\varphi_{\lambda,1,k}(x,\theta) = \frac{1}{\sqrt{\zeta_{k,\lambda}(\theta)}}\Big(\varphi_{\lambda,1} - \varphi_{\lambda,1,k}(x,\theta)\Big).$$

By this, for any $u \in \text{Ł}^2(\mathbb{T}^*)$, we have

$$\mathcal{T}_{\varphi_{\lambda,1}}(u) = \mathcal{T}_{\widetilde{\varphi}_{\lambda,1,k}}(u) + \mathcal{T}_{\delta\varphi_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda}}u). \tag{3.9}$$

For $v \in \mathcal{H}^2_\theta = \{v \in H^2_{\rm loc}(\mathbb{R}^d); v(\cdot-\gamma) = e^{-i\gamma\cdot\theta}u(\cdot)\}$ one defines the following norms:

$$\sup_{\theta\in\mathbb{T}^*}\{\|v(\cdot,\theta)\|^2_{L^2(C_0)}\} = \|v\|^2_{1,\infty}.$$

and

$$\sup_{\theta\in\ \mathbb{T}^*}\Big(\|H_{\lambda,\omega^-}(\theta)u(\cdot,\theta)\|^2_{L^2(C_0)} + ||v(\cdot,\theta)||^2_{L^2(C_0)}\Big) = ||v||_{H_{\lambda,\omega^-},\infty}.$$

Remark 3.1 *The functions $\widetilde{\varphi}_{\lambda,1,k}$ and $\delta\varphi_{\lambda,1,k}$ are well defined and*

$$\|\widetilde{\varphi}_{\lambda,1,k}\|_{1,\infty},\ \|\widetilde{\varphi}_{\lambda,1,k}\|_{H_{\lambda,\omega^-},\infty}\ \|\delta\widetilde{\varphi}_{\lambda,1,k}\|_{1,\infty},\ \text{and}\ \|\delta\widetilde{\varphi}_{\lambda,1,k}\|_{H_{\lambda,\omega^-},\infty};$$

are finished (See [4]).

The following Lemma is of use. It will be proven at the end of this section.

Lemma 3.2 *For $\theta_k, \theta_{k'} \in \mathbb{T}^*$ and $\varphi \in L^2(\mathbb{T}^*, \mathcal{H}^2)$ let $\varphi_k = e^{i(\theta-\theta_k)x}\varphi(x,\theta_k)$, $\varphi_{k'} = e^{i(\theta-\theta_{k'})x}\varphi(x,\theta_{k'})$ and $a_{\varphi,k,k'}(x) = f(x)\varphi(x,\theta_k)\varphi(x,\theta_{k'})$. If $\|\varphi\|_{1,\infty} < \infty$ (resp. $\|\varphi\|_{H_{\lambda,\omega^-},\infty} < \infty$) then $\mathcal{T}_\varphi \in \mathcal{L}(L^2(\mathbb{T}^*), L^2(\mathbb{R}^d))$ (resp. $V_{\widetilde{\omega}}\cdot\mathcal{T}_\varphi \in \mathcal{L}(L^2(\mathbb{T}^*), L^2(\mathbb{R}^d))$) and there exist $C, \beta > 0$ such that for all $u, v \in L^2(\mathbb{T}^*)$, we have*

$$\Big|\langle V_{\widetilde{\omega}}\mathcal{T}_{\varphi_k}(u), \mathcal{T}_{\varphi_{k'}}(v)\rangle - \Big(\int_{C_0} a_{\varphi,k,k'}(x)dx\Big)\cdot\sum_{\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma\hat{u}(\gamma)\overline{\hat{v}(\gamma)}\Big|$$

$$\leq C\beta\sum_{\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma(|\hat{u}(\gamma)|^2+|\hat{v}(\gamma)|^2)+C(1+1/\beta)(\langle\zeta_{k,\lambda}u,u\rangle_{L^2(\mathbb{T}^*)}+\langle\zeta_{k',\lambda}v,v\rangle_{L^2(\mathbb{T}^*)}). \tag{3.10}$$

$$\|V_{\widetilde{\omega}}\mathcal{T}_{\varphi_k}(u)\|^2_{L^2(\mathbb{R}^d)} \le C\Big(\sum_{\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma|\hat{u}(\gamma)|^2 + \langle\zeta_{k,\lambda}u,u\rangle_{L^2(\mathbb{T}^*)}\Big). \tag{3.11}$$

