On Self-adjoint and J-self-adjoint Dirac-type Operators: A Case Study

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Abstract. We provide a comparative treatment of some aspects of spectral theory for self-adjoint and non-self-adjoint (but J-self-adjoint) Dirac-type operators connected with the defocusing and focusing nonlinear Schrödinger equation, of relevance to nonlinear optics.

In addition to a study of Dirac and Hamiltonian systems, we also introduce the concept of Weyl–Titchmarsh half-line m-coefficients (and 2×2 matrix-valued M-matrices) in the non-self-adjoint context and derive some of their basic properties. We conclude with an illustrative example showing that crossing spectral arcs in the non-self-adjoint context imply the blowup of the norm of spectral projections in the limit where the crossing point is approached.

1. Introduction

The principal part of this paper is devoted to a comparative study of Dirac-type operators of the formally self-adjoint type

$$\widehat{\mathcal{D}} = i \begin{pmatrix} d/dx & -q(x) \\ \overline{q(x)} & -d/dx \end{pmatrix}, \quad x \in \mathbb{R}, \tag{1.1}$$

and the formally non-self-adjoint (but formally J-self-adjoint cf. (2.13)) Dirac-type operators of the form

$$\widetilde{\mathcal{D}} = i \begin{pmatrix} d/dx & -q(x) \\ -\overline{q(x)} & -d/dx \end{pmatrix}, \quad x \in \mathbb{R}, \tag{1.2}$$

where q is locally integrable on $\mathbb{R}$. Interest in these two particular Dirac-type operators stems from the fact that both are intimately connected with applications to nonlinear optics. In fact, the differential expression $\widehat{\mathcal{D}}$ gives rise to the Lax operator of the *defocusing* nonlinear Schrödinger equation (NLS$_+$), while the differential expression $\widetilde{\mathcal{D}}$ defines the Lax operator for the *focusing* nonlinear Schrödinger equation (NLS$_-$). In appropriate units, the propagation equation for a pulse envelope $q(x,t)$

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in a monomode optical fiber in the plane-wave limit neglecting loss is given by the nonlinear Schrödinger equation

$$\mathrm{NLS}_{\pm}(q) := iq_x \mp \tfrac{1}{2}q_{tt} + |q|^2 q = 0 \tag{1.3}$$

(assuming weak nonlinearity of the medium and weak dispersion). The focusing nonlinear Schrödinger equation admits a one-soliton solution that propagates without change of shape and more generally admits "bright" soliton solutions. The defocusing Schrödinger equation shows a very different behavior since pulses undergo enhanced broadening (to be used as optical pulse compression), thereby yielding "dark" solitons. For pertinent general references of this fascinating area we refer the reader, for instance, to [**1**], [**2**], [**10**], [**12**], [**14**], [**21**], [**22**], [**36**], [**40**].

While typical applications to quantum mechanical problems in connection with Schrödinger and Dirac equations require the study of self-adjoint boundary value problems, many applications of completely integrable systems most naturally lead to non-self-adjoint Lax operators underlying the integrable system. The prime example in this connection is the nonlinear Schrödinger equation (1.3). With this background in mind, we embarked upon a more systematic study of the spectral properties of operator realizations of (1.1) and especially, (1.2), in $L^2(\mathbb{R})^2$.

There exists a large body of results on spectral and inverse spectral theory of self-adjoint and non-self-adjoint Dirac-type operators, especially, in the periodic and certain quasi-periodic cases (we refer, e.g., to [**3**, Ch. 5], [**4**]–[**9**], [**13**], [**14**, Ch. 3], [**15**], [**18**]–[**20**], [**24**], [**25**], [**26**], [**28**], [**29**], [**31**], [**33**]–[**35**], [**37**], [**41**], [**42**]). In this paper, however, we offer a different treatment focusing on a comparative study of self-adjoint and non-self-adjoint (but J-self-adjoint) Dirac-type operators with emphasis on Weyl–Titchmarsh-type results. For basic results on J-self-adjoint operators we refer, for instance, to [**11**, Sect. III.5], [**17**, Sects. 21–24], [**27**], [**38**], [**46**].

In Section 2, we begin by considering the general Dirac-type expression

$$\mathcal{D} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{d}{dx} + Q(x), \quad Q = \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix} \in L^1_{\mathrm{loc}}(\mathbb{R})^{2\times 2}, \ x \in \mathbb{R}. \tag{1.4}$$

Introducing the conjugate linear operator acting upon $\mathbb{C}^2$, described by

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}, \quad \mathcal{J}^2 = I_2, \tag{1.5}$$

where $\mathcal{C}$ denotes the operator of conjugation acting on $\mathbb{C}^2$ by

$$\mathcal{C}(a,b)^{\top} = (\overline{a}, \overline{b})^{\top}, \quad a, b \in \mathbb{C}, \tag{1.6}$$

with $(a,b)^{\top}$ denoting transposition of the vector (a,b), we show that $\mathcal{D}$ is formally J-self-adjoint (implying the same property for $\widehat{\mathcal{D}}$ and $\widetilde{\mathcal{D}}$). In particular, we show that

$$\mathcal{J}\mathcal{D}\mathcal{J} = \mathcal{D}^*, \tag{1.7}$$

if and only if $Q_{1,1} = Q_{2,2}$ a.e. on $\mathbb{R}$.

Since Dirac-type operators are often studied in Hamiltonian form, we also introduce the unitarily equivalent Hamiltonian form $\mathcal{H}$ of $\mathcal{D}$ given by

$$\mathcal{H} = U\mathcal{D}U^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + B(x), \quad B(x) = UQ(x)U^{-1}, \ x \in \mathbb{R} \tag{1.8}$$

with the constant 2×2 matrix U given by (2.36). Green's matrices are described both for $\mathcal{D}$ and its unitarily equivalent Hamiltonian form $\mathcal{H}$.

Section 3 is devoted to a study of the maximally defined $L^2(\mathbb{R})^2$-realization $\widehat{D}$ of the special case $\widehat{\mathcal{D}}$ in (1.1) and its unitarily equivalent Hamiltonian version $\widehat{H} = U\widehat{D}U^{-1}$. $\widehat{D}$ (and hence $\widehat{H}$) is known to be self-adjoint for all $q \in L^1_{\rm loc}(\mathbb{R})$, (cf. [**6**]). We determine the Green's matrices of $\widehat{H}$ and $\widehat{D}$ and recall some elements of the Weyl–Titchmarsh theory associated with $\widehat{H}$. Due to the unitary equivalence of $\widehat{H}$ and $\widehat{D}$, we show that the Weyl–Titchmarsh formalism for $\widehat{D}$ can be set up in such a manner that the half-line Weyl–Titchmarsh m-coefficients (and hence the 2×2 matrix-valued full-line Weyl–Titchmarsh M-matrices) for $\widehat{H}$ and $\widehat{D}$ coincide. The latter appears to be new as Weyl–Titchmarsh theory, to the best of our knowledge, is typically formulated in connection with the Hamiltonian version $\widehat{H}$. Moreover, we provide a streamlined derivation of the 2×2 matrix-valued spectral functions of $\widehat{H}$ and $\widehat{D}$ starting from the corresponding families of spectral projections. This section is concluded with the simple constant coefficient example $q(x) = q_0 \in \mathbb{C}$ a.e.

Our final Section 4 then deals with a study of the maximally defined $L^2(\mathbb{R})^2$-realization $\widetilde{D}$ of the special case $\widetilde{\mathcal{D}}$ in (1.2) and its unitarily equivalent Hamiltonian version $\widetilde{H} = U\widetilde{D}U^{-1}$. $\widetilde{D}$ (and hence $\widetilde{H}$) is known to be J-self-adjoint for all $q \in L^1_{\rm loc}(\mathbb{R})$, (cf. [**5**]). We determine the Green's matrices of $\widetilde{H}$ and $\widetilde{D}$, and develop some basic cornerstones of the analog of the Weyl–Titchmarsh theory in the self-adjoint context of Section 3 for the non-self-adjoint (but J-self-adjoint) operator $\widetilde{H}$. Again, due to the unitary equivalence of $\widetilde{H}$ and $\widetilde{D}$, we show that the Weyl–Titchmarsh formalism for $\widetilde{D}$ can be set up in such a manner that the half-line Weyl–Titchmarsh m-coefficients (and hence the 2×2 matrix-valued full-line Weyl–Titchmarsh M-matrices) for $\widetilde{H}$ and $\widetilde{D}$ coincide. In addition, we indicate the link between the spectral projections of $\widetilde{H}$ (and hence of $\widetilde{D}$) and a 2×2 matrix-valued spectral function of $\widetilde{H}$ (and $\widetilde{D}$) determined from the corresponding full-line Weyl–Titchmarsh M-matrix away from spectral singularities of $\widetilde{H}$. This section also supplies the illustrative constant coefficient example $q(x) = q_0 \in \mathbb{C}\backslash\{0\}$ a.e. In this case, the spectrum of $\widetilde{H}$ consists of the real axis and the line segment from $-i|q_0|$ to $+i|q_0|$ along the imaginary axis. In other words, this is presumably the simplest differential operator with crossing spectral arcs. We conclude this section with a proof of the fact that the norm of the spectral projection in this example associated with an interval of the type (λ_1, λ_2), $0 < \lambda_1 < \lambda_2$, blows up in the limit $\lambda_1 \downarrow 0$, that is, when λ_1 approches the crossing point $\lambda = 0$ of the spectral arcs of $\widetilde{H}$. The material developed in this section represents the major new results in this paper.

2. A comparison of Dirac and Hamiltonian Systems

2.1. Dirac differential expressions. Throughout this paper for a matrix A with complex-valued entries, $A^\top$ denotes the transposition of A; $\overline{A}$ denotes the matrix with complex conjugate entries; and A^* denotes the adjoint matrix, that is, the conjugate transpose of A, $A^* = \overline{A}^\top$. We will have occasion in our discussion to consider the following 2×2 matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.1}$$

Moreover, we subsequently denote by $\sigma(A)$ and $\rho(A)$ the spectrum and resolvent set of a closed densely defined linear operator A in a separable complex Hilbert space $\mathcal{H}$.

We now consider *whole-line Dirac differential expressions* of the form

$$\mathcal{D} = i\sigma_3 \frac{d}{dx} + Q(x), \quad Q = \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix} \in L^1_{\text{loc}}(\mathbb{R})^{2\times 2}, \tag{2.2}$$

that is, Q is a 2×2 matrix with complex-valued entries that are locally integrable on $\mathbb{R}$. In particular, we shall be concerned with the formally self-adjoint differential expression that arises when

$$\widehat{\mathcal{D}} = i\sigma_3 \frac{d}{dx} + \widehat{Q}(x), \quad \widehat{Q} = i\begin{pmatrix} 0 & -q \\ \overline{q} & 0 \end{pmatrix} \in L^1_{\text{loc}}(\mathbb{R})^{2\times 2}, \tag{2.3}$$

and the formally non-self-adjoint differential expression arising when

$$\widetilde{\mathcal{D}} = i\sigma_3 \frac{d}{dx} + \widetilde{Q}(x), \quad \widetilde{Q} = i\begin{pmatrix} 0 & -q \\ -\overline{q} & 0 \end{pmatrix} \in L^1_{\text{loc}}(\mathbb{R})^{2\times 2}. \tag{2.4}$$

By the *formal adjoint* of the differential expression $\mathcal{D}$ given in (2.2), we shall mean the differential expression $\mathcal{D}^*$, for which

$$\int_a^b dx\, \Psi(x)^* (\mathcal{D}\Phi)(x) = \Psi(x)^* i\sigma_3 \Phi(x)\big|_a^b + \int_a^b dx\, \left(\mathcal{D}^*\Psi\right)(x)^* \Phi(x), \tag{2.5}$$

for all $a, b \in \mathbb{R}$ and all

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad \Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \quad \psi_j,\ \phi_j \in \text{AC}([a,b]),\ j = 1,2, \tag{2.6}$$

with $\text{AC}([a,b])$ the set of absolutely continuous functions on $[a,b]$. Hence, $\mathcal{D}^*$ is given by

$$\mathcal{D}^* = i\sigma_3 \frac{d}{dx} + Q(x)^*, \quad x \in \mathbb{R}. \tag{2.7}$$

In particular, we note that $\widehat{\mathcal{D}}^* = \widehat{\mathcal{D}}$ while $\widetilde{\mathcal{D}}^* \neq \widetilde{\mathcal{D}}$. Moreover, by the *formal real adjoint* of the differential expression $\mathcal{D}$ given by (2.2), we shall mean the differential expression, $\mathcal{D}^\dagger$, where for all $a, b \in \mathbb{R}$,

$$\int_a^b dx\, \Psi(x)^\top (\mathcal{D}\Phi)(x) = \Psi(x)^\top i\sigma_3 \Phi(x)\big|_a^b + \int_a^b dx\, \left(\mathcal{D}^\dagger\Psi\right)(x)^\top \Phi(x). \tag{2.8}$$

Hence, $\mathcal{D}^\dagger$ is given by

$$\mathcal{D}^\dagger = -i\sigma_3 \frac{d}{dx} + Q(x)^\top. \tag{2.9}$$

Associated with the Dirac differential expression (2.2) is the *homogeneous Dirac system* given by

$$(\mathcal{D}\Psi)(z,x) = i\sigma_3 \Psi'(z,x) + Q(x)\Psi(z,x) = z\Psi(z,x), \tag{2.10}$$

for a.e. $x \in \mathbb{R}$, where z plays the role of the spectral parameter and

$$\Psi(z,x) = \begin{pmatrix} \psi_1(z,x) \\ \psi_2(z,x) \end{pmatrix}, \quad \psi_j(z,\cdot) \in \text{AC}_{\text{loc}}(\mathbb{R}),\ j = 1,2 \tag{2.11}$$

with $AC_{\text{loc}}(\mathbb{R})$ denoting the set of locally absolutely continuous functions on $\mathbb{R}$. By analogy, one obtains Dirac systems associated with the differential expressions in

(2.3), (2.4), (2.7), and (2.9). Solutions of (2.10) are said to be *z-wave functions of* $\mathcal{D}$.

The Wronskian of two elements $F = (f_1\ f_2)^\top, G = (g_1\ g_2)^\top \in C(\mathbb{R})^2$ is defined as usual by

$$\begin{aligned} W(F(x), G(x)) &= F(x)^\top \sigma_4 G(x) = f_1(x)g_2(x) - f_2(x)g_1(x) \\ &= \det\left(\begin{pmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{pmatrix}\right), \quad x \in \mathbb{R}. \end{aligned} \tag{2.12}$$

The differential expressions (2.3) and (2.4), which will be the focus of our study, each exhibit the property of *formal J-self-adjointness*; a property that is manifest in the following relations:

$$\mathcal{J}\widehat{\mathcal{D}}\mathcal{J} = \widehat{\mathcal{D}}^* = \widehat{\mathcal{D}}, \quad \mathcal{J}\widetilde{\mathcal{D}}\mathcal{J} = \widetilde{\mathcal{D}}^* \tag{2.13}$$

or equivalently, by

$$\sigma_3\widehat{\mathcal{D}}\sigma_3 = \widehat{\mathcal{D}}^* = \widehat{\mathcal{D}}, \quad \sigma_3\widetilde{\mathcal{D}}\sigma_3 = \widetilde{\mathcal{D}}^*, \tag{2.14}$$

where $\mathcal{J}$ is defined in (1.5), and where the equalities hold a.e. on $\mathbb{R}$. While not all Dirac differential expressions described in (2.2) are formally J-self-adjoint, those which are can be characterized as follows:

THEOREM 2.1. *Let $\mathcal{D}$ be the Dirac differential expression* (2.2)*. Then the following statements are equivalent:*

(i) *$\mathcal{D}$ is formally J-self-adjoint: $\mathcal{J}\mathcal{D}\mathcal{J} = \mathcal{D}^*$, where equality holds a.e. on $\mathbb{R}$.*

(ii) *$Q_{1,1} = Q_{2,2}$ a.e. on $\mathbb{R}$ in the matrix Q of the differential expression $\mathcal{D}$.*

(iii) *The Wronskian is a nonzero constant for any pair of linearly independent z-wave functions of the Dirac system* (2.10)*.*

PROOF. The equivalence of statements (i) and (ii) follows from (2.7) and the fact that

$$\mathcal{J}\mathcal{D}\mathcal{J} = i\sigma_3\frac{d}{dx} + \begin{pmatrix} \overline{Q}_{2,2} & \overline{Q}_{2,1} \\ \overline{Q}_{1,2} & \overline{Q}_{1,1} \end{pmatrix}. \tag{2.15}$$

The equivalence of statements (ii) and (iii) follows from the observation that if $\Psi_j(z,x)$, $j = 1,2$ represent two independent z-wave functions of the Dirac system (2.10), then

$$\frac{d}{dx}W(\Psi_1(z,x), \Psi_2(z,x)) = i[Q_{1,1}(x) - Q_{2,2}(x)]W(\Psi_1(z,x), \Psi_2(z,x)), \tag{2.16}$$

and hence

$$W(\Psi_1(z,x), \Psi_2(z,x)) = W(\Psi_1(z,0), \Psi_2(z,0)) \exp\left(i\int_0^x ds\, [Q_{1,1}(s) - Q_{2,2}(s)]\right). \tag{2.17}$$

□

From (2.17), we obtain

COROLLARY 2.2. *The Wronskian has nonzero constant magnitude for any pair of independent z-wave functions of the Dirac system* (2.10) *if and only if*

$$\operatorname{Im}[Q_{1,1}(x) - Q_{2,2}(x)] = 0 \ \text{ for a.e. } x \in \mathbb{R}. \tag{2.18}$$

In light of Theorem 2.1 and the earlier observation that our study will focus upon the two examples of J-self-adjoint differential expressions provided by $\widehat{\mathcal{D}}$ and $\widetilde{\mathcal{D}}$, we make the following hypothesis for the remainder of this paper:

HYPOTHESIS 2.3. *We assume that the Dirac differential expression* $\mathcal{D}$ *given in* (2.2) *is formally* J*-self-adjoint, that is,* $Q_{1,1}(x) = Q_{2,2}(x)$ *holds for a.e.* $x \in \mathbb{R}$.

2.2. Green's matrices and Dirac operators. Assuming the existence of a whole-line Green's matrix for a J-self-adjoint Dirac system (2.10), we can associate a Dirac operator D on $\mathbb{R}$ in the following way: Let $f \in L^2(\mathbb{R})^2$, assume $\rho \subset \mathbb{C}$ is open and nonempty, and consider the inhomogeneous Dirac system given by

$$(\mathcal{D}\Psi)(z,x) = i\sigma_3\Psi'(z,x) + Q(x)\Psi(z,x) = z\Psi(z,x) + f(x), \quad z \in \rho. \tag{2.19}$$

If $G^D(z,x,x')$, $z \in \rho$, $x, x' \in \mathbb{R}$, denotes the unique Green's matrix associated with (2.10), then (2.19) has a unique solution, $\Psi(z,\cdot) \in L^2(\mathbb{R})^2 \cap \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^2$ given by

$$\Psi(z,x) = \int_{-\infty}^{\infty} dx'\, G^D(z,x,x')f(x'), \quad z \in \rho,\ x \in \mathbb{R}. \tag{2.20}$$

The Dirac operator D in $L^2(\mathbb{R})^2$ associated with (2.10) is then defined by

$$((D-z)^{-1}f)(x) = \int_{-\infty}^{\infty} dx'\, G^D(z,x,x')f(x'), \quad z \in \rho,\ f \in L^2(\mathbb{R})^2. \tag{2.21}$$

In terms of the differential expression (2.2), D is explicitly defined by

$$\begin{aligned} D &= i\sigma_3\frac{d}{dx} + Q, \\ \mathrm{dom}(D) &= \big\{\Psi \in L^2(\mathbb{R})^2 \cap \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^2 \,\big|\, \mathcal{D}\Psi \in L^2(\mathbb{R})^2\big\}. \end{aligned} \tag{2.22}$$

REMARK 2.4. Construction of a unique whole-line Green's matrix for (2.10) in association with the operator D is equivalent to the existence of unique (up to constant multiples) Weyl–Titchmarsh-type solutions $\Psi_\pm^{\mathcal{D}}(z,\cdot) \in L^2([0,\pm\infty))^2$, $z \in \rho$ of (2.10). Such solutions are known to exist for Dirac systems associated with (2.3) (cf. [**6**]), and (2.4) (cf. [**4**]). Hence by the construction above, one can describe the operator $\widehat{D}$ in association with (2.3) and the operator $\widetilde{D}$ in association with (2.4). As special cases of (2.2) satisfying Hypothesis 2.3, both $\widehat{\mathcal{D}}$ and $\widetilde{\mathcal{D}}$ are formally J-self-adjoint differential expressions. Moreover, it has been proved in [**6**] that $\widehat{D}$, maximally defined as in (2.22), is self-adjoint,

$$\widehat{D} = \widehat{D}^*. \tag{2.23}$$

In addition, it was shown in [**4**] that $\widetilde{D}$, maximally defined as in (2.22), is J-self-adjoint,

$$\mathcal{J}\widetilde{D}\mathcal{J} = \widetilde{D}^*. \tag{2.24}$$

In the self-adjoint context (2.3), the existence of unique Weyl–Titchmarsh-type solutions is of course equivalent to the limit point case of $\widehat{\mathcal{D}}$ at $\pm\infty$. In the context of (2.4), the existence of unique Weyl–Titchmarsh-type solutions, or equivalently, the existence of a unique Green's function, is then the proper analog of the limit point case in this non-self-adjoint situation.

