Cyclic thermodynamic processes and entropy production

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Abstract

We study the time evolution of a periodically driven quantum-mechanical system coupled to several reservoirs of free fermions at different temperatures and chemical potentials. This is a paradigm of a cyclic thermodynamic process. We introduce the notion of a Floquet Liouvillean as the generator of the dynamics of the coupled system on an extended Hilbert space. We show that the time-periodic state which the true state of the coupled system converges to after very many periods corresponds to a zero-energy resonance of the Floquet Liouvillean. We then show that the entropy production per cycle is (strictly) positive, a property that implies Carnot's formulation of the second law of thermodynamics.

1 Introduction

During the past several years, there has been substantial progress in the program of deriving the fundamental laws of thermodynamics from nonequilibrium quantum statistical mechanics (see [A-SF1] for a synopsis). In this paper, we make a contribution to this program by studying Carnot's formulation of the second law of thermodynamics from the point of view of quantum statistical mechanics. For the sake of concreteness, we consider

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a periodically driven two-level quantum-mechanical system, Σ, coupled to $n \geq 2$ reservoirs of free fermions, $\mathcal{R}_1, \cdots, \mathcal{R}_n$. Our analysis can be generalized to a system composed of an arbitrary quantum-mechanical system with a finite-dimensional Hilbert space coupled to several reservoirs of (free) bosons or fermions at different temperatures and chemical potentials.[1] In order to study the time evolution of the coupled system, we extend Floquet theory for periodically driven quantum systems at zero temperature (see for example [Ho, Ya1, Ya2]) to apply to systems at positive temperatures.

In particular, we introduce an operator, the Floquet Liouvillean, generating the dynamics of the system on an extended Hilbert space, and we show that the time-periodic state, which the true state of the system converges to after very many periods, corresponds to a zero-energy resonance of the Floquet Liouvillean. We also prove (strict) positivity of entropy production per cycle, which amounts to Carnot's formulation of the second law of thermodynamics. For weak enough coupling of the small system, Σ, to the reservoirs, the time-periodic state corresponding to the zero-energy resonance can be expanded in powers of the coupling constant. This is of considerable practical importance, since it enables one to explicitly compute the *degree of efficiency* of the coupled system. Further discussion of the second law of thermodynamics and another proof of convergence to time-periodic states using methods of scattering theory will appear in [A-SF1]. As far as we know, all previous investigations of Carnot's formulation of the second law of thermodynamics from the point of view of quantum statistical mechanics assumed that the coupling is switched on at time $t = t_0$, and then switched off at a later time $t = t_0 + \tau$ (see [PW, BR] and references therein). The novelty of our approach is to prove that the state of the coupled system converges to a time-periodic state with the same period as the one of the interaction, and that entropy production is (strictly) positive; (see also [FMSUe]).

The organization of this paper is as follows. In section 2, we recall some basic notions from quantum statistical mechanics, in particular, time-dependent perturbations of C^*-dynamical systems, relative entropy, and Carnot's formulation of the second law of thermodynamics. We also discuss sufficient conditions to prove strict positivity of entropy production per cycle. These conditions are satisfied in the concrete example we consider in the following section. In section 3, we discuss a concrete model, and we state the as-

[1] Bosonic reservoirs are more difficult, technically, since the interaction term coupling Σ to the bosonic reservoirs is generally an unbounded operator.

sumptions we make on the interaction between the small system, Σ, and the reservoirs, $\mathcal{R}_1, \cdots, \mathcal{R}_n$. In section 4, we introduce the Floquet Liouvillean, whose spectrum we study using complex spectral deformation techniques. In section 5, we use results on the Floquet Liouvillean to prove convergence of the true state of the coupled system to a time-periodic state. This is one of the main results of our paper. We then prove strict positivity of entropy production per cycle in section 6, and we discuss how to compute the degree of efficiency of the coupled system for weak enough coupling. The main ingredients of our analysis are a concrete representation of the fermionic reservoirs (Araki-Wyss representation), a spectral approach to cyclic thermodynamic processes using the so called Floquet Liouvillean, and complex spectral deformation techniques.

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2 General considerations

In this section, we recall some of the basic notions of quantum statistical mechanics, and we introduce the notion of time-periodic states. Although some of the material is standard, it is presented in this section in order to make our exposition reasonably self-contained.

In the algebraic formulation of quantum statistical mechanics, a physical system is described by a C^* or W^*- dynamical system. Since we only consider fermionic reservoirs in this paper, we restrict our attention to the discussion of C^*-dynamical systems. However, our analysis can be generalized to W^*-dynamical systems; (see for example [DJP]).

A C^*-dynamical system is a pair $(\mathcal{O}, \alpha)$, where $\mathcal{O}$ is the kinematical algebra of the system, a C^*-algebra with identity, and α, which specifies the time evolution, is a norm-continuous one-parameter group of $*$-automorphisms of $\mathcal{O}$. A physical state of the system is described by a positive, linear functional ω, with $\omega(\mathbf{1}) = 1$. The set $E(\mathcal{O})$ of all states is a convex, weak-* compact subset of the dual $\mathcal{O}^*$.

Physically relevant states of (isolated) thermal reservoirs are assumed to be normal to equilibrium states characterized by the Kubo-Martin-Schwinger (KMS) condition. An equilibrium state of $(\mathcal{O}, \alpha)$ at inverse temperature

β, ω_β, is an (α, β)-KMS state satisfying

$$\omega_\beta(a\alpha^t(b)) = \omega_\beta(\alpha^{t-i\beta}(b)a) \ ,$$

for all $a, b \in \mathcal{O}^0$, where $\mathcal{O}^0$ is norm-dense in $\mathcal{O}$.

We briefly recall the perturbation theory for C^*-dynamical systems; (for further details, see for example [BR]). Let δ be the generator of α, ie, $\alpha^t = e^{t\delta}, t \in \mathbf{R}$. The domain of the derivation δ, $\mathcal{D}(\delta)$, is a $*$-subalgebra of $\mathcal{O}$, and for all $a, b \in \mathcal{D}(\delta)$,

$$\delta(a)^* = \delta(a^*), \ \delta(ab) = \delta(a)b + a\delta(b) \ .$$

Consider the time-dependent family of perturbations, $\{gV(t)\}_{t\in\mathbf{R}}$, with selfadjoint elements $gV(t) \in \mathcal{O}$. Then α_g, the perturbed time evolution, is a norm-continuous one-parameter family of $*$-automorphisms of $\mathcal{O}$ satisfying

$$\frac{d}{dt}\alpha_g^t(a) = \alpha_g^t(\delta(a) + ig[V(t), a]) \ , \tag{1}$$

and $\alpha_g^0(a) = a$, for all $a \in \mathcal{O}$. Explicitly,

$$\alpha_g^t(a) = \alpha^t(a) + \sum_{n\geq 1} i^n g^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n [\alpha^{t_n}(V(t_n)), \cdots, [\alpha^{t_1}(V(t_1)), \alpha^t(a)], \cdots]. \tag{2}$$

In the standard interaction picture,

$$\alpha_g^t(a) = \Gamma_g^t \alpha^t(a) \Gamma_g^{t*},$$

where Γ_g is a unitary element of $\mathcal{O}$ which satisfies

$$\frac{d}{dt}\Gamma_g^t = i\Gamma_g^t \alpha^t(V(t)), \tag{3}$$

and $\Gamma_g^0 = 1$; ie,

$$\Gamma_g^t = \mathbf{1} + \sum_{n\geq 1} i^n g^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \alpha^{t_n}(V(t_n)) \cdots \alpha^{t_1}(V(t_1)).$$

Next, we discuss the notion of relative entropy and of entropy production. Assume that there exists a reference C^*-dynamics σ_ω on $\mathcal{O}$ and a state

ω with the property that ω is an $(\sigma_\omega, -1)$-KMS state. (Equivalently, at inverse temperature $\beta \geq 0, \sigma^t_{\omega,\beta} = \sigma_\omega^{-t/\beta}$). Let δ_ω be the generator of σ_ω, and let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation of the kinematical algebra $\mathcal{O}$ associated to the state ω. (For further discussion of the GNS construction see, for example, [BR].)

A state $\eta \in E(\mathcal{O})$ is called ω-normal if there exists a density matrix ρ_η on $\mathcal{H}_\omega$, such that, for all $a \in \mathcal{O}$,

$$\eta(a) = Tr(\rho_\eta \pi_\omega(a)),$$

where Tr is the trace over $\mathcal{H}_\omega$. We will denote by $\mathcal{N}_\omega$ the set of all ω-normal states in $E(\mathcal{O})$.

For $\eta \in \mathcal{N}_\omega$, denote by $Ent(\eta|\omega)$ the relative entropy of Araki [Ar,D]. For finite systems,

$$Ent(\eta|\omega) = Tr(\eta \log \omega - \eta \log \eta).$$

If $\eta \notin \mathcal{N}_\omega$, set $Ent(\eta|\omega) = -\infty$.

For a selfadoint perturbation $gV(t) \in \mathcal{D}(\delta_\omega)$ of the dynamical system, as discussed above, we define the rate of entropy production in a state $\eta \in E(\mathcal{O})$ relative to a reference state ω as

$$Ep(\eta) := \eta(\delta_\omega(gV(t))); \tag{4}$$

see for example [BR,JP4]. Note that the state η in (4) can be time-dependent.

It is instructive to see how one obtains this expression for entropy production as the thermodynamic limit of quantities referring to finitely extended reservoirs.[A-SF2] Consider a quantum system composed of a small system, Σ, coupled to n reservoirs, $\mathcal{R}_1, \cdots, \mathcal{R}_n$, at inverse temperatures $\beta_1, \cdots, \beta_n$, respectively. We first treat the reservoirs as finitely extended systems and then take the thermodynamic limit of suitable quantities. The total Hamiltonian of the *finite* coupled system is

$$H(t) = H^\Sigma + \sum_{i=1}^n H^{\mathcal{R}_i} + gV(t),$$

where H^Σ is the Hamiltonian of the uncoupled small system, $H^{\mathcal{R}_i}$ is the Hamiltonian of the i^{th} uncoupled reservoir, and $gV(t)$ is the interaction term coupling Σ to the reservoirs. Let the reference state of the reservoirs be $\omega^{\mathcal{R}}$ with corresponding density matrix $\rho^{\mathcal{R}}$ given by

$$\rho^{\mathcal{R}} = \rho^{\mathcal{R}_1} \otimes \cdots \otimes \rho^{\mathcal{R}_n},$$

where $\rho^{\mathcal{R}_i}$ is the density matrix corresponding to the equilibrium state of the reservoir $\mathcal{R}_i$ at inverse temperature β_i, which is given by

$$\rho^{\mathcal{R}_i} = \frac{e^{-\beta_i H^{\mathcal{R}_i}}}{Tr_{\mathcal{R}_i}(e^{-\beta_i H^{\mathcal{R}_i}})},$$

where $Tr_{\mathcal{R}_i}$ is the trace over the Hilbert space of the reservoir $\mathcal{R}_i$.

Define $\omega_t := \omega \circ \alpha_g^t$, where ω is the initial $(\sigma_\omega, -1)$-KMS state, and let ρ^ω be the density matrix corresponding to ω. For a finite system,

$$\begin{aligned} Ent(\omega_t|\omega^{\mathcal{R}}) &= Tr(\rho^{\omega_t} \log \rho^{\mathcal{R}}) - Tr(\rho^{\omega_t} \log \rho^{\omega_t}) && (5) \\ &= Tr(\rho^{\omega_t} \log \rho^{\mathcal{R}}) - Tr(\rho^{\omega} \log \rho^{\omega}), && (6) \end{aligned}$$

and hence

$$\frac{d}{dt} Ent(\omega_t|\omega^{\mathcal{R}}) = -iTr([H(t), \rho^{\omega_t}] \log \rho^{\mathcal{R}}).$$

By cyclicity of the trace we have that

$$\begin{aligned} \frac{d}{dt} Ent(\omega_t|\omega^{\mathcal{R}}) &= -i \sum_{i=1}^{n} \beta_i Tr(\rho^{\omega_t}[H(t), H^{\mathcal{R}_i}]) \\ &= -i \sum_{i=1}^{n} \beta_i Tr(\rho^{\omega_t}[gV(t), H^{\mathcal{R}_i}]) \\ &= -\omega \circ \alpha_g^t(\delta_\omega(gV(t))), \end{aligned}$$

where we have used in the last equation that $\delta_\omega = -\sum_{i=1}^n \beta_i \delta_i$, and $\delta_i = i[H^{\mathcal{R}_i}, \cdot]$. Note that the thermodynamic limit of the entropy production rate is well-defined.

One may relate the entropy production to the heat flux from the reservoirs. The heat flux from reservoir $\mathcal{R}_i, i = 1, \cdots, n$, at time t is

$$\begin{aligned} \Phi^{\mathcal{R}_i}(t) &= -\frac{d}{dt} \omega \circ \alpha_g^t(H^{\mathcal{R}_i}) && (7) \\ &= -i\omega(\alpha_g^t([V(t), H^{\mathcal{R}_i}])) && (8) \\ &= \omega \circ \alpha_g^t(\delta_i(V(t))). && (9) \end{aligned}$$

It follows that

$$\sum_{i=1}^{n} \beta_i \Phi^{\mathcal{R}_i}(t) = -Ep(\omega \circ \alpha_g^t), \qquad (10)$$

which, for $n = 1$ and reversible processes, is a familiar equation.