3.1 If $A(0)$ is positive-definite

We set

$$H^0_{\lambda,\omega} = \Pi_{\lambda,0}H_{\lambda,\widetilde{\omega}}\Pi_{\lambda,0} = \Pi_{\lambda,0}H_{\lambda,\omega^-}\Pi_{\lambda,0} + \Pi_{\lambda,0}V_{\widetilde{\omega}}\Pi_{\lambda,0}.$$

Theorem 3.3 *Assume that the matrix $A(0)$ is positive-definite. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in [0,\lambda_0]$ we have:*

$$\Pi_{\lambda,0}\Big(H_{\lambda,\omega} - E_{\lambda,\omega^-}\Big)\Pi_{\lambda,0} \quad \textit{is a positive operator.}$$

The proof of Theorem 3.3 is the object of the following section.

3.1.1 The proof of Theorem 3.3

Using (3.8), we get that $H^0_{\lambda,\omega}$ is unitarily equivalent to the operator

$$h^0_{\lambda,\omega} = \mathcal{T}^*_{\varphi_{\lambda,1}}H^0_{\lambda,\omega}\mathcal{T}_{\varphi_{\lambda,1}},$$

acting on $L^2(\mathbb{T}^*)$ and written as

$$h^0_{\lambda,\omega} = h^0_{\lambda,\omega^-} + \lambda V^0_{\lambda,\widetilde{\omega}}.$$

With h^0_{λ,ω^-} is the multiplication operator by $E_1(\lambda,\theta)$ and $V^0_{\lambda,\widetilde{\omega}}$ is an integral operator with the kernel

$$V_{\lambda,\widetilde{\omega}}(\theta,\theta') = \langle V_{\widetilde{\omega}}\varphi_{\lambda,1}(\cdot,\theta),\varphi_{\lambda,1}(\cdot,\theta')\rangle.$$

Let V_k be a neighborhood of θ_k, such that if $\theta_{k'} \in Z$ and $k \neq k'$ then $\theta \notin \overline{V_k}$ and $V_k \cap V_{k'} = \emptyset$. As $\mathbb{T}^*$ is compact, one can cover it by $(V_k)_{1\le k\le m}$ (i.e $\cup_{1\le k\le m}V_k = \mathbb{T}^*$). For $1 \le k \le m$ let χ_k be the characteristic function of V_k. For simplicity for $u \in L^2(\mathbb{T}^*)$, we will denote $\chi_k u$ as u_k in the following. We consider u as a system of m columns denoted by $(u_k)_{1\le k\le m}$. We endow $L^2(\mathbb{T}^*)\otimes\mathbb{C}^m$ with the scalar product generating the following Euclidean norm:

$$\|u\|^2_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^m} = \sum_{k=1}\|u_k\|^2_{L^2(\mathbb{T}^*)}.$$

3.1.2 The lower bound of h^0_{λ,ω^-}

Proposition 3.4 *There exists $C > 0$ such that for any $u \in L^2(\mathbb{T}^*)$, we have*

$$\langle h^0_{\lambda,\omega^-} u, u\rangle \geq \sum_{1\leq k\leq m} E_1(\lambda, \theta_k(\lambda))\|u_k\|^2_{L^2(\mathbb{T}^*)} + \frac{1}{C}\sum_{1\leq k\leq m}\langle \zeta_{k,\lambda} u_k, u_k\rangle_{L^2(\mathbb{T}^*)}. \tag{3.12}$$

Proof: For $u \in L^2(\mathbb{T}^*)$, one computes

$$\langle h^0_{\lambda,\omega^-} u, u\rangle = \int_{\mathbb{T}^*} E_1(\lambda,\theta)|u(\theta)|^2 d\theta = \sum_{1\leq k\leq m}\int_{\mathbb{T}^*} E_1(\lambda,\theta)\chi_k(\theta)|u(\theta)|^2 d\theta,$$