In the next Lemma, and under the presumption of the existence of half-line square integrable solutions, we describe the whole-line Green's matrix for the Dirac system (2.10) in association with the operator D defined in (2.22).

LEMMA 2.5. *Let $\rho \subset \mathbb{C}$ be open and nonempty. Suppose that for all $z \in \rho$, $\Psi_\pm^{\mathcal{D}}(z,\cdot) = \begin{pmatrix} \psi_{\pm,1}^{\mathcal{D}}(z,\cdot) \\ \psi_{\pm,2}^{\mathcal{D}}(z,\cdot) \end{pmatrix} \in L^2([0,\pm\infty))^2$ represent a basis of solutions of the Dirac system given by* (2.10). *Then the whole-line Green's matrix for this system is given by*

(2.25)

$$\begin{aligned} G^D(z,x,x') &= C(z)\begin{cases} \Psi_-^{\mathcal{D}}(z,x)\Psi_+^{\mathcal{D}}(z,x')^\top \sigma_1, & x < x', \\ \\ \Psi_+^{\mathcal{D}}(z,x)\Psi_-^{\mathcal{D}}(z,x')^\top \sigma_1, & x > x' \end{cases} \\ &= C(z)\begin{cases} \begin{pmatrix} \psi_{-,1}^{\mathcal{D}}(z,x)\psi_{+,2}^{\mathcal{D}}(z,x') & \psi_{-,1}^{\mathcal{D}}(z,x)\psi_{+,1}^{\mathcal{D}}(z,x') \\ \psi_{-,2}^{\mathcal{D}}(z,x)\psi_{+,2}^{\mathcal{D}}(z,x') & \psi_{-,2}^{\mathcal{D}}(z,x)\psi_{+,1}^{\mathcal{D}}(z,x') \end{pmatrix}, & x < x', \\ \begin{pmatrix} \psi_{+,1}^{\mathcal{D}}(z,x)\psi_{-,2}^{\mathcal{D}}(z,x') & \psi_{+,1}^{\mathcal{D}}(z,x)\psi_{-,1}^{\mathcal{D}}(z,x') \\ \psi_{+,2}^{\mathcal{D}}(z,x)\psi_{-,2}^{\mathcal{D}}(z,x') & \psi_{+,2}^{\mathcal{D}}(z,x)\psi_{-,1}^{\mathcal{D}}(z,x') \end{pmatrix}, & x > x', \end{cases} \\ & \hspace{20em} z \in \rho, \end{aligned}$$

where

$$\text{(2.26)} \qquad \begin{aligned} C(z) &= -i[W(\Psi_+^{\mathcal{D}}(z,x), \Psi_-^{\mathcal{D}}(z,x))]^{-1} \\ &= -i\big[\psi_{+,1}^{\mathcal{D}}(z,x)\psi_{-,2}^{\mathcal{D}}(z,x) - \psi_{-,1}^{\mathcal{D}}(z,x)\psi_{+,2}^{\mathcal{D}}(z,x)\big]^{-1} \end{aligned}$$

is constant with respect to $x \in \mathbb{R}$.

PROOF. Note the following unitary equivalence of differential expressions associated with (2.2) and (2.9):

$$\text{(2.27)} \qquad \mathcal{D} - z = \sigma_1(\mathcal{D}^\dagger - z)\sigma_1.$$

As a consequence, if

$$\text{(2.28)} \qquad \Psi_\pm^{\mathcal{D}^\dagger}(z,\cdot) = \sigma_1 \Psi_\pm^{\mathcal{D}}(z,\cdot) \in L^2([0,\pm\infty))^2, \quad z \in \rho,$$

then $\Psi_\pm^{\mathcal{D}^\dagger}(z,x)$ represent half-line square integrable solutions of the associated real adjoint system

$$\text{(2.29)} \qquad (\mathcal{D}^\dagger \Psi)(z,x) = -i\sigma_3 \Psi'(z,x) + Q(x)^\top \Psi(z,x) = z\Psi(z,x).$$

Hence, the Green's matrix ansatz given in (2.25) can be written as

$$\text{(2.30)} \qquad G^D(z,x,x') = \begin{cases} C\Psi_-^{\mathcal{D}}(z,x)\Psi_+^{\mathcal{D}^\dagger}(z,x')^\top, & x < x', \\ C\Psi_+^{\mathcal{D}}(z,x)\Psi_-^{\mathcal{D}^\dagger}(z,x')^\top, & x > x'. \end{cases}$$

To verify the ansatz, let $f \in L^2(\mathbb{R})^2$ and note that

$$\begin{aligned} \text{(2.31)} \quad & (\mathcal{D} - z)\int_{-\infty}^{\infty} dx'\, G^D(z,x,x')f(x') \\ &= (\mathcal{D} - z)\int_{-\infty}^{x} dx'\, G^D(z,x,x')f(x') + (\mathcal{D} - z)\int_{x}^{\infty} dx'\, G^D(z,x,x')f(x') \\ &= iC\sigma_3\big[\Psi_+^{\mathcal{D}}(z,x)\Psi_-^{\mathcal{D}^\dagger}(z,x)^\top - \Psi_-^{\mathcal{D}}(z,x)\Psi_+^{\mathcal{D}^\dagger}(z,x)^\top\big]f(x) \end{aligned}$$

with the last equality following from the fact that $(\mathcal{D}-z)G^D(z,x,x')=0$ for $x\neq x'$. Given (2.28), we see that

$$\begin{aligned}&(\mathcal{D}-z)\int_{-\infty}^{\infty}dx'\,G^D(z,x,x')f(x')\\ &=iC\begin{pmatrix}(\psi^{\mathcal{D}}_{+,1}\psi^{\mathcal{D}}_{-,2}-\psi^{\mathcal{D}}_{-,1}\psi^{\mathcal{D}}_{+,2})(z,x) & 0\\ 0 & (\psi^{\mathcal{D}}_{-,2}\psi^{\mathcal{D}}_{+,1}-\psi^{\mathcal{D}}_{+,2}\psi^{\mathcal{D}}_{-,1})(z,x)\end{pmatrix}f(x).\end{aligned}\tag{2.32}$$

Given that the Wronksian

$$\begin{aligned}W(\Psi^{\mathcal{D}}_{+}(z,x),\Psi^{\mathcal{D}}_{-}(z,x))&=\Psi^{\mathcal{D}}_{-}(z,x)^{\top}J\Psi^{\mathcal{D}}_{+}(z,x)\\ &=\psi^{\mathcal{D}}_{+,1}(z,x)\psi^{\mathcal{D}}_{-,2}(z,x)-\psi^{\mathcal{D}}_{-,1}(z,x)\psi^{\mathcal{D}}_{+,2}(z,x)\end{aligned}\tag{2.33}$$

is a nonzero constant for $x\in\mathbb{R}$, we obtain

$$f(x)=(\mathcal{D}-z)\int_{-\infty}^{\infty}dx'\,G^D(z,x,x')f(x')\tag{2.34}$$

when $C=-i(\psi^{\mathcal{D}}_{+,1}\psi^{\mathcal{D}}_{-,2}-\psi^{\mathcal{D}}_{-,1}\psi^{\mathcal{D}}_{+,2})^{-1}$. □

2.3. Hamiltonian Systems and Green's matrices. Associated with the whole-line formally J-self-adjoint Dirac differential expression (2.2) is the unitarily equivalent differential expression in Hamiltonian form given by

$$\mathcal{H}=U\mathcal{D}U^{-1}=-\sigma_4\frac{d}{dx}+B(x),\quad B\in L^1_{\text{loc}}(\mathbb{R})^{2\times 2},\tag{2.35}$$

in terms of the unitary matrix

$$U=\frac{1}{2}\begin{pmatrix}-1+i & -1+i\\ 1+i & -1-i\end{pmatrix},\quad U^*=U^{-1}.\tag{2.36}$$

In particular, we note that

$$B=UQU^{-1}=\frac{1}{2}\begin{pmatrix}Q_{2,1}+Q_{1,2}+2Q_{1,1} & i(Q_{2,1}-Q_{1,2})\\ i(Q_{2,1}-Q_{1,2}) & -Q_{2,1}-Q_{1,2}+2Q_{1,1}\end{pmatrix}.\tag{2.37}$$

With $U\in\mathbb{C}^{2\times 2}$ defined in (2.36), we note also that

$$i\sigma_4=U\sigma_3U^{-1},\quad U\sigma_1U^{\top}=-iI_2,\tag{2.38}$$

and observe that the property of formal J-self-adjointness for $\mathcal{D}$ is now manifest in the unitarily equivalent Hamiltonian differential expression $\mathcal{H}$ by the following relationships:

$$(i\mathcal{C})\mathcal{H}(i\mathcal{C})=(i\sigma_4)\mathcal{H}(i\sigma_4)=\mathcal{H}^*,\tag{2.39}$$

where $\mathcal{C}$ again represents the conjugation operator (1.6) acting on $\mathbb{C}^2$.

We note that in association with (2.3), one obtains the unitarily equivalent formally self-adjoint differential expression $\widehat{\mathcal{H}}$ given by

$$\widehat{\mathcal{H}}=U\widehat{\mathcal{D}}U^{-1}=J\frac{d}{dx}+\widehat{B}(x),\quad \widehat{B}=U\widehat{Q}U^{-1}=\begin{pmatrix}\operatorname{Im}(q) & -\operatorname{Re}(q)\\ -\operatorname{Re}(q) & -\operatorname{Im}(q)\end{pmatrix},\tag{2.40}$$

while in association with (2.4) one obtains the unitarily equivalent formally non-self-adjoint differential expression $\widetilde{\mathcal{H}}$ given by

$$\widetilde{\mathcal{H}}=U\widetilde{\mathcal{D}}U^{-1}=-\sigma_4\frac{d}{dx}+\widetilde{B}(x),\quad \widetilde{B}=U\widetilde{Q}U^{-1}=i\begin{pmatrix}-\operatorname{Re}(q) & -\operatorname{Im}(q)\\ -\operatorname{Im}(q) & \operatorname{Re}(q)\end{pmatrix}.\tag{2.41}$$

As special cases of (2.35), both $\widehat{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$ satisfy the relationship given in (2.39).

In addition to providing unitarily equivalent differential expressions, the unitary matrix U exhibits another notable feature: It preserves the Wronskian.

Lemma 2.6. *Let* $\eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$, $\xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \in \mathbb{C}^2$. *Then*

$$\det(\eta\ \ \xi) = \det(U\eta\ \ U\xi) = \det(U^{-1}\eta\ \ U^{-1}\xi). \tag{2.42}$$

Proof. We note that $\det(\eta\ \ \xi) = \eta^\top \sigma_4 \xi$, where $\sigma_4 \in \mathbb{C}^{2\times 2}$ is defined in (2.1). Similarly, we note that $\det(U\eta\ \ U\xi) = \eta^\top U^\top \sigma_4 U\xi$, and that $\det(U^{-1}\eta\ \ U^{-1}\xi) = \eta^\top (U^{-1})^\top J U^{-1}\xi$. Upon verifying that $\sigma_4 = U^\top \sigma_4 U = (U^{-1})^\top \sigma_4 U^{-1}$, the result follows. $\square$

Through the unitary equivalence in (2.35) of the differential expressions $\mathcal{D}$ and $\mathcal{H}$, we can define an operator H, in association with the homogeneous Hamiltonian system given by

$$(\mathcal{H}\Psi)(z,x) = -\sigma_4 \Psi'(z,x) + B(x)\Psi(z,x) = z\Psi(z,x), \tag{2.43}$$

for a.e. $x \in \mathbb{R}$, where z plays the role of the spectral parameter, and where

$$\Psi(z,x) = \begin{pmatrix}\psi_1(z,x)\\ \psi_2(z,x)\end{pmatrix}, \quad \psi_j(z,\cdot) \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}),\ j=1,2. \tag{2.44}$$

Namely,

$$H = UDU^{-1} = -\sigma_4 \frac{d}{dx} + B \tag{2.45}$$

$$\operatorname{dom}(H) = \big\{\Psi \in L^2(\mathbb{R})^2 \cap \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^2 \,\big|\, \mathcal{H}\Psi \in L^2(\mathbb{R})^2\big\}. \tag{2.46}$$

The presumptive existence of half-line square integrable solutions of the Dirac system (2.10) yields the existence of half-line square integrable solutions of the associated Hamiltonian system (2.43) by

$$\Psi_\pm^{\mathcal{H}}(z,x) = U\Psi_\pm^{\mathcal{D}}(z,x). \tag{2.47}$$

Then, by Lemma 2.5, we obtain a description of the whole-line Green's matrix for the Hamiltonian system (2.43).

Lemma 2.7. *Let $\rho \subset \mathbb{C}$ be open and nonempty. Suppose that for all $z \in \rho$, $\Psi_\pm^{\mathcal{D}}(z,\cdot) \in L^2([0,\pm\infty))^2$ represent a basis of solutions of the Dirac system given by* (2.10). *Then, with U given in* (2.36), $\Psi_\pm^{\mathcal{H}}(z,\cdot) = U\Psi_\pm^{\mathcal{D}}(z,x) \in L^2([0,\pm\infty))^2$, *represent a basis of solutions of the Hamiltonian system given by* (2.43), *and the whole-line Green's matrix for the Hamiltonian system* (2.43) *is given by*

$$\begin{aligned} G^H(z,x,x') &= UG^D(z,x,x')U^{-1} \\ &= K(z)\Psi_\mp^{\mathcal{H}}(z,x)\Psi_\pm^{\mathcal{H}}(z,x')^\top, \quad x \lesseqgtr x',\ z \in \rho, \end{aligned} \tag{2.48}$$

where

$$K(z) = [W(\Psi_+^{\mathcal{H}}(z,x), \Psi_-^{\mathcal{H}}(z,x))]^{-1} = [W(\Psi^{\mathcal{D}}(z,x)_+, \Psi_-^{\mathcal{D}}(z,x))]^{-1} \tag{2.49}$$

is constant with respect to $x \in \mathbb{R}$.

Proof. Let $z \in \rho$. By the unitary equivalence of $\mathcal{D}$ and $\mathcal{H}$ seen in (2.45), and with U defined in (2.36) and $\sigma_1 \in \mathbb{R}^{2\times 2}$ defined in (2.1), it follows that

$$\begin{aligned} G^H(z,x,x') &= UG^D(z,x,x')U^{-1} \\ &= CU\Psi_{\mp}^{\mathcal{D}}(z,x)\Psi_{\pm}^{\mathcal{D}}(z,x')^{\top}\sigma_1 U^{-1}, \quad x \lessgtr x', \\ &= C\Psi_{\mp}^{\mathcal{H}}(z,x)\Psi_{\pm}^{\mathcal{H}}(z,x')^{\top}(U^{-1})^{\top}\sigma_1 U^{-1}, \quad x \lessgtr x', \\ &= iC\Psi_{\mp}^{\mathcal{H}}(z,x)\Psi_{\pm}^{\mathcal{H}}(z,x')^{\top}, \quad x \lessgtr x', \end{aligned} \tag{2.50}$$

where by Lemmas 2.5 and 2.6,

$$\begin{aligned} C &= -i[W(\Psi_+^{\mathcal{D}}, \Psi_-^{\mathcal{D}})]^{-1} \\ &= -i[W(U^{-1}\Psi_+^{\mathcal{H}}, U^{-1}\Psi_-^{\mathcal{H}})]^{-1} = -i[W(\Psi_+^{\mathcal{H}}, \Psi_-^{\mathcal{H}})]^{-1}. \end{aligned} \tag{2.51}$$

□

3. Self-adjoint Dirac and Hamiltonian Systems

As developed in the previous section, the Dirac operator D defined in (2.22) corresponding to the Dirac system (2.10), is unitarily equivalent to the operator H in (2.45) associated with the Hamiltonian system (2.43). In this section, we focus upon self-adjoint realizations for each of these operators, specifically, the operator $\widehat{D}$, maximally defined by (2.22) associated with the special case of (2.2) given by (2.3), and the operator $\widehat{H}$ maximally defined by (2.45) corresponding to the special case of (2.35) given by (2.40).

3.1. Weyl–Titchmarsh coefficients. Let $N^{\widehat{\mathcal{H}}}(z,\pm\infty)$ and $N^{\widehat{\mathcal{D}}}(z,\pm\infty)$, $z \in \mathbb{C}$, denote the spaces defined for the differential expressions $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{D}}$, respectively, by

$$N^{\widehat{\mathcal{H}}}(z,\pm\infty) = \big\{\Psi \in L^2([0,\pm\infty))^2 \,\big|\, (\widehat{\mathcal{H}} - z)\Psi = 0\big\}, \tag{3.1}$$

$$N^{\widehat{\mathcal{D}}}(z,\pm\infty) = \big\{\Psi \in L^2([0,\pm\infty))^2 \,\big|\, (\widehat{\mathcal{D}} - z)\Psi = 0\big\}. \tag{3.2}$$

By [**6**, Lemma 2.15],

$$\dim\big(N^{\widehat{\mathcal{H}}}(z,\pm\infty)\big) = 1, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.3}$$

and hence by the unitary equivalence given in (2.35),

$$\dim\big(N^{\widehat{\mathcal{D}}}(z,\pm\infty)\big) = 1, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.4}$$

In particular, one has the following result.

Theorem 3.1 ([**6**], [**30**], [**43**]). *The operator $\widehat{H}$, maximally defined in* (2.45)*, is self-adjoint,*

$$\widehat{H} = \widehat{H}^* \tag{3.5}$$

and the operator $\widehat{D}$, maximally defined in (2.22)*, is self-adjoint,*

$$\widehat{D} = \widehat{D}^*. \tag{3.6}$$

Moreover, $\widehat{H}$ and $\widehat{D}$ are unitarily eqivalent,

$$\widehat{H} = U\widehat{D}U^{-1}. \tag{3.7}$$

Proof. Equation (3.5) has been proven in [**6**]. The rest follows from the unitary equivalence (2.35) via the constant unitary matrix U. □

Self-adjoint half-line operators associated with the differential expressions $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{D}}$ are defined by

$$\begin{aligned} &\widehat{H}_{\pm}(\alpha) = -\sigma_4 \frac{d}{dx} + \widehat{B}, \\ &\operatorname{dom}(\widehat{H}_{\pm}(\alpha)) = \big\{\Psi \in L^2([0,\pm\infty))^2 \,\big|\, \Psi \in \mathrm{AC}_{\mathrm{loc}}([0,\pm\infty))^2, \\ &\qquad\qquad \alpha\Psi(0) = 0,\ \widehat{\mathcal{H}}\Psi \in L^2([0,\pm\infty))^2\big\}, \end{aligned} \tag{3.8}$$

where $\alpha = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$, and by

$$\begin{aligned} &\widehat{D}_{\pm}(\beta) = i\sigma_3 \frac{d}{dx} + \widehat{Q}, \\ &\operatorname{dom}(\widehat{D}_{\pm}(\beta)) = \big\{\Psi \in L^2([0,\pm\infty))^2 \,\big|\, \Psi \in \mathrm{AC}_{\mathrm{loc}}([0,\pm\infty))^2, \\ &\qquad\qquad \beta\Psi(0) = 0,\ \widehat{\mathcal{D}}\Psi \in L^2([0,\pm\infty))^2\big\}, \end{aligned} \tag{3.9}$$

where

$$\beta = \alpha U = [(-1+i)/2](e^{-i\theta}, e^{i\theta}), \quad \theta \in [0, 2\pi). \tag{3.10}$$

$\widehat{H}_{\pm}(\alpha)$ is unitarily equivalent to $\widehat{D}_{\pm}(\beta)$, given (2.40) and the fact that the unitary 2×2 matrix U naturally defines a unitary mapping of $L^2([0,\pm\infty))^2$ onto itself, again for simplicity denoted by U, which maps $\operatorname{dom}(\widehat{D}_{\pm}(\beta))$ onto $\operatorname{dom}(\widehat{H}_{\pm}(\alpha))$. The later fact can be seen by noting that

$$0 = \beta\Psi^{\widehat{\mathcal{D}}}(z,0) = \beta U^{-1} U \Psi^{\widehat{\mathcal{D}}}(z,0) = \alpha\Psi^{\widehat{\mathcal{H}}}(z,0). \tag{3.11}$$

Thus,

$$\widehat{H}_{\pm}(\alpha) = U\widehat{D}_{\pm}(\beta)U^{-1}, \quad \beta = \alpha U. \tag{3.12}$$

In passing, we note that (3.3) and (3.4) prove that both $\widehat{H}_{\pm}(\alpha)$ and $\widehat{D}_{\pm}(\beta)$ are in the limit point case at $\pm\infty$.