Next, we exhibit a connection to Carnot's formulation of the second law of thermodynamics. Consider a cyclic thermodynamic process in which Σ is coupled to two reservoirs, $\mathcal{R}_1$ and $\mathcal{R}_2$, at temperatures T_1 and T_2, respectively, with $T_1 > T_2$. Reservoir 1 acts as a heat source and reservoir 2 as a heat sink.

Assume that the perturbation $gV(t)$ is time periodic with period τ, and is norm-differentiable, for $t > 0$. For $s \in [0, \tau]$, define the time-periodic state[2] $\omega^+_{g,s}$ as

$$\omega^+_{g,s} := \lim_{n\to\infty} \omega \circ \alpha_g^{n\tau+s}. \tag{11}$$

Recall that the generator of the free dynamics of reservoir $\mathcal{R}_i$ is δ_i, $i = 1, 2$. Since $V(t)$ is norm differentiable, for $t > 0$, it follows that

$$\frac{d}{dt}\delta_i \Gamma^t_g = \delta_i \frac{d}{dt}\Gamma^t_g,$$

for $i = 1, 2$. Using the fact that δ_i, $i = 1, 2$, is a $*$-derivation which commutes with α^t and equation (3), it follows that

$$\frac{d}{dt}\omega(\Gamma^t_g \delta_i(\Gamma^t_g)) = i\omega \circ \alpha^t_g(\delta_i(gV(t))).$$

Therefore, the heat energy flowing from reservoir $\mathcal{R}_i$ into system Σ during the time interval $[0, t]$ is

$$Q_i(t) = -\int_0^t dt\omega \circ (\delta_i(gV(t))) \tag{12}$$

$$= -i\omega(\Gamma^t_g \delta_i(\Gamma^{t*}_g)) \ , \tag{13}$$

$i = 1, 2$. Moreover, integrating (10), we have

$$\beta_1 Q_1(t) + \beta_2 Q_2(t) = Ent(\omega \circ \alpha^t_g | \omega) \leq 0, \tag{14}$$

since the relative entropy $Ent(\omega \circ \alpha^t_g|\omega) \leq 0$.[3]

[2] A weaker definition of a time-periodic state is

$$\omega^+_{g,s} := \lim_{n\to\infty} \frac{1}{n\tau} \int_0^{n\tau} dt\omega \circ \alpha_g^{t+s}.$$

[3] This follows from a general trace inequality, see for example [BR].

If the system does work (ie, it is a *heat engine*), then the total heat energy released by the two reservoirs in a thermodynamic process is $Q(t) \geq 0$. Together with (14), and the fact that $\beta_1 < \beta_2$, it follows that $Q_1(t) \geq 0$.

Now define the heat flow per cycle from each reservoir $\mathcal{R}_i$ into Σ as

$$\Delta Q_i := \lim_{n\to\infty} [Q_i((n+1)\tau) - Q_i(n\tau)], i = 1, 2. \tag{15}$$

If this limit exists, then $\Delta Q_1 \geq 0$, since $Q_1(t) \geq 0$. Together with (14), this implies that the degree of efficiency

$$\begin{aligned} \eta &:= \frac{\Delta Q_1 + \Delta Q_2}{\Delta Q_1} \qquad (16) \\ &\leq \frac{T_1 - T_2}{T_1} =: \eta^{Carnot}, \qquad (17) \end{aligned}$$

which is nothing but Carnot's formulation of the second law of thermodynamics. In certain situations, one can show that

$$\lim_{n\to\infty} (Ent(\omega \circ \alpha_g^{(n+1)\tau}) - Ent(\omega \circ \alpha_g^{n\tau})) < 0,$$

which holds for the model we consider in this paper, and hence $\eta < \eta^{Carnot}$ if the heat engine does work, ie, if $(\Delta Q_1 + \Delta Q_2) > 0$. The following proposition states sufficient conditions for this to be the case.

Proposition 2.1 *For $s \in [0, \tau]$, suppose that*

(a) $\omega_{g,s}^+ \notin \mathcal{N}_\omega$, *and that*

(b) $\sup_{T\in\mathbf{R}^+} | \int_0^T dt\{\omega_{g,t\ mod\ \tau}^+(\delta_\omega(gV(t))) - \omega \circ \alpha_g^t(\delta_\omega(gV(t)))\}| < C$, *where C is a finite, nonnegative constant.*

Then $Ep(\omega_{g,s}^+) > 0$.

Proof. Suppose that $Ep(\omega_{g,s}^+) = 0$. Then

$$\begin{aligned} Ent(\omega \circ \alpha_g^t|\omega) &= -\int_0^t dt'\omega \circ \alpha_g^{t'}(\delta_\omega(gV(t'))) \\ &= -\int_0^t dt'\{\omega \circ \alpha_g^{t'}(\delta_\omega(gV(t'))) - \omega_{g,t'\ mod\ \tau}^+(\delta_\omega(gV(t')))\} \\ &\geq -C. \end{aligned}$$

In particular,

$$Ent(\omega^+_{g,s}|\omega) = \lim_{n\to\infty} Ent(\omega \circ \alpha_g^{n\tau+s}|\omega) \geq -C.$$

Let $\mathcal{M} = \pi_\omega(\mathcal{O})''$, the double commutant of $\pi_\omega(\mathcal{O})$, and let $\mathcal{M}_*$ be its predual. The set of all states $\gamma \in \mathcal{N}_\omega$ such that $Ent(\gamma|\omega) \geq -C$ is $\sigma(\mathcal{M}_*, \mathcal{M})$-compact (see [BR,D]). It follows that $\omega^+_{g,s} \in \mathcal{N}_\omega$, which contradicts assumption (a). $\square$

Next, we discuss a concrete model of a thermodynamic system to be studied subsequently.

3 The Model

As an example, we consider a two-level quantum system Σ coupled to n reservoirs $,\mathcal{R}_1, \cdots, \mathcal{R}_n, n \geq 2$, of free fermions in thermal equilibrium at inverse temperatures $\beta_1, \cdots, \beta_n$, and chemical potentials $\mu_1, \cdots, \mu_n$.

3.1 The small system

The kinematical algebra of Σ is $\mathcal{O}^\Sigma = \mathcal{M}(\mathbf{C}^2)$, the algebra of complex 2×2 matrices over the Hilbert space $\mathcal{H}^\Sigma = \mathbf{C}^2$. Its Hamiltonian is given by $H^\Sigma = \omega_0 \sigma_3$, where $\sigma_i, i = 1, 2, 3$, are the Pauli matrices. When the system Σ is not coupled to the reservoirs, its dynamics in the Heisenberg picture is given by

$$\alpha^t_\Sigma(a) := e^{iH^\Sigma t} a e^{-iH^\Sigma t} \ , \tag{18}$$

for $a \in \mathcal{O}^\Sigma$.

A physical state of the small system is described by a density matrix ρ_Σ. The operator $\kappa_\Sigma = \rho_\Sigma^{1/2}$ belongs to the space of Hilbert-Schmidt operators, which is isomorphic to $\mathcal{H}^\Sigma \otimes \mathcal{H}^\Sigma$. Two commuting representations of $\mathcal{O}^\Sigma$ on $\mathcal{H}^\Sigma \otimes \mathcal{H}^\Sigma$ are given by

$$\pi_\Sigma(a) := a \otimes \mathbf{1}^\Sigma \ , \tag{19}$$

$$\pi^\#_\Sigma(a) := \mathbf{1}^\Sigma \otimes C^\Sigma a C^\Sigma \ , \tag{20}$$

where C^Σ is an antiunitary involution on $\mathcal{H}^\Sigma$ corresponding to complex conjugation; (see for example [BFS]).

The generator of the free dynamics on the Hilbert space $\mathcal{H}^\Sigma \otimes \mathcal{H}^\Sigma$ is the standard Liouvillean

$$\mathcal{L}^\Sigma = H^\Sigma \otimes \mathbf{1}^\Sigma - \mathbf{1}^\Sigma \otimes H^\Sigma \, . \tag{21}$$

The spectrum of $\mathcal{L}^\Sigma$ is $\sigma(\mathcal{L}^\Sigma) = \{-2\omega_0, 0, 2\omega_0\}$, with double degeneracy at zero.

Let ω^Σ be the initial state of the small system Σ, with corresponding vector $\Omega^\Sigma \in \mathcal{H}^\Sigma \otimes \mathcal{H}^\Sigma$. The modular operator associated with ω_Σ is $\Delta^\Sigma = \omega_\Sigma \otimes \overline{\omega_\Sigma}^{-1}$, and the modular conjugation operator, J^Σ, is given by

$$J^\Sigma(\phi \otimes \psi) = \overline{\psi} \otimes \overline{\phi},$$

for $\phi, \psi \in \mathcal{H}^\Sigma$. If ω_Σ corresponds to the trace state, then $\Delta^\Sigma = \mathbf{1}^\Sigma \otimes \mathbf{1}^\Sigma$.

3.2 The reservoirs

Each thermal reservoir is formed of free fermions. It is infinitely extended and *dispersive*. We assume that the Hilbert space of a single fermion is $\mathbf{h} = L^2(\mathbf{R}^+, m(u)du; \mathcal{B})$, where $\mathcal{B}$ is an auxiliary Hilbert space, and $m(u)du$ is a measure on $\mathbf{R}^+$. We also assume that the single-fermion Hamiltonian, h, corresponds to the operator of multiplication by $u \in \mathbf{R}^+$. For instance, for reservoirs formed of nonrelativistic fermions in $\mathbf{R}^3$, the auxiliary Hilbert space $\mathcal{B}$ is $L^2(S^2, d\sigma)$, where S^2 is the unit sphere in $\mathbf{R}^3, d\sigma$ is the uniform measure on S^2, and $u = |\vec{k}|^2$, where $\vec{k} \in \mathbf{R}^3$ is the particle's momentum. In the latter case, the measure on $\mathbf{R}^+$ is choosen to be $m(u)du = \frac{1}{2}\sqrt{u}du$.

Let b and b^* be the annihilation-and creation operators on the Fermionic Fock space $\mathcal{F}(L^2(\mathbf{R}^+, \mathcal{B}))$. They satisfy the CAR

$$\{b^\#(f), b^\#(g)\} = 0 \, , \tag{22}$$

$$\{b(f), b^*(g)\} = (f, g)\mathbf{1} \, , \tag{23}$$

where $b^\#$ stands for b or b^*, $f, g \in L^2(\mathbf{R}^+; \mathcal{B})$, and $(\cdot, \cdot)$ denotes the scalar product in $L^2(\mathbf{R}^+; \mathcal{B})$. Moreover, let $\Omega^\mathcal{R}$ denote the vacuum state in $\mathcal{F}(L^2(\mathbf{R}^+; \mathcal{B}))$.

The kinematical algebra, $\mathcal{O}^{\mathcal{R}_i}$, of the i^{th} reservoir $\mathcal{R}_i, i = 1, \cdots, n$, is generated by $b_i^\#$ and the identity $\mathbf{1}^{\mathcal{R}_i}$. The free dynamics of each reservoir (before the systems are coupled) is given by

$$\alpha^t_{\mathcal{R}_i}(b_i^\#(f)) = b_i^\#(e^{itu} f) \, , \tag{24}$$

for $i = 1, \cdots, n, f \in L^2(\mathbf{R}^+; \mathcal{B})$. For a nonzero chemical potential, μ_i, of reservoir $\mathcal{R}_i$, an auxiliary free dynamics is generated by $\tilde{H}^{\mathcal{R}_i} = d\Gamma_i(h - \mu_i)$; see for example [BR].

The $(\alpha^t_{\mathcal{R}_i}, \beta_i, \mu_i)$-KMS state, $\omega^{\mathcal{R}_i}$, of each reservoir $\mathcal{R}_i, i = 1, \cdots, n$, at inverse temperature β_i and chemical potential μ_i, is the gauge invariant, quasi-free state uniquely determined by the two-point function

$$\omega^{\mathcal{R}_i}(b_i^*(f) b_i(f)) = (f, \rho_{\beta_i,\mu_i}(\cdot) f) \,, \tag{25}$$

where $\rho_{\beta_i,\mu_i}(u) := \frac{1}{e^{\beta_i(u-\mu_i)}+1}$.