As for any θ in V_k the support of χ_k, there exists $C > 0$ such that we have

$$E_1(\lambda, \theta_k(\lambda)) + \frac{1}{C}\zeta_{k,\lambda}(\theta) \leq E_1(\lambda,\theta),$$

we get the result. □

3.1.3 The lower bound of $V^0_{\lambda,\widetilde{\omega}}$

Proposition 3.5 *There exists $C_1, C_2 > 0$ and λ_0 such that for all $\lambda \in [0, \lambda_0]$ and $u \in L^2(\mathbb{T}^*)$ we have*

$$\langle V^0_{\lambda,\widetilde{\omega}} u, u\rangle \geq C_1 \sum_{1\leq k\leq m, \gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma |(\hat{u}_k)(\gamma)|^2 - C_2 \sum_{1\leq k\leq m}\langle \zeta_{k,\lambda} u_k, u_k\rangle_{L^2(\mathbb{T}^*)}. \tag{3.13}$$

The proof of Theorem 3.3

Let us notice that, by combining the results of Proposition 3.4 and 3.5, one gets that there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$ and for any $u \in L^2(\mathbb{T}^*$ we have,

$$\begin{aligned}
\langle h^0_{\lambda,\omega} u, u\rangle \quad &\geq \quad \sum_{1\leq k\leq m} E_1(\lambda,\theta_k(\lambda))\|u_k\|^2_{L^2(\mathbb{T}^*)} + \frac{1}{C}\Big(\sum_{1\leq k\leq m}\langle\zeta_{k,\lambda}u_k, u_k\rangle_{L^2(\mathbb{T}^*)} + \\
&\qquad \lambda \sum_{1\leq k\leq m}\sum_{\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma |\hat{u}_k(\gamma)|^2\Big) \\
&\geq \quad E_{\lambda,\omega^-}\|u\|^2_{L^2(\mathbb{T}^*)} + \frac{1}{C}\Big(\sum_{1\leq k\leq m}\langle\zeta_{k,\lambda}u_k, u_k\rangle_{L^2(\mathbb{T}^*)} + \\
&\qquad \lambda \sum_{1\leq k\leq m}\sum_{\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma |\hat{u}_k(\gamma)|^2\Big).
\end{aligned}$$

This gives that,

$$\langle (h^0_{\lambda,\widetilde{\omega}} - E_{\lambda,\omega^-})u, u\rangle \geq \frac{1}{C}\Big(\sum_{1\leq k\leq m} \langle \zeta_{k,\lambda} u_k, u_k\rangle_{L^2(\mathbb{T}^*)} + \lambda \sum_{1\leq k\leq m}\sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma |\hat{u_k}(\gamma)|^2\Big). \quad (3.14)$$

This ends the proof of Theorem 3.3. □

Remark 3.6 *We notice that even if we know that the bottom of the spectrum of H_ω coincides with the bottom of the spectrum of H_{λ,ω^-} we cannot consider $\Pi_{\lambda,0}\Big(H_{\lambda,\omega} - H_{\lambda,\omega^-}\Big)\Pi_{\lambda,0}$ as a positive operator.*

The proof of proposition 3.5:
Let us start by expanding $\langle V^0_{\lambda,\widetilde{\omega}} u, u\rangle$,

$$\begin{aligned}\langle V^0_{\lambda,\widetilde{\omega}} u, u\rangle &= \sum_{1\leq k,k'\leq m} \langle V^0_{\lambda,\widetilde{\omega}} \mathcal{T}_{\widetilde{\varphi}_{\lambda,1,k}}(u_k), \mathcal{T}_{\widetilde{\varphi}_{\lambda,1,k'}}(u_{k'})\rangle_{L^2(\mathbb{R}^d)}\\ &+ \sum_{1\leq k,k'\leq m} \langle V^0_{\lambda,\widetilde{\omega}} \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda}} u_k), \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda}} u_{k'})\rangle_{L^2(\mathbb{R}^d)}\\ &+ 2\sum_{1\leq k,k'\leq m} \Re\Big(\langle V^0_{\lambda,\widetilde{\omega}} \mathcal{T}_{\widetilde{\varphi}_{\lambda,1,k}}(u_k), \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda}} u_{k'})\rangle_{L^2(\mathbb{R}^d)}\Big).\end{aligned}$$