Next, let a fundamental system of solutions of the self-adjoint Hamiltonian system $\widehat{\mathcal{H}}\Psi = z\Psi$ be given by

$$\Theta^{\widehat{\mathcal{H}}}(z,\cdot,\alpha),\ \Phi^{\widehat{\mathcal{H}}}(z,\cdot,\alpha) \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^2, \quad z \in \mathbb{C} \tag{3.13}$$

such that

$$\Theta^{\widehat{\mathcal{H}}}(z,0,\alpha) = \alpha^*, \quad \Phi^{\widehat{\mathcal{H}}}(z,0,\alpha) = -\sigma_4\alpha^*, \tag{3.14}$$

where $\alpha \in \mathbb{C}^2$ and where

$$\alpha\alpha^* = 1, \quad \alpha\sigma_4\alpha^* = 0. \tag{3.15}$$

By Theorem 2.1, $W(\Theta^{\widehat{\mathcal{H}}}(z,x,\alpha), \Phi^{\widehat{\mathcal{H}}}(z,x,\alpha))$ is constant for $x \in \mathbb{R}$. If in addition, we require that

$$W(\Theta^{\widehat{\mathcal{H}}}(z,0,\alpha), \Phi^{\widehat{\mathcal{H}}}(z,0,\alpha)) = -\det(\alpha^*\ \sigma_4\alpha^*) = 1, \tag{3.16}$$

then it can be shown that $\alpha \in \mathbb{R}^2$. Thus, (3.8) and (3.9) yield the only self-adjoint half-line operators consistent with (3.15) and (3.16). Hence, for the remainder of this section, we let $\alpha = \alpha(\theta) = (\cos(\theta), \sin(\theta))$, for $\theta \in [0, 2\pi)$, and let

$$\Theta^{\widehat{\mathcal{H}}}(z,0,\alpha) = (\cos(\theta), \sin(\theta))^\top, \quad \Phi^{\widehat{\mathcal{H}}}(z,0,\alpha) = (-\sin(\theta), \cos(\theta))^\top. \tag{3.17}$$

In [**6**], it is shown that $\Phi^{\widehat{\mathcal{H}}}(z,\cdot,\alpha) \notin L^2([0,\pm\infty))^2$ for $z \in \mathbb{C}\backslash\mathbb{R}$. Then, as a consequence of (3.3), let $m_\pm^{\widehat{\mathcal{H}}}(z,\alpha)$ denote the half-line Weyl–Titchmarsh coefficients; that is, the unique coefficients such that

$$\Psi_\pm^{\widehat{\mathcal{H}}}(z,\cdot,\alpha) = \Theta^{\widehat{\mathcal{H}}}(z,\cdot,\alpha) + m_\pm^{\widehat{\mathcal{H}}}(z,\alpha)\Phi^{\widehat{\mathcal{H}}}(z,\cdot,\alpha) \in L^2([0,\pm\infty))^2, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.18}$$

A corresponding development for the self-adjoint Dirac system $\widehat{\mathcal{D}}\Psi(z,x) = z\Psi(z,x)$ begins with its fundamental system of solutions

$$\Theta^{\widehat{\mathcal{D}}}(z,\cdot,\beta),\ \Phi^{\widehat{\mathcal{D}}}(z,\cdot,\beta) \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^2, \quad z \in \mathbb{C} \tag{3.19}$$

for $\beta = \alpha U$, where $\alpha = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$ (cf. (3.17)), $U \in \mathbb{C}^{2\times 2}$ is given in (2.36), and hence,

$$\beta = \alpha U = [(-1+i)/2](e^{-i\theta}, e^{i\theta}), \quad \theta \in [0, 2\pi). \tag{3.20}$$

Specifically, for $\theta \in [0, 2\pi)$, let

$$\Theta^{\widehat{\mathcal{D}}}(z,0,\beta) = i\sigma_1\beta^\top = -[(1+i)/2](e^{i\theta}, e^{-i\theta})^\top, \tag{3.21}$$

$$\Phi^{\widehat{\mathcal{D}}}(z,0,\beta) = i\sigma_3\beta^* = [(1-i)/2](e^{i\theta}, -e^{-i\theta})^\top. \tag{3.22}$$

In particular, we see that

$$\Theta^{\widehat{\mathcal{D}}}(z,x,\beta) = U^{-1}\Theta^{\widehat{\mathcal{H}}}(z,x,\alpha), \quad \Phi^{\widehat{\mathcal{D}}}(z,x,\beta) = U^{-1}\Phi^{\widehat{\mathcal{H}}}(z,x,\alpha). \tag{3.23}$$

As a consequence, $\Phi^{\widehat{\mathcal{D}}}(z,\cdot,\beta) \notin L^2([0,\pm\infty))^2$. As before, given (3.2), let $m_\pm^{\widehat{\mathcal{D}}}(z,\beta)$ denote the unique coefficients such that

$$\Psi_\pm^{\widehat{\mathcal{D}}}(z,\cdot,\beta) = \Theta^{\widehat{\mathcal{D}}}(z,\cdot,\beta) + m_\pm^{\widehat{\mathcal{D}}}(z,\beta)\Phi^{\widehat{\mathcal{D}}}(z,\cdot,\beta) \in L^2([0,\pm\infty))^2, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.24}$$

In summary, we have the following result.

LEMMA 3.2. *Let $\alpha = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$, and let $\beta = \alpha U$ with U defined in* (2.36). *Let $\Theta^{\widehat{\mathcal{H}}}$, $\Phi^{\widehat{\mathcal{H}}}$ represent the fundamental system of solutions of the Hamiltonian system $\widehat{\mathcal{H}}\Psi = z\Psi$ satisfying* (3.17)*, and let $\Theta^{\widehat{\mathcal{D}}}$, $\Phi^{\widehat{\mathcal{D}}}$ represent the fundamental system of solutions of the Dirac system $\widehat{\mathcal{D}}\Psi = z\Psi$ satisfying* (3.21)*, and* (3.22)*. Then, with $\Psi_\pm^{\widehat{\mathcal{H}}}$ defined in* (3.18) *and with $\Psi_\pm^{\widehat{\mathcal{D}}}$ defined in* (3.24)*, one infers that*

$$\Psi_\pm^{\widehat{\mathcal{D}}}(z,x,\beta) = U^{-1}\Psi_\pm^{\widehat{\mathcal{H}}}(z,x,\alpha), \quad z \in \mathbb{C}\backslash\mathbb{R},\ x \in \mathbb{R}, \tag{3.25}$$

and in particular, that

$$m_\pm^{\widehat{\mathcal{D}}}(z,\beta) = m_\pm^{\widehat{\mathcal{H}}}(z,\alpha), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.26}$$

PROOF. By (3.23)

$$\begin{aligned} U^{-1}\Psi_\pm^{\widehat{\mathcal{H}}}(z,\cdot,\alpha) &= U^{-1}\Theta^{\widehat{\mathcal{H}}}(z,\cdot,\alpha) + m_\pm^{\widehat{\mathcal{H}}}(z,\alpha)U^{-1}\Phi^{\widehat{\mathcal{H}}}(z,\cdot,\alpha) \\ &= \Theta^{\widehat{\mathcal{D}}}(z,\cdot,\beta) + m_\pm^{\widehat{\mathcal{H}}}(z,\alpha)\Phi^{\widehat{\mathcal{D}}}(z,\cdot,\beta) \in L^2([0,\pm\infty))^2. \end{aligned} \tag{3.27}$$

By the uniqueness of the representation for the combination given in (3.24), equation (3.26) follows. ☐

REMARK 3.3. In light of Lemma 3.2, in the future we shall represent both $m_\pm^{\widehat{\mathcal{H}}}(z,\alpha)$ and $m_\pm^{\widehat{\mathcal{D}}}(z,\beta)$ by $\widehat{m}_\pm(z,\gamma)$, where it is understood that $\beta=\alpha U$. Here

$$\gamma=\begin{cases}\text{represents } \alpha \text{ in the context of } \widehat{\mathcal{H}},\\ \text{represents } \beta=\alpha U \text{ in the context of } \widehat{\mathcal{D}},\end{cases} \tag{3.28}$$

and we keep this convention in similar contexts in the following.

Of course, $\pm\widehat{m}_\pm(\cdot,\gamma)$ are well-known to be Herglotz functions (i.e., analytic functions mapping the open complex upper half-plane into itself) and

$$\widehat{m}_\pm(\cdot,\gamma) \text{ are analytic on } \rho(\widehat{H}_\pm(\alpha))=\rho(\widehat{D}_\pm(\beta)). \tag{3.29}$$

3.2. Green's matrices. Before describing Green's matrices for self-adjoint Hamiltonian and Dirac systems, we introduce two matrices. First, for the fundamental system of solutions defined in (3.13) and satisfying (3.17), let $\mathcal{F}^{\widehat{\mathcal{H}}}(z,\cdot,\alpha)$ denote the associated fundamental matrix given by

$$\mathcal{F}^{\widehat{\mathcal{H}}}(z,x,\alpha)=\big(\Theta^{\widehat{\mathcal{H}}}(z,x,\alpha)\ \ \Phi^{\widehat{\mathcal{H}}}(z,x,\alpha)\big), \quad z\in\mathbb{C},\ x\in\mathbb{R}. \tag{3.30}$$

Next, we introduce the matrix $\widehat{\Gamma}(z,\gamma)$ (we recall the meaning of γ as introduced in Remark 3.3), where

$$\widehat{\Gamma}(z,\gamma)=\begin{pmatrix}\frac{1}{\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)} & \frac{\widehat{m}_-(z,\gamma)}{\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)}\\ \frac{\widehat{m}_+(z,\gamma)}{\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)} & \frac{\widehat{m}_-(z,\gamma)\widehat{m}_+(z,\gamma)}{\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)}\end{pmatrix}, \quad z\in\mathbb{C}\backslash\mathbb{R}. \tag{3.31}$$

Then, as a consequence of Lemma 2.7 we obtain the following result.

LEMMA 3.4. *With $\Psi_\pm^{\widehat{\mathcal{H}}}(z,\cdot,\alpha)$ representing the half-line Weyl–Titchmarsh solutions defined in* (3.18) *for the Hamiltonian system $\widehat{\mathcal{H}}\Psi=z\Psi$, the associated whole-line Green's matrix associated with the operator $\widehat{H}$ is given by*

$$G^{\widehat{H}}(z,x,x')=K(z,\alpha)\Psi_\mp^{\widehat{\mathcal{H}}}(z,x,\alpha)\Psi_\pm^{\widehat{\mathcal{H}}}(z,x',\alpha)^\top, \quad x\lesseqgtr x', \tag{3.32}$$

$$=\begin{cases}\mathcal{F}^{\widehat{\mathcal{H}}}(z,x,\alpha)\widehat{\Gamma}(z,\gamma)^\top\mathcal{F}^{\widehat{\mathcal{H}}}(z,x'\alpha)^\top, & x<x',\\ \mathcal{F}^{\widehat{\mathcal{H}}}(z,x,\alpha)\widehat{\Gamma}(z,\gamma)\mathcal{F}^{\widehat{\mathcal{H}}}(z,x'\alpha)^\top, & x>x',\end{cases} \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{3.33}$$

where

$$K(z,\alpha)=\big[W(\Psi_+^{\widehat{\mathcal{H}}}(z,x,\alpha),\Psi_-^{\widehat{\mathcal{H}}}(z,x,\alpha))\big]^{-1}\big|_{x=0}=[\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)]^{-1}. \tag{3.34}$$

PROOF. Equation (3.32) follows from Lemma 2.7 for the operator $\widehat{H}$ defined by (2.45), but in association with the special case of (2.35) given by (2.40). Moreover, it follows that $K=[W(\Psi_+^{\widehat{\mathcal{H}}},\Psi_-^{\widehat{\mathcal{H}}})]^{-1}$. Then, by (3.18) one notes that

$$W(\Psi_+^{\widehat{\mathcal{H}}},\Psi_-^{\widehat{\mathcal{H}}})=\begin{pmatrix}1\\ \widehat{m}_-\end{pmatrix}^\top\begin{pmatrix}(\Theta^{\widehat{\mathcal{H}}})^\top J\Theta^{\widehat{\mathcal{H}}} & (\Theta^{\widehat{\mathcal{H}}})^\top J\Phi^{\widehat{\mathcal{H}}}\\ (\Phi^{\widehat{\mathcal{H}}})^\top J\Theta^{\widehat{\mathcal{H}}} & (\Phi^{\widehat{\mathcal{H}}})^\top J\Phi^{\widehat{\mathcal{H}}}\end{pmatrix}\begin{pmatrix}1\\ \widehat{m}_+\end{pmatrix}. \tag{3.35}$$

However, $\eta^\top J\eta=0$, and $\eta^\top J\xi=-\xi^\top J\eta$ for every $\eta,\ \xi\in\mathbb{C}^2$. As a consequence,

$$\begin{aligned}W(\Psi_+^{\widehat{\mathcal{H}}},\Psi_-^{\widehat{\mathcal{H}}})&=[\widehat{m}_--\widehat{m}_+](\Phi^{\widehat{\mathcal{H}}})^\top J\Theta^{\widehat{\mathcal{H}}}\\ &=[\widehat{m}_--\widehat{m}_+]W(\Theta^{\widehat{\mathcal{H}}},\Phi^{\widehat{\mathcal{H}}})\\ &=\widehat{m}_--\widehat{m}_+,\end{aligned} \tag{3.36}$$

where the last equality follows from the normalization (3.16).

The description of $G^{\widehat{H}}(z,x,x')$ given in (3.33) follows from (3.18), (3.32), and the fact that

$$\Psi_\pm^{\widehat{\mathcal{H}}}(z,\cdot,\alpha) = \mathcal{F}^{\widehat{\mathcal{H}}}(z,\cdot,\alpha)\begin{pmatrix} 1 \\ \widehat{m}_\pm(z,\alpha)\end{pmatrix}. \tag{3.37}$$

□

Following as an immediate consequence of the unitary equivalence of $\widehat{H}$ and $\widehat{D}$, together with Lemmas 2.5, 2.7, and 3.4, one infers the following fact.

LEMMA 3.5. *With $\Psi_\pm^{\widehat{\mathcal{H}}}(z,\cdot,\alpha)$ and $\Psi_\pm^{\widehat{\mathfrak{D}}}(z,\cdot,\beta)$ defined in* (3.18) *and* (3.24)*, the whole-line Green's matrix for the self-adjoint Dirac system $\widehat{\mathfrak{D}}\Psi = z\Psi$ is given by*

$$\begin{aligned} G^{\widehat{D}}(z,x,x') &= C(z,\beta)\Psi_\mp^{\widehat{\mathfrak{D}}}(z,x,\beta)\Psi_\pm^{\widehat{\mathfrak{D}}^\dagger}(z,x',\beta)^\top, \quad x \lessgtr x', \\ &= C(z,\beta)\Psi_\mp^{\widehat{\mathfrak{D}}}(z,x,\beta)\Psi_\pm^{\widehat{\mathfrak{D}}}(z,x',\beta)^\top\sigma_1, \quad x \lessgtr x', \\ &= \begin{cases} -i\mathcal{F}^{\widehat{\mathfrak{D}}}(z,x,\beta)\widehat{\Gamma}(z,\gamma)^\top\mathcal{F}^{\widehat{\mathfrak{D}}}(z,x',\beta)^\top, & x<x', \\ -i\mathcal{F}^{\widehat{\mathfrak{D}}}(z,x,\beta)\widehat{\Gamma}(z,\gamma)\mathcal{F}^{\widehat{\mathfrak{D}}}(z,x',\beta)^\top, & x>x', \end{cases} \quad z\in\mathbb{C}\backslash\mathbb{R}, \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} C(z,\beta) &= -i\big[W(\Psi_+^{\widehat{\mathfrak{D}}}(z,x,\beta),\Psi_-^{\widehat{\mathfrak{D}}}(z,x,\beta))\big]^{-1}\big|_{x=0} \\ &= -i\big[W(\Psi_+^{\widehat{\mathcal{H}}}(z,x,\alpha),\Psi_-^{\widehat{\mathcal{H}}}(z,x,\alpha))\big]^{-1}\big|_{x=0} \\ &= -i[\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)]^{-1}. \end{aligned} \tag{3.39}$$

Here $\mathcal{F}^{\widehat{\mathfrak{D}}}(z,\cdot,\beta)$ is the fundamental matrix of solutions of the Dirac system $\widehat{\mathfrak{D}}\Psi = z\Psi$ given by

$$\mathcal{F}^{\widehat{\mathfrak{D}}}(z,x,\beta) = \big(\Theta^{\widehat{\mathfrak{D}}}(z,x,\beta)\;\; \Phi^{\widehat{\mathfrak{D}}}(z,x,\beta)\big), \quad z\in\mathbb{C},\ x\in\mathbb{R}, \tag{3.40}$$

and $\Theta^{\widehat{\mathfrak{D}}}(z,\cdot,\beta)$, $\Phi^{\widehat{\mathfrak{D}}}(z,\cdot,\beta)$ represent the fundamental system of solutions defined in (3.19) *for the self-adjoint Dirac system.*

Of course, the Green's matrices (3.32) and (3.38) extend to analytic 2×2 matrix-valued functions with respect to $z\in\rho(\widehat{H}) = \rho(\widehat{D})$.

3.3. Spectral matrices. In preparation for the description of the spectral matrix associated with the operator $\widehat{H}$, we now introduce two matrices and a transformation.

We denote by $\widehat{M}(z,\gamma)\in\mathbb{C}^{2\times 2}$, $z\in\mathbb{C}\backslash\mathbb{R}$, the whole-line Weyl–Titchmarsh M-function of the operator $\widehat{H}$ defined in (2.45) in association with the special case given by (2.40), namely,

$$\begin{aligned} \widehat{M}(z,\gamma) &= \big(\widehat{M}_{\ell,\ell'}(z,\gamma)\big)_{\ell,\ell'=0,1} = \frac{1}{2}[\widehat{\Gamma}(z,\gamma)+\widehat{\Gamma}(z,\gamma)^\top] = \Gamma(z,\gamma)+\frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)} & \frac{1}{2}\frac{\widehat{m}_-(z,\gamma)+\widehat{m}_+(z,\gamma)}{\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)} \\ \frac{1}{2}\frac{\widehat{m}_-(z,\gamma)+\widehat{m}_+(z,\gamma)}{\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)} & \frac{\widehat{m}_-(z,\gamma)\widehat{m}_+(z,\gamma)}{\widehat{m}_-(z,\gamma)-\widehat{m}_+(z,\gamma)} \end{pmatrix}, \quad z\in\mathbb{C}\backslash\mathbb{R}, \end{aligned} \tag{3.41}$$

where by Remark 3.3, $\widehat{m}_\pm(z,\gamma) = m_\pm^{\widehat{\mathcal{H}}}(z,\alpha) = m_\pm^{\widehat{\mathfrak{D}}}(z,\beta)$ for $\beta = \alpha U$, and $\alpha = (\cos(\theta),\sin(\theta))$, $\theta\in[0,2\pi)$. Again, (3.41) extends to an analytic 2×2 matrix-valued function with respect to $z\in\rho(\widehat{H}) = \rho(\widehat{D})$.