Next, we introduce $\mathcal{F}_i^{AW} := \mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+; \mathcal{B})) \otimes \mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+; \mathcal{B}))$, the GNS Hilbert space for the Araki-Wyss representation of each fermionic reservoir $\mathcal{R}_i$ associated with the state $\omega^{\mathcal{R}_i}$, [ArWy]. Let $\tilde{b}_i$ and $\tilde{b}_i^*$ denote the annihilation- and creation operators on $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+; \mathcal{B}))$ satisfying the CAR, and denote by $\Omega^{\mathcal{R}_i}$ the vacuum state in $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+; \mathcal{B}))$, with $\tilde{b}_i \Omega^{\mathcal{R}_i} = 0$. The Araki-Wyss representation, π_i, of the kinematical algebra $\mathcal{O}^{\mathcal{R}_i}, i = 1, \cdots, n$, on $\mathcal{F}_i^{AW}$ is given by

$$\begin{aligned} \pi_i(b_i(f)) \;&:=\; \tilde{b}_i(\sqrt{1-\rho_{\beta_i,\mu_i}}\, f) \otimes \mathbf{1}^{\mathcal{R}_i} + (-1)^{N_i} \otimes \tilde{b}_i^*(\sqrt{\rho_{\beta_i,\mu_i}}\, \overline{f}) \,, \qquad (26) \\ \pi_i^{\#}(b_i(f)) \;&:=\; \tilde{b}_i^*(\sqrt{\rho_{\beta_i,\mu_i}} f)(-1)^{N_i} \otimes (-1)^{N_i} + \mathbf{1}^{\mathcal{R}_i} \otimes (-1)^{N_i} \tilde{b}_i(\sqrt{1-\rho_{\beta_i,\mu_i}}\, \overline{f}) \,, \end{aligned}$$

where $N_i = d\Gamma_i(1)$ is the particle number operator for reservoir $\mathcal{R}_i$. Furthermore, $\Omega^{\mathcal{R}_i} \otimes \Omega^{\mathcal{R}_i} \in \mathcal{F}_i^{AW}$ corresponds to the equilibrium KMS state $\omega^{\mathcal{R}_i}$ of reservoir $\mathcal{R}_i$.

The free dynamics on the GNS Hilbert space $\mathcal{F}_i^{AW}$ of each reservoir $\mathcal{R}_i$ is generated by the standard Liouvillean $\mathcal{L}^{\mathcal{R}_i}$. The modular operator associated with $(\mathcal{O}^{\mathcal{R}_i}, \omega^{\mathcal{R}_i})$ is given by

$$\Delta^{\mathcal{R}_i} = e^{-\beta_i \mathcal{L}^{\mathcal{R}_i}} \,,$$

and the modular conjugation is given by

$$J^{\mathcal{R}_i}(\Psi \otimes \Phi) = (-1)^{N_i(N_i-1)/2} \overline{\Phi} \otimes (-1)^{N_i(N_i-1)/2} \overline{\Psi},$$

for $\Psi, \Phi \in \mathcal{F}_i^{AW}$; (see, for example, [BR]).

In order to apply the complex translation method developed in [JP1,2,3], we map $\mathcal{F}_i^{AW} = \mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+; \mathcal{B})) \otimes \mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+; \mathcal{B}))$ to $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}; \mathcal{B}))$ using the isomorphism between $L^2(\mathbf{R}^+; \mathcal{B}) \oplus L^2(\mathbf{R}^+; \mathcal{B})$ and $L^2(\mathbf{R}; \mathcal{B})$. To every $f \in L^2(\mathbf{R}^+; \mathcal{B})$, we associate functions $f_{\beta,\mu}, f_{\beta,\mu}^{\#} \in L^2(\mathbf{R}; \mathcal{B})$ by setting

$$f_{\beta,\mu}(u, \sigma) := \begin{cases} \sqrt{m(u)} \sqrt{1 - \rho_{\beta,\mu}(u)} f(u, \sigma) \,, & u \geq 0 \\ \sqrt{m(-u)} \sqrt{\rho_{\beta,\mu}(-u)}\, \overline{f}(-u, \sigma) \,, & u < 0 \end{cases} \,, \tag{27}$$

and

$$
\begin{aligned}
f^{\#}_{\beta,\mu}(u,\sigma) &:= \begin{cases} \sqrt{m(u)}i\sqrt{\rho_{\beta,\mu}(u)}f(u,\sigma), u \geq 0 \\ \sqrt{m(-u)}i\sqrt{1-\rho_{\beta,\mu}(-u)}\;\overline{f}(-u,\sigma), u < 0 \end{cases} \\
&= i\overline{f}_{\beta,\mu}(-u,\sigma),
\end{aligned} \tag{28}
$$

where $m(u)du$ is the measure on $\mathbf{R}^+$, see eq. (26). (For a discussion of this map, see Appendix A.)

Let a_i and a_i^* be the annihilation and creation operators on $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}, du; \mathcal{B}))$. Then

$$\pi_i(b_i^{\#}(f)) \to a_i^{\#}(f_{\beta_i,\mu_i}), \tag{29}$$

$$\pi_i^{\#}(b_i^{\#}(f)) \to i(-1)^{N_i} a_i^{\#}(f^{\#}_{\beta_i,\mu_i}); \tag{30}$$

$$\Omega^{\mathcal{R}_i} \otimes \Omega^{\mathcal{R}_i} \to \tilde{\Omega}^{\mathcal{R}_i}, \tag{31}$$

where $a_i^{\#}$ stands for a_i or a_i^*, and $\tilde{\Omega}^{\mathcal{R}_i}$ is the vacuum state in $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}, \mathcal{B}))$. Using eqs. (25) and (26), one readily verifies that

$$
\begin{aligned}
\langle \tilde{\Omega}^{\mathcal{R}_i}, a_i^*(f_{\beta_i,\mu_i}) a_i(f_{\beta_i,\mu_i}) \tilde{\Omega}^{\mathcal{R}_i} \rangle &= \langle \tilde{\Omega}^{\mathcal{R}_i}, a_i^*(f^{\#}_{\beta_i,\mu_i}) a_i(f^{\#}_{\beta_i,\mu_i}) \tilde{\Omega}^{\mathcal{R}_i} \rangle \\
&= \omega^{\mathcal{R}_i}(b_i^*(f) b_i(f)) \\
&= (f, \rho_{\beta_i,\mu_i}(\cdot) f) \,.
\end{aligned}
$$

Moreover, the free Liouvillean on $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}; \mathcal{B}))$ for the reservoir $\mathcal{R}_i$ is mapped to

$$\mathcal{L}^{\mathcal{R}_i} = d\Gamma_i(u_i) \,, \tag{32}$$

where $u_i \in \mathbf{R}$.

3.3 The coupled system

The kinematical algebra of the total system, $\Sigma \vee \mathcal{R}_1 \vee \cdots \vee \mathcal{R}_n$, is given by

$$\mathcal{O} = \mathcal{O}^{\Sigma} \otimes \mathcal{O}^{\mathcal{R}_1} \otimes \cdots \otimes \mathcal{O}^{\mathcal{R}_n} \,, \tag{33}$$

and the Heisenberg-picture dynamics of the uncoupled system is given by

$$\alpha_0^t = \alpha_{\Sigma}^t \otimes \alpha_{\mathcal{R}_1}^t \otimes \cdots \otimes \alpha_{\mathcal{R}_n}^t \,. \tag{34}$$

The representation of $\mathcal{O}$ on $\mathcal{H} := \mathcal{H}^\Sigma \otimes \mathcal{H}^\Sigma \otimes \mathcal{F}^{\mathcal{R}_1}(L^2(\mathbf{R};\mathcal{B})) \otimes \cdots \otimes \mathcal{F}^{\mathcal{R}_n}(L^2(\mathbf{R};\mathcal{B}))$, determined by the initial state

$$\omega = \omega^\Sigma \otimes \omega^{\mathcal{R}_1} \otimes \cdots \otimes \omega^{\mathcal{R}_n} \tag{35}$$

by the GNS construction, is given by

$$\pi = \pi_\Sigma \otimes \pi_{\beta_1} \otimes \cdots \otimes \pi_{\beta_n}, \tag{36}$$

and an anti-representation commuting with π by

$$\pi^\# = \pi^\#_\Sigma \otimes \pi^\#_{\beta_1} \otimes \cdots \otimes \pi^\#_{\beta_n} \ . \tag{37}$$

Moreover, let $\Omega := \Omega^\Sigma \otimes \tilde{\Omega}^{\mathcal{R}}_1 \otimes \cdots \otimes \tilde{\Omega}^{\mathcal{R}_n}$ denote the vector in $\mathcal{H}$ corresponding to the state ω. Denote the double commutant of $\pi(\mathcal{O})$ by $\mathcal{M} := \pi(\mathcal{O})''$, which is the smallest von Neumann algebra containing $\pi(\mathcal{O})$.

For $a \in \mathcal{O}$, we abreviate $\pi(a)$ by a whenever there is no danger of confusion. The modular operator of the total system is

$$\Delta = \Delta^\Sigma \otimes \Delta^{\mathcal{R}_1} \otimes \cdots \otimes \Delta^{\mathcal{R}_n},$$

and the modular conjugation is

$$J = J^\Sigma \otimes J^{\mathcal{R}_1} \otimes \cdots \otimes J^{\mathcal{R}_n}.$$

According to Tomita-Takesaki theory,

$$J\mathcal{M}J = \mathcal{M}' \ , \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \ ,$$

for $t \in \mathbf{R}$; (see for example [BR]). Furthermore, for $a \in \mathcal{M}$,

$$J\Delta^{1/2}a\Omega = a^*\Omega. \tag{38}$$

The Liouvillean of the total uncoupled system is given by

$$\mathcal{L}_0 = \mathcal{L}^\Sigma + \sum_{i=1}^n \mathcal{L}^{\mathcal{R}_i} \ . \tag{39}$$

This defines a selfadjoint operator on $\mathcal{H}$.

The system Σ is coupled to the reservoirs $\mathcal{R}_1, \cdots, \mathcal{R}_n$, through an interaction $gV(t)$, where $V(t) \in \mathcal{O}$ is given by

$$V(t) = \sum_{i=1}^{n} \{\sigma_- \otimes b_i^*(f_i(t)) + \sigma_+ \otimes b_i(f_i(t))\} \ , \tag{40}$$

where $\sigma_\pm = \sigma_1 \pm i\sigma_2$, and $f_i \in L^2(\mathbf{R}^+; \mathcal{B}), i = 1, \cdots, n$, are form factors.[4]

The *standard* Liouvillean of the interacting system acting on the GNS Hilbert space $\mathcal{H}$ is given by

$$\mathcal{L}_g(t) = \mathcal{L}_0 + gI(t) \ , \tag{41}$$

where the unperturbed Liouvillean is defined in (39), and the interaction Liouvillean determined by the operator $V(t)$ is given by

$$\begin{aligned} I(t) &= \{V(t) - JV(t)J\} \\ &= \sum_{i=1}^{n} \{\sigma_- \otimes \mathbf{1}^\Sigma \otimes a_i^*(f_{i,\beta_i,\mu_i}(t)) + \sigma_+ \otimes \mathbf{1}^\Sigma \otimes a_i(f_{i,\beta_i,\mu_i}(t)) \\ &- i\mathbf{1}^\Sigma \otimes \sigma_- \otimes (-1)^{N_i} a_i^*(f^\#_{i,\beta_i,\mu_i}(t)) - i\mathbf{1}^\Sigma \otimes \sigma_+ \otimes (-1)^{N_i} a_i(f^\#_{i,\beta_i,\mu_i}(t))\}, \end{aligned} \tag{42}$$

where a_i, a_i^* are the annihilation and creation operators on the fermionic Fock space $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}; \mathcal{B}))$. Note that since the perturbation is bounded, the domain of $\mathcal{L}_g(t)$ is $\mathcal{D}(\mathcal{L}_g(t)) = \mathcal{D}(\mathcal{L}_0)$.

Let $\overline{U}_g$ be the propagator generated by the standard Liouvillean. It satisfies

$$\partial_t \overline{U}_g(t) = -i\mathcal{L}_g(t)\overline{U}_g(t) \ ; \overline{U}_g(0) = 1 \ . \tag{43}$$

The Heisenberg-picture evolution is given by

$$\alpha_g^t(a) = \overline{U}_g^*(t) a \overline{U}_g(t) \ , \tag{44}$$

for $a \in \mathcal{O}$.

Generally, the kernel of $\mathcal{L}_g(t), Ker\ \mathcal{L}_g$, is expected to be empty when at least two of the reservoirs have different temperatures; (see section 6)[5].

[4] Note that this form of an interaction preserves the total number of fermions in $\Sigma \vee \mathcal{R}_1 \vee \cdots \vee \mathcal{R}_n$, as required by gauge invariance of the first kind.

[5] This is consistent with the fact that the coupled system is not expected to possess the property of return to equilibrium if the reservoirs have different temperatures (or chemical potentials). One can verify that, indeed, this is the case when assumptions (A1)-(A3), below, are satisfied.

This motivates introducing the so called C-Liouvillean, L_g , which generates dynamics on a Banach space contained in $\mathcal{H}$ (isomorphic to $\mathcal{O}$) and which, *by construction*, has a non-trivial kernel.

Consider the Banach space

$$\mathcal{C}(\mathcal{O},\Omega) := \{a\Omega : a \in \mathcal{O}\},$$

with norm $\|a\Omega\|_\infty = \|a\|$. Since Ω is separating for $\mathcal{O}$, the norm $\|a\Omega\|_\infty$ is well-defined, and since Ω is cyclic for $\mathcal{O}$, $\mathcal{C}(\mathcal{O},\Omega)$ is dense in $\mathcal{H}$.