We start by estimating the three sums of the last equation.
For the second sum, using Cauchy Schwartz inequality and Lemma 3.2, we get that for any $1 \leq k, k' \leq m$, there exists $C > 0$ such that we have,

$$\begin{aligned}&\Big|\langle V^0_{\lambda,\widetilde{\omega}} \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda}} u_k), \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda}} u_{k'})\rangle_{L^2(\mathbb{R}^d)}\Big|\\ &\leq \frac{1}{2}\Big(\|V^0_{\lambda,\widetilde{\omega}} \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda}} u_k)\|^2_{L^2(\mathbb{R}^d)} + \|\mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda}} u_{k'})\|^2_{L^2(\mathbb{R}^d)}\Big)\\ &\leq C\cdot \Big(\langle \zeta_{k,\lambda} u_k, u_k\rangle_{L^2(\mathbb{T}^*)} + \langle \zeta_{k',\lambda} u_{k'}, u_{k'}\rangle_{L^2(\mathbb{T}^*)}\Big).\end{aligned}$$

So there exists $C > 0$ such that we have

$$\sum_{1\leq k,k'\leq m} \Big|\langle V^0_{\lambda,\widetilde{\omega}} \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k}}(\sqrt{\zeta_{k,\lambda}} u_k), \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda}} u_{k'})\rangle_{L^2(\mathbb{R}^d)}\Big| \leq C\sum_{1\leq k\leq m} \langle \zeta_{k,\lambda} u_k, u_k\rangle. \quad (3.15)$$

For the third sum, using the Cauchy-Schwartz inequality once more, we get that for $1 \le k \le m$, there exists $\beta > 0$ such that we have

$$\begin{aligned}\Big|\langle V^0_{\lambda,\widetilde{\omega}} \mathcal{T}_{\widetilde{\varphi}_{\lambda,1,k}}(u_k), \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda}} u_{k'})\rangle_{L^2(\mathbb{R}^d)}\Big| \\ \le \beta \|V^0_{\widetilde{\omega}} \mathcal{T}_{\widetilde{\varphi}_{\lambda,1,k}}(u_k)\|^2_{L^2(\mathbb{R}^d)} + 1/(4\beta) \|\mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k'}}(\sqrt{\zeta_{k,\lambda}} u_{k'})\|^2_{L^2(\mathbb{R}^d)}. \quad (3.16)\end{aligned}$$

Using equation (3.11), one gets that there exist $\widetilde{C}_1, \widetilde{C}_2 > 0$ such that

$$\begin{aligned}\sum_{1\le k,k'\le m} \Big|\langle V^0_{\lambda,\widetilde{\omega}} \mathcal{T}_{\widetilde{\varphi}_{\lambda,1,k}}(u_k), \mathcal{T}_{\delta\widetilde{\varphi}_{\lambda,1,k'}}(\sqrt{\zeta_{k',\lambda}} u_{k'})\rangle_{L^2(\mathbb{R}^d)}\Big| \\ \le \widetilde{C}_1 \beta \sum_{1\le k\le m} \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma |\hat{u_k}(\gamma)|^2 + \widetilde{C}_2(\beta + 1/\beta) \sum_{1\le k\le m} \langle \zeta_{\lambda,k} u_k, u_k\rangle_{L^2(\mathbb{T}^*)}. \quad (3.17)\end{aligned}$$

For the first sum, using (3.10), we get that there exist $C'_1, C'_2 > 0$ such that

$$\begin{aligned}\Big|\sum_{1\le k,k'\le m} \langle V^0_{\widetilde{\omega}} \mathcal{T}_{\widetilde{\varphi}_{\lambda,1,k}}(u_k), \mathcal{T}_{\widetilde{\varphi}_{\lambda,1,k'}}(u_{k'})\rangle_{L^2(\mathbb{R}^d)} \\ - \sum_{1\le k,k'\le m} \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma \Big(\int_{C_0} a_{\varphi_{\lambda,1,k,k'}}(x) dx\Big)(\hat{u_k})(\gamma)\overline{(\hat{u_{k'}})(\gamma)}\Big| \\ \le C'_1 \beta \sum_{1\le k\le m} \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma |\hat{u_k}(\gamma)|^2 + C'_2(1 + 1/\beta) \sum_{1\le k\le m} \langle \zeta_{\lambda,k} u_k, u_k\rangle_{L^2(\mathbb{T}^*)}. \quad (3.18)\end{aligned}$$