From $\operatorname{Im}\big(\widehat M(z,\gamma)\big) = (2i)^{-1}\big[\widehat M(z,\gamma) - \widehat M(z,\gamma)^*\big]$ one infers that

$$\operatorname{Im}\big(\widehat M(z,\gamma)\big) = \begin{pmatrix} \operatorname{Im}\Big(\frac{1}{\widehat m_-(z,\gamma)-\widehat m_+(z,\gamma)}\Big) & \frac{1}{2}\operatorname{Im}\Big(\frac{\widehat m_-(z,\gamma)+\widehat m_+(z,\gamma)}{\widehat m_-(z,\gamma)-\widehat m_+(z,\gamma)}\Big) \\ \frac{1}{2}\operatorname{Im}\Big(\frac{\widehat m_-(z,\gamma)+\widehat m_+(z,\gamma)}{\widehat m_-(z,\gamma)-\widehat m_+(z,\gamma)}\Big) & \operatorname{Im}\Big(\frac{\widehat m_-(z,\gamma)\widehat m_+(z,\gamma)}{\widehat m_-(z,\gamma)-\widehat m_+(z,\gamma)}\Big) \end{pmatrix}. \tag{3.42}$$

Associated with $\widehat M(z,\gamma)$ we introduce the measure $d\widehat\Omega(\lambda,\gamma)$ by

$$\widehat\Omega((\lambda_1.\lambda_2],\gamma) = \frac{1}{\pi}\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\, \operatorname{Im}\big(\widehat M(\lambda+i\varepsilon,\gamma)\big), \quad \lambda_j\in\mathbb{R},\ j=1,2,\ \lambda_1<\lambda_2, \tag{3.43}$$

and use the abbreviation

$$\big(T_0^{\widehat{\mathcal H}}(\alpha)f\big)(\lambda) = \int_{\mathbb{R}} dx\, \mathcal{F}^{\widehat{\mathcal H}}(\lambda,x,\alpha)^\top f(x), \quad \lambda\in\mathbb{R},\ f\in C_0^\infty(\mathbb{R})^2. \tag{3.44}$$

Henceforth we also abbreviate the scalar product in $L^2((a,b))^2$ by $\langle\cdot,\cdot\rangle_{L^2((a,b))^2}$ (chosen to be linear in the second place), where $-\infty\le a<b\le\infty$.

Theorem 3.6. *Let $\big\{E_{\widehat H}(\lambda)\big\}_{\lambda\in\mathbb{R}}$ denote the spectral family associated with the operator $\widehat H$. Then, for $f,\, g\in C_0^\infty(\mathbb{R})^2$ and $\lambda_1<\lambda_2$,*

$$\langle f, E_{\widehat H}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} = \int_{(\lambda_1,\lambda_2]} \Big(\big(T_0^{\widehat{\mathcal H}}(\alpha)f\big)(\lambda)\Big)^* d\widehat\Omega(\lambda,\gamma)\big(T_0^{\widehat{\mathcal H}}(\alpha)g\big)(\lambda). \tag{3.45}$$

Proof. For simplicity we will suppress the α (resp., γ) dependence of all quantities involved in this proof. We follow the strategy of proof employed in connection with one-dimensional Schrödinger operators in [**16**] (see also [**23**]). Then, by Stone's formula (cf. [**44**, p. 191]),

$$\begin{aligned} &\langle f, E_{\widehat H}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} \\ &\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\,\big\langle f,\ \big[(\widehat H-(\lambda+i\varepsilon))^{-1} - (\widehat H-(\lambda-i\varepsilon))^{-1}\big]g\big\rangle_{L^2(\mathbb{R})^2} \\ &\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\int_{\mathbb{R}} dx\int_{\mathbb{R}} dx'\big\{f(x)^*G^{\widehat H}(\lambda+i\varepsilon,x,x')g(x') \\ &\qquad\qquad - f(x)^*G^{\widehat H}(\lambda-i\varepsilon,x,x')g(x')\big\}. \end{aligned} \tag{3.46}$$

Using the fact that $\widehat m_\pm(\lambda-i\varepsilon) = \overline{\widehat m_\pm(\lambda+i\varepsilon)}$, one concludes that $\widehat\Gamma(\lambda-i\varepsilon) = \overline{\widehat\Gamma(\lambda+i\varepsilon)}$, where $\widehat\Gamma(z)$ is defined in (3.31). Consequently, using the description of

$G^{\widehat H}(z,x,x')$ given in (3.33), (3.46) implies

$$\begin{aligned}
&\langle f, E_{\widehat H}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda \int_{\mathbb{R}} dx \\
&\quad\times\bigg\{\int_{-\infty}^{x} dx'\, f(x)^* \mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x)\Big[\widehat\Gamma(\lambda+i\varepsilon)-\overline{\widehat\Gamma(\lambda+i\varepsilon)}\,\Big]\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x')^\top g(x') \\
&\qquad+\int_{x}^{\infty} dx'\, f(x)^* \mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x)\Big[\widehat\Gamma(\lambda+i\varepsilon)^\top-\overline{\widehat\Gamma(\lambda+i\varepsilon)^\top}\,\Big]\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x')^\top g(x')\bigg\} \\
&\quad=\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda \int_{\mathbb{R}} dx \\
&\qquad\times\bigg\{\int_{-\infty}^{x} dx'\, f(x)^* \mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x)\mathrm{Im}\big(\widehat M(\lambda+i\varepsilon)\big)\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x')^\top g(x') \\
&\qquad\quad+\int_{x}^{\infty} dx'\, f(x)^* \mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x)\mathrm{Im}\big(\widehat M(\lambda+i\varepsilon)\big)\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x')^\top g(x')\bigg\},
\end{aligned}\tag{3.47}$$

since

$$\widehat\Gamma(\lambda+i\varepsilon)-\overline{\widehat\Gamma(\lambda+i\varepsilon)}=\widehat\Gamma(\lambda+i\varepsilon)^\top-\overline{\widehat\Gamma(\lambda+i\varepsilon)^\top}=2i\,\mathrm{Im}\big(\widehat M(\lambda+i\varepsilon)\big),\quad \lambda\in\mathbb{R},\ \varepsilon>0.\tag{3.48}$$

To arrive at equation (3.47) we used the fact that for fixed $x\in\mathbb{R}$, $\mathcal{F}^{\widehat{\mathcal{H}}}(z,x)$ is entire with respect to z, that $\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x)$ is real-valued for $\lambda\in\mathbb{R}$, that $\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x)^\top\in AC_{\mathrm{loc}}(\mathbb{R})^2$, and hence that

$$\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda\pm i\varepsilon,x)\underset{\varepsilon\downarrow 0}{=}\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x)\pm i\varepsilon(d/dz)\mathcal{F}^{\widehat{\mathcal{H}}}(z,x)|_{z=\lambda}+O(\varepsilon^2),\tag{3.49}$$

with $O(\varepsilon^2)$ being uniform with respect to (λ,x) as long as (λ,x) vary in compact subsets of $\mathbb{R}^2$. Moreover, we used that

$$\begin{aligned}
&\varepsilon|\widehat M_{\ell,\ell'}(\lambda+i\varepsilon,\gamma)|\le C(\lambda_1,\lambda_2,\varepsilon_0),\quad \lambda\in[\lambda_1,\lambda_2],\ 0<\varepsilon\le\varepsilon_0,\ \ell,\ell'=0,1,\\
&\varepsilon|\mathrm{Re}(\widehat M_{\ell,\ell'}(\lambda+i\varepsilon,\gamma))|\underset{\varepsilon\downarrow 0}{=}o(1),\quad \lambda\in\mathbb{R},\ \ell,\ell'=0,1,
\end{aligned}\tag{3.50}$$

which follow from the properties of Herglotz functions since $\widehat M_{\ell,\ell}$, $\ell=0,1$, are Herglotz and $\widehat M_{0,1}=\widehat M_{1,0}$ have Herglotz-type representations by decomposing the associated complex measure $d\widehat\Omega_{0,1}$ into $d\widehat\Omega_{0,1}=d(\omega_1-\omega_2)+id(\omega_3-\omega_4)$, with $d\omega_k$, $k=1,\dots,4$, nonnegative measures. Finally, we also used (for $\lambda\in\mathbb{R}$, $\varepsilon>0$)

$$\begin{aligned}
\widehat\Gamma(\lambda+i\varepsilon)+\overline{\widehat\Gamma(\lambda+i\varepsilon)}&=2\,\mathrm{Re}\big(\widehat M(\lambda+i\varepsilon)\big)+\begin{pmatrix}0&1\\-1&0\end{pmatrix},\\
\widehat\Gamma(\lambda+i\varepsilon)^\top+\overline{\widehat\Gamma(\lambda+i\varepsilon)^\top}&=2\,\mathrm{Re}\big(\widehat M(\lambda+i\varepsilon)\big)+\begin{pmatrix}0&-1\\1&0\end{pmatrix}.
\end{aligned}\tag{3.51}$$

Thus,

$$\begin{aligned}
&\langle f, E_{\widehat H}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2}\\
&\quad=\int_{(\lambda_1,\lambda_2]}\int_{\mathbb{R}} dx\int_{\mathbb{R}} dx'\, f(x)^*\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x)d\widehat\Omega(\lambda)\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x')^\top g(x').
\end{aligned}\tag{3.52}$$

Equation (3.45) then follows from the fact that $\mathcal{F}^{\widehat{\mathcal{H}}}(\bar{z},\cdot) = \overline{\mathcal{F}^{\widehat{\mathcal{H}}}(z,\cdot)}$, $z\in\mathbb{C}$, and hence that

$$\Big(\big(T_0^{\widehat{\mathcal{H}}} f\big)(\lambda)\Big)^* = \int_{\mathbb{R}} dx f(x)^* \overline{\mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x)} = \int_{\mathbb{R}} dx f(x)^* \mathcal{F}^{\widehat{\mathcal{H}}}(\lambda,x), \quad \lambda\in\mathbb{R}. \tag{3.53}$$

□

The proof of Theorem 3.6 shows that $T_0^{\widehat{\mathcal{H}}}(\alpha)$ represents a linear operator (denoted by the same symbol),

$$T_0^{\widehat{\mathcal{H}}}(\alpha)\colon \begin{cases} C_0^\infty(\mathbb{R})^2 \to L^2(\mathbb{R}; d\widehat{\Omega}(\lambda,\gamma)) \\ \qquad f \mapsto T_0^{\widehat{\mathcal{H}}}(\alpha) f = \int_{\mathbb{R}} dx\, \mathcal{F}^{\widehat{\mathcal{H}}}(\cdot,x,\alpha)^\top f(x). \end{cases} \tag{3.54}$$

(For some subtleties of L^2-spaces with matrix-valued measures we refer to the discussion in [**16**] and the references cited therein.) Moreover, as recently discussed in the analogous context of Schrödinger operators in [**16**], $T_0^{\widehat{\mathcal{H}}}(\alpha)$ extends to a bounded operator from $L^2(\mathbb{R})^2$ to $L^2(\mathbb{R}; d\widehat{\Omega}(\lambda,\gamma))$, which we denote by $T^{\widehat{\mathcal{H}}}(\alpha)$. This then immediately leads to the following extension of Theorem 3.6.

Theorem 3.7. *Let $\{E_{\widehat{H}}(\lambda)\}_{\lambda\in\mathbb{R}}$ denote the spectral family associated with the operator $\widehat{H}$. Then, for f, $g\in L^2(\mathbb{R})^2$ and $\lambda_1<\lambda_2$,*

$$\langle f, E_{\widehat{H}}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} = \int_{(\lambda_1,\lambda_2]} \Big(\big(T^{\widehat{\mathcal{H}}}(\alpha) f\big)(\lambda)\Big)^* d\widehat{\Omega}(\lambda,\gamma) \big(T^{\widehat{\mathcal{H}}}(\alpha) g\big)(\lambda). \tag{3.55}$$

As a corollary, we obtain the corresponding result for the operator $\widehat{D}$.

Corollary 3.8. *Let $\{E_{\widehat{D}}(\lambda)\}_{\lambda\in\mathbb{R}}$ denote the spectral family associated with the operator $\widehat{D}$. Then, for f, $g\in L^2(\mathbb{R})^2$ and $\lambda_1<\lambda_2$,*

$$\langle f, E_{\widehat{D}}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} = \int_{(\lambda_1,\lambda_2]} \Big(\big(T^{\widehat{\mathcal{H}}}(\alpha) U f\big)(\lambda)\Big)^* d\widehat{\Omega}(\lambda,\gamma) \big(T^{\widehat{\mathcal{H}}}(\alpha) U g\big)(\lambda). \tag{3.56}$$

Proof. This follows immediately from the observation that

$$\begin{aligned} &\langle f, E_{\widehat{D}}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} \\ &\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda \, \big\langle f, U^{-1}\big[(\widehat{H}-(\lambda+i\varepsilon))^{-1} - (\widehat{H}-(\lambda-i\varepsilon))^{-1}\big] U g\big\rangle_{L^2(\mathbb{R})^2}. \end{aligned} \tag{3.57}$$

□

3.4. Examples. We now consider the calculation of quantities discussed in the previous section for $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{D}}$ for the special case where $q(x)=q_0\in\mathbb{C}$ is constant. In this case we denote $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{D}}$ by $\widehat{\mathcal{H}}_{q_0}$ and $\widehat{\mathcal{D}}_{q_0}$, etc. But first we consider the case $q_0=0$ and denote $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{D}}$ by $\widehat{\mathcal{H}}_0$ and $\widehat{\mathcal{D}}_0$, etc.

(i) The case $q_0 = 0$:
By direct calculation for general $\alpha = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$, and $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\Psi_{\pm}^{\widehat{\mathcal{H}}_0}(z,x,\alpha) = \begin{cases} a_{\pm} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} e^{\pm izx}, & \operatorname{Im}(z) > 0, \\ a_{\pm} \begin{pmatrix} 1 \\ \mp i \end{pmatrix} e^{\mp izx}, & \operatorname{Im}(z) < 0 \end{cases} \tag{3.58}$$

for some $a_{\pm} \in \mathbb{C}$. As noted earlier,

$$\Psi_{\pm}^{\widehat{\mathcal{D}}_0}(z,x,\beta) = U^{-1} \Psi_{\pm}^{\widehat{\mathcal{H}}_0}(z,x,\alpha) \tag{3.59}$$

for the corresponding general $\beta = \alpha U = [(-1+i)/2](e^{-i\theta}, e^{i\theta})$, $\theta \in [0, 2\pi)$, where U is defined in (2.36). Explicitly,

$$\Psi_{+}^{\widehat{\mathcal{D}}_0}(z,x,\beta) = b_{+} \begin{pmatrix} 0 \\ 1+i \end{pmatrix} e^{izx}, \quad \Psi_{-}^{\widehat{\mathcal{D}}_0}(z,x,\beta) = b_{-} \begin{pmatrix} 1+i \\ 0 \end{pmatrix} e^{-izx}, \quad \operatorname{Im}(z) > 0, \tag{3.60}$$

$$\Psi_{+}^{\widehat{\mathcal{D}}_0}(z,x,\beta) = b_{+} \begin{pmatrix} 1+i \\ 0 \end{pmatrix} e^{-izx}, \quad \Psi_{-}^{\widehat{\mathcal{D}}_0}(z,x,\beta) = b_{-} \begin{pmatrix} 0 \\ 1+i \end{pmatrix} e^{izx}, \quad \operatorname{Im}(z) < 0 \tag{3.61}$$

for some $b_{\pm} \in \mathbb{C}$.

In particular, for $\alpha = \alpha_0 = (1, 0)$ we see that $\mathcal{F}^{\widehat{\mathcal{H}}_0}(z, 0, \alpha_0) = I_2$ and hence by (3.37) that

$$\Psi_{\pm}^{\widehat{\mathcal{H}}_0}(z,0,\alpha_0) = \begin{pmatrix} 1 \\ \widehat{m}_{0,\pm}(z,\gamma_0) \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.62}$$

From this we conclude that $a_{\pm} = 1$ for $\alpha = \alpha_0$ and that

$$\widehat{m}_{0,\pm}(z,\gamma_0) = m_{\pm}^{\widehat{\mathcal{H}}_0}(z,\alpha_0) = m_{\pm}^{\widehat{\mathcal{D}}_0}(z,\beta_0) = \begin{cases} \pm i, & \operatorname{Im}(z) > 0, \\ \mp i, & \operatorname{Im}(z) < 0. \end{cases} \tag{3.63}$$

As a consequence, the whole-line Weyl–Titchmarsh M-function defined in (3.41) is given by

$$\widehat{M}_0(z,\gamma_0) = \pm(i/2) I_2, \quad \operatorname{Im}(z) \gtrless 0. \tag{3.64}$$

Hence, for $q_0 = 0$, the spectral measure for $\widehat{H}_0$, as described in Theorem 3.6, is given by

$$d\widehat{\Omega}_0(\lambda,\gamma_0) = [1/(2\pi)] I_2 \, d\lambda. \tag{3.65}$$

By (3.32), we see that

$$G^{\widehat{H}_0}(z,x,x') = \frac{1}{2} \begin{pmatrix} i & \mp 1 \\ \pm 1 & i \end{pmatrix} e^{\pm iz(x'-x)}, \quad x \lessgtr x', \quad \operatorname{Im}(z) > 0. \tag{3.66}$$

For the corresponding Green's matrix $G^{\widehat{D}_0}(z,x,x') = U^{-1}G^{\widehat{H}_0}(z,x,x')U$ one obtains

$$G^{\widehat{D}_0}(z,x,x') = \begin{cases} i\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{iz(x'-x)}, & x < x', \\ \\ i\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e^{-iz(x'-x)}, & x > x', \end{cases} \quad \operatorname{Im}(z) > 0. \tag{3.67}$$

Similarly,

$$G^{\widehat{H}_0}(z,x,x') = \frac{1}{2}\begin{pmatrix} -i & \mp 1 \\ \pm 1 & -i \end{pmatrix} e^{\mp iz(x'-x)}, \quad x \lessgtr x', \quad \operatorname{Im}(z) < 0, \tag{3.68}$$

and

$$G^{\widehat{D}_0}(z,x,x') = \begin{cases} -i\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e^{-iz(x'-x)}, & x < x', \\ \\ -i\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{iz(x'-x)}, & x > x', \end{cases} \quad \operatorname{Im}(z) < 0. \tag{3.69}$$

The spectra of $\widehat{H}_0$ and $\widehat{D}_0$ are purely absolutely continous of uniform multiplicity two and given by

$$\sigma(\widehat{H}_0) = \sigma(\widehat{D}_0) = \mathbb{R}. \tag{3.70}$$

(ii) The case $q_0 \in \mathbb{C}\backslash\{0\}$:
In considering the case where $q = q_0$ is a nonzero complex constant, we first define $\widehat{S}_{q_0}(z)$ to be a function that is analytic with positive imaginary part on the split plane

$$\widehat{\mathbb{P}}_{q_0} = \mathbb{C}\backslash\{\lambda \in \mathbb{R} \,|\, |\lambda| \geqslant |q_0|\}, \tag{3.71}$$

such that

$$\widehat{S}_{q_0}(z) = \sqrt{z^2 - |q_0|^2}, \quad \operatorname{Im}\big(\widehat{S}_{q_0}(z)\big) > 0, \quad z \in \widehat{\mathbb{P}}_{q_0}. \tag{3.72}$$

Thus,

$$\widehat{S}_{q_0}(\overline{z}) = -\overline{\widehat{S}_{q_0}(z)}, \quad z \in \widehat{\mathbb{P}}_{q_0}, \tag{3.73}$$

$$\widehat{S}_{q_0}(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} \widehat{S}_{q_0}(\lambda \pm i\varepsilon) = \begin{cases} \pm\sqrt{\lambda^2 - |q_0|^2}, & \lambda \geqslant |q_0|, \\ \mp\sqrt{\lambda^2 - |q_0|^2}, & \lambda \leqslant -|q_0|. \end{cases} \tag{3.74}$$

(If $q_0 = 0$, this convention amounts to defining $\sqrt{z^2} = \pm z$ for $\operatorname{Im}(z) \gtrless 0$.)