We set $\phi(a) = a\Omega$, and define a propagator $U_g(t,t')$ by

$$\phi(\alpha_g^{t,t'}(a)) = U_g(t,t')a\Omega. \tag{45}$$

Then

$$\partial_t U_g(t,t') = iL_g(t)U_g(t,t') \text{ with } U_g(t,t) = 1, \tag{46}$$

and

$$U_g(t',t)\Omega = \Omega. \tag{47}$$

Differentiating (45) with respect to t, setting $t = t'$, and using (46), (44) and (38), one obtains

$$\begin{aligned}
[(\mathcal{L}_0 + gV(t))a - a(\mathcal{L}_0 + gV(t))]\Omega &= [(\mathcal{L}_0 + gV(t))a - (V(t)a^*)^*]\Omega \\
&= (\mathcal{L}_0 + gV(t) - gJ\Delta^{1/2}V(t)\Delta^{-1/2}J)a\Omega \\
&\equiv L_g(t)a\Omega\ .
\end{aligned}$$

Hence, the C-Liouvillean is given by

$$L_g(t) := \mathcal{L}_0 + gV(t) - gJ\Delta^{1/2}V(t)\Delta^{-1/2}J\ . \tag{48}$$

Note that, *by construction*,

$$L_g(t)\Omega = 0,$$

for all $t \in \mathbf{R}$.

Next, we discuss the assumptions on the interaction. For $\delta > 0$, we define the strips in the complex plane

$$I(\delta) := \{z \in \mathbf{C} : |Imz| < \delta\}$$

and

$$I^-(\delta) := \{z \in \mathbf{C} : -\delta < Imz < 0\}. \tag{49}$$

Moreover, for every function $f \in L^2(\mathbf{R}^+;\mathcal{B})$, we define a function $\tilde{f}$ by setting

$$\tilde{f} := \begin{cases} \sqrt{m(u)} f(u,\sigma), u \geq 0 \\ \sqrt{m(|u|)}\ \overline{f}(|u|,\sigma), u < 0 \end{cases}, \tag{50}$$

where $m(u)du$ is the measure on $\mathbf{R}^+$. Denote by $H^2(\delta,\mathcal{B})$ the Hardy class of analytic functions

$$g : I(\delta) \to \mathcal{B},$$

with

$$\|g\|_{H^2(\delta,\mathcal{B})} := \sup_{|\theta|<\delta} \int_{\mathbf{R}} \|g(u+i\theta)\|_{\mathcal{B}}^2 du < \infty.$$

We require the following basic assumptions on the interaction term.

(A1) *Periodicity.*

The interaction term $V(t)$ is periodic with period $\tau < \infty$: $V(t) = V(t+\tau)$.

(A2) *Regularity of the form factors.*

Assume that $\exists \delta > 0$, independent of t and $i \in \{1, \cdots, n\}$, such that

$$e^{-\beta_i u_i/2} \tilde{f}_i(u,t) \in H^2(\delta,\mathcal{B}),$$

where f_i is as in eq. (40) and $\tilde{f}_i$ is defined in (50).

(A3) *Fermi Golden Rule.*

We assume that

$$\sum_{i=1}^{n} \|\tilde{f}_i(2\omega_0,t)\|_{\mathcal{B}}^2 > 0,$$

which means that the coupling of Σ to the reservoirs is non-vanishing in second order perturbation theory.[6]

---

[6]For instance, when the reservoirs are formed of nonrelativistic fermions in $\mathbf{R}^3$, an example of a form factor satisfying assumptions (A1)-(A3) is given by

$$f_i(u_i,t) = h_i(t) e^{-|u_i|^2},$$

where $h_i(t)$ is a bounded, periodic function of $t \in \mathbf{R}$.

Let $\hat{f}$ be the Fourier transform of $\tilde{f}$ given by

$$\hat{f}_m(u,\omega) = \frac{1}{\tau}\int_0^\tau dt e^{-i\omega t}\tilde{f}(u,t), \tag{51}$$

where $\omega = \frac{2\pi}{\tau}$, and τ is the period of the interaction term (see assumption (A1)). Then $\tilde{f}(u,t) = \sum_{m\in\mathbf{Z}} e^{i\omega t}\hat{f}_m(u,\omega)$. It follows from (A2) and Parseval's theorem that

$$\sum_{m\in\mathbf{Z}} \|\hat{f}_m(u+m\omega,\omega)\|_{\mathcal{B}}^2 < \infty, \tag{52}$$

for $u \in \mathbf{R}$.

Let $\tilde{U}_g$ be the propagator generated by the adjoint of the C-Liouvillean, ie,

$$\begin{aligned} \partial_t\tilde{U}_g(t,t') &= -i\tilde{U}_g(t,t')L_g^*(t), && (53)\\ \tilde{U}_g(t,t) &= 1. && (54) \end{aligned}$$

Assumption (A2) implies that the perturbation is bounded, and hence the domain of $L_g^\#$, where $L_g^\#$ stands for L_g or L_g^*, is

$$\mathcal{D}(L_g^\#) = \mathcal{D}(\mathcal{L}_0),$$

and $U_g, \tilde{U}_g$ are bounded and strongly continuous in t and t'.

4 The Floquet Liouvillean

In this section, we extend Floquet theory for periodically driven quantum systems at zero temperature to a theory for systems at positive temperatures. The goal is to investigate whether the true state of the coupled system converges to a time-periodic state. We introduce the Floquet Liouvillean, which generates the dynamics on a suitable Banach space, and we show in the following section that the time-periodic state to which the true state of the system converges after very many periods is related to a *zero-energy resonance* of the Floquet Liouvillean.

We consider the extended Hilbert space $\tilde{\mathcal{H}} := L^2([0,\tau]) \otimes \mathcal{H}$, where τ is the period of the perturbation appearing in (40) and (A1), and we introduce the Floquet Liouvillean

$$K_g^* := -i\partial_t \otimes \mathbf{1} + \mathbf{1} \otimes L_g^*(t), \tag{55}$$

with periodic boundary conditions in t. Note that, under assumption (A2), K_g^* is a closed operator with domain $\mathcal{D} = \mathcal{D}(i\partial_t \otimes \mathbf{1}) \cap \mathcal{D}(\mathbf{1} \otimes \mathcal{L}_0)$.

By Fourier transformation, $\tilde{\mathcal{H}}$ is isomorphic to

$$\bigoplus_{n\in\mathbf{Z}} \langle e^{in\omega t}\rangle \otimes \mathcal{H} = \bigoplus_{n\in\mathbf{Z}} \mathbf{h}^{(n)} \otimes \mathcal{H},$$

where $\mathbf{h}^{(n)} := \langle e^{in\omega t}\rangle$ and $\omega = \frac{2\pi}{\tau}$.

According to Floquet theory of quantum mechanical systems driven by periodic perturbation [Ho,Ya1,Ya2], the semi-group generated by K_g^* is given by

$$(e^{-i\sigma K_g^*} f)(t) = \tilde{U}_g(t, t-\sigma) f(t-\sigma) \; , \tag{56}$$

where $f \in \tilde{\mathcal{H}}$ and $\sigma \in \mathbf{R}$. Relation (56) can be seen by differentiating both sides with respect to σ and setting $\sigma = 0$ (see [Ho]). (Alternatively, one may use the Trotter product formula, [RS1].)

Note that if

$$K_g^* \phi = \lambda \phi \; , \tag{57}$$

for $\phi \equiv \phi(t) \in \tilde{\mathcal{H}}$ and $\lambda \in \mathbf{C}$, then $\phi(t)$ satisfies

$$\tilde{U}_g(t,0)\phi(0) = e^{-i\lambda t}\phi(t) \; . \tag{58}$$

Conversely, if

$$\tilde{U}_g(\tau,0)\phi(0) = e^{-i\lambda \tau}\phi_0 \; , \tag{59}$$

then

$$\phi(t) = e^{i\lambda t}\tilde{U}_g(t,0)\phi_0 \tag{60}$$

is an eigenfunction of K_g^* with eigenvalue λ.

We now study the spectrum of K_g^* using complex spectral deformation techniques as developed in [HP,JP1,2,3]; (see also [A-S]).

Let $\mathbf{u}_i$ be the unitary transformation generating translations in energy for the i^{th} reservoir, $i = 1, \cdots, n$. More precisely, for $f_i \in L^2(\mathbf{R}; \mathcal{B})$,

$$\mathbf{u}_i(\theta) f_i(u) = f_i^{\theta}(u) = f_i(u+\theta).$$

Moreover, let

$$U_i(\theta) := \Gamma_i(\mathbf{u}_i(\theta))$$

denote the second quantization of $\mathbf{u}_i(\theta)$.

Explicitly, $U_i(\theta) = e^{i\theta A_i}$, where $A_i := id\Gamma_i(\partial_{u_i})$ is the second quantization of the generator of energy translations for the i^{th}reservoir, $i = 1, \cdots, n$. We set

$$U(\theta) := \mathbf{1}^\Sigma \otimes \mathbf{1}^\Sigma \otimes U_1(\theta) \otimes \cdots \otimes U_n(\theta) \ . \tag{61}$$

Let

$$K_g^*(\theta) := U(\theta) K_g^* U(-\theta) \tag{62}$$

$$= -i\partial_t + L_g^*(\theta, t) \ , \tag{63}$$

where $L_g^*(t, \theta)$ is given by

$$L_g^*(t, \theta) := U(\theta) L_g^*(t) U(-\theta) = \mathcal{L}_0 + N\theta + g\tilde{V}^{tot}(t, \theta) \ , \tag{64}$$

$\mathcal{L}_0 = \mathcal{L}^\Sigma + \sum_i \mathcal{L}^{\mathcal{R}_i}$, $\mathcal{L}^{\mathcal{R}_i} = d\Gamma(u_i), i = 1, \cdots, n$, and

$$\begin{aligned}\tilde{V}^{tot}(t, \theta) = \sum_i \{ &\sigma_+ \otimes \mathbf{1}^\Sigma \otimes a_i(f^{(\theta)}_{i,\beta_i,\mu_i}(t)) + \sigma_- \otimes \mathbf{1}^\Sigma \otimes a_i^*(f^{(\theta)}_{i,\beta_i,\mu_i}(t)) \\ &- i\mathbf{1}^\Sigma \otimes \sigma_- \otimes (-1)^{N_i} (a_i(e^{\beta_i(u_i - \mu_i)/2} f^{\#(\theta)}_{i,\beta_i,\mu_i}(t)) \\ &- i\mathbf{1}^\Sigma \otimes \sigma_+ \otimes (-1)^{N_i} a_i^*(e^{-\beta_i(u_i - \mu_i)/2} f^{\#(\theta)}_{i,\beta_i,\mu_i}(t)) \} \ .\end{aligned}$$

It follows from assuption (A2) that, for $\theta \in I(\delta)$, $\tilde{V}_g^{tot}(t, \theta)$ is a bounded operator. Hence $K_g^*(t, \theta)$ is well-defined and closed on the domain $\mathcal{D} := \mathcal{D}(i\partial_t) \cap \mathcal{D}(N) \cap \mathcal{D}(\mathcal{L}^{\mathcal{R}_1}) \cap \cdots \cap \mathcal{D}(\mathcal{L}^{\mathcal{R}_n})$. When the coupling $g = 0$, the pure point spectrum of $\mathcal{L}_0$ is $\sigma_{pp}(\mathcal{L}_0) = \{-2\omega_0, 0, 2\omega_0\}$, with double degeneracy at 0, and the continuous spectrum of $\mathcal{L}_0$ is $\sigma_{cont}(\mathcal{L}_0) = \mathbf{R}$. It follows that

$$\sigma_{pp}(K_0) = \{E_j^{(k)}(g = 0) = E_j + k\omega : j = 0, \cdots, 3, k \in \mathbf{Z}\}, \tag{65}$$

where $E_{0,1} = 0, E_2 = -2\omega_0$, and $E_3 = 2\omega_0$, and $\sigma_{cont}(K_0) = \mathbf{R}$. Let

$$K^\Sigma := -i\partial_t + \mathcal{L}^\Sigma.$$

Clearly, $\sigma(K^\Sigma) = \sigma_{pp}(K_0)$. We have the following two easy lemmas.

Lemma 4.1 *For $\theta \in \mathbf{C}$, the following holds.*

(i) For any $\psi \in \mathcal{D}$, one has

$$\|K_0(\theta)\psi\|^2 = \|K_0(Re\theta)\psi\|^2 + |Im\theta|^2 \|N\psi\|^2 \ . \tag{66}$$

(ii) If $Im\theta \neq 0$, then $K_0(\theta)$ is a normal operator satisfying

$$K_0(\theta)^* = K_0(\overline{\theta}) \ , \tag{67}$$

and $\mathcal{D}(K_0(\theta)) = \mathcal{D}$.

(iii) The spectrum of $K_0(\theta)$ is

$$\sigma_{cont}(K_0(\theta)) = \{n\theta + s : n \in \mathbf{N}\backslash\{0\} \text{ and } s \in \mathbf{R}\}, \tag{68}$$

$$\sigma_{pp}(K_0(\theta)) = \{k\omega + E_j : j = 0, \cdots, 3, k \in \mathbf{Z}\}, \tag{69}$$

where $E_{0,1} = 0, E_2 = -2\omega_0$ and $E_3 = 2\omega_0$, (the eigenvalues of $\mathcal{L}^{\Sigma}$), and $\omega = \frac{2\pi}{\tau}$.