Now equations (3.15), (3.17) and (3.18) give that there exist $K_1, K_2 > 0$ such that

$$\begin{aligned}\Big|\langle V^0_{\lambda,\widetilde{\omega}} u, u\rangle - \sum_{1\le k,k'\le m} \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma \Big(\int_{C_0} a_{\varphi_{\lambda,1,k,k'}}(x) dx\Big)(\hat{u_k})(\gamma)\overline{(\hat{u_{k'}})(\gamma)}\Big| \le \\ K_1 \beta \sum_{1\le k\le m} \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma |(\hat{u_k})(\gamma)|^2 + K_2(1 + 1/\beta) \sum_{1\le k\le m} \langle \zeta_{k,\lambda} u_k, u_k\rangle_{L^2(\mathbb{T}^*)}. \quad (3.19)\end{aligned}$$

Now, if the matrix

$$A(0) = \Big(\int_{C_0} a_{\varphi_{0,1,k,k'}}(x) dx\Big)_{1\le k,k'\le m},$$

is positive-definite, one gets that $\inf \sigma(A(0)) = C > 0$ satisfies

$$C I_m \le A.$$

Let $A(\lambda)$ be the matrix,

$$\Big(\int_{C_0} a_{\varphi_{\lambda,1},k,k'}(x)dx\Big)_{1\leq k,k'\leq m}.$$

Notice that for any $1 \leq k, k' \leq m$, the functions

$$f_{k,k'} : \lambda \to \int_{C_0} a_{\varphi_{\lambda,1},\theta_k(\lambda),\theta_{k'}(\lambda)}(x)dx,$$

are continuous in λ. So there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$

$$\frac{C}{2} I_m \leq A(\lambda).$$

This gives that for any $u \in L^2(\mathbb{T}^*)$

$$\begin{aligned}\frac{C}{2}&\sum_{1\leq k\leq m}\sum_{\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma|(\hat{u_k})(\gamma)|^2\\ &\leq \sum_{1\leq k,k'\leq m}\sum_{\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma\Big(\int_{C_0} a_{\varphi_{\lambda,1},k,k'}(x)dx\Big)(\hat{u_k})(\gamma)\overline{(\hat{u_{k'}})(\gamma)}. \end{aligned}\tag{3.20}$$

Now using the expanation of $\langle V^0_{\lambda,\widetilde{\omega}} u, u\rangle$ and equation (3.19), we get that there exist $K_1, K_2 > 0$ and $\beta > 0$ such that

$$\begin{aligned}&\langle V^0_{\lambda,\widetilde{\omega}} u, u\rangle \geq\\ &(\frac{C_1'}{2} - K_1\beta)\sum_{1\leq k\leq m;\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma|(\hat{u_k})(\gamma)|^2 - K_2(1+1/\beta)\sum_{1\leq k\leq m}\langle\zeta_{k,\lambda}u_k, u_k\rangle_{L^2(\mathbb{T}^*)}.\end{aligned}\tag{3.21}$$

So, for $\beta > 0$, well chosen we get that there exist constants $C_1, C_2 > 0$ such that

$$\langle V^0_{\lambda,\widetilde{\omega}} u, u\rangle \geq \frac{C_1}{3}\sum_{1\leq k\leq m;\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma|(\hat{u_k})(\gamma)|^2 - C_2\sum_{1\leq k\leq m}\langle\zeta_{k,\lambda}u_k, u_k\rangle_{L^2(\mathbb{T}^*)}. \tag{3.22}$$

This ends the proof of Proposition 3.5. $\square$

3.2 If $A(0)$ is negative-definite

Let H_{λ,ω^+} be the following operator,

$$H_{\lambda,\omega^+} = -\Delta + W_{\text{per}} + \lambda \sum_{\gamma\in\mathbb{Z}^d} \omega^+ f(\cdot - \gamma).$$