A direct calculation shows for general $\alpha = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$, that

$$\Psi_\pm^{\widehat{\mathcal{H}}_{q_0}}(z,x,\alpha) = a_\pm \begin{pmatrix} 1 \\ \frac{-\operatorname{Re}(q_0) \pm i\widehat{S}_{q_0}(z)}{z + \operatorname{Im}(q_0)} \end{pmatrix} e^{\pm i\widehat{S}_{q_0}(z)x}, \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{3.75}$$

for some $a_\pm \in \mathbb{C}$. For the corresponding general $\beta = \alpha U = [(-1+i)/2](e^{-i\theta}, e^{i\theta})$, $\theta \in [0, 2\pi)$, we see by direct calculation that

$$\Psi_\pm^{\widehat{\mathcal{D}}_{q_0}}(z,x,\beta) = b_\pm \begin{pmatrix} 1 \\ \frac{i}{q_0}\big[z \pm \widehat{S}_{q_0}(z)\big] \end{pmatrix} e^{\pm i\widehat{S}_{q_0}(z)x}, \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{3.76}$$

for some $b_\pm \in \mathbb{C}$, and alternatively that

$$\begin{aligned}\Psi_\pm^{\widehat{\mathcal{D}}_{q_0}}(z,x,\beta) &= U^{-1}\Psi_\pm^{\widehat{\mathcal{H}}_{q_0}}(z,x,\alpha) \\ &= \frac{a_\pm(1+i)}{2(z+\operatorname{Im}(q_0))}\begin{pmatrix} -z+iq_0 \pm \widehat{S}_{q_0}(z) \\ -z-i\overline{q}_0 \mp \widehat{S}_{q_0}(z)\end{pmatrix} e^{\pm i S_{q_0}(z)x}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned} \tag{3.77}$$

In particular, for $\alpha = \alpha_0 = (1,0)$ we see that $\mathcal{F}^{\widehat{H}_{q_0}}(z,0,\alpha_0) = I_2$ and hence by (3.37) that

$$\Psi_\pm^{\widehat{\mathcal{H}}_{q_0}}(z,0,\alpha_0) = \begin{pmatrix} 1 \\ \widehat{m}_{q_0,\pm}(z,\gamma_0)\end{pmatrix} = a_\pm \begin{pmatrix} 1 \\ \frac{-\operatorname{Re}(q_0) \pm i\widehat{S}_{q_0}(z)}{z+\operatorname{Im}(q_0)}\end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.78}$$

From this we conclude that $a_\pm = 1$ for $\alpha = \alpha_0$ and that

$$\widehat{m}_{q_0,\pm}(z,\gamma_0) = m_\pm^{\widehat{\mathcal{H}}_{q_0}}(z,\alpha_0) = m_\pm^{\widehat{\mathcal{D}}_{q_0}}(z,\beta_0) = \frac{-\operatorname{Re}(q_0) \pm i\widehat{S}_{q_0}(z)}{z+\operatorname{Im}(q_0)}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.79}$$

For $\operatorname{Re}(q_0) \neq 0$, we note that $\widehat{m}_{q_0,\pm}(z,\gamma_0)$ is analytic for $z \in (-|q_0|,|q_0|)$ with the possible exception of $z = -\operatorname{Im}(q_0)$. In this case, the z-wave functions of $\widehat{\mathcal{H}}_{q_0}$ corresponding to $z = -\operatorname{Im}(q_0)$ are given by

$$\Psi^{\widehat{\mathcal{H}}_{q_0}}(x) = \psi_1^{\widehat{\mathcal{H}}_{q_0}}(0)\begin{pmatrix} e^{x\operatorname{Re}(q_0)} \\ 2\left(\frac{\operatorname{Im}(q_0)}{\operatorname{Re}(q_0)}\right)\sinh(x\operatorname{Re}(q_0))\end{pmatrix} + \psi_2^{\widehat{\mathcal{H}}_{q_0}}(0)\begin{pmatrix} 0 \\ e^{-x\operatorname{Re}(q_0)}\end{pmatrix}. \tag{3.80}$$

As a consequence, we see that while $z = -\operatorname{Im}(q_0)$ is not an eigenvalue for $\widehat{H}_{q_0}$, it is an eigenvalue for $\widehat{H}_{q_0,\pm}(\alpha_0)$ corresponding to a simple pole for $\widehat{m}_{q_0,\pm}(z,\gamma_0)$ for $\operatorname{Re}(q_0) \gtrless 0$. We also note that $z = -\operatorname{Im}(q_0)$ corresponds to a removable singularity for $\widehat{m}_{q_0,\mp}(z,\gamma_0)$ for $\operatorname{Re}(q_0) \gtrless 0$.

However, for $\operatorname{Re}(q_0) = 0$, $z = -\operatorname{Im}(q_0)$ corresponds to an endpoint of the spectral gap $(-|q_0|,|q_0|)$ and the z-wave functions of $\widehat{\mathcal{H}}_{q_0}$ are given by

$$\Psi^{\widehat{\mathcal{H}}_{q_0}}(x) = \psi_1^{\widehat{\mathcal{H}}_{q_0}}(0)\begin{pmatrix} 1 \\ 2x\operatorname{Im}(q_0)\end{pmatrix} + \psi_2^{\widehat{\mathcal{H}}_{q_0}}(0)\begin{pmatrix} 0 \\ 1\end{pmatrix}. \tag{3.81}$$

In this case, $\Psi^{\widehat{\mathcal{H}}_{q_0}}(x)$ is neither in $L^2(\mathbb{R})^2$ nor in $L^2([0,\pm\infty))^2$; hence $z = -\operatorname{Im}(q_0)$ is not an eigenvalue for $\widehat{H}_{q_0}$, or for $\widehat{H}_{q_0,\pm}(\alpha_0)$.

As a consequence, the whole-line Weyl–Titchmarsh M-function defined in (3.41) is now given by

$$\widehat{M}_{q_0}(z,\alpha_0) = \frac{i}{2\widehat{S}_{q_0}(z)}\begin{pmatrix} z+\operatorname{Im}(q_0) & -\operatorname{Re}(q_0) \\ -\operatorname{Re}(q_0) & z-\operatorname{Im}(q_0)\end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.82}$$

Hence, the spectral measure for $\widehat{H}_{q_0}$, as described in Theorem 3.6 by $d\widehat{\Omega}(\lambda,\gamma)$ in (3.43), is determined by $\lim_{\varepsilon\downarrow 0}\operatorname{Im}\big(\widehat{M}_{q_0}(\lambda+i\varepsilon,\gamma_0)\big)$ and, in light of (3.72), found to be

$$d\widehat{\Omega}_{q_0}(\lambda,\gamma_0) = d\lambda \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}, & |\lambda| < |q_0| \\ \dfrac{\pm 1}{2\pi\widehat{S}_{q_0}(\lambda+i0)}\begin{pmatrix} \lambda+\operatorname{Im}(q_0) & -\operatorname{Re}(q_0) \\ -\operatorname{Re}(q_0) & \lambda-\operatorname{Im}(q_0)\end{pmatrix}, & \pm\lambda \geqslant |q_0|. \end{cases} \tag{3.83}$$

By (3.32) we see that

(3.84)

$$G^{\widehat{H}_{q_0}}(z,x,x') = \frac{i}{2\widehat{S}_{q_0}(z)} \begin{pmatrix} z+\mathrm{Im}(q_0) & -\mathrm{Re}(q_0) \pm i\widehat{S}_{q_0}(z) \\ -\mathrm{Re}(q_0) \mp i\widehat{S}_{q_0}(z) & z-\mathrm{Im}(q_0) \end{pmatrix} e^{\pm i\widehat{S}_{q_0}(z)(x'-x)}, \quad x \lessgtr x', \quad z\in\mathbb{C}\backslash\mathbb{R}.$$

Using either (3.38) or the fact that $G^{\widehat{D}_{q_0}}(z,x,x') = U^{-1}G^{\widehat{H}_{q_0}}(z,x,x')U$, we obtain

(3.85)

$$G^{\widehat{D}_{q_0}}(z,x,x') = \frac{1}{2\widehat{S}_{q_0}(z)} \begin{pmatrix} i\big[z\pm\widehat{S}_{q_0}(z)\big] & q_0 \\ -\overline{q_0} & i\big[z\mp\widehat{S}_{q_0}(z)\big] \end{pmatrix} e^{\pm i\widehat{S}_{q_0}(z)(x'-x)}, \quad x \lessgtr x', \quad z\in\mathbb{C}\backslash\mathbb{R}.$$

The spectra of $\widehat{H}_{q_0}$ and $\widehat{D}_{q_0}$ are purely absolutely continous of uniform multiplicity two and given by

$$\sigma(\widehat{H}_{q_0}) = \sigma(\widehat{D}_{q_0}) = (-\infty,-|q_0|]\cup[|q_0|,\infty). \tag{3.86}$$

4. Non-self-adjoint Dirac and Hamiltonian Systems

In this section, we focus upon J-self-adjoint realizations for D and its unitarily equivalent H, specifically, the operator $\widetilde{D}$, defined by (2.22) corresponding to the special case of (2.2) given by (2.4), and the operator $\widetilde{H}$ defined by (2.45) associated with the special case of (2.35) given by (2.41). Some spectral theory for the non-self-adjoint operator $\widetilde{D}$, and of necessity for its unitary equivalent $\widetilde{H}$, has been developed in [**4**]. However, it remains incomplete by comparison with their self-adjoint counterparts $\widehat{D}$ and $\widehat{H}$ as described in the previous section.

4.1. Weyl–Titchmarsh coefficients. We now turn to the development in the non-self-adjoint setting of the analog for the Weyl–Titchmarsh coefficient defined and discussed in Section 3.1. This subsection details (and partially corrects) Remark 5.6 in [**4**] which anticipated the introduction of half-line Weyl–Titchmarsh m-functions associated with $\widetilde{\mathfrak{D}}$.

Hypothesis 4.1. *Throughout this section, we assume that the resolvent set $\rho(\widetilde{D})$ of $\widetilde{D}$ (and hence that of $\widetilde{H}$) is nonempty.*

To begin, we note the fundamental result established in [**4**, Theorem 5.4] which states that

$$\dim\big(N^{\widetilde{\mathfrak{D}}}(z,\pm\infty)\big) = 1, \quad z\in\rho(\widetilde{D}), \tag{4.1}$$

and hence by the unitary equivalence given in (2.35),

$$\dim\big(N^{\widetilde{\mathcal{H}}}(z,\pm\infty)\big) = 1, \quad z\in\rho(\widetilde{H}) = \rho(\widetilde{D}). \tag{4.2}$$

In particular, one has the following result.

THEOREM 4.2 ([**4**]). *The operator $\widetilde{D}$, maximally defined in* (2.22)*, is J-self-adjoint since*

$$\mathcal{J}\widetilde{D}\mathcal{J} = \widetilde{D}^*, \tag{4.3}$$

where $\mathcal{J}$ is defined in (1.5)*, and the operator $\widetilde{H}$, maximally defined in* (2.45)*, is J-self-adjoint since*

$$\widetilde{\mathcal{J}}\widetilde{H}\widetilde{\mathcal{J}} = \widetilde{H}^*, \tag{4.4}$$

where $\widetilde{\mathcal{J}}$ denotes the conjugate linear involution

$$\widetilde{\mathcal{J}} = i\mathcal{C}I_2. \tag{4.5}$$

Moreover, $\widetilde{H}$ and $\widetilde{D}$ are unitarily equivalent, i.e.

$$\widetilde{H} = U\widetilde{D}U^{-1}. \tag{4.6}$$

PROOF. Equation (4.4) has been proven in [**4**]. The rest follows from the unitary equivalence (2.35) via the constant unitary matrix U. □

As in the self-adjoint setting, one defines the half-line operator $\widetilde{D}_{\pm}(\beta)$ in association with the differential expression $\widetilde{\mathcal{D}}$ found in (2.4) by

$$\begin{aligned} &\widetilde{D}_{\pm}(\beta) = i\sigma_3\frac{d}{dx} + \widetilde{Q}, \\ &\operatorname{dom}(\widetilde{D}_{\pm}(\beta)) = \big\{\Psi \in L^2([0,\pm\infty))^2 \,\big|\, \Psi \in \mathrm{AC}_{\mathrm{loc}}([0,\pm\infty))^2, \\ &\qquad\qquad\qquad \beta\Psi(0) = 0,\ \widetilde{\mathcal{D}}\Psi \in L^2([0,\pm\infty))^2\big\}, \end{aligned} \tag{4.7}$$

where

$$\beta = [(-1+i)/2](e^{-i\theta}, e^{i\theta}), \quad \theta \in [0, 2\pi), \tag{4.8}$$

and where $\beta\Psi(0) = 0$ represents a J-self-adjoint boundary condition for $\widetilde{D}_{\pm}(\beta)$ using the conjugation $\mathcal{J}$. One also defines the half-line operator $\widetilde{H}_{\pm}(\alpha)$ in association with the differential expression $\widetilde{\mathcal{H}}$ found in (2.41) by

$$\begin{aligned} &\widetilde{H}_{\pm}(\alpha) = -\sigma_4\frac{d}{dx} + \widetilde{B}, \\ &\operatorname{dom}(\widetilde{H}_{\pm}(\alpha)) = \big\{\Psi \in L^2([0,\pm\infty))^2 \,\big|\, \Psi \in \mathrm{AC}_{\mathrm{loc}}([0,\pm\infty))^2, \\ &\qquad\qquad\qquad \alpha\Psi(0) = 0,\ \widetilde{\mathcal{H}}\Psi \in L^2([0,\pm\infty))^2\big\}, \end{aligned} \tag{4.9}$$

where $\alpha = \beta U^{-1} = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$, and where $\alpha\Psi(0) = 0$ represents a J-self-adjoint boundary condition for $\widetilde{H}_{\pm}(\alpha)$ using the conjugation $\widetilde{\mathcal{J}}$. In fact, $\widetilde{D}_{\pm}(\beta)$ and $\widetilde{H}_{\pm}(\alpha)$ are also J-self-adjoint,

$$\mathcal{J}\widetilde{D}_{\pm}(\beta)\mathcal{J} = \widetilde{D}_{\pm}(\beta)^*, \quad \widetilde{\mathcal{J}}\widetilde{H}_{\pm}(\alpha)\widetilde{\mathcal{J}} = \widetilde{H}_{\pm}(\alpha)^*. \tag{4.10}$$

To prove (4.10) one first notes that apart from the J-self-adjoint boundary condition imposed at $x = 0$, $\widetilde{D}_{\pm}(\beta)$ and $\widetilde{H}_{\pm}(\alpha)$ are maximally defined and one only needs to check the corresponding $L^2([0,\pm\infty))^2$ condition in a neighborhood of $\pm\infty$. But the latter immediately follows from (4.3) and (4.4). As in the self-adjoint context (cf. (3.7)) one infers that

$$\widetilde{H}_{\pm}(\alpha) = U\widetilde{D}_{\pm}(\beta)U^{-1}, \quad \beta = \alpha U, \tag{4.11}$$

holds in addition to (4.6).

REMARK 4.3. Also exploited in [**4**, Lemma 5.2, Theorem 5.4] is a feature that distinguishes $\widetilde{D}$ from $\widehat{D}$: The bijection $\mathcal{K}$ acting upon $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^2$ and described by

$$\mathcal{K} = \sigma_4 \mathcal{C}, \quad \mathcal{K}^2 = -I_2, \tag{4.12}$$

where $\mathcal{C}$ is the conjugation operator acting on $\mathbb{C}^2$ defined in (1.6), maps z-wave functions of $\widetilde{\mathfrak{D}}$ to $\bar{z}$-wave functions of $\widetilde{\mathfrak{D}}$. By this, we mean that

$$(\widetilde{\mathfrak{D}}\Psi)(z,x) = z\Psi(z,x) \text{ if and only if } (\widetilde{\mathfrak{D}}\mathcal{K}\Psi)(z,x) = \bar{z}\mathcal{K}\Psi(z,x), \quad z \in \mathbb{C},\ x \in \mathbb{R}. \tag{4.13}$$

By contrast, $\mathcal{K}$ fails to map z-wave functions to $\bar{z}$-wave functions of $\widehat{\mathfrak{D}}$. Distinguishing $\widehat{\mathfrak{D}}$ from $\widetilde{\mathfrak{D}}$ is the fact that rather than $\mathcal{K}$, it is the operator $\mathcal{J}$, defined in (1.5) and acting as a bijection on $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^2$, that serves to map z-wave functions to $\bar{z}$-wave functions of $\widehat{\mathfrak{D}}$:

$$(\widehat{\mathfrak{D}}\Psi)(z,x) = z\Psi(z,x) \text{ if and only if } (\widehat{\mathfrak{D}}\mathcal{J}\Psi)(z,x) = \bar{z}\mathcal{J}\Psi(z,x), \quad z \in \mathbb{C},\ x \in \mathbb{R}. \tag{4.14}$$

As before, we introduce the fundamental system of solutions of $\widetilde{\mathfrak{H}}\Psi = z\Psi$ by

$$\Theta^{\widetilde{\mathfrak{H}}}(z,\cdot,\alpha),\ \Phi^{\widetilde{\mathfrak{H}}}(z,\cdot,\alpha) \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^2, \quad z \in \mathbb{C}, \tag{4.15}$$

and the matrix-valued function $\mathcal{F}^{\widetilde{\mathfrak{H}}}(z,\cdot,\alpha)$ given by

$$\mathcal{F}^{\widetilde{\mathfrak{H}}}(z,x,\alpha) = \big(\Theta^{\widetilde{\mathfrak{H}}}(z,x,\alpha) \ \ \Phi^{\widetilde{\mathfrak{H}}}(z,x,\alpha)\big), \tag{4.16}$$

where for $\theta \in [0,2\pi)$,

$$\Theta^{\widetilde{\mathfrak{H}}}(z,0,\alpha) = (\cos(\theta),\sin(\theta))^\top, \quad \Phi^{\widetilde{\mathfrak{H}}}(z,0,\alpha) = (-\sin(\theta),\cos(\theta))^\top. \tag{4.17}$$

We also introduce the related fundamental system of z-wave functions of $\widetilde{\mathfrak{D}}$ given by $\Theta^{\widetilde{\mathfrak{D}}}(z,\cdot,\beta)$ and $\Phi^{\widetilde{\mathfrak{D}}}(z,\cdot,\beta)$, as well as the matrix-valued function

$$\mathcal{F}^{\widetilde{\mathfrak{D}}}(z,x,\beta) = \big(\Theta^{\widetilde{\mathfrak{D}}}(z,x,\beta) \ \ \Phi^{\widetilde{\mathfrak{D}}}(z,x,\beta)\big) = U^{-1}\mathcal{F}^{\widetilde{\mathfrak{H}}}(z,x,\alpha), \tag{4.18}$$

where $\beta = \alpha U$. Thus, for $\theta \in [0,2\pi)$,

$$\Theta^{\widehat{\mathfrak{D}}}(z,0,\beta) = [(1+i)/2](e^{i\theta}, e^{-i\theta})^\top, \quad \Phi^{\widehat{\mathfrak{D}}}(z,0,\beta) = [(1-i)/2](e^{i\theta}, -e^{-i\theta})^\top. \tag{4.19}$$

Analogous to the self-adjoint setting, a Weyl-Titchmarsh coefficient can be defined for values of $z \in \mathbb{C}$ that lie in the compliment of the combined spectrum for $\widetilde{D}$ and $\widetilde{D}_\pm(\beta)$. The fact that $\Psi_\pm^{\widetilde{\mathfrak{D}}}(z,\cdot,\beta)$ form a basis for the z-wave functions of $\widetilde{\mathfrak{D}}$ implies that $\Psi_\pm^{\widetilde{\mathfrak{D}}}(z,\cdot,\beta)$ are not scalar multiples of $\Phi^{\widetilde{\mathfrak{D}}}(z,\cdot,\beta)$ for $z \in \rho(\widetilde{D}) \cap \rho(\widetilde{D}_\pm(\beta))$ and hence similarly that $\Psi_\pm^{\widetilde{\mathfrak{H}}}(z,\cdot,\alpha)$ are not scalar multiples of $\Phi^{\widetilde{\mathfrak{H}}}(z,\cdot,\alpha)$ for $z \in \ \rho(\widetilde{D}) \cap \rho(\widetilde{D}_\pm(\beta)) = \rho(\widetilde{H}) \cap \rho(\widetilde{H}_\pm(\alpha))$. This being the case, $\Psi_\pm^{\widetilde{\mathfrak{H}}}$ and $\Psi_\pm^{\widetilde{\mathfrak{D}}}$ have the unique representations given by

$$\Psi_\pm^{\widetilde{\mathfrak{H}}}(z,\cdot,\alpha) = \Theta^{\widetilde{\mathfrak{H}}}(z,\cdot,\alpha) + m_\pm^{\widetilde{\mathfrak{H}}}(z,\alpha)\Phi^{\widetilde{\mathfrak{H}}}(z,\cdot,\alpha) \in L^2([0,\pm\infty))^2, \tag{4.20}$$

$$\Psi_\pm^{\widetilde{\mathfrak{D}}}(z,\cdot,\beta) = \Theta^{\widetilde{\mathfrak{D}}}(z,\cdot,\beta) + m_\pm^{\widetilde{\mathfrak{D}}}(z,\beta)\Phi^{\widetilde{\mathfrak{D}}}(z,\cdot,\beta) \in L^2([0,\pm\infty))^2. \tag{4.21}$$

In complete analogy to Lemma 3.2, and by completely analogous proof, one obtains the following result:

Lemma 4.4. *Let* $\alpha = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$, *and let* $\beta = \alpha U$ *with* U *defined in* (2.36). *Let* $\Theta^{\widetilde{\mathcal{H}}}$, $\Phi^{\widetilde{\mathcal{H}}}$ *represent the fundamental system of solutions of the Hamiltonian system* $\widetilde{\mathcal{H}}\Psi = z\Psi$ *satisfying* (4.17), *and let* $\Theta^{\widetilde{\mathcal{D}}}$, $\Phi^{\widetilde{\mathcal{D}}}$ *represent the fundamental system of solutions of the Dirac system* $\widetilde{\mathcal{D}}\Psi = z\Psi$ *satisfying* (4.19). *Then, for* $z \in \rho(\widetilde{D}) \cap \rho(\widetilde{D}_\pm(\beta)) = \rho(\widetilde{H}) \cap \rho(\widetilde{H}_\pm(\alpha))$, *with* $\Psi_\pm^{\widetilde{\mathcal{H}}}$ *defined in* (4.20) *and with* $\Psi_\pm^{\widetilde{\mathcal{D}}}$ *defined in* (4.21), *one infers that*

$$\Psi_\pm^{\widetilde{\mathcal{D}}}(z, x, \beta) = U^{-1}\Psi_\pm^{\widetilde{\mathcal{H}}}(z, x, \alpha), \quad x \in \mathbb{R}, \tag{4.22}$$

and in particular, that

$$m_\pm^{\widetilde{\mathcal{D}}}(z, \beta) = m_\pm^{\widetilde{\mathcal{H}}}(z, \alpha). \tag{4.23}$$

Remark 4.5. As in Remark 3.3, we denote $\widetilde{m}_\pm(z, \gamma) = m_\pm^{\widetilde{\mathcal{D}}}(z, \beta) = m_\pm^{\widetilde{\mathcal{H}}}(z, \alpha)$, where γ represents α in the context of $\widetilde{\mathcal{H}}$ and $\beta = \alpha U$ in the context of $\widetilde{\mathcal{D}}$.