Proof. The first claim follows directly by looking at the sector where $N = n\mathbf{1}$, since $K_0(\theta)$ restricted to this sector is reduced to

$$K_0^{(n)}(\theta) = K^{\Sigma} + s_1 + \cdots + s_n + n\theta \ , \tag{70}$$

where $s_1, \cdots, s_n$ are interpreted as one-particle multiplication operators. For $Im\theta \neq 0$, it also follows from (70) that

$$\mathcal{D} = \{\psi = \{\psi^{(n)}\} : \psi^{(n)} \in \mathcal{D}(K_0^{(n)}(\theta)) \text{ and } \sum_n \|K_0^{(n)}(\theta)\psi^{(n)}\|^2 < \infty\},$$

and hence $K_0(\theta)$ is a closed normal operator on $\mathcal{D}$. Claims (ii) and (iii) follow from the corresponding statements on $K_0^{(n)}(\theta)$. $\square$

Lemma 4.2

Suppose (A1)-(A3) hold, and assume that $(g, \theta) \in \mathbf{C} \times I^-(\delta)$. Then the following holds.

(i) $\mathcal{D}(K_g^*(\theta)) = \mathcal{D}$ and $(K_g^*(\theta))^* = K_{\overline{g}}(\overline{\theta})$.

(ii) The map $(g, \theta) \to K_g^*(\theta)$ from $\mathbf{C} \times I^-(\delta)$ to the set of closed operators on $\tilde{\mathcal{H}}$ is an analytic family (of type A) in each variable separately; (see [Ka1], chapter V, section 3.2).

(iii) For $g \in \mathbf{R}$ and Imz large enough,

$$s - \lim_{Im\theta \uparrow 0} (K_g^*(\theta) - z)^{-1} = (K_g^*(Re\theta) - z)^{-1} \ . \tag{71}$$

Proof. The first claim (i) follows from the fact that $g\tilde{V}^{tot}(t,\theta)$ is bounded for $\theta \in I(\delta)$. It also follows from assumption (A2) that $(g,\theta) \to K_g^*(\theta)$ is analytic in $\theta \in I^-(\delta)$. Analyticity in g is obvious from (63). We still need to prove claim (iii). Without loss of generality, assume that $Re\theta = 0$. It follows from assumption (A2) that the resolvent formula

$$(K_g^*(\theta) - z)^{-1} = (K_0^*(\theta) - z)^{-1}(1 + g\tilde{V}^{tot}(\cdot,\theta)(K_0^*(\theta) - z)^{-1})^{-1} \ , \tag{72}$$

holds for small g, as long as z belongs to the half-plane $\{z \in \mathbf{C} : 0 < c < Imz\}$. Since $(K_0^*(\theta) - z)^{-1}$ is uniformly bounded as $Im\theta \uparrow 0$ for $g \in \mathbf{R}$ and Imz large enough, and $\tilde{V}^{tot}(\theta)$ is bounded and analytic in θ, claim (iii) follows from the Neumann series expansion of the resolvent of $K_g^*(\theta)$. □

Next, we apply degenerate perturbation theory, as developed in [HP], to compute the spectrum of $K_g^*(\theta)$. Using contour integration, one may define the projection onto the perturbed eigenstates of $K_g^*(\theta)$, for $Im\theta \in I^-(\delta)$. Let

$$\tilde{P}_{g,(k)}(\theta) := \oint_{\gamma_k} \frac{dz}{2\pi i}(z - K_g^*(\theta))^{-1} \ , \tag{73}$$

where γ_k is a contour that encloses the eigenvalues $E_j^{(k)}(g=0), j = 0, \cdots, 3, k \in \mathbf{Z}$, at a distance $d > 0$, such that, for sufficiently small $|g|$ (to be specified below) the contour also encloses $E_j^{(k)}(g)$.

Moreover, let $\tilde{T}_{g,(k)} := \tilde{P}_{0,(k)}\tilde{P}_{g,(k)}(\theta)\tilde{P}_{0,(k)}$.[7] We show in Theorem 4.3 that the isomorphism

$$\tilde{S}_{g,(k)}(\theta) := \tilde{T}_{g,(k)}^{-1/2}\tilde{P}_{0,(k)}\tilde{P}_{g,(k)}(\theta) : Ran(\tilde{P}_{g,(k)}(\theta)) \to \mathbf{h}^{(k)} \otimes \mathcal{H}^\Sigma \otimes \mathcal{H}^\Sigma \tag{74}$$

has an inverse

$$\tilde{S}_{g,(k)}^{-1}(\theta) := \tilde{P}_{g,(k)}(\theta)\tilde{P}_{0,(k)}\tilde{T}_{g,(k)}^{-1/2}(t) : \mathbf{h}^{(k)} \otimes \mathcal{H}^\Sigma \otimes \mathcal{H}^\Sigma \to Ran(\tilde{P}_{g,(k)}(\theta)). \tag{75}$$

We set

$$\tilde{M}_{g,(k)}(t) := \tilde{P}_{0,(k)}\tilde{P}_{g,(k)}(\theta)K_g^*(\theta)\tilde{P}_{g,(k)}(\theta)\tilde{P}_{0,(k)} \ , \tag{76}$$

and define the quasi-Floquet Liouvillean by

$$\tilde{\Sigma}_{g,(k)} := \tilde{S}_{g,(k)}(\theta)\tilde{P}_{g,(k)}(\theta)K_g^*(\theta)\tilde{P}_{g,(k)}(\theta)\tilde{S}_{g,(k)}^{-1}(\theta) = \tilde{T}_{g,(k)}^{-1/2}\tilde{M}_{g,(k)}\tilde{T}_{g,(k)}^{-1/2} \ . \tag{77}$$

[7]Note that $K_0^*(\theta)$ is a normal operator, and hence $\tilde{P}_{0,(k)}$ is well-defined by the spectral theorem.

Let $k = min\{\delta, \frac{\pi}{\beta_1}, \cdots, \frac{\pi}{\beta_n}\}$, where δ appears in assumption (A2), section 3, and $\beta_1, \cdots, \beta_n$, are the inverse temperatures of the reservoirs $\mathcal{R}_1, \cdots, \mathcal{R}_n$, respectively. For $\theta \in I^-(k)$ (see (49)), we choose a parameter ν such that

$$-k < \nu < 0 \ \ and \ -k < Im\theta < -\frac{k+|\nu|}{2}. \tag{78}$$

We also choose a constant $g_1 > 0$ such that

$$g_1 C < (k - |\nu|)/2, \tag{79}$$

where

$$C := \sup_{\theta \in I(\delta), t \in \mathbf{R}} \|\tilde{V}^{tot}(t, \theta)\| < \infty. \tag{80}$$

Theorem 4.3

Suppose that assumptions (A1)-(A3) hold. Then, for $g_1 > 0$ satisfying (79), $\theta \in I^-(k)$ and ν satisfying (78), the following holds.

(i) If $|g| < g_1$, the essential spectrum of the operator $K_g^*(\theta)$ is contained in the half-plane $\mathbf{C} \backslash \Xi(\nu)$, where $\Xi(\nu) := \{z \in \mathbf{C} : Imz \geq \nu\}$. Moreover, the discrete spectrum of $K_g^*(\theta)$ is independent of $\theta \in I^-(k)$. If $|g| < \frac{1}{2} g_1$, then the spectral projections $\tilde{P}_{g,(k)}(\theta), k \in \mathbf{Z}$, associated to the spectrum of $K_g^*(\theta)$ in the half-plane $\Xi(\nu)$, are analytic in g and satisfy the estimate

$$\|\tilde{P}_{g,(k)}(\theta) - \tilde{P}_{0,(k)}\| < 1 \ . \tag{81}$$

(ii) If $|g| < \frac{g_1}{2}$, then the quasi-Floquet Liouvillean $\tilde{\Sigma}_{g,(k)}$ defined in (77) depends analytically on g, and has a Taylor expansion

$$\tilde{\Sigma}_{g,(k)} = K_{(k)}^{\Sigma} + \sum_{j=1}^{\infty} g^{2j} \tilde{\Sigma}_{(k)}^{(2j)} \tag{82}$$

where

$$K_{(k)}^{\Sigma} := k\omega + \mathcal{L}^{\Sigma} \ , k \in \mathbf{Z}. \tag{83}$$

The first non-trivial coefficient in (82) is

$$\tilde{\Sigma}_{(k)}^{(2)} = \frac{1}{2} \oint_{\gamma} \frac{dz}{2\pi i} (\xi_{(k)}(z)(z - K_{(k)}^{\Sigma})^{-1} + (z - K_{(k)}^{\Sigma})^{-1} \xi_{(k)}(z)) \ ,$$

where $\xi_{(k)}(z) = \tilde{P}_{0,(k)} \tilde{V}^{tot}(\theta)(z - K_0(\theta))^{-1} \tilde{V}^{tot}(\theta) \tilde{P}_{0,(k)}$.

Proof. (i) The resolvent formula

$$(K_g^*(\theta)-z)^{-1}=(K_0^*(\theta)-z)^{-1}(1+g\tilde{V}^{tot}(\cdot,\theta)(K_0^*(\theta)-z)^{-1})^{-1}\ , \tag{84}$$

holds for small g and z in the half-plane $\{z\in\mathbf{C}:0<c<Imz\}$. We extend the domain of validity of (84) by refining the estimate on $g\tilde{V}^{tot}(t,\theta)(K_0^*(\theta)-z)^{-1}$.

Note that

$$\begin{aligned}\|g\tilde{V}^{tot}(\cdot,\theta)(K_0^*(\theta)-z)^{-1}\| &\le |g|C\|(K_0(\theta)-z)^{-1}\|\\ &\le |g|C\frac{1}{dist(z,\eta(K_0^*(\theta)))},\end{aligned}$$

where C is given by (80) and $\eta(K_0^*(\theta))$ is the closure of the numerical range of K_0^*. Fix g_1 such that it satisfies (79), and choose ϵ such that $\epsilon>\frac{k-|\nu|}{2}>0$. Let

$$G(\nu,\epsilon):=\{z\in\mathbf{C}:Imz>\nu;dist(z,\eta(K_0^*(\theta))>\epsilon\}.$$

Then

$$\sup_{z\in G(\nu,\epsilon)}\|g\tilde{V}^{tot}(\cdot,\theta)(K_0^*(\theta)-z)^{-1}\|\le\frac{|g|}{g_1}.$$

If $|g|<g_1$, the resolvent formula (84) holds on $G(\nu,\epsilon)$, and, for $m\ge 1$,

$$\sup_{z\in G(\nu,\epsilon)}\|(z-K_g^*(\theta))^{-1}-\sum_{j=0}^{m-1}(z-K_0^*(\theta))^{-1}(g\tilde{V}^{tot}(\cdot,\theta)(z-K_0^*(\theta))^{-1})^j\|\le\frac{(\frac{|g|}{g_1})^m}{1-\frac{|g|}{g_1}}. \tag{85}$$

It follows that

$$\bigcup_{\epsilon>\frac{k-|\nu|}{2}}G(\nu,\epsilon)\subset\rho(K_g^*(\theta))\ , \tag{86}$$

where $\rho(K_g^*(\theta))$ is the resolvent set of $K_g^*(\theta)$. Moreover, setting $m=1$ in (85), it follows that, for $|g|<g_1/2$,

$$\|\tilde{P}_{g,(k)}(\theta)-\tilde{P}_{0,(k)}\|<1,$$

and hence $\tilde{P}_{g,(k)}$ is analytic in g. We still need to prove the independence of $\sigma_{pp}(K_g^*(\theta))$ of $\theta\in I^-(k)$.

Fix $(g_0,\theta_0)\in\mathbf{C}\times I^-(\delta)$ such that $|g_0|<g_1$. The discrete eigenvalues of $K_{g_0}^*(\theta)$ are analytic functions with at most algebraic singularities in the

neighbourhood of θ_0, since $K^*_{g_0}(\theta)$ is analytic in θ. Moreover, since $K^*_{g_0}(\theta_0)$ and $K^*_{g_0}(\theta)$ are unitarily equivalent if $(\theta-\theta_0)\in\mathbf{R}$, it follows that the pure point spectrum of $K^*_{g_0}(\theta)$ is independent of θ.
(ii) Analyticity of $\tilde{T}_{g,(k)}$ directly follows from (i) and the definition of $\tilde{T}_{g,(k)}$. Since $\|\tilde{T}_{g,(k)}-1\|<1$ for $|g|<g_1/2$, $\tilde{T}^{-1/2}_{g,(k)}$ is also analytic in g. Inserting the Neumann series for the resolvent of $K^*_g(\theta)$, gives

$$\tilde{T}_{g,(k)} = 1+\sum_{j=1}^{\infty} g^j \tilde{T}^{(j)}_{(k)} \ , \tag{87}$$

with

$$\tilde{T}^{(j)}_{(k)} = \oint_{\gamma_k}\frac{dz}{2\pi i}(z-K^{\Sigma})^{-1}\tilde{P}_{0,(k)}\tilde{V}^{tot}(\cdot,\theta)((z-K^*_0(\theta))^{-1}\tilde{V}^{tot}(\cdot,\theta))^{j-1}\tilde{P}_{0,(k)}(z-K^{\Sigma})^{-1} \ . \tag{88}$$

Similarly,

$$\tilde{M}_{g,(k)} = K^{\Sigma}+\sum_{j=1}^{\infty} g^j \tilde{M}^{(j)}_{(k)} \ , \tag{89}$$

with

$$\tilde{M}^{(j)}_{(k)} = \oint_{\gamma}\frac{dz}{2\pi i}z(z-K^{\Sigma})^{-1}\tilde{P}_{0,(k)}\tilde{V}^{tot}(\cdot,\theta)((z-K^*_0(\theta))^{-1}\tilde{V}^{tot}(\cdot,\theta))^{j-1}\tilde{P}_{0,(k)}(z-K^{\Sigma})^{-1} \ . \tag{90}$$