As H_{λ,ω^+} is a $\mathbb{Z}^d$-periodic operator, the analysis given in subsection 2.1 for H_{λ,ω^+} is still true in the present case. For $(E_j(\lambda,\theta))_{j\in\mathbb{N}^*}$, the Floquet eigenvalue of H_{λ,ω^+} let us set $E_{\lambda,\omega^+} = \inf_{\theta\in\mathbb{T}^*} E_1(\lambda,\theta)$.

Theorem 3.7 *Assume that the matrix $A(0)$ is negative-definite. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in [0,\lambda_0]$ we have:*

$$\Pi_{\lambda,0}\Big(H_{\lambda,\omega} - E_{\lambda,\omega^+}\Big)\Pi_{\lambda,0} \quad \textit{is a positive operator.}$$

The result of Theorem 3.7 can be proved in the same way as we did for Theorem 1.3 in the previous subsection. Indeed, $H_{\lambda,\omega}$ can be seen as a perturbation of H_{λ,ω^+} as follow,

$$H_{\lambda,\omega} = H_{\lambda,\omega^+} + V_{\overline{\omega}}.$$

With $V_{\overline{\omega}}(\cdot) = \sum_{\gamma\in\mathbb{Z}^d} \overline{\omega}_\gamma f(\cdot - \gamma)$ and for any $\gamma \in \mathbb{Z}^d$ $\overline{\omega}_\gamma = \omega_\gamma - \omega^+$. Notice that in this case $(\overline{\omega}_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a family of bounded and negative random variables. using the analogous unitary transformation, one gets that $H^0_{\lambda,\omega}$ is unitarly equivalent to

$$h^0_{\lambda,\omega} = h^0_{\lambda,\omega^+} + V^0_{\lambda,\overline{\omega}}.$$

The lower bound of h_{λ,ω^+} can be derived easily. As all arguments used to lower bound $V^0_{\tilde{\omega}}$ remain valid; we lower bound $V^0_{\lambda,\overline{\omega}}$ using the same computation done in subsection 3.1.
So we get that there exist $K_1, K_2 > 0$ such that

$$\Big|\langle V^0_{\lambda,\overline{\omega}} u, u\rangle - \sum_{1\le k,k'\le m}\sum_{\gamma\in\mathbb{Z}^d} \overline{\omega}_\gamma \Big(\int_{C_0} a_{\varphi_{\lambda,1},k,k'}(x)dx\Big)(\hat{u_k})(\gamma)\overline{(\hat{u_{k'}})(\gamma)}\Big| \le$$
$$K_1\beta(\omega^+ - \omega^-)\sum_{1\le k\le m}\sum_{\gamma\in\mathbb{Z}^d} |(\hat{u_k})(\gamma)|^2 + K_2(1+1/\beta)\sum_{1\le k\le m}\langle \zeta_{k,\lambda} u_k, u_k\rangle_{L^2(\mathbb{T}^*)}. \tag{3.23}$$

When $A(0)$ is negative-definite, there exists $C<0$ and $\lambda_0>0$ such that for any $\lambda \in [0,\lambda_0]$, we have

$$A(\lambda) \leq C I_m.$$

As the random variables $(\overline{\omega}_\gamma)_{\gamma\in\mathbb{Z}^d}$ are negative, we get that

$$\begin{aligned} C \sum_{1\leq k\leq m}\sum_{\gamma\in\mathbb{Z}^d} \overline{\omega}_\gamma |(\hat{u_k})(\gamma)|^2 \\ \leq \sum_{1\leq k,k'\leq m}\sum_{\gamma\in\mathbb{Z}^d} \overline{\omega}_\gamma \Big(\int_{C_0} a_{\varphi_{\lambda,1},k,k'}(x)dx\Big)(\hat{u_k})(\gamma)\overline{(\hat{u_{k'}})(\gamma)}. \quad (3.24)\end{aligned}$$

This and equation (3.23) give the sought result on the lower bound of $V_{\overline{\omega}}$. This ends the proof of Theorem 3.7. $\square$