It is well-known that $\overline{\widehat{m}_\pm(z, \gamma)} = \widehat{m}_\pm(\overline{z}, \gamma)$, $z \in \mathbb{C}\backslash\mathbb{R}$, for the self-adjoint differential expression $\widehat{\mathcal{D}}$. By contrast, this is not the case in the non-self-adjoint setting for $\widetilde{\mathcal{D}}$ as seen in the next result.

Lemma 4.6. *Let* $\alpha = \alpha(\theta) = (\cos(\theta), \sin(\theta))$, $\beta = \beta(\theta) = [(-1+i)/2](e^{-i\theta}, e^{i\theta})$, *and* $\gamma(\theta)$ *defined as in Remark 4.5,* $\theta \in [0, 2\pi)$, *and* $z \in \rho(\widetilde{D}) \cap \rho(\widetilde{D}_\pm(\beta))$. *Then,*

$$\overline{\widetilde{m}_\pm(z, \gamma(\theta))} = -[\widetilde{m}_\pm(\overline{z}, \gamma(\theta))]^{-1}, \tag{4.24}$$

$$\overline{m_\pm^{\widetilde{\mathcal{D}}}(z, \beta(\theta))} = m_\pm^{\widetilde{\mathcal{D}}}(\overline{z}, \beta(\theta - \pi/2)), \tag{4.25}$$

$$\overline{m_\pm^{\widetilde{\mathcal{H}}}(z, \alpha(\theta))} = m_\pm^{\widetilde{\mathcal{H}}}(\overline{z}, \alpha(\theta - \pi/2)). \tag{4.26}$$

Proof. As defined in (4.12), $\mathcal{K}$ is an isometry on $L^2([0, \pm\infty))^2$ which by (4.13) maps z-wave functions to $\overline{z}$-wave functions of $\widetilde{\mathcal{D}}$. As a consequence of (4.1), $\mathcal{K}\Psi_\pm^{\widetilde{\mathcal{D}}}(z, \cdot, \beta) = c_\pm \Psi_\pm^{\widetilde{\mathcal{D}}}(\overline{z}, \cdot, \beta)$ for some $c_\pm \in \mathbb{C}$. Moreover, for the fundamental system defined in (4.19), $\mathcal{K}\Theta^{\widetilde{\mathcal{D}}}(z, 0, \beta) = -\Phi^{\widetilde{\mathcal{D}}}(\overline{z}, 0, \beta)$, $\mathcal{K}\Phi^{\widetilde{\mathcal{D}}}(z, 0, \beta) = \Theta^{\widetilde{\mathcal{D}}}(\overline{z}, 0, \beta)$, and hence

$$\mathcal{K}\Theta^{\widetilde{\mathcal{D}}}(z, x, \beta) = -\Phi^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta), \quad \mathcal{K}\Phi^{\widetilde{\mathcal{D}}}(z, x, \beta) = \Theta^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta), \quad x \in \mathbb{R}. \tag{4.27}$$

Given the unique representations provided by (4.21), we obtain for $\theta_1, \theta_2 \in [0, 2\pi)$, that

$$\begin{aligned} &c_\pm(\Theta^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta(\theta_1)) + m_\pm^{\widetilde{\mathcal{D}}}(\overline{z}, \beta(\theta_1))\Phi^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta(\theta_1))) \\ &\quad = \mathcal{K}(\Theta^{\widetilde{\mathcal{D}}}(z, x, \beta(\theta_2)) + m_\pm^{\widetilde{\mathcal{D}}}(z, \beta(\theta_2))\Phi^{\widetilde{\mathcal{D}}}(z, x, \beta(\theta_2))) \\ &\quad = -\Phi^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta(\theta_2)) + \overline{m_\pm^{\widetilde{\mathcal{D}}}(z, \beta(\theta_2))}\Theta^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta(\theta_2)). \end{aligned} \tag{4.28}$$

From this we see that

$$c_\pm \begin{pmatrix} 1 \\ m_\pm^{\widetilde{\mathcal{D}}}(\overline{z}, \beta(\theta_1)) \end{pmatrix} = \mathcal{F}^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta(\theta_1))^{-1}\mathcal{F}^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta(\theta_2)) \begin{pmatrix} \overline{m_\pm^{\widetilde{\mathcal{D}}}(z, \beta(\theta_2))} \\ -1 \end{pmatrix}. \tag{4.29}$$

Then, (4.24) follows from $\mathcal{F}^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta(\theta))^{-1}\mathcal{F}^{\widetilde{\mathcal{D}}}(\overline{z}, x, \beta(\theta)) = I_2$; (4.25) follows from $\mathcal{F}^{\widetilde{\mathcal{D}}}(\overline{z}, 0, \beta(\theta - \pi/2))^{-1}\mathcal{F}^{\widetilde{\mathcal{D}}}(\overline{z}, 0, \beta(\theta)) = J$; and (4.26) follows from (4.23). $\square$

REMARK 4.7. The same argument used to prove (4.24) can be used in the self-adjoint context to prove that

$$\overline{\widehat{m}_\pm(z,\gamma)} = \widehat{m}_\pm(\overline{z},\gamma), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{4.30}$$

where the operator $\mathcal{J}$ which maps z-wave functions to $\overline{z}$-wave functions of $\widehat{\mathfrak{D}}$ is used rather than $\mathcal{K}$.

4.2. Green's matrices. Given the existence of the half-line Weyl–Titchmarsh solutions $\Psi^{\widetilde{\mathfrak{D}}}_\pm(z,\cdot,\beta) \in L^2([0,\pm\infty))^2$ for $\widetilde{\mathfrak{D}}\Psi = z\Psi$, and the corresponding solutions $\Psi^{\widetilde{\mathcal{H}}}_\pm(z,\cdot,\alpha) = U\Psi^{\widetilde{\mathfrak{D}}}_\pm(z,\cdot,\beta) \in L^2([0,\pm\infty))^2$ of $\widetilde{\mathcal{H}}\Psi = z\Psi$ for $z \in \rho(\widetilde{H}) = \rho(\widetilde{D})$, we now obtain, as special cases of Lemmas 2.5 and 2.7, descriptions for the Green's matrices corresponding to the whole-line operators $\widetilde{D}$ and $\widetilde{H}$:

LEMMA 4.8. *With $z \in \rho(\widetilde{H})$, let $\Psi^{\widetilde{\mathcal{H}}}_\pm(z,\cdot,\alpha) \in L^2([0,\pm\infty))^2$ represent a basis of solutions for the Hamiltonian system $\widetilde{\mathcal{H}}\Psi = z\Psi$. Then, the whole-line Green's matrix associated with the operator $\widetilde{H}$ is given by*

$$G^{\widetilde{H}}(z,x,x') = K(z,\alpha)\Psi^{\widetilde{\mathcal{H}}}_\mp(z,x,\alpha)\Psi^{\widetilde{\mathcal{H}}}_\pm(z,x',\alpha)^\top, \quad x \lessgtr x', \tag{4.31}$$

where

$$K(z,\alpha) = \big[W(\Psi^{\widetilde{\mathcal{H}}}_+(z,x,\alpha),\Psi^{\widetilde{\mathcal{H}}}_-(z,x,\alpha))\big]^{-1} = \big[W(\Psi^{\widetilde{\mathfrak{D}}}_+(z,x,\beta),\Psi^{\widetilde{\mathfrak{D}}}_-(z,x,\beta))\big]^{-1} \tag{4.32}$$

$$= [\widetilde{m}_-(z,\gamma) - \widetilde{m}_+(z,\gamma)]^{-1} \tag{4.33}$$

is constant with respect to $x \in \mathbb{R}$.

LEMMA 4.9. *With $z \in \rho(\widetilde{D})$, let $\Psi^{\widetilde{\mathfrak{D}}}_\pm(z,\cdot,\beta) \in L^2([0,\pm\infty))^2$ represent a basis of z-wave functions of $\widetilde{\mathfrak{D}}$. Then, the whole-line Green's matrix associated with the operator $\widetilde{D}$ is given by*

$$G^{\widetilde{D}}(z,x,x') = C(z,\beta)\Psi^{\widetilde{\mathfrak{D}}}_\mp(z,x,\beta)\Psi^{\widetilde{\mathfrak{D}}}_\pm(z,x',\beta)^\top\sigma_1, \quad x \lessgtr x' \tag{4.34}$$

where σ_1 is given in (1.5) *and where*

$$C(z,\beta) = -i\big[W(\Psi^{\widetilde{\mathfrak{D}}}_+(z,x,\beta),\Psi^{\widetilde{\mathfrak{D}}}_-(z,x,\beta))\big]^{-1} \tag{4.35}$$

is constant with respect to $x \in \mathbb{R}$.

With $\widetilde{m}_\pm(\cdot,\gamma)$ defined in the non-self-adjoint settings of $\widetilde{\mathfrak{D}}$ and the unitarily equivalent $\widetilde{\mathcal{H}}$, we define $\widetilde{\Gamma}(\cdot,\gamma)$ by substituting $\widetilde{m}_\pm(\cdot,\gamma)$ for its corresponding $\widehat{m}_\pm(\cdot,\gamma)$ in the definition of $\widehat{\Gamma}(\cdot,\gamma)$ given in (3.31). That is,

$$\widetilde{\Gamma}(z,\gamma) = \begin{pmatrix} \frac{1}{\widetilde{m}_-(z,\gamma)-\widetilde{m}_+(z,\gamma)} & \frac{\widetilde{m}_-(z,\gamma)}{\widetilde{m}_-(z,\gamma)-\widetilde{m}_+(z,\gamma)} \\ \frac{\widetilde{m}_+(z,\gamma)}{\widetilde{m}_-(z,\gamma)-\widetilde{m}_+(z,\gamma)} & \frac{\widetilde{m}_-(z,\gamma)\widetilde{m}_+(z,\gamma)}{\widetilde{m}_-(z,\gamma)-\widetilde{m}_+(z,\gamma)} \end{pmatrix}, \tag{4.36}$$
$$z \in \rho(\widetilde{D}) \cap \rho(\widetilde{D}_\pm(\beta)) = \rho(\widetilde{H}) \cap \rho(\widetilde{H}_\pm(\alpha)).$$

With this definition one obtains alternative expressions for the Green's matrices $G^{\widetilde{D}}(z,x,x')$ and $G^{\widetilde{H}}(z,x,x')$ given in Lemmas 4.9 and 4.8 which are analogous to those given for $G^{\widehat{D}}(z,x,x')$ and $G^{\widehat{H}}(z,x,x')$ in Lemmas 3.4 and 3.5, that is, for $z \in \rho(\widetilde{D}) \cap \rho(\widetilde{D}_\pm(\beta)) = \rho(\widetilde{H}) \cap \rho(\widetilde{H}_\pm(\alpha))$,

$$G^{\widetilde{H}}(z,x,x') = \begin{cases} \mathcal{F}^{\widetilde{\mathcal{H}}}(z,x,\alpha)\widetilde{\Gamma}(z,\gamma)^\top \mathcal{F}^{\widetilde{\mathcal{H}}}(z,x',\alpha)^\top, & x < x', \\ \mathcal{F}^{\widetilde{\mathcal{H}}}(z,x,\alpha)\widetilde{\Gamma}(z,\gamma)\mathcal{F}^{\widetilde{\mathcal{H}}}(z,x',\alpha)^\top, & x > x', \end{cases} \tag{4.37}$$

$$G^{\widetilde{D}}(z,x,x') = \begin{cases} -i\mathcal{F}^{\widetilde{\mathcal{D}}}(z,x,\beta)\widetilde{\Gamma}(z,\gamma)^\top \mathcal{F}^{\widetilde{\mathcal{D}}}(z,x',\beta)^\top, & x < x', \\ -i\mathcal{F}^{\widetilde{\mathcal{D}}}(z,x,\beta)\widetilde{\Gamma}(z,\gamma)\mathcal{F}^{\widetilde{\mathcal{D}}}(z,x',\beta)^\top, & x > x'. \end{cases} \tag{4.38}$$

LEMMA 4.10. *Let $\alpha = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$, and let $\beta = \alpha U$ with U defined in* (2.36). *Then,*

$$\widetilde{m}_\pm(\cdot,\gamma) \ \text{ are analytic on } \ \rho(\widetilde{H}_\pm(\alpha)) = \rho(\widetilde{D}_\pm(\beta)). \tag{4.39}$$

PROOF. In analogy to (4.34), the half-line Green's matrix of $\widetilde{H}_+(\alpha)$ is of the form

$$G^{\widetilde{H}_+(\alpha)}(z,x,x') = \Phi^{\widetilde{\mathcal{H}}}(z,x,\alpha)\Psi_+^{\widetilde{\mathcal{H}}}(z,x',\alpha)^\top, \quad 0 \le x < x', \ z \in \rho(\widetilde{H}_+(\alpha)). \tag{4.40}$$

Writing

$$\Phi^{\widetilde{\mathcal{H}}} = (\phi_1^{\widetilde{\mathcal{H}}} \ \phi_2^{\widetilde{\mathcal{H}}})^\top, \quad \Theta^{\widetilde{\mathcal{H}}} = (\theta_1^{\widetilde{\mathcal{H}}} \ \theta_2^{\widetilde{\mathcal{H}}})^\top, \quad \Psi_+^{\widetilde{\mathcal{H}}} = (\psi_{+,1}^{\widetilde{\mathcal{H}}} \ \psi_{+,2}^{\widetilde{\mathcal{H}}})^\top, \tag{4.41}$$

we next pick $z_0 \in \rho(\widetilde{H}_+(\alpha))$ and choose $f, g \in C_0^\infty((0,\infty))$ such that

$$\operatorname{supp}(f) \subseteq [a,b], \quad \operatorname{supp}(g) \subseteq [c,d], \quad 0 < a < b < c < d, \tag{4.42}$$

and

$$\int_a^b dx\, \overline{f(x)}\phi_1^{\widetilde{\mathcal{H}}}(z_0,x,\alpha) \neq 0, \quad \int_c^d dx'\, \phi_1^{\widetilde{\mathcal{H}}}(z_0,x',\alpha)g(x') \neq 0. \tag{4.43}$$

Since for fixed $x \in \mathbb{R}$, $\phi_1^{\widetilde{\mathcal{H}}}(z,x,\alpha)$ and $\theta_1^{\widetilde{\mathcal{H}}}(z,x,\alpha)$ are entire with respect to z and for fixed $z \in \mathbb{C}$ locally absolutely continuous in $x \in \mathbb{R}$, continuity with respect to z then yields

$$\int_a^b dx\, \overline{f(x)}\phi_1^{\widetilde{\mathcal{H}}}(z,x,\alpha) \neq 0, \quad \int_c^d dx'\, \phi_1^{\widetilde{\mathcal{H}}}(z,x',\alpha)g(x') \neq 0, \quad z \in \mathcal{U}(z_0), \tag{4.44}$$

where $\mathcal{U}(z_0) \subset \rho(\widetilde{H}_+(\alpha))$ is a sufficiently small open neighborhood of z_0.

Next, one computes for $z \in \mathcal{U}(z_0)$,

$$\begin{aligned} &\left\langle \begin{pmatrix} f \\ 0 \end{pmatrix}, (\widetilde{H}_+(\alpha) - z)^{-1} \begin{pmatrix} g \\ 0 \end{pmatrix} \right\rangle_{L^2(\mathbb{R})^2} \\ &\quad = \int_a^b dx\, \overline{f(x)}\phi_1^{\widetilde{\mathcal{H}}}(z,x,\alpha) \int_c^d dx' \left[\theta_1^{\widetilde{\mathcal{H}}}(z,x',\alpha) + \widetilde{m}_+(z,\gamma)\phi_1^{\widetilde{\mathcal{H}}}(z,x',\alpha)\right] g(x'). \end{aligned} \tag{4.45}$$

Since the left-hand side of (4.45) is analytic on $\mathcal{U}(z_0)$, one concludes that $\widetilde{m}_+(\cdot,\gamma)$ is analytic on $\mathcal{U}(z_0)$. Since $z_0 \in \rho(\widetilde{H}_+(\alpha))$ was arbitrary, $\widetilde{m}_+(\cdot,\gamma)$ (and hence $G^{\widetilde{H}_+(\alpha)}(\cdot,x,x')$, $0 \le x < x'$) is analytic on $\rho(\widetilde{H}_+(\alpha))$. The analogous proof applies to $\widetilde{m}_-(\cdot,\gamma)$ (and $G^{\widetilde{H}_-(\alpha)}(\cdot,x,x')$, $0 \le x' < x$). $\square$

The argument in the proof of Lemma 4.10 is a simple variant of the proof of Lemma 9.1 in [**45**] in the context of self-adjoint higher-order matrix-valued differential operators adapted to the present case of non-self-adjoint Dirac-type operators.

This strategy of proof markedly differs from the usual approach in the self-adjoint case which is based on uniform convergence of sequences of Weyl–Titchmarsh functions lying in nesting Weyl circles. The latter approach generally fails in the non-self-adjoint context.

A proof analogous to that of Lemma 4.10 also applies to the full-line operators $\widetilde{H}$ and $\widetilde{D}$ and we turn to that next.

Lemma 4.11. *Let $\alpha = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$, and let $\beta = \alpha U$ with U defined in* (2.36)*. Then,*

$$\widetilde{\Gamma}(\cdot, \gamma) \text{ is analytic on } \rho(\widetilde{H}) = \rho(\widetilde{D}). \tag{4.46}$$

Thus, for $x, x' \in \mathbb{R}$, $x \neq x'$, the Green's matrices $G^{\widetilde{H}}(z,x,x')$ and $G^{\widetilde{D}}(z,x,x')$ are analytic on $\rho(\widetilde{H}) = \rho(\widetilde{D})$ and hence (4.36)*–*(4.38) *extend to $\rho(\widetilde{H}) = \rho(\widetilde{D})$.*

Proof. For simplicity we only consider $G^{\widetilde{H}}(z,x,x')$ for $x < x'$. The case $x > x'$ and the corresponding results for $G^{\widetilde{D}}(z,x,x')$ follow in an analogous manner.