The odd terms in the above two expansions are zero due to the fact that $\tilde{P}_{0,(k)}$ projects onto the $N=0$ sector. The first non-trivial coefficient in the Taylor series of $\tilde{\Sigma}_{g(k)}$ is

$$\tilde{\Sigma}^{(2)}_{(k)} = \tilde{M}^{(2)}_{(k)} - \frac{1}{2}(\tilde{T}^{(2)}_{(k)}K^{\Sigma}+K^{\Sigma}\tilde{T}^{(2)}_{(k)}) \tag{91}$$

$$= \frac{1}{2}\oint_{\gamma_k}\frac{dz}{2\pi i}(\xi_k(z)(z-K^{\Sigma})^{-1}+(z-K^{\Sigma})^{-1}\xi_k(z)) \ , \tag{92}$$

with

$$\xi_k(z) = \tilde{P}_{0(k)}\tilde{V}^{tot}_g(\cdot,\theta)(z-K^*_0(\theta))^{-1}\tilde{V}^{tot}_g(\cdot,\theta)\tilde{P}_{0(k)}.$$

□

We explicitly compute the discrete spectrum of $K^*_g(\theta)$ to second order in the coupling constant. Let $e_{1,2}\in\mathcal{H}^{\Sigma}$ denote the vectors of spin up and down respectively. Then the states in $\tilde{\mathcal{H}}$ corresponding to the eigenvalues $E_j^{(k)}(g=0), j=0,\cdots,3, k\in \mathbf{Z}$, are

$$\begin{aligned}\phi_k^0 &= e^{ik\omega}\otimes e_1\otimes e_1\otimes \tilde{\Omega}^{\mathcal{R}_1}\otimes\cdots\otimes\tilde{\Omega}^{\mathcal{R}_n},\\ \phi_k^1 &= e^{ik\omega}\otimes e_2\otimes e_2\otimes \tilde{\Omega}^{\mathcal{R}_1}\otimes\cdots\otimes\tilde{\Omega}^{\mathcal{R}_n},\\ \phi_k^2 &= e^{ik\omega}\otimes e_2\otimes e_1\otimes \tilde{\Omega}^{\mathcal{R}_1}\otimes\cdots\otimes\tilde{\Omega}^{\mathcal{R}_n},\\ \phi_k^3 &= e^{ik\omega}\otimes e_1\otimes e_2\otimes \tilde{\Omega}^{\mathcal{R}_1}\otimes\cdots\otimes\tilde{\Omega}^{\mathcal{R}_n},\end{aligned}$$

where $\Omega^{\mathcal{R}_i}$ is the vacuum state in $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R};\mathcal{B}))$.

We apply perturbation theory to calculate $E_j^{(k)}(g)$. We know that

$$\begin{aligned}\tilde{\Sigma}_{(k)}^{(2)} &= \frac{1}{2}\oint_{\gamma_k}\frac{dz}{2\pi i}\{\tilde{P}_{0,(k)}\tilde{V}^{tot}(\cdot,\theta)(z-K_0(\theta))^{-1}\tilde{V}^{tot}(\cdot,\theta)\tilde{P}_{0,(k)}(z-K_{(k)}^{\Sigma})^{-1}\\ &+ (z-K_{(k)}^{\Sigma})\tilde{P}_{0,(k)}\tilde{V}^{tot}(\cdot,\theta)(z-K_0(\theta))^{-1}\tilde{V}^{tot}(\cdot,\theta)\tilde{P}_{0,(k)}\}\ . \qquad (93)\end{aligned}$$

For $f_{\beta,\mu}$ as in (27), we let its Fourier transform be

$$\hat{f}_{\beta,\mu,m}(u,\omega) := \frac{1}{\tau}\int_0^\tau dt e^{-im\omega t} f_{\beta,\mu}(u,t). \qquad (94)$$

Similarly, for $f_{\beta,\mu}^{\#}$ as in (28), we let

$$\hat{f}_{\beta,\mu,m}^{\#}(u,\omega) := \frac{1}{\tau}\int_0^\tau dt e^{-im\omega t} f_{\beta,\mu}^{\#}(u,t). \qquad (95)$$

Consider first the nondegenerate eigenvalue $E_3^{(k)}$. Applying the Cauchy integration formula to (93), and using the facts that

$$\lim_{\epsilon\searrow 0} Re\frac{1}{x-i\epsilon} = \mathcal{PV}\frac{1}{x},\ and\ \lim_{\epsilon\searrow 0} Im\frac{1}{x-i\epsilon} = \pi\delta(x),$$

where $\mathcal{PV}$ denotes the Cauchy principal value, it follows that

$$\begin{aligned}Re \ll \phi_k^3,\tilde{\Sigma}_3^{k(2)}\phi_k^3 \gg &= \sum_{m\in\mathbf{Z}}\sum_{i=1}^n \mathcal{PV}\int_{\mathbf{R}} du\frac{\|\hat{f}_{\beta_i,\mu_i,m}(u,\omega)\|_{\mathcal{B}}^2}{2\omega_0-(k-m)\omega-u}\ ,\\ Im \ll \phi_k^3,\tilde{\Sigma}_3^{k(2)}\phi_k^3 \gg &= -\pi\sum_{m\in\mathbf{Z}}\sum_{i=1}^n \|\hat{f}_{\beta_i,\mu_i,m}(2\omega_0-(k-m)\omega,\omega)\|_{\mathcal{B}}^2\ ,\end{aligned}$$

where $\ll \cdot, \cdot \gg$ is the scalar product on $\tilde{\mathcal{H}}$. Therefore,

$$E_3^{(k)}(g) = k\omega + 2\omega_0 + g^2 \sum_{m\in\mathbf{Z}} \sum_{i=1}^{n} \mathcal{P}V \int_{\mathbf{R}} du \frac{\|\hat{f}_{\beta_i,\mu_i,m}(u,\omega)\|_{\mathcal{B}}^2}{2\omega_0 - (k-m)\omega - u} -$$
$$- i\pi g^2 \sum_{m\in\mathbf{Z}} \sum_{i=1}^{n} \|\hat{f}_{\beta_i,\mu_i,m}(2\omega_0 - (k-m)\omega, \omega)\|_{\mathcal{B}}^2 + \mathcal{O}(g^4).$$

Similarly,

$$E_2^{(k)}(g) = k\omega - 2\omega_0 - g^2 \sum_{m\in\mathbf{Z}} \sum_{i=1}^{n} \mathcal{P}V \int_{\mathbf{R}} du \frac{\|\hat{f}_{\beta_i,\mu_i,m}(u,\omega)\|_{\mathcal{B}}^2}{2\omega_0 - (k-m)\omega - u} -$$
$$- i\pi g^2 \sum_{m\in\mathbf{Z}} \sum_{i=1}^{n} \|\hat{f}_{\beta_i,\mu_i,m}(2\omega_0 - (k-m)\omega, \omega)\|_{\mathcal{B}}^2 + \mathcal{O}(g^4).$$

Next we use degenerate perturbation theory to calculate $E_j^{(k)}(g), j = 0, 1$. Applying the Cauchy integration formula to (93) and using the definitions of $f_{\beta,\mu}$ and $f_{\beta,\mu}^{\#}$, it follows that

$$Re \ll \phi_k^{0,1}, \tilde{\Sigma}_3^{k(2)} \phi_k^{0,1} \gg = -Re \ll \phi_k^{1,0}, \tilde{\Sigma}_3^{k(2)} \phi_k^{0,1} \gg$$
$$= \pm \sum_{m\in\mathbf{Z}} \sum_{i=1}^{n} \mathcal{P}V \int_{\mathbf{R}} du \frac{\|\hat{f}_{\beta_i,\mu_i,m}(u,\omega)\|_{\mathcal{B}}^2}{2\omega_0 - (k-m)\omega - u} ,$$
$$Im \ll \phi_k^{0,1}, \tilde{\Sigma}_3^{k(2)} \phi_k^{0,1} \gg = -Im \ll \phi_k^{1,0}, \tilde{\Sigma}_3^{k(2)} \phi_k^{0,1} \gg$$
$$= -\pi \sum_{m\in\mathbf{Z}} \sum_{i=1}^{n} \|\hat{f}_{\beta_i,\mu_i,m}(2\omega_0 - (k-m)\omega, \omega)\|_{\mathcal{B}}^2.$$

Therefore,

$$E_{0,1}^{(k)}(g) = k\omega + g^2 a_{0,1} + O(g^4) , \tag{96}$$

where $a_{0,1}$ are the eigenvalues of the 2×2 matrix

$$-i\pi \sum_{m\in\mathbf{Z}} \sum_{i=1}^{n} \|\hat{f}_{\beta_i,\mu_i,m}(2\omega_0 - (k-m)\omega, \omega)\|_{\mathcal{B}}^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} . \tag{97}$$

By construction, $K_g e^{ik\omega t}\Omega = k\omega e^{ik\omega t}\Omega$ and $U(\theta)e^{ik\omega t}\Omega = e^{ik\omega t}\Omega$, so $\{k\omega : k \in \mathbf{Z}\}$ are also isolated eigenvalues of $K_g^*(\theta), \theta \in I^-(\delta)$. This can be seen by defining the spectral projections corresponding to the real isolated eigenvalues of

$K_g(\theta)$, using the resolvent, and taking the adjoint to define the corresponding spectral projections for the real isolated eigenvalues of $K_g^*(\theta)$.[8]

The vector $\psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigenvector corresponding the eigenvalue 0 of $\tilde{\Sigma}^2_{g,(k)}$. Hence,

$$E_0^{(k)}(g,t) = k\omega \ , \tag{98}$$

$$E_1^{(k)}(g,t) = k\omega - 2\pi i g^2 \sum_{m\in\mathbf{Z}} \sum_{i=1}^{n} \|\hat{f}_{\beta_i,\mu_i,m}(2\omega_0 - (k-m)\omega)\|^2_{\mathcal{B}} + O(g^4) \ . \tag{99}$$

Note that due to assumption (A3), $Im E_j^{(k)} < 0$, for $j = 1,2,3$, while $Im E_0^{(k)} = 0$.[9] We have proven the following corollary to Theorem 4.3.

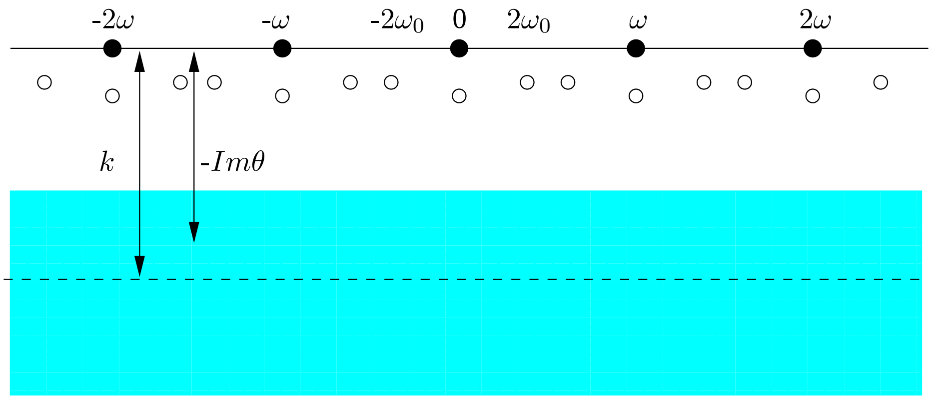

Figure 1. Spectrum of $K_g^*(\theta)$

Corollary 4.4 *Suppose assumptions (A1)-(A3) hold. Then, for* $\theta \in I^-(k)$

[8]Suppose α is an isolated and real eigenvalue of a closed operator A. Then the spectral projection corresponding to α is

$$P = \frac{1}{2\pi i}\oint_{\gamma_\alpha} (z-A)^{-1} dz,$$

where γ_α is a contour enclosing α only. Since α is real and isolated, it is also an eigenvalue of A^* with corresponding projection P^*. (Using Cauchy's integration formula, one may readily verify that $A^*P^* = P^*A^* = \alpha P^*$ and that $(P^*)^2 = P^*$.)

[9]Alternatively, one can use the Feshbach map (see [BFS]) to compute the perturbation of the discrete spectrum of $K_g^*(\theta)$, which gives the same result.

and $|g| < g_1/2$, *where* g_1 *satisfies (79),* $K_g^*(\theta)$ *has countably many eigenvalues* $\{k\omega\}_{k\in\mathbf{Z}}$, *on the real axis, where* $\omega = \frac{2\pi}{\tau}$.

We will use the results of Theorem 4.3 and Corollary 4.4 to prove that the true state of the coupled system converges to a time-periodic state.

5 Convergence to time-periodic states

The following theorem claims that, under suitable assumptions, the true state of the system converges to a time-periodic state with the period τ of the perturbation.

Choose $k = \min(\delta, \frac{\pi}{\beta_1}, \cdots, \frac{\pi}{\beta_n})$ (as in section 4), define $\mathbf{h}^{test} := D(e^{k\sqrt{p^2+1}})$, and let $\mathcal{O}^{test,\mathcal{R}}$ be the $*$-algebra generated by $b^{\#}(f), f \in \mathbf{h}^{test}$, and by $\mathbf{1}^{\mathcal{R}}$. Note that $\mathcal{O}^{test,\mathcal{R}}$ is norm-dense in $\mathcal{O}^{\mathcal{R}}$. We define a $*$-algebra

$$\mathcal{C} := \mathcal{O}^{\Sigma} \otimes \mathcal{O}^{\mathcal{R}_1,test} \otimes \cdots \otimes \mathcal{O}^{\mathcal{R}_n,test} \, , \tag{100}$$

which is dense in $\mathcal{O}$.