The proof of Lemma 3.2:

As $V_{\widetilde{\omega}}$ is H_{λ,ω^-}-relatively bound uniformly on $\widetilde{\omega}_\gamma$, there exists $c>0$ such that for any $u\in L^2(\mathbb{T}^*)$ we have

$$\begin{aligned} \|V_{\widetilde{\omega}}\mathcal{T}_\varphi(u)\| &\leq c\Big(\|H_{\lambda,\omega^-}\mathcal{T}_\varphi(u)\|^2 + \|\mathcal{T}_\varphi(u)\|^2\Big) \\ &\leq c\int_{\mathbb{T}^*}\Big(\|H_{\lambda,\omega^-}(\theta)\varphi(\cdot,\theta)\|^2_{L^2(C_0)} + \|\varphi(\cdot,\theta)\|^2_{L^2(C_0)}\Big)|u(\theta)|^2 d\theta. \\ &\leq c \quad \|\varphi\|^2_{H_{\lambda,\omega^-},\infty}\cdot\|u\|^2_{L^2(\mathbb{T}^*)}.\end{aligned}$$

One computes

$\langle V^0_{\widetilde{\omega}}\mathcal{T}_{\varphi_k}(u), \mathcal{T}_{\varphi_{k'}}(v)\rangle_{L^2(\mathbb{R}^d)}$

$$\int_{\mathbb{R}^d} V_{\widetilde{\omega}}(x)\varphi_k(x,\theta_k)\overline{\varphi_{k'}(x,\theta_{k'})}\cdot\Big(\int_{\mathbb{T}^*} e^{(\theta-\theta_k)x}v\theta)d\theta\cdot\overline{\int_{\mathbb{T}^*} e^{(\theta-\theta_{k'})x}u(\theta)d\theta}\Big)dx$$

$$\begin{aligned}\sum_{\gamma\in\mathbb{Z}^d}\widetilde{\omega}_\gamma\int_{C_0} f(x)\varphi_k(x,\theta_k)\overline{\varphi_{k'}(x,\theta_{k'})}\cdot\Big(\int_{\mathbb{T}^*} e^{(\theta-\theta_k)x}e^{i\gamma\cdot\theta}u(\theta)d\theta\cdot \\ \overline{\int_{\mathbb{T}^*} e^{(\theta-\theta_{k'})x}e^{i\gamma\cdot\theta}u(\theta)d\theta}\Big)dx.\end{aligned}$$

Let

$$\hat{u}(\gamma) = \int_{\mathbb{T}^*} e^{i\gamma\cdot\theta}u(\theta)d\theta.$$

For any $(x,\theta) \in \mathbb{R}^d \times \mathbb{T}^*$ and $1 \le k \le m$, we set

$$g_k(x,\theta) = \frac{e^{i(\theta-\theta_k)\cdot x} - 1}{\sqrt{\zeta_{k,\lambda}(\theta)}}. \tag{3.25}$$

As $\theta_k(\lambda)$ is the only zero of $\zeta_{k,\lambda}$ and as it is nondegenerate, there exist $C > 0$ such that, for $(x,\theta) \in \mathbb{R} \times \mathbb{R}^d$, and $1 \le k \le m$, we have

$$|g_k(x,\theta)| \le C(1 + |x|). \tag{3.26}$$

We have $e^{i(\theta-\theta_k)x} = \sqrt{\zeta_{k,\lambda}} g_k(x,\theta) + 1$. So using this and expanding $\langle V_{\widetilde{\omega}} \mathcal{T}_{\varphi_k}(u), \mathcal{T}_{\varphi_{k'}}(v)\rangle_{L^2(\mathbb{R}^d)}$, we get