Recalling our notation in (4.41) and suppressing α, β, and γ for simplicity, we start by noting that (4.37) yields for the $(1,1)$-element of $G^{\widetilde{H}}(z,x,x')$,

$$\begin{aligned} G^{\widetilde{H}}_{1,1}(z,x,x') &= \frac{1}{\widetilde{m}_-(z) - \widetilde{m}_+(z)} \big[\theta_1^{\widetilde{\mathcal{H}}}(z,x)\theta_1^{\widetilde{\mathcal{H}}}(z,x') + \widetilde{m}_+(z)\theta_1^{\widetilde{\mathcal{H}}}(z,x)\phi_1^{\widetilde{\mathcal{H}}}(z,x') \\ &\qquad + \widetilde{m}_-(z)\phi_1^{\widetilde{\mathcal{H}}}(z,x)\theta_1^{\widetilde{\mathcal{H}}}(z,x') + \widetilde{m}_-(z)\widetilde{m}_+(z)\phi_1^{\widetilde{\mathcal{H}}}(z,x)\phi_1^{\widetilde{\mathcal{H}}}(z,x')\big] \\ &= \begin{pmatrix} \theta_1^{\widetilde{\mathcal{H}}}(z,x) \\ \phi_1^{\widetilde{\mathcal{H}}}(z,x) \end{pmatrix}^{\top} \widetilde{\Gamma}(z) \begin{pmatrix} \theta_1^{\widetilde{\mathcal{H}}}(z,x') \\ \phi_1^{\widetilde{\mathcal{H}}}(z,x') \end{pmatrix} \\ &= \sum_{j,k=0}^{1} \psi_j(x)\widetilde{\Gamma}_{j,k}(z)\psi_k(z), \quad x < x', \end{aligned} \tag{4.47}$$

where

$$\psi_j(z,x) = \begin{cases} \theta_1^{\widetilde{\mathcal{H}}}(z,x), & j = 0, \\ \phi_1^{\widetilde{\mathcal{H}}}(z,x), & j = 1, \end{cases} \quad (z,x) \in \mathbb{C} \times \mathbb{R}. \tag{4.48}$$

Next we choose $f_\ell, g_\ell \in C_0^\infty(\mathbb{R})$, $\ell = 0,1$, such that

$$\operatorname{supp}(f_\ell) \subseteq [a,b], \quad \operatorname{supp}(g_\ell) \subseteq [c,d], \quad a < b < c < d, \ \ell = 0,1, \tag{4.49}$$

and introduce the 2×2 matrices

$$A(z) = \left(A_{\ell,m}(z) = \left\langle \begin{pmatrix} f_\ell \\ 0 \end{pmatrix}, (\widetilde{H} - z)^{-1} \begin{pmatrix} g_m \\ 0 \end{pmatrix} \right\rangle_{L^2(\mathbb{R})^2} \right)_{\ell,m=0,1}, \quad z \in \rho(\widetilde{H}), \tag{4.50}$$

$$B(z) = \big(B_{\ell,j}(z) = \langle f_\ell, \psi_j(z)\rangle_{L^2(\mathbb{R})}\big)_{\ell,j=0,1}, \quad z \in \mathbb{C}, \tag{4.51}$$

$$C(z) = \big(C_{k,m}(z) = \langle \psi_k(z), g_m\rangle_{L^2(\mathbb{R})}\big)_{k,m=0,1}, \quad z \in \mathbb{C}, \tag{4.52}$$

where, in obvious notation, $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R})}$ denotes the scalar product in $L^2(\mathbb{R})$ (linear in the second place). In addition, we let $z_0 \in \rho(\widetilde{H})$ and suppose that f_ℓ and g_ℓ, $\ell = 0,1$, are chosen such that

$$\det(B(z_0)) \neq 0, \quad \det(C(z_0)) \neq 0. \tag{4.53}$$

Since for fixed $x \in \mathbb{R}$, $\psi_j(z,x)$, $j = 0,1$, are entire with respect to z, and for fixed $z \in \mathbb{C}$, locally absolutely continuous in $x \in \mathbb{R}$, one infers by continuity with respect to z that

$$\det(B(z)) \neq 0, \quad \det(C(z)) \neq 0, \quad z \in \mathcal{U}(z_0), \tag{4.54}$$

where $\mathcal{U}(z_0) \subset \rho(\widetilde{H})$ is a sufficiently small open neighborhood of z_0. Thus, combining (4.47) and (4.50)–(4.52) one computes

$$A(z) = B(z)\widetilde{\Gamma}(z)C(z), \quad z \in \mathcal{U}(z_0). \tag{4.55}$$

Since A is analytic on $\mathcal{U}(z_0)$, B and C are entire and invertible on $\mathcal{U}(z_0)$, one concludes that $\widetilde{\Gamma}$ is analytic on $\mathcal{U}(z_0)$. Since $z_0 \in \rho(\widetilde{H})$ was arbitrary, this proves analyticity of $\widetilde{\Gamma}$ on $\rho(\widetilde{H})$.

Since for fixed $x \in \mathbb{R}$, $\mathcal{F}^{\widetilde{\mathcal{H}}}(\cdot,x,\alpha)$ and $\mathcal{F}^{\widetilde{\mathcal{D}}}(\cdot,x,\alpha)$ are entire, the claims for $G^{\widetilde{H}}(\cdot,x,x')$ and $G^{\widetilde{D}}(\cdot,x,x')$ are immediate from (4.37), (4.38), and (4.46). $\square$

4.3. General spectral properties. In this subsection we recall some of the spectral properties of $\widetilde{D}$ recorded in [**4**].

In the following, $\sigma_{\rm a}(\widetilde{D}), \sigma_{\rm p}(\widetilde{D}), \sigma_{\rm c}(\widetilde{D}), \sigma_{\rm e}(\widetilde{D})$, and $\sigma_{\rm r}(\widetilde{D})$, denote the approximate point, point, continuous, essential, and residual spectra of $\widetilde{D}$, respectively, while $\pi(\widetilde{D})$ denotes the regularity domain and $\rho(\widetilde{D})$ the resolvent set for $\widetilde{D}$. Moreover, for $\omega \subset \mathbb{C}$, the complex conjugate of ω is denoted by

$$\omega^* = \{\overline{\lambda} \in \mathbb{C} \,|\, \lambda \in \omega\}. \tag{4.56}$$

We begin by noting a result which holds for general J-self-adjoint operators and hence in particular for $\widetilde{D}$ and its unitarily equivalent $\widetilde{H}$. (Of course, it also applies to the self-adjoint operators $\widehat{D}$ and $\widehat{H}$, cf. (2.13).)

THEOREM 4.12 ([**4**]). *Let $\widetilde{D}$ be maximally defined as in* (2.22)*. Then,*

$$\sigma(\widetilde{D}) = \sigma_{\rm p}(\widetilde{D}) \cup \sigma_{\rm c}(\widetilde{D}) \tag{4.57}$$
$$= \sigma_{\rm p}(\widetilde{D}) \cup \sigma_{\rm e}(\widetilde{D}), \tag{4.58}$$
$$\sigma_{\rm r}(\widetilde{D}) = \emptyset, \tag{4.59}$$
$$\sigma_{\rm p}(\widetilde{D}) = \sigma_{\rm p}(\widetilde{D}^*)^*, \tag{4.60}$$
$$\sigma_{\rm a}(\widetilde{D}) = \sigma(\widetilde{D}), \tag{4.61}$$
$$\pi(\widetilde{D}) = \rho(\widetilde{D}). \tag{4.62}$$

Remark 4.3 is a crucial ingredient in the proof of the next result which details spectral properties specific to $\widetilde{D}$ but which extend to $\widetilde{H}$ by the unitary equivalence found in (4.6).

THEOREM 4.13 ([**4**]). *Let $\widetilde{D}$ be maximally defined as in* (2.22)*. Then,*

$$\sigma(\widetilde{D})^* = \sigma(\widetilde{D}), \tag{4.63}$$
$$\sigma_{\rm c}(\widetilde{D}) \supseteq \mathbb{R}, \tag{4.64}$$
$$\sigma_{\rm e}(\widetilde{D}) \supseteq \mathbb{R}, \tag{4.65}$$
$$\sigma_{\rm p}(\widetilde{D}) \cap \mathbb{R} = \emptyset. \tag{4.66}$$

Thus, the spectrum for $\widetilde{D}$ is symmetric with respect to $\mathbb{R}$, the continuous spectrum contains $\mathbb{R}$, and the point spectrum is disjoint from $\mathbb{R}$. Non-real continuous and essential spectrum can occur as is seen in the example to follow in which *crossing spectral arcs* are an essential feature. Contrasted with this is the fact that the spectrum for the self-adjoint operator $\widehat{D}$ is of course confined to $\mathbb{R}$.

To underscore the relevance of the Weyl–Titchmarsh coefficients $\widetilde{m}_\pm(z,\gamma)$ for spectral theoretic questions concerning the non-self-adjoint operators $\widetilde{D}$ and $\widetilde{H}$, we now present a calculation analogous to that of Theorem 3.6. Much more remains to be done in this context and the remainder of this subsection offers just a preliminary glimpse at the difficulties imposed by non-self-adjoint Dirac-type operators.

By analogy with the self-adjoint case discussed in Subsection 3.3, we denote by $\widetilde{M}(\cdot,\gamma) \in \mathbb{C}^{2\times 2}$, $z \in \rho(\widetilde{H})$, the whole-line Weyl–Titchmarsh M-function of the operator $\widetilde{H}$ defined in (2.45) in association with the special case given by (2.41),

$$\begin{aligned}\widetilde{M}(z,\gamma) &= \big(\widetilde{M}_{\ell,\ell'}(z,\gamma)\big)_{\ell,\ell'=0,1} = \frac{1}{2}[\widetilde{\Gamma}(z,\gamma) + \widetilde{\Gamma}(z,\gamma)^\top] = \widetilde{\Gamma}(z,\gamma) + \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\widetilde{m}_-(z,\gamma)-\widetilde{m}_+(z,\gamma)} & \frac{1}{2}\frac{\widetilde{m}_-(z,\gamma)+\widetilde{m}_+(z,\gamma)}{\widetilde{m}_-(z,\gamma)-\widetilde{m}_+(z,\gamma)} \\ \frac{1}{2}\frac{\widetilde{m}_-(z,\gamma)+\widetilde{m}_+(z,\gamma)}{\widetilde{m}_-(z,\gamma)-\widetilde{m}_+(z,\gamma)} & \frac{\widetilde{m}_-(z,\gamma)\widetilde{m}_+(z,\gamma)}{\widetilde{m}_-(z,\gamma)-\widetilde{m}_+(z,\gamma)} \end{pmatrix}, \quad z \in \rho(\widetilde{H}) = \rho(\widetilde{D}). \end{aligned} \tag{4.67}$$

Here, by Remark 4.5, $\widetilde{m}_\pm(z,\gamma) = m_\pm^{\widetilde{\mathcal{H}}}(z,\alpha) = m_\pm^{\widetilde{\mathcal{D}}}(z,\beta)$ for $\beta = \alpha U$, and $\alpha = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$. By (4.46),

$$\widetilde{M}(\cdot,\gamma) \text{ is analytic on } \rho(\widetilde{H}) = \rho(\widetilde{D}). \tag{4.68}$$

Given $\widetilde{M}(\cdot,\gamma)$, we introduce the set function $\widehat{\Omega}(\cdot,\gamma)$ on intervals $(\lambda_1,\lambda_2] \subset \mathbb{R}$, $\lambda_1 < \lambda_2$, by

$$\widetilde{\Omega}((\lambda_1.\lambda_2],\gamma) = \frac{1}{2\pi i}\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\, \big[\widetilde{M}(\lambda+i\varepsilon,\gamma) - \widetilde{M}(\lambda-i\varepsilon,\gamma)\big]. \tag{4.69}$$

To proceed as in the self-adjoint case in Subsetion 3.3, we now make the following set of assumptions.

Hypothesis 4.14. *Let* $[\lambda_1,\lambda_2] \subset \mathbb{R}$, $\lambda_1 < \lambda_2$.
(i) *Suppose no spectral component of* $\widetilde{H}$ *other than* $[\lambda_1,\lambda_2]$ *intersects the set*

$$\{z \in \mathbb{C} \,|\, \lambda_1 \le \mathrm{Re}(z) \le \lambda_2,\, 0 \le |\mathrm{Im}(z)| \le \varepsilon_0\}, \tag{4.70}$$

for some fixed $\varepsilon_0 > 0$.
(ii) *Assume that* (4.69) *defines a measure on the Borel subsets of* $[\lambda_1,\lambda_2]$.
(iii) *Suppose that*

$$\begin{aligned} &\varepsilon|\widetilde{M}_{\ell,\ell'}(\lambda+i\varepsilon,\gamma)| \le C(\lambda_1,\lambda_2,\varepsilon_0), \quad \lambda\in[\lambda_1,\lambda_2],\ 0<\varepsilon\le\varepsilon_0,\ \ell,\ell'=0,1, \\ &\varepsilon|\widetilde{M}_{\ell,\ell'}(\lambda+i\varepsilon,\gamma) + \widetilde{M}_{\ell,\ell'}(\lambda-i\varepsilon,\gamma)| \underset{\varepsilon\downarrow 0}{=} o(1), \quad \lambda\in[\lambda_1,\lambda_2],\ \ell,\ell'=0,1. \end{aligned} \tag{4.71}$$

We also use the abbreviation

$$\big(T_0^{\widetilde{\mathcal{H}}}(\alpha)f\big)(\lambda) = \int_{\mathbb{R}} dx\, \mathcal{F}^{\widetilde{\mathcal{H}}}(\lambda,x,\alpha)^\top f(x), \quad \lambda\in[\lambda_1,\lambda_2],\ f\in C_0^\infty(\mathbb{R})^2. \tag{4.72}$$

Analogous definitions and hypotheses apply, of course, to other parts of the spectrum, assuming one can separate a (complex) neighborhood of the spectral arc in question from the rest of the spectrum of $\widetilde{H}$ similarly to (4.70). (We note that this excludes the possibility of crossing spectral arcs, cf. [**15**] and Lemma 4.18). The extent to which Hypothesis 4.14 applies to general J-self-adjoint operators studied in this section is beyond the scope of this paper and will be taken up elsewhere. Typical examples we have in mind are periodic and certain classes of quasi-periodic operators $\widetilde{H}$, where the spectrum is known to consist of piecewise analytic arcs.

Given an interval $(\lambda_1, \lambda_2]$ with properties as in Hypothesis 4.14, we define the analog of the spectral projection (3.46) in the self-adjoint case, now denoted by $E_{\widetilde{H}}((\lambda_1, \lambda_2])$, associated with $\widetilde{H}$ and $(\lambda_1, \lambda_2]$, by

$$\begin{aligned}
&\langle f, E_{\widetilde{H}}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} \\
&\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\,\big\langle f, \big[(\widetilde{H}-(\lambda+i\varepsilon))^{-1} \\
&\qquad\qquad -(\widetilde{H}-(\lambda-i\varepsilon))^{-1}\big]g\big\rangle_{L^2(\mathbb{R})^2}, \quad f,g\in C_0^\infty(\mathbb{R})^2.
\end{aligned} \tag{4.73}$$

In the present non-self-adjoint context, it is far from obvious that $E_{\widetilde{H}}((\lambda_1, \lambda_2])$ extends to a bounded operator, let alone, a bounded projection, on $L^2(\mathbb{R})^2$. A careful study of this question is again beyond the scope of this paper and hence we introduce the following hypothesis for now and postpone a detailed discussion of the properties of $E_{\widetilde{H}}((\lambda_1, \lambda_2])$ to a future investigation:

Hypothesis 4.15. *Given an interval $(\lambda_1, \lambda_2]$ with properties as in Hypothesis 4.14, we suppose that $E_{\widetilde{H}}((\lambda_1, \lambda_2])$, as defined in* (4.73)*, extends to a bounded projection operator on $L^2(\mathbb{R})^2$.*

We note that Hypotheses 4.14 and 4.15 can be verified in some special cases. For instance, in the case of periodic Schrödinger operators, one can successfully apply Floquet theory and verify Hypothesis 4.14 in connection with parts of spectral arcs which are not intersected by other spectral arcs (cf. [**15**]). On the other hand, Hypothesis 4.15 is known to fail in the presence of crossings of spectral arcs as shown in [**15**]. This is also underscored in Lemma 4.18 in connection with the simple constant coefficient Dirac-type operator $\widetilde{D}_{q_0}$, which exhibits the crossing of spectral arcs at the origin. We will return to this circle of ideas elsewhere.

Theorem 4.16. *Assume Hypotheses 4.14 and 4.15 and $f, g \in C_0^\infty(\mathbb{R})^2$. Then,*

$$\langle f, E_{\widetilde{H}}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} = \int_{(\lambda_1,\lambda_2]} \Big(\big(T_0^{\widetilde{\mathcal{H}}}(\alpha_0)\overline{f}\big)(\lambda)\Big)^\top d\widetilde{\Omega}(\lambda,\gamma_0)\big(T_0^{\widetilde{\mathcal{H}}}(\alpha_0)g\big)(\lambda). \tag{4.74}$$

Proof. For simplicity we will suppress the α (resp., γ) dependence of all quantities involved. We closely follow the strategy of proof employed in connection with

Theorem 3.6. Then,

(4.75)

$$
\begin{aligned}
&\langle f, E_{\widetilde{H}}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} \\
&\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\,\big\langle f,\ \big[(\widetilde{H}-(\lambda+i\varepsilon))^{-1}-(\widetilde{H}-(\lambda-i\varepsilon))^{-1}\big]g\big\rangle_{L^2(\mathbb{R})^2} \\
&\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\int_{\mathbb{R}}dx\int_{\mathbb{R}}dx'\big\{f(x)^*G^{\widetilde{H}}(\lambda+i\varepsilon,x,x')g(x') \\
&\qquad\qquad\qquad\qquad - f(x)^*G^{\widetilde{H}}(\lambda-i\varepsilon,x,x')g(x')\big\} \\
&\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\int_{\mathbb{R}}dx\bigg\{\int_{-\infty}^{x}dx'\big\{f(x)^*G^{\widetilde{H}}(\lambda+i\varepsilon,x,x')g(x') \\
&\qquad\qquad\qquad\qquad - f(x)^*G^{\widetilde{H}}(\lambda-i\varepsilon,x,x')g(x')\big\} \\
&\qquad\qquad + \int_{x}^{\infty}dx'\big\{f(x)^*G^{\widetilde{H}}(\lambda+i\varepsilon,x,x')g(x') \\
&\qquad\qquad\qquad\qquad - f(x)^*G^{\widetilde{H}}(\lambda-i\varepsilon,x,x')g(x')\big\}\bigg\} \\
&\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\int_{\mathbb{R}}dx \\
&\quad\times\bigg\{\int_{-\infty}^{x}dx'\,f(x)^*\mathcal{F}^{\widetilde{\mathcal{H}}}(\lambda,x)\Big[\widetilde{\Gamma}(\lambda+i\varepsilon)-\widetilde{\Gamma}(\lambda-i\varepsilon)\Big]\mathcal{F}^{\widetilde{\mathcal{H}}}(\lambda,x')^\top g(x') \\
&\qquad + \int_{x}^{\infty}dx'\,f(x)^*\mathcal{F}^{\widetilde{\mathcal{H}}}(\lambda,x)\Big[\widetilde{\Gamma}(\lambda+i\varepsilon)^\top-\widetilde{\Gamma}(\lambda-i\varepsilon)^\top\Big]\mathcal{F}^{\widetilde{\mathcal{H}}}(\lambda,x')^\top g(x')\bigg\}.
\end{aligned}
$$

Here we used conditions (4.71) to pass to the last line in (4.75). By means of the fundamental identity given in (4.24), a calculation shows that

$$
\widetilde{\Gamma}(\lambda+i\varepsilon)-\widetilde{\Gamma}(\lambda-i\varepsilon) = [\widetilde{\Gamma}(\lambda+i\varepsilon)-\widetilde{\Gamma}(\lambda-i\varepsilon)]^\top, \tag{4.76}
$$

and by (4.67) that

$$
\widetilde{M}(\lambda+i\varepsilon)-\widetilde{M}(\lambda-i\varepsilon) = \widetilde{\Gamma}(\lambda+i\varepsilon)-\widetilde{\Gamma}(\lambda-i\varepsilon). \tag{4.77}
$$

As a consequence,

$$
\begin{aligned}
&\langle f, E_{\widetilde{H}}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} \\
&\quad = \int_{(\lambda_1,\lambda_2]}\int_{\mathbb{R}}dx\int_{\mathbb{R}}dx'\,f(x)^*\mathcal{F}^{\widetilde{\mathcal{H}}}(\lambda,x)d\widetilde{\Omega}(\lambda)\mathcal{F}^{\widetilde{\mathcal{H}}}(\lambda,x')^\top g(x'),
\end{aligned} \tag{4.78}
$$

from which (4.74) then follows. □

As a corollary, we obtain the corresponding result for the operator $\widetilde{D}$.