We make the following additional assumption.

(A4) The interaction Hamiltonian $V(t)$ belongs to $\mathcal{C}$, for $t \in \mathbf{R}$.

Let

$$D := \mathbf{1}^{\Sigma} \otimes \mathbf{1}^{\Sigma} \otimes e^{-k\sqrt{A^2_{\mathcal{R}_1}+1}} \otimes \cdots \otimes e^{-k\sqrt{A^2_{\mathcal{R}_n}+1}} \, , \tag{101}$$

where $A_{\mathcal{R}_i} = d\Gamma(i\partial_{u_i}), i = 1, \cdots, n$, is the second quantization of the generator of energy translations for the i^{th} reservoir. The operator D is positive such that $D\Omega = \Omega$, and $RanD$ is dense in $\mathcal{H}$. We have the following theorem.

Theorem 5.1 (Convergence to time-periodic states).

Assume assumptions (A1)-(A4) hold. Assume further that $a \in \mathcal{C}$. *Then there is a constant* $g_1 > 0$ *satisfying (79), such that, for* $|g| < g_1/2$, *the following holds*

$$\lim_{n\to\infty} \langle \Omega, \alpha_g^{n\tau+t}(a)\Omega\rangle = \langle \tilde{\Omega}_{g,(0)}, D^{-1}\alpha_g^t(a)\Omega\rangle \, , \tag{102}$$

where $\tilde{\Omega}_{g,(0)}$ *corresponds to the zero-energy resonance of the adjoint of the Floquet Liouvillean,* K_g^*, *and* D *is given by (101).*

Proof. First note that by using a Dyson series expansion, it follows from assumption (A4) and the fact that $a \in \mathcal{C}$ that $\alpha_g^t(a) \in \mathcal{C}$, and hence $\alpha_g^t(a)\Omega \in \mathcal{D}(D^{-1})$.

The remainder of the proof relies on the result of Theorem 4.3, Corollary 4.4, and equation (56) (section 4). It follows from (56) and the time periodicity of $f \in \tilde{\mathcal{H}} = L^2([0,\tau]) \otimes \mathcal{H}$ that

$$(e^{-iK_g^* n\tau}\mathbf{1} \otimes \Omega)(0) = \tilde{U}_g(n\tau, 0)(\mathbf{1} \otimes \Omega)(0) = \tilde{U}_g(n\tau, 0)\Omega \, . \tag{103}$$

Let $\mathbf{1} \otimes \Omega =: \overline{\Omega} \in \tilde{\mathcal{H}}$, and $\overline{D} := \mathbf{1} \otimes D$.

Without loss of generality, we assume that $\omega \equiv \frac{2\pi}{\tau} \neq 2\omega_0$; (if $\frac{2\pi}{\tau} = 2\omega_0$, the state of the system typically oscillates between two resonance states until it finally converges to a time-periodic state; see remark 1). Using the dynamics on $C(\mathcal{O}, \Omega)$ and (103), it follows that

$$\lim_{n\to\infty} \langle \Omega, \alpha_g^{n\tau+t}(a)\Omega\rangle = \lim_{n\to\infty} \langle \tilde{U}_g(n\tau,0)\Omega, \alpha_g^t(a)\Omega\rangle \tag{104}$$

$$= \lim_{n\to\infty} \langle (e^{-iK_g^* n\tau}\overline{\Omega})(0), \alpha_g^t(s)\Omega\rangle \tag{105}$$

Using the regulator $\overline{D}$ and complex spectral translation,

$$\lim_{n\to\infty} \langle \Omega, \alpha_g^{n\tau+t}(a)\Omega\rangle = \lim_{n\to\infty} \langle (\overline{D}U(-\theta)e^{-iK_g^*(\theta)n\tau}U(\theta)\overline{D}\,\overline{\Omega})(0), D^{-1}\alpha_g^t(a)\Omega\rangle \tag{106}$$

$$= \lim_{n\to\infty} \langle (\overline{D}U(-\theta)\int_{-\infty}^{\infty} du (u + i\eta - K_g^*(\theta))^{-1} e^{-i(u+i\eta)n\tau}$$
$$\overline{\Omega})(0), D^{-1}\alpha_g^t(a)\Omega\rangle. \tag{107}$$

We split the above integration into two terms,

$$\lim_{n\to\infty} \langle (\overline{D}U(-\theta)\int_{-\infty}^{\infty} du (u + i\eta - K_g^*(\theta))^{-1} e^{-i(u+i\eta)n\tau}$$
$$\overline{\Omega})(0), D^{-1}\alpha_g^t(a)\Omega\rangle \tag{108}$$
$$= \lim_{n\to\infty} \langle (\overline{D}U(-\theta)\oint_\gamma dz (z - K_g^*(\theta))^{-1} e^{-izn\tau}\overline{\Omega})(0), D^{-1}\alpha_g^t(a)\Omega\rangle +$$
$$+ \lim_{n\to\infty} \langle (\overline{D}U(-\theta)\int_{-\infty}^{\infty} du (u + i(\mu-\epsilon) - K_g^*(\theta))^{-1} e^{-i(u+i(\mu-\epsilon))n\tau}\overline{\Omega})(0),$$
$$D^{-1}\alpha_g^t(a)\Omega\rangle, \tag{109}$$

where $\eta > 0, 0 < \epsilon < \mu$, and γ is the contour enclosing the point spectrum of $K_g^*(\theta)$ only.

Using the results of Theorem 4.3 and Corollary 4.4, the first term converges to a time-periodic expression,

$$\lim_{n\to\infty} \langle (\overline{D}U(-\theta) \oint_\gamma dz (z - K_g^*(\theta))^{-1} e^{-izn\tau} \overline{\Omega})(0), D^{-1}\alpha_g^t(a)\Omega\rangle \tag{110}$$

$$= \sum_{k\in\mathbf{Z}} \lim_{n\to\infty} \langle (\overline{D}U(-\theta)\tilde{S}_{g,(k)}^{-1}(\theta) e^{-i\tilde{\Sigma}_{g,(k)}(\theta)n\tau} \tilde{S}_{g,(k)}(\theta)\overline{\Omega})(0), D^{-1}\alpha_g^t(a)\Omega\rangle \tag{111}$$

$$= \sum_{k\in\mathbf{Z}} \langle (\overline{D}\tilde{P}_{g,(k)}(\theta)\overline{\Omega})(0), \alpha_g^t(a)\Omega\rangle \tag{112}$$

Let

$$\tilde{\Omega}_{g,(k)} := \overline{D}\tilde{P}_{g,(k)}(\theta)(e^{ik\omega t} \otimes \Omega) , \tag{113}$$

and denote by $\ll \cdot,\cdot \gg$ the scalar product on $\tilde{\mathcal{H}}$. Then

$$\overline{D}\tilde{P}_{g,(k)}(\theta)\overline{D} = \ll (e^{ik\omega t} \otimes \Omega), \cdot \gg \tilde{\Omega}_{g,(k)} . \tag{114}$$

Therefore,

$$\overline{D}\tilde{P}_{g,(k)}(\theta)\overline{\Omega} = \ll e^{ik\omega t} \otimes \Omega, \mathbf{1} \otimes \Omega \gg \tilde{\Omega}_{g,(k)} \tag{115}$$

$$= \tilde{\Omega}_{g,(0)}\delta_{k,0} , \tag{116}$$

where $\delta_{k,0}$ is the Kronecker delta, and hence

$$\sum_{k\in\mathbf{Z}} \langle (\overline{D}\tilde{P}_{g,(k)}\overline{\Omega})(0), D^{-1}\alpha_g^t(a)\Omega\rangle = \langle (\tilde{\Omega}_{g,(0)})(0), D^{-1}\alpha_g^t(a)\Omega\rangle , \tag{117}$$

where $\tilde{\Omega}_{g,(0)}$ is the zero-energy resonance of the Floquet Liouvillean. The second term in (109) converges exponentially fast to zero since

$$\langle (\overline{D}U(-\theta) \int_{-\infty}^{\infty} du (u + i(\mu-\epsilon) - K_g^*(\theta))^{-1} e^{-i(u+i(\mu-\epsilon))n\tau} \overline{\Omega})(0), D^{-1}\alpha_g^t(a)\Omega\rangle = O(e^{-(\mu-\epsilon')n\tau}) , \tag{118}$$

where $0 < \epsilon' < \epsilon < \mu$; (see [Rud], chapter 19). □

Remarks.

(1) When $\omega = \frac{2\pi}{\tau} = 2\omega_0$, the system exhibits the phenomenon of resonance: The state of the system oscillates between two resonances until it finally converges to the time periodic state corresponding to $\tilde{\Omega}_{g(0)}$. This can be verified by a second order time-dependent perturbation theory calculation; (see also [Ya2]).

(2) By a standard argument, the result of Theorem 5.1 can be extended to any initial state normal to ω (see for example [JP3]).

(3) Note that

$$\langle (\tilde{\Omega}_{g,(0)})(0), D^{-1}\alpha_g^t(a)\Omega\rangle = \langle (\tilde{\Omega}_{g,(k)})(0), D^{-1}\alpha_g^t(a)e^{ik\omega t}\Omega\rangle \ ,$$

where $\tilde{\Omega}_{g,(k)}$ is the state corresponding to the $k\omega$-energy resonance of the adjoint of the Floquet Liouvillean. In other words, all $k\omega$-energy resonances belong to the same class of time-periodic states.

In the next section, we discuss the positivity of entropy production per cycle and Carnot's formulation of the second law of thermodynamics.

6 Positivity of entropy production

We consider a small system coupled to two fermionic reservoirs at the same chemical potential μ, yet at two different temperatures β_1 and β_2, with $\beta_1 < \beta_2$. Together with assumptions (A1)-(A4), we assume that the perturbation is differentiable in t, for $t > 0$. The first reservoir acts as a heat source, and the second reservoir as a heat sink. We want to show that, after the true state of the system has converged to a time-periodic state, the entropy production per cycle is *strictly* positive. We first prove that the time-periodic state, which the true state of the system converges to, is not normal to the initial state. [10]

We introduce the *standard* Floquet Liouvillean,

$$\tilde{K}_g := -i\partial_t + \mathcal{L}_g(t) \ , \tag{119}$$

[10]In this section, we will use standard results about von Neumann algebras and Tomita-Takesaki modular theory. We refer the reader to [BR], chapters 2.4 and 2.5 for an exposition of these results.

acting on the extended Hilbert space $L^2([0,\tau])\otimes\mathcal{H}$, with periodic boundary conditions in t, where $\mathcal{L}_g(t)=\mathcal{L}_0+gV(t)-gJV(t)J$, is the standard Liouvillean. We study the spectrum of the standard Floquet Liouvillean using complex spectral translations. Since the proof of the following proposition is very similar to the analysis in section 4, we only sketch the main steps of the proof.

Proposition 6.1

Suppose assumptions (A1)-(A3) (section 3) hold. Then there exists a positive constant g_2, such that, for $|g|<g_2$, the spectrum of $\tilde{K}_g$ is absolutely continuous and

$$\sigma(\tilde{K}_g)=\sigma_{ac}(\tilde{K}_g)=\mathbf{R}.$$

Sketch of proof.