$$\begin{aligned}
&\langle V_{\widetilde{\omega}} \mathcal{T}_{\varphi_k}(u), \mathcal{T}_{\varphi_{k'}}(v)\rangle_{L^2(\mathbb{R}^d)} - \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma \Big(\int_{C_0} a_{\varphi,k,k'}(x)dx \Big) \cdot |u(\gamma)|^2 \\
&\quad = \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma \int_{C_0} a_{\varphi,k,k'}(x) \cdot \Big(\int_{\mathbb{T}^*} e^{i\gamma\cdot\theta} g_k(x,\theta) \sqrt{\zeta_{k,\lambda}(\theta)} u(\theta) d\theta \cdot \\
&\qquad\qquad \overline{\int_{\mathbb{T}^*} e^{i\gamma\cdot\theta} g_{k'}(x,\theta) \sqrt{\zeta_{k',\lambda}(\theta)} v(\theta) d\theta} \Big) dx \\
&\qquad + \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma \hat{u}(\gamma) \int_{C_0} a_{\varphi,k,k'}(x) \cdot \overline{\int_{\mathbb{T}^*} e^{i\gamma\cdot\theta} g_{k'}(x,\theta) \sqrt{\zeta_{k',\lambda}(\theta)} v(\theta) d\theta dx} \\
&\qquad + \sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma \overline{\hat{v}(\gamma)} \int_{C_0} a_{\varphi,k,k'}(x) \cdot \int_{\mathbb{T}^*} e^{i\gamma\cdot\theta} g_k(x,\theta) \sqrt{\zeta_{k,\lambda}(\theta)} u(\theta) d\theta dx.
\end{aligned} \tag{3.27}$$

Now using the fact that the family $(\widetilde{\omega}_\gamma)_{\gamma\in\mathbb{Z}^d}$ is bounded, Cauchy-Schwartz inequality and Perseval identity and equation (3.26), we get that there exists $C > 0$ such that

$$\begin{aligned}
&\sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma \int_{C_0} a_{\varphi,k,k'}(x) \cdot \Big(\int_{\mathbb{T}^*} e^{i\gamma\cdot\theta} g_k(x,\theta) \sqrt{\zeta_{k,\lambda}(\theta)} u(\theta) d\theta \cdot \\
&\qquad\qquad \overline{\int_{\mathbb{T}^*} e^{i\gamma\cdot\theta} g_{k'}(x,\theta) \sqrt{\zeta_{k',\lambda}(\theta)} v(\theta) d\theta} \Big) dx \\
&\qquad\qquad\qquad \le C \langle \zeta_{k,\lambda} u, u\rangle_{L^2(\mathbb{T}^*)} + \langle \zeta_{k'+\lambda} v, v\rangle_{L^2(\mathbb{T}^*)}.
\end{aligned} \tag{3.28}$$

And $\beta > 0$,

$$\sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma \hat{u}(\gamma) \int_{C_0} a_{\varphi,k,k'}(x) \cdot \overline{\int_{\mathbb{T}^*} e^{i\gamma\cdot\theta} g_{k'}(x,\theta) \sqrt{\zeta_{k',\lambda}(\theta)} v(\theta) d\theta} dx \leq$$

$$\beta \sum_{\gamma\in\mathbb{Z}^d} |\hat{u}(\gamma)|^2 + \frac{1}{4\beta} \langle \zeta_{k',\lambda} v, v \rangle_{L^2(\mathbb{T}^*)}. \quad (3.29)$$

The same argument gives

$$\sum_{\gamma\in\mathbb{Z}^d} \widetilde{\omega}_\gamma \overline{\hat{v}(\gamma)} \int_{C_0} a_{\varphi,k,k'}(x) \cdot \int_{\mathbb{T}^*} e^{i\gamma\cdot\theta} g_k(x,\theta) \sqrt{\zeta_{k,\lambda}(\theta)} u(\theta) d\theta dx.$$

$$\leq \beta \sum_{\gamma\in\mathbb{Z}^d} |\hat{v}(\gamma)|^2 + \frac{1}{4\beta} \langle \zeta_{k,\lambda} u, u \rangle_{L^2(\mathbb{T}^*)}. \quad (3.30)$$

So from (3.27), (3.28), (3.29) and (3.30) we get (3.10).
The proof of (3.11); follows by changing $\mathcal{T}_{\varphi_{\lambda,1}}(u)$ using (3.9) and following the same steps as (3.10).
This ends the proof of Lemme 3.2 $\square$

Acknowledgements. *The author would like to thank Frederic Klopp for his precious comments and remarks.*

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