Corollary 4.17. *Assume Hypotheses 4.14 and 4.15 and $f, g \in C_0^\infty(\mathbb{R})^2$. Then,*

(4.79)

$$
\langle f, E_{\widetilde{D}}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} = \int_{(\lambda_1,\lambda_2]}\Big(\big(T_0^{\widetilde{\mathcal{H}}}(\alpha_0)\overline{Uf}\big)(\lambda)\Big)^\top d\widetilde{\Omega}(\lambda,\gamma_0)\big(T_0^{\widetilde{\mathcal{H}}}(\alpha_0)Ug\big)(\lambda).
$$

PROOF. This follows immediately from the observation that

$$\begin{aligned}\langle f, E_{\widetilde{D}}((\lambda_1,\lambda_2])g\rangle_{L^2(\mathbb{R})^2} &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\,\big\langle f, U^{-1}\big[(\widetilde{H}-(\lambda+i\varepsilon))^{-1} \\ &\quad -(\widetilde{H}-(\lambda-i\varepsilon))^{-1}\big]Ug\big\rangle_{L^2(\mathbb{R})^2}.\end{aligned} \tag{4.80}$$

□

4.4. An Example. Next we turn to then the calculation of quantities discussed in the previous section for $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{D}}$ in the case where $q(x) = q_0 \in \mathbb{C}\backslash\{0\}$ is constant. (The self-adjoint case $q_0 = 0$ has already been discussed in Subsection 3.4.) In this case we denote $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{D}}$ by $\widetilde{\mathcal{H}}_{q_0}$ and $\widetilde{\mathcal{D}}_{q_0}$, etc. Without much exaggeration, the example to follow describes probably the simplest periodic (even constant coefficient) differential operator with crossing spectral arcs.

In consideration of this case, we define the function $\widetilde{S}_{q_0}(z)$ to be a function that is analytic with positive imaginary part on the split plane

$$\widetilde{\mathbb{P}}_{q_0} = \mathbb{C}\backslash(\mathbb{R}\cup\{z\in\mathbb{C}\,|\,z=it,\, t\in[-|q_0|,|q_0|]\}) \tag{4.81}$$

such that

$$\widetilde{S}_{q_0}(z) = \sqrt{z^2+|q_0|^2}, \quad z\in\widetilde{\mathbb{P}}_{q_0}. \tag{4.82}$$

Given this definition, the following conventions are used:

$$\widetilde{S}_{q_0}(\overline{z}) = -\overline{\widetilde{S}_{q_0}(z)}, \quad z\in\widetilde{\mathbb{P}}_{q_0}, \tag{4.83}$$

$$\widetilde{S}_{q_0}(\lambda\pm i0) = \lim_{\varepsilon\downarrow 0}\widetilde{S}_{q_0}(\lambda\pm i\varepsilon) = \begin{cases}\pm\sqrt{\lambda^2+|q_0|^2}, & \lambda>0,\\ \mp\sqrt{\lambda^2+|q_0|^2}, & \lambda<0,\end{cases} \tag{4.84}$$

$$\widetilde{S}_{q_0}(it\pm 0) = \lim_{\lambda\downarrow 0}\widetilde{S}_{q_0}(it\pm\lambda) = \begin{cases}\pm\sqrt{|q_0|-t^2}, & 0<t\leqslant|q_0|,\\ \mp\sqrt{|q_0|-t^2}, & -|q_0|\leqslant t<0.\end{cases} \tag{4.85}$$

A direct calculation shows for general $\alpha = (\cos(\theta),\sin(\theta))$, $\theta\in[0,2\pi)$, that

$$\Psi_{\pm}^{\widetilde{\mathcal{H}}_{q_0}}(z,x,\alpha) = a_{\pm}\begin{pmatrix}1\\ \frac{\mathrm{Im}(q_0)\mp\widetilde{S}_{q_0}(z)}{iz+\mathrm{Re}(q_0)}\end{pmatrix}e^{\pm i\widetilde{S}_{q_0}(z)x}, \quad z\in\widetilde{\mathbb{P}}_{q_0}. \tag{4.86}$$

For the the corresponding general $\beta = \alpha U = [(-1+i)/2](e^{-i\theta}, e^{i\theta})$, $\theta\in[0,2\pi)$, we see by direct calculation that

$$\Psi_{\pm}^{\widehat{\mathcal{D}}_{q_0}}(z,x,\beta) = b_{\pm}\begin{pmatrix}1\\ \frac{i}{q_0}(z\pm\widetilde{S}_{q_0}(z))\end{pmatrix}e^{\pm i\widetilde{S}_{q_0}(z)x}, \quad z\in\widetilde{\mathbb{P}}_{q_0} \tag{4.87}$$

for some $b_{\pm}\in\mathbb{C}$, and alternatively that

$$\begin{aligned}\Psi_{\pm}^{\widetilde{\mathcal{D}}_{q_0}}(z,x,\beta) &= U^{-1}\Psi_{\pm}^{\widetilde{\mathcal{H}}_{q_0}}(z,x,\alpha)\\ &= \frac{a_{\pm}(-1+i)}{2(iz+\mathrm{Re}(q_0))}\begin{pmatrix}-z+iq_0\pm\widetilde{S}_{q_0}(z)\\ -z+i\overline{q}_0\mp\widetilde{S}_{q_0}(z)\end{pmatrix}e^{\pm i\widetilde{S}_{q_0}(z)x},\\ &\qquad z\in\widetilde{\mathbb{P}}_{q_0}\end{aligned} \tag{4.88}$$

for some $a_{\pm}\in\mathbb{C}$.

In particular, for $\alpha=\alpha_0=(1,0)$ and $z\in\widetilde{\mathbb{P}}_{q_0}$,

$$\Psi_\pm^{\widetilde{\mathcal{H}}_{q_0}}(z,0,\alpha_0)=\mathcal{F}^{\widetilde{\mathcal{H}}_{q_0}}(z,0,\alpha_0)\begin{pmatrix}1\\ m_\pm^{\widetilde{\mathcal{H}}_{q_0}}(z,\gamma_0)\end{pmatrix}=a_\pm\begin{pmatrix}1\\ \frac{\mathrm{Im}(q_0)\mp\widetilde{S}_{q_0}(z)}{iz+\mathrm{Re}(q_0)}\end{pmatrix}. \tag{4.89}$$

Because $\mathcal{F}^{\widetilde{\mathcal{H}}_{q_0}}(z,0,\alpha_0)=I_2$, we conclude that $a_\pm=1$ and that

$$\widetilde{m}_{q_0,\pm}(z,\gamma_0)=m_\pm^{\widetilde{\mathcal{H}}_{q_0}}(z,\alpha_0)=m_\pm^{\widetilde{\mathcal{D}}_{q_0}}(z,\beta_0)=\frac{\mathrm{Im}(q_0)\mp\widetilde{S}_{q_0}(z)}{iz+\mathrm{Re}(q_0)},\quad z\in\widetilde{\mathbb{P}}_{q_0}. \tag{4.90}$$

By (4.31) we see that

$$\begin{aligned}G^{\widetilde{H}_{q_0}}&(z,x,x')\\ &=\frac{1}{2\widetilde{S}_{q_0}(z)}\begin{pmatrix}iz+\mathrm{Re}(q_0) & \mathrm{Im}(q_0)\mp\widetilde{S}_{q_0}(z)\\ \mathrm{Im}(q_0)\pm\widetilde{S}_{q_0}(z) & iz-\mathrm{Re}(q_0)\end{pmatrix}e^{\pm i\widetilde{S}_{q_0}(z)(x'-x)},\\ &\qquad\qquad x\lesseqgtr x',\ z\in\widetilde{\mathbb{P}}_{q_0}.\end{aligned} \tag{4.91}$$

Using either (4.34) or the fact that $G^{\widetilde{D}_{q_0}}(z,x,x')=U^{-1}G^{\widetilde{H}_{q_0}}(z,x,x')U$, we see that

$$\begin{aligned}G^{\widetilde{D}_{q_0}}(z,x,x')=\frac{1}{2\widetilde{S}_{q_0}(z)}\begin{pmatrix}i\big[z\pm\widetilde{S}_{q_0}(z)\big] & q_0\\ \overline{q_0} & i\big[z\mp\widetilde{S}_{q_0}(z)\big]\end{pmatrix}e^{\pm i\widetilde{S}_{q_0}(z)(x'-x)},\\ x\lesseqgtr x',\ z\in\widetilde{\mathbb{P}}_{q_0}.\end{aligned} \tag{4.92}$$

The spectra of $\widetilde{H}_{q_0}$ and $\widetilde{D}_{q_0}$ are purely continous and given by

$$\sigma(\widetilde{H}_{q_0})=\sigma(\widetilde{D}_{q_0})=\mathbb{R}\cup\{z\in\mathbb{C}\,|\,z=it,\ t\in[-|q_0|,|q_0|]\}, \tag{4.93}$$

that is, the spectrum consists of the real axis and the interval from $-|q_0|$ to $|q_0|$ along the imaginary axis. Since $q_0\in\mathbb{C}\backslash\{0\}$, the origin is a crossing point of the two spectral arcs.

In contrast to the self-adjoint example $q_0\in\mathbb{C}\backslash\{0\}$ discussed in Section 3.4, the potential pole for $\widetilde{m}_{q_0,\pm}(z,\gamma_0)$ given by $z=i\mathrm{Re}(q_0)$ now lies in the continuous spectrum for $\widetilde{H}$. For $z=i\mathrm{Re}(q_0)$ and $\mathrm{Im}(q_0)\neq 0$, the z-wave functions for $\widetilde{\mathcal{H}}_{q_0}$ are given by

$$\Psi^{\widetilde{\mathcal{H}}_{q_0}}(x)=\psi_1(0)\begin{pmatrix}e^{ix\mathrm{Im}(q_0)}\\ 2\left(\frac{i\mathrm{Re}(q_0)}{\mathrm{Im}(q_0)}\right)\sin(x\,\mathrm{Im}(q_0))\end{pmatrix}+\psi_2(0)\begin{pmatrix}0\\ e^{-ix\mathrm{Im}(q_0)}\end{pmatrix}, \tag{4.94}$$

and for $z=i\mathrm{Re}(q_0)$ and $\mathrm{Im}(q_0)=0$, the z-wave functions for $\widetilde{\mathcal{H}}_{q_0}$ are given by

$$\Psi(x)=\psi_1(0)\begin{pmatrix}1\\ -2ix\,\mathrm{Re}(q_0)\end{pmatrix}+\psi_2(0)\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{4.95}$$

Consequently, $z=i\mathrm{Re}(q_0)$ is neither an eigenvalue of $\widetilde{H}_{q_0}$ nor an eigenvalue of $\widetilde{H}_{q_0,\pm}(\alpha_0)$.

4.5. Nonspectrality. The principal result of this subsection illustrates that for all the similarities developed thus far, $\widehat{D}$ and $\widetilde{D}$ bear the following stark difference: $\widehat{D}$, being self-adjoint, is always a spectral operator of scalar type in the sense of Dunford and Schwartz while $\widetilde{D}$ cannot be expected to be a spectral operator whenever there are crossing spectral arcs in the spectrum of $\widetilde{D}$.

In the case of periodic Schrödinger operators, this result has recently been proved in [**15**]. Here we confine ourselves to a study of the constant coefficient operator $\widetilde{D}_{q_0}$ but on the basis of [**15**] it is natural to expect this result extends to all periodic Dirac-type operators $\widetilde{D}$ with crossing spectral arcs.

Applying Corollary 4.17 to the concrete example $q(x) = q_0 \in \mathbb{C}\backslash\{0\}$ treated in the previous subsection, one can rewrite (4.79) to obtain

(4.96)

$$\begin{aligned}
&\langle f, E_{\widetilde{D}_{q_0}}((\lambda_1,\lambda_2))g\rangle_{L^2(\mathbb{R})^2} \\
&\quad = \int_{(\lambda_1,\lambda_2)} \Big(\big(T_0^{\widetilde{\mathcal{H}}_{q_0}}(\alpha)\overline{Uf}\big)(\lambda)\Big)^{\top} d\widetilde{\Omega}_{q_0}(\lambda,\gamma)\big(T_0^{\widetilde{\mathcal{H}}_{q_0}}(\alpha)Ug\big)(\lambda) \\
&\quad = \frac{1}{2}\int_{\lambda_1}^{\lambda_2} d\lambda \Bigg[\begin{pmatrix} \hat f_1(\sqrt{\lambda^2+|q_0|^2}) \\ \hat f_2(\sqrt{\lambda^2+|q_0|^2}) \end{pmatrix}^{*} \\
&\qquad\qquad \times \begin{pmatrix} \big[\lambda/\sqrt{\lambda^2+|q_0|^2}\,\big]+1 & -iq_0/\sqrt{\lambda^2+|q_0|^2} \\ -i\overline{q_0}/\sqrt{\lambda^2+|q_0|^2} & \big[\lambda/\sqrt{\lambda^2+|q_0|^2}\,\big]-1 \end{pmatrix} \begin{pmatrix} \hat g_1(\sqrt{\lambda^2+|q_0|^2}) \\ \hat g_2(\sqrt{\lambda^2+|q_0|^2}) \end{pmatrix} \\
&\qquad\quad + \begin{pmatrix} \check f_1(\sqrt{\lambda^2+|q_0|^2}) \\ \check f_2(\sqrt{\lambda^2+|q_0|^2}) \end{pmatrix}^{*} \begin{pmatrix} \big[\lambda/\sqrt{\lambda^2+|q_0|^2}\,\big]-1 & -iq_0/\sqrt{\lambda^2+|q_0|^2} \\ -i\overline{q_0}/\sqrt{\lambda^2+|q_0|^2} & \big[\lambda/\sqrt{\lambda^2+|q_0|^2}\,\big]+1 \end{pmatrix} \\
&\qquad\qquad \times \begin{pmatrix} \check g_1(\sqrt{\lambda^2+|q_0|^2}) \\ \check g_2(\sqrt{\lambda^2+|q_0|^2}) \end{pmatrix} \Bigg], \quad 0<\lambda_1<\lambda_2, \ f,g \in C_0^\infty(\mathbb{R})^2,
\end{aligned}$$

where

$$\hat h(p) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} dx\, e^{ipx} h(x), \quad \check h(p) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} dx\, e^{-ipx} h(x), \quad p\in\mathbb{R}, \ h \in C_0^\infty(\mathbb{R}). \tag{4.97}$$

We note that the spectrum of $\widetilde{D}_{q_0}$ is purely continuous and so the distinction between the intervals $(\lambda_1,\lambda_2]$ and (λ_1,λ_2) becomes irrelevant throughout this subsection.

In the following, $\mathcal{B}(\mathcal{H})$ denotes the Banch space of bounded linear operators in a Hilbert space $\mathcal{H}$.

Lemma 4.18. *Let $[\lambda_1,\lambda_2] \subset (0,\infty)$. Then,*

$$\lim_{\lambda_1 \downarrow 0} \|E_{\widetilde{D}_{q_0}}((\lambda_1,\lambda_2))\|_{\mathcal{B}(L^2(\mathbb{R})^2)} = \infty. \tag{4.98}$$

Proof. We choose f,g of the form $f = (h,0)^\top$, $g = (0,h)^\top$, $h \in C_0^\infty(\mathbb{R})$. Then,

$$\langle f, E_{\widetilde{D}_{q_0}}((\lambda_1,\lambda_2))g\rangle_{L^2(\mathbb{R})^2} \tag{4.99}$$

$$= -\int_{\lambda_1}^{\lambda_2} d\lambda \frac{iq_0}{2\sqrt{\lambda^2+|q_0|^2}} \big[|\hat h(\sqrt{\lambda^2+|q_0|^2})|^2 + |\check h(\sqrt{\lambda^2+|q_0|^2})|^2 \big]. \tag{4.100}$$

The change of variables

$$\mu = \sqrt{\lambda^2 + |q_0|^2} \geq |q_0|, \quad d\lambda = \frac{\mu d\mu}{\sqrt{\mu^2 - |q_0|^2}} \tag{4.101}$$

then yields
(4.102)

$$\langle f, E_{\widetilde{D}_{q_0}}((\lambda_1, \lambda_2))g\rangle_{L^2(\mathbb{R})^2} = -\frac{iq_0}{2}\int_{\sqrt{\lambda_1^2+|q_0|^2}}^{\sqrt{\lambda_2^2+|q_0|^2}} \frac{d\mu}{\sqrt{\mu^2 - |q_0|^2}}\big[|\hat{h}(\mu)|^2 + |\check{h}(\mu)|^2\big].$$

It suffices to study the first term on the right-hand side of (4.102). (The second term is handled in exactly the same manner.) For this purpose we now introduce in $L^2(\mathbb{R}; d\mu)$ the maximally defined operator $T(\lambda_1, \lambda_2)$ of multiplication by the function

$$t((\lambda_1, \lambda_2), \mu) = \frac{1}{\sqrt{\mu^2 - |q_0|^2}}\chi_{[\sqrt{\lambda_1^2+|q_0|^2}, \sqrt{\lambda_2^2+|q_0|^2}]}(\mu), \quad \mu \in \mathbb{R}, \tag{4.103}$$

where χ_ω denotes the characteristic function of the set $\omega \subset \mathbb{R}$. We recall that

$$\|T(\lambda_1, \lambda_2)\|_{\mathcal{B}(L^2(\mathbb{R};d\mu))} = \|t((\lambda_1, \lambda_2), \cdot)\|_{L^\infty(\mathbb{R};d\mu)} \tag{4.104}$$

(cf. [**44**, p. 51–54]).

Next, we note that

$$\int_{\sqrt{\lambda_1^2+|q_0|^2}}^{\sqrt{\lambda_2^2+|q_0|^2}} \frac{d\mu}{\sqrt{\mu^2 - |q_0|^2}}|\hat{h}(\mu)|^2 = \|T(\lambda_1, \lambda_2)\hat{h}\|^2_{L^2(\mathbb{R};d\mu)}. \tag{4.105}$$

Thus, as long as $0 < \lambda_1 < \lambda_2$, one infers that $\mu > |q_0|$ and hence that

$$\|T(\lambda_1, \lambda_2)\|_{\mathcal{B}(L^2(\mathbb{R};d\mu))} = \|t((\lambda_1, \lambda_2), \cdot)\|_{L^\infty(\mathbb{R};d\mu)} < \infty, \quad 0 < \lambda_1 < \lambda_2. \tag{4.106}$$

However, since $\mu \downarrow q_0$ as $\lambda \downarrow 0$, one obtains

$$\|T(\lambda_1, \lambda_2)\|_{\mathcal{B}(L^2(\mathbb{R};d\mu))} = \|t((\lambda_1, \lambda_2), \cdot)\|_{L^\infty(\mathbb{R};d\mu)} \uparrow \infty \text{ as } \lambda_1 \downarrow 0. \tag{4.107}$$

Since $\|\hat{h}\|_{L^2(\mathbb{R})} = \|\check{h}\|_{L^2(\mathbb{R})} = \|h\|_{L^2(\mathbb{R})}$, there exists a $C > 0$ such that

$$\int_{\sqrt{\lambda_1^2+|q_0|^2}}^{\sqrt{\lambda_2^2+|q_0|^2}} \frac{d\mu}{\sqrt{\mu^2 - |q_0|^2}}|\hat{h}(\mu)|^2 \leq C\|h\|^2_{L^2(\mathbb{R})} \tag{4.108}$$

for all $h \in C_0^\infty(\mathbb{R})$ if and only if $T(\lambda_1, \lambda_2) \in \mathcal{B}(L^2(\mathbb{R}; d\mu))$ and hence if and only if $t((\lambda_1, \lambda_2), \cdot) \in L^\infty(\mathbb{R}; d\mu)$. The blowup in (4.107) then shows that (4.98) holds. $\square$

Thus, the crossing of spectral arcs at the point $\lambda = 0$ prevents the operator $\widetilde{D}_{q_0}$ (and hence $\widetilde{H}_{q_0}$) to have a uniformly bounded family of spectral projections. This is remarkable since the corresponding Green's matrices (4.91) and (4.92) exhibit no singularity at $z = 0$.

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