Let $U(\theta)$ as in section 4. We define the complex deformed standard Floquet Liouvillean by

$$\begin{aligned}\tilde{K}_g(t,\theta) &:= U(\theta)\tilde{K}_gU(-\theta) && (120)\\ &= -i\partial_t+\mathcal{L}_g(t,\theta)\ , && (121)\end{aligned}$$

where

$$\mathcal{L}_g(t,\theta)=\mathcal{L}_0+\theta N+gV^{tot}(t,\theta)\ ,$$

and

$$\begin{aligned}V^{tot}(t,\theta) &= U(\theta)(V(t)-JV(t)J)U(-\theta)\\ &=\sum_{i=1}^n\{\sigma_-\otimes\mathbf{1}^\Sigma\otimes a^*(f^{(\theta)}_{i,\beta_i,\mu}(t))+\sigma_+\otimes\mathbf{1}^\Sigma\otimes a(f^{(\theta)}_{i,\beta_i,\mu}(t))\\ &-i\mathbf{1}^\Sigma\otimes\sigma_-\otimes(-1)^{N_i}a^*(f^{\#(\theta)}_{i,\beta_i,\mu}(t))-i\mathbf{1}^\Sigma\otimes\sigma_+\otimes(-1)^{N_i}a(f^{\#(\theta)}_{i,\beta_i,\mu}(t))\}\ .\end{aligned}$$

It follows from assumption (A2) that $V^{tot}(t,\theta)$ is bounded for $\theta\in I(\delta)$. Let

$$\tilde{C}:=\sup_{t\in\mathbf{R},\theta\in I(\delta)}\|V^{tot}(t,\theta)\|. \qquad (122)$$

For $\theta\in I^-(k)$, choose ν such that $0>\nu>-k$ and $-k<Im\theta<-(k+|\nu|)/2$. Choose $g_2>0$ such that

$$g_2\tilde{C}<(k+\nu)/4. \qquad (123)$$

Then, using an argument which is similar to the proof of Theorem 4.3, one can show that, for $|g| < g_2$, the essential spectrum of $\tilde{K}_g(\theta)$ is contained in the half-plane $\{z \in \mathbf{C} : Imz < \nu\}$, and that it discrete spectrum

$$\sigma_{pp}(\tilde{K}_g) = \{\tilde{E}_j^{(k)}(g) : k \in \mathbf{Z}, j = 0, \cdots, 3\},$$

where (to second order in perturbation theory)

$$\tilde{E}_{2,3}^{(k)}(g) = k\omega \mp 2\omega_0 \mp g^2 \sum_{m\in\mathbf{Z}} \sum_{i=1}^{2} \mathcal{P}V \int_{\mathbf{R}} du \frac{\|\hat{f}_{\beta_i,\mu_i,m}(u,\omega)\|_{\mathcal{B}}^2}{2\omega_0 - (k-m)\omega - u} -$$

$$- i\pi g^2 \sum_{m\in\mathbf{Z}} \sum_{i=1}^{2} \|\hat{f}_{\beta_i,\mu_i,m}(2\omega_0 - (k-m)\omega, \omega)\|_{\mathcal{B}}^2 + \mathcal{O}(g^4),$$

while

$$\tilde{E}_{0,1}^{(k)}(g) = k\omega + g^2 a_{0,1} + O(g^4) \ , \tag{124}$$

where $a_{0,1}$ are the eigenvalues of the 2×2 matrix

$$-i\pi \sum_{m\in\mathbf{Z}} \sum_{i=1}^{2} \|\hat{f}_{\beta_i,\mu_i,m}(2\omega_0-(k-m)\omega,\omega)\|_{\mathcal{B}}^2 \begin{pmatrix} 1 & -e^{-\beta_i(2\omega_0-(k-m)\omega-\mu)/2} \\ -e^{\beta_i(2\omega_0-(k-m)\omega-\mu)/2} & 1 \end{pmatrix} . \tag{125}$$

Note that, to second order in the coupling g, the discrete spectrum is below the real axis. The claim of the theorem follows by noting that

$$s - \lim_{Im\theta\uparrow 0} (\tilde{K}_g(\theta) - z)^{-1} = (\tilde{K}_g(Re\theta) - z)^{-1}$$

for large enough Imz. $\square$

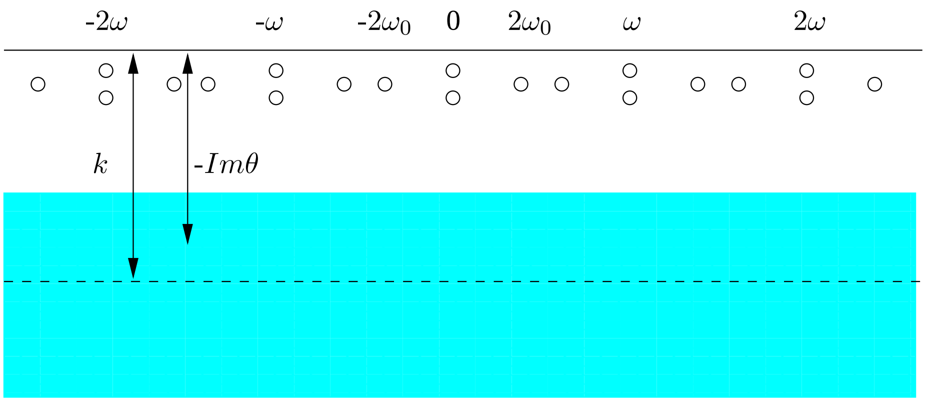

Figure 2. Spectrum of $\tilde{K}_g(\theta)$

This result is sufficient to show that the time periodic state ω_g^+ is not normal to the initial state ω. Let $g_3 := \min\{g_1/2, g_2\}$, where g_1 satisfies (79) (section 4) and g_2 satisfies (123).

Theorem 6.2

Suppose assumptions (A1)-(A3) (section 3) and (A4) (section 5) hold. Then, for $|g| < g_3$, the time-periodic state $\omega_{g,s}^+$, for $s \in [0, \tau]$, *does not belong to $\mathcal{N}_\omega$.*

Proof.

First note that under the assumptions of this theorem, the results of Proposition 6.1 hold. In particular, $\tilde{K}_g$ has no real eigenvalues.

By construction,

$$\omega_{g,s}^+ \circ \alpha_g^{\pm\tau} = \omega_{g,s}^+. \tag{126}$$

Since we are assuming that $\omega_{g,s}^+ \in \mathcal{N}_\omega$, there exists a unique vector $\Omega_{g,s}^+$ in the natural positive cone $\mathcal{P} := \{aJa\Omega : a \in \mathcal{M}\}$ associated to the pair $(\mathcal{M}, \omega)$, such that

$$\omega_{g,s}^+(a) = \langle \Omega_{g,s}^+, a\Omega_{g,s}^+ \rangle \ , \tag{127}$$

for $a \in \mathcal{M}$.[11] Now, (44), (126), and (127) imply that

$$\langle \overline{U}_g(\tau,0)\Omega_{g,s}^+, a\overline{U}_g(\tau,0)\Omega_{g,s}^+ \rangle = \langle \Omega_{g,s}^+, a\Omega_{g,s}^+ \rangle, \tag{128}$$

for $a \in \mathcal{M}$. Using the facts that

$$[J, \overline{U}_g(t,t')] = 0 \ , \tag{129}$$

where J is the modular conjugation and $\overline{U}_g$ is the propagator generated by the standard Liouvillean, $J\Omega_{g,s}^+ = \Omega_{g,s}^+$ since $\Omega_{g,s}^+ \in \mathcal{P}$, $J^* = J$, and $J\mathcal{M}J = \mathcal{M}'$, it follows that (128) also holds for $a \in \mathcal{M}'$. Furthermore, we know that[12]

$$\overline{U}_g \mathcal{P} \subset \mathcal{P},$$

and that the linear span of $\mathcal{P}$ is dense in $\mathcal{H}$. Let $\mathsf{P}_{g,s}^+$ be the orthogonal projection onto $\Omega_{g,s}^+$, then $\mathsf{P}_{g,s}^+ \in \mathcal{M} \vee \mathcal{M}'$. Moreover, since $\pi = \pi^\Sigma \otimes \pi^{\mathcal{R}_1} \otimes$

[11] For a proof of this statement, see [BR], Theorems 2.5.31 and 2.3.19.
[12] see Corollary 2.5.32 in [BR].

$\cdots \otimes \pi^{\mathcal{R}_n}$, in the Araki-Wyss representation, $\mathcal{M}$ is a factor (of type III_1; see [ArWy]), ie, $\mathcal{M} \cap \mathcal{M}' = \{\mathbf{C1}\}$.[13] Suppose that

$$\overline{U}_g(\tau, 0)\Omega^+_{g,s} = c_1 \Omega^+_{g,s} + c_2 \Psi,$$

where $\Psi \in Ran(1 - \mathsf{P}^+_{g,s})$, and $c_{1,2}$ are complex numbers to be determined. Choosing $a = (1 - \mathsf{P}^+_{g,s})$ in (128) gives

$$|c_2|^2 = 0.$$

Together with the fact that

$$\langle \overline{U}_g(\tau, 0)\Omega^+_{g,s}, \overline{U}_g(\tau, 0)\Omega^+_{g,s}\rangle = \langle \Omega^+_{g,s}, \Omega^+_{g,s}\rangle = 1,$$

it follows that $|c_1|^2 = 1$. This implies that there exists $\lambda \in \mathbf{R}$ such that

$$\overline{U}_g(\tau, 0)\Omega^+_{g,s} = e^{-i\lambda\tau}\Omega^+_{g,s}. \tag{130}$$

For each fixed $s \in [0, \tau]$, we define

$$\phi_s(t) := e^{i\lambda t}\overline{U}_g(t, 0)\Omega^+_{g,s}, \tag{131}$$

then ϕ_s is an eigenfunction of $\tilde{K}_g$ with eigenvalue λ. (This can be checked by looking at $\tilde{K}_g \phi_s$.) However, this is in *contradiction* with the result of Proposition 6.1, and hence $\omega^+_{g,s} \notin \mathcal{N}_\omega$. $\square$

It also follows from assumptions (A1)-(A4) and Theorem 5.1, that

$$\sup_{T \in \mathbf{R}^+} \left| \int_0^T dt \{\omega^+_{g,t\ mod\ \tau}(\delta_\omega(gV(t))) - \omega \circ \alpha^t_g(\delta_\omega(gV(t)))\} \right| < \infty.$$

Therefore, $\omega^+_{g,s}$ satisfies the assumptions of Proposition 2.1, and the entropy production rate after the true state of the coupled system has converged to a time-periodic state is strictly positive.

Regarding the explicit computation of entropy production per cycle,

$$\Delta Ent := \int_0^\tau dt \omega^+_{g,t}(\delta_\omega(gV(t))) > 0, \tag{132}$$

we would like to make the following remark. Since it follows from Theorem 4.3 that $\tilde{\Omega}_{g,(0)}$ is analytic in g for $g < g_1/2$, one can expand $\omega^+_{g,s}$ to any order in the coupling g, and compute an explicit expression for ΔEnt given in (132) up to this order in the coupling constant.

[13] Using the isomorphism between $\mathcal{F}^{\mathcal{R}_1}(L^2) \otimes \cdots \otimes \mathcal{F}^{\mathcal{R}_n}(L^2)$ and $\mathcal{F}(L^2_{\mathcal{R}_1} \oplus \cdots \oplus L^2_{\mathcal{R}_n})$, one can proceed to show that $\mathcal{M}$ is a factor as in the case of the Araki-Wyss representation for a single reservoir of free fermions.

7 Appendix A

Glued Hilbert space representation

We want to show that

$$\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))\otimes\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))\cong\mathcal{F}(L^2(\mathbf{R};\mathcal{B}))\ .$$

Let Ω be the vacuum state in the fermionic Fock space $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$. For fermionic creation/annihilation operators on $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$,

$$b^{\#}(f):=\int m(u)dud\sigma f(u,\sigma)b^{\#}(u,\sigma)\ ,f\in L^2(\mathbf{R}^+;\mathcal{B}),$$

define the creation/annihilation operators on $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))\otimes\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$ as

$$\begin{aligned} b_l^{\#}(f) &:= b^{\#}(f)\otimes\mathbf{1}\ ;\\ b_r^{\#}(f) &:= (-1)^N\otimes b^{\#}(\overline{f})\ , \end{aligned}$$

where $\bar{\cdot}$ corresponds to complex conjugation. Note that b_l and b_r anti-commute. Let $\tilde{a}$ and $\tilde{a}^*$ be the annihilation and creation operators on the fermionic Fock space $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B}))$, such that they satisfy the usual CAR, and let $\tilde{\Omega}$ be the vacuum state in $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B}))$. An isomorphism between $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))\otimes\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$ and $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B}))$ follows by the identification

$$\begin{aligned} b_l^{\#}(f) &\cong \tilde{a}^{\#}((f,0)),\\ b_r^{\#}(g) &\cong \tilde{a}^{\#}((0,g))\ ,\\ \Omega\otimes\Omega &\cong \tilde{\Omega}. \end{aligned}$$

Now we claim that $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B}))$ is isomorphic to $\mathcal{F}(L^2(\mathbf{R},du;\mathcal{B}))$. For $\phi,\psi\in\mathbf{R}$, consider the mapping

$$j_{\phi,\psi}:L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B})\ni(f,g)\to h\in L^2(\mathbf{R}du;\mathcal{B})\ ,$$

such that

$$h(u,\sigma):=\begin{cases} e^{i\phi}\sqrt{m(u)}f(u,\sigma)\ ,u\geq 0\\ e^{i\psi}\sqrt{m(|u|)}g(|u|,\sigma)\ ,u<0 \end{cases}\ .$$

This mapping is an isometry, since

$$\begin{aligned}
\|h\|^2_{L^2(\mathbf{R},du;\mathcal{B})} &= \|(f,g)\|^2_{L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B})} \\
&= \int_{\mathbf{R}^+;\mathcal{B}} dud\sigma m(u)|f(u,\sigma)|^2 + \int_{\mathbf{R}^+;\mathcal{B}} dud\sigma m(u)|g(u,\sigma)|^2 \\
&= \|f\|^2_{L^2(\mathbf{R}^+;\mathcal{B})} + \|g\|^2_{L^2(\mathbf{R}^+;\mathcal{B})} \ .
\end{aligned}$$

Moreover, the mapping $j_{\phi,\psi}$ is an isomorphism, since, for given $h \in L^2(\mathbf{R};\mathcal{B})$, there exists a mapping $j^{-1}_{\phi,\psi} : h \to (f,g) \in L^2(\mathbf{R}^+;\mathcal{B}) \oplus L^2(\mathbf{R}^+;\mathcal{B})$, such that

$$\begin{aligned}
f(u,\sigma) &:= \frac{e^{-i\phi}}{\sqrt{m(u)}} h(u,\sigma), u > 0 \ , \\
g(u,\sigma) &:= \frac{e^{-i\psi}}{\sqrt{m(|u|)}} h(|u|,\sigma), u < 0 \ .
\end{aligned}$$